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TWISTOR LINES ON CUBIC SURFACES[*]

Abstract. It is shown that there exist non-singular cubic surfaces in $\mathbb{CP}^3$ containing 5 twistor lines. This is the maximum number of twistor fibres that a non-singular cubic can contain. Cubic surfaces in $\mathbb{CP}^3$ with 5 twistor lines are classified up to transformations preserving the conformal structure of S^4.

Introduction

The twistor space, Z, of an oriented Riemannian 4-manifold M is the bundle of almost-complex structures on M compatible with the metric and orientation. The 6-dimensional total space of the twistor space admits a canonical almost-complex structure which is integrable whenever the 4-manifold is half conformally flat.

The definition of the twistor space does not require the full Riemannian metric; it only depends upon the conformal structure of the manifold. The idea of studying the twistor space is that, on a half conformally flat manifold, the conformal geometry of M is encoded into the complex geometry of Z.

As an example, the condition that an almost-complex structure on M compatible with the conformal structure is *integrable* can be interpreted as saying that the corresponding section of Z defines a *holomorphic* submanifold of Z.

The basic example of twistor space is that of the 4-sphere S^4, which itself may be identified with the quaternionic projective line $\mathbb{HP}^1$, and is topologically $\mathbb{R}^4 \cup \infty$. The twistor space in this case is biholomorphic to $\mathbb{CP}^3$, and the associated bundle structure $\mathbb{CP}^3 \to \mathbb{HP}^1$ is the Hopf fibration. Following the work of Penrose, Ward and Atiyah, it was used to great effect in classifying instanton bundles on S^4 [3, 2].

Combining these two facts, we see that complex hypersurfaces in $\mathbb{CP}^3$ locally give rise to integrable complex structures on S^4 compatible with the metric. For topological reasons there are no global almost-complex structures on S^4, so no hypersurface in $\mathbb{CP}^3$ can intersect every fibre of the Hopf fibration in exactly one point.

One can try to investigate the algebraic geometry of surfaces in $\mathbb{CP}^3$ from this twistor perspective. In this paper, we take the opportunity to revisit some of the beautiful results on cubic surfaces discovered by geometers in the nineteenth century. A brief history of their discoveries can be found in [9].

A natural question when studying complex surfaces from this point of view is to classify surfaces in $\mathbb{CP}^3$ of degree d up to a conformal transformation of the base space S^4. Various conformal invariants of a surface can be defined immediately. The fibres of the Hopf fibration are complex projective lines in $\mathbb{CP}^3$, and the number of fibres that

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lie entirely in the suface is an invariant of the surface up to conformal transformation.

Closely related invariants arise from the topology of the discriminant locus. A generic fibre, intersecting the surface transversely, will contain d points. This is simply because the defining polynomial of the surface, when restricted to the fibre, gives a polynomial of degree d. The set of points where the discriminant of this polynomial vanishes is called the discriminant locus. It can be thought of as the set of fibres that are not transverse to the surface at each point of intersection, or as the set of points of S^4 where we cannot locally define a complex structure corresponding to the surface.

In [18], quadric surfaces are classified up to conformal transformation in considerable detail. The table below shows all the possible topologies of the discriminant locus in this case and how they correspond to the number of twistor lines. A preliminary question to ask when trying to study the conformal geometry of surfaces of higher degree is: what is the maximum number of twistor lines on surfaces of that degree?

No of twistor lines	Topology of discriminant locus
0	Torus
1	Torus with two points pinched together
2	Torus with two pairs of points pinched together
∞	Circle

Before restricting to twistor lines, it is worth reviewing pure algebro-geometric results on the number of projective lines on a surface of given degree. Since twistor lines are fibres of a fibration they must be skew (i.e., mutually disjoint), so we will also review the maximum number of skew lines on a surface of degree d.

Dimension counting alone leads one to expect that a quadric surface will contain an infinite number of lines, a cubic surface a finite number of lines and a higher degree surface will generically contain no lines at all.

A startling result is the celebrated Cayley–Salmon theorem: all non-singular cubic surfaces contain precisely 27 lines. Moreover a non-singular cubic surface contains precisely 72 sets of 6 skew lines.

The situation for higher degree curves is less well understood. The state of knowledge about the number of lines on surfaces of degree d was both reviewed and advanced in [4]. We summarize these findings next.

Define N_d to be the maximum number of lines on a smooth projective surface of degree d. Then:

- there are always 27 lines on a cubic,
- $N_4 = 64$ (see [20]),
- $N_d \leqslant (d-2)(11d-6)$ (see [20]),
- $N_d \geqslant 3d^2$ (see [6]),
- $N_6 \geqslant 180, N_8 \geqslant 352, N_12 \geqslant 864, N_{20} \geqslant 1600$ (see [6, 4]).

Here are the bounds on S_d, the maximum number of skew lines:

- there are always 6 skew lines on a cubic,
- $S_4 = 16$ (see [15]),
- $S_d \leqslant 2d(d-2)$ when $d \geqslant 4$ (see [13]),
- $S_d \geqslant d(d-2)+2$ (see [17]),
- $S_d \geqslant d(d-2)+4$ when $d \geqslant 7$ is odd (see [4]).

Specializing to the case of twistor lines, it was noted in the first version of [18] that the number of twistor lines is at most d^2 when $d \geqslant 3$ and that there exists a quartic surface containing exactly 8 twistor lines.

In this paper, we determine the maximum number of twistor lines on a smooth cubic surface. We shall show that in fact there are at most 5 twistor lines, and we shall give a detailed classification of all cubic surfaces with 5 twistor lines. In particular we shall prove the

THEOREM. *Any set of 5 points on a 2-sphere, no 4 of which lie on a circle, determines a one-parameter family of non-singular cubic surfaces with 5 twistor lines. All cubics in the family are projectively, but not conformally, equivalent. Two such cubic surfaces are projectively equivalent if and only if the sets of 5 points on the 2-sphere are conformally equivalent. All cubic surfaces with 5 twistor lines arise in this way.*

One would like explicit examples of such surfaces. We provide the necessary formulae and find the most symmetrical examples. In particular, we shall show that the cubic surface with 5 twistor lines which has the largest conformal symmetry group is projectively, but not conformally, equivalent to the Fermat cubic. There are various choices one can make for a twistor structure on $\mathbb{CP}^3$ that give the Fermat cubic 5 twistor lines, and the set of such structures has 54 connected components.

The paper begins with a brief review of the twistor fibration of $\mathbb{CP}^3$ and then moves on to discuss cubic surfaces. We review the classical results on cubic surfaces and demonstrate how the same ideas can be used to prove results about the twistor geometry.

1. The twistor fibration

To identify S^4 with $\mathbb{HP}^1$, we define two equivalence relations on $\mathbb{H} \times \mathbb{H}$:

$$[q_1, q_2] \sim_{\mathbb{H}} [\lambda q_1, \lambda q_2], \qquad \lambda \in \mathbb{H}^*,$$

$$[q_1, q_2] \sim_{\mathbb{C}} [\lambda q_1, \lambda q_2], \qquad \lambda \in \mathbb{C}^*.$$

By definition, the quotient of $\mathbb{H} \times \mathbb{H}$ by the first equivalence relation is the quaternionic projective line. Since $\mathbb{H} \times \mathbb{H} \cong \mathbb{C}^4$, the quotient by the second relation is isomorphic to the complex projective space $\mathbb{CP}^3$.

Thus we can define a map $\pi : \mathbb{CP}^3 \to S^4$ by sending a complex 1-dimensional subspace of $\mathbb{C}^4$ to the quaternionic 1-dimensional subspace of $\mathbb{H}^2$ that it spans. The map π is equivalent to the more general twistor fibration defined on an arbitrary oriented Riemannian 4-manifold as the total space of the bundle of almost-complex structures compatible with the metric and orientation.

On any twistor fibration one can define a map j which sends an almost-complex structure J to $-J$. In our case, applying j can be thought of as the action of multiplying a 1-dimensional complex subspace of $\mathbb{C}^4$ by the quaternion j in order to get a new 1-dimensional subspace.

The map j is an anti-holomorphic involution of the twistor space to itself with no fixed points. Starting with such a map j, one can recover the twistor fibration: given a point z in $\mathbb{CP}^3$ there is a unique projective line connecting z and $j(z)$. These lines form the fibres. We will call an anti-holomorphic involution on $\mathbb{CP}^3$ obtained by conjugating j by a projective transformation a *twistor structure*. The standard twistor structure on $\mathbb{CP}^3$ is given by

$$[z_1, z_2, z_3, z_4] \longmapsto [-\overline{z_2}, \overline{z_1}, -\overline{z_4}, \overline{z_3}].$$

The conformal symmetries of S^4 can be represented by quaternionic Möbius transformations

$$q \longmapsto (qc + d)^{-1}(qa + b), \qquad q \in \mathbb{H}$$

(see, for example, [10]). These correspond to the projective transformations of $\mathbb{CP}^3$ that preserve j. Thus we will say that two complex submanifolds of $\mathbb{CP}^3$ are *conformally equivalent* if they are projectively equivalent by a transformation that preserves j.

As an example, consider lines in $\mathbb{CP}^3$. If both lines are fibres of π then they are conformally equivalent by an isometry of S^4 sending the image of one line under π to the image of the other line. If a line is not a fibre of π then its image will be a round 2-sphere in S^4 (corresponding to a 2-sphere or a 2-plane in $\mathbb{R}^4$). Given such a 2-sphere in S^4, there are in fact two projective lines lying above it in $\mathbb{CP}^3$. Therefore, a line in $\mathbb{CP}^3$ is given by either an oriented 2-sphere or a point in S^4. Moreover, any two such 2-spheres are conformally equivalent. This geometrical correspondence is described in detail by Shapiro [21].

As another example, consider planes in $\mathbb{CP}^3$. A plane in $\mathbb{CP}^3$ cannot be transverse to every fibre of π because it would then define a complex structure on the whole of S^4, which is a topological impossibility. Thus a plane always contains at least one twistor fibre. Twistor fibres are always skew, whereas two lines in a plane always meet. Therefore a plane always contains exactly one twistor fibre. If one picks another line in the plane transverse to the fibre, its image under π will be a 2-sphere. We can find a conformal transformation of S^4 mapping any 2-sphere with a marked point to any other 2-sphere with a marked point. We deduce that any two planes in $\mathbb{CP}^3$ are conformally equivalent.

The case of quadric surfaces is considered in detail in [18] and is much more complicated. The aim of this paper is to make a start on the analogous question for cubic surfaces.

2. The Schläfli graph

Before looking at the twistor geometry of cubic surfaces. Let us review the classical results about the lines on twistor surfaces.

The Cayley–Salmon theorem states that every non-singular cubic surface contains exactly 27 lines [7]. Schläfli discovered that the intersection properties of these 27 lines are the same for all cubics [19]. This means that we can define the Schläfli graph of a cubic surface to be a graph with 27 vertices corresponding to each line on the cubic and and with an edge between the two vertices whenever the corresponding lines *do not* intersect. This graph will be independent of the choice of non-singular cubic surface. This definition is the standard one used by graph theorists, but from our point of view the complement of the Schläfli graph showing which lines *do* intersect is more natural. It is shown in Figure 1
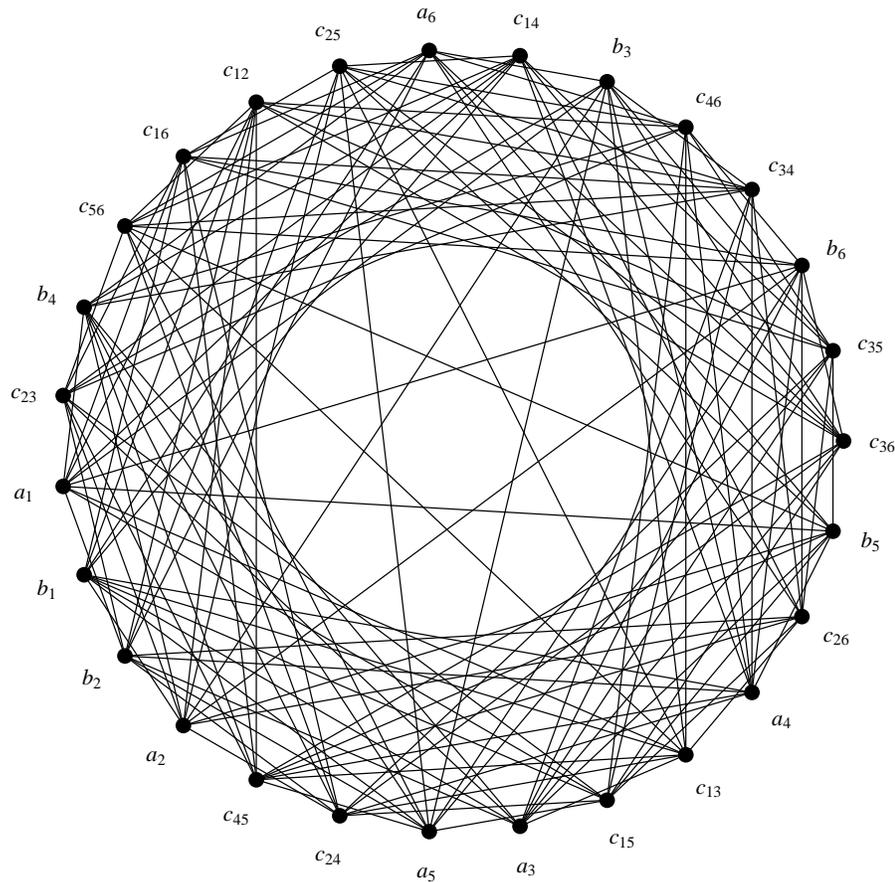
Figure 1: The complement of the Schlafli graph emphasizing a symmetry of order 9

Understanding the Schläfli graph provides a good deal of insight into the geometry of cubic surfaces. It is interesting from a purely graph theoretic point of view. Among its many properties, one particularly nice one is that it is 4-*ultrahomogeneous*. A graph is said to be k-ultrahomogeneous if every isomorphism between subgraphs with at most k vertices extends to an automorphism of the entire graph. If a graph is 5-ultrahomogenous it is k-ultrahomogeneous for any k. It turns out that the Schläfli graph and its complement are the only 4-ultrahomogeneous connected graphs that are not 5-ultrahomogeneous [5].

Although our picture of the Schläfli graph is pretty, it is not very practical. Schläfli devised a notation that allows one to understand the graph more directly, and we shall now describe this.

Among the 27 lines one can always find a set of 6 skew lines. We label these a_1, a_2, a_3, a_4, a_5 and a_6. Having chosen these labels, there will now be 6 more skew lines b_i ($1 \leqslant i \leqslant 6$) with each b_i intersecting all of the a lines except for a_i. We thereby obtain the configuration shown in Figure 2.
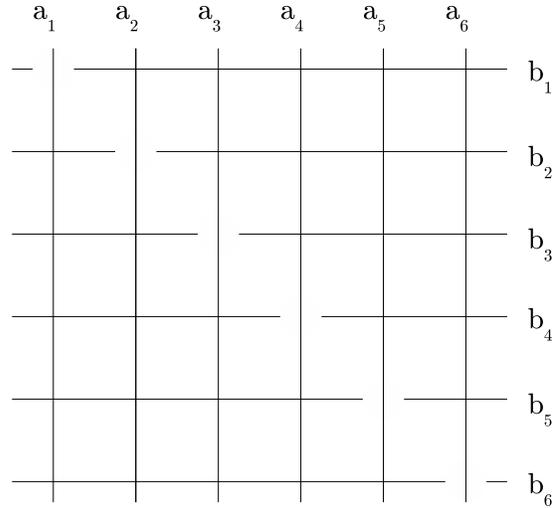
Figure 2: A "double six"

The remaining 15 lines are labelled c_{ij} with ($1 \leqslant i < j \leqslant 6$). The line c_{ij} is defined by the property that it intersects a_i and a_j, but no other a lines. The full intersection rules for distinct lines on the cubic surface are:

- a_i never intersects a_j.

- a_i intersects b_j iff $i \neq j$.

- a_i intersects c_{jk} iff $i \in \{j, k\}$.

- b_i never intersects b_j.

- b_i intersects c_{jk} iff $i \in \{j, k\}$.

- c_{ij} intersects c_{kl} iff $\{i, j\} \cap \{k, l\} = \varnothing$.

Another graphical representation of these properties was given in [16] and is reproduced in Figure 3. This is intended to be used rather like the tables of distances between towns that are used to be found in road atlases. A red/darker square indicates that the two lines intersect and a cyan/lighter ones indicates that the lines are skew. In this case we have chosen the ordering of the lines to show that this figure can be constructed using only a small number of different types of tile of size 3×3. The grouping is indicated with black lines.
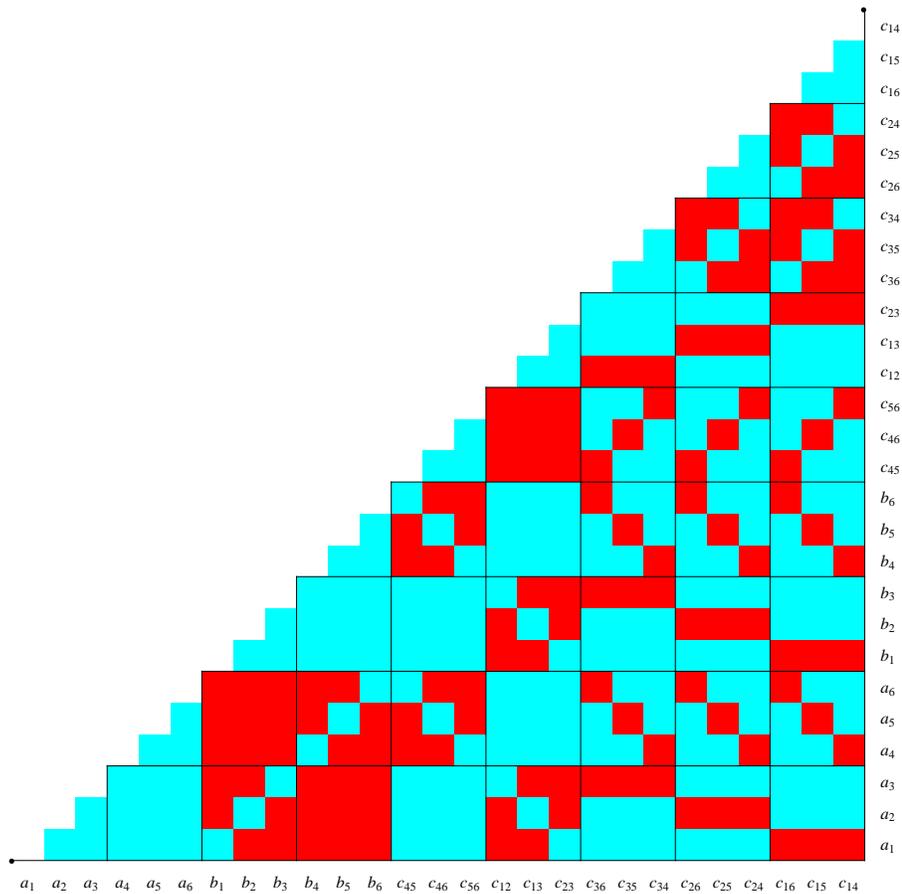
Figure 3: A tabular representation of the Schäfli graph

Being 4-ultrahomogeneous, it is no surprise that the Schläfli graph has a large group of automorphisms. Each choice of 6 skew lines from the 27 will give us a different way of labelling the lines as a_i, b_i and c_{ij}. Schläfli used the following notation for each choice:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

A set of 12 lines with these intersection properties is called a "double-six". In this notation here are the forms of the other double sixes:

$$\begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}$$

By varying the indices this gives a total of 36 double sixes on any cubic surface — hence 72 choices of a set of six disjoint lines and $72 \cdot 6! = 51840$ automorphisms of the Schläfli graph.

This large automorphism group is in fact the Weyl group of the exceptional Lie group E_6, and 27 is the dimension of the smallest non-trivial irreducible representation of E_6. Relative to the isotropy subgroup $SU(6) \times_{\mathbb{Z}_2} SU(2)$ of the corresponding Wolf space [22], this representation decomposes into irreducible subspaces of dimension 12 and 15, namely $\mathbb{C}^2 \otimes \mathbb{C}^6$ and $\wedge^2 \overline{\mathbb{C}^6}$. This is the algebraic interpretation of a double six.

As Schläfli discovered, consideration of the arrangement of the 27 lines on a non-singular cubic surface rapidly leads to a classification of cubic surfaces up to projective transformation [19]. By applying the same ideas, we can find a similar classification of cubic surfaces with sufficiently many twistor lines.

3. Classifying cubic surfaces with 5 twistor lines

As a first application of the Schläfli graph to the study of twistor lines on cubic surfaces we prove

LEMMA 1. *If a non-singular cubic surface in $\mathbb{CP}^3$ contains four twistor lines* a_1, a_2, a_3, a_4 *then, in Schläfli's notation,* $jb_5 = b_6$.

Proof. Since the Schläfli graph is 4 ultrahomogenoeus and twistor lines are always skew, we can assume that the first four lines are indeed those of a double six.

In Schläfli's notation, the line b_5 intersects a_1, a_2, a_3 and a_4. Therefore jb_5 intersects $ja_1 = a_1$, $ja_2 = a_2$, $ja_3 = a_3$ and $ja_4 = a_4$. Since it jb_5 is a line and since it intersects the cubic surface in 4 points, it must lie in the cubic surface. Since j has no fixed points, the points of intersection of b_5 and jb_5 with the line a_1 must be distinct. So $jb_5 \neq b_5$, Given this and the fact that it intersects a_1, a_2, a_3 and a_4 we deduce that $jb_5 = b_6$. □

COROLLARY 1. *If a non-singular cubic surface contains five twistor lines, and we label the first four a_1, a_2, a_3, a_4 in Schläfli's notation then the fifth line is c_{56}.*

Proof. Since the fifth twistor line must be skew to a_1, a_2, a_3 and a_4, it must be one of a_5, a_6, c_{56}. Suppose that the fifth line is a_6. This means it intersects b_5 so $ja_6 = a_6$ intersects $jb_5 = b_6$, which is a contradiction. The same argument shows that the fifth line cannot be a_5. □

The arrangement of lines described by Corollary 1 is illustrated in Figure 4. The twistor lines are shown as roughly vertical.
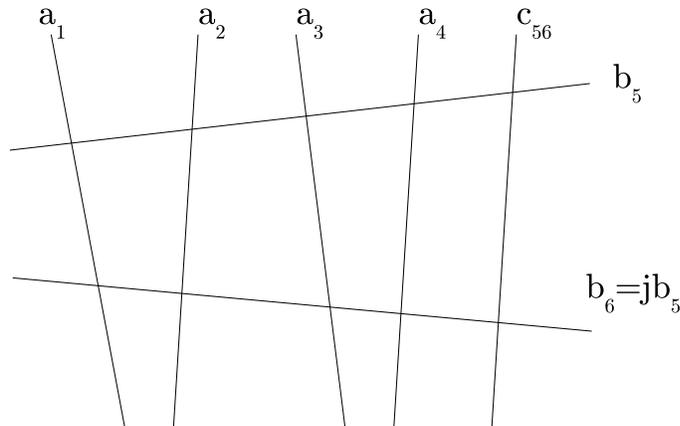
Figure 4: Seven lines

In particular we have proved:

THEOREM 1. *A non-singular cubic surface contains at most five twistor lines.*

This raises the question of whether or not we can find cubic surfaces containing 5 twistor lines. Simple dimension counting suggests it should be easy to find cubic surfaces which contain 4 twistor lines. Simply select any four twistor lines and apply the well-known

PROPOSITION 1. *Four lines in $\mathbb{CP}^3$ always lie on a (possibly singular) cubic surface.*

Proof. Pick 4 points on each line to get a total of 16 points. If a cubic surface has 4 points in common with a line, then it contains the entire line. So if we can find a cubic containing all 16 points, it will contain all 4 lines.

The general equation for a cubic surface has 20 coefficients, since this is the dimension of $S^3(\mathbb{C}^4)$. Putting the coordinates of these 16 points into the equation for

the cubic surface gives us 16 linear equations in the 20 unknown coefficients, so non-trivial solutions exist. □

It will become clear later that if we choose everything generically, the cubic surface will be non-singular. A corollary of this is that *four generic lines in $\mathbb{CP}^3$ have two lines intersecting all four of them*. This observation can be proved easily enough without appealing to the theory of cubics — for example one can use 2-forms to represent points of the Klein quadric, or Schubert calculus. Indeed, in [12] this observation is used as the starting point to establish the existence of the 27 lines on a cubic!

The same dimension counting argument tells us that 5 lines (let alone twistor ones) do not generically all lie on a cubic surface. Let us understand geometrically when 5 lines *do* all lie on a cubic surface.

PROPOSITION 2. *Five lines in $\mathbb{CP}^3$ lie on a (possibly singular) cubic surface if they are collinear, that is, there exists a fifth line intersecting all four.*

Proof. Let $\ell_1, \ldots, \ell_5$ be the lines and let k be another line intersecting all five in the points p_i. Choose 3 other points on each of the lines to get a set P of 20 points.

The condition on the coefficients of a cubic surface for it to contain all the points in P except for p_5 is represented by 19 linear equations in 20 unknowns. So we can find a cubic surface passing through all the points marked in black in Figure 5. This cubic surface has 4 points in common with k so it contains k. In particular it contains p_5. So it actually contains all 5 of the ℓ lines. □
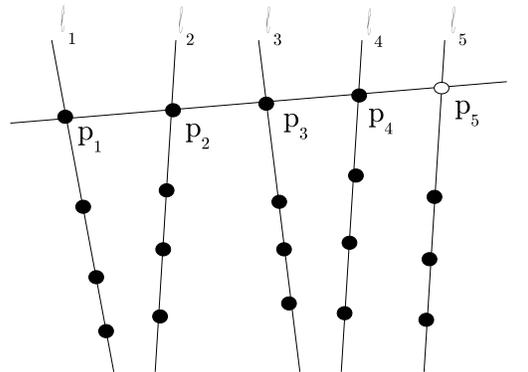
Figure 5: Nineteen points plus one

As a partial converse to Proposition 2, we remark that if 5 *skew* lines lie on a cubic surface then they are necessarily collinear. For example, if the cubic is non-singular then the Schläfli graph guarantees that the 5 lines are collinear; we can label them $a_1, \ldots, a_5$ (all intersecting just a_6) or $a_1, a_2, a_3, a_4, c_{56}$ (all intersecting b_5 and b_6).

The situation that is of interest for us is the second, in which we have 5 lines that are collinear in two different ways. The dimension counting argument above now shows that if one has 5 lines that are collinear in two ways, there will be a one-parameter family of (possibly singular) cubic surfaces containing all the lines.

Another way of seeing why there is a one-parameter family of cubics through such a configuration of lines is to observe that there is a one-parameter family of projective transformations that fixes all the lines. To see this observe that you can choose coordinates such that b_5 is given by the equations $z_1 = z_3 = 0$ and b_6 is given by the equations $z_2 = z_4 = 0$. Since the lines $a_1, a_2, a_3, a_4, c_{56}$ are all skew, their intersections with b_5 are distinct. So we can make a Möbius transformation of z_2 and z_4 so that in the inhomogenous coordinate z_2/z_4 the intersection points of a_1, a_2, c_{56} with b_5 are respectivly 0, 1, ∞. Similarly we can choose our coordinates such that the intersections of a_1, a_2, c_{56} with b_6 correspond to $z_1/z_3 = 0$, 1, ∞. With these specifications, one can choose to independently rescale the coordinate pairs (z_1, z_3), (z_2, z_4) and one will still have coordinates with these properties. This choice of coordinates gives us a one-parameter family of projective transformations that fix all the lines.

Given two non-singular cubics $(\mathscr{C}_1, \mathscr{C}_2)$ that each contain all 7 lines, we can construct the projective transformation mapping $\mathscr{C}_1$ to $\mathscr{C}_2$ directly from the geometry of the cubics.

Indeed, given a point $p \in \mathbb{CP}^3$ away from b_5 and b_6 there is a unique line ℓ_p passing through b_5, b_6 and p. This line intersects each $\mathscr{C}_i$ in 3 points, so for generic p there is a unique projective transformation of ℓ_p fixing the points where ℓ_p intersects b_5 and b_6, and mapping the remaining point of $\ell_p \cap \mathscr{C}_1$ to that of $\ell_p \cap \mathscr{C}_2$. If we define Φ to map p to the image of p under this projective transformation, then we see that, so long as it is defined, Φ maps $\mathscr{C}_1$ to $\mathscr{C}_2$. If we can show that Φ extends to a biholomorphism, then we will have shown that Φ is a projective transformation. This is not too difficult to prove directly, but we will postpone the proof to the next section when it falls out from general theory.

We have just shown that any two non-singular cubics that contain all 7 lines will be projectively equivalent by a projective transformation that fixes all 7 lines.

We have already seen that there is only a one-parameter family of projective transformations that fix all the lines, so there is at most a parameter family of non-singular cubics containing all 7 lines. Since non-singularity is an open condition on the space of cubic surfaces, there is at most a one-parameter family of cubics containing all 7 lines if there are any non-singular cubics containing all 7 lines.

Putting all of this information together, we end up with a classification of non-singular cubic surfaces. To make things explicit, write $(l_1, l_2; l_3, l_4)_k$ for the cross ratio of the intersection points of four lines l_i meeting on a fifth line k. We can then define four invariants associated with the configuration of lines as follows:

(1)
$$\begin{aligned}
\alpha &= (c_{56}, a_1; a_2, a_3)_{b_5} \\
\alpha' &= (c_{56}, a_1; a_2, a_3)_{b_6} \\
\beta &= (c_{56}, a_1; a_2, a_4)_{b_5} \\
\beta' &= (c_{56}, a_1; a_2, a_4)_{b_6}
\end{aligned}$$

With this notation, we state

THEOREM 2. *If b_5 and b_6 are skew lines in $\mathbb{CP}^3$ and a_1, a_2, a_3, a_4, c_{56} are five other skew lines each passing through b_5 and b_6 then consider the pencil of cubics spanned by the following two polynomials in z_1, z_2, z_3, z_4:*

$$
\begin{aligned}
\mathscr{C}_1 \;=\; & \left[(-\alpha\beta\beta' + \beta^2\beta' + \beta\alpha'\beta' - \beta^2\alpha'\beta' - \beta(\beta')^2 + \alpha\beta(\beta')^2)z_3\right]z_2^2 \\
& + \left[\left(\alpha\beta\alpha' - \beta^2\alpha' - \alpha\alpha'\beta' + \beta^2\alpha'\beta' + \alpha(\beta')^2 - \alpha\beta(\beta')^2\right)z_1 \right. \\
& \qquad \left. + \left(-\beta\alpha' + \beta^2\alpha' + \alpha\beta' - \beta^2\beta' - \alpha(\beta')^2 + \beta(\beta')^2\right)z_3\right]z_2 z_4 \\
& + \left[(\beta\alpha' - \alpha\beta\alpha' - \alpha\beta' + \alpha\beta\beta' + \alpha\alpha'\beta' - \beta\alpha'\beta')z_1\right]z_4^2
\end{aligned}
$$

(2)

$$
\begin{aligned}
\mathscr{C}_2 \;=\; & \left[(-\beta\alpha' + \alpha\beta\alpha' + \alpha\beta' - \alpha\beta\beta' - \alpha\alpha'\beta' + \beta\alpha'\beta')z_2\right]z_3^2 \\
& + \left[\left(\beta(\alpha')^2 - \alpha\beta(\alpha')^2 - \alpha^2\beta' + \alpha\beta\beta' + \alpha^2\alpha'\beta' - \beta\alpha'\beta'\right)z_2 \right. \\
& \qquad \left. + \left(-\alpha^2\alpha' + \beta\alpha' + \alpha(\alpha')^2 - \beta(\alpha')^2 - \alpha\beta' + \alpha^2\beta'\right)z_4\right]z_1 z_3 \\
& + \left[\left(\alpha^2\alpha' - \alpha\beta\alpha' - \alpha(\alpha')^2 + \alpha\beta(\alpha')^2 + \alpha\alpha'\beta' - \alpha^2\alpha'\beta'\right)z_4\right]z_1^2
\end{aligned}
$$

(3)

All the cubics in this pencil contain all 7 lines. If there is a non-singular cubic containing all 7 lines, then all cubics containing the 7 lines lie in the pencil. All the non-singular surfaces in the pencil are projectively isomorphic.

All non-singular cubics arise in this way.

Proof. We have proved all of this already, except the explicit formulae.

One approach to proving this is brute force. Write down the 18×20 matrix corresponding to the 18 equations in 20 unknowns. One can then compute its kernel in order to find the two equations. This is not as tedious as one might expect; it can be done by hand, and is the work of moments for a computer algebra system. We have included the formulae for completeness, but will not use them directly, so we will omit the details.

It is interesting, however, to understand the general form of these equations, and this is something we will take advantage of.

We have seen that the projective transformations

$$
\phi(u,v) : [z_1, z_2, z_3, z_4] \longrightarrow [uz_1, vz_2, uz_3, vz_4]
$$

(4)

will preserve the 7 lines. We therefore look for cubic surfaces which are linear in z_1, z_3 and quadratic in z_2, z_4. In other words cubics of the form:

$$
(az_1 + bz_3)z_2^2 + (cz_1 + dz_3)z_2 z_4 + (ez_1 + fz_3)z_4^2 = 0
$$

(5)

The justification for considering such surfaces is that they will always contain b_5 and b_6 and will be invariant under $\phi(u,v)$.

Suppose that $(0, w_2, 0, w_4)$ and $(w_1, 0, w_3, 0)$ are points on b_5 and b_6. The general point on the line between these points is:

$$
[\lambda w_1, \mu w_2, \lambda w_3, \mu w_4]
$$

with λ, μ in $\mathbb{C}$. When we put the coordinates of this point into equation (5), we get a common factor of $\lambda\mu^2$. Hence the line lies on this cubic surface if and only if a single point on the line away from b_5 and b_6 does.

If we choose a generic plane transverse to b_5 and b_6 it will intersect the lines a_1, a_2, a_3, a_4, c_{56} in 5 points. Plugging the coordinates of these intersection points into equation (5) we get 5 linear equations in the 6 unknowns $a, b, \ldots, f$. So there is a cubic surface of the given form that contains all 7 lines. $\square$

We have chosen this presentation of the classification of cubic surfaces because it yields the following classification for twistor lines on cubic surfaces.

THEOREM 3. *For a generic set of 5 points lying on a 2-sphere in S^4, there exists a one-parameter family of projectively isomorphic but conformally non-isomorphic non-singular cubic surfaces with 5 twistor lines corresponding to the 5 points.*

All cubic surfaces with 5 twistor lines arise in this way. Given such a surface, one can label the twistor lines a_1, a_2, a_3, a_4, c_{56} and the two transversals b_5, b_6.

One can associate a real invariant ξ to a labelled cubic surface with five twistor lines in such a way that labelled cubic surfaces containing 5 twistor lines are conformally isomorphic if and only if the points on the sphere are conformally isomorphic and the values for ξ are equal.

Proof. A 2-sphere in S^4 lifts to two projective lines b and jb in $\mathbb{CP}^3$. The choice of 5 points on the sphere determines 5 collinear lines in $\mathbb{CP}^3$. It follows from above that there exists a one-parameter family of cubics containing all 7 lines. We shall show later that if the 5 points are chosen generically then the general cubic in this family is non-singular. This being the case, we can label the twistor fibres a_1, a_2, a_4, a_4, c_{56}, and the transversals $b = b_5$, $jb = b_6$.

The bijection $b_6 \to b_5$ is determined by its action on 3 points, so in the coordinates used in our study we have $j[z_1, 0, z_3, 0] = [0, \overline{z_1}, 0, \overline{z_3}]$. The action of j on all of $\mathbb{CP}^3$ follows by anti-linearity.

Since j maps the intersection of a_i and b_5 to the intersection of a_i with b_6, it follows that $\alpha = \overline{\alpha'}$ and $\beta = \overline{\beta'}$ in (1).

In general, a projective transformation (4) of $\mathbb{CP}^3$ which fixes all 7 labelled lines will not correspond to a conformal transformation of S^4. It will do so if and only if it preserves j. This will be the case if and only if $|u| = |v|$.

The general cubic surface containing all 7 lines is given by a linear combination of $\mathscr{C}_1$ and $\mathscr{C}_2$ as defined in equation (2) and (3). Given such a cubic, define $M \in \mathbb{R}$ to be the coefficient of $z_1 z_4^2$ and $N \in \mathbb{R}$ to be the coefficient of $z_3 z_2^2$. Define $\xi = |M/N|$.

We need to check that neither M nor N is zero. We know that M is a non-zero multiple of the corresponding coefficient in the polynomial (3). Suppose that this coefficient were equal to zero. This would mean that any cubic surface containing the 7 lines would depend only linearly upon z_1 since this is the only non-linear term in z_1 in either (3) or (2). This would mean that the cubic was ruled and hence singular. Similarly, we see that N is non-zero.

By construction, ξ is invariant under transformations $\phi(u,v)$ with $|u| = |v|$. Thus it is well defined solely in terms of the cubic surface and the labelling. Since ξ changes in proportion to u/v, ξ will always distinguish conformally inequivalent surfaces. □

COROLLARY 2. *Given the* 27 *lines on a non-singular cubic surface there is an algorithm to determine whether it has a twistor structure such that* 5 *of the* 27 *are twistor fibres.*

Proof. Run through all pairs of skew lines, compute the cross ratios of the intersection points and check whether $\alpha = \overline{\alpha'}$ and $\beta = \overline{\beta'}$ □

We can summarize by saying that a cubic surface depends up to projective transformation upon a choice of 4 complex parameters $\alpha, \beta, \alpha', \beta'$ determined by the cross ratios of the line intersection points. These 4 complex parameters do depend upon a labelling of the lines in the cubic — so we have an action of the graph isomorphism group of the Schläfli graph on the space of such parameters. This is the Weyl group $W(E_6)$ of the exceptional Lie group E_6. Thus the moduli space of cubic surfaces up to projective isomorphism is given by an open subset of $\mathbb{C}^4$ quotiented by $W(E_6)$. For more details, we refer the reader to [14] and [1].

In the case of conformal isomorphism classes of cubic surfaces with 5 twistor lines we have a choice of 2 complex parameters α, β and one real parameter ξ. In addition we have a choice of labelling for the 5 twistor lines and a labelling of the two lines collinear to all the twistor lines. So the moduli space of cubic surfaces with 5 twistor lines up to conformal isomorphism is given by an open subset of $\mathbb{C}^2 \times \mathbb{R}$ quotiented by $S_5 \times \mathbb{Z}_2$.

In both cases we can write down an explicit equation for a cubic surface with given values for the parameters by choosing appropriate multiples of polynomials (2) and (3).

4. Identifying non-singular cubic surfaces with 5 twistor lines

Modern treatments of the classification of cubic surfaces usually state that non-singular cubic surfaces are given by blowing up 6 points in $\mathbb{CP}^2$ in general position, the latter meaning that no 3 points are collinear and that the 6 points do not all lie on a conic.

This perspective highlights the intrinsic complex geometry of the cubic surfaces — it ostensibly describes cubic surfaces up to biholomorphism rather than up to projective transformation. However, these two classifications are equivalent. This is guaranteed by the fact that any automorphism of a smooth hypersurface of $\mathbb{CP}^n$ ($n \geqslant 3$) of degree $d \neq n+1$ is induced by a projective transformation. This in turn follows from the general correspondence between maps to projective space and sections of complex line bundles, see [11].

Because we are interested in the classification up to conformal transformation, we have emphasized the embedding of the cubic surface om $\mathbb{CP}^3$. Let us review the connection between this and the intrinsic geometry.

Given a cubic surface $\mathscr{C}$, let ψ_1 be the biholomorphism $\mathbb{CP}^1 \to b_5$ mapping 0, 1, ∞ to the intersection points of b_5 with a_1, a_2, c_{56} respectively. Similarly define $\psi_2 : \mathbb{CP}^1 \to b_6$ by sending 0, 1, ∞ to points on a_1, a_2, c_{56}. One can now define a rational map $\psi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathscr{C}$ by defining $\psi(z_1, z_2)$ to be the intersection point of the line containing $\psi_1(z_1)$, $\psi_2(z_2)$ with the surface $\mathscr{C}$.

The map ψ will be well defined for general points (z_1, z_2) in $\mathbb{CP}^1 \times \mathbb{CP}^1$. If we incorporate the multiplicity of the intersection into our definition, it is clear that we have a map ψ well defined at all points except: $(0,0)$, $(1,1)$, (α, α'), (β, β'), (∞, ∞) where the α's and β's are the cross-ratio invariants defined earlier. These points correspond to the lines a_1, a_2, a_3, a_4, c_{56} respectively, and are indicated in Figure 6. It turns out that ψ extends to a biholomorphism from $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up at these five points to the cubic surface $\mathscr{C}$.
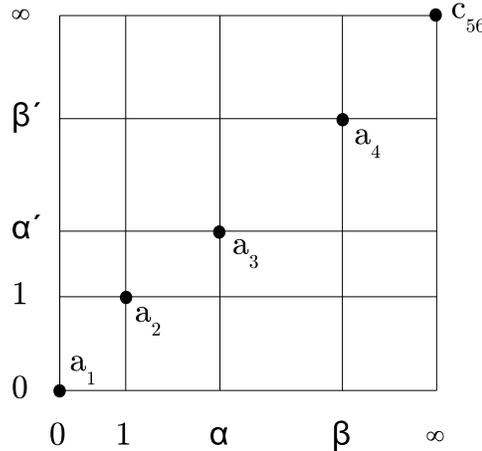
Figure 6: The five points to blow up on $\mathbb{CP}^1 \times \mathbb{CP}^1$

Now, $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be thought of as $\mathbb{CP}^2$ with two points at infinity blown up and then the line at infinity blown down. This allows us to think of the blow up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at 5 points as being the blow up of $\mathbb{CP}^2$ at 6 points corresponding to a_1, a_2, a_3, a_4, a_5, a_6. Now, c_{56} corresponds to the line at infinity.

To be very concrete, blowing up the two points $[1, 0, 0]$ and $[0, 1, 0]$ at infinity and then blowing down the proper transform of the line at infinity is given by the rational map $(z_1, z_2) \to (z_1, z_2)$. The left hand side should be viewed as giving inhomogeneous coordinates for $\mathbb{CP}^2$, the right hand side as giving inhomogeneous coordinates for each factor of $\mathbb{CP}^1 \times \mathbb{CP}^1$. We define a rational map $\tilde{\psi}$ from $\mathbb{CP}^2$ to our cubic by $\tilde{\psi}(z_1, z_2) = \psi(z_1, z_2)$.

Since any four points in general position in $\mathbb{CP}^3$ are projectively equivalent, we see that a choice of 6 points to blow up corresponds to the 4 cross ratios α, α', β, β'.

Notice that the lines on the cubic surface are easily understood in terms of the blow-up picture. The *a* lines correspond to the points that have been blown up. The line c_{ij} corresponds to the straight line in $\mathbb{CP}^2$ passing through the points a_i and a_j. The line b_i corresponds to the conic passing through all the blown-up points except a_i. One can immediately read off the intersection properties of all the lines when they are thought of in terms of this picture. This gives a particularly nice way of remembering the structure of the Schläfli graph.

This intrinsic view of cubic surfaces allows us to tie up a loose end we left dangling in the previous section. Recall that given two cubics $\mathscr{C}_1$ and $\mathscr{C}_2$ containing the lines $a_1, a_2, a_3, a_4, c_{56}$ we constructed a rational map $\Phi : \mathbb{CP}^3 \to \mathbb{CP}^3$ sending $\mathscr{C}_1$ to $\mathscr{C}_2$. We claimed that this map could was in fact a biholomorphism. Identifying each of $\mathscr{C}_1$ and $\mathscr{C}_2$ with $\mathbb{CP}^2$ blown up at six points, we see that Φ restricted to $\mathscr{C}_1$ is essentially the identity — hence it is certainly a biholomorphism.

The most important feature of this intrinsic view of cubic surfaces from the perspective of the twistor geometry is the criteria it gives for determining whether a cubic surface is non-singular. The blow-up of $\mathbb{CP}^2$ at 6 points is obviously smooth, so if we can find a cubic surface corresponding to the 6 points, that cubic surface will be non-singular. To construct a cubic surface given 6 points in general position, one considers the vector space C of cubic *curves* in $\mathbb{CP}^2$ that pass through all 6 points. This space will be 4-dimensional so long as no 6 points lie on a conic and no 3 points lie on a line. One then defines a rational map sending a point $z \in \mathbb{CP}^2$ to the projectized dual space $\mathbb{P}(C^*)$ by mapping a cubic polynomial to its value at z. This rational map is a biholomorphism of the blow up at 6 points to a cubic surface in $\mathbb{P}(C^*)$. This result was first discoved by Clebsch in [8]. Details of the proof can be found in [11].

We saw in the previous section that given 5 points lying on a 2-sphere in S^4 we can find a family of cubic surfaces with 5 twistor lines corresponding to these five points. It follows from the discussion above that if the 5 points on the 2-sphere are chosen in general position then the cubic surfaces will be non-singular. We would like to identify more clearly what "in general position" actually means in this case.

THEOREM 4. *Given* 5 *points lying on a 2-sphere in* S^4*, there is a non-singular cubic surface with* 5 *twistor lines corresponding to these points if and only if no* 4 *of the points lie on a circle.*

Proof. There are two lines in $\mathbb{CP}^3$ lying above S^4 under the twistor correspondence. Label one of them b_5 and the other b_6.

To each of the five points on b_5, there is a unique twistor line over that point. We label these lines arbitrarily as a_1, a_2, a_3, a_4 and c_{56}.

Three distinct points on a 2-sphere are conformally equivalent, and so always in general position. We choose an inhomogeneous coordinate z_1 for b_5 and z_2 for b_6 by requiring that the intersections of a_1, a_2, a_3, a_4 and c_{56} with b_5 are given by 0, 1, α, β and ∞. Similarly we choose an inhomogeneous coordinate z_2 for b_6 such that the intersection points are 0, 1, $\overline{\alpha}, \overline{\beta}$ and ∞.

We now have an unambiguously defined rational map ϕ from $\mathbb{CP}^2$ to $b_5 \times b_6$

given in inhomogenous coordinates by $\phi(z_1, z_2) = (z_1, z_2)$. Corresponding to each of the lines a_1, a_2, a_3, a_4 and c_{56} we can define points a'_1, a'_2, a'_3, a'_4 and c'_{56} in $b_5 \times b_6$ given by sending a line to its intersection point with each b_i. We can then define six points in $\mathbb{CP}^2$ as follows:

$$
\begin{aligned}
A_1 &= \phi^{-1}(a'_1) = [0,0,1] \\
A_2 &= \phi^{-1}(a'_2) = [1,1,1] \\
A_3 &= \phi^{-1}(a'_3) = [\alpha, \overline{\alpha}, 1] \\
A_4 &= \phi^{-1}(a'_4) = [\beta, \overline{\beta}, 1] \\
A_5 &= [1,0,0] \\
A_6 &= [0,1,0]
\end{aligned}
$$

Note that c_{56} corresponds to the line at infinity in $\mathbb{CP}^2$. A_5 and A_6 correspond to the two lines at infinity in $b_5 \times b_6$. The setup is precisely as summarized in Figure 6.

The point we are making is that these points in $\mathbb{CP}^2$ are determined entirely by the 5 points on the sphere and the choice of labelling: we do not need there to be a non-singular cubic through the five twistor lines in order to construct the A_i.

The blow up of $\mathbb{CP}^2$ at the A_i corresponds to a smooth cubic if and only if these A_i are in general position (meaning no three collinear and no conic through all 6). This cubic must then be biholomorphic to one of the cubic curves in the pencil generated by (2) and (3). We deduce that there is a smooth cubic with 5 twistor lines corresponding to the five points on S^2 if and only if these six points A_i in $\mathbb{CP}^2$ are in general position.

A_1, A_2 and A_3 are collinear if and only if $\alpha = \overline{\alpha}$. This is equivalent to saying that 0, 1, α and ∞ all lie on the real line. In invariant terms this is equivalent to requiring that a_1, a_2, a_3 and c_{56} all lie on a circle in S^2.

We deduce that there is a smooth cubic corresponding to the five points on S^2 only if no four of the points lie on a circle.

It is a simple calculation to check that the condition that no four points lie on a circle implies that no three of the A_i lie on a line. We also need to confirm that the same condition implies that there is no conic through all 6 of the A_i.

Suppose for a contradiction that there is such a conic and so the A_i form an "inscribed hexagon". Pascal's theorem implies the intersection points

$$
A_1 A_2 \cap A_4 A_5, \quad A_2 A_3 \cap A_5 A_6, \quad A_3 A_4 \cap A_6 A_1
$$

are collinear. These points can be computed using the vector cross product; the first is $(A_1 \times A_2) \times (A_4 \times A_5)$ with a slight abuse of notation. The collinearity condition is then

$$
\det \begin{pmatrix} \overline{\beta} & \overline{\beta} & 1 \\ \alpha - 1 & \overline{\alpha} - 1 & 0 \\ 0 & \beta\overline{\alpha} - \alpha\overline{\beta} & \beta - \alpha \end{pmatrix} = 0,
$$

which gives

$$
|\alpha|^2 (\beta - \overline{\beta}) - |\beta|^2 (\alpha - \overline{\alpha}) + \alpha\overline{\beta} - \overline{\alpha}\beta = 0.
$$

But this is easily seen to be exactly the condition that $0, 1, \alpha, \beta$ lie on a circle in $\mathbb{C}$. $\quad\square$

On first reading this proof one may wonder where the asymmetry between the a_i and c_{56} arises. It can be traced directly to the choice to associate c_{56} with the points $z_1 = \infty$ and $z_2 = \infty$.

For a coordinate-free explanation of why the cubic must be singular if four of the 5 points in S^2 lie in a circle Γ, recall that by by [18, Theorem 3.10] that there is a quadric surface $\mathcal{Q}$ in $\mathbb{CP}^3$ containing $\pi^{-1}(\Gamma)$ (where π is the twistor projection). Suppose that there is also non-singular cubic $\mathcal{C}$ for which the twistor fibres a_1, a_2, a_3, a_4 from part of a double six (a_i, b_j). In this case, $\mathcal{C}$ must be the only cubic containing the double six. But each b_j intersects at least three of a_1, a_2, a_3, a_4 and therefore lies in $\mathcal{Q}$. The latter must now contain a_5, a_6 as well. But then the union of $\mathcal{Q}$ and any plane is a cubic containing the double six, which is a contradiction.

As an application of our theorem, we observe that the well known Clebsch diagonal surface does not have 5 twistor lines irrespective of the twistor structure j one places on $\mathbb{CP}^3$. The Clebsch diagonal surface is the complex surface in $\mathbb{CP}^4$ defined by the two equations

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 = 0$$
$$z_1 + z_2 + z_3 + z_4 + z_5 = 0.$$

It is biholomorphic to the surface in $\mathbb{CP}^3$ given by the single equation:

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 = (z_1 + z_2 + z_3 + z_4)^3,$$

and is the only cubic surface with symmetry group S_5. It has the nice property that all 27 lines on the cubic surface are real lines. This immediately means that it admits no twistor structure j such that it has five twistor lines. Simply note that the cross ratios of all the intersection points on the lines must be real. Therefore any four points on one of its lines must lie on a circle when that line is viewed as the Riemann sphere.

5. The Fermat cubic

Having found a large family of cubic surfaces with 5 twistor lines we would like to ask if there are any particularly nice examples. In particular what is the most symmetrical cubic surface with 5 twistor lines?

A conformal transformation of S^4 that induces a symmetry of a cubic surface with 5 twistor lines must leave the 2-sphere image of b_5 and b_6 fixed. If the conformal transformation leaves the image of the 5 twistor lines fixed, then the associated projective transformation must swap b_5 and b_6. Otherwise the conformal transformation must permute the 5 points on the 2-sphere.

Therefore let us first choose the most symmetrical arrangement of 5 points on a 2-sphere no four of which lie on a circle. If we have a rotation of the sphere that permutes n of the points then those points must all lie on a circle. So $n \leqslant 3$. So any rotation fixes at least two points. Either those two points lie on the axis of rotation, or the rotation is a rotation through 180 degrees and the fixed points all lie on a circle. We deduce that the largest possible symmetry group for the five points is $\mathbb{Z}_3 \times \mathbb{Z}_2$ and,

up to conformal transformation of the 2-sphere, we can assume that our five points are 0, ∞ and the three cube roots of unity. The $\mathbb{Z}_3$ action rotates the three cube roots into each other. The $\mathbb{Z}_2$ action swaps 0 and ∞.

Setting α and β to be complex cube roots of unity and α' and β' to be their conjugates, polynomials (2) and (3) simplify to:

$$-3i\sqrt{3}\left(z_2 z_3^2 - z_1^2 z_4\right),$$
$$3i\sqrt{3}\left(-z_2^2 z_3 + z_1 z_4^2\right).$$

We now wish to choose a linear combination of these polynomials that will remain fixed under the transformation that swaps the lines b_5 and b_6. This corresponds to the projective transformation $z_1 \mapsto z_2$, $z_2 \mapsto z_1$, $z_3 \mapsto z_4$, $z_4 \mapsto z_3$.

Hence there is, up to conformal transformation, a unique non-singular cubic surface with 5 twistor lines and symmetry group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It is defined by

(6) $$z_1 z_4^2 + z_2 z_3^2 - z_3 z_2^2 - z_4 z_1^2 = 0.$$

One can make a conformal transformation (corresponding to using the cube roots of -1 rather than those of 1) to replace the two minus signs with plus signs. In any case, it is projectively, but not conformally, equivalent to a familar example: the Fermat cubic, which is defined by the equation

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0.$$

One can prove that these surfaces are projectively equivalent by calculating the cross ratio invariants we defined earlier. This approach allows one to write down an explicit linear transformation sending the Fermat cubic to the surface (6).

A more pleasing approach is to use the symmetries of the Fermat cubic to deduce that there must be some twistor structure that gives it five twistor lines. To see how this is done, first choose a complex cube root of unity ω and label 7 of the lines on the Fermat cubic as follows:

Label	Line
b_5	$z_1 + \omega z_2 = z_3 + \omega^2 z_4 = 0$
b_6	$z_1 + \omega^2 z_2 = z_3 + \omega z_4 = 0$
a_1	$z_1 + \omega z_2 = z_3 + \omega z_4 = 0$
a_2	$z_1 + z_4 = z_2 + z_3 = 0$
a_3	$z_1 + \omega z_4 = z_2 + \omega z_3 = 0$
a_4	$z_1 + \omega^2 z_4 = z_2 + \omega^2 z_3 = 0$
c_{56}	$z_1 + \omega^2 z_2 = z_3 + \omega^2 z_4 = 0$

Consider the symmetry of the cubic given by

$$(z_1, z_2, z_3, z_4) \longmapsto (z_1, z_2, \omega z_3, \omega z_4).$$

This generates a $\mathbb{Z}_3$ action that fixing the lines b_5, b_6, a_1 and c_{56} and permuting a_2, a_3, a_4. Thus we have a $\mathbb{Z}_3$ symmetry of b_5 fixing the intersection points with a_1 and c_{56} and permuting the intersection points with a_2, a_3 and a_4. Therefore these 5 points on b_5 are conformally equivalent to the points 0, ∞, 1, ω and $\overline{\omega}$ on the Riemann sphere. The same applies to the 5 intersection points with b_6. This implies that the invariants (1) satisfy $\alpha = \overline{\beta}$ and $\alpha' = \overline{\beta'}$

Now consider the symmetry:

$$(z_1, z_2, z_3, z_4) \longmapsto (z_3, z_4, z_1, z_2).$$

This swaps b_5 and b_6, swaps a_3 and a_4 and fixes a_1, a_2 and c_{56}. We deduce that:

$$\alpha = (c_{56}, a_1; a_2, a_4)_{b_5} = (c_{56}, a_1; a_2, a_3)_{b_6} = \beta' = \overline{\alpha'}.$$

Thus the cross ratios of the intersection points on b_5 and b_6 are the same as the cross ratios for the intersection points on the cubic surface (6). We conclude that the cubic given by (6) is projectively isomorphic to the Fermat cubic.

THEOREM 5. *The set of twistor structures on $\mathbb{CP}^3$ for which the Fermat cubic has 5 twistor lines is a real 1-manifold with 54 components.*

Proof. The surface (6) has 12 conformal symmetries. The Fermat cubic has symmetry group $S_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ given by permutations of the coordinates and by multiplying various coordinates by cube roots of unity. Thus there are $4! \cdot 27/12 = 54$ twistor structures on $\mathbb{CP}^3$ such that the Fermat cubic is isomorphic to surface (6) with the standard twistor structure. We can then vary the invariant ξ to get a one-parameter family of conformally inequivalent twistor structures.

We need to check that there are no other twistor structures that give the Fermat cubic five twistor lines. We we gave an algorithm to do this earlier: run through all pairs of skew lines and compute cross ratios. We can speed this up significantly using the symmetries of the Fermat cubic. The general line on the Fermat cubic is

$$z_i + \eta_1 z_j = z_k + \eta_2 z_l = 0$$

where $\{i, j, k, l\}$ is a permutation of $\{1, 2, 3, 4\}$ and η_1 and η_2 are cube roots of unity. So given two skew lines on the Fermat cubic, using the cubic's symmetries we can assume that the first line is:

$$z_1 + z_2 = z_3 + z_4 = 0$$

and the second line is one of:

$$z_1 + \eta_1 z_2 = z_3 + \eta_2 z_4 = 0,$$
$$z_1 + \eta_1 z_3 = z_2 + \eta_2 z_4 = 0.$$

In the first case there is a $\mathbb{Z}_3$ symmetry preserving both lines — so if we have 5 twistor lines it will be one of the cases already considered. In the second case we can further assume that $\eta_1 = 1$ and, since the lines are skew, $\eta_2 \neq 1$. Therefore we just need to

show that the cross ratios of the intersection points of the 5 lines on the Fermat cubic meeting $z_1 + z_2 = z_3 + z_4 = 0$ and $z_1 + z_3 = z_2 + \omega z_4 = 0$ do not form complex conjugate pairs. This is easily done. □

There is a lot more one could ask about the twistor geometry of the Fermat cubic. For example: what is the topology of its discriminant locus? How does this vary as one varies the choice of twistor structure? We will consider these questions in another paper.

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