DOI:
10.1112/plms.12206

Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

When does the norm of a Fourier multiplier dominate its L^∞ norm?

Alexei Karlovich and Eugene Shargorodsky

Abstract

One can define Fourier multipliers on a Banach function space by using the direct and inverse Fourier transforms on $L^2(\mathbb{R}^n)$ or by using the direct Fourier transform on $S(\mathbb{R}^n)$ and the inverse one on $S'(\mathbb{R}^n)$. In the former case, one assumes that the Fourier multipliers belong to $L^\infty(\mathbb{R}^n)$, while in the latter one this requirement may or may not be included in the definition. We provide sufficient conditions for those definitions to coincide as well as examples when they differ. In particular, we prove that if a Banach function space $X(\mathbb{R}^n)$ satisfies a certain weak doubling property, then the space of all Fourier multipliers $\mathcal{M}_{X(\mathbb{R}^n)}$ is continuously embedded into $L^\infty(\mathbb{R}^n)$ with the best possible embedding constant one. For weighted Lebesgue spaces $L^p(\mathbb{R}^n, w)$, the weak doubling property is much weaker than the requirement that w is a Muckenhoupt weight, and our result implies that $\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{L^p(\mathbb{R}^n,w)}}$ for such weights. This inequality extends the inequality for $n = 1$ from [**3**, Theorem 2.3], where it is attributed to J. Bourgain. We show that although the weak doubling property is not necessary, it is quite sharp. It allows the weight w in $L^p(\mathbb{R}^n, w)$ to grow at any subexponential rate. On the other hand, the space $L^p(\mathbb{R}, e^x)$ has plenty of unbounded Fourier multipliers.

1. *Introduction*

Let $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ denote the Schwartz spaces of rapidly decreasing functions and of tempered distributions on $\mathbb{R}^n$, respectively. The action of a distribution $a \in S'(\mathbb{R}^n)$ on a function $u \in S(\mathbb{R}^n)$ is denoted by $\langle a, u\rangle := a(u)$. A Fourier multiplier on $\mathbb{R}^n$ with symbol $a \in S'(\mathbb{R}^n)$ is defined as the operator $u \mapsto F^{-1}aFu$, where

$$(Fu)(\xi) := \widehat{u}(\xi) := \int_{\mathbb{R}^n} u(x)e^{-ix\xi}\,dx$$

is the Fourier transform of $u \in S(\mathbb{R}^n)$, F^{-1} denotes the inverse Fourier transform, and $x\xi$ denotes the scalar product of $x, \xi \in \mathbb{R}^n$. We observe that since $u \in S(\mathbb{R}^n)$ and $a \in S'(\mathbb{R}^n)$, the function Fu belongs to the space $S(\mathbb{R}^n)$ and aFu is a tempered distribution. Thus $F^{-1}aFu$ is well defined and it belongs to $S'(\mathbb{R}^n)$. In fact, we have $F^{-1}aFu = (F^{-1}a) * u$, and therefore, $F^{-1}aFu \in C^\infty_{\rm poly}(\mathbb{R}^n)$ (see, e.g., [**18**, Theorem 2.3.20] or [**29**, Theorem 7.19(b)]). Here and in what follows $C^\infty_{\rm poly}(\mathbb{R}^n)$ denotes the set of all smooth polynomially bounded functions, i.e., the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for every $\alpha \in \mathbb{Z}^n_+$ there exist $m_\alpha \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and $C_\alpha > 0$ satisfying $|\partial^\alpha_x f(x)| \le C_\alpha(1 + |x|)^{m_\alpha}$ for all $x \in \mathbb{R}^n$. Thus, if $u \in S(\mathbb{R}^n)$ and $a \in S'(\mathbb{R}^n)$, then $F^{-1}aFu$ is a regular tempered distribution, whose action on $v \in S(\mathbb{R}^n)$ is evaluated as follows:

$$\langle F^{-1}aFu, v\rangle = \int_{\mathbb{R}^n} (F^{-1}aFu)(x)v(x)\,dx \quad \text{for all} \quad v \in S(\mathbb{R}^n).$$

2000 *Mathematics Subject Classification* 42B15, 46E30.

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

Let $C_0^\infty(\mathbb{R}^n)$ denote the space of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports and let $\mathcal{D}'(\mathbb{R}^n)$ be the space of distributions, that is, the dual space of $C_0^\infty(\mathbb{R}^n)$. Suppose $X(\mathbb{R}^n)$ is a Banach space continuously embedded into the space of distributions $\mathcal{D}'(\mathbb{R}^n)$. We say that a distribution $a \in S'(\mathbb{R}^n)$ belongs to the set $\mathcal{M}_{X(\mathbb{R}^n)}$ of Fourier multipliers on $X(\mathbb{R}^n)$ if

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} := \sup\left\{ \frac{\|F^{-1}aFu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} : u \in (S(\mathbb{R}^n) \cap X(\mathbb{R}^n)) \setminus \{0\} \right\} < \infty.$$

Many authors adopt the following alternative definition of Fourier multipliers (see, e.g., [**5**, p. 368], [**6**, p. 323], [**12**, p. 28], [**15**, p. 7], [**28**, p. 199]). A function $a \in L^\infty(\mathbb{R}^n)$ is said to belong to the set $\mathcal{M}^0_{X(\mathbb{R}^n)}$ of Fourier multipliers on $X(\mathbb{R}^n)$ if

$$\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} := \sup\left\{ \frac{\|F^{-1}aFu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} : u \in \left(L^2(\mathbb{R}^n) \cap X(\mathbb{R}^n)\right) \setminus \{0\} \right\} < \infty.$$

Here $F^{\pm 1}$ are understood as mappings on $L^2(\mathbb{R}^n)$. Since $S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, it is clear that

$$\mathcal{M}^0_{X(\mathbb{R}^n)} \subseteq \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n) \subseteq \mathcal{M}_{X(\mathbb{R}^n)} \tag{1.1}$$

and

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \le \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}}. \tag{1.2}$$

We feel that insufficient attention has been paid so far to the relationship between the above classes of Fourier multipliers. In this paper, we confine ourselves to the Fourier multipliers acting on so-called Banach function spaces, which are defined below, and provide sufficient conditions for equalities to hold in (1.1) (see Theorem 6.1, Subsection 2.2, and Theorem 1.3) as well as examples when they do not hold (see Theorem 6.2). We pay particular attention to the question of existence of a constant D_X such that $\|a\|_{L^\infty(\mathbb{R}^n)} \le D_X \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$.

Our initial motivation came from the following result that appeared in the paper by E. Berkson and T. A. Gillespie [**3**], where it was attributed to J. Bourgain. A measurable function $w : \mathbb{R}^n \to [0,\infty]$ is referred to as a weight if $0 < w(x) < \infty$ a.e. on $\mathbb{R}^n$. The weighted Lebesgue space $L^p(\mathbb{R}^n, w)$, $1 \le p \le \infty$, is the set of all measurable complex-valued functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^p(\mathbb{R}^n,w)} := \|fw\|_{L^p(\mathbb{R}^n)} < \infty.$$

Recall that a weight $w : \mathbb{R}^n \to [0,\infty]$ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w^p(x)\,dx\right)^{1/p} \left(\frac{1}{|Q|}\int_Q w^{-p'}(x)\,dx\right)^{1/p'} < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

THEOREM 1.1 ([**3**, Theorem 2.3]). *Suppose that $1 < p < \infty$ and $w \in A_p(\mathbb{R})$. Then there exists a constant $D_{p,w} > 0$ depending on p and w such that for all $a \in \mathcal{M}_{L^p(\mathbb{R},w)} \cap L^\infty(\mathbb{R})$,*

$$\|a\|_{L^\infty(\mathbb{R})} \le D_{p,w} \|a\|_{\mathcal{M}_{L^p(\mathbb{R},w)}}.$$

The proof of Theorem 1.1 relies on the deep result on a.e. convergence of Fourier integrals, that is, the transplanted version of the celebrated Carleson theorem on the a.e. convergence of Fourier series (see, e.g., [**19**, Theorem 6.1.1] or [**24**]). Theorem 1.1 was extended by the first author [**21**, Theorem 1] to the case of weighted Banach function spaces $X(\mathbb{R}, w)$, in which the Cauchy singular integral operator (the Hilbert transform) is bounded.

In this paper, we provide a more elementary proof that the estimate

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$$

holds with the (optimal) constant equal to 1 for a large class of Banach function spaces $X(\mathbb{R}^n)$ and arbitrary $n \ge 1$. In particular, it holds for all weighted Lebesgue spaces $L^p(\mathbb{R}^n, w)$ with $1 < p < \infty$ and Muckenhoupt weights $w \in A_p(\mathbb{R}^n)$.

We need several definitions to state our main result. The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}^n$ is denoted by $\mathfrak{M}(\mathbb{R}^n)$. Let $\mathfrak{M}^+(\mathbb{R}^n)$ be the subset of functions in $\mathfrak{M}(\mathbb{R}^n)$ whose values lie in $[0,\infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}^n$ is denoted by χ_E and the Lebesgue measure of E is denoted by $|E|$.

Following [**1**, Chap. 1, Definition 1.1], a mapping $\rho : \mathfrak{M}^+(\mathbb{R}^n) \to [0,\infty]$ is called a Banach function norm if, for all functions f, g, f_j $(j \in \mathbb{N})$ in $\mathfrak{M}^+(\mathbb{R}^n)$, for all constants $a \ge 0$, and for all measurable subsets E of $\mathbb{R}^n$, the following properties hold:

$$\begin{array}{ll}
\text{(A1)} & \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \quad \rho(af) = a\rho(f), \quad \rho(f+g) \le \rho(f) + \rho(g), \\
\text{(A2)} & 0 \le g \le f \text{ a.e.} \ \Rightarrow\ \rho(g) \le \rho(f) \quad \text{(the lattice property)}, \\
\text{(A3)} & 0 \le f_j \uparrow f \text{ a.e.} \ \Rightarrow\ \rho(f_j) \uparrow \rho(f) \quad \text{(the Fatou property)}, \\
\text{(A4)} & |E| < \infty \Rightarrow \rho(\chi_E) < \infty, \\
\text{(A5)} & |E| < \infty \Rightarrow \displaystyle\int_E f(x)\,dx \le C_E \rho(f)
\end{array}$$

with $C_E \in (0,\infty)$ that may depend on E and ρ but is independent of f.

When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R}^n)$ of all functions $f \in \mathfrak{M}(\mathbb{R}^n)$ for which $\rho(|f|) < \infty$ becomes a Banach space under the norm

$$\|f\|_{X(\mathbb{R}^n)} := \rho(|f|)$$

and under the natural linear space operations (see [**1**, Chap. 1, Theorems 1.4 and 1.6]). It is called a *Banach function space*.

If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{R}^n)$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{R}^n} f(x)g(x)\,dx \ :\ f \in \mathfrak{M}^+(\mathbb{R}^n),\ \rho(f) \le 1 \right\}.$$

It is a Banach function norm itself [**1**, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{R}^n)$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{R}^n)$. The Lebesgue space $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, is the archetypical example of Banach function spaces. Other classical examples of Banach function spaces are Orlicz spaces, rearrangement-invariant spaces, and variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

Let $X(\mathbb{R}^n)$ be a Banach function space. We say that $f \in X_{\rm loc}(\mathbb{R}^n)$ if $f\chi_E \in X(\mathbb{R}^n)$ for every measurable set $E \subset \mathbb{R}^n$ of finite measure. If $w : \mathbb{R}^n \to [0,\infty]$ is a weight satisfying $w \in X_{\rm loc}(\mathbb{R}^n)$ and $1/w \in X'_{\rm loc}(\mathbb{R}^n)$, then

$$X(\mathbb{R}^n, w) := \{ f \in \mathfrak{M}(\mathbb{R}^n) \ :\ fw \in X(\mathbb{R}^n) \}$$

becomes a Banach function space when it is equipped with the norm

$$\|f\|_{X(\mathbb{R}^n,w)} := \|fw\|_{X(\mathbb{R}^n)},$$

and $[X(\mathbb{R}^n, w)]' = X'(\mathbb{R}^n, w^{-1})$ (see [**22**, Lemma 2.4]). It is clear that if $w \in A_p(\mathbb{R}^n)$, then $w \in L^p_{\rm loc}(\mathbb{R}^n)$ and $1/w \in L^{p'}_{\rm loc}(\mathbb{R}^n)$, whence $L^p(\mathbb{R}^n, w)$ is a Banach function space.

For $y \in \mathbb{R}^n$ and $R > 0$, let $B(y, R) := \{x \in \mathbb{R}^n : |x - y| < R\}$ be the open ball of radius R centered at y.

Definition 1.2. *We say that a Banach function space $X(\mathbb{R}^n)$ satisfies the weak doubling property if there exists a number $\tau > 1$ such that*

$$\liminf_{R\to\infty}\left(\inf_{y\in\mathbb{R}^n}\frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}}\right)<\infty.$$

Theorem 1.3 (Main result). *Let $n \ge 1$ and $X(\mathbb{R}^n)$ be a Banach function space satisfying the weak doubling property. If $a \in \mathcal{M}_{X(\mathbb{R}^n)} \subset S'(\mathbb{R}^n)$, then $a \in L^\infty(\mathbb{R}^n)$ and*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}. \tag{1.3}$$

The constant 1 on the right-hand side of (1.3) is best possible.

The paper is organized as follows. Section 2 contains auxiliary results. For a Banach function space, we introduce the bounded L^2-approximation property and the norm fundamental property, study relations between them, and give examples of Banach function spaces, which do not satisfy these properties. Further, we prove a variant of a well-known lemma on approximation at Lebesgue points, which is an important ingredient in the proof of our main result.

The weak doubling property is discussed in Section 3. In particular, we prove that a Muckenhoupt-type condition A_X implies the weak doubling property and show that weighted Banach function spaces $X(\mathbb{R}, w_j)$, built upon a translation-invariant Banach function space $X(\mathbb{R})$ and exponential weights $w_1(x) = e^{cx}$ and $w_2(x) = e^{c|x|}$ with $c > 0$, fail to have the weak doubling property. On the other hand, we also show that $Y(\mathbb{R}^n, w)$ satisfies the weak doubling condition for weights w that can grow at any subexponential rate.

Section 4 contains the proof of our main result. We divide it into two parts. The first part of the proof is developed for Fourier multipliers belonging to a weighted Lebesgue space $L_{1,\sigma}(\mathbb{R}^n)$. Our arguments at this step are similar to those used in the proof of [**3**, Theorem 2.3] with the important difference that we substitute the application of the theorem on a.e. convergence of Fourier integrals by a simpler lemma on approximation at Lebesgue points proved in Section 2. Further, we approximate an arbitrary Fourier multiplier $a \in \mathcal{M}_{X(\mathbb{R}^n)}$ by $a * \psi_\varepsilon \in C^\infty_{\rm poly}(\mathbb{R}^n)$ with suitably chosen functions $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$. Note that $C^\infty_{\rm poly}(\mathbb{R}^n)$ is contained in $L_{1,\sigma}(\mathbb{R}^n)$ for some $\sigma \in \mathbb{R}$, which allows us to complete the proof of Theorem 1.3. We conclude this section the proof of a multi-dimensional analogue of Theorem 1.1.

In Section 5, we discuss the optimality of the requirement of the weak doubling property in Theorem 1.3. In particular, we show that for an arbitrary translation-invariant Banach function space $Y(\mathbb{R})$ and the weight $w_1(x) = e^{cx}$ with any $c > 0$, the weighted Banach function space $Y(\mathbb{R}, w_1)$ admits many unbounded Fourier multipliers.

In Section 6, we discuss the classes of Fourier multipliers $\mathcal{M}^0_{X(\mathbb{R}^n)}$ and $\mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ and prove that they coincide if $X(\mathbb{R}^n)$ satisfies the bounded L^2-approximation property. We also construct an example showing that $\mathcal{M}^0_{X(\mathbb{R}^n)}$ and $\mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ may differ. We show that the classes $\mathcal{M}^0_{X(\mathbb{R}^n)}$ and $\mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ are normed algebras and that the normed space $\mathcal{M}_{X(\mathbb{R}^n)}$ and the normed algebra $\mathcal{M}^0_{X(\mathbb{R}^n)}$ are not complete, in general.

The weak doubling property is of course by no means necessary for the conclusion of Theorem 1.3 to hold. Using duality and interpolation as in [**20**] (see also [**8**, Lemma 6]), one can prove the estimate (1.3) for arbitrary reflexive reflection-invariant Banach function spaces. This is done in Section 7 with the help of the interpolation theorem for Calderón products $(X_0^{1-\theta}X_1^\theta)(\mathbb{R}^n)$ and Lozanovskiĭ's formula $(X^{1/2}(X')^{1/2})(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Here we do not assume that the space $X(\mathbb{R}^n)$ satisfies the weak doubling property. We also show that the estimate

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \quad \text{for all} \quad a \in \mathcal{M}^0_{X(\mathbb{R}^n)}$$

holds if $X(\mathbb{R}^n)$ is an arbitrary, not necessarily reflexive, reflection-invariant Banach function space.

In Section 8, we extend J. Löfström's result [**25**] and show that there are no non-trivial Fourier multipliers on the weighted Banach function space $Y(\mathbb{R}^n, w)$ built upon a translation-invariant Banach function space $Y(\mathbb{R}^n)$ in the case of a weight w growing superexponentially in all directions: $\mathcal{M}_{Y(\mathbb{R}^n,w)} = \mathbb{C}$. On the other hand, we show that there are non-trivial Fourier multipliers in $\mathcal{M}_{Y(\mathbb{R}^n,w)}$ in the case of (sub)exponentially growing weights like $w(x) = \exp(c|x|^\alpha)$ for $x \in \mathbb{R}^n$ with some constants $c > 0$ and $\alpha \in (0, 1]$.

2. *Auxiliary results*

2.1. *Translation-invariant Banach function spaces*

We say that a Banach function space $X(\mathbb{R}^n)$ is *translation-invariant* if for all $y \in \mathbb{R}^n$ and for all functions $u \in X(\mathbb{R}^n)$, one has

$$\|\tau_y u\|_{X(\mathbb{R}^n)} = \|u\|_{X(\mathbb{R}^n)},$$

where the translation operator τ_y is defined by $(\tau_y u)(x) := u(x - y)$ for all $x \in \mathbb{R}^n$.

Lemma 2.1. *Let $X(\mathbb{R}^n)$ be a Banach function space and $X'(\mathbb{R}^n)$ be its associate space. Then $X(\mathbb{R}^n)$ is translation-invariant if and only if $X'(\mathbb{R}^n)$ is translation-invariant.*

Proof. Suppose $X(\mathbb{R}^n)$ is translation-invariant, $g \in X'(\mathbb{R}^n)$, and $y \in \mathbb{R}^n$. Then for every $f \in X(\mathbb{R}^n)$ with $\|f\|_{X(\mathbb{R}^n)} \le 1$, we have $\|\tau_{-y} f\|_{X(\mathbb{R}^n)} = \|f\|_{X(\mathbb{R}^n)} \le 1$. By Hölder's inequality (see [**1**, Chap. 1, Theorem 2.4]), $g(\tau_{-y} f) \in L^1(\mathbb{R}^n)$. Changing variables, we get

$$\int_{\mathbb{R}^n} g(x)(\tau_{-y} f)(x)\, dx = \int_{\mathbb{R}^n} (\tau_y g)(x) f(x)\, dx.$$

Then, in view of [**1**, Chap. 1, Lemma 2.8],

$$\begin{aligned}\|\tau_y g\|_{X'(\mathbb{R}^n)} &= \sup\left\{\left|\int_{\mathbb{R}^n} (\tau_y g)(x) f(x)\, dx\right| \;:\; f \in X(\mathbb{R}^n),\ \|f\|_{X(\mathbb{R}^n)} \le 1\right\} \\ &= \sup\left\{\left|\int_{\mathbb{R}^n} g(x) f(x)\, dx\right| \;:\; f \in X(\mathbb{R}^n),\ \|f\|_{X(\mathbb{R}^n)} \le 1\right\} = \|g\|_{X'(\mathbb{R}^n)},\end{aligned}$$

that is, $X'(\mathbb{R}^n)$ is translation-invariant. The reverse implication follows from what was proved above and the Lorentz-Luxemburg theorem (see [**1**, Chap. 1, Theorem 2.7]). □

2.2. *The bounded L^2-approximation property*

Definition 2.2. *We will say that a Banach function space $X(\mathbb{R}^n)$ satisfies the bounded L^2-approximation property if for every function $u \in L^2(\mathbb{R}^n) \cap X(\mathbb{R}^n)$, there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that*

$$\lim_{j\to\infty} \|u - u_j\|_{L^2(\mathbb{R}^n)} = 0, \quad \limsup_{j\to\infty} \|u_j\|_{X(\mathbb{R}^n)} \le \|u\|_{X(\mathbb{R}^n)}. \tag{2.1}$$

Following [**1**, Chap. 1, Definition 3.1], a function f in a Banach function space $X(\mathbb{R}^n)$ is said to have *absolutely continuous norm* in $X(\mathbb{R}^n)$ if $\|f\chi_{E_j}\|_{X(\mathbb{R}^n)} \to 0$ as $j \to \infty$ for every sequence $\{E_j\}_{j\in\mathbb{N}}$ of measurable sets in $\mathbb{R}^n$ satisfying $\chi_{E_j} \to 0$ a.e. on $\mathbb{R}^n$ as $j \to \infty$. The set of all functions of absolutely continuous norm in $X(\mathbb{R}^n)$ is denoted by $X_a(\mathbb{R}^n)$. If $X_a(\mathbb{R}^n) = X(\mathbb{R}^n)$, then the space $X(\mathbb{R}^n)$ itself is said to have absolutely continuous norm.

THEOREM 2.3. *Let $X(\mathbb{R}^n)$ be a Banach function space with absolutely continuous norm. Then $X(\mathbb{R}^n)$ has the bounded L^2-approximation property.*

Proof. It is clear that $L^2(\mathbb{R}^n) \cap X(\mathbb{R}^n)$ is a Banach function space when it is equipped with the norm

$$\|f\|_{L^2(\mathbb{R}^n)\cap X(\mathbb{R}^n)} = \max\left\{\|f\|_{L^2(\mathbb{R}^n)}, \|f\|_{X(\mathbb{R}^n)}\right\}.$$

It is easy to see that it has absolutely continuous norm. Then for every $u \in L^2(\mathbb{R}^n) \cap X(\mathbb{R}^n)$, there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} \|u - u_j\|_{L^2(\mathbb{R}^n)\cap X(\mathbb{R}^n)} = 0$$

(a proof for the case $n = 1$ can be found in [**22**, Lemma 2.10(b)], it can be easily extended to $n \in \mathbb{N}$). Hence (2.1) holds. ☐

Now let $\varrho(x) = e^{1/(|x|^2-1)}$ if $|x| < 1$ and $\varrho(x) = 0$ if $|x| \ge 1$. Consider the sequence

$$\varrho_j(x) := \frac{j^n \varrho(xj)}{\int_{\mathbb{R}^n} \varrho(x)\,dx}, \quad j \in \mathbb{N}. \tag{2.2}$$

As usual, let $\operatorname{supp} u$ denote the support of a function $u \in \mathfrak{M}(\mathbb{R}^n)$.

THEOREM 2.4. *Let $Y(\mathbb{R}^n)$ be a translation-invariant Banach function space and let w be a continuous function such that $w(x) > 0$ for all $x \in \mathbb{R}^n$. Then $Y(\mathbb{R}^n, w)$ has the bounded L^2-approximation property.*

Proof. Take any function $u \in L^2(\mathbb{R}^n) \cap Y(\mathbb{R}^n, w)$ and any $\varepsilon > 0$. There exists $R > 0$ such that the function $v := \chi_{B(0,R)} u$ satisfies

$$\|u - v\|_{L^2(\mathbb{R}^n)} < \varepsilon/2.$$

It is clear that $\operatorname{supp} v \subseteq B(0, R)$ and $\|v\|_{Y(\mathbb{R}^n,w)} \le \|u\|_{Y(\mathbb{R}^n,w)}$ in view of axiom (A2).

Since $w > 0$ is continuous, there exists $j_0 \in \mathbb{N}$ such that for all $x \in B(0, R)$ and $y \in B(0, 1/j_0)$,

$$\frac{(\tau_{-y} w)(x)}{w(x)} = \frac{w(x + y)}{w(x)} \le 1 + \varepsilon.$$

Then taking into account that $Y(\mathbb{R}^n)$ is translation-invariant, one gets for all $y \in B(0, 1/j_0)$,

$$\begin{aligned} \|\tau_y v\|_{Y(\mathbb{R}^n,w)} &= \|w\tau_y v\|_{Y(\mathbb{R}^n)} = \left\|\tau_y\Big((\tau_{-y} w)v\Big)\right\|_{Y(\mathbb{R}^n)} = \|(\tau_{-y} w)v\|_{Y(\mathbb{R}^n)} \\ &\le (1 + \varepsilon)\|wv\|_{Y(\mathbb{R}^n)} = (1 + \varepsilon)\|v\|_{Y(\mathbb{R}^n,w)}. \end{aligned} \tag{2.3}$$

Let $v_j := \varrho_j * v$, where $\varrho_j \in C_0^\infty(\mathbb{R}^n)$ are the functions defined by (2.2). Then $v_j \in C_0^\infty(\mathbb{R}^n)$ and one can choose $j_1 \ge j_0$ such that for all $j \ge j_1$,

$$\|v - v_j\|_{L^2(\mathbb{R}^n)} < \varepsilon/2,$$

(see, e.g., [**7**, Theorem 4.22]). Hence, for all $j \ge j_1$,

$$\|u - v_j\|_{L^2(\mathbb{R}^n)} < \varepsilon.$$

Since $w \in Y_{\rm loc}(\mathbb{R}^n)$ and $1/w \in Y'_{\rm loc}(\mathbb{R}^n)$, $Y(\mathbb{R}^n, w)$ is a Banach function space and $Y'(\mathbb{R}^n, w^{-1})$ is its associate space in view of [**22**, Lemma 2.4]. Using Hölder's inequality for Banach function spaces (see [**1**, Chap. 1, Theorem 2.4]) and (2.4), one gets for all $g \in Y'(\mathbb{R}^n, w^{-1})$ and all $j \geq j_1$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} v_j(x)g(x)\,dx\right| &\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\varrho_j(y)||v(x-y)|\,dy\right)|g(x)|\,dx\\ &= \int_{\mathbb{R}^n}|\varrho_j(y)|\left(\int_{\mathbb{R}^n}|v(x-y)||g(x)|\,dx\right)dy\\ &\leq \int_{B(0,1/j)}|\varrho_j(y)|\,\|\tau_y v\|_{Y(\mathbb{R}^n,w)}\,\|g\|_{Y'(\mathbb{R}^n,w^{-1})}\,dy\\ &\leq (1+\varepsilon)\,\|v\|_{Y(\mathbb{R}^n,w)}\,\|g\|_{Y'(\mathbb{R}^n,w^{-1})}\int_{\mathbb{R}^n}|\varrho_j(y)|\,dy\\ &= (1+\varepsilon)\,\|v\|_{Y(\mathbb{R}^n,w)}\,\|g\|_{Y'(\mathbb{R}^n,w^{-1})}\,.\end{aligned}$$

By [**1**, Chap. 1, Theorem 2.7 and Lemma 2.8], the above inequality implies that for all $j \geq j_1$,

$$\begin{aligned}\|v_j\|_{Y(\mathbb{R}^n,w)} &= \sup\left\{\left|\int_{\mathbb{R}^n} v_j(x)g(x)\,dx\right| :\ g\in Y'(\mathbb{R}^n,w^{-1}),\ \|g\|_{Y'(\mathbb{R}^n,w^{-1})}\leq 1\right\}\\ &\leq (1+\varepsilon)\,\|v\|_{Y(\mathbb{R}^n,w)}\,,\end{aligned}$$

which completes the proof, since $\varepsilon > 0$ is arbitrary. □

Theorem 2.6 below shows that one cannot drop the requirement of continuity of the weight w in Theorem 2.4 (as well as in [**8**, Lemma 2]). The construction of our counterexample is based on the fact that there exist compact sets of positive Lebesgue measure with empty interior (see, e.g., [**4**, Example 1.7.6] or [**14**, Chap. 12, Exercise 9]).

LEMMA 2.5. *Let $G \subset \mathbb{R}^n$ be a compact set of positive measure with empty interior and let*

$$w_G(x) := \begin{cases} 1, & x \in G,\\ 2, & x \in \mathbb{R}^n \setminus G.\end{cases} \tag{2.4}$$

Suppose $\psi \in C(\mathbb{R}^n)$ and $\|\psi\|_{L^\infty(\mathbb{R}^n,w_G)} \leq 1$. Then for all $x \in \mathbb{R}^n$,

$$|\psi(x)| \leq 1/2. \tag{2.5}$$

Proof. For every point $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|\psi(y)| \geq |\psi(x)| - \varepsilon$ for each $y \in B(x,\delta)$. Since G is a closed set with empty interior, $B(x,\delta)\setminus G$ is a nonempty open set. It follows from (2.4) and the condition $\|\psi\|_{L^\infty(\mathbb{R}^n,w_G)} \leq 1$ that $2|\psi(y)| \leq 1$ for almost all $y \in B(x,\delta)\setminus G$. Hence $|\psi(x)| - \varepsilon \leq 1/2$ for all $\varepsilon > 0$. Passing in this inequality to the limit as $\varepsilon \to 0$, we arrive at (2.5). □

THEOREM 2.6. *Let $G \subset \mathbb{R}^n$ be a compact set of positive measure with empty interior and let the weight w_G be defined by (2.4). Then the Banach function space $L^\infty(\mathbb{R}^n, w_G)$ does not satisfy the bounded L^2-approximation property.*

Proof. Let $u := \frac{3}{4}\chi_G$. Then $u \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n, w_G)$ and $\|u\|_{L^\infty(\mathbb{R}^n,w_G)} \leq \frac{3}{4}$. It follows from Lemma 2.5 that if $\psi \in C(\mathbb{R}^n)$ and $\|\psi\|_{L^\infty(\mathbb{R}^n,w_G)} \leq 1$, then

$$\|u-\psi\|_{L^2(\mathbb{R}^n)} \geq \left(\int_G |u(x)-\psi(x)|^2\,dx\right)^{1/2} \geq \left(\int_G\left(\frac{3}{4}-\frac{1}{2}\right)^2 dx\right)^{1/2} \geq \frac{1}{4}\,|G|^{1/2}.$$

This inequality implies that there is no sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that (2.1) is fulfilled for $X(\mathbb{R}^n) = L^\infty(\mathbb{R}^n, w_G)$. □

For a set $F \subset \mathbb{R}^n$, we denote by F^* the closure of the set $\{x + y \in \mathbb{R}^n \ : \ x \in F,\ y \in B(0,1)\}$. The next result is well known. its proof is included for the reader's convenience.

LEMMA 2.7. *For every function $f \in L^\infty(\mathbb{R}^n)$ with compact support there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} v_j \subseteq (\operatorname{supp} f)^*$, $\|v_j\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}$ for all $j \in \mathbb{N}$, and $v_j \to f$ a.e. on $\mathbb{R}^n$ as $j \to \infty$.*

Proof. Let $\{\varrho_k\}_{k\in\mathbb{N}}$ be the sequence defined by (2.2). By [**7**, Theorem 4.22], the sequence $\nu_k := \varrho_k * f$ converges to f in $L^1(\mathbb{R}^n)$ as $k \to \infty$. In view of [**7**, Proposition 4.18], we have $\operatorname{supp} \nu_k \subseteq (\operatorname{supp} f)^*$. By the Young inequality for convolutions (see, e.g., [**7**, Theorem 4.15]), one has $\|\nu_k\|_{L^\infty(\mathbb{R}^n)} \le \|\varrho_k\|_{L^1(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)}$ for all $k \in \mathbb{N}$. Since $\|\nu_k - f\|_{L^1(\mathbb{R}^n)} \to 0$ as $k \to \infty$, there exists a subsequence $\{\nu_{k_j}\}_{j\in\mathbb{N}}$ of the sequence $\{\nu_k\}_{k\in\mathbb{N}}$ such that $\nu_{k_j} \to f$ a.e. on $\mathbb{R}^n$ as $j \to \infty$. Then the required sequence $\{v_j\}_{j\in\mathbb{N}}$ is defined by $v_j := \nu_{k_j}$ for $j \in \mathbb{N}$. □

We finish this subsection with a result that we will use in the next one.

LEMMA 2.8. *Suppose a Banach function space $X(\mathbb{R}^n)$ has the bounded L^2-approximation property. Then for every function $f \in L^\infty(\mathbb{R}^n)$ with compact support there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that*

$$\operatorname{supp} v_j \subseteq (\operatorname{supp} f)^*, \quad \|v_j\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)} + 1, \quad \limsup_{j\to\infty} \|v_j\|_{X(\mathbb{R}^n)} \le \|f\|_{X(\mathbb{R}^n)},$$

and $v_j \to f$ a.e. on $\mathbb{R}^n$ as $j \to \infty$.

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ be such that

$$\lim_{k\to\infty} \|f - u_k\|_{L^2(\mathbb{R}^n)} = 0, \quad \limsup_{k\to\infty} \|u_k\|_{X(\mathbb{R}^n)} \le \|f\|_{X(\mathbb{R}^n)}.$$

Consider a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \zeta(x) \le 1$ for $x \in \mathbb{R}^n$, $\zeta(x) = 1$ for $x \in \operatorname{supp} f$, and $\operatorname{supp} \zeta \subseteq (\operatorname{supp} f)^*$. Then $\zeta f = f$. Further, let $\eta : \mathbb{C} \to \mathbb{C}$ be a function, which can be represented for $z = y_1 + iy_2 \in \mathbb{C}$ as $\eta(z) = \eta(y_1 + iy_2) = U(y_1, y_2) + iV(y_1, y_2)$ with real-valued functions $U, V \in C_0^\infty(\mathbb{R}^2)$. Assume that $\eta(z) = z$ for all $z \in \mathbb{C}$ with $|z| \le \|f\|_{L^\infty(\mathbb{R}^n)}$, $|\eta(z)| \le |z|$ for all $z \in \mathbb{C}$, and

$$\max_{z\in\mathbb{C}} |\eta(z)| \le \|f\|_{L^\infty(\mathbb{R}^n)} + 1.$$

Then $\eta \circ (\zeta f) = \eta \circ f = f$.

Set $\nu_k := \eta \circ (\zeta u_k)$ for $k \in \mathbb{N}$. Then $\nu_k \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \nu_k \subseteq \operatorname{supp} \zeta \subseteq (\operatorname{supp} f)^*$,

$$\|\nu_k\|_{L^\infty(\mathbb{R}^n)} \le \max_{z\in\mathbb{C}} |\eta(z)| \le \|f\|_{L^\infty(\mathbb{R}^n)} + 1,$$

and

$$|\nu_k(x)| = |\eta(\zeta(x)u_k(x))| \le |\zeta(x)u_k(x)| \le |u_k(x)| \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Hence

$$\limsup_{k\to\infty} \|\nu_k\|_{X(\mathbb{R}^n)} \le \limsup_{k\to\infty} \|u_k\|_{X(\mathbb{R}^n)} \le \|f\|_{X(\mathbb{R}^n)}.$$

Further, by the mean value theorem, there exists a constant C_η depending on the maxima of the partial derivatives of the functions U and V such that

$$\begin{aligned}\|f - \nu_k\|_{L^2(\mathbb{R}^n)} &= \|\eta \circ (\zeta f) - \eta \circ (\zeta u_k)\|_{L^2(\mathbb{R}^n)} \le C_\eta \|\zeta f - \zeta u_k\|_{L^2(\mathbb{R}^n)} \\ &\le C_\eta \|f - u_k\|_{L^2(\mathbb{R}^n)} \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence the sequence $\{\nu_k\}_{k\in\mathbb{N}}$ has a subsequence $\{\nu_{k_j}\}_{j\in\mathbb{N}}$ such that $\nu_{k_j} \to f$ a.e. on $\mathbb{R}^n$ as $j \to \infty$ (see, e.g., [**1**, Chap. 1, Theorem 1.7(vi)]). Then the required sequence $\{v_j\}_{j\in\mathbb{N}}$ is defined by $v_j := \nu_{k_j}$ for $j \in \mathbb{N}$. □

2.3. *The norm fundamental property*

Definition 2.9. *We say that a Banach function space $X(\mathbb{R}^n)$ satisfies the norm fundamental property if for every $f \in X(\mathbb{R}^n)$,*

$$\|f\|_{X(\mathbb{R}^n)} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)\psi(x)\,dx\right| \;:\; \psi \in C_0^\infty(\mathbb{R}^n),\ \|\psi\|_{X'(\mathbb{R}^n)} \le 1\right\}.$$

Let $S_0(\mathbb{R}^n)$ denote the set of all simple compactly supported functions.

Lemma 2.10. *Let $X(\mathbb{R}^n)$ be a Banach function space and $X'(\mathbb{R}^n)$ be its associate space. For every $f \in X(\mathbb{R}^n)$,*

$$\|f\|_{X(\mathbb{R}^n)} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}^n),\ \|s\|_{X'(\mathbb{R}^n)} \le 1\right\}. \tag{2.6}$$

Proof. By [**1**, Theorem 2.7 and Lemma 2.8], for every $f \in X(\mathbb{R}^n)$,

$$\|f\|_{X(\mathbb{R}^n)} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \;:\; g \in X'(\mathbb{R}^n),\ \|g\|_{X'(\mathbb{R}^n)} \le 1\right\}. \tag{2.7}$$

It follows from the inclusion $S_0(\mathbb{R}^n) \subset X'(\mathbb{R}^n)$ and equality (2.7) that

$$\|f\|_{X(\mathbb{R}^n)} \ge \sup\left\{\left|\int_{\mathbb{R}^n} f(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}^n),\ \|s\|_{X'(\mathbb{R}^n)} \le 1\right\}. \tag{2.8}$$

Fix $g \in X'(\mathbb{R}^n)$ such that $\|g\|_{X'(\mathbb{R}^n)} \le 1$. Then there exists a sequence $\{s_j\}_{j\in\mathbb{N}} \subset S_0(\mathbb{R}^n)$ such that $0 \le |s_1| \le |s_2| \le \cdots \le |g|$ and $s_j \to g$ a.e. on $\mathbb{R}^n$ as $j \to \infty$. Therefore, $fs_j \to fg$ as $j \to \infty$ and $|fs_j| \le |fg|$ for all $j \in \mathbb{N}$ a.e. on $\mathbb{R}^n$. By Hölder's inequality (see [**1**, Chap. 1, Theorem 2.4]), $fg \in L^1(\mathbb{R}^n)$. Hence, in view of the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} f(x)s_j(x)\,dx = \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

On the other hand, inequality $|s_j| \le |g|$ implies that $\|s_j\|_{X'(\mathbb{R}^n)} \le \|g\|_{X'(\mathbb{R}^n)} \le 1$ for all $j \in \mathbb{N}$. Thus, for all $g \in X'(\mathbb{R}^n)$ satisfying $\|g\|_{X'(\mathbb{R}^n)} \le 1$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &= \lim_{j\to\infty}\left|\int_{\mathbb{R}^n} f(x)s_j(x)\,dx\right| \le \sup_{j\in\mathbb{N}}\left|\int_{\mathbb{R}^n} f(x)s_j(x)\,dx\right| \\ &\le \sup\left\{\left|\int_{\mathbb{R}^n} f(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}^n),\ \|s\|_{X'(\mathbb{R}^n)} \le 1\right\}.\end{aligned}$$

This inequality and equality (2.7) imply that

$$\|f\|_{X(\mathbb{R}^n)} \le \sup\left\{\left|\int_{\mathbb{R}^n} f(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}^n),\ \|s\|_{X'(\mathbb{R}^n)} \le 1\right\}. \tag{2.9}$$

Combining inequalities (2.8) and (2.9), we arrive at equality (2.6). □

Theorem 2.11. *If $X'(\mathbb{R}^n)$ satisfies the bounded L^2-approximation property, then $X(\mathbb{R}^n)$ has the norm fundamental property.*

Proof. Lemma 2.10 implies that it is sufficient to prove the inequality

$$\left|\int_{\mathbb{R}^n} \varphi(x)s(x)\,dx\right| \le \sup\left\{\left|\int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx\right| : \psi \in C_0^\infty(\mathbb{R}^n),\ \|\psi\|_{X'(\mathbb{R}^n)} \le 1\right\} \tag{2.10}$$

for any $\varphi \in X(\mathbb{R}^n)$ and any $s \in S_0(\mathbb{R}^n)$ with $\|s\|_{X'(\mathbb{R}^n)} \le 1$. According to Lemma 2.8, there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\psi_j \subseteq (\operatorname{supp} s)^*, \quad \|\psi_j\|_{L^\infty(\mathbb{R}^n)} \le \|s\|_{L^\infty(\mathbb{R}^n)} + 1, \quad \limsup_{j\to\infty} \|\psi_j\|_{X'(\mathbb{R}^n)} \le \|s\|_{X'(\mathbb{R}^n)},$$

and $\psi_j \to s$ a.e. on $\mathbb{R}^n$ as $j \to \infty$. Take an arbitrary $\varepsilon \in (0,1)$. Then $(1-\varepsilon)\|\psi_j\|_{X'(\mathbb{R}^n)} \le 1$ for all sufficiently large $j \in \mathbb{N}$. Axiom (A5) and the Lebesgue dominated convergence theorem imply that

$$\begin{aligned}(1-\varepsilon)\left|\int_{\mathbb{R}^n} \varphi(x)s(x)\,dx\right| &= (1-\varepsilon)\lim_{j\to\infty}\left|\int_{\mathbb{R}^n} \varphi(x)\psi_j(x)\,dx\right| \\ &\le \sup\left\{\left|\int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx\right| : \psi \in C_0^\infty(\mathbb{R}^n),\ \|\psi\|_{X'(\mathbb{R}^n)} \le 1\right\}.\end{aligned}$$

Since $\varepsilon \in (0,1)$ is arbitrary, (2.10) follows. □

COROLLARY 2.12. *If $X(\mathbb{R}^n)$ is a Banach function space such that its associate space $X'(\mathbb{R}^n)$ has absolutely continuous norm, then $X(\mathbb{R}^n)$ satisfies the norm fundamental property.*

Proof. This follows from Theorems 2.11 and 2.3. □

Note that a Banach function space $X(\mathbb{R}^n)$ may satisfy the norm fundamental property even if $(X')_a(\mathbb{R}^n) = \{0\}$. For instance, if $X(\mathbb{R}^n) = L^1(\mathbb{R}^n)$, then $(X')_a(\mathbb{R}^n) = (L^\infty)_a(\mathbb{R}^n) = \{0\}$ in view of [**1**, Chap. 3, Theorem 5.5(b)]. However, the following result is true.

COROLLARY 2.13. *Let $Y(\mathbb{R}^n)$ be a translation-invariant Banach function space and let $w \in C(\mathbb{R}^n)$ be a function such that $w(x) > 0$ for all $x \in \mathbb{R}^n$. Then $X(\mathbb{R}^n) = Y(\mathbb{R}^n, w)$ has the norm fundamental property.*

Proof. In view of Lemma 2.1, the space $Y'(\mathbb{R}^n)$ is translation-invariant. On the other hand, $w^{-1} \in C(\mathbb{R}^n)$ and $w^{-1} > 0$. It follows from [**22**, Lemma 2.4(c)] that $X'(\mathbb{R}^n) = Y'(\mathbb{R}^n, w^{-1})$. By Theorem 2.4, the space $Y'(\mathbb{R}^n, w^{-1})$ satisfies the bounded L^2-approximation property. Then the space $Y(\mathbb{R}^n, w)$ has the norm fundamental property due to Theorem 2.11. □

Similarly to Theorem 2.4, one cannot drop the requirement of continuity of the weight w in Corollary 2.13.

LEMMA 2.14. *Let $G \subset \mathbb{R}^n$ be a compact set of positive measure with empty interior and let the weight w_G be defined by (2.4). Suppose $f \in L^1(\mathbb{R}^n, w_G^{-1})$, $f \ge 0$, and $\operatorname{supp} f \cap G$ has positive measure. Then*

$$\|f\|_{L^1(\mathbb{R}^n, w_G^{-1})} > \sup\left\{\left|\int_{\mathbb{R}^n} f(x)\psi(x)\,dx\right| : \psi \in C(\mathbb{R}^n),\ \|\psi\|_{L^\infty(\mathbb{R}^n, w_G)} \le 1\right\}.$$

Proof. It follows from Lemma 2.5 that

$$\left|\int_{\mathbb{R}^n} f(x)\psi(x)\,dx\right| \le \frac{1}{2}\int_{\mathbb{R}^n} f(x)\,dx$$

for any function $\psi \in C(\mathbb{R}^n)$ with $\|\psi\|_{L^\infty(\mathbb{R}^n,w_G)} \le 1$. On the other hand,

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^n,w_G^{-1})} &= \int_G f(x)w_G^{-1}(x)\,dx + \int_{\mathbb{R}^n\setminus G} f(x)w_G^{-1}(x)\,dx \\ &= \int_G f(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^n\setminus G} f(x)\,dx > \frac{1}{2}\int_{\mathbb{R}^n} f(x)\,dx,\end{aligned}$$

since supp $f \cap G$ has positive measure. □

COROLLARY 2.15. *Let $G \subset \mathbb{R}^n$ be a compact set of positive measure with empty interior and let the weight w_G be defined by (2.4). Then the Banach function space $L^1(\mathbb{R}^n, w_G^{-1})$ does not have the norm fundamental property.*

Corollary 2.15 and Theorem 2.11 provide an alternative proof of Theorem 2.6.

2.4. *Lemma on approximation at Lebesgue points*

Given $\delta > 0$ and a function ψ on $\mathbb{R}^n$, we define the function ψ_δ by

$$\psi_\delta(\xi) := \delta^{-n}\psi(\xi/\delta), \quad \xi \in \mathbb{R}^n.$$

Recall that a point $x \in \mathbb{R}^n$ is said to be a Lebesgue point of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if

$$\lim_{R\to 0^+} \frac{1}{|B(x,R)|}\int_{B(x,R)} |f(y) - f(x)|\,dy = 0.$$

For $\sigma \in \mathbb{R}$, we will say that a measurable function f belongs to the space $L_{1,\sigma}(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} (1+|\xi|)^{-\sigma}|f(\xi)|\,d\xi < \infty.$$

LEMMA 2.16. *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ be such that $\sigma_2 \ge \sigma_1$ and $\sigma_2 > n$. Suppose ψ is a measurable function on $\mathbb{R}^n$ satisfying*

$$|\psi(\xi)| \le C(1+|\xi|)^{-\sigma_2} \quad \text{for almost all} \quad \xi \in \mathbb{R}^n \tag{2.11}$$

with some constant $C \in (0,\infty)$. Then for every Lebesgue point $\eta \in \mathbb{R}^n$ of a function a belonging to the space $L_{1,\sigma_1}(\mathbb{R}^n)$, one has

$$\int_{\mathbb{R}^n} |a(\xi) - a(\eta)|\,|\psi_\delta(\eta - \xi)|\,d\xi \to 0 \quad \text{as} \quad \delta \to 0.$$

Proof. The proof is similar to the proof of [**31**, Chap. I, Theorem 1.25] (see also [**10**, Chap. II, Lemma 1]). We give it here for the convenience of the reader.

Take an arbitrary $\varepsilon > 0$. Since $\eta \in \mathbb{R}^n$ is a Lebesgue point of a, there exists a $\rho > 0$ such that

$$r^{-n}\int_{|\zeta|<r} |a(\eta-\zeta) - a(\eta)|\,d\zeta \le \varepsilon \quad \text{for all} \quad r \in (0,\rho]. \tag{2.12}$$

Substituting $\eta - \xi$ with ζ and splitting the integral, we get for any $\delta > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} |a(\xi) - a(\eta)|\,|\psi_\delta(\eta - \xi)|\,d\xi = &\int_{|\zeta|<\rho} |a(\eta-\zeta) - a(\eta)|\,|\psi_\delta(\zeta)|\,d\zeta \\ &+ \int_{|\zeta|\ge\rho} |a(\eta-\zeta) - a(\eta)|\,|\psi_\delta(\zeta)|\,d\zeta \\ =&:I_1(\delta) + I_2(\delta). \end{aligned} \tag{2.13}$$

Let $\mathbb{S}^{n-1} = \{\vartheta \in \mathbb{R}^n : |\vartheta| = 1\}$ be the unit sphere in $\mathbb{R}^n$ and let

$$g(r) := \int_{\mathbb{S}^{n-1}} |a(\eta - r\vartheta) - a(\eta)|\,d\vartheta,$$

where $d\vartheta$ is an element of the surface area on $\mathbb{S}^{n-1}$. Then condition (2.12) is equivalent to

$$G(r) := \int_0^r s^{n-1} g(s)\,ds \le \varepsilon r^n \quad \text{for all} \quad r \in (0, \rho] \tag{2.14}$$

(see, e.g., [**17**, Theorem 2.49]). Let

$$\phi(r) := C(1+r)^{-\sigma_2}, \quad \phi_{(\delta)}(r) := \delta^{-n}\phi(r/\delta), \quad r \ge 0.$$

Then (2.11) implies that

$$I_1(\delta) \le \int_{|\zeta|<\rho} |a(\eta-\zeta) - a(\eta)|\phi_{(\delta)}(|\zeta|)\,d\zeta = \int_0^\rho r^{n-1} g(r)\phi_{(\delta)}(r)\,dr.$$

Integrating by parts twice and taking into account (2.14) and the inequalities $\phi'_{(\delta)} < 0$ and $\sigma_2 > n$, we obtain

$$\begin{aligned} I_1(\delta) &\le \left[G(r)\phi_{(\delta)}(r)\right]_0^\rho - \int_0^\rho G(r)\phi'_{(\delta)}(r)\,dr \le \varepsilon\rho^n\phi_{(\delta)}(\rho) - \int_0^\rho \varepsilon r^n \phi'_{(\delta)}(r)\,dr \\ &= n\varepsilon \int_0^\rho r^{n-1}\phi_{(\delta)}(r)\,dr \le n\varepsilon \int_0^\infty r^{n-1}\phi_{(\delta)}(r)\,dr = n\varepsilon\int_0^\infty s^{n-1}\phi(s)\,ds \\ &\le Cn\varepsilon \int_0^\infty (1+s)^{n-1-\sigma_2}\,ds = \frac{Cn}{\sigma_2 - n}\,\varepsilon =: A\varepsilon. \end{aligned} \tag{2.15}$$

Since $a \in L_{1,\sigma_1}(\mathbb{R}^n)$, we get for all $\delta > 0$,

$$\begin{aligned} I_2(\delta) &\le \int_{|\zeta|\ge\rho} |a(\eta-\zeta) - a(\eta)|\phi_{(\delta)}(|\zeta|)\,d\zeta \\ &\le \left(\int_{\mathbb{R}^n} |a(\eta-\zeta)|(1+|\eta-\zeta|)^{-\sigma_1}\,d\zeta\right) \sup_{|\zeta|\ge\rho} \left((1+|\eta-\zeta|)^{\sigma_1}\phi_{(\delta)}(|\zeta|)\right) \\ &\quad + |a(\eta)| \int_{|\zeta|\ge\rho} \phi_{(\delta)}(|\zeta|)\,d\zeta \\ &\le \|a\|_{1,\sigma_1} \sup_{|\zeta|\ge\rho} \left((1+|\eta-\zeta|)^{\sigma_2}\phi_{(\delta)}(|\zeta|)\right) + |a(\eta)|\,\omega_n \int_\rho^\infty r^{n-1}\phi_{(\delta)}(r)\,dr, \end{aligned} \tag{2.16}$$

where ω_n is the surface area of $\mathbb{S}^{n-1}$. It is clear that for $|\zeta| \ge \rho$,

$$\begin{aligned} (1+|\eta-\zeta|)^{\sigma_2}\phi_{(\delta)}(|\zeta|) &\le C(1+|\eta|+|\zeta|)^{\sigma_2}\delta^{-n}(1+|\zeta|/\delta)^{-\sigma_2} \\ &= C\frac{(1+|\eta|+|\zeta|)^{\sigma_2}}{(\delta+|\zeta|)^{\sigma_2}}\delta^{\sigma_2-n} < C\left(\frac{1+|\eta|}{|\zeta|}+1\right)^{\sigma_2}\delta^{\sigma_2-n} \\ &\le C\left(\frac{1+|\eta|}{\rho}+1\right)^{\sigma_2}\delta^{\sigma_2-n}. \end{aligned} \tag{2.17}$$

Further,

$$\int_{\rho}^{\infty} r^{n-1}\phi_{(\delta)}(r)\,dr = \int_{\rho/\delta}^{\infty} s^{n-1}\phi(s)\,ds \le C\int_{\rho/\delta}^{\infty}(1+s)^{n-1-\sigma_2}\,ds$$
$$= \frac{C}{\sigma_2-n}\,(1+\rho/\delta)^{n-\sigma_2} \le \frac{C}{(\sigma_2-n)\rho^{\sigma_2-n}}\,\delta^{\sigma_2-n}. \qquad (2.18)$$

It follows from (2.16)–(2.18) that

$$I_2(\delta) \le C\left(\|a\|_{1,\sigma_1}\left(\frac{1+|\eta|}{\rho}+1\right)^{\sigma_2} + \frac{\omega_n|a(\eta)|}{(\sigma_2-n)\rho^{\sigma_2-n}}\right)\delta^{\sigma_2-n}.$$

Hence there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$I_2(\delta) < \varepsilon \quad \text{for all} \quad \delta \in (0,\delta_0),$$

and inequality (2.15) implies that

$$I_1(\delta) + I_2(\delta) < (A+1)\varepsilon \quad \text{for all} \quad \delta \in (0,\delta_0).$$

Combining this estimate with (2.13), we arrive at the desired result. □

3. *Weak doubling property*

3.1. *The infimum of the doubling constants*

For a Banach function space $X(\mathbb{R}^n)$ and $\tau > 1$, consider the doubling constant

$$D_{X,\tau} := \liminf_{R\to\infty}\left(\inf_{y\in\mathbb{R}^n}\frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}}\right). \qquad (3.1)$$

We immediately deduce from the lattice property (Axiom (A2) in the definition of a Banach function space) that $1 \le D_{X,\tau_1} \le D_{X,\tau_2}$ for all $1 < \tau_1 \le \tau_2$. Therefore,

$$\inf_{\tau>1} D_{X,\tau} \ge 1.$$

Lemma 3.1. *If a Banach function space $X(\mathbb{R}^n)$ satisfies the weak doubling property, then*

$$\inf_{\tau>1} D_{X,\tau} = 1.$$

Proof. Since $X(\mathbb{R}^n)$ satisfies the weak doubling property, there exists a number $\varrho > 1$ such that $D_{X,\varrho} < \infty$. Assume, contrary to the hypothesis, that

$$D := \inf_{\tau>1} D_{X,\tau} > 1.$$

Take an arbitrary $N \in \mathbb{N}$ and consider $\tau = \varrho^{1/N}$. Since

$$D_{X,\tau} \ge D > D_0 := \frac{D+1}{2} > 1,$$

it follows from the definition of $D_{X,\tau}$ that there exists a number $R_0 > 0$ such that for all $R \ge R_0$,

$$\inf_{y\in\mathbb{R}^n}\frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}} \ge D_0.$$

Hence, for all $y \in \mathbb{R}^n$ and all $R \geq R_0$,

$$\frac{\|\chi_{B(y,\varrho R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}} = \prod_{j=1}^{N} \frac{\|\chi_{B(y,\tau^j R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,\tau^{j-1} R)}\|_{X(\mathbb{R}^n)}} \geq D_0^N.$$

Therefore, $D_{X,\varrho} \geq D_0^N$ for all $N \in \mathbb{N}$, which is impossible since $D_0 > 1$ and $D_{X,\varrho} < \infty$. The obtained contradiction completes the proof. □

3.2. *The doubling property and the A_X-condition*

DEFINITION 3.2. *We say that a Banach function space $X(\mathbb{R}^n)$ satisfies the (strong) doubling property if there exist a number $\tau > 1$ and a constant $C_\tau > 0$ such that for all $R > 0$ and $y \in \mathbb{R}^n$,*

$$\frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}} \leq C_\tau. \tag{3.2}$$

The doubling property is considerably stronger than the weak doubling property. Indeed, it is easy to see that a Banach function space $X(\mathbb{R}^n)$ satisfies the weak doubling property if and only if there exist a number $\tau > 1$, a constant $C_\tau > 0$, a sequence $\{R_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ satisfying $R_j \to \infty$ as $j \to \infty$, and a sequence $\{y_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\frac{\|\chi_{B(y_j,\tau R_j)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y_j,R_j)}\|_{X(\mathbb{R}^n)}} \leq C_\tau \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.3}$$

So, the difference between the doubling property and the weak doubling property is that the former requires estimate (3.2) to hold for all balls, while the latter requires it to hold only for some sequence of balls with radii going to infinity. We will return to this comparison in Subsection 3.5.

Now we give a sufficient condition guaranteeing that a Banach function space $X(\mathbb{R}^n)$ satisfies the doubling property. We say that a Banach function space $X(\mathbb{R}^n)$ satisfies the A_X-condition if

$$\sup_Q \frac{1}{|Q|} \|\chi_Q\|_{X(\mathbb{R}^n)} \|\chi_Q\|_{X'(\mathbb{R}^n)} < \infty, \tag{3.4}$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes. This condition goes back to E. I. Berezhnoi [**2**].

LEMMA 3.3. *If $X(\mathbb{R}^n)$ is a Banach function space satisfying the A_X-condition, then $X(\mathbb{R}^n)$ satisfies the doubling property.*

Proof. It is well-known that there exist constants $0 < m_n < M_n < \infty$ such that for every ball B in $\mathbb{R}^n$ and the corresponding inscribed and circumscribed cubes Q and P one has

$$m_n|P| \leq |B| \leq M_n|Q|.$$

Then it follows from Axiom (A2) in the definition of a Banach function function norm that condition (3.4) is equivalent to the condition

$$C_X := \sup_B \frac{1}{|B|} \|\chi_B\|_{X(\mathbb{R}^n)} \|\chi_B\|_{X'(\mathbb{R}^n)} < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$. Then, for every $y \in \mathbb{R}^n$, $\tau > 1$ and $R > 0$,

$$\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)} \leq C_X \frac{|B(y,\tau R)|}{\|\chi_{B(y,\tau R)}\|_{X'(\mathbb{R}^n)}} \leq C_X \frac{|B(y,\tau R)|}{\|\chi_{B(y,R)}\|_{X'(\mathbb{R}^n)}} = C_X \tau^n \frac{|B(y,R)|}{\|\chi_{B(y,R)}\|_{X'(\mathbb{R}^n)}}.$$

It follows from the above inequality and Hölder's inequality for Banach function spaces (see [**1**, Chap. 1, Theorem 2.4]) that

$$\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)} \le C_X \tau^n \frac{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}\|\chi_{B(y,R)}\|_{X'(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X'(\mathbb{R}^n)}} = C_X\tau^n\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}.$$

Applying this inequality, we immediately get (3.2) with $C_\tau = C_X\tau^n$ for every $\tau > 1$, which completes the proof. □

3.3. *Translation-invariant Banach function spaces satisfy the doubling property*

LEMMA 3.4. *Let $X(\mathbb{R}^n)$ be a translation-invariant Banach function space.*
(a) *There exist constants $C_1, C_2 > 0$ such that for all $R > 0$ and $y \in \mathbb{R}^n$,*

$$C_1 \min\{1, R^n\} \le \|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)} \le C_2 \max\{1, R^n\}. \tag{3.5}$$

(b) *For all $\tau > 1$, $R > 0$, and $y \in \mathbb{R}^n$,*

$$\frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}} \le \left(4\sqrt{n}\,\tau\right)^n. \tag{3.6}$$

Proof. (a) All cubes in this proof are assumed to be closed and to have sides parallel to the coordinate axes. Let $Q(x,a)$ denote the cube centered at x of side length a. Since the space $X(\mathbb{R}^n)$ is translation-invariant, we have $\|\chi_{B(x,R)}\|_{X(\mathbb{R}^n)} = \|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)}$ and $\|\chi_{Q(x,a)}\|_{X(\mathbb{R}^n)} = \|\chi_{Q(y,a)}\|_{X(\mathbb{R}^n)}$ for all $x, y \in \mathbb{R}^n$ and $a, R > 0$. Therefore, we may simply write B_R and Q_a for arbitrary open balls of radius R and arbitrary cubes of side length a, respectively.

Let $a > 0$ and $\mathcal{F}$ be the family of 2^n cubes Q_a with pairwise disjoint interiors obtained from a fixed cube Q_{2a} by dividing each its side in two segments of equal length: $Q_{2a} = \cup_{Q_a\in\mathcal{F}} Q_a$. Then

$$\|\chi_{Q_{2a}}\|_{X(\mathbb{R}^n)} = \left\|\sum_{Q_a\in\mathcal{F}} \chi_{Q_a}\right\|_{X(\mathbb{R}^n)} \le \sum_{Q_a\in\mathcal{F}} \|\chi_{Q_a}\|_{X(\mathbb{R}^n)} = 2^n \|\chi_{Q_a}\|_{X(\mathbb{R}^n)}. \tag{3.7}$$

Using inequality (3.7) m times, one gets for all $m \in \mathbb{N}$,

$$\|\chi_{Q_{2^m}}\|_{X(\mathbb{R}^n)} \le 2^{mn} \|\chi_{Q_1}\|_{X(\mathbb{R}^n)}, \quad \|\chi_{Q_1}\|_{X(\mathbb{R}^n)} \le 2^{mn} \left\|\chi_{Q_{2^{-m}}}\right\|_{X(\mathbb{R}^n)}, \tag{3.8}$$

and hence

$$\left\|\chi_{Q_{2^{-m}}}\right\|_{X(\mathbb{R}^n)} \ge 2^{-mn}\|\chi_{Q_1}\|_{X(\mathbb{R}^n)}. \tag{3.9}$$

If $R \ge 1$, there exists $m \in \mathbb{N}$ such that $2^{m-2} < R \le 2^{m-1}$. Then B_R is contained in a cube Q_{2^m} of side length 2^m and it follows from the first inequality in (3.8) that

$$\begin{aligned}\|\chi_{B_1}\|_{X(\mathbb{R}^n)} &\le \|\chi_{B_R}\|_{X(\mathbb{R}^n)} \le \|\chi_{Q_{2^m}}\|_{X(\mathbb{R}^n)}\\ &\le 2^{mn} \|\chi_{Q_1}\|_{X(\mathbb{R}^n)} < \left(4^n \|\chi_{Q_1}\|_{X(\mathbb{R}^n)}\right) R^n.\end{aligned} \tag{3.10}$$

If $R \le 1$, there exists $m \in \mathbb{N}\cup\{0\}$ such that $2^{-m-1} \le R/\sqrt{n} < 2^{-m}$. Then it is easy to see that B_R contains a cube $Q_{2^{-m}}$ of side length 2^{-m} and it follows from (3.9) that

$$\begin{aligned}\|\chi_{B_1}\|_{X(\mathbb{R}^n)} &\ge \|\chi_{B_R}\|_{X(\mathbb{R}^n)} \ge \left\|\chi_{Q_{2^{-m}}}\right\|_{X(\mathbb{R}^n)}\\ &\ge 2^{-mn}\|\chi_{Q_1}\|_{X(\mathbb{R}^n)} > \left(n^{-n/2} \|\chi_{Q_1}\|_{X(\mathbb{R}^n)}\right) R^n.\end{aligned} \tag{3.11}$$

Estimates (3.10) and (3.11) imply (3.5) with

$$C_1 = \min\left\{\|\chi_{B_1}\|_{X(\mathbb{R}^n)},\, n^{-n/2} \|\chi_{Q_1}\|_{X(\mathbb{R}^n)}\right\}, \quad C_2 = \max\left\{\|\chi_{B_1}\|_{X(\mathbb{R}^n)},\, 4^n \|\chi_{Q_1}\|_{X(\mathbb{R}^n)}\right\}.$$

Part (a) is proved.

(b) For any $R > 0$, there exists $m \in \mathbb{Z}$ such that $2^{m-2} < \tau R \le 2^{m-1}$. Then any ball $B_{\tau R}$ is contained in a cube Q_{2^m} of side length 2^m. Let $m_0 := [\log_2(\tau\sqrt{n})] + 1$. It is easy to see that $2^{m-m_0-2} < R/\sqrt{n}$ and B_R contains a cube $Q_{2^{m-m_0-1}}$ of side length 2^{m-m_0-1}. Hence

$$\frac{\|\chi_{B_{\tau R}}\|_{X(\mathbb{R}^n)}}{\|\chi_{B_R}\|_{X(\mathbb{R}^n)}} \le \frac{\|\chi_{Q_{2^m}}\|_{X(\mathbb{R}^n)}}{\|\chi_{Q_{2^{m-m_0-1}}}\|_{X(\mathbb{R}^n)}} \le 2^{(m_0+1)n} \le \left(4\sqrt{n}\,\tau\right)^n,$$

where the second inequality is obtained by applying (3.7) $m_0 + 1$ times. □

Lemma 3.4(b) immediately yields the following.

COROLLARY 3.5. *If $X(\mathbb{R}^n)$ is a translation-invariant Banach function space, then it satisfies the doubling property.*

3.4. *Translation-invariant spaces with exponential weights fail the weak doubling property*

THEOREM 3.6. *Suppose that $X(\mathbb{R})$ is a translation-invariant Banach function space. If $w(x) := e^{cx}$ for $x \in \mathbb{R}$ with a constant $c > 0$, then the weighted Banach function space $X(\mathbb{R}, w)$ does not satisfy the weak doubling property.*

Proof. Let $\tau > 1$. By the second inequality in (3.5), for every $y \in \mathbb{R}$ and every $R \ge 1$, one has

$$\|\chi_{B(y,R)}\|_{X(\mathbb{R},w)} \le e^{c(y+R)}\|\chi_{B(y,R)}\|_{X(\mathbb{R})} \le C_2 e^{c(y+R)} R.$$

Set

$$r := \frac{\tau - 1}{4} R, \qquad z := y + \left(\frac{3}{4}\tau + \frac{1}{4}\right) R.$$

It is easy to see that $B(z, r) \subset B(y, \tau R)$ and $x \ge y + \frac{\tau+1}{2} R$ for all $x \in B(z, r)$. Then these observations and the first inequality in (3.5) imply that

$$\begin{aligned}\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R},w)} &\ge \|\chi_{B(z,r)}\|_{X(\mathbb{R},w)} \ge e^{c\left(y+\frac{\tau+1}{2}R\right)}\|\chi_{B(z,r)}\|_{X(\mathbb{R})} \\ &\ge e^{c\left(y+\frac{\tau+1}{2}R\right)} C_1 \min\{1, r\} = C_1 e^{c\left(y+\frac{\tau+1}{2}R\right)} \min\left\{1, \frac{\tau-1}{4} R\right\}.\end{aligned}$$

Hence

$$\inf_{y\in\mathbb{R}^n} \frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R},w)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R},w)}} \ge \frac{C_1}{C_2} \frac{e^{c\,\frac{\tau-1}{2} R}}{R} \to \infty \ \text{ as } \ R \to \infty,$$

and $X(\mathbb{R}, w)$ does not satisfy the weak doubling property. □

THEOREM 3.7. *Suppose that $X(\mathbb{R}^n)$ is a translation-invariant Banach function space. If $w(x) := e^{c|x|}$ for $x \in \mathbb{R}^n$ with a constant $c > 0$, then the weighted Banach function space $X(\mathbb{R}^n, w)$ does not satisfy the weak doubling property.*

Proof. The proof is similar to that of Theorem 3.6. Let $\tau > 1$. It follows from the second inequality in (3.5) that for every $y \in \mathbb{R}^n$ and every $R \ge 1$, one has

$$\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n,w)} \le e^{c(|y|+R)}\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n)} \le C_2 e^{c(|y|+R)} R^n.$$

Set

$$r := \frac{\tau-1}{4}R, \quad z := \begin{cases} y + \left(\frac{3}{4}\tau + \frac{1}{4}\right) R\frac{y}{|y|}, & y \neq 0, \\ \left(\frac{3}{4}\tau + \frac{1}{4}\right) Re_1, & y = 0, \end{cases}$$

where $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^n$. It is not difficult to see that $B(z, r) \subset B(y, \tau R)$ and

$$|x| \geq |y| + \frac{\tau+1}{2}R \quad \text{for all} \quad x \in B(z, r).$$

Hence, taking into account the first inequality in (3.5), we obtain

$$\begin{aligned} \|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n,w)} &\geq \|\chi_{B(z,r)}\|_{X(\mathbb{R}^n,w)} \geq e^{c\left(|y|+\frac{\tau+1}{2}R\right)}\|\chi_{B(z,r)}\|_{X(\mathbb{R}^n)} \\ &\geq e^{c\left(|y|+\frac{\tau+1}{2}R\right)}C_1 \min\{1, r^n\} = C_1 e^{c\left(|y|+\frac{\tau+1}{2}R\right)} \min\left\{1, \left(\frac{\tau-1}{4}R\right)^n\right\}. \end{aligned}$$

Thus,

$$\inf_{y\in\mathbb{R}^n} \frac{\|\chi_{B(y,\tau R)}\|_{X(\mathbb{R}^n,w)}}{\|\chi_{B(y,R)}\|_{X(\mathbb{R}^n,w)}} \geq \frac{C_1}{C_2} \frac{e^{c\,\frac{\tau-1}{2}R}}{R^n} \to \infty \quad \text{as} \quad R \to \infty,$$

and $X(\mathbb{R}^n, w)$ does not satisfy the weak doubling property. □

3.5. *Comparison of the doubling property and the weak doubling property*

LEMMA 3.8. *If $X(\mathbb{R}^n)$ is a Banach function space satisfying the doubling property, then the function*

$$f_X(R) := \|\chi_{B(0,R)}\|_{X(\mathbb{R}^n)}, \quad R \in (0, \infty), \tag{3.12}$$

cannot grow faster than polynomially as $R \to +\infty$.

Proof. The proof is analogous to the proof of [**30**, Lemma 5.2.4]. Suppose there exist $\tau > 1$ and $C_\tau > 0$ such that (3.2) holds for $y = 0$ and any $R > 1$. Then there exists $m \in \mathbb{N}$ such that $\tau^{m-1} < R \leq \tau^m$. Applying (3.2) m times, one gets

$$f_X(R) \leq f_X(\tau^m) \leq C_\tau^m f_X(1) = \tau^{m \log_\tau C_\tau} f_X(1) < R^{\log_\tau C_\tau} \tau^{\log_\tau C_\tau} f_X(1) = C_\tau f_X(1) R^{\log_\tau C_\tau},$$

which completes the proof. □

On the other hand, we will show that the weak doubling property of a Banach function space $X(\mathbb{R}^n)$ allows the function f_X given by (3.12) to grow at any subexponential rate as $R \to +\infty$. In fact, we will show that if a weight w grows at a subexponential rate in an open cone and $Y(\mathbb{R}^n)$ is a translation-invariant Banach function space, then the weighted Banach function space $X(\mathbb{R}^n) = Y(\mathbb{R}^n, w)$ satisfies the weak doubling property.

Inequalities (3.3) and (3.6) yield the following.

LEMMA 3.9. *Let $Y(\mathbb{R}^n)$ be a translation-invariant Banach function space. Suppose that $w : \mathbb{R}^n \to [0, \infty]$ is a weight satisfying $w, 1/w \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. If there exist a number $\tau > 1$, a constant $c_\tau > 0$, a sequence $\{R_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ satisfying $R_j \to \infty$ as $j \to \infty$, and a sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that*

$$\frac{\operatorname*{ess\,sup}_{x\in B(y_j,\tau R_j)} w(x)}{\operatorname*{ess\,inf}_{x\in B(y_j,R_j)} w(x)} \leq c_\tau \quad \textit{for all} \quad j \in \mathbb{N}, \tag{3.13}$$

then the weighted Banach function space $X(\mathbb{R}^n) = Y(\mathbb{R}^n, w)$ satisfies the weak doubling property.

LEMMA 3.10. *Let $Y(\mathbb{R}^n)$ be a translation-invariant Banach function space and φ be a nonincreasing function such that $\varphi(r) \to 0$ as $r \to +\infty$ and $r\varphi(r)$ is nondecreasing for $r \geq 1$. Suppose $w : \mathbb{R}^n \to [0, \infty]$ is a weight satisfying $w, 1/w \in L^\infty_{\rm loc}(\mathbb{R}^n)$ and there exist $C_0, C_1 > 0$, $\gamma > 0$, $\theta \in \mathbb{R}^n$ with $|\theta| = 1$ such that*

$$C_0 \exp(|x|\varphi(|x|)) \leq w(x) \leq C_1 \exp(|x|\varphi(|x|)) \quad \text{when} \quad \left|\frac{x}{|x|} - \theta\right| < \gamma, \; |x| \geq 1. \tag{3.14}$$

Then $X(\mathbb{R}^n) = Y(\mathbb{R}^n, w)$ satisfies the weak doubling property and there exists a constant $C > 0$ such that

$$f_X(R) \geq C \exp\left(R\varphi(R)\right) \tag{3.15}$$

for all sufficiently large R, where the function $f_X : (0, \infty) \to (0, \infty)$ is given by (3.12).

Proof. Let us show that (3.13) is satisfied for $R_j = \varphi^{-1/2}(j)$ and $y_j = (j + m)\theta$ with a sufficiently large $m > 0$. Indeed, since

$$\frac{R_j}{j} = \frac{\varphi^{1/2}(j)}{j\varphi(j)} \leq \frac{\varphi^{1/2}(j)}{\varphi(1)} \to 0 \quad \text{as} \quad j \to \infty,$$

the balls $B(y_j, \tau R_j)$ lie in the cone $\left|\frac{x}{|x|} - \theta\right| < \gamma$ provided m is sufficiently large. Then for all $j \in \mathbb{N}$,

$$\begin{aligned}
\frac{\operatorname*{ess\,sup}_{x \in B(y_j, \tau R_j)} w(x)}{\operatorname*{ess\,inf}_{x \in B(y_j, R_j)} w(x)} &\leq \frac{C_1 \exp\left[\left(j + m + \tau\varphi^{-1/2}(j)\right) \varphi\left(j + m + \tau\varphi^{-1/2}(j)\right)\right]}{C_0 \exp\left[\left(j + m - \varphi^{-1/2}(j)\right) \varphi\left(j + m - \varphi^{-1/2}(j)\right)\right]} \\
&\leq \frac{C_1 \exp\left[\left(j + m + \tau\varphi^{-1/2}(j)\right) \varphi(j + m)\right]}{C_0 \exp\left[\left(j + m - \varphi^{-1/2}(j)\right) \varphi(j + m)\right]} \\
&= \frac{C_1}{C_0} \exp\left[(\tau + 1)\omega^{-1/2}(j)\varphi(j + m)\right] \\
&\leq \frac{C_1}{C_0} \exp\left[(\tau + 1)\varphi^{1/2}(j)\right] \leq \frac{C_1}{C_0} \exp\left[(\tau + 1)\varphi^{1/2}(1)\right].
\end{aligned}$$

By Lemma 3.9, $X(\mathbb{R}^n) = Y(\mathbb{R}^n, w)$ satisfies the weak doubling property. Since $Y(\mathbb{R}^n)$ is translation-invariant and $r\varphi(r)$ is nonincreasing, it follows from (3.14) that

$$\begin{aligned}
f_X(R) &\geq \|\chi_{B((R-1)\theta,1)}\|_{Y(\mathbb{R}^n,w)} \geq C_0 \exp((R - 2)\varphi(R - 2))\|\chi_{B((R-1)\theta,1)}\|_{Y(\mathbb{R}^n)} \\
&= C_0 \exp((R - 2)\varphi(R - 2))\|\chi_{B(0,1)}\|_{Y(\mathbb{R}^n)} \geq C_0 \exp(R\varphi(R))\|\chi_{B(0,1)}\|_{Y(\mathbb{R}^n)}
\end{aligned}$$

for all sufficiently large R, i.e. (3.15) is satisfied with $C = C_0\|\chi_{B(0,1)}\|_{Y(\mathbb{R}^n)}$. □

Let $1 \leq p \leq \infty$. It follows from Lemma 3.9 that $X(\mathbb{R}) = L^p(\mathbb{R}, w)$ has the weak doubling property if $w : \mathbb{R} \to [0, \infty]$ satisfies $w, 1/w \in L^\infty_{\rm loc}(\mathbb{R})$ and $w(x)$ is equal to, e.g., $(1 + x)^\alpha$, $\alpha > 0$; $\exp\left(x^\beta\right)$, $\beta \in (0, 1)$; or $\exp\left(\frac{x}{\log\log(3+x)}\right)$ for $x > 0$. On the other hand, Theorems 3.6 and 3.7 imply that $L^p(\mathbb{R}, w)$ does not satisfy the weak doubling property for $w(x) = e^{cx}$ or $w(x) = e^{c|x|}$, $x \in \mathbb{R}$, with any $c > 0$.

4. *Proof of the main result*

4.1. *The case of $a \in L_{1,\sigma}(\mathbb{R}^n)$ for some $\sigma \in \mathbb{R}$*

THEOREM 4.1. *Let $X(\mathbb{R}^n)$ be a Banach function space satisfying the weak doubling property. If $\sigma \in \mathbb{R}$ and $a \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L_{1,\sigma}(\mathbb{R}^n)$, then $a \in L^\infty(\mathbb{R}^n)$ and*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}. \tag{4.1}$$

Proof. Let $D_{X,\varrho}$ be defined for all $\varrho > 1$ by (3.1). If, for some $\varrho > 1$, the quantity $D_{X,\varrho}$ is infinite, then it is obvious that

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le D_{X,\varrho}\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}. \tag{4.2}$$

Since $X(\mathbb{R}^n)$ satisfies the weak doubling property, there exists $\varrho > 1$ such that $D_{X,\varrho} < \infty$. Take an arbitrary Lebesgue point $\eta \in \mathbb{R}^n$ of the function a. Let an even function $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy the following conditions:

$$0 \le \varphi \le 1, \quad \varphi(x) = 1 \text{ for } |x| \le 1, \quad \varphi(x) = 0 \text{ for } |x| \ge \varrho.$$

Let

$$f_{\delta,\eta}(x) := e^{i\eta x}\varphi(\delta x), \quad x \in \mathbb{R}^n, \quad \delta > 0,$$

and

$$f_{\delta,\eta,y}(x) := f_{\delta,\eta}(x-y), \quad y \in \mathbb{R}^n.$$

Then

$$\begin{aligned}(Ff_{\delta,\eta,y})(\xi) &= e^{-i\xi y}(Ff_{\delta,\eta})(\xi) = e^{-i\xi y}\delta^{-n}(F\varphi)\left(\frac{\xi-\eta}{\delta}\right)\\ &= e^{-i\xi y}(F\varphi)_\delta(\xi-\eta) = e^{-i\xi y}(F\varphi)_\delta(\eta-\xi)\end{aligned}$$

and

$$\begin{aligned}\left(F^{-1}aFf_{\delta,\eta,y}\right)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}a(\xi)(F\varphi)_\delta(\eta-\xi)\,d\xi,\\ a(\eta)f_{\delta,\eta,y}(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}a(\eta)(F\varphi)_\delta(\eta-\xi)\,d\xi.\end{aligned}$$

Hence, for all $x, y \in \mathbb{R}^n$ and $\delta > 0$,

$$\begin{aligned}\left|\left(F^{-1}aFf_{\delta,\eta,y}\right)(x) - a(\eta)f_{\delta,\eta,y}(x)\right| &= \frac{1}{(2\pi)^n}\left|\int_{\mathbb{R}^n} e^{i(x-y)\xi}(a(\xi)-a(\eta))(F\varphi)_\delta(\eta-\xi)\,d\xi\right|\\ &\le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|a(\xi)-a(\eta)|\,|(F\varphi)_\delta(\eta-\xi)|\,d\xi.\end{aligned}$$

Since $F\varphi \in S(\mathbb{R}^n)$ and η is a Lebesgue point of a, it follows from Lemma 2.16 that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $x, y \in \mathbb{R}^n$ and all $\delta \in (0, \delta_\varepsilon)$,

$$\left|\left(F^{-1}aFf_{\delta,\eta,y}\right)(x) - a(\eta)f_{\delta,\eta,y}(x)\right| < \varepsilon.$$

It is clear that $|f_{\delta,\eta,y}|\chi_{B(y,1/\delta)} = \chi_{B(y,1/\delta)}$. Then the above inequality implies that for all $y \in \mathbb{R}^n$ and $\delta \in (0, \delta_\varepsilon)$,

$$|a(\eta)|\chi_{B(y,1/\delta)} \le \left|F^{-1}aFf_{\delta,\eta,y}\right| + \varepsilon\chi_{B(y,1/\delta)}.$$

Hence

$$\begin{aligned}|a(\eta)|\,\left\|\chi_{B(y,1/\delta)}\right\|_{X(\mathbb{R}^n)} &\le \left\|F^{-1}aFf_{\delta,\eta,y}\right\|_{X(\mathbb{R}^n)} + \varepsilon\left\|\chi_{B(y,1/\delta)}\right\|_{X(\mathbb{R}^n)}\\ &\le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\left\|f_{\delta,\eta,y}\right\|_{X(\mathbb{R}^n)} + \varepsilon\left\|\chi_{B(y,1/\delta)}\right\|_{X(\mathbb{R}^n)}\\ &\le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\left\|\chi_{B(y,\varrho/\delta)}\right\|_{X(\mathbb{R}^n)} + \varepsilon\|\chi_{B(y,1/\delta)}\|_{X(\mathbb{R}^n)}. \end{aligned}\tag{4.3}$$

Since $D_{X,\varrho} < \infty$, the definition of $D_{X,\varrho}$ given in (3.1) implies that there exist $\delta \in (0, \delta_\varepsilon)$ and $y \in \mathbb{R}^n$ such that

$$\frac{\left\|\chi_{B(y,\varrho/\delta)}\right\|_{X(\mathbb{R}^n)}}{\left\|\chi_{B(y,1/\delta)}\right\|_{X(\mathbb{R}^n)}} \le D_{X,\varrho} + \varepsilon.$$

Choosing these δ and y, and dividing both sides of inequality (4.3) by $\left\|\chi_{B(y,1/\delta)}\right\|_{X(\mathbb{R}^n)}$, we get

$$|a(\eta)| \le (D_{X,\varrho} + \varepsilon)\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} + \varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

Hence, for all Lebesgue points $\eta \in \mathbb{R}^n$ of the function a, we have

$$|a(\eta)| \le D_{X,\varrho}\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}.$$

Since $a \in L_{1,\sigma}(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$, almost all points $\eta \in \mathbb{R}^n$ are Lebesgue points of the function a in view of the Lebesgue differentiation theorem (see, e.g., [**18**, Corollary 2.1.16 and Exercise 2.1.10]). Therefore $a \in L^\infty(\mathbb{R}^n)$ and inequality (4.2) is fulfilled for all $\varrho > 1$. It is now left to apply Lemma 3.1. □

4.2. *Proof of Theorem 1.3 for arbitrary* $a \in \mathcal{M}_{X(\mathbb{R}^n)} \subset S'(\mathbb{R}^n)$

For a function $w \in S(\mathbb{R}^n)$, we will use the following notation

$$\widehat{w} := Fw, \quad \check{w} := F^{-1}w, \quad \widetilde{w}(\xi) := w(-\xi), \quad (\tau_\zeta w)(\xi) := w(\xi - \zeta), \quad e_\zeta(x) := e^{i\zeta x}.$$

Let a nonnegative even function $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfy the condition

$$\int_{\mathbb{R}^n} \psi(\xi)\,d\xi = 1,$$

and let

$$\psi_\varepsilon(\xi) := \varepsilon^{-n}\psi(\xi/\varepsilon), \quad \varepsilon > 0.$$

Fix $\varepsilon > 0$ and take arbitrary functions $u \in S(\mathbb{R}^n) \cap X(\mathbb{R}^n)$ and $v \in C_0^\infty(\mathbb{R}^n)$. Then we have $\widehat{u}, \widehat{v} \in S(\mathbb{R}^n)$. In view of [**29**, Theorem 7.19(b)] or [**18**, Theorem 2.3.20], we observe that

$$a * \psi_\varepsilon \in C^\infty_{\rm poly}(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \tag{4.4}$$

because $a \in S'(\mathbb{R}^n)$ and $\psi_\varepsilon \in S(\mathbb{R}^n)$. Then $(a * \psi_\varepsilon)\widehat{u} \in S(\mathbb{R}^n)$. By [**18**, Theorem 2.2.14],

$$\begin{aligned}\int_{\mathbb{R}^n}\big(F^{-1}(a*\psi_\varepsilon)Fu\big)(x)v(x)\,dx &= \int_{\mathbb{R}^n}\big(F^{-1}(a*\psi_\varepsilon)\widehat{u}\big)\widehat{\ }(\xi)\check{v}(\xi)\,d\xi\\ &= \int_{\mathbb{R}^n}(a*\psi_\varepsilon)(\xi)\widehat{u}(\xi)\check{v}(\xi)\,d\xi.\end{aligned}\tag{4.5}$$

Observe that $\widehat{u}\check{v} \in S(\mathbb{R}^n)$ in view of [**18**, Proposition 2.2.7]. Then, taking into account that $a \in S'(\mathbb{R}^n)$ and $\psi_\varepsilon \in S(\mathbb{R}^n)$, it follows from [**18**, Theorem 2.2.14] or [**29**, Theorem 7.19(d)] that

$$\begin{aligned}\int_{\mathbb{R}^n}(a*\psi_\varepsilon)(\xi)\widehat{u}(\xi)\check{v}(\xi)\,d\xi &= \big((a*\psi_\varepsilon)*(\widehat{u}\check{v})^\sim\big)(0) = \big((a*(\widehat{u}\check{v})^\sim)*\psi_\varepsilon\big)(0)\\ &= \int_{\mathbb{R}^n}\left(a*(\widehat{u}\check{v})^\sim\right)(\zeta)\widetilde{\psi}_\varepsilon(\zeta)\,d\zeta = \int_{\mathbb{R}^n}\langle a, \tau_\zeta(\widehat{u}\check{v})\rangle\,\psi_\varepsilon(\zeta)\,d\zeta.\end{aligned}\tag{4.6}$$

Since $\tau_\zeta \widehat{u} \in S(\mathbb{R}^n) \subset C^\infty_{\rm poly}(\mathbb{R}^n)$, by the definition of multiplication of $a \in S'(\mathbb{R}^n)$ by a function in $C^\infty_{\rm poly}(\mathbb{R}^n)$, we have

$$\langle a, \tau_\zeta(\widehat{u}\check{v})\rangle = \langle a, \tau_\zeta\widehat{u} \cdot \tau_\zeta\check{v}\rangle = \langle a\tau_\zeta\widehat{u}, \tau_\zeta\check{v}\rangle. \tag{4.7}$$

It is easy to see that $\tau_\zeta\widehat{u} = F(e_\zeta u)$ and $\tau_\zeta\check{v} = F^{-1}(e_{-\zeta}v)$. Then

$$\langle a, \tau_\zeta(\widehat{u}\check{v})\rangle = \langle aF(e_\zeta u), F^{-1}(e_{-\zeta}v)\rangle. \tag{4.8}$$

By the definition of the inverse Fourier transform of $aF(e_\zeta u) \in S'(\mathbb{R}^n)$, we have

$$\langle aF(e_\zeta u), F^{-1}(e_{-\zeta}v)\rangle = \langle F^{-1}aF(e_\zeta u), e_{-\zeta}v\rangle. \tag{4.9}$$

Combining (4.5)–(4.9), we arrive at the following equality:

$$\int_{\mathbb{R}^n} \big(F^{-1}(a * \psi_\varepsilon)Fu\big)(x)v(x)\,dx = \int_{\mathbb{R}^n} \langle F^{-1}aF(e_\zeta u), e_{-\zeta}v\rangle\psi_\varepsilon(\zeta)\,d\zeta. \tag{4.10}$$

Let $s \in S_0(\mathbb{R}^n)$ be such that $\|s\|_{X'(\mathbb{R}^n)} \le 1$. Put $K := \operatorname{supp} s$ and consider a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \phi \le 1$. By Lemma 2.7, there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} v_j \subseteq K^*$ and $\|v_j\|_{L^\infty(\mathbb{R}^n)} \le \|s\|_{L^\infty(\mathbb{R}^n)}$ for all $j \in \mathbb{N}$ and $v_j \to s$ a.e. on $\mathbb{R}^n$ as $j \to \infty$. Since $(a*\psi_\varepsilon)Fu$ belongs to $S(\mathbb{R}^n)$, we have $F^{-1}(a*\psi_\varepsilon)Fu \in S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Therefore $\phi \cdot F^{-1}(a*\psi_\varepsilon)Fu \cdot s$ also belongs to $L^1(\mathbb{R}^n)$ and it follows from the Lebesgue dominated convergence theorem that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} \phi(x)\big(F^{-1}(a*\psi_\varepsilon)Fu\big)(x)v_j(x)\,dx = \int_{\mathbb{R}^n} \phi(x)\big(F^{-1}(a*\psi_\varepsilon)Fu\big)(x)s(x)\,dx. \tag{4.11}$$

Further, $|v_j| \le \|s\|_{L^\infty(\mathbb{R}^n)}\chi_{K^*}$ for all $j \in \mathbb{N}$. Since $a \in \mathcal{M}_{X(\mathbb{R}^n)}$, one has $F^{-1}aF(e_{-\zeta}u) \in X(\mathbb{R}^n)$, and it follows from axiom (A5) that $\big(F^{-1}aF(e_\zeta u)\big)\chi_{K^*} \in L^1(\mathbb{R}^n)$. Hence, in view of the Lebesgue dominated convergence theorem, for all $\zeta \in \mathbb{R}^n$,

$$\lim_{j\to\infty}\langle \phi F^{-1}aF(e_\zeta u), e_{-\zeta}v_j\rangle = \int_{\mathbb{R}^n} \phi(x)\left(F^{-1}aF(e_\zeta u)\right)(x)\,(e_{-\zeta}s)\,(x)\,dx. \tag{4.12}$$

Hölder's inequality for $X(\mathbb{R}^n)$ (see [**1**, Chap. 1, Theorem 2.4]) implies for all $\zeta \in \mathbb{R}^n$,

$$\begin{aligned}\big|\langle \phi F^{-1}aF(e_\zeta u), e_{-\zeta}v_j\rangle\big| &\le \|s\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\big|\left(F^{-1}aF(e_\zeta u)\right)(x)\big|\,\chi_{K^*}(x)\,dx\\ &\le \|s\|_{L^\infty(\mathbb{R}^n)}\|F^{-1}aF(e_\zeta u)\|_{X(\mathbb{R}^n)}\|\chi_{K^*}\|_{X'(\mathbb{R}^n)}\\ &\le \|s\|_{L^\infty(\mathbb{R}^n)}\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\|e_\zeta u\|_{X(\mathbb{R}^n)}\|\chi_{K^*}\|_{X'(\mathbb{R}^n)}\\ &= \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\|u\|_{X(\mathbb{R}^n)}\|s\|_{L^\infty(\mathbb{R}^n)}\|\chi_{K^*}\|_{X'(\mathbb{R}^n)}.\end{aligned} \tag{4.13}$$

Taking into account (4.12)–(4.13) and using the Lebesgue dominated convergence theorem again, one gets

$$\begin{aligned}&\lim_{j\to\infty}\int_{\mathbb{R}^n}\langle \phi F^{-1}aF(e_\zeta u), e_{-\zeta}v_j\rangle\psi_\varepsilon(\zeta)\,d\zeta\\ &\quad= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\phi(x)\left(F^{-1}aF(e_\zeta u\right)(x)\,(e_{-\zeta}s)\,(x)\,dx\right)\psi_\varepsilon(\zeta)\,d\zeta.\end{aligned} \tag{4.14}$$

It follows from (4.11), (4.14), and (4.10) with ϕv_j in place of v that

$$\begin{aligned}&\int_{\mathbb{R}^n}\phi(x)\big(F^{-1}(a*\psi_\varepsilon)Fu\big)(x)s(x)\,dx\\ &\quad= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\phi(x)\left(F^{-1}aF(e_\zeta u\right)(x)\,(e_{-\zeta}s)\,(x)\,dx\right)\psi_\varepsilon(\zeta)\,d\zeta.\end{aligned} \tag{4.15}$$

Similarly to (4.13) one gets

$$\left|\int_{\mathbb{R}^n} \phi(x) \left(F^{-1}aF(e_\zeta u)\,(x)\,(e_{-\zeta}s)\,(x)\,dx\right| \le \|F^{-1}aF(e_\zeta u)\|_{X(\mathbb{R}^n)}\|e_{-\zeta}s\|_{X'(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_X(\mathbb{R}^n)}\|u\|_{X(\mathbb{R}^n)}\|s\|_{X'(\mathbb{R}^n)}. \tag{4.16}$$

Equality (4.15) and inequality (4.16) immediately yield that for all $u \in S(\mathbb{R}^n) \cap X(\mathbb{R}^n)$, all $s \in S_0(\mathbb{R}^n)$ satisfying $\|s\|_{X'(\mathbb{R}^n)} \le 1$, and all $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $0 \le \phi \le 1$, one has

$$\left|\int_{\mathbb{R}^n} \phi(x)\big(F^{-1}(a*\psi_\varepsilon)Fu\big)(x)s(x)\,dx\right| \le \|a\|_{\mathcal{M}_X(\mathbb{R}^n)}\|u\|_{X(\mathbb{R}^n)}\|s\|_{X'(\mathbb{R}^n)}. \tag{4.17}$$

Now, take functions $\phi_j \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \phi_j \le 1$ and $\phi_j(x) = 1$ for all $|x| \le j$ and all $j \in \mathbb{N}$. Since $F^{-1}(a*\psi_\varepsilon)Fu \in S(\mathbb{R}^n)$, we have $\phi_j F^{-1}(a*\psi_\varepsilon)Fu \in C_0^\infty(\mathbb{R}^n)$ for all $j \in \mathbb{N}$. Inequality (4.17) implies that for all $s \in S_0(\mathbb{R}^n)$ satisfying $\|s\|_{X'(\mathbb{R}^n)} \le 1$ and all $j \in \mathbb{N}$, one has

$$\left|\int_{\mathbb{R}^n} \phi_j(x)F^{-1}(a*\psi_\varepsilon)Fu(x)s(x)\,dx\right| \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\|u\|_{X(\mathbb{R}^n)}\|s\|_{X'(\mathbb{R}^n)}.$$

Hence it follows from Lemma 2.10 that for all $j \in \mathbb{N}$,

$$\|\phi_j F^{-1}(a*\psi_\varepsilon)Fu\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\|u\|_{X(\mathbb{R}^n)}.$$

Since the functions $\phi_j F^{-1}(a*\psi_\varepsilon)Fu$ converge to $F^{-1}(a*\psi_\varepsilon)Fu$ everywhere as $j \to \infty$, Fatou's lemma (see [**1**, Chap. 1, Lemma 1.5]) implies that $F^{-1}(a*\psi_\varepsilon)Fu \in X(\mathbb{R}^n)$ and

$$\|F^{-1}(a*\psi_\varepsilon)Fu\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\|u\|_{X(\mathbb{R}^n)}$$

for all $u \in S(\mathbb{R}^n) \cap X(\mathbb{R}^n)$. Thus

$$\|a*\psi_\varepsilon\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}.$$

Since $C^\infty_{\rm poly}(\mathbb{R}^n) \subset L_{1,\sigma}(\mathbb{R}^n)$ for some $\sigma \in \mathbb{R}$ and $a*\psi_\varepsilon \in C^\infty_{\rm poly}(\mathbb{R}^n)$, it follows from Theorem 4.1 that $a*\psi_\varepsilon \in L^\infty(\mathbb{R}^n)$ and

$$\|a*\psi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \|a*\psi_\varepsilon\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \quad \text{for all} \quad \varepsilon > 0.$$

It is not difficult to see that $w*\psi_\varepsilon$ converges to w in $S(\mathbb{R}^n)$ as $\varepsilon \to 0$ for every $w \in S(\mathbb{R}^n)$ (see, e.g., [**18**, Exercise 2.3.2]). Then, for all $w \in S(\mathbb{R}^n)$,

$$|\langle a, w\rangle| = \lim_{\varepsilon\to 0}|\langle a, w*\psi_\varepsilon\rangle| = \lim_{\varepsilon\to 0}|\langle a*\psi_\varepsilon, w\rangle| = \lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^n}(a*\psi_\varepsilon)(x)w(x)\,dx\right| \le \limsup_{\varepsilon\to 0}\|a*\psi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|w\|_{L^1(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}\|w\|_{L^1(\mathbb{R}^n)}.$$

Hence a can be extended to a bounded linear functional on $L^1(\mathbb{R}^n)$, i.e., it can be identified with a function in $L^\infty(\mathbb{R}^n)$ and $\|a\|_{L^\infty} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$ holds.

Suppose there exists a constant $D_X > 0$ such that $\|a\|_{L^\infty(\mathbb{R}^n)} \le D_X\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$ for all $a \in \mathcal{M}_{X(\mathbb{R}^n)}$. Then taking $a \equiv 1$, one gets $D_X \ge 1$. So, the constant $D_X = 1$ in the estimate $\|a\|_{L^\infty} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$ is best possible, which completes the proof of Theorem 1.3.

4.3. *Multidimensional analogue of Theorem 1.1*

COROLLARY 4.2. *Suppose $n \ge 1$ and $X(\mathbb{R}^n)$ is a Banach function space satisfying the A_X-condition. If $a \in \mathcal{M}_{X(\mathbb{R}^n)} \subset S'(\mathbb{R}^n)$, then $a \in L^\infty(\mathbb{R}^n)$ and*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}.$$

The constant 1 on the right-hand side in the above inequality is best possible.

Proof. This statement follows from Theorem 1.3 and Lemma 3.3. □

We conclude this section with the proof of the following multidimensional analogue of Theorem 1.1.

COROLLARY 4.3. *Let $n \geq 1$ and $1 < p < \infty$. If $w \in A_p(\mathbb{R}^n)$ and $a \in \mathcal{M}_{L^p(\mathbb{R}^n,w)} \subset S'(\mathbb{R}^n)$, then $a \in L^\infty(\mathbb{R}^n)$ and*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \leq \|a\|_{\mathcal{M}_{L^p(\mathbb{R}^n,w)}}.$$

The constant 1 on the right-hand side in the above inequality is best possible.

Proof. Since $w \in A_p(\mathbb{R}^n)$, we have

$$\begin{aligned}\|w\|_{A_p(\mathbb{R}^n)} &= \sup_Q \frac{1}{|Q|}\|\chi_Q\|_{L^p(\mathbb{R}^n,w)}\|\chi_Q\|_{L^{p'}(\mathbb{R}^n,w^{-1})} \\ &= \sup_Q \left(\frac{1}{|Q|}\int_Q w^p(x)\,dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q w^{-p'}(x)\,dx\right)^{1/p'} < \infty,\end{aligned}$$

where $1/p + 1/p' = 1$ and the supremum is taken over all cubes with sides parallel to the axes. Thus, the Banach function space $X(\mathbb{R}^n) = L^p(\mathbb{R}^n, w)$ satisfies the A_X-condition. It remains to apply Corollary 4.2. □

We would like to stress again that the A_p condition implies the doubling property (see Lemma 3.3), which places much stronger restrictions on the behaviour of the weight w at infinity than the weak doubling property (see Subsection 3.5). Note also that the latter puts no restrictions on the local behaviour of the weight w. When dealing with weighted function spaces $Y(\mathbb{R}^n, w)$, we usually assume that $w \in Y_{\mathrm{loc}}(\mathbb{R}^n)$ and $1/w \in Y'_{\mathrm{loc}}(\mathbb{R}^n)$. It is instructive to compare the latter conditions with $w \in A_p(\mathbb{R}^n)$ in the case $n = 1$, $Y(\mathbb{R}) = L^p(\mathbb{R})$ and

$$w(x) = \begin{cases} |x|^\alpha, & -1 \leq x < 0, \\ x^\beta, & 0 \leq x \leq 1, \\ 1, & |x| > 1, \end{cases} \qquad \alpha, \beta \in \mathbb{R}.$$

It is easy to see that $w \in L^p_{\mathrm{loc}}(\mathbb{R})$ and $1/w \in L^{p'}_{\mathrm{loc}}(\mathbb{R})$ if and only if $-1/p < \alpha, \beta < 1 - 1/p$. On the other hand, $w \notin A_p(\mathbb{R})$ if $\alpha \neq \beta$ (see [**5**, Example 2.6]).

5. *On optimality of the requirement of the weak doubling property in Theorem 1.3*

5.1. *Estimates for convolutions*

THEOREM 5.1 (Cf. [**25**, p. 91]). *Let w^* be a weight such that $w^* \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let $\Omega \subseteq \mathbb{R}^n$ be a set of positive measure, and let*

$$L^1_\Omega(\mathbb{R}, w^*) := \left\{h \in L^1(\mathbb{R}, w^*) \ : \ \operatorname{supp} h \subseteq \Omega\right\}.$$

Suppose $Y(\mathbb{R}^n)$ is a translation-invariant Banach function space and w is a weight satisfying $w \in Y_{\mathrm{loc}}(\mathbb{R}^n)$ and $1/w \in Y'_{\mathrm{loc}}(\mathbb{R}^n)$. If

$$w^*(y)w(x-y)w^{-1}(x) \geq 1 \quad \textit{for all} \quad x \in \mathbb{R}^n,\ y \in \Omega, \tag{5.1}$$

then

$$\|\kappa * f\|_{Y(\mathbb{R}^n,w)} \leq \|\kappa\|_{L^1(\mathbb{R}^n,w^*)}\|f\|_{Y(\mathbb{R}^n,w)} \quad \textit{for all} \quad f \in Y(\mathbb{R}^n, w),\ \kappa \in L^1_\Omega(\mathbb{R}, w^*).$$

Proof. Since $w \in Y_{\rm loc}(\mathbb{R}^n)$ and $1/w \in Y'_{\rm loc}(\mathbb{R}^n)$, we recall that $Y(\mathbb{R}^n, w)$ is a Banach function space and $Y'(\mathbb{R}^n, w^{-1})$ is its associate space in view of [**22**, Lemma 2.4]. Using (5.1) and Hölder's inequality for Banach function spaces (see [**1**, Chap. 1, Theorem 2.4]) and taking into account that $Y(\mathbb{R}^n)$ is translation-invariant, one gets for all $g \in Y'(\mathbb{R}^n, w^{-1})$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(\kappa * f)(x)g(x)\,dx\right| &\le \int_{\mathbb{R}^n}\left(\int_\Omega |\kappa(y)|\,|f(x-y)|\,dy\right)|g(x)|\,dx\\ &\le \int_\Omega w^*(y)|\kappa(y)|\left(\int_{\mathbb{R}^n} w(x-y)|f(x-y)|\cdot w^{-1}(x)|g(x)|\,dx\right)dy\\ &\le \int_{\mathbb{R}^n} w^*(y)|\kappa(y)|\|\tau_y(wf)\|_{Y(\mathbb{R}^n)}\left\|w^{-1}g\right\|_{Y'(\mathbb{R}^n)}\,dy\\ &= \|wf\|_{Y(\mathbb{R}^n)}\left\|w^{-1}g\right\|_{Y'(\mathbb{R}^n)}\int_{\mathbb{R}^n} w^*(y)|\kappa(y)|\,dy\\ &= \|\kappa\|_{L^1(\mathbb{R}^n,w^*)}\|f\|_{Y(\mathbb{R}^n,w)}\|g\|_{Y'(\mathbb{R}^n,w^{-1})}.\end{aligned}$$

By [**1**, Chap. 1, Theorem 2.7 and Lemma 2.8], the above inequality implies that

$$\begin{aligned}\|\kappa * f\|_{Y(\mathbb{R}^n,w)} &= \sup\left\{\left|\int_{\mathbb{R}^n}(\kappa * f)(x)g(x)\,dx\right| \;:\; g \in Y'(\mathbb{R}^n, w^{-1}),\ \|g\|_{Y'(\mathbb{R}^n,w^{-1})} \le 1\right\}\\ &\le \|\kappa\|_{L^1(\mathbb{R}^n,w^*)}\|f\|_{Y(\mathbb{R}^n,w)},\end{aligned}$$

which completes the proof. □

COROLLARY 5.2. *Suppose $Y(\mathbb{R})$ is a translation-invariant Banach function space.*

(a) *Let $w_1(x) = e^{cx}$ for $x \in \mathbb{R}$, with some constant $c > 0$. Then for all $\kappa \in L^1(\mathbb{R}, w_1)$ and all $f \in Y(\mathbb{R}, w_1)$,*

$$\|\kappa * f\|_{Y(\mathbb{R},w_1)} \le \|\kappa\|_{L^1(\mathbb{R},w_1)}\|f\|_{Y(\mathbb{R},w_1)}.$$

(b) *Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a function such that $\varphi(x) = x$ for $x \le 0$, and*

$$\frac{\varphi(z)-\varphi(x)}{z-x} \ge 1 \quad \text{for all} \quad z > x \ge 0. \tag{5.2}$$

Let $w_2(x) := e^{c\varphi(x)}$ for $x \in \mathbb{R}$, with some constant $c > 0$. Then for all $\kappa \in L^1_{(-\infty,0]}(\mathbb{R}, w_2)$ and all $f \in Y(\mathbb{R}, w_2)$,

$$\|\kappa * f\|_{Y(\mathbb{R},w_2)} \le \|\kappa\|_{L^1(\mathbb{R},w_2)}\|f\|_{Y(\mathbb{R},w_2)}.$$

Proof. (a) Since

$$w_1(y)w_1(x-y)w_1^{-1}(x) = e^{c(y+(x-y)-x)} = 1 \quad \text{for all} \quad x, y \in \mathbb{R},$$

the conditions of Theorem 5.1 are satisfied for $w = w^* = w_1$ and $\Omega = \mathbb{R}$.

(b) Take any $y < 0$ and any $x \in \mathbb{R}$. If $x \ge 0$, then it follows from (5.2) that

$$\frac{\varphi(x-y)-\varphi(x)}{(x-y)-x} \ge 1 \implies y + \varphi(x-y) - \varphi(x) \ge 0.$$

If $y < x < 0$, then

$$\frac{\varphi(x-y)-\varphi(0)}{(x-y)-0} \ge 1 \implies y + \varphi(x-y) - x \ge 0.$$

Finally, if $x \le y < 0$, then

$$\varphi(y) + \varphi(x-y) - \varphi(x) = y + (x-y) - x = 0.$$

Hence

$$w_2(y)w_2(x-y)w_2^{-1}(x) = e^{c(\varphi(y)+\varphi(x-y)-\varphi(x))} \geq 1 \quad \text{for all} \quad x \in \mathbb{R} \tag{5.3}$$

and all $y < 0$. It is clear that (5.3) holds for $y = 0$ as well.

Since w_2 and $1/w_2$ are locally bounded, we have $w_2 \in Y_{\mathrm{loc}}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R})$ and $1/w_2 \in Y'_{\mathrm{loc}}(\mathbb{R})$. Hence the conditions of Theorem 5.1 are satisfied for $w = w^* = w_2$ and $\Omega = (-\infty, 0]$. □

5.2. *Banach function spaces with unbounded Fourier multipliers*

Let $Y(\mathbb{R})$ be a translation-invariant Banach function space and w be a weight. We know that the weak doubling property of the space $X(\mathbb{R}) = Y(\mathbb{R}, w)$ allows the weight w to grow at any subexponential rate (see Subsection 3.5). It is natural to ask whether Theorem 1.3 still holds for $X(\mathbb{R}) = Y(\mathbb{R}, w)$ with the exponential weight $w(x) = e^{cx}$ with $c > 0$. We show in this subsection that the answer is negative and that $\mathcal{M}_{Y(\mathbb{R},w)}$ contains many unbounded functions in this case. This means that the weak doubling property is optimal in a sense. (This also provides an alternative indirect proof of Theorem 3.6).

THEOREM 5.3. *Let $Y(\mathbb{R})$ be a translation-invariant Banach function space and the weights w_1 and w_2 be the same as in Corollary 5.2. Then*

$$F\left(S'(\mathbb{R}) \cap L^1(\mathbb{R}, w_1)\right) \subseteq \mathcal{M}_{Y(\mathbb{R},w_1)}, \quad F\big(S'(\mathbb{R}) \cap L^1_{(-\infty,0]}(\mathbb{R}, w_2)\big) \subseteq \mathcal{M}_{Y(\mathbb{R},w_2)}. \tag{5.4}$$

Proof. If $a \in F\left(S'(\mathbb{R}) \cap L^1(\mathbb{R}, w_1)\right)$, then $F^{-1}a \in L^1(\mathbb{R}, w_1)$. It follows from Corollary 5.2 that for every function $u \in S(\mathbb{R}) \cap Y(\mathbb{R}, w_1)$,

$$\|F^{-1}aFu\|_{Y(\mathbb{R},w_1)} = \|(F^{-1}a) * u\|_{Y(\mathbb{R},w_1)} \leq \|F^{-1}a\|_{L^1(\mathbb{R},w_1)}\|u\|_{Y(\mathbb{R},w_1)}. \tag{5.5}$$

Hence

$$\|a\|_{\mathcal{M}_{Y(\mathbb{R},w_1)}} \leq \|F^{-1}a\|_{L^1(\mathbb{R},w_1)}. \tag{5.6}$$

The same argument allows one to show that if $a \in F\big(S'(\mathbb{R}) \cap L^1_{(-\infty,0]}(\mathbb{R}, w_2)\big)$, then

$$\|a\|_{\mathcal{M}_{Y(\mathbb{R},w_2)}} \leq \|F^{-1}a\|_{L^1(\mathbb{R},w_2)}. \tag{5.7}$$

Inequalities (5.6)–(5.7) imply embeddings (5.4). □

LEMMA 5.4. *Let $Y(\mathbb{R})$ be a translation-invariant Banach function space and the weights w_1 and w_2 be the same as in Corollary 5.2.*

(a) *If $a \in F\left(L^1(\mathbb{R}) \cap L^1(\mathbb{R}, w_1)\right)$, then*

$$\|a\|_{\mathcal{M}^0_{Y(\mathbb{R},w_1)}} \leq \|F^{-1}a\|_{L^1(\mathbb{R},w_1)}. \tag{5.8}$$

(b) *If $a \in F\big(L^1_{(-\infty,0]}(\mathbb{R})\big) = F\big(L^1(\mathbb{R}) \cap L^1_{(-\infty,0]}(\mathbb{R}, w_2)\big)$, then*

$$\|a\|_{\mathcal{M}^0_{Y(\mathbb{R},w_2)}} \leq \|F^{-1}a\|_{L^1(\mathbb{R},w_2)}. \tag{5.9}$$

Proof. If $a \in F\left(L^1(\mathbb{R}) \cap L^1(\mathbb{R}, w_1)\right)$, then (5.5) holds for every $u \in L^2(\mathbb{R}) \cap Y(\mathbb{R}, w_1)$, which implies (5.8) and completes the proof of part (a). The proof of part (b) is analogous. □

In view of Theorem 5.3, $\mathcal{M}_{Y(\mathbb{R},w_j)}$, $j = 1, 2$ contain many unbounded functions. Let us give some concrete examples.

Let $\mathbb{C}_+ := \{z = x + iy \in \mathbb{C} : y > 0\}$ and $H^2(\mathbb{C}_+)$ be the Hardy space of all functions f analytic in $\mathbb{C}_+$ such that $|f(x + iy)|^2$ is integrable for each $y > 0$ and

$$\sup_{y>0} \int_{\mathbb{R}} |f(x + iy)|^2 \, dx < \infty.$$

If $f \in H^2(\mathbb{C}_+)$, then the boundary function

$$f(x) = \lim_{y \to 0+} f(x + iy)$$

exists a.e. on $\mathbb{R}$ and belongs to $L^2(\mathbb{R})$ (see, e.g., the corollary of [**13**, Theorem 1.1]). Let $H^2_+(\mathbb{R})$ be the Hardy space of the boundary functions of the functions $f \in H^2(\mathbb{C}_+)$.

COROLLARY 5.5. *Let $Y(\mathbb{R})$ be a translation-invariant Banach function space and w be one of the weights w_1 and w_2 in Corollary 5.2.*

(a) *The Hardy space $H^2_+(\mathbb{R})$ corresponding to the upper complex half-plane is a subset of the space $\mathcal{M}_{Y(\mathbb{R},w)}$.*

(b) *Let*

$$a_{-\alpha}(\xi) := \lim_{\varepsilon \to 0+} (\xi + i\varepsilon)^{-\alpha}, \quad \xi \in \mathbb{R} \setminus \{0\}, \ 0 < \alpha < 1,$$

where $z^{-\alpha}$ denotes a branch analytic in the complex plane cut along the negative half-line. Then $a_{-\alpha} \in \mathcal{M}_{Y(\mathbb{R},w)}$.

Proof. (a) It follows from Hölder's inequality that any function in $L^2(\mathbb{R})$, which vanishes on $(0, \infty)$, belongs to $L^1(\mathbb{R}, w)$. By the Paley-Wiener theorem (see, e.g., [**13**, Corollary to Theorem 11.9]), if $u \in H^2_+(\mathbb{R})$, then $Fu(\xi)$ vanishes for almost all $\xi < 0$. Hence $F^{-1}u(\xi)$ vanishes for almost all $\xi > 0$. Therefore $F^{-1}u \in L^1(\mathbb{R}, w)$ and $u \in FL^1(\mathbb{R}, w)$. Now we can apply Theorem 5.3 to complete the proof of part (a).

(b) Since $0 < \alpha < 1$, it follows from [**16**, Example 2.3, formula (2.41′)] that

$$a_{-\alpha} = k_\alpha F^{-1} f^+_{\alpha-1} = k_\alpha F f^-_{\alpha-1},$$

where $k_\alpha \in \mathbb{C}$ is some constant depending on α and

$$f^+_{\alpha-1}(x) := \begin{cases} x^{\alpha-1} & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad f^-_{\alpha-1}(x) := \begin{cases} 0 & \text{if } x > 0, \\ |x|^{\alpha-1} & \text{if } x < 0, \end{cases}$$

define regular distributions in $S'(\mathbb{R})$ (see [**16**, Example 1.6]). It is clear that $f^-_{\alpha-1} \in L^1(\mathbb{R}, w)$, whence $a_{-\alpha} \in F\left(S'(\mathbb{R}) \cap L^1(\mathbb{R}, w)\right)$. It remains to apply Theorem 5.3. □

6. *Classes $\mathcal{M}^0_{X(\mathbb{R}^n)}$ and $\mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$*

6.1. *Two classes of Fourier multipliers coincide in the case of a nice underlying space*

THEOREM 6.1. *If a Banach function space $X(\mathbb{R}^n)$ satisfies the bounded L^2-approximation property, then*

$$\mathcal{M}^0_{X(\mathbb{R}^n)} = \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n) \quad \text{and} \quad \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} = \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}}. \tag{6.1}$$

Proof. According to (1.1) and (1.2), we only need to prove that

$$\mathcal{M}^0_{X(\mathbb{R}^n)} \supseteq \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n) \quad \text{and} \quad \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \geq \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}}.$$

Choose any function $a \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$. Take any function $u \in L^2(\mathbb{R}^n) \cap X(\mathbb{R}^n)$ and consider a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ satisfying (2.1). Since $a \in L^\infty(\mathbb{R}^n)$ and $\mathcal{M}^0_{L^2(\mathbb{R}^n)} = L^\infty(\mathbb{R}^n)$ (see, e.g., [**18**, Theorem 2.5.10]), we have

$$\lim_{j\to\infty} \|F^{-1}aFu - F^{-1}aFu_j\|_{L^2(\mathbb{R}^n)} = 0.$$

Then there exists a subsequence $\{F^{-1}aFu_{j_k}\}_{k\in\mathbb{N}}$ of the sequence $\{F^{-1}aFu_j\}_{j\in\mathbb{N}}$ that converges to $F^{-1}aFu$ almost everywhere. Since $a \in \mathcal{M}_{X(\mathbb{R}^n)}$, from the inequality in (2.1) we get

$$\begin{aligned}\liminf_{k\to\infty} \|F^{-1}aFu_{j_k}\|_{X(\mathbb{R}^n)} &\le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \liminf_{k\to\infty} \|u_{j_k}\|_{X(\mathbb{R}^n)} \\ &\le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \limsup_{j\to\infty} \|u_j\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|u\|_{X(\mathbb{R}^n)}.\end{aligned}$$

Fatou's lemma (see [**1**, Chap. 1, Lemma 1.5]) and the above inequality imply that the function $F^{-1}aFu$ belongs to $X(\mathbb{R}^n)$ and for $u \in L^2(\mathbb{R}^n) \cap X(\mathbb{R}^n)$,

$$\|F^{-1}aFu\|_{X(\mathbb{R}^n)} \le \liminf_{k\to\infty} \|F^{-1}aFu_{j_k}\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|u\|_{X(\mathbb{R}^n)}.$$

Hence $a \in \mathcal{M}^0_X(\mathbb{R}^n)$ and $\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$. □

6.2. *Two classes of Fourier multipliers are different in general*

The following theorem shows that equalities in (6.1) do not always hold.

THEOREM 6.2. *For a compact set $G \subset [0,1]^n$ of positive measure with empty interior and a sequence $b = \{b_m\}_{m\in\mathbb{N}} \subset (0,1)$ satisfying*

$$\lim_{m\to\infty} b_m = 0, \tag{6.2}$$

consider the sequence

$$\begin{aligned}G_m &:= (2m, 0, \dots, 0) + G \\ &= \{(y_1 + 2m, y_2, \dots, y_n) \in \mathbb{R}^n \ : \ y = (y_1, y_2, \dots, y_n) \in G\}, \quad m \in \mathbb{N},\end{aligned} \tag{6.3}$$

and define the weight $w_{G,b}$ for $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ satisfying $y_1 \ge 0$ by

$$w_{G,b}(y) := \begin{cases} b_m, & y \in G_m,\ m \in \mathbb{N}, \\ 1, & y \notin \bigcup_{m\in\mathbb{N}} G_m, \end{cases} \tag{6.4}$$

and for $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ satisfying $y_1 < 0$ by

$$w_{G,b}(y) := w_{G,b}(-y). \tag{6.5}$$

Then there exists $a \in S(\mathbb{R}^n)$ such that $a \in \mathcal{M}_{L^\infty(\mathbb{R}^n, w_{G,b})} \setminus \mathcal{M}^0_{L^\infty(\mathbb{R}^n, w_{G,b})}$.

Proof. Let $x^{(0)} \in G$ be a Lebesgue point of the function χ_G and let $\{\varrho_j\}_{j\in\mathbb{N}}$ be the sequence defined by (2.2). It follows from Lemma 2.16 that there exists $j \in \mathbb{N}$ for which

$$\int_{\mathbb{R}^n} |\chi_G(y) - \chi_G(x^{(0)})| \varrho_j(x^{(0)} - y)\, dy \le \frac{1}{2}.$$

Hence,

$$\int_{\mathbb{R}^n} \varrho_j(x^{(0)} - y)\, dy - \int_{\mathbb{R}^n} \chi_G(y) \varrho_j(x^{(0)} - y)\, dy \le \frac{1}{2}.$$

Therefore, $1 - (\varrho_j * \chi_G)(x^{(0)}) \le 1/2$, whence

$$(\varrho_j * \chi_G)(x^{(0)}) \ge \frac{1}{2}. \tag{6.6}$$

Let $a := F\varrho_j \in S(\mathbb{R}^n)$. Take any $u \in S(\mathbb{R}^n)$ with $\|u\|_{L^\infty(\mathbb{R}^n,w_{G,b})} \le 1$. The same argument as in the proof of Lemma 2.5 shows that $|u(y)| \le 1$ for all $y \in \mathbb{R}^n$. Then by [**29**, Theorem 7.8(b)],

$$\big|(F^{-1}aFu)(x)\big| = |(\varrho_j * u)(x)| \le \int_{\mathbb{R}^n} \varrho_j(y)\,dy = 1 \quad \text{for all} \quad x \in \mathbb{R}^n$$

and $\|F^{-1}aFu\|_{L^\infty(\mathbb{R}^n,w_{G,b})} \le 1$. Hence $a \in \mathcal{M}_{L^\infty(\mathbb{R}^n,w_{G,b})}$ and $\|a\|_{\mathcal{M}_{L^\infty(\mathbb{R}^n,w_{G,b})}} \le 1$.

On the other hand, consider

$$u_m := b_m^{-1}\,\chi_{G_m}, \quad m \in \mathbb{N}.$$

It is clear that $\|u_m\|_{L^\infty(\mathbb{R}^n,w_{G,b})} = 1$. Since $u_m \in L^1(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ and $a = F\varrho_j \in S(\mathbb{R}^n)$, it follows from [**29**, Theorem 7.19(c)] that $(\varrho_j * u_m)\widehat{\ } = \widehat{\varrho_j}\widehat{u}_m = aFu_m$. This equality and [**7**, Propositions 4.18, 4.20] imply that $F^{-1}aFu_m = \varrho_j * u_m \in C_0^\infty(\mathbb{R}^n)$ because $\varrho_j \in C_0^\infty(\mathbb{R}^n)$ and $u_m \in L^\infty(\mathbb{R}^n)$ has compact support. Therefore,

$$v_m := \|F^{-1}aFu_m\|^{-1}_{L^\infty(\mathbb{R}^n,w_{G,b})} F^{-1}aFu_m \in C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$$

and $\|v_m\|_{L^\infty(\mathbb{R}^n,w_{G,b})} = 1$. Then, as above, $|v_m(x)| \le 1$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$, i.e.

$$\big|F^{-1}aFu_m(x)\big| \le \|F^{-1}aFu_m\|_{L^\infty(\mathbb{R}^n,w_{G,b})} \quad \text{for all} \quad x \in \mathbb{R}^n,\ m \in \mathbb{N}. \tag{6.7}$$

Let

$$x^{(m)} := (2m, 0, \dots, 0) + x^{(0)}, \quad m \in \mathbb{N}.$$

Then, taking into account (6.6), we get for all $m \in \mathbb{N}$,

$$F^{-1}aFu_m\big(x^{(m)}\big) = b_m^{-1}\,(\varrho_j * \chi_{G_m})\,\big(x^{(m)}\big) = b_m^{-1}\,(\varrho_j * \chi_G)\,\big(x^{(0)}\big) \ge \frac{1}{2b_m}. \tag{6.8}$$

Now it follows from (6.7), (6.8) and (6.2) that

$$\|F^{-1}aFu_m\|_{L^\infty(\mathbb{R}^n,w_{G,b})} \ge \frac{1}{2b_m} \to \infty \quad \text{as} \quad m \to \infty,$$

while $\|u_m\|_{L^\infty(\mathbb{R}^n,w_{G,b})} = 1$. Hence $a \notin \mathcal{M}^0_{L^\infty(\mathbb{R}^n,w_{G,b})}$. □

6.3. *Normed algebras of Fourier multipliers*

LEMMA 6.3. *Let $X(\mathbb{R}^n)$ be a Banach function space. Then the set $\mathcal{M}^0_{X(\mathbb{R}^n)}$ is a normed algebra with respect to the norm $\|\cdot\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}}$ and*

$$\|ab\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \le \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \|b\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \quad \textit{for all} \quad a, b \in \mathcal{M}^0_{X(\mathbb{R}^n)}.$$

Proof. The proof is straightforward. □

The proof of the following result requires a bit more effort.

THEOREM 6.4. *Let $X(\mathbb{R}^n)$ be a Banach function space. Then the set $\mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ is a normed algebra with respect to the norm $\|\cdot\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$ and*

$$\|ab\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|b\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \quad \textit{for all} \quad a, b \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n). \tag{6.9}$$

Proof. It is clear that $\mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ is a normed space, so one only needs to prove that $ab \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ for all $a, b \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n)$ and that (6.9) holds.

Take any function $u \in S(\mathbb{R}^n) \cap X(\mathbb{R}^n)$. Then $F^{-1}bFu \in X(\mathbb{R}^n)$. Since $u \in S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and $b \in L^\infty(\mathbb{R}^n)$, we have $F^{-1}bFu \in L^2(\mathbb{R}^n)$. On the other hand, in view of [**29**, Theorem 7.19(a)], $u \in S(\mathbb{R}^n)$ and $b \in S'(\mathbb{R}^n)$ imply that $F^{-1}bFu = \big(F^{-1}b\big) * u \in C^\infty(\mathbb{R}^n)$. Hence $F^{-1}bF$ maps $S(\mathbb{R}^n) \cap X(\mathbb{R}^n)$ into $X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

Now take a function $v \in X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and consider the sequence $v_m := \phi_m v$, where $\phi_m \in C_0^\infty(\mathbb{R}^n)$, $0 \le \phi_m \le 1$, and $\phi_m(x) = 1$ for $|x| \le m$ and $m \in \mathbb{N}$. Since $|v_m| \le |v|$, it follows from axiom (A2) that $v_m \in X(\mathbb{R}^n)$ and $\|v_m\|_{X(\mathbb{R}^n)} \le \|v\|_{X(\mathbb{R}^n)}$ for all $m \in \mathbb{N}$. Further, $v_m \in C_0^\infty(\mathbb{R}^n)$ and $\|v_m - v\|_{L^2(\mathbb{R}^n)} \to 0$ as $m \to \infty$. Then it follows as in the proof of Theorem 6.1 that $F^{-1}aFv \in X(\mathbb{R}^n)$ and

$$\|F^{-1}aFv\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|v\|_{X(\mathbb{R}^n)}.$$

Taking $v = F^{-1}bFu$, one gets $F^{-1}abFu = F^{-1}aF(F^{-1}bFu) \in X(\mathbb{R}^n)$ and

$$\|F^{-1}abFu\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|F^{-1}bFu\|_{X(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|b\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|u\|_{X(\mathbb{R}^n)}$$

for all $u \in S(\mathbb{R}^n) \cap X(\mathbb{R}^n)$, which immediately implies (6.9). □

Theorem 6.4 allows one to prove that if there exists a constant $D_X > 0$ such that

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le D_X \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \quad \text{for all} \quad a \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n), \tag{6.10}$$

then in fact

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \quad \text{for all} \quad a \in \mathcal{M}_{X(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n). \tag{6.11}$$

Indeed, one can apply (6.10) and Theorem 6.4 to the function a^m with $m \in \mathbb{N}$ to get

$$\|a\|^m_{L^\infty(\mathbb{R}^n)} = \|a^m\|_{L^\infty(\mathbb{R}^n)} \le D_X \|a^m\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \le D_X \|a\|^m_{\mathcal{M}_{X(\mathbb{R}^n)}}.$$

Taking $m \to \infty$ in the inequality

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le D_X^{1/m} \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}},$$

one gets (6.11). One can use this observation instead of Lemma 3.1 to derive (4.1) from (4.2) under the assumption that a Banach function space $X(\mathbb{R}^n)$ satisfies the weak doubling property. In particular, this implies that [**3**, Theorem 2.3] holds with the constant $K_{p,C} = 1$. It is also clear that the implication (6.10) ⇒ (6.11) holds with $\mathcal{M}^0_{X(\mathbb{R}^n)}$ in place of $\mathcal{M}_{X(\mathbb{R}^n)}$.

6.4. *Normed spaces of Fourier multipliers are not complete in general*

Theorem 6.5. *Let $Y(\mathbb{R})$ be a translation-invariant Banach function space and w be one of the weights w_1 and w_2 in Corollary 5.2.*

(a) *The normed space $\mathcal{M}_{Y(\mathbb{R},w)}$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{M}_{Y(\mathbb{R},w)}}$.*
(b) *The normed algebra $\mathcal{M}^0_{Y(\mathbb{R},w)}$ is not complete with respect to the norm $\|\cdot\|_{\mathcal{M}^0_{Y(\mathbb{R},w)}}$.*

Proof. (a) Consider the function $g_0(x) := e^{-cx/2}\phi_0(x)$, where the constant $c > 0$ is from the definition of the weights w_1 and w_2, and a function $\phi_0 \in C^\infty(\mathbb{R})$ is such that $\phi_0(x) = 0$ for $x \ge 0$ and $\phi_0(x) = 1$ for $x \le -1$. It is easy to see that $g_0 \in L^1(\mathbb{R}, w)$, whence it may be identified with the distribution in $\mathcal{D}'(\mathbb{R})$. On the other hand, $g_0 \notin S'(\mathbb{R})$ (cf. [**29**, Chap. 7, Exercise 3]).

Consider $g_k(x) := \phi_k(x)g_0(x)$, where $\phi_k \in C_0^\infty(\mathbb{R})$, $0 \le \phi_k \le 1$, and $\phi_k(x) = 1$ for $|x| \le k$ and all $k \in \mathbb{N}$. The Lebesgue dominated convergence theorem implies that $\|g_k - g_0\|_{L^1(\mathbb{R},w)} \to 0$ as $k \to \infty$. Hence $\{g_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\mathbb{R}, w)$. Let $a_k := Fg_k$ for $k \in \mathbb{N}$.

It follows from (5.6)–(5.7) that

$$\|a_k - a_m\|_{\mathcal{M}_{Y(\mathbb{R},w)}} \le \|F^{-1}(a_k - a_m)\|_{L^1(\mathbb{R},w)} = \|g_k - g_m\|_{L^1(\mathbb{R},w)} \quad \text{for all} \quad k, m \in \mathbb{N}.$$

Therefore, $\{a_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_{Y(\mathbb{R},w)}$. Suppose it converges to a limit a_0 in $\mathcal{M}_{Y(\mathbb{R},w)}$. Then the sequence $F^{-1}a_kFu = g_k * u$ converges to $F^{-1}a_0Fu = (F^{-1}a_0) * u$ in $Y(\mathbb{R},w)$ as $k \to \infty$ for any function $u \in C_0^\infty(\mathbb{R}) \subset S(\mathbb{R}) \cap Y(\mathbb{R},w)$.

On the other hand, Corollary 5.2 implies that $g_k * u$ converges to $g_0 * u$ in $Y(\mathbb{R},w)$. Hence $(F^{-1}a_0) * u = g_0 * u$ for any $u \in C_0^\infty(\mathbb{R})$. Since $(F^{-1}a_0) * u = g_0 * u$ are continuous functions (see, e.g., [**29**, Theorem 6.30(b)]), one gets

$$\langle F^{-1}a_0, \widetilde{u}\rangle = \big((F^{-1}a_0) * u\big)(0) = (g_0 * u)(0) = \langle g_0, \widetilde{u}\rangle \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}).$$

Hence the distributions $F^{-1}a_0 \in S'(\mathbb{R})$ and $g_0 \in \mathcal{D}'(\mathbb{R}) \setminus S'(\mathbb{R})$ are equal to each other. This contradiction shows that $\{a_k\}_k \in \mathbb{N}$ does not converge to a limit in $\mathcal{M}_{Y(\mathbb{R},w)}$.

(b) Consider the functions $g_m(x) := e^{x/m} f_{\alpha-1}^-(x)$, $m \in \mathbb{N}$, where $f_{\alpha-1}^-$ is the same as in the proof of Corollary 5.5(b). The Lebesgue dominated convergence theorem implies that

$$\|g_m - f_{\alpha-1}^-\|_{L^1(\mathbb{R},w)} \to 0 \quad \text{as} \quad m \to \infty, \qquad \|g_m - g_k\|_{L^1(\mathbb{R},w)} \to 0 \quad \text{as} \quad m,k \to \infty.$$

Then it follows from (5.6)–(5.9) and Corollary 5.5(b) that

$$\|a_m - a_{-\alpha}\|_{\mathcal{M}_{Y(\mathbb{R},w)}} \le k_\alpha \|g_m - f_{\alpha-1}^-\|_{L^1(\mathbb{R},w)} \to 0 \quad \text{as} \quad m \to \infty \tag{6.12}$$

and

$$\|a_m - a_k\|_{\mathcal{M}^0_{Y(\mathbb{R},w)}} \le k_\alpha \|g_m - g_k\|_{L^1(\mathbb{R},w)} \to 0 \quad \text{as} \quad m,k \to \infty, \tag{6.13}$$

where

$$a_m(\xi) := k_\alpha F g_m(\xi) = k_\alpha F f_{\alpha-1}^-(\xi + i/m) = (\xi + i/m)^{-\alpha},$$

k_α is the constant from the proof of Corollary 5.5(b), and

$$a_{-\alpha}(\xi) := k_\alpha F f_{\alpha-1}^-(\xi) = \lim_{\varepsilon\to 0+} (\xi + i\varepsilon)^{-\alpha}.$$

Since $\{g_m\}_{m\in\mathbb{N}}$ is convergent in $L^1(\mathbb{R},w)$, it follows from (6.13) that $\{a_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^0_{Y(\mathbb{R},w)}$. If it had a limit there, then inequality (1.2) would imply that it converges to the same limit in $\mathcal{M}_{Y(\mathbb{R},w)}$. On the other hand, we know from (6.12) that $\{a_m\}_{m\in\mathbb{N}}$ converges to $a_{-\alpha} \notin L^\infty(\mathbb{R})$. Hence $\{a_m\}_{m\in\mathbb{N}}$ cannot converge to a limit in $\mathcal{M}^0_{Y(\mathbb{R},w)}$. □

7. *Fourier multipliers on reflection-invariant Banach function spaces*

7.1. *Interpolation in Calderón products of Banach function spaces*

Let $X_0(\mathbb{R}^n)$ and $X_1(\mathbb{R}^n)$ be Banach function spaces and $0 < \theta < 1$. The Calderón product $(X_0^{1-\theta}X_1^\theta)(\mathbb{R}^n)$ (see [**9**, p. 123]) consists of all measurable functions f such that a.e. pointwise inequality $|f| \le \lambda |f_0|^{1-\theta}|f_1|^\theta$ holds for some $\lambda > 0$ and elements f_j in $X_j(\mathbb{R}^n)$ with $\|f_j\|_{X_j(\mathbb{R}^n)} \le 1$ for $j = 0,1$. The norm of f in $(X_0^{1-\theta}X_1^\theta)(\mathbb{R}^n)$ is defined to be the infimum of all values λ appearing in the above inequality. We will need an interpolation theorem, which follows immediately from [**32**, Theorem 1] and [**1**, Chap. 1, Theorem 2.7].

THEOREM 7.1. *Let $X_0(\mathbb{R}^n)$ and $X_1(\mathbb{R}^n)$ be Banach function spaces. Let A be a linear operator bounded on $X_0(\mathbb{R}^n)$ and $X_1(\mathbb{R}^n)$. Then A is bounded on $(X_0^{1-\theta}X_1^\theta)(\mathbb{R}^n)$ and*

$$\|A\|_{\mathcal{B}((X_0^{1-\theta}X_1^\theta)(\mathbb{R}^n))} \le \|A\|^{1-\theta}_{\mathcal{B}(X_0(\mathbb{R}^n))}\|A\|^\theta_{\mathcal{B}(X_1(\mathbb{R}^n))}.$$

The following result is contained in [**26**, Theorem 5] in a slightly different form.

LEMMA 7.2. *If $X(\mathbb{R}^n)$ is a Banach function space and $X'(\mathbb{R}^n)$ is its associate space, then*

$$(X^{1/2}(X')^{1/2})(\mathbb{R}^n) = L^2(\mathbb{R}^n)$$

with equality of the norms.

The above lemma is a consequence of the more general Lozanovskiĭ's formula [**26**, Theorem 2] (see also [**11**, Theorem 7.2]):

$$(X_0^{1-\theta}X_1^{\theta})'(\mathbb{R}^n) = ((X_0')^{1-\theta}(X_1')^{\theta})(\mathbb{R}^n), \quad 0 < \theta < 1,$$

which is valid with equality of the norms. We refer to Maligranda's book [**27**, p. 185] for the proof of Lozanovskiĭ's Lemma 7.2.

COROLLARY 7.3. *Let $X(\mathbb{R}^n)$ be a Banach function space and $X'(\mathbb{R}^n)$ be its associate space. If A is a linear operator bounded on $X(\mathbb{R}^n)$ and on $X'(\mathbb{R}^n)$, then A is bounded on $L^2(\mathbb{R}^n)$ and*

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \|A\|^{1/2}_{\mathcal{B}(X(\mathbb{R}^n))}\|A\|^{1/2}_{\mathcal{B}(X'(\mathbb{R}^n))}.$$

This result follows immediately from Theorem 7.1 and Lemma 7.2.

7.2. *Fourier multipliers on reflexive reflection-invariant Banach function spaces are bounded*

We say that a Banach function space $X(\mathbb{R}^n)$ is reflection-invariant if $\|f\|_{X(\mathbb{R}^n)} = \|\widetilde{f}\|_{X(\mathbb{R}^n)}$ for every $f \in X(\mathbb{R}^n)$, where $\widetilde{f}$ denotes the reflection of a function f defined by $\widetilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$.

LEMMA 7.4. *A Banach function space $X(\mathbb{R}^n)$ is reflection-invariant if and only if its associate space $X'(\mathbb{R}^n)$ is reflection-invariant.*

This statement follows immediately from [**1**, Chap. 1, Theorem 2.7 and Lemma 2.8].

LEMMA 7.5. *If a Banach function space $X(\mathbb{R}^n)$ is reflection-invariant, then*

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} = \|\widetilde{a}\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \quad \text{for all} \quad a \in \mathcal{M}_{X(\mathbb{R}^n)},$$
$$\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} = \|\widetilde{a}\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \quad \text{for all} \quad a \in \mathcal{M}^0_{X(\mathbb{R}^n)}.$$

Proof. If $u \in (S(\mathbb{R}^n) \cap X(\mathbb{R}^n)) \setminus \{0\}$, then $\widetilde{u} \in (S(\mathbb{R}^n) \cap X(\mathbb{R}^n)) \setminus \{0\}$ and

$$F^{-1}\widetilde{a}Fu = F^{-1}(a(Fu)^{\sim})^{\sim} = (F^{-1}aF\widetilde{u})^{\sim}.$$

Therefore, taking into account that $X(\mathbb{R}^n)$ is reflection-invariant, we see that

$$\begin{aligned}
\|\widetilde{a}\|_{\mathcal{M}_{X(\mathbb{R}^n)}} &= \sup\left\{\frac{\|F^{-1}\widetilde{a}Fu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} \ :\ u \in (S(\mathbb{R}^n)\cap X(\mathbb{R}^n))\setminus\{0\}\right\}\\
&= \sup\left\{\frac{\|(F^{-1}aF\widetilde{u})^{\sim}\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} \ :\ u \in (S(\mathbb{R}^n)\cap X(\mathbb{R}^n))\setminus\{0\}\right\}\\
&= \sup\left\{\frac{\|F^{-1}aF\widetilde{u}\|_{X(\mathbb{R}^n)}}{\|\widetilde{u}\|_{X(\mathbb{R}^n)}} \ :\ u \in (S(\mathbb{R}^n)\cap X(\mathbb{R}^n))\setminus\{0\}\right\}\\
&= \sup\left\{\frac{\|F^{-1}aFu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} \ :\ u \in (S(\mathbb{R}^n)\cap X(\mathbb{R}^n))\setminus\{0\}\right\}\\
&= \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}.
\end{aligned}$$

Replacing $S(\mathbb{R}^n)$ with $L^2(\mathbb{R}^n)$ one gets a proof for $\mathcal{M}^0_{X(\mathbb{R}^n)}$. □

LEMMA 7.6. *Let $X(\mathbb{R}^n)$ be a Banach function space and $X'(\mathbb{R}^n)$ be its associate space.*

(a) *If $\widetilde{a} \in \mathcal{M}^0_{X'(\mathbb{R}^n)}$, then $a \in \mathcal{M}^0_{X(\mathbb{R}^n)}$ and*

$$\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \le \|\widetilde{a}\|_{\mathcal{M}^0_{X'(\mathbb{R}^n)}}.$$

(b) *If $a \in \mathcal{M}^0_{X(\mathbb{R}^n)}$, then $\widetilde{a} \in \mathcal{M}^0_{X'(\mathbb{R}^n)}$ and*

$$\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \ge \|\widetilde{a}\|^0_{\mathcal{M}_{X'(\mathbb{R}^n)}}.$$

Proof. If $u, v \in L^2(\mathbb{R}^n)$ and $a \in L^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} (F^{-1}aFu)(x)v(x)\,dx &= \langle F^{-1}aFu, v\rangle = \langle aFu, F^{-1}v\rangle\\
&= \langle a, Fu\cdot F^{-1}v\rangle = \langle a, F^{-1}\widetilde{u}\cdot F\widetilde{v}\rangle\\
&= \langle a, (F^{-1}u)^{\sim}\cdot (Fv)^{\sim}\rangle = \langle \widetilde{a}, F^{-1}u\cdot Fv\rangle\\
&= \langle \widetilde{a}Fv, F^{-1}u\rangle = \langle F^{-1}\widetilde{a}Fv, u\rangle\\
&= \int_{\mathbb{R}^n} (F^{-1}\widetilde{a}Fv)(x)u(x)\,dx. \qquad (7.1)
\end{aligned}$$

(a) Take $u \in (L^2(\mathbb{R}^n)\cap X(\mathbb{R}^n))\setminus\{0\}$. Then, in view of equality (7.1), Lemma 2.10, Hölder's inequality (see [**1**, Chap. 1, Theorem 2.4]), and the embedding $S_0(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)\cap X'(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|F^{-1}aFu\|_{X(\mathbb{R}^n)} &= \sup\left\{ \frac{\left|\int_{\mathbb{R}^n} (F^{-1}aFu)(x)s(x)\,dx\right|}{\|s\|_{X'(\mathbb{R}^n)}} \;:\; s \in S_0(\mathbb{R}^n)\setminus\{0\} \right\} \\
&= \sup\left\{ \frac{\left|\int_{\mathbb{R}^n} (F^{-1}\widetilde{a}Fs)(x)u(x)\,dx\right|}{\|s\|_{X'(\mathbb{R}^n)}} \;:\; s \in S_0(\mathbb{R}^n)\setminus\{0\} \right\} \\
&\le \sup\left\{ \frac{\|F^{-1}\widetilde{a}Fs\|_{X'(\mathbb{R}^n)}\|u\|_{X(\mathbb{R}^n)}}{\|s\|_{X'(\mathbb{R}^n)}} \;:\; s \in S_0(\mathbb{R}^n)\setminus\{0\} \right\} \\
&\le \|u\|_{X(\mathbb{R}^n)} \sup\left\{ \frac{\|F^{-1}\widetilde{a}Fv\|_{X'(\mathbb{R}^n)}}{\|v\|_{X'(\mathbb{R}^n)}} \;:\; v \in (L^2(\mathbb{R}^n)\cap X'(\mathbb{R}^n))\setminus\{0\} \right\} \\
&= \|\widetilde{a}\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \|u\|_{X(\mathbb{R}^n)},
\end{aligned}$$

whence

$$\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} = \sup\left\{ \frac{\|F^{-1}aFu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} \;:\; u \in (L^2(\mathbb{R}^n)\cap X(\mathbb{R}^n))\setminus\{0\} \right\} \le \|\widetilde{a}\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}},$$

which completes the proof of part (a).

(b) By the Lorentz-Luxemburg theorem (see [**1**, Chap. 1, Theorem 2.7]), $X(\mathbb{R}^n) = X''(\mathbb{R}^n)$ with the equality of the norms. Then $(\widetilde{a})^{\sim} = a \in \mathcal{M}^0_{X(\mathbb{R}^n)} = \mathcal{M}^0_{X''(\mathbb{R}^n)}$. Hence, part (b) follows from part (a). □

Lemma 7.7. *Let $X(\mathbb{R}^n)$ be a Banach function space and $X'(\mathbb{R}^n)$ be its associate space.*

(a) *Suppose the space $X(\mathbb{R}^n)$ satisfies the norm fundamental property. If $\widetilde{a} \in \mathcal{M}_{X'(\mathbb{R}^n)}$, then $a \in \mathcal{M}_{X(\mathbb{R}^n)}$ and*

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \le \|\widetilde{a}\|_{\mathcal{M}_{X'(\mathbb{R}^n)}}.$$

(b) *Suppose the space $X'(\mathbb{R}^n)$ satisfies the norm fundamental property. If $a \in \mathcal{M}_{X(\mathbb{R}^n)}$, then $\widetilde{a} \in \mathcal{M}_{X'(\mathbb{R}^n)}$ and*

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \ge \|\widetilde{a}\|_{\mathcal{M}_{X'(\mathbb{R}^n)}}.$$

Proof. The proof is similar to that of Lemma 7.6. Interpreting $\langle\cdot,\cdot\rangle$ in (7.1) as the $(S'(\mathbb{R}^n), S(\mathbb{R}^n))$ rather than $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ or $(L^\infty(\mathbb{R}^n), L^1(\mathbb{R}^n))$ duality, one gets for any $u, v \in S(\mathbb{R}^n)$ and $a \in S'(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (F^{-1}aFu)(x)v(x)\,dx = \int_{\mathbb{R}^n} (F^{-1}\widetilde{a}Fv)(x)u(x)\,dx. \tag{7.2}$$

(a) Take $u \in (S(\mathbb{R}^n) \cap X(\mathbb{R}^n)) \setminus \{0\}$. Then, in view of equality (7.2), Definition 2.9, and Hölder's inequality (see [**1**, Chap. 1, Theorem 2.4]), we obtain

$$\begin{aligned}
\|F^{-1}aFu\|_{X(\mathbb{R}^n)} &= \sup \left\{ \frac{\left| \int_{\mathbb{R}^n} (F^{-1}aFu)(x)\psi(x)\,dx \right|}{\|\psi\|_{X'(\mathbb{R}^n)}} \ : \ \psi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\} \right\} \\
&= \sup \left\{ \frac{\left| \int_{\mathbb{R}^n} (F^{-1}\widetilde{a}F\psi)(x)u(x)\,dx \right|}{\|\psi\|_{X'(\mathbb{R}^n)}} \ : \ \psi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\} \right\} \\
&\le \sup \left\{ \frac{\|F^{-1}\widetilde{a}F\psi\|_{X'(\mathbb{R}^n)} \|u\|_{X(\mathbb{R}^n)}}{\|\psi\|_{X'(\mathbb{R}^n)}} \ : \ \psi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\} \right\} \\
&\le \|u\|_{X(\mathbb{R}^n)} \sup \left\{ \frac{\|F^{-1}\widetilde{a}Fv\|_{X'(\mathbb{R}^n)}}{\|v\|_{X'(\mathbb{R}^n)}} \ : \ v \in (S(\mathbb{R}^n) \cap X'(\mathbb{R}^n)) \setminus \{0\} \right\} \\
&= \|\widetilde{a}\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \|u\|_{X(\mathbb{R}^n)},
\end{aligned}$$

whence

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} = \sup \left\{ \frac{\|F^{-1}aFu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} \ : \ u \in (S(\mathbb{R}^n) \cap X(\mathbb{R}^n)) \setminus \{0\} \right\} \le \|\widetilde{a}\|_{\mathcal{M}_{X(\mathbb{R}^n)}},$$

which completes the proof of part (a).

(b) By the Lorentz-Luxemburg theorem (see [**1**, Chap. 1, Theorem 2.7]), $X(\mathbb{R}^n) = X''(\mathbb{R}^n)$ with the equality of the norms. Then $(\widetilde{a})^{\sim} = a \in \mathcal{M}_{X(\mathbb{R}^n)} = \mathcal{M}_{X''(\mathbb{R}^n)}$. Hence, part (b) follows from part (a). □

THEOREM 7.8. *Let $X(\mathbb{R}^n)$ be a reflection-invariant Banach function space and $X'(\mathbb{R}^n)$ be its associate space.*

(a) *We have $\mathcal{M}^0_{X(\mathbb{R}^n)} = \mathcal{M}^0_{X'(\mathbb{R}^n)}$ and $\|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} = \|a\|_{\mathcal{M}^0_{X'(\mathbb{R}^n)}}$ for all $a \in \mathcal{M}^0_{X(\mathbb{R}^n)}$.*

(b) *If both $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$ satisfy the norm fundamental property, then we have $\mathcal{M}_{X(\mathbb{R}^n)} = \mathcal{M}_{X'(\mathbb{R}^n)}$ and $\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} = \|a\|_{\mathcal{M}_{X'(\mathbb{R}^n)}}$ for all $a \in \mathcal{M}_{X(\mathbb{R}^n)}$.*

Proof. We prove part (b). The proof of part (a) is almost exactly the same. By Lemma 7.4, both $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$ are reflection-invariant Banach function spaces. If $a \in \mathcal{M}_{X'(\mathbb{R}^n)}$, then $\widetilde{a} \in \mathcal{M}_{X'(\mathbb{R}^n)}$ and

$$\|a\|_{\mathcal{M}_{X'(\mathbb{R}^n)}} = \|\widetilde{a}\|_{\mathcal{M}_{X'(\mathbb{R}^n)}} \ge \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \tag{7.3}$$

in view of Lemmas 7.5 and 7.7(a). On the other hand, if $a \in \mathcal{M}_{X(\mathbb{R}^n)}$, then $\widetilde{a} \in \mathcal{M}_{X'(\mathbb{R}^n)}$ and

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} \ge \|\widetilde{a}\|_{\mathcal{M}_{X'(\mathbb{R}^n)}} = \|(\widetilde{a})^{\sim}\|_{\mathcal{M}_{X'(\mathbb{R}^n)}} = \|a\|_{\mathcal{M}_{X'(\mathbb{R}^n)}} \tag{7.4}$$

in view of Lemmas 7.7(b) and 7.5. Combining inequalities (7.3)–(7.4), we arrive at the desired result. □

Now we will show that one cannot drop the norm fundamental property in Theorem 7.8(b).

THEOREM 7.9. *Suppose $G \subset [0,1]^n$ is a compact set of positive measure with empty interior, $b = \{b_m\}_{m\in\mathbb{N}} \subset (0,1)$ is a sequence satisfying (6.2), and $w_{G,b}$ is the weight given by (6.3)–(6.5). Then the reflection-invariant space $L^1(\mathbb{R}^n, w_{G,b}^{-1})$ does not satisfy the norm-fundamental property and there exists $a \in S(\mathbb{R}^n)$ such that $a \in \mathcal{M}_{L^\infty(\mathbb{R}^n, w_{G,b})} \setminus \mathcal{M}_{L^1(\mathbb{R}^n, w_{G,b}^{-1})}$.*

Proof. It follows from (6.5) that $w_{G,b}(y) = w_{G,b}(-y)$ for all $y \in \mathbb{R}^n$. Therefore, $L^1(\mathbb{R}^n, w_{G,b}^{-1})$ is a reflection-invariant Banach function space. One can prove, similarly to Corollary 2.15, that $L^1(\mathbb{R}^n, w_{G,b}^{-1})$ does not satisfy the norm-fundamental property.

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be an even function such that $\rho \geq 0$ and $\rho(y) = 1$ when $|y| \leq \sqrt{n+3}$, and let $a := F\rho$. Then $a \in S(\mathbb{R}^n)$ and $\widetilde{a} = a$.

Take any $u \in S(\mathbb{R}^n)$ with $\|u\|_{L^\infty(\mathbb{R}^n, w_{G,b})} \leq 1$. The same argument as in the proof of Lemma 2.5 shows that $|u(y)| \leq 1$ for all $y \in \mathbb{R}^n$. Then

$$\big|(F^{-1}aFu)(x)\big| = |(\rho * u)(x)| \leq \int_{\mathbb{R}^n} \rho(y)\, dy = \|\rho\|_{L^1(\mathbb{R}^n)} < \infty \quad \text{for all} \quad x \in \mathbb{R}^n$$

and $\|F^{-1}aFu\|_{L^\infty(\mathbb{R}^n, w_{G,b})} \leq \|\rho\|_{L^1(\mathbb{R}^n)}$. Hence $a \in \mathcal{M}_{L^\infty(\mathbb{R}^n, w_{G,b})}$ and

$$\|a\|_{\mathcal{M}_{L^\infty(\mathbb{R}^n, w_{G,b})}} \leq \|\rho\|_{L^1(\mathbb{R}^n)}.$$

On the other hand, consider $v_m \in C_0^\infty(\mathbb{R}^n)$ such that $v_m \geq 0$, $v_m(y) = 1$ for $|y - y^{(m)}| \leq 1/4$ and $v_m(y) = 0$ for $|y - y^{(m)}| \geq 1/2$, where $y^{(m)} := \left(2m - \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ is the centre of the cube $Q_m := (2m-1, 0, \ldots, 0) + [0,1]^n$, $m \in \mathbb{N}$. Then $\operatorname{supp} v_m \subset Q_m$ and $w_{G,b}(x) = 1$ for $x \in Q_m$, whence

$$\|v_m\|_{L^1(\mathbb{R}^n, w_{G,b}^{-1})} = \int_{Q_m} v_m(y)\, dy \leq 1.$$

Since the distance from any point of G_m to any point of Q_m is less than or equal to $\sqrt{2^2 + 1^2 + \cdots + 1^2} = \sqrt{n+3}$, it follows from the definition of ρ that for all $x \in G_m$,

$$\begin{aligned} |F^{-1}aFv_m(x)| = |(\rho * v_m)(x)| &= \int_{\mathbb{R}^n} v_m(y)\rho(x-y)\, dy \\ &= \int_{Q_m} v_m(y)\rho(x-y)\, dy = \int_{Q_m} v_m(y)\, dy \geq 4^{-n}\Omega_n, \end{aligned}$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. Hence

$$\|F^{-1}aFv_m\|_{L^1(\mathbb{R}^n, w_{G,b}^{-1})} \geq b_m^{-1} \int_{G_m} |F^{-1}aFv_m(x)|\, dx \geq \frac{4^{-n}\Omega_n |G|}{b_m} \to \infty \quad \text{as} \quad m \to \infty,$$

while $\|v_m\|_{L^1(\mathbb{R}^n, w_{G,b}^{-1})} \leq 1$. Hence $a \notin \mathcal{M}_{L^1(\mathbb{R}^n, w_{G,b}^{-1})}$. □

For Banach spaces E_0, E_1 and a number $\theta \in (0,1)$, let $[E_0, E_1]_\theta$ denote the space obtained by the (lower) complex method of interpolation (see, e.g., [**9**] or [**23**, Chap. IV, §1.4]).

THEOREM 7.10. *Let $X(\mathbb{R}^n)$ be a Banach function space and $X'(\mathbb{R}^n)$ be its associate space. If $a \in \mathcal{M}^0_{X(\mathbb{R}^n)} \cap \mathcal{M}^0_{X'(\mathbb{R}^n)}$, then*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \leq \|a\|^{1/2}_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \|a\|^{1/2}_{\mathcal{M}^0_{X'(\mathbb{R}^n)}}. \tag{7.5}$$

Proof. Since $L^2(\mathbb{R}^n) = (X^{1/2}(X')^{1/2})(\mathbb{R}^n)$ (see Lemma 7.2) and $L^2(\mathbb{R}^n)$ has absolutely continuous norm, by [**23**, Chap. IV, Theorem 1.14], we have $L^2(\mathbb{R}^n) = [X(\mathbb{R}^n), X'(\mathbb{R}^n)]_{1/2}$ with equality of the norms.

Let $\overline{X}(\mathbb{R}^n)$ and $\overline{X'}(\mathbb{R}^n)$ denote the closures of $X(\mathbb{R}^n) \cap X'(\mathbb{R}^n)$ in the spaces $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$ respectively. Then $L^2(\mathbb{R}^n) = [\overline{X}(\mathbb{R}^n), \overline{X'}(\mathbb{R}^n)]_{1/2}$ with equality of the norms (see the discussion after the proof of [**23**, Chap. IV, Theorem 1.3]).

By [**23**, Chap. IV, Theorem 1.3], $X(\mathbb{R}^n) \cap X'(\mathbb{R}^n) \subset [X(\mathbb{R}^n), X'(\mathbb{R}^n)]_{1/2} = L^2(\mathbb{R}^n)$. Since $a \in \mathcal{M}^0_{X(\mathbb{R}^n)} \cap \mathcal{M}^0_{X'(\mathbb{R}^n)}$, the operator $W_a : f \mapsto F^{-1}aFf$, defined initially on $X(\mathbb{R}^n) \cap X'(\mathbb{R}^n)$, can be extended to bounded linear operators

$$W_a : \overline{X}(\mathbb{R}^n) \to X(\mathbb{R}^n), \quad W_a : \overline{X'}(\mathbb{R}^n) \to X'(\mathbb{R}^n).$$

Then, by the interpolation theorem for the complex method of interpolation (see, e.g., [**23**, Chap. IV, Theorem 1.2]),

$$\begin{aligned} \|W_a\|_{\mathcal{B}(L^2(\mathbb{R}^n))} &= \|W_a\|_{\mathcal{B}\left([\overline{X}(\mathbb{R}^n),\overline{X'}(\mathbb{R}^n)]_{1/2},[X(\mathbb{R}^n),X'(\mathbb{R}^n)]_{1/2}\right)} \\ &\le \|W_a\|^{1/2}_{\mathcal{B}\left(\overline{X}(\mathbb{R}^n),X(\mathbb{R}^n)\right)} \|W_a\|^{1/2}_{\mathcal{B}\left(\overline{X'}(\mathbb{R}^n),X'(\mathbb{R}^n)\right)}. \end{aligned} \tag{7.6}$$

Since $X(\mathbb{R}^n) \cap X'(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap \overline{X}(\mathbb{R}^n)$ and $X(\mathbb{R}^n) \cap X'(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap \overline{X'}(\mathbb{R}^n)$, we conclude that $L^2(\mathbb{R}^n) \cap \overline{X}(\mathbb{R}^n)$ is dense in $\overline{X}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n) \cap \overline{X'}(\mathbb{R}^n)$ is dense in $\overline{X'}(\mathbb{R}^n)$. Hence

$$\begin{aligned} \|W_a\|_{\mathcal{B}\left(\overline{X}(\mathbb{R}^n),X(\mathbb{R}^n)\right)} &= \sup\left\{ \frac{\|F^{-1}aFu\|_{X(\mathbb{R}^n)}}{\|u\|_{X(\mathbb{R}^n)}} \ :\ u \in (L^2(\mathbb{R}^n) \cap \overline{X}(\mathbb{R}^n)) \setminus \{0\} \right\} \\ &\le \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}} \end{aligned} \tag{7.7}$$

and

$$\begin{aligned} \|W_a\|_{\mathcal{B}\left(\overline{X'}(\mathbb{R}^n),X'(\mathbb{R}^n)\right)} &= \sup\left\{ \frac{\|F^{-1}aFu\|_{X'(\mathbb{R}^n)}}{\|u\|_{X'(\mathbb{R}^n)}} \ :\ u \in (L^2(\mathbb{R}^n) \cap \overline{X'}(\mathbb{R}^n)) \setminus \{0\} \right\} \\ &\le \|a\|_{\mathcal{M}^0_{X'(\mathbb{R}^n)}}. \end{aligned} \tag{7.8}$$

It is well known (see, e.g., [**18**, Theorem 2.5.10]) that $\mathcal{M}_{L^2(\mathbb{R}^n)} = L^\infty(\mathbb{R}^n)$ and

$$\|W_a\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|a\|_{\mathcal{M}_{L^2(\mathbb{R}^n)}} = \|a\|_{L^\infty(\mathbb{R}^n)}. \tag{7.9}$$

Combining (7.6)–(7.9), we arrive at (7.5). □

We are now we are in a position to prove the main result of this section.

THEOREM 7.11. *Let $X(\mathbb{R}^n)$ be a reflection-invariant Banach function space.*

(a) *If $a \in \mathcal{M}^0_{X(\mathbb{R}^n)}$, then*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}^0_{X(\mathbb{R}^n)}}. \tag{7.10}$$

(b) *If $X(\mathbb{R}^n)$ is reflexive and $a \in \mathcal{M}_{X(\mathbb{R}^n)} \subset S'(\mathbb{R}^n)$, then $a \in L^\infty(\mathbb{R}^n)$ and*

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}. \tag{7.11}$$

The constant 1 *in the right-hand sides of* (7.10) *and* (7.11) *is best possible.*

Proof. Part (a) follows from Theorems 7.8(a) and 7.10.

(b) By [**1**, Chap. 1, Corollary 4.4], a Banach function space $X(\mathbb{R}^n)$ is reflexive if and only if both $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$ have absolutely continuous norm. Then both $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$ satisfy the norm fundamental property (see Corollary 2.12). If $a \in \mathcal{M}_{X(\mathbb{R}^n)}$, then $a \in \mathcal{M}_{X'(\mathbb{R}^n)}$ and

$$\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}} = \|a\|_{\mathcal{M}_{X'(\mathbb{R}^n)}} \tag{7.12}$$

in view of Theorem 7.8(b). It follows from [**22**, Lemma 2.12(b)], that the set $C_0^\infty(\mathbb{R}^n)$ is dense in the spaces $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$. Hence the convolution operator $W_a : u \mapsto F^{-1}aFu$ defined initially on $C_0^\infty(\mathbb{R}^n)$ extends to a bounded linear operator on both $X(\mathbb{R}^n)$ and $X'(\mathbb{R}^n)$. Moreover,

$$\|W_a\|_{\mathcal{B}(X(\mathbb{R}^n))} = \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}, \quad \|W_a\|_{\mathcal{B}(X'(\mathbb{R}^n))} = \|a\|_{\mathcal{M}_{X'(\mathbb{R}^n)}}. \tag{7.13}$$

By Corollary 7.3, W_a is bounded on $L^2(\mathbb{R}^n)$ and

$$\|W_a\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \|W_a\|^{1/2}_{\mathcal{B}(X(\mathbb{R}^n))}\|W_a\|^{1/2}_{\mathcal{B}(X'(\mathbb{R}^n))}. \tag{7.14}$$

It is well known (see, e.g., [**18**, Theorem 2.5.10]) that $\mathcal{M}_{L^2(\mathbb{R}^n)} = L^\infty(\mathbb{R}^n)$ and

$$\|W_a\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|a\|_{\mathcal{M}_{L^2(\mathbb{R}^n)}} = \|a\|_{L^\infty(\mathbb{R}^n)}. \tag{7.15}$$

Combining (7.12)–(7.15), we see that

$$\|a\|_{L^\infty(\mathbb{R}^n)} \le \|a\|^{1/2}_{\mathcal{M}_{X(\mathbb{R}^n)}}\|a\|^{1/2}_{\mathcal{M}_{X'(\mathbb{R}^n)}} = \|a\|^{1/2}_{\mathcal{M}_{X(\mathbb{R}^n)}}\|a\|^{1/2}_{\mathcal{M}_{X(\mathbb{R}^n)}} = \|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}},$$

which completes the proof of (7.11).

Suppose now that there exists a constant $D_X > 0$ such that $\|a\|_{L^\infty(\mathbb{R}^n)} \le D_X\|a\|_{\mathcal{M}_{X(\mathbb{R}^n)}}$ for all $a \in \mathcal{M}_{X(\mathbb{R}^n)}$. Then taking $a \equiv 1$, one gets $D_X \ge 1$. So, the constant $D_X = 1$ in (7.11) is best possible. The same can be proved similarly for (7.10). □

Unfortunately, we have not been able to answer the following question.

QUESTION 7.12. *Can one drop the reflexivity requirement in Theorem 7.11(b)?*

8. *Concluding remarks*

Let $Y(\mathbb{R})$ be a translation-invariant Banach function space. We have seen that for subexponentially growing weights w like (3.14), $Y(\mathbb{R}, w)$ has the weak doubling property in view of Lemma 3.10, and hence $\mathcal{M}_{Y(\mathbb{R},w)} \subseteq L^\infty(\mathbb{R})$ according to Theorem 1.3. This inclusion holds also for reflexive transaltion-invariant Banach function spaces $Y(\mathbb{R})$ and symmetric weights $w = \widetilde{w}$ that may grow arbitrarily fast (see Theorem 7.11). On the other hand, $\mathcal{M}_{Y(\mathbb{R},w)}$ may contain unbounded functions if w grows at least exponentially as $x \to +\infty$ and decays to 0 exponentially as $x \to -\infty$ (see Corollaries 5.2 and 5.5). It is natural to ask whether there are any unbounded Fourier multipliers in the case of weights like

$$w(x) = \begin{cases} \exp(|x|^{\alpha_1}), & x < 0, \\ \exp(x^{\alpha_2}), & x \ge 0, \end{cases} \qquad \alpha_1, \alpha_2 > 1. \tag{8.1}$$

It turns out that there are no non-trivial Fourier multipliers in the case of weights like (8.1) and, more generally, of weights on $\mathbb{R}^n$ that grow superexponentially in all directions: $\mathcal{M}_{Y(\mathbb{R}^n,w)} = \mathbb{C}$. This fact was observed first by Löfström [**25**] in the case $Y(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. For the convenience of the reader, we present here a slightly modified argument from [**25**] in the case of arbitrary translation-invariant Banach function spaces.

THEOREM 8.1 (Cf. [**25**, p. 93]). *Suppose $X(\mathbb{R}^n)$ is a Banach function space such that for every $x_0 \in \mathbb{R}^n \setminus \{0\}$ there exist $\varepsilon > 0$ and a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ satisfying the condition*

$$\frac{\|\chi_{B(x_k,\rho)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(x_k-x_0,\varepsilon)}\|_{X(\mathbb{R}^n)}} \to \infty \quad \text{as} \quad k \to \infty \tag{8.2}$$

for every $\rho \in (0, \varepsilon]$. If κ is a distribution such that

$$\|\kappa * f\|_{X(\mathbb{R}^n)} \le C\|f\|_{X(\mathbb{R}^n)} \quad \text{for all} \quad f \in C_0^\infty(\mathbb{R}^n), \tag{8.3}$$

with some constant $C > 0$, then $\kappa = c\delta$ with some constant $c \in \mathbb{C}$, where δ is the Dirac measure.

Proof. Take any $x_0 \in \mathbb{R}^n \setminus \{0\}$. Suppose $x_0 \in \operatorname{supp} \kappa$. Then there exists $f \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \widetilde{f} \subseteq B(x_0, \varepsilon)$ and

$$\kappa * f(0) = \langle \kappa, \widetilde{f} \rangle = 1.$$

Since $\kappa * f$ is continuous (see, e.g., [**29**, Theorem 6.30(b)]), there exists $\rho \in (0, \varepsilon]$ such that $|\kappa * f(x)| > 1/2$ for all $x \in B(0, \rho)$. Then

$$\left|\big(\kappa * (\tau_{x_k} f)\big)(x + x_k)\right| = \left|\big(\tau_{-x_k}(\kappa * (\tau_{x_k} f))\big)(x)\right| = |\kappa * f(x)| > \frac{1}{2}$$

for all $x \in B(0, \rho)$. Hence it follows from (8.3) that

$$\begin{aligned}\frac{1}{2}\|\chi_{B(x_k,\rho)}\|_{X(\mathbb{R}^n)} &\le \left\|\chi_{B(x_k,\rho)}\big(\kappa * (\tau_{x_k} f)\big)\right\|_{X(\mathbb{R}^n)} \le \|\kappa * (\tau_{x_k} f)\|_{X(\mathbb{R}^n)} \\ &\le C\|\tau_{x_k} f\|_{X(\mathbb{R}^n)} \le C\|f\|_{L^\infty(\mathbb{R}^n)}\|\chi_{B(x_k - x_0,\varepsilon)}\|_{X(\mathbb{R}^n)},\end{aligned}$$

since $\operatorname{supp}(\tau_{x_k} f) = x_k + \operatorname{supp} f \subseteq x_k + B(-x_0, \varepsilon) = B(x_k - x_0, \varepsilon)$. So,

$$\frac{\|\chi_{B(x_k,\rho)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(x_k - x_0,\varepsilon)}\|_{X(\mathbb{R}^n)}} \le 2C\|f\|_{L^\infty(\mathbb{R}^n)} \quad \text{for all} \quad k \in \mathbb{N},$$

which contradicts (8.2). This means that $x_0 \in \mathbb{R}^n \setminus \{0\}$ cannot belong to the support of κ, i.e. $\operatorname{supp} \kappa = \{0\}$. Hence κ is a linear combination of δ and its partial derivatives (see, e.g., [**29**, Theorems 6.24(d) and 6.25]). It is easy to see that then (8.3) implies the equality $\kappa = c\delta$ with some constant $c \in \mathbb{C}$. □

THEOREM 8.2 (Cf. [**25**, p. 91-93]). *Let $Y(\mathbb{R}^n)$ be a translation-invariant Banach function space and $w : \mathbb{R}^n \to [0, \infty]$ be a weight such that $w \in Y_{\mathrm{loc}}(\mathbb{R}^n)$ and $1/w \in Y'_{\mathrm{loc}}(\mathbb{R}^n)$. Suppose for every $x_0 \in \mathbb{R}^n \setminus \{0\}$ there exist $\varepsilon > 0$ and a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ satisfying the condition*

$$\inf_{|x|\le\varepsilon,\ |y|\le\varepsilon} \frac{w(x_k + x)}{w(x_k - x_0 + y)} \to \infty \quad \text{as} \quad k \to \infty. \tag{8.4}$$

Then $\mathcal{M}_{Y(\mathbb{R}^n,w)} = \mathbb{C}$.

Proof. Since $Y(\mathbb{R}^n)$ is translation-invariant, it follows from Lemma 3.4(a) that there exist constants $C_1, C_2 > 0$ such that for all $k \in \mathbb{N}$ and all $\rho \in (0, \varepsilon]$ one has

$$\frac{\|\chi_{B(x_k,\rho)}\|_{Y(\mathbb{R}^n)}}{\|\chi_{B(x_k - x_0,\varepsilon)}\|_{Y(\mathbb{R}^n)}} \ge \frac{C_1 \min\{1, \rho^n\}}{C_2 \max\{1, \varepsilon^n\}} =: C(\rho, \varepsilon).$$

Hence

$$\begin{aligned}\frac{\|\chi_{B(x_k,\rho)}\|_{X(\mathbb{R}^n)}}{\|\chi_{B(x_k - x_0,\varepsilon)}\|_{X(\mathbb{R}^n)}} &= \frac{\|w\chi_{B(x_k,\rho)}\|_{Y(\mathbb{R}^n)}}{\|w\chi_{B(x_k - x_0,\varepsilon)}\|_{Y(\mathbb{R}^n)}} \ge \frac{\inf_{|x|\le\varepsilon} w(x_k + x)}{\sup_{|y|\le\varepsilon} w(x_k - x_0 + y)} \frac{\|\chi_{B(x_k,\rho)}\|_{Y(\mathbb{R}^n)}}{\|\chi_{B(x_k - x_0,\varepsilon)}\|_{Y(\mathbb{R}^n)}} \\ &\ge C(\rho, \varepsilon) \inf_{|x|\le\varepsilon,\ |y|\le\varepsilon} \frac{w(x_k + x)}{w(x_k - x_0 + y)} \to \infty \quad \text{as} \quad k \to \infty.\end{aligned}$$

Thus, the conditions of Theorem 8.1 are satisfied for $X(\mathbb{R}^n) = Y(\mathbb{R}^n, w)$.

If $a \in \mathcal{M}_{Y(\mathbb{R},w)}$, then

$$\|F^{-1}a * u\|_{Y(\mathbb{R}^n,w)} = \|F^{-1}aFu\|_{Y(\mathbb{R}^n,w)} \le \|a\|_{\mathcal{M}_{Y(\mathbb{R}^n,w)}}\|u\|_{Y(\mathbb{R}^n,w)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. By Theorem 8.1, $F^{-1}a = c\delta$. Therefore $a \in \mathbb{C}$. □

COROLLARY 8.3. *Let $Y(\mathbb{R})$ be a translation-invariant Banach function space and $w : \mathbb{R} \to (0, \infty)$ be the weight given by (8.1). Then $\mathcal{M}_{Y(\mathbb{R},w)} = \mathbb{C}$.*

Proof. Let $\alpha > 1$ and $\epsilon > 0$. Using the mean value theorem, one gets

$$(\rho+\epsilon)^\alpha - \rho^\alpha \geq \alpha\rho^{\alpha-1}\epsilon \to +\infty \quad \text{as} \quad \rho \to +\infty.$$

Let w be the weight defined by (8.1). Take any $x_0 \in \mathbb{R} \setminus \{0\}$. Let $\varepsilon := |x_0|/3$, $x_k := (k+1)x_0$, $k \in \mathbb{N}$. If $x_0 < 0$, then it follows from the above that

$$\begin{aligned}\inf_{|x|\leq\varepsilon,\ |y|\leq\varepsilon} \frac{w(x_k+x)}{w(x_k-x_0+y)} &\geq \frac{\exp\left(\left((k+1-1/3)|x_0|\right)^{\alpha_1}\right)}{\exp\left(\left((k+1/3)|x_0|\right)^{\alpha_1}\right)}\\ &= \exp\left(\left((k+2/3)^{\alpha_1} - (k+1/3)^{\alpha_1}\right)|x_0|^{\alpha_1}\right) \to \infty \quad \text{as} \quad k \to \infty,\end{aligned}$$

that is, condition (8.4) is satisfied. Similarly, it can be shown that it is also satisfied if $x_0 > 0$. Since $w \in L^\infty_{\text{loc}}(\mathbb{R}) \subset Y_{\text{loc}}(\mathbb{R})$ and $1/w \in L^\infty_{\text{loc}}(\mathbb{R}) \subset Y'_{\text{loc}}(\mathbb{R})$, it remains to apply Theorem 8.2. □

It might be instructive to contrast the above nonexistence results of non-trivial Fourier multipliers with the following statements.

THEOREM 8.4. *Let $Y(\mathbb{R}^n)$ be a translation-invariant Banach function space and w be a weight such that $w \in Y_{\text{loc}}(\mathbb{R}^n)$ and $1/w \in Y'_{\text{loc}}(\mathbb{R}^n)$. Suppose there exist $R > 0$, $\varepsilon > 0$ and $C_\varepsilon > 0$ such that*

$$\frac{w(x+y)}{w(x)} \leq C_\varepsilon \quad \text{for all} \quad |x| \geq R,\ |y| \leq \varepsilon. \tag{8.5}$$

Then there exists a constant $C > 0$ such that for any $\kappa \in L^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\kappa \subseteq B(0,\varepsilon)$ and any $f \in Y(\mathbb{R}^n, w)$ one has

$$\|\kappa * f\|_{Y(\mathbb{R}^n,w)} \leq C\|\kappa\|_{L^\infty(\mathbb{R}^n)}\|f\|_{Y(\mathbb{R}^n,w)}. \tag{8.6}$$

Proof. Since $w \in Y_{\text{loc}}(\mathbb{R}^n)$ and $1/w \in Y'_{\text{loc}}(\mathbb{R}^n)$, we see that $Y(\mathbb{R}^n, w)$ is a Banach function space and $Y'(\mathbb{R}^n, w^{-1})$ is its associate space in view of [**22**, Lemma 2.4].

If $|x| \leq R + \varepsilon$, then

$$\kappa * f(x) = \int_{\mathbb{R}^n} \kappa(x-y)f(y)\,dy = \int_{B(0,R+2\varepsilon)} \kappa(x-y)f(y)\,dy, \tag{8.7}$$

since $\operatorname{supp}\kappa \subseteq B(0,\varepsilon)$.

If $|x| > R + \varepsilon$, then

$$\begin{aligned}\kappa * f(x) &= \int_{\mathbb{R}^n} \kappa(y)f(x-y)\,dy = \int_{B(0,\varepsilon)} \kappa(y)f(x-y)\,dy\\ &= \int_{B(0,\varepsilon)} \kappa(y)\chi_{\mathbb{R}^n\setminus B(0,R)}(x-y)f(x-y)\,dy.\end{aligned} \tag{8.8}$$

Further, axiom (A5) implies the existence of a constant $C_{R,\varepsilon} > 0$ such that

$$\int_{B(0,R+2\varepsilon)} |f(y)|\,dy \leq C_{R,\varepsilon}\|f\|_{Y(\mathbb{R}^n,w)} \quad \text{for all} \quad f \in Y(\mathbb{R}^n, w). \tag{8.9}$$

It is clear that

$$\|\kappa * f\|_{Y(\mathbb{R}^n,w)} \leq \|\chi_{B(0,R+\varepsilon)}\kappa * f\|_{Y(\mathbb{R}^n,w)} + \|\chi_{\mathbb{R}^n\setminus B(0,R+\varepsilon)}\kappa * f\|_{Y(\mathbb{R}^n,w)}. \tag{8.10}$$

It follows from (8.7), (8.9), and axiom (A4) that

$$\begin{aligned}\|\chi_{B(0,R+\varepsilon)}\kappa * f\|_{Y(\mathbb{R}^n,w)} &\leq \left\|\chi_{B(0,R+\varepsilon)}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^1(B(0,R+2\varepsilon))}\right\|_{Y(\mathbb{R}^n,w)}\\ &\leq C_{R,\varepsilon}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\|f\|_{Y(\mathbb{R}^n,w)}\left\|\chi_{B(0,R+\varepsilon)}\right\|_{Y(\mathbb{R}^n,w)}\\ &=: C'_{R,\varepsilon}\|\kappa\|_{L^\infty(\mathbb{R}^n)}\|f\|_{Y(\mathbb{R}^n,w)}.\end{aligned} \tag{8.11}$$

Taking into account that $Y(\mathbb{R}^n)$ is translation-invariant and using (8.5), one gets for all $y \in B(0,\varepsilon)$,

$$\begin{aligned}
&\left\|\tau_y\left(\chi_{\mathbb{R}^n\setminus B(0,R)}f\right)\right\|_{Y(\mathbb{R}^n,w)} = \left\|w\tau_y\left(\chi_{\mathbb{R}^n\setminus B(0,R)}f\right)\right\|_{Y(\mathbb{R}^n)}\\
&\quad = \left\|\tau_y\left((\tau_{-y}w)\left(\chi_{\mathbb{R}^n\setminus B(0,R)}f\right)\right)\right\|_{Y(\mathbb{R}^n)} = \left\|(\tau_{-y}w)\left(\chi_{\mathbb{R}^n\setminus B(0,R)}f\right)\right\|_{Y(\mathbb{R}^n)}\\
&\quad \le C_\varepsilon\left\|w\left(\chi_{\mathbb{R}^n\setminus B(0,R)}f\right)\right\|_{Y(\mathbb{R}^n)} = C_\varepsilon\left\|\chi_{\mathbb{R}^n\setminus B(0,R)}f\right\|_{Y(\mathbb{R}^n,w)}. \qquad (8.12)
\end{aligned}$$

Using (8.8), (8.12) and Hölder's inequality for Banach function spaces (see [**1**, Chap. 1, Theorem 2.4]), and taking into account that $Y(\mathbb{R}^n)$ is translation-invariant, one gets for all $g \in Y'(\mathbb{R}^n, w^{-1})$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}(\chi_{\mathbb{R}^n\setminus B(0,R+\varepsilon)}\kappa * f)(x)g(x)\,dx\right|\\
&\quad\le \int_{\mathbb{R}^n}\chi_{\mathbb{R}^n\setminus B(0,R+\varepsilon)}(x)\left(\int_{B(0,\varepsilon)}|\kappa(y)|\left|\chi_{\mathbb{R}^n\setminus B(0,R)}f\right|(x-y)\,dy\right)|g(x)|\,dx\\
&\quad\le \int_{B(0,\varepsilon)}|\kappa(y)|\left(\int_{\mathbb{R}^n}\left|\chi_{\mathbb{R}^n\setminus B(0,R)}f\right|(x-y)|g(x)|\,dx\right)dy\\
&\quad\le \int_{B(0,\varepsilon)}|\kappa(y)|\left\|\tau_y\left(\chi_{\mathbb{R}^n\setminus B(0,R)}f\right)\right\|_{Y(\mathbb{R}^n,w)}\|g\|_{Y'(\mathbb{R}^n,w^{-1})}\,dy\\
&\quad\le C_\varepsilon\|\chi_{\mathbb{R}^n\setminus B(0,R)}f\|_{Y(\mathbb{R}^n,w)}\|g\|_{Y'(\mathbb{R}^n,w^{-1})}\int_{B(0,\varepsilon)}|\kappa(y)|\,dy\\
&\quad\le C_\varepsilon\|\kappa\|_{L^1(B(0,\varepsilon))}\|f\|_{Y(\mathbb{R}^n,w)}\|g\|_{Y'(\mathbb{R}^n,w^{-1})}.
\end{aligned}$$

By [**1**, Chap. 1, Theorem 2.7 and Lemma 2.8], the above inequality implies that

$$\begin{aligned}
&\|\chi_{\mathbb{R}^n\setminus B(0,R+\varepsilon)}\kappa * f\|_{Y(\mathbb{R}^n,w)}\\
&\quad= \sup\left\{\left|\int_{\mathbb{R}^n}(\chi_{\mathbb{R}^n\setminus B(0,R+\varepsilon)}\kappa * f)(x)g(x)\,dx\right| \;:\; g \in Y'(\mathbb{R}^n,w^{-1}),\ \|g\|_{Y'(\mathbb{R}^n,w^{-1})}\le 1\right\}\\
&\quad\le C_\varepsilon\|\kappa\|_{L^1(B(0,\varepsilon))}\|f\|_{Y(\mathbb{R}^n,w)} \le C_\varepsilon|B(0,1)|\varepsilon^n\|\kappa\|_{L^\infty(\mathbb{R}^n)}\|f\|_{Y(\mathbb{R}^n,w)}. \qquad (8.13)
\end{aligned}$$

Combining (8.10), (8.11), and (8.13), one gets (8.6) with $C = C'_{R,\varepsilon} + C_\varepsilon|B(0,1)|\varepsilon^n$. □

Corollary 8.5. *Let $c > 0$, $0 < \alpha \le 1$, and $w(x) = \exp(c|x|^\alpha)$ for $x \in \mathbb{R}^n$. If $Y(\mathbb{R}^n)$ is a translation-invariant Banach function space, then there exist non-trivial Fourier multipliers in $\mathcal{M}_{Y(\mathbb{R}^n,w)}$.*

Proof. Fix $\varepsilon > 0$. Then for all $x \in \mathbb{R}^n$ and $|y| \le \varepsilon$,

$$\frac{w(x+y)}{w(x)} \le \frac{\exp\left(c(|x|+\varepsilon)^\alpha\right)}{\exp\left(c|x|^\alpha\right)} = \exp\left(c\big((|x|+\varepsilon)^\alpha - |x|^\alpha\big)\right) \le \exp\left(cM(\alpha,\varepsilon)\right),$$

where, by the mean value theorem,

$$M(\alpha,\varepsilon) := \max_{0\le\rho<\infty}\left((\rho+\varepsilon)^\alpha - \rho^\alpha\right) < +\infty.$$

Then w satisfies condition (8.5). There exists $j \in \mathbb{N}$ such that the function $\varrho_j \in C_0^\infty(\mathbb{R}^n)$ given by (2.2) satisfies $\operatorname{supp}\varrho_j \subseteq B(0,\varepsilon)$. Put $a := F\varrho_j$. By Theorem 8.4, there exists a constant $C > 0$ such that for all $u \in S(\mathbb{R}^n) \cap Y(\mathbb{R}^n,w)$, one has

$$\|F^{-1}aFu\|_{Y(\mathbb{R}^n,w)} = \|\varrho_j * u\|_{Y(\mathbb{R}^n,w)} \le C\|\varrho_j\|_{L^\infty(\mathbb{R}^n)}\|u\|_{Y(\mathbb{R}^n,w)}.$$

Therefore, $a \in \mathcal{M}_{Y(\mathbb{R}^n,w)}$. □

References

1. C. Bennett and R. Sharpley, *Interpolation of operators* (Academic Press, Boston etc., 1988).

2. E. I. Berezhnoi, 'Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces' *Proc. Amer. Math. Soc.* 127 (1999) 79–87.

3. E. Berkson and T. A. Gillespie, 'Multipliers for weighted L^p-spaces, transference, and the q-variation of functions', *Bull. Sci. Math.* 122 (1998) 427–454.

4. V. I. Bogachev, *Measure theory. Vol. I* (Springer, Berlin, 2007).

5. A. Böttcher and Yu. I. Karlovich, *Carleson curves, Muckenhoupt weights, and Toeplitz operators* (Birkhäuser, Basel, 1997).

6. A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky, *Convolution operators and factorization of almost periodic matrix functions* (Operator Theory: Advances and Applications, vol. 131. Birkhäuser, Basel, 2002).

7. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations* (Springer, New York, 2011).

8. V. I. Burenkov and M. Sh. Tuyakbaev, 'On Fourier integral multipliers in weighted L_p-spaces with an exponential weight. II', *Proc. Steklov Inst. Math.* 204 (1994) 69-96.

9. A. P. Calderón, 'Intermediate spaces and interpolation, the complex method', *Studia Math.* 24 (1964) 113–190.

10. A. P. Calderón and A. Zygmund, 'On the existence of certain singular integrals', *Acta Math.* 88 (1952) 85–139.

11. M. Cwikel and P. G. Nilsson, 'Interpolation of weighted Banach lattices', *Mem. Amer. Math. Soc.* 165 (2003) no. 787, 1–105.

12. R. Duduchava, *Integral equations with fixed singularities* (Teubner Verlagsgesellschaft, Leipzig, 1979).

13. P. L. Duren, *Theory of H^p spaces* (Academic Press, New York and London, 1970).

14. P. Duren, *Invitation to classical analysis* (American Mathematical Society, Providence, RI, 2012).

15. R. E. Edwards and G. I. Gaudry, *Littlewood-Paley and multiplier theory* (Springer, Berlin, 1977).

16. G. I. Èskin, *Boundary value problems for elliptic pseudodifferential equations* (AMS, Providence, R.I., 1981).

17. G. Folland, *Real analysis* (John Wiley and Sons, New York, 1999).

18. L. Grafakos, *Classical Fourier analysis.* 3rd ed. (Springer, New York, 2014).

19. L. Grafakos, *Modern Fourier analysis.* 3rd ed. (Springer, New York, 2014).

20. L. Hörmander, 'Estimates for translation invariant operators in L^p spaces', *Acta Math.* 104 (1960) 93–140.

21. A. Yu. Karlovich, 'Banach algebra of the Fourier multipliers on weighted Banach function spaces', *Concr. Oper.* 2 (2015) 27–36.

22. A. Yu. Karlovich and I. M. Spitkovsky, 'The Cauchy singular integral operator on weighted variable Lebesgue spaces', In: *Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation.* Operator Theory: Advances and Applications, vol. 236, pp. 275–291, Birkhäuser/Springer, Basel, 2014.

23. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of linear operators* (AMS, Providence, R.I., 1982).

24. M. Lacey, 'Carleson's theorem: proof, complements, variations', *Publ. Mat.* 48 (2004) 251–307.

25. J. Löfström, 'A non-existence theorem for translation invariant operators on weighted L_p-spaces', *Math. Scand.* 53 (1983) 88–96.

26. G. Ja. Lozanovskiĭ, 'Certain Banach lattices', *Siberian Math. J.* 10 (1969) 419–431.

27. L. Maligranda, *Orlicz spaces and interpolation* (Seminars in Mathematics 5, Univ. Estadual de Campinas, Campinas SP, Brazil, 1989).

28. S. Roch, P. Santos and B. Silbermann, *Non-commutative Gelfand theories. A tool-kit for operator theorists and numerical analysts* (Springer, Berlin, 2011).

29. W. Rudin, *Functional analysis* (McGraw-Hill, New York, 1991).

30. L. Saloff-Coste, *Aspects of Sobolev-type inequalities* (London Mathematical Society Lecture Note Series 289, Cambridge University Press, Cambridge, 2002).

31. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Princeton University Press, Princeton, 1971).

32. P. P. Zabreiko, 'Interpolation theorem for linear operators', *Math. Notes* 2 (1967) 853-855.

Alexei Karlovich
Centro de Matemática e Aplicações,
Departamento de Matemática,
Faculdade de Ciências e Tecnologia,
Universidade Nova de Lisboa,
Quinta da Torre,
2829–516 Caparica,
Portugal

oyk@fct.unl.pt

Eugene Shargorodsky
Department of Mathematics
King's College London
Strand, London WC2R 2LS
United Kingdom

eugene.shargorodsky@kcl.ac.uk