Conformal Defects in Two-Dimensional Conformal Field Theories

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Abstract

We study conformal defects in two-dimensional conformal field theories (CFTs). These are one-dimensional objects across which the difference between the holomorphic and antiholomorphic parts of the stress-energy tensor is continuous. Such defects may exist within a CFT as well as between two different CFTs. There are two subclasses of conformal defects that are well-known: topological defects, which preserve the holomorphic and antiholomorphic parts of the stress-energy tensor separately, and factorising defects, which can be considered as products of conformal boundary conditions separating the theory along the defect. In this thesis, we call conformal defects, which do not fall into either of the aforementioned subclasses, non-trivial conformal defects.

The primary focus of this thesis is studying the non-trivial conformal defect present in a unitary Virasoro minimal model which was first predicted by Kormos, Runkel, and Watts [98]. As a first step, we calculate the reflection and transmission coefficients, which were first defined in [91], of these defects using the leading-order perturbative calculation. We then consider conformal defects in the tri-critical Ising model as a concrete example. We revisit the construction of super-conformal defects proposed by Gang and Yamaguchi [94] and give a more systematic construction of such defects using super W-algebras. In addition, we propose a topological interface separating the super-conformal and bosonic theories, from which conformal defects in the latter theory can be obtained from the former one. Using the topological interfaces and superconformal defects, we obtain non-topological and non-factorising defects in the bosonic tri-critical Ising model.

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1. Unfortunately, as far as I know, EPSRC and STFC do not fund non-EU PhD students, and JSPS only funds PhD students studying in Japan or those who graduated from Japanese universities wishing to study abroad.

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Chapter 1

Introduction

Conformal field theory (CFT) is a quantum field theory (QFT) with conformal symmetry; roughly speaking, it is a theory in which physical observables are invariant under coordinate transformations that preserve local angles. In particular, conformal transformations contain scale transformations, and therefore, CFTs have their applications in scale invariant theories, for example, systems undergoing second order phase transitions and world-sheet description of string theory.

In two dimensions, CFTs have many features that are not available in higher dimensions. One of the most important differences is that the symmetry algebra becomes infinite dimensional, and many quantities of a theory can be derived by considering conformal symmetry alone. Not only for CFTs, but usual power counting shows that any ϕ^n-interaction with $n \geq 3$ is renormalisable in two-dimensional QFTs, which allows us to have interesting theories with higher spin conserved currents. From the mathematical point of view, two-dimensional CFT is one of the most well understood QFTs. In certain 2d CFTs, conformal symmetry completely determines local operator algebras, and, in fact, Lagrangian descriptions are less important in 2d CFTs.

History

Modern investigation of two-dimensional conformal fields theory was initiated by Belavin, Polyakov, and Zamolodchikov in their seminal 1984 paper[4]. One of the important discoveries of their paper was the so-called Virasoro minimal models where operator contents and their correlation functions can be calculated from representation theory of the Virasoro algebra—the symmetry algebra of local conformal transformations in two dimensions. The Virasoro minimal models exist for a certain range of the parameter c, which is called the central charge as we shall explain in the next chapter, and they cover many interesting 2d CFTs, in fact, they correspond to the scaling limits of certain two-dimensional lattice statistical models including the Ising model.

Extensions of conformal symmetry lead to 2d CFTs with extended symmetries. It was pioneered by Zamolodchikov in his 1985 paper[11] in which theories with spin-3 symmetry generators were considered. The extended conformal algebra with a spin-3 field is known as the W_3 algebra, and generalisations are called W-algebras. In general, W-algebras are not necessarily Lie algebras, and this leads to many interesting phenomena. Some W-algebras exist for generic values of the central charge c and such algebras are called deformable, while so-called non-deformable W-algebras only exist at isolated points of c. Similar to the Virasoro algebra case, some W-algebras admit minimal series in which CFTs can be constructed from finite number of W-algebra representations. During the 90s, one of the main motivations to study W-algebras was to classify 2d CFTs using W-algebras.

Recently, there is renewed interest in W-algebras from the AdS/CFT correspondence[103].

From the physical perspective, it is interesting to study CFTs on surfaces with boundaries that describe surface critical behaviours. Boundary conformal field theory (BCFT) was developed by Cardy in his 1984 paper[5] and subsequent publications. In string theory, open strings that satisfy Dirichlet boundary conditions are attached to extended objects known as the D-branes. Polchinski's 1995 paper[53] showed that D-branes can be studied from BCFT, which attracted interest in BCFT from the string community.

Defects and Interfaces

In two-dimensional conformal field theory, a conformal defect is a line of inhomogeneity in a theory, across which the values of correlators may change or become singular. As an example, conformal defects can be realised as continuum limits of lattice models with defect lines. The concept of conformal defects generalises to interfaces between two different CFTs. A class of conformal defects called topological defects have gained much interest recently not only in condensed matter physics but also in string theory. Topological defects implement internal symmetries of a CFT as well as order-disorder type dualities [83] [89]. For another class of conformal defects called factorising defects, two CFTs separated by a defect line decouple completely, and classification of such defects becomes that of conformal boundaries in each of the CFTs. As we can view the bulk fields of a CFT as the defect fields on the identity defect, which is also called the invisible defect, conformal defects can be regarded as a natural generalisation of bulk CFTs, conformal boundaries, and topological defects.

The so-called AGT correspondence of [99] relates two-dimensional CFTs and four-dimensional supersymmetric gauge theories. While the original conjecture involves Liouville CFTs in two-dimensions, there is a version of this relation which involves A-series Virasoro minimal models [112]. From the AGT correspondence it is possible to relate certain quantities in two-dimensional CFTs and those of four-dimensional gauge theories. In [101], it is shown that loop operators and domain wall operators in four-dimensional $N = 2$ supersymmetric gauge theories correspond to topological defect operators in Liouville and Toda CFTs in two-dimensions. Therefore, study of conformal defects in two-dimension is also interesting from the perspective of four-dimensional supersymmetric gauge theories.

For topological defects in rational CFTs, there is a systematic way to study them in many different theories. From the topological field theory (TFT) approach of [76] and the sequels, it is possible to compute various important quantities involving topological defects, which include correlation functions. On the other hand, there has not been found a systematic way to approach general conformal defects.

One way to study conformal defects within a CFT is to 'fold' the theory along the defect line, and consider the corresponding conformal boundary condition in the doubled theory. As the central charge doubles after folding, studying the corresponding boundary theory is difficult in general. Conformal defects in the Ising model are classified by identifying the doubled theory as an orbifolded free boson theory in [59]. For the cases where the sums of central charges of two Virasoro minimal models are again minimal values—namely, Lee–Yang $\times$ Lee–Yang, Lee–Yang $\times$ Ising, and Lee–Yang $\times$ $M(2,7)$—conformal defects are classified in [91] by identifying the partition functions of doubled theories. While the paper uses methods from the TFT approach to identify the partition functions, one of the important observations one can makes is that the folded theories have $\mathcal{W}(2,2)$ symmetries—another spin-2 current other than the stress-energy tensor. In [94], a set of conformal defects in the tri-critical Ising model is proposed by using the fact that the doubled theory has the central charge which is minimal with respect to the $N=1$ super-Virasoro algebra. While this idea is very interesting, the analysis of superconformal boundary conditions in the folded model and mapping to conformal defects in the tri-critical Ising model is not so transparent, and we believe some important details are missing. One of the motivations of this thesis is to revisit this idea and give a more systematic treatment.

Another way of obtaining conformal defects is by defect perturbations. Just like bulk or boundary perturbations, we can consider perturbations of a topological defect by taking a defect field and integrating it over the defect. As in the Ising model [59] and other A-series Virasoro minimal models [98], defect flows may generate conformal defects that are not topological nor factorising. While [98] predicts the presence of at least one such conformal defect in a unitary Virasoro minimal model, much of their nature—such as exact defect g-values, or reflection and transmission coefficients—remains unknown. In this thesis, we calculate their reflection and transmission coefficients using the leading-order perturbation calculation.

If two CFTs are related in some way, it may be possible to construct a conformal interface between them. It has been done for some CFTs related by renormalisation group (RG) flows. RG interfaces are constructed by Gaiotto for A-series Virasoro minimal models in [106], for $N=1$ super-Virasoro minimal models in [114], and for $N=2$ super-Virasoro minimal models in [93]. The defect entropy as well as reflection and transmission coefficients of Gaiotto's RG defect are calculated in [111] and [116].

Another relation between CFTs, which may be exploited to construct conformal interfaces, is extended conformal symmetries. In Chapter 5, we construct topological interfaces between the $c=\frac{1}{2}$ free fermion theory and the Ising model, which is taken as a bosonic theory, and also between the bosonic and $N=1$ super-Virasoro tri-critical Ising model at $c=\frac{7}{10}$. For bosonic extended symmetries, topological interfaces can be constructed from the TFT construction in [89]. Examples of these interfaces include those between A-series and D-series Virasoro minimal models with the same central charges. The case for $c=\frac{4}{5}$, that is the interface between the tetra-critical Ising model and the three-state Potts model, is discussed in Section 6 of [89].

Motivation and Summary

The main motivation of this thesis is to study the nature of non-topological and non-factorising defect in the diagonal Virasoro minimal models predicted in [98]. In diagonal Virasoro minimal models, conformal boundary conditions and topological defects are labelled by unique representations of the Virasoro algebra, and therefore we can use Kac labels (r, s) to denote them. The paper [98] considers perturbations of the topological defect $(1, 2)$ by a linear combination of chiral defect fields

$$\lambda\phi(x) + \bar{\lambda}\bar{\phi}(x) \ , \tag{1.1}$$

where ϕ and $\bar{\phi}$ are the chiral defect fields on the topological defect $(1, 2)$ with the conformal weights $(h_{1,3}, 0)$ and $(0, h_{1,3})$, respectively. For a minimal model $M(p, p+1)$ with $p > 3$, the endpoints of these flows are: the identity defect in the directions of $\lambda = 0$ and $\bar{\lambda} < 0$, or $\lambda < 0$ and $\bar{\lambda} = 0$; the topological defect $(2, 1)$ in the directions of $\lambda = 0$ and $\bar{\lambda} > 0$, or $\lambda > 0$ and $\bar{\lambda} = 0$; the linear combinations of factorising defects, denoted by F, in the direction of $\lambda = \bar{\lambda} < 0$; a new conformal defect, denoted by C, in the direction of $\lambda = \bar{\lambda} > 0$.

One of the initial approaches we took to study these conformal defects was to apply the lattice mean-field approach. In [104], the phase spaces of boundary flows for the Ising model and the tri-critical Ising model were analysed by applying the mean-field theory to the underlying classical square lattice models. In particular, it was shown that the number and directions of relevant boundary flows for any given boundary condition of these models can be obtained by this method.

Let us outline how this method works. One starts from a classical square lattice action describing a system with a defect line, for example, the one given in (B.7). By substituting some of the classical spin variables $\sigma_{i,j}$ with mean magnetisation per site, which is defined as

$$M_{i,j} = \langle \sigma_{i,j} \rangle = \sigma_{i,j} - \delta\sigma_{i,j}, \tag{1.2}$$

one obtains the mean-field action $\mathcal{E}_{\text{MF}}$, in which the local spins interact with neighbours only through the mean-field. In mean-field theory, contributions from $(\delta\sigma)^2$ are assumed to be negligible, and this simplifies the matter greatly. Then, the mean-field partition function

$$Z_{\text{MF}} = \sum_{\{\sigma\}} e^{-\beta\mathcal{E}_{\text{MF}}} \tag{1.3}$$

can be written as the product of $Z_{i,j}$, from which we can obtain the free energies per site as

$$f_{i,j} = -\frac{1}{\beta} \log Z_{i,j}. \tag{1.4}$$

Since the nearest neighbour couplings are different along the defect, we need to distinguish defect free energies from bulk free energies. The magnetisation per site is given by

$$M_{i,j} = -\frac{\partial}{\partial h} f_{i,j}, \tag{1.5}$$

where h is the external magnetic field, therefore, we obtain the bulk and defect mean-field consistency equations. The equations of motion can be obtained by inverting the consistency equations, and the bulk and defect critical parameters can be determined. By integrating the equations of motion, we may obtain the effective potentials. Analysing the defect potential at and around the critical parameter values, it should be possible to obtain some information on the phase space of defect flows.

We have attempted to apply this method to the Ising model with a defect line, however, it turned out that it is not straightforward to construct a lattice model, which captures the properties of the Ising defects predicted from CFT and compatible with the mean-field analysis. For example, we have not found an appropriate way of assigning coupling constants and external magnetic fields at defect sites. As seen in Section B.2, in the Ising model, marginal defect fields with scaling dimension $\Delta = 1$ are always present for all the conformal defects. This may be posing some problems for the mean-field analysis.

One of the objectives of this thesis is to calculate the reflection and transmission coefficients[91] of the new conformal defects. In the folded model, these coefficients can be obtained from the quantity

$$\omega_D = \frac{\langle W||D\rangle\!\rangle}{\langle 0||D\rangle\!\rangle} , \tag{1.6}$$

where $\|D\rangle\!\rangle$ is the boundary state corresponding to the defect, and $|W\rangle$ is the Virasoro primary state with the conformal weight $h = 2$ corresponding to the generator of $\mathcal{W}(2, 2)$. In addition, we emphasise that the topological and factorising defects correspond to the boundary states that preserve the $\mathcal{W}(2, 2)$ symmetry.

Next, we calculate the reflection and transmission coefficients of the predicted conformal defects in minimal models using the leading-order perturbation calculation. This provides a first insight into the nature of new conformal defects. We find the reflection coefficient of the conformal defect C to be

$$\mathcal{R} = \frac{9\pi^2 y^4}{8} + O(y^5) , \tag{1.7}$$

where $y := 1 - h_{1,3}$. The transmission coefficient is given by $\mathcal{T} = 1 - \mathcal{R}$.

We then focus on the tri-critical Ising model $M(4, 5)$ which has $c = \frac{7}{10}$. We construct topological interfaces between the $N = 1$ supersymmetric theory and the bosonic theory at $c = \frac{7}{10}$. By using the topological interfaces, we obtain conformal defects in the bosonic tri-critical Ising model from the supersymmetric theory. As a warm-up, we also give topological interfaces between the free fermion theory and the Ising model.

We then study topological defects and conformal boundary conditions in $N = 1$ super-Virasoro theories and construct a consistent theory of superconformal defects and boundaries at $c = \frac{7}{10}$. Unlike in most of the literature, we do not take the GSO projected boundary states. As we shall explain later, we take fermionic theories that are local i.e. take the Neveu–Schwarz sectors only and relax the requirement of modular invariant partition functions.

In order to obtain conformal defects in the supersymmetric tri-critical Ising model, we study superconformal boundary conditions of the doubled model, which has $c = \frac{7}{5}$. Our construction of these boundary states differ from the one given in [94]. It turns out that it is important to analyse carefully the embeddings of the super-Virasoro algebra $\mathcal{SV}\mathrm{ir}$ into $\mathcal{SV}\mathrm{ir} \oplus \mathcal{SV}\mathrm{ir}$ in order to identify the boundaries with the defects. We also analyse the boundary conditions and corresponding defects in terms of the super W-algebra $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$, and discuss their fusion rules.

Finally, using the topological interfaces, we obtain the entropies and transmission coefficients of non-topological and non-factorising defects in the bosonic tri-critical Ising model. The results suggest that there are two non-topological and non-factorising defects that we denote $\mathcal{D}_t^{M(4,5)}$ and $\mathcal{D}_f^{M(4,5)}$ from which we can obtain the other defects by actions of topological defects.

Outline

In Chapter 2, we summarise background materials in two-dimensional conformal fields theories. These include the Virasoro and W-algebras, conformal boundary conditions, and conformal defects. In Chapter 3, we discuss relation between conformal defects and extended conformal symmetries in the folded theories, and explain the reflection and transmission coefficients in terms of W-algebras. In Chapter 4, we calculate reflection and transmission coefficients in the diagonal Virasoro minimal models using the leading-order perturbation calculation. In Chapter 5, we construct topological interfaces for the free fermion−Ising and bosonic−supersymmetric tri-critical Ising model cases. In Chapter 6, we construct superconformal defects in the tri-critical Ising model and project these to the bosonic theory.

The results presented in Chapter 4, 5, and 6 are published jointly with G. Watts as

[117] I. Makabe and G. M. T. Watts. "Defects in the Tri-critical Ising model". In: *Journal of High Energy Physics* 09 (2017), p. 013. arXiv: `1703.09148 [hep-th]`

[118] I. Makabe and G. M. T. Watts. "The reflection coefficient for minimal model conformal defects from perturbation theory". In: (2017). arXiv: `1712.07234 [hep-th]`

Chapter 2

Background from Two-Dimensional Conformal Field Theory

In this chapter, we present an introduction to two-dimensional conformal field theory (CFT). This is meant to be a summary of material necessary for this thesis rather than an exhaustive survey on the topic of CFT. Most discussions given in this chapter can be found in the literature, and references are not always given explicitly in the main body of this chapter.

Section 2.1 discusses conformal transformations in two-dimensional Euclidean spaces without boundaries or defects, and their implications for correlation functions and the field content of a theory. Discussions are based on the seminal paper by Belavin, Polyakov, and Zamolodchikov[4]; review papers by Ginsparg[22] and by Alvarez-Gaumé, Sierra, and Gomez[27]; and books by Di Francesco, Mathieu, and Sénéchal[56] and by Blumenhagen and Plauschinn[96].

In Section 2.2, analysis of representations of the Virasoro algebra is based on the book by Kac and Raina[20]. Presentation of W-algebras as meromorphic CFTs is based on review papers by Bouwknegt and Schoutens[44] and by Watts[60] while some discussions on chiral vertex operators are also based on the review paper by Gaberdiel[69]. We follow Honecker's paper[46] for the discussion on automorphisms of W-algebras while calculations on automorphisms of $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ are original. For systematic study of W- and super W-algebras, we refer to [38] and [40] as well as [41] and [45] for their representation theories. For $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ at $c = \frac{7}{5}$, references are [35, 39, 42]. For various aspects of the Ramond algebra we refer to [21, 72, 80] for representation theory, and to [47, 95] for fusion rules.

2.1 Bulk Conformal Field Theories

As we shall see in this section, conformal symmetries fix the forms of two- and three-point correlation functions. Together with the notion of operator product expansions (OPEs), conformal symmetries are a very powerful tool to solve a theory for correlation functions. Especially, in two-dimensions, the algebra of conformal symmetry becomes infinite-dimensional, and a theory can be solved by considering the symmetries alone in certain cases. In this section, we focus on conformal field theories without boundaries or defects. For brevity, we call such theories bulk CFTs.

2.1.1 Conformal Transformations

Consider a d-dimensional Euclidean space $\mathbb{E}^d$ with the metric $g_{\mu\nu}$, which is not necessarily flat. Under a coordinate change $x \mapsto x'$, the metric transforms as

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \; . \tag{2.1}$$

A conformal transformation is a coordinate transformation which leaves the metric invariant up to a scale factor Ω^2, that is $g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x)$.

Under an infinitesimal local coordinate transformation $x \mapsto x' = x + \varepsilon(x)$, the metric transforms to $g'_{\mu\nu}(x') = g_{\mu\nu}(x) - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$. If this corresponds to a conformal transformation, the second term must be proportional to $g_{\mu\nu}$, and one can write $\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) g_{\mu\nu}$ for some function f. Multiplying this by $g^{\mu\nu}$, one obtains $f(x) = 2d^{-1}(\partial \cdot \varepsilon)$. Thus, for conformal transformations

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial \cdot \varepsilon) g_{\mu\nu} \quad \text{and} \quad \Omega^2(x) = 1 - \frac{2}{d}(\partial \cdot \varepsilon) \; . \tag{2.2}$$

Differentiating (2.2) by ∂^μ and ∂^ν, one obtains $\Box(\partial \cdot \varepsilon) = 0$ unless $d = 1$. Furthermore, it implies that ε is at most quadratic in x for $d > 2$. We can interpret the solutions of (2.2) up to the quadratic order as the following:

- ε constant, which corresponds to
  - translations $\varepsilon^\mu(x) = a^\mu$,
- ε linear in x, which corresponds to
  - rotations $\varepsilon^\mu(x) = \omega^\mu{}_\nu x^\nu$ (ω antisymmetric) or
  - scale transformations (dilations) $\varepsilon^\mu(x) = \lambda x^\mu$,
- and ε quadratic in x, which corresponds to
  - special conformal transformations $\varepsilon^\mu(x) = b^\mu x^2 - 2x^\mu(b \cdot x)$.

Among the solutions above, special conformal transformations need a little explanation. By considering a finite special conformal transformation

$$x^\mu \mapsto x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2(b \cdot x) + b^2 x^2} \; , \tag{2.3}$$

we can understand this as an inversion $x^\mu \mapsto -x^\mu/x^2$ followed by a translation $x^\mu \mapsto x^\mu - b^\mu$ and another inversion. If b^μ is non-zero, there is one point in $\mathbb{E}^d$ which is mapped to infinity. Therefore, if we require finite special conformal transformations to be globally defined, we need to compactify $\mathbb{E}^d$ by including one point at infinity.

From the solutions of (2.2), one can define the infinitesimal generators of conformal transformations as

$$P_\mu = -i\partial_\mu \qquad \text{(translations)}, \tag{2.4}$$

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad \text{(rotations)}, \tag{2.5}$$

$$D = -i(x \cdot \partial) \qquad \text{(dilations), and} \tag{2.6}$$

$$K_\mu = i(x^2 \partial_\mu - 2x_\mu (x \cdot \partial)) \qquad \text{(special conformal transformations)}. \tag{2.7}$$

They form a basis of the Lie algebra of $\mathrm{SO}(1, d+1)$ over $\mathbb{R}$. Thus, it is straightforward to deduce that the conformal group in a d-dimensional Euclidean space is isomorphic to $\mathrm{SO}(1, d+1)$ for $d > 2$. In a Minkowski space $\mathbb{R}^{p,q}$, the conformal group is isomorphic to $\mathrm{SO}(p+1, q+1)$.

In two-dimensions, (2.2) still holds, however ε is no longer at most quadratic in x. We can see this by introducing the complex coordinates[1]

$$z = x^1 + ix^2 \quad \text{and} \quad \bar{z} = x^1 - ix^2 \quad \text{with} \quad \partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \ . \tag{2.8}$$

Then, (2.2) becomes the Cauchy–Riemann equations $\bar{\partial}\varepsilon = 0$ and $\partial\bar{\varepsilon} = 0$, where $\varepsilon = \varepsilon^1 + i\varepsilon^2$ and $\bar{\varepsilon} = \varepsilon^1 - i\varepsilon^2$. Therefore, $\varepsilon(z)$ is a holomorphic function of z. In fact, any holomorphic function with nowhere-vanishing derivative $f : M \to \mathbb{C}$, where M is an open subset of $\mathbb{C}$, is an orientation-preserving conformal transformation with $\Omega^2 = |\partial f|^2$.

Although $\bar{z}$ is defined to be the complex conjugate of z, if we consider $\bar{z}$ as an independent coordinate[2], we can treat $\bar{\varepsilon}(\bar{z})$ as a holomorphic function of $\bar{z}$. Expanding $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ in the Laurent series

$$\varepsilon(z) = \sum_{n\in\mathbb{Z}} \varepsilon_n(-z^{n+1}) \quad \text{and} \quad \bar{\varepsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n(-\bar{z}^{n+1}) \ , \tag{2.9}$$

where $\varepsilon_n, \ \bar{\varepsilon}_n \in \mathbb{C}$ are constants, one obtains the sets of orthogonal solutions of $\bar{\partial}\varepsilon = 0$ and $\partial\bar{\varepsilon} = 0$. For each ε_n and $\bar{\varepsilon}_n$, we can define the infinitesimal generators of conformal transformations as

$$l_n = -z^{n+1}\partial \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \ . \tag{2.10}$$

Their commutators satisfy

$$[l_n, l_m] = (n-m)l_{n+m} \ , \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \ , \quad \text{and} \quad [l_n, \bar{l}_m] = 0 \ . \tag{2.11}$$

Each of the sets $\{l_n : n \in \mathbb{Z}\}$ and $\{\bar{l}_m : m \in \mathbb{Z}\}$ forms a basis of the infinite-dimensional complex Lie algebra called the Witt algebra. It is isomorphic to the complexification of the Lie algebra of real vector fields on the circle S^1.

As we have seen before, we need to compactify $\mathbb{R}^2 \simeq \mathbb{C}$ in order to define finite special conformal transformations globally. The resulting space is the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Even on $\mathbb{CP}^1$, not all the generators l_n are globally defined. Considering regularity at $z = 0$ and $z = \infty$, we see that the algebra of globally defined conformal transformations is generated by $\{l_{-1}, \ l_0, \ l_1\}$, whose commutation relations are that of the complex Lie algebra[3] $\mathrm{sl}(2)$. By imposing $\bar{z}$ to be the complex conjugate of z, these

1. By ∂ or ∂_z, we mean $\partial/\partial z$ where it is obvious. Sometimes, by abusing notation, we also use ∂ to denote $\mathrm{d}/\,\mathrm{d}z$.
2. It is customary to consider $(z, \bar{z}) \in \mathbb{C}^2$ and impose $\bar{z}$ to be the complex conjugate of z when necessary.
3. By $\mathrm{sl}(2)$ we denote the three-dimensional simple Lie algebra over $\mathbb{C}$. In addition, $\mathrm{sl}(2)$ is the complexification of the real Lie algebra $\mathrm{su}(2)$.

generators can be understood as

$$P = l_{-1}\,, \quad \bar{P} = \bar{l}_{-1} \qquad \text{(translations)}, \tag{2.12}$$

$$L = i(l_0 - \bar{l}_0) \qquad \text{(rotation)}, \tag{2.13}$$

$$D = l_0 + \bar{l}_0 \qquad \text{(dilation), and} \tag{2.14}$$

$$K = l_1\,, \quad \bar{K} = \bar{l}_1 \qquad \text{(special conformal transformations)}. \tag{2.15}$$

Using (2.8), the generators above can be written in terms of the real coordinates. This reminds ourselves that, in two-dimension, finite conformal transformations have six real parameters that can be considered as three complex parameters. From the $\mathrm{sl}(2)$ generators, finite conformal transformations can be written as

$$z \mapsto z' = \frac{az+b}{cz+d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \cong \mathrm{PSL}(2,\mathbb{C})\ . \tag{2.16}$$

Therefore, the conformal group of the Riemann sphere $\mathbb{CP}^1$ is the Möbius group $\mathrm{PSL}(2,\mathbb{C})$, and we refer to the globally defined conformal transformations on $\mathbb{CP}^1$ as the Möbius transformations. As we have seen before, Möbius transformations consist of the following global transformations: translations, rotations, dilations, and special conformal transformations.

2.1.2 Transformations of Fields

In a conformal field theory, fields are classified by their local transformation properties under a given conformal transformation. As before, we take the complex coordinates (2.8) in two-dimension. If a field $\varphi(z,\bar{z})$ transforms as

$$\varphi(z,\bar{z}) \mapsto \varphi'(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \varphi(w,\bar{w})\ , \tag{2.17}$$

under any local conformal transformations $z \mapsto w(z)$, then it is called a primary field. If (2.17) only holds for Möbius transformations, it is called a quasiprimary field. By considering a scale transformation $w = \lambda z$ and a rotation $w = e^{i\alpha} z$, where $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, we find that $\varphi(z,\bar{z})$ has scaling dimension $\Delta = h + \bar{h}$ and spin $s = h - \bar{h}$. The two real numbers h and $\bar{h}$ are called the conformal dimensions of $\varphi(z,\bar{z})$. In a conformal field theory we consider in this thesis[4], all the fields can be expressed as linear combinations of quasiprimary fields and their derivatives. Derivatives of quasiprimary fields are no longer quasiprimary and they are called secondary fields.

2.1.3 Forms of Correlation Functions

In a conformal field theory, correlation functions must be invariant under conformal transformations. This means, for quasiprimary fields φ_I with conformal dimensions h_I and $\bar{h}_I$,

4. We only consider CFTs where L_0 operators (see the section on the Virasoro algebra) are diagonalisable and energies are bounded from below on a given representation space. This excludes so-called logarithmic CFTs where L_0 operators can be brought to the Jordan normal forms only.

the following has to hold

$$\langle \varphi_1(z_1,\bar{z}_1)\dots\varphi_n(z_n,\bar{z}_n)\rangle = \prod_{I=1}^{n}\left(\frac{\partial w}{\partial z}\right)_{w=w_I}^{h_I}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)_{\bar{w}=\bar{w}_I}^{\bar{h}_I}\langle \varphi_1(w_1,\bar{w}_1)\dots\varphi_n(w_n,\bar{w}_n)\rangle\ , \tag{2.18}$$

where $z \mapsto w(z)$ is a Möbius transformation. This equation determines two- and three-point functions up to some constants, and four-point functions up to some functions.

From translation invariance, one-point functions must be constants. In addition, considering scale transformations, one-point functions have to vanish unless $h = 0$ and $\bar{h} = 0$. Therefore, only the identity field has the non-vanishing one-point function. We assume the bulk vacuum state is unique, and denote the identity field by $\mathbf{1}$. Since it is natural to normalise the identity field to one, we take $\langle \mathbf{1}\rangle = 1$. If translation invariance is broken in some direction by introduction of a boundary or a defect, one-point functions may acquire position dependence. We will discuss this later.

For two-point functions, translation invariance and scale invariance determine their functional forms. Furthermore, considering a special conformal transformation $w = -1/z$, we obtain

$$\langle \varphi_1(z_1,\bar{z}_1)\varphi_2(z_2,\bar{z}_2)\rangle = \frac{d_{12}}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}}\ , \tag{2.19}$$

where $d_{12} = 0$ if $h_1 \neq h_2$ or $\bar{h}_1 \neq \bar{h}_2$, and d_{12} is called a structure constant or the two-point coupling of φ_1 and φ_2. For the bulk fields, we can take a basis of fields such that

$$d_{IJ} = d_I \delta_{J,I^+}\ , \tag{2.20}$$

where the field φ_{I^+} is the charge conjugate[5] of φ_I, and we have $h_{I^+} = h_I$ and $\bar{h}_{I^+} = \bar{h}_I$. Clearly, the identity field is self-conjugate as it is the only bulk field with $h = 0$ and $\bar{h} = 0$. Abstractly, we can think of charge conjugation as an automorphism $I \mapsto I^+$ on the set of labels of the bulk fields which leaves the conformal weights invariant.

Three-point functions can be analysed similarly. From translation invariance, they must be the functions of $z_{IJ} := z_I - z_J$. Considering their scale invariance and the same special conformal transformation as before, we obtain

$$\langle \varphi_1(z_1,\bar{z}_1)\varphi_2(z_2,\bar{z}_2)\varphi_3(z_3,\bar{z}_3)\rangle = \frac{C_{123}}{z_{12}^{h_{123}}z_{23}^{h_{231}}z_{13}^{h_{132}}\ \bar{z}_{12}^{\bar{h}_{123}}\bar{z}_{23}^{\bar{h}_{231}}\bar{z}_{13}^{\bar{h}_{132}}}\ , \tag{2.21}$$

where

$$h_{IJK} := h_I + h_J - h_K \tag{2.22}$$

and the structure constant C_{123} is also called the three-point couplings of φ_1, φ_2, and φ_3.

Calculations for four-point functions are slightly involved but the principles do not change. Using Möbius transformations, we find

$$\langle \varphi_1(z_1,\bar{z}_1)\dots\varphi_4(z_4,\bar{z}_4)\rangle = f_{1234}(\eta,\bar{\eta})\prod_{I<J}^{4} z_{IJ}^{h/3-h_I-h_J}\ \bar{z}_{IJ}^{\bar{h}/3-\bar{h}_I-\bar{h}_J}\ , \tag{2.23}$$

5. As we shall see later, a bulk field label can be regarded as $I = (i,\bar{\imath})$, where i and $\bar{\imath}$ label representations of the chiral algebra. Then, its conjugate means $I^+ = (i^+,\bar{\imath}^+)$, where i^+ and $\bar{\imath}^+$ labels the representations conjugate to i and $\bar{\imath}$, respectively. In addition, we have $(I^+)^+ = I$, which comes from the fact that the charge conjugation matrix squares to the identity matrix[51].

where

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{2.24}$$

is called the cross ratio and $h := h_1 + h_2 + h_3 + h_4$. As we shall see later, it is useful to define the function

$$\begin{aligned} G_{1234}(z,\bar{z}) &:= \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\langle\varphi_1(z_1,\bar{z}_1)\varphi_2(1,1)\varphi_3(z,\bar{z})\varphi_4(0,0)\rangle \\ &= f_{1234}(z,\bar{z})\,(1-z)^{h/3-h_2-h_3}z^{h/3-h_3-h_4}\,(1-\bar{z})^{\bar{h}/3-\bar{h}_2-\bar{h}_3}\bar{z}^{\bar{h}/3-\bar{h}_3-\bar{h}_4}\ , \end{aligned} \tag{2.25}$$

and work with G_{1234} rather than f_{1234}.

In principle, any n-point functions can be reduced to one-point functions by operations known as the operator product expansions (OPEs) which will be discussed in Subsection 2.1.7. In order to do this, one needs to determine the OPE constants involved in the calculations. As we shall see later, this can be carried out by considering different ways of expressing a four-point function in terms of OPEs.

2.1.4 Radial Quantisation

In two-dimensional conformal field theories, explicit forms of action functionals are not as important as in other quantum field theories, and correlation functions are usually calculated using the operator formalism rather than the path integral methods.

Consider the Riemann sphere with a coordinate z. In radial quantisation, we consider compactified space, and equal-time surfaces are given by constant $|z|$. Then, time flows along the radial direction, and $z = 0$ is taken to be the infinite past and $z = \infty$ corresponds to the infinite future. The space of states $\mathcal{H}$ contains the vacuum state $|0\rangle$, and for each field operator $\varphi_I(z,\bar{z})$, there is the corresponding state given by

$$|\varphi_I\rangle = \lim_{z,\bar{z}\to 0}\varphi_I(z,\bar{z})|0\rangle\ . \tag{2.26}$$

Conversely, we can think as, for each vector $|\varphi_I\rangle \in \mathcal{H}$, there is a map[6] $V : \mathcal{H}\times\mathbb{C}\times\mathbb{C}\to \mathrm{End}(\mathcal{H})$ and the local field is given by

$$V(|\varphi_I\rangle; z,\bar{z}) := \varphi_I(z,\bar{z})\ , \tag{2.27}$$

which satisfies (2.26). This is called the state-field correspondence. If we expand $\varphi_I(z,\bar{z})$ in terms of the modes as

$$\varphi_I(z,\bar{z}) = \sum_{n,\bar{n}\in\mathbb{Z}}(\varphi_I)_{n,\bar{n}}\, z^{-n-h_I}\bar{z}^{-\bar{n}-\bar{h}_I}\ , \tag{2.28}$$

then regularity of the limit (2.26) demands

$$(\varphi_I)_{n,\bar{n}}|0\rangle = 0 \quad \text{for} \quad n > -h_I \quad \text{or} \quad \bar{n} > -\bar{h}_I\ . \tag{2.29}$$

6. This map has to satisfy certain conditions, for example $V(|0\rangle; z,\bar{z}) = \mathrm{id}_{\mathcal{H}}$ and $V(|v\rangle; z,\bar{z})|0\rangle|_{z=0,\bar{z}=0} = |v\rangle$ for any $|v\rangle\in\mathcal{H}$, which is know as the vacuum axiom. The definition of a vertex algebra can be found, for example, in [58] and [82].

We define the hermitian conjugate of $\varphi_I(z,\bar{z})$ as

$$\varphi_I^\dagger(z,\bar{z}) \coloneqq \bar{z}^{-2h_I} z^{-2\bar{h}_I} \varphi_{I^+}(\bar{z}^{-1}, z^{-1}) \,, \tag{2.30}$$

where we assume the basis of bulk fields in which (2.20) holds. In terms of the modes, this definition of hermitian conjugation means

$$(\varphi_I^\dagger)_{n,\bar{n}} = (\varphi_{I^+})_{-n,-\bar{n}} \,. \tag{2.31}$$

Using hermitian conjugation, we can write the dual vector $(|\varphi_I\rangle)^\dagger =: \langle\varphi_I| \in \mathcal{H}^*$ as

$$\langle\varphi_I| = \lim_{z,\bar{z}\to 0} \langle 0|\varphi_I^\dagger(z,\bar{z}) = \lim_{w,\bar{w}\to\infty} w^{2h_I} \bar{w}^{2\bar{h}_I} \langle 0|\varphi_{I^+}(w,\bar{w}) \,, \tag{2.32}$$

where $w = \bar{z}^{-1}$. From (2.31), we find

$$\langle 0|(\varphi_{I^+})_{n,\bar{n}} = 0 \quad \text{for} \quad n < h_I \quad \text{or} \quad \bar{n} < \bar{h}_I \,, \tag{2.33}$$

in order for (2.32) to be regular. Recalling that $z = 0$ is the infinite past and $z = \infty$ is the infinite future in radial quantisation, we can view state vectors as "in-states" and their duals as "out-states".

In radial quantisation, time ordering becomes what is called radial ordering, which is given by

$$\mathcal{R}\left(\varphi_1(z_1,\bar{z}_1)\varphi_2(z_2,\bar{z}_2)\right) = \begin{cases} \varphi_1(z_1,\bar{z}_1)\varphi_2(z_2,\bar{z}_2) & \text{if} \quad |z_1| > |z_2| \,, \\ \epsilon_{12}\, \varphi_2(z_2,\bar{z}_2)\varphi_1(z_1,\bar{z}_1) & \text{if} \quad |z_2| > |z_1| \,, \end{cases} \tag{2.34}$$

where $\epsilon_{12} = -1$ if both φ_1 and φ_2 are fermions[7], otherwise $\epsilon_{12} = 1$. Operators must be radially ordered within correlation functions to make any sense, and we can write

$$\langle \varphi_1(z_1,\bar{z}_1)\ldots\varphi_n(z_n,\bar{z}_n)\rangle = \langle 0|\varphi_1(z_1,\bar{z}_1)\ldots\varphi_n(z_n,\bar{z}_n)|0\rangle \quad \text{for} \quad |z_1| > \cdots > |z_n| > 0 \,. \tag{2.35}$$

In this thesis, strings of operators like $\varphi_1(z_1,\bar{z}_1)\ldots\varphi_n(z_n,\bar{z}_n)$ are understood to be radially ordered, and the symbol $\mathcal{R}$ will not always be written explicitly.

2.1.5 Ward Identities

Consider a two-dimensional quantum field theory described by the action functional $S[\varphi]$ with local fields $\varphi_i(x_i)$, where $x \in \mathbb{R}^2$. One may define a symmetric stress-energy tensor $T_{\mu\nu}$ by

$$\delta S = \frac{1}{2}\int \delta g^{\mu\nu} T_{\mu\nu} \sqrt{|g|}\, \mathrm{d}^2 x \,, \tag{2.36}$$

where $g_{\mu\nu}$ is the metric and g is its determinant. If conformal transformations are symmetries of this theory, classically we should have $\delta S = 0$ for such transformations. Taking $\delta g^{\mu\nu}$ to be an infinitesimal conformal transformation (2.2), then $\delta g^{\mu\nu} = -(\partial\cdot\varepsilon)g^{\mu\nu}$ and $\delta S = 0$ implies $g^{\mu\nu}T_{\mu\nu} = 0$, that is, the stress-energy tensor is traceless. Furthermore, we

7. In this thesis, we assume bulk fields have integer (bosons) or half-integer (fermions) spins.

can also write $\delta g^{\mu\nu} = -(\partial^\mu \varepsilon^\nu + \partial^\nu \varepsilon^\mu)$ and using the fact that $T_{\mu\nu}$ is symmetric, one can define the classical conserved currents as $j_\mu = \varepsilon^\nu T_{\mu\nu}$.

If the conformal symmetry is preserved at the quantum level, the Ward identities involving $j_\mu = \varepsilon^\nu T_{\mu\nu}$ should hold. In particular, we have

$$\int \langle \partial^\mu (\varepsilon^\nu T_{\mu\nu}(x)) \varphi(y) \rangle \, \mathrm{d}^2 x = \langle \delta\varphi(y) \rangle \, . \tag{2.37}$$

Since the classical conservation equation $\partial^\mu j_\mu = 0$ holds away from the field insertion at y, it suffices to evaluate the integral for a small disc Σ around y. Using Stokes's theorem

$$\int_\Sigma \partial^\mu (\varepsilon^\nu T_{\mu\nu}) \, \mathrm{d}^2 x = \oint_{\partial\Sigma} \left(\varepsilon^\nu T_{1\nu} \, \mathrm{d}x^2 - \varepsilon^\nu T_{2\nu} \, \mathrm{d}x^1 \right) \tag{2.38}$$

and introducing the complex coordinates defined by (2.8), the Ward identity becomes

$$\frac{1}{2\pi i} \oint_w \varepsilon(z) \langle T(z) \varphi(w, \bar{w}) \rangle \, \mathrm{d}z + \frac{1}{2\pi i} \oint_{\bar{w}} \bar{\varepsilon}(\bar{z}) \langle \overline{T}(\bar{z}) \varphi(w, \bar{w}) \rangle \, \mathrm{d}\bar{z} = \langle \delta\varphi(w, \bar{w}) \rangle \, , \tag{2.39}$$

where $T(z) = 2\pi T_{zz}(z)$ and $\overline{T}(\bar{z}) = 2\pi T_{\bar{z}\bar{z}}(\bar{z})$. These contours are defined to encircle the singularities only at $z = w$ and $\bar{z} = \bar{w}$. Since these contour integrations are around the field insertion, the classical conservation equation $\partial^\mu j_\mu = 0$ still holds, and this implies $T_{z\bar{z}} = T_{\bar{z}z} = 0$ and $\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$. Therefore, for infinitesimal local conformal transformations, we may treat transformations of $\varphi(w, \bar{w})$ for w and $\bar{w}$ separately, and write

$$\begin{aligned} \frac{1}{2\pi i} \oint_w \varepsilon(z) \langle T(z) \varphi(w, \bar{w}) \rangle \, \mathrm{d}z &= \langle \delta_\varepsilon \varphi(w, \bar{w}) \rangle \quad \text{and} \\ \frac{1}{2\pi i} \oint_{\bar{w}} \bar{\varepsilon}(\bar{z}) \langle \overline{T}(\bar{z}) \varphi(w, \bar{w}) \rangle \, \mathrm{d}\bar{z} &= \langle \delta_{\bar{\varepsilon}} \varphi(w, \bar{w}) \rangle \, . \end{aligned} \tag{2.40}$$

If $\varphi(w, \bar{w})$ is a primary field, we can substitute $w \mapsto w + \varepsilon$ and $\bar{w} \mapsto \bar{w} + \bar{\varepsilon}$ into (2.17) and write $\delta\varphi = \varphi' - \varphi$ as

$$\delta_\varepsilon \varphi(w, \bar{w}) = (h(\partial\varepsilon) + \varepsilon\partial) \, \varphi(w, \bar{w}) \quad \text{and} \quad \delta_{\bar{\varepsilon}} \varphi(w, \bar{w}) = \left(\bar{h}(\bar{\partial}\bar{\varepsilon}) + \bar{\varepsilon}\bar{\partial} \right) \varphi(w, \bar{w}) \, . \tag{2.41}$$

Then, we can substitute these into the right hand sides of the Ward identities and use the residue theorem to obtain the singular terms

$$\begin{aligned} T(z) \varphi(w, \bar{w}) &= \frac{h \varphi(w, \bar{w})}{(z - w)^2} + \frac{\partial \varphi(w, \bar{w})}{z - w} + \text{reg.} \quad \text{and} \\ \overline{T}(\bar{z}) \varphi(w, \bar{w}) &= \frac{\bar{h} \varphi(w, \bar{w})}{(\bar{z} - \bar{w})^2} + \frac{\bar{\partial} \varphi(w, \bar{w})}{\bar{z} - \bar{w}} + \text{reg.} \, , \end{aligned} \tag{2.42}$$

where we have used the fact that $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are non-singular as $z \to w$ and $\bar{z} \to \bar{w}$. These equations are understood to hold inside correlators, and they can be viewed as a way of expressing the product of two fields at $(z, \bar{z})$ and $(w, \bar{w})$ as a series of local fields at $(w, \bar{w})$. This is an example of operator product expansions. Another way of characterising primary fields is that their OPEs with T and $\overline{T}$ are of the form (2.42).

2.1.6 Virasoro Algebra

For an infinitesimal conformal transformation $z \mapsto z + \varepsilon(z)$, the corresponding chiral conserved charge is given by

$$Q = \frac{1}{2\pi i}\oint_0 \varepsilon(z)T(z)\,\mathrm{d}z \ . \tag{2.43}$$

Here, equal-time surfaces are given by constant $|z|$ as we are working in radial quantisation. By expanding $T(z)$ in terms of the modes as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad L_n = \frac{1}{2\pi i}\oint_0 T(z)\, z^{n+1}\,\mathrm{d}z \ , \tag{2.44}$$

and using the mode expansion of $\varepsilon(z)$ given in (2.9), we can write

$$Q = \sum_{n\in\mathbb{Z}} -\varepsilon_n L_n \ . \tag{2.45}$$

Thus, the modes of the stress-energy tensor L_n can be viewed as the infinitesimal generators of conformal transformations. Similarly, we can view $\bar{L}_n$ as the infinitesimal generators of transformations of the form $\bar{z} \mapsto \bar{z} + \bar{\varepsilon}(\bar{z})$. Classically, these generators form two copies of the Witt algebra (2.11), however, upon quantisation, they become the central extension known as the Virasoro algebra, which satisfies

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ , \\
&[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0} \ , \quad \text{and} \\
&[L_n, \bar{L}_m] = 0 \ ,
\end{aligned} \tag{2.46}$$

where $c,\ \bar{c} \in \mathbb{C}$ is called the central charge[8]. We denote the Virasoro algebra by $\mathcal{V}\mathrm{ir}$. By convention, the form of the central terms in (2.46) is chosen to ensure the commutators involving L_{-1}, L_0, and L_1 do not yield central terms. As before, $\{L_{-1},\ L_0,\ L_1\}$ generates $\mathrm{sl}(2)$, which corresponds to the Möbius transformations. From the condition (2.29), these generators annihilate the vacuum state as they should; the vacuum state is invariant under global conformal transformations. Because of radial quantisation, scale transformations are regarded as translations in time and rotations correspond to translations in space. Therefore, the Hamiltonian and momentum operators on the plane are given by

$$H = L_0 + \bar{L}_0 \quad \text{and} \quad P = i(L_0 - \bar{L}_0) \ , \tag{2.47}$$

respectively.

From the OPEs (2.42) involving the components of the stress-energy tensor and a primary field $\varphi(z,\bar{z})$ with conformal weights h and $\bar{h}$, we can calculate

$$\begin{aligned}
&[L_n, \varphi(z,\bar{z})] = \big(h(n+1)z^n + z^{n+1}\partial\big)\,\varphi(z,\bar{z}) \quad \text{and} \\
&[\bar{L}_n, \varphi(z,\bar{z})] = \big(\bar{h}(n+1)\bar{z}^n + \bar{z}^{n+1}\bar{\partial}\big)\,\varphi(z,\bar{z}) \ .
\end{aligned} \tag{2.48}$$

8. To be precise, the Virasoro algebra is the central extension of the Witt algebra by the one-dimensional centre $\mathbb{C}\hat{c}$, and it has the Cartan subalgebra spanned by L_0 and $\hat{c}$. Therefore, a highest weight vector, which is defined to be annihilated by the action of L_n for $n > 0$, is characterised by the L_0- and $\hat{c}$-eigenvalues h and c respectively. In a highest weight representation of the Virasoro algebra, $\hat{c}$ can be treated as the number c.

In terms of the modes (2.28) of $\varphi(z,\bar{z})$, these equations can be written as

$$[L_n, \varphi_{m,\bar{m}}] = ((h-1)n - m)\, \varphi_{n+m,\bar{m}} \quad \text{and}$$
$$[\bar{L}_n, \varphi_{m,\bar{m}}] = \big((\bar{h}-1)n - \bar{m}\big)\, \varphi_{m,n+\bar{m}} \; . \tag{2.49}$$

Using these commutators together with the state-field correspondence (2.26) and (2.29), we find actions of L_n and $\bar{L}_n$ on $|\varphi\rangle$ as

$$L_n|\varphi\rangle = 0 \quad \text{and} \quad \bar{L}_n|\varphi\rangle = 0 \quad \text{for} \quad n > 0\,, \quad L_0|\varphi\rangle = h|\varphi\rangle\,, \quad \bar{L}_0|\varphi\rangle = \bar{h}|\varphi\rangle\,,$$
$$L_{-1}|\varphi\rangle = \lim_{z,\bar{z}\to 0} \partial\varphi(z,\bar{z})|0\rangle\,, \quad \text{and} \quad \bar{L}_{-1}|\varphi\rangle = \lim_{z,\bar{z}\to 0} \bar{\partial}\varphi(z,\bar{z})|0\rangle\,. \tag{2.50}$$

These observations tell us that $|\varphi\rangle$ acts as a highest weight state for L_n and $\bar{L}_n$ with the weight $(h,\bar{h})$. In addition, we see that the state $L_{-1}|\varphi\rangle$ corresponds to the field $\partial\varphi$. Since L_n and $\bar{L}_n$ commute, we can think of $|\varphi\rangle$ as the highest weight vector of $\mathcal{H}_h \otimes \overline{\mathcal{H}}_{\bar{h}}$, where $\mathcal{H}_h$ and $\overline{\mathcal{H}}_{\bar{h}}$ are the irreducible highest weight modules of $\mathcal{V}\mathrm{ir}$ and $\overline{\mathcal{V}\mathrm{ir}}$ with the highest weights h and $\bar{h}$, respectively. Then, other vectors in $\mathcal{H}_h \otimes \overline{\mathcal{H}}_{\bar{h}}$ are generated by repeated actions of lowering operators, L_n and $\bar{L}_n$ with $n < 0$, on the state $|\varphi\rangle$ modulo null vectors. If $|v\rangle \in \mathcal{H}_h \otimes \overline{\mathcal{H}}_{\bar{h}}$ is one of these vectors, the corresponding field $V(|v\rangle; z, \bar{z})$ is called a descendant field.

It is important to note that the components of the stress-energy tensor $T(z)$ and $\overline{T}(\bar{z})$ are quasiprimary fields but they are not primary fields. Under local conformal transformations $z \mapsto w$, they transform as

$$T(z) = \left(\frac{\partial w}{\partial z}\right)^2 T(w) + \frac{c}{12}\{w; z\}\,, \tag{2.51}$$

where

$$\{w; z\} := \frac{(\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2}{(\partial_z w)^2} \tag{2.52}$$

is called the Schwarzian derivative. In addition, from the commutation relations (2.46), we can calculate the TT OPE as

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}\,. \tag{2.53}$$

For the antiholomorphic component $\overline{T}(\bar{z})$, the same equations hold with c, z, and w replaced by $\bar{c}$, $\bar{z}$, and $\bar{w}$.

One of the classes of CFTs we consider in this thesis is the Virasoro minimal models. One way to characterise a Virasoro minimal model is that it has a space of states $\mathcal{H}$, which decomposes into a finite number of tensor products of $\mathcal{V}\mathrm{ir}$ and $\overline{\mathcal{V}\mathrm{ir}}$ irreducible modules. We may label primary bulk fields—fields that are not on boundaries or defects—as $\varphi_I(z,\bar{z})$ where $I = (i,\bar{i})$ with i and $\bar{i}$ labelling $\mathcal{V}\mathrm{ir}$ and $\overline{\mathcal{V}\mathrm{ir}}$ irreducible modules, respectively. If there is more than one bulk primary field carrying the same Virasoro representations, we need to introduce multiplicity labels, say α, and write $I_\alpha = (i,\bar{i};\alpha)$ to distinguish these fields. Then, the space of states for the bulk fields can be written as

$$\mathcal{H} = \bigoplus_{I\in\mathcal{S}} \mathcal{H}_i \otimes \overline{\mathcal{H}}_{\bar{i}}\,, \tag{2.54}$$

where $\mathcal{S}$ is the set of labels for the bulk fields, which is also called the spectrum of the bulk theory. A bulk spectrum is not completely arbitrary; it can be constrained by requiring the torus partition function to be modular invariant. There is a generalisation of this idea to arbitrary chiral algebras. Even if a theory is not minimal with respect to the Virasoro algebra, it may be possible to write the space of states of the form (2.54) if we consider representations of a larger chiral algebra, which contains $\mathcal{V}\text{ir}$. Such theories are called rational conformal field theories (RCFTs).

2.1.7 Operator Product Expansions

Operator product expansions can be regarded as ways of expanding two nearby fields in terms of local fields at a point which is also close to these two fields. Since we are only considering CFTs in which all the fields are quasiprimary fields and their derivatives, we can assume OPEs of two primary fields involve primary fields and their descendants only. Therefore, if we denote quasiprimary fields and their derivatives by $\varphi_\alpha(z,\bar{z}),\ \varphi_\beta(z,\bar{z})\,,\ldots,$ OPEs can be expressed as

$$\varphi_\alpha(z,\bar{z})\varphi_\beta(w,\bar{w}) = \sum_\gamma C^\gamma_{\alpha\beta}(z-w)^{h_\gamma-h_\alpha-h_\beta}(\bar{z}-\bar{w})^{\bar{h}_\gamma-\bar{h}_\alpha-\bar{h}_\beta}\varphi_\gamma(w,\bar{w})\ , \tag{2.55}$$

where $C^\gamma_{\alpha\beta}$ are structure constants, and the summation runs over the set of quasiprimary fields and their derivatives.

If we specialise to the Virasoro case, the OPE of two primary fields φ_I and φ_J can be written as

$$\varphi_I(z,\bar{z})\varphi_J(w,\bar{w}) = \sum_P \sum_{\{k,\bar{k}\}} C^P_{IJ}\frac{\beta^{P,\{k\}}_{IJ}\bar{\beta}^{P,\{\bar{k}\}}_{IJ}\varphi^{\{k,\bar{k}\}}_P(w,\bar{w})}{(z-w)^{h_{IJP}-K}(\bar{z}-\bar{w})^{\bar{h}_{IJP}-\bar{K}}}\ , \tag{2.56}$$

where $\varphi^{\{k,\bar{k}\}}_P$ is a descendant of the primary field φ_P and the multi-index $\{k,\bar{k}\}$ specifies descendant fields by

$$\varphi^{\{k,\bar{k}\}}_P(w,\bar{w}) = V(|v\rangle;w,\bar{w}) \quad \text{with} \quad |v\rangle = L_{-k_1}\ldots L_{-k_n}\bar{L}_{-\bar{k}_1}\ldots\bar{L}_{-\bar{k}_{\bar{n}}}|\varphi_P\rangle\ . \tag{2.57}$$

In addition, $K=\sum_i k_i$, $\bar{K}=\sum_i \bar{k}_i$, and h_{IJK} is the same as in the three-point function (2.21). The summation over descendant fields includes the primary field as well, and denote it by $\varphi^{\{\emptyset,\emptyset\}}_P := \varphi_P$. Then, the couplings to descendant fields are normalised as $\beta^{P,\{\emptyset\}}_{IJ} = \bar{\beta}^{P,\{\emptyset\}}_{IJ} = 1$, so that C^P_{IJ} gives the couplings between the primary fields. The constants C^P_{IJ} are also called OPE structure constants. The constants $\beta^{P,\{k\}}_{IJ}$ and $\bar{\beta}^{P,\{\bar{k}\}}_{IJ}$ can be calculated from the Ward identities as given in Appendix B of [4] or Section 6.6.3 of [56], but the OPE structure constants are yet to be determined. In general, couplings to descendants are not completely fixed by chiral algebras alone which leads to, for example, even and odd fusion rules of the $N=1$ super-Virasoro algebra[16, 24].

OPE structure constants are related to two- and three-point couplings discussed in Subsection 2.1.3. We can express a two-point function (2.19) of primary fields using the OPE (2.56). Since one-point functions vanish except for the identity field, we obtain

$$C^{\mathbf{1}}_{IJ} = d_{IJ}\ . \tag{2.58}$$

Moreover, from (2.20), $C_{II^+}^{\mathbf{1}} = d_I$ is the only non-zero coupling to the identity field. In addition, we can calculate

$$\langle \varphi_I | \varphi_I \rangle = \lim_{w,\bar{w}\to\infty} w^{2h_I} \bar{w}^{2\bar{h}_I} \langle 0 | \varphi_{I^+}(w,\bar{w}) \varphi_I(0,0) | 0 \rangle = d_I \ . \tag{2.59}$$

Therefore, the non-vanishing two-point couplings d_I give the normalisation of fields and corresponding state vectors. In order to relate three-point couplings and OPE structure constants, consider the following correlator obtained from (2.21)

$$\langle \varphi_{I^+} | \varphi_J(z,\bar{z}) | \varphi_K \rangle = \lim_{w,\bar{w}\to\infty} w^{2h_I} \bar{w}^{2\bar{h}_I} \langle \varphi_I(w,\bar{w}) \varphi_J(z,\bar{z}) \varphi_K(0,0) \rangle = \frac{C_{IJK}}{z^{h_{JKI}} \bar{z}^{\bar{h}_{JKI}}} \ . \tag{2.60}$$

Comparing this with the result obtained by evaluating the OPEs

$$\lim_{w,\bar{w}\to\infty} w^{2h_I} \bar{w}^{2\bar{h}_I} \langle \varphi_I(w,\bar{w}) \left(\varphi_J(z,\bar{z}) \varphi_K(0,0) \right) \rangle = \sum_P \frac{d_{IP}\, C_{JK}^P}{z^{h_{JKI}} \bar{z}^{\bar{h}_{JKI}}} \ , \tag{2.61}$$

where we have used $h_P = h_I$ and $\bar{h}_P = \bar{h}_I$, we find

$$C_{IJK} = \sum_P C_{IJ}^P \, d_{PK} \ . \tag{2.62}$$

From (2.20), this simplifies to $C_{IJK} = C_{IJ}^{K^+} C_{K^+K}^{\mathbf{1}}$. If we normalise all the bulk fields as $d_I = 1$, then $C_{IJK} = C_{IJ}^{K^+}$. Furthermore, if all the bulk fields are self-conjugate, there is no need to distinguish up-indices and down-indices; $C_{IJK} = C_{IJ}^K$.

In principle, we can obtain any correlation function by evaluating OPEs if the necessary structure constants are known. So-called completely solvable theories are the ones in which we can obtain all the OPE structure constants.

2.1.8 Crossing Constraints

By considering different ways of expressing a four-point function using OPEs, we can obtain relations among OPE constants. Consider the four-point function $G_{1234}(z,\bar{z})$ given in (2.25). Assuming $0 < |z| \ll 1$, this can be rewritten as

$$\begin{aligned} G_{1234}(z,\bar{z}) &= \langle \varphi_{1^+} | \varphi_2(1,1) \varphi_3(z,\bar{z}) | \varphi_4 \rangle \\ &= \sum_P \sum_{\{k,\bar{k}\}} C_{34}^P \frac{\beta_{34}^{P,\{k\}} \bar{\beta}_{34}^{P,\{\bar{k}\}}}{z^{h_{34P}-K} \bar{z}^{\bar{h}_{34P}-\bar{K}}} \langle \varphi_{1^+} | \varphi_2(1,1) L_{-k_1} \dots L_{-k_n} \bar{L}_{-\bar{k}_1} \dots \bar{L}_{-\bar{k}_{\bar{n}}} | \varphi_P \rangle \ , \end{aligned} \tag{2.63}$$

where we have used the OPE for φ_3 and φ_4. If we define

$$\mathcal{F}_{1^+234}^P(z) := \frac{1}{z^{h_{34P}}} \sum_{\{k\}} z^K \beta_{34}^{P,\{k\}} \frac{\langle \varphi_{1^+} | \varphi_2(1,1) L_{-k_1} \dots L_{-k_n} | \varphi_P \rangle}{\langle \varphi_{1^+} | \varphi_2(1,1) | \varphi_P \rangle} \quad \text{and}$$

$$\bar{\mathcal{F}}_{1^+234}^P(\bar{z}) := \frac{1}{\bar{z}^{\bar{h}_{34P}}} \sum_{\{\bar{k}\}} z^{\bar{K}} \bar{\beta}_{34}^{P,\{\bar{k}\}} \frac{\langle \varphi_{1^+} | \varphi_2(1,1) \bar{L}_{-\bar{k}_1} \dots \bar{L}_{-\bar{k}_{\bar{n}}} | \varphi_P \rangle}{\langle \varphi_{1^+} | \varphi_2(1,1) | \varphi_P \rangle} \ , \tag{2.64}$$

then, the four-point function simplifies to

$$G_{1234}(z,\bar{z}) = \sum_P C_{12P} C^P_{34}\, \mathcal{F}^P_{1+234}(z) \overline{\mathcal{F}}^P_{1+234}(\bar{z}) \ . \tag{2.65}$$

These functions $\mathcal{F}^P_{1234}(z)$ and $\overline{\mathcal{F}}^P_{1234}(\bar{z})$ encoding coordinate dependence of $G_{1234}(z,\bar{z})$ are called four-point conformal blocks. By convention, four-point blocks are normalised to give

$$\mathcal{F}^P_{1234}(z) = \frac{1}{z^{h_{34P}}}(1+\mathcal{O}(z)) \tag{2.66}$$

when $z \to 0$. In the Virasoro case, three-point blocks inside a four-point block are determined up to an overall constant, therefore we can normalise four-point blocks as above. In general, this is not true in other chiral algebras, for example chiral three- and four-point blocks of the W_3 algebra are discussed in [48].

Under any Möbius transformation, the right hand side of (2.25) should be invariant. If we consider a transformation $z \mapsto 1-z$, it exchanges the order of φ_2 and φ_4 in the correlation function, and we obtain

$$G_{1234}(z,\bar{z}) = (-1)^{3\Delta_1+\Delta_2+\Delta_3+\Delta_4+4(s_2s_3+s_2s_4+s_3s_4)} G_{1432}(1-z, 1-\bar{z}) \ , \tag{2.67}$$

where the sign comes from the Jacobian factors and reordering of operators[9] due to the radial ordering (2.34) with $\epsilon_{IJ} = (-1)^{4s_I s_J}$. In terms of the four-point conformal blocks (2.64), this means that we are changing the bases of four-point blocks from the one with $z \sim 0$ to the other with $1-z \sim 1$. This linear transformation can be written as

$$\mathcal{F}^p_{ijkl}(z) = \sum_q \mathsf{F}_{pq}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \mathcal{F}^q_{ilkj}(1-z) \ , \tag{2.68}$$

where F is called the fusing matrix[10]. As it is clear from the way four-point blocks are constructed in (2.64), the lower case Roman letters, $i, j, k, \ldots$ label Virasoro representations, and the fusing matrices in (2.68) are for the Virasoro algebra cases. As we shall see later, we can similarly define four-point blocks and fusing matrices for other chiral algebras.

2.2 Chiral Algebras

Chiral fields are fields that do not depend on the antiholomorphic coordinates. Other than the identity field, there is at least one chiral field in any CFT with $c \neq 0$, which is the component of the stress-energy tensor $T(z)$. The set of chiral fields is closed under OPEs, and the modes of chiral fields form what is called the chiral algebra, which we denote by $\mathcal{A}$. The chiral algebra $\bar{\mathcal{A}}$ for the antiholomorphic coordinates is defined similarly. If a

9. In the most literature, this sign is missing, however this agrees with [42].

10. So far, we have used $\mathcal{F}^P_{IJKL}(z)$ and $\overline{\mathcal{F}}^P_{IJKL}(\bar{z})$ to denote four-point blocks, however, since they are chiral quantities, we may also write $\mathcal{F}^p_{ijkl}(z) := \mathcal{F}^P_{IJKL}(z)$ and $\mathcal{F}^{\bar{p}}_{\bar{i}\bar{j}\bar{k}\bar{l}}(\bar{z}) := \overline{\mathcal{F}}^P_{IJKL}(\bar{z})$, where it is understood $I = (i, \bar{i})$, etc. We assume the same convention for other chiral quantities, for example, $h_i := h_I$ and $h_{\bar{i}} := \bar{h}_I$. In addition, by writing $I = (i, \bar{i})$, we are assuming that the bulk fields can be uniquely labelled by the chiral algebra representation labels i and $\bar{i}$, and the fusion coefficients $N^k_{ij} \in \{0, 1\}$, which may not be always true. In such cases, we may write $I_\alpha = (i, \bar{i}; \alpha)$, where α is the multiplicity label, in order to avoid ambiguities.

theory does not contain chiral fields other than $T(z)$, then $\mathcal{A} = \mathcal{V}\mathrm{ir}$. So-called W-algebras are the extensions of the Virasoro algebra by chiral primary fields of integer or half-integer conformal weights, and they describe extended symmetries of CFTs.

As we have seen in the previous section, the space of bulk states (2.54) consists of tensor products of $\mathcal{A}$ and $\overline{\mathcal{A}}$ irreducible modules. Therefore, we are interested in irreducible highest weight modules of $\mathcal{A}$ and operators acting on them.

2.2.1 Representations of Virasoro Algebra

In this subsection, we follow the book[20] by Kac and Raina for the analysis of representations of the Virasoro algebra.

A highest weight representation of the Virasoro algebra is a complex vector space $\mathcal{H}$ with a non-zero vector $|h\rangle$, which is called the highest weight vector, and the representation map $\rho : \mathcal{V}\mathrm{ir} \to \mathrm{End}(\mathcal{H})$. The highest weight vector satisfies

$$\rho(L_0)|h\rangle = h|h\rangle \quad \text{and} \quad \rho(c)|h\rangle = c|h\rangle \; . \tag{2.69}$$

Furthermore, $\mathcal{H}$ is the linear span of vectors of the form

$$\rho(L_{n_1})\rho(L_{n_2})\cdots\rho(L_{n_k})|h\rangle \quad \text{with} \quad n_1 \le n_2 \le \cdots \le n_k < 0 \; , \tag{2.70}$$

which implies $\rho(L_n)|h\rangle = 0$ for $n > 0$.

If all the vectors of the form (2.70) are linearly independent, the highest weight representation is called a Verma representation. Verma modules can be constructed as universal highest weight representations of $\mathcal{V}\mathrm{ir}$. The universal enveloping algebra $\mathrm{U}(\mathcal{V}\mathrm{ir})$ of the Virasoro algebra is an associative algebra with unit, whose elements are formal power series in the elements of $\mathcal{V}\mathrm{ir}$ with the identification $[x, y] = xy - yx$ for all $x, y \in \mathcal{V}\mathrm{ir}$. Let $\mathfrak{b}$ denote a Borel subalgebra of $\mathcal{V}\mathrm{ir}$ given by

$$\mathfrak{b} := \bigoplus_{n\in\mathbb{Z}_{\ge 0}} \mathbb{C}L_n \oplus \mathbb{C}c \; . \tag{2.71}$$

Since $\mathrm{U}(\mathcal{V}\mathrm{ir})$ is a bimodule of itself, it is a right $\mathrm{U}(\mathfrak{b})$-module as well. Then, we can construct a left $\mathrm{U}(\mathcal{V}\mathrm{ir})$-module M as the induced module

$$\mathrm{M} = \mathrm{Ind}_{\mathfrak{b}}^{\mathcal{V}\mathrm{ir}}(\mathrm{B}) := \mathrm{U}(\mathcal{V}\mathrm{ir}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathrm{B} \; , \tag{2.72}$$

where B is a one-dimensional left $\mathrm{U}(\mathfrak{b})$-module constructed form the vector $|h\rangle$, which satisfies

$$L_0|h\rangle = h|h\rangle \; , \quad \text{and} \quad L_n|h\rangle = 0 \quad \text{for all} \quad n > 0 \; . \tag{2.73}$$

As a left $\mathrm{U}(\mathcal{V}\mathrm{ir})$-module, M is a representation of the Virasoro algebra, and it is spanned by vectors of the form

$$L_{n_1}L_{n_2}\cdots L_{n_k}|h\rangle \quad \text{with} \quad n_1 \le n_2 \le \cdots \le n_k < 0 \; , \tag{2.74}$$

that are linearly independent from the Poincaré–Birkhoff–Witt theorem. Therefore, M is a Verma representation of the Virasoro algebra.

Using Hermitian conjugation defined in (2.31), an "inner product"[11] $\langle \cdot | \cdot \rangle : \mathrm{M} \times \mathrm{M} \to \mathbb{C}$ on the Verma module M can be defined as

$$\langle v|w\rangle := \langle h|L_{-n_k} \cdots L_{-n_1} L_{m_1} \cdots L_{m_l}|h\rangle \ , \tag{2.75}$$

where

$$v = L_{n_1} \cdots L_{n_k}|h\rangle \ , \quad w = L_{m_1} \cdots L_{m_l}|h\rangle \quad \in \mathrm{M} \ . \tag{2.76}$$

If a highest weight module $\mathcal{H}$ is unitary, that is, $\langle v|v\rangle > 0$ for all non-zero $v \in \mathcal{H}$, then it is irreducible; since every vector in $\mathcal{H}$ cannot be orthogonal to itself and repeated actions of L_n with $n < 0$ on the highest weight vector $|h\rangle$ generate $\mathcal{H}$, there is no non-trivial invariant subspace of $\mathcal{H}$.

Verma modules can be decomposed into the L_0-eigenspaces as

$$\mathrm{M} = \bigoplus_{N \in \mathbb{Z}_{\geq 0}} \mathrm{M}_N \ , \tag{2.77}$$

where M_N is spanned by vectors of the form (2.74) with $n_1 + \cdots + n_k = -N$ and it has the L_0-eigenvalue of $h + N$. These subspaces are mutually orthogonal with respect to the inner product (2.75). From (2.74), the dimension of M_N is given by the number $p(N)$ of the integer partitions of N. This decomposition is useful as each M_N is a finite-dimensional vector space even though M is infinite-dimensional.

One of the important properties of Verma modules is that, for a given weight (c, h), the Verma representation is unique, and any other highest weight module with the same highest weight can be obtained by a quotient of the Verma module. In addition, a Verma module M is indecomposable, that is, there are no non-trivial submodules V and W such that $\mathrm{M} = \mathrm{V} \oplus \mathrm{W}$, and the Verma module has a unique maximal proper submodule J. An irreducible highest weight module is given by the quotient $\mathcal{H} = \mathrm{M}/\mathrm{J}$. Since M is indecomposable, any $v \in \mathrm{J}$ must be orthogonal to every vector in M including v itself. Such a vector v is called a null vector. From this observation, we can identify the maximal proper submodule J as

$$\mathrm{J} = \ker \langle \cdot | \cdot \rangle \ , \tag{2.78}$$

where

$$\ker \langle \cdot | \cdot \rangle := \{ v \in \mathrm{M} : \langle v|w\rangle = 0 \quad \forall w \in \mathrm{M} \} \ . \tag{2.79}$$

Since L_0-eigenspaces are mutually orthogonal, it suffices to consider the Gram matrix of a subspace M_N, whose elements are given by

$$(\Gamma_N)_{ij} := \langle v_i | v_j \rangle \ , \tag{2.80}$$

where $v_i, v_j \in \mathrm{M}_N$ are the basis vectors of the form (2.74) and $i, j \in \{1, 2, \ldots, \dim \mathrm{M}_N\}$. If J is not trivial, $\det \Gamma_N$ becomes zero at certain level N. For the first two levels, $\det \Gamma_N$ can be calculated as

$$\det \Gamma_1 = 2h \quad \text{and} \tag{2.81}$$

$$\det \Gamma_2 = 2h \left((4h-1)^2 + (2h+1)(c-1) \right) \ . \tag{2.82}$$

11. This "inner product" is not necessarily positive-definite. To be mathematically precise, this should be called an Hermitian form[20].

Recall that $h = 0$ corresponds to the vacuum module, and the above calculations show that the basis vector $L_{-1}|h\rangle$ of M_1 is null if $h = 0$, from which we recover $\mathrm{sl}(2)$ invariance of the vacuum vector. If M has a null vector at level 2, we obtain a polynomial relation in h and c, which constrains their values. In general, $\det \Gamma_N$ is given by the Kac determinant formula

$$\det \Gamma_N = K_N \prod_{\substack{r,s\in\mathbb{Z}_{>0} \\ rs\leq N}} \left(h - h_{r,s}\right)^{p(N-rs)} \ , \tag{2.83}$$

where $h_{r,s}$ is a function of c, and K_N is a positive constant given by

$$K_N = \prod_{\substack{r,s\geq 1 \\ rs\leq N}} \left((2r)^s s!\right)^{p(N-rs)-p(l-r(s+1))} \ , \tag{2.84}$$

which depends only on N. By analysing the explicit expression for $h_{r,s}$, it can be shown that Verma representations of the Virasoro algebra are unitary for $c > 1$ and $h > 0$, which means they are irreducible for this range. In addition, irreducible highest weight modules are unitary for $c \geq 1$ and $h \geq 0$. For $c < 1$, it is useful to introduce the parameters $p, q \in \mathbb{Z}$ that are coprime and satisfy $1 < p < q$, and write

$$c(p,q) = 1 - \frac{6(q-p)^2}{pq} \quad \text{and} \quad h_{r,s} = \frac{(qr-ps)^2 - (q-p)^2}{4pq} \ , \tag{2.85}$$

where $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$. Then, for a given $c = c(p,q)$, a Verma module with $h = h_{r,s}$ has the first null vector at level $N = rs$, and therefore it is reducible. Unless $q = p + 1$, the irreducible quotients corresponding to the highest weights given in (2.85) are not unitary. The Virasoro characters (A.1) of Verma modules and irreducible highest weight modules encode their vector space structures, and they are discussed in Appendix A.1.

The presence of null vectors in a Verma module gives rise to differential equations for correlation functions. For example, consider the level 2 null vector in a Verma module with $h = h_{1,2}$. In general, an explicit expression for the level $N = rs$ null vector $|\chi_{r,s}\rangle$ can be obtained by writing

$$|\chi_{r,s}\rangle = \left((L_{-1})^N + \alpha_1 \, L_{-2}(L_{-1})^{N-2} + \cdots + \alpha_k \, L_{-N}\right)|h_{r,s}\rangle \ , \tag{2.86}$$

where $k = \dim \mathrm{M}_N - 1$, and solving $L_n|\chi_{r,s}\rangle = 0$ for $1 \leq n \leq N$ in order to determine the coefficients. For this case, we get

$$|\chi_{1,2}\rangle = \left((L_{-1})^2 - \frac{2}{3}(2h_{1,2}+1)L_{-2}\right)|h_{1,2}\rangle \ . \tag{2.87}$$

Consider a three-point function (2.60) of bulk primary fields φ_I, φ_J, and φ_K with $h_K = h_{1,2}$. Then, the existence of the level 2 null vector gives

$$\begin{aligned} 0 &= \langle \varphi_{I^+}|\varphi_J(z,\bar{z}) \left((L_{-1})^2 - \frac{2}{3}(2h_{1,2}+1)L_{-2}\right)|\varphi_K\rangle \\ &= \left(\partial^2 - \frac{2}{3}(2h_{1,2}+1)(h_J z^{-2} - z^{-1}\partial)\right)\langle \varphi_{I^+}|\varphi_J(z,\bar{z})|\varphi_K\rangle \ . \end{aligned} \tag{2.88}$$

Substituting in (2.60), we obtain a constraint on h_I and h_J for non-vanishing C_{IJK}

$$h_I = \frac{1}{6}\left(1 + 6h_J + 2h_{1,2} \pm \sqrt{(1-4h_{1,2})^2 - 24h_J(1+2h_{1,2})}\right) \; . \tag{2.89}$$

If $h_J = h_{r,s}$ as in (2.85), the above equation determines h_I to be $h_{r,s-1}$ or $h_{r,s+1}$. Since C_{IJK} has to be non-vanishing in order for C^I_{JK} to be non-zero, this result gives a fusion rule

$$(r,s) \otimes (1,2) = (r,s-1) \oplus (r,s+1) \; , \tag{2.90}$$

which can be interpreted as the OPE of two primary fields with holomorphic conformal weights $h_{r,s}$ and $h_{1,2}$ only involves the primary fields with $h_{r,s-1}$ and $h_{r,s+1}$, and their descendants. Similarly considering other null vectors, the fusion rules for the Virasoro primary fields with h_{r_1,s_1} and h_{r_2,s_2} are given by

$$(r_1,s_1) \otimes (r_2,s_2) = \bigoplus_{\substack{r_3=1+|r_1-r_2| \\ r_1+r_2+r_3\in 1+2\mathbb{Z}}}^{r_{3\,\max}} \; \bigoplus_{\substack{s_3=1+|s_1-s_2| \\ s_1+s_2+s_3\in 1+2\mathbb{Z}}}^{s_{3\,\max}} (r_3,s_3) \; , \tag{2.91}$$

where

$$\begin{aligned} r_{3\,\max} &= \min(r_1+r_2-1,\; 2p-1-r_1-r_2) \quad \text{and} \\ s_{3\,\max} &= \min(s_1+s_2-1,\; 2q-1-s_1-s_2) \; . \end{aligned} \tag{2.92}$$

This shows that, for a given $c = c(p,q)$, the set of bulk primary fields with $h = h_{r,s}$ and $\bar{h} = h_{r',s'}$, where the range of (r,s) and (r',s') is given by (2.85), is closed under fusion. A theory constructed from the set of conformal weights $h_{r,s}$ given in (2.85) and the central charge $c = c(p,q)$ is called a Virasoro minimal model $M(p,q)$. As the number of bulk primary fields in $M(p,q)$ is finite, it has the space of bulk states given by (2.54) wherein the irreducible highest weight modules can be labelled by the Kac labels (r,s). Since $h_{r,s} = h_{p-r,q-s}$, we have the identification $(r,s) \sim (p-r,q-s)$.

For four-point functions, consider a conformal block $\mathcal{F}^p_{ijkl}(z)$, which was defined in (2.64). It can be non-zero if $N^i_{jp}N^p_{kl} \neq 0$, where the fusion coefficient $N^k_{ij} \in \mathbb{Z}_{\geq 0}$ counts how many times the representation labelled by k appear in the fusion rule for $i \otimes j$. Furthermore, if $l = (r,s)$, we can use the null vector $|\chi_{r,s}\rangle$ at level $N = rs$, and obtain an N-th order homogeneous linear ordinary differential equation for $\mathcal{F}^p_{ijkl}(z)$. For example, if it involves a null vector at level 2, we can solve the differential equation and express $\mathcal{F}^p_{ijkl}(z)$ in terms of hypergeometric functions.

2.2.2 W-Algebras

Let $\mathcal{W}(2, h_1, \ldots, h_N)$ denote an extension of the Virasoro algebra by chiral Virasoro primary fields $W^{(i)}(z)$ of integer or half-integer conformal weights h_i. We are interested in various OPEs involving T and $W^{(i)}$. Since $W^{(i)}$ is a Virasoro primary field, $TW^{(i)}$ OPEs are given by

$$T(z)W^{(i)}(w) = \frac{h_i W^{(i)}(w)}{(z-w)^2} + \frac{\partial W^{(i)}(w)}{z-w} + \text{reg.} \; . \tag{2.93}$$

Using the mode expansion

$$W^{(i)}(z) = \sum_{n\in\mathbb{Z}-h_i} W_n^{(i)} z^{-n-h_i} \quad \text{and} \quad W_n^{(i)} = \frac{1}{2\pi i}\oint_0 W^{(i)}(z)\, z^{n+h_i-1}\,\mathrm{d}z\ , \tag{2.94}$$

the OPE (2.93) is equivalent to the commutation relation

$$[L_n, W_m^{(i)}] = (n(h_i-1)-m)W_{n+m}^{(i)}\ . \tag{2.95}$$

Before working out the remaining OPEs, we need a few more ingredients.

The space of states $\mathcal{H}_0^{\mathcal{W}}$, on which $T(z)$ and $W^{(i)}$ act, contains the vacuum vector $|0\rangle$, and it is spanned by vectors of the form

$$W_{-l_c}^{(N)}\cdots W_{-l_1}^{(N)}\cdots\cdots W_{-m_b}^{(1)}\cdots W_{-m_1}^{(1)} L_{-n_a}\cdots L_{-n_1}|0\rangle \tag{2.96}$$

with $2\leq n_1\leq\cdots\leq n_a$, $h_1\leq m_1\leq\cdots\leq m_b$ (or $h_1\leq m_1<\cdots<m_b$ if h_1 is half-integer), and so on. Since null fields decouple from a theory, $\mathcal{H}_0^{\mathcal{W}}$ should be the vacuum irreducible highest weight module of $\mathcal{W}(2,h_1,\ldots,h_N)$, however it is sometimes useful to keep the null vectors and consider the Verma module $\mathrm{M}_0^{\mathcal{W}}$. From the level N subspace $(\mathrm{M}_0^{\mathcal{W}})_N$ of the Verma module, the subspace $(\mathcal{H}_0^{\mathcal{W}})_N$ of the irreducible module can be obtained by taking the maximal number of basis vectors (2.96) which makes the determinant of the Gram matrix non-zero.

From the state-field correspondence, the vertex operator $V:\mathcal{H}_0^{\mathcal{W}}\times\mathbb{C}\to\mathrm{End}(\mathcal{H}_0^{\mathcal{W}})$ maps a state $|\psi\rangle\in\mathcal{H}_0^{\mathcal{W}}$ to a field $V(|\psi\rangle,z):=\psi(z)$. The vertex operator has to satisfy the following conditions[44, 60]:

(1) $V(|\psi\rangle,z)|0\rangle = e^{zL_{-1}}|\psi\rangle$,

(2) $\langle\psi_1|V(|\psi\rangle,z)|\psi_2\rangle$ is a meromorphic function of z,

(3) $\langle\psi_1|V(|\psi\rangle,z)V(|\chi\rangle,w)|\psi_2\rangle$ is a holomorphic function for $|z|>|w|$, and

(4) $\langle\psi_1|V(|\psi\rangle,z)V(|\chi\rangle,w)|\psi_2\rangle = \epsilon_{\psi\chi}\langle\psi_1|V(|\chi\rangle,w)V(|\psi\rangle,z)|\psi_2\rangle$ by analytic continuation. (2.97)

Condition (1) tells us that L_{-1} is the infinitesimal translation operator, and we have $V(L_{-1}|\psi\rangle,z)=\partial_z V(|\psi\rangle,z)$. Using these axioms, one can derive the duality relation

$$V(|\psi\rangle,z)V(|\phi\rangle,w) = V(V(|\psi\rangle,z-w)|\phi\rangle,w)\ , \tag{2.98}$$

from which we can obtain the operator product expansion. For a chiral field $V(|\psi\rangle,z)=\psi(z)$, its mode expansion[12] is given by

$$\psi(z) = \sum_{n\in\mathbb{Z}-h_\psi}\psi_n z^{-n-h_\psi} \quad \text{and} \quad \psi_n = \frac{1}{2\pi i}\oint_0 \psi(z)\, z^{n+h_\psi-1}\,\mathrm{d}z\ . \tag{2.99}$$

12. Since $\psi_n\in\mathrm{End}(\mathcal{H}_0^{\mathcal{W}})$, the vertex operator $V(\,\cdot\,,z)$ is a map to the formal Laurent series $\mathrm{End}(\mathcal{H}_0^{\mathcal{W}})[\![z^{\pm1}]\!]$.

We also use the notation $V_n(|\psi\rangle) = \psi_n$. Then, using the duality relation (2.98) and the mode expansion, the OPE of $\psi(z)$ and $\phi(w)$ can be written as

$$\psi(z)\phi(w) = \sum_{\substack{n\in\mathbb{Z}-h_\psi \\ n\leq h_\phi}} V(\psi_n|\phi\rangle, w)\,(z-w)^{-n-h_\psi} \;. \tag{2.100}$$

The normal ordered product of $\psi(z)$ and $\phi(w)$ is defined as the constant term in the OPE (2.100), that is

$$(\psi\phi)(z) = V(\psi_{-h_\psi}|\phi\rangle, z) \;, \tag{2.101}$$

whose modes are given[13] by

$$(\psi\phi)_n = \sum_{\substack{m\in\mathbb{Z}-h_\psi \\ m\leq -h_\psi}} \psi_m\phi_{n-m} + \epsilon_{\psi\phi} \sum_{\substack{m\in\mathbb{Z}-h_\psi \\ m> -h_\psi}} \phi_{n-m}\psi_m \;, \tag{2.102}$$

where $n \in \mathbb{Z} - h_\psi - h_\phi$. The supercommutator of the modes ψ_n and ϕ_m is related to the OPE (2.100) since

$$\begin{aligned}
[\psi_n, \phi_m] &\coloneqq \psi_n\phi_m - \epsilon_{\psi\phi}\,\phi_m\psi_n \\
&= \Bigg(\underbrace{\oint_0 \frac{\mathrm{d}z}{2\pi i}\oint_0 \frac{\mathrm{d}w}{2\pi i}}_{|z|>|w|} - \underbrace{\oint_0 \frac{\mathrm{d}w}{2\pi i}\oint_0 \frac{\mathrm{d}z}{2\pi i}}_{|w|>|z|} \Bigg) z^{n+h_\psi-1} w^{m+h_\phi-1}\psi(z)\phi(w) \\
&= \oint_0 \frac{\mathrm{d}w}{2\pi i}\oint_w \frac{\mathrm{d}z}{2\pi i}\, z^{n+h_\psi-1} w^{m+h_\phi-1}\psi(z)\phi(w) \;,
\end{aligned} \tag{2.103}$$

where we have used the definition of modes (2.99) and deformed the contour for integration over z.

Using the form of OPEs given in (2.100), we can write the OPE of $W^{(i)}(z)$ and $W^{(j)}(w)$ as

$$W^{(i)}(z)W^{(j)}(w) = \sum_{\substack{n\in\mathbb{Z} \\ n\leq h_i+h_j}} V(W^{(i)}_{n-h_i}W^{(j)}_{-h_j}|0\rangle, w)\,(z-w)^{-n} \;, \tag{2.104}$$

in which the terms with $n \leq 0$ are regular as $z - w \to 0$. Evaluating the contour integral (2.103) with the OPE (2.104), the supercommutator of $W^{(i)}_n$ and $W^{(j)}_m$ is obtained as

$$[W^{(i)}_n, W^{(j)}_m] = \sum_{k=1}^{h_i+h_j} \binom{n+h_i-1}{n+h_i-k} V_{n+m}(W^{(i)}_{k-h_i}W^{(j)}_{-h_j}|0\rangle) \;. \tag{2.105}$$

In order to determine the fields appearing in the OPE (2.104) and the supercommutator (2.105), we need to express $W^{(i)}_{n-h_i}W^{(j)}_{-h_j}|0\rangle$, where $1 \leq n \leq h_i + h_j$, in terms of the basis vectors (2.96). Since it has the conformal weight $h_i + h_j - n$, we only need the basis vectors in the level $N = h_i + h_j - n$ subspace $(\mathcal{H}^{\mathcal{W}}_0)_N$ where $0 \leq N \leq h_i + h_j - 1$. Let $|l, N\rangle \in (\mathcal{H}^{\mathcal{W}}_0)_N$ be a basis vector of the form (2.96) and $1 \leq l \leq \dim(\mathcal{H}^{\mathcal{W}}_0)_N$, and write

$$W^{(i)}_{n-h_i}W^{(j)}_{-h_j}|0\rangle = \sum_l \alpha^{i,j;N}_l |l, N\rangle \;. \tag{2.106}$$

13. This definition of normal ordering in terms of the modes results in normal ordered products that are not necessary quasiprimary. There is a way to define ‘quasiprimary normal ordered products’ which can be found, for example, in [38].

We need to determine the coefficients $\alpha_l^{i,j;N}$. By taking the normalisation $\langle 0|0\rangle = 1$, the coefficient at level 0 is given by

$$\alpha_1^{i,j;0} = \langle 0|W_{h_j}^{(i)} W_{-h_j}^{(j)}|0\rangle \ , \tag{2.107}$$

which vanishes for $h_i > h_j$. In addition, from symmetry or antisymmetry of the supercommutator (2.105), this coefficient should vanish for $h_i < h_j$. Therefore, $\alpha_1^{i,j;0}$ is non-zero only for $h_i = h_j$. Since $\alpha_1^{i,i;0}$ gives the normalisation of the field $W^{(i)}(z)$ and it is arbitrary, we take $\alpha_1^{i,i;0} = d_{W^{(i)}} = \frac{c}{h_i}$ by convention. In general, other coefficients can be determined by acting L_m with $0 \le m \le N$ on the both sides of (2.106) and using the commutators (2.95), by checking symmetry or antisymmetry of the supercommutator (2.105), and by demanding the Jacobi identity to hold for the supercommutator. It turns out that these conditions force some W-algebras to exist only for certain values of the central charges.

If a W-algebra has an outer automorphism Ω, it is possible to impose boundary conditions on the generators $W^{(i)}(z)$ as

$$W^{(i)}(e^{2\pi i}z) = \Omega(W^{(i)}(z)) \ , \tag{2.108}$$

which results in so-called twisted representations of the W-algebra. For simplicity, consider $\mathcal{W}(2,\delta)$ with the generator $W(z)$ of conformal dimension $\delta \in \frac{1}{2}\mathbb{Z}_{>0}$ and the vanishing self-coupling constant $C_{WW}^{W} = 0$. In this case, the outer automorphism Ω is given by

$$\Omega(T(z)) = T(z) \quad \text{and} \quad \Omega(W(z)) = -W(z) \ , \tag{2.109}$$

which is an involution. Due to the condition (2.108) and the automorphism (2.109), the mode expansion of $W(z)$ becomes

$$W(z) = \sum_{n\in\mathbb{Z}-\delta+\frac{1}{2}} W_n z^{-n-\delta} \ . \tag{2.110}$$

Then, the OPE of $W(z)$ with another chiral field $\phi(w)$ becomes

$$W(z)\phi(w) = \sum_{\substack{n\in\mathbb{Z}+\frac{1}{2} \\ n\le\delta+h_\phi}} V(W_{n-\delta}|\phi\rangle, w)\,(z-w)^{-n} \ , \tag{2.111}$$

which picks up the phase factor of -1 when $W(z)$ is rotated by 2π around w. Twisted representations are constructed from the modes (2.110), and corresponding primary fields are non-local with respect to $W(z)$ as in (2.111). In general, an automorphism of the kind (2.109) exists if and only if the self-coupling C_{WW}^{W} vanishes[46]. If $W(z)$ is fermionic, its self-coupling vanishes due to the fermion number conservation, and the twisted sector is called the Ramond sector.

For bosonic $\mathcal{W}(2,\delta)$ algebras, the zero modes L_0, W_0, and $\hat{c}$ form the Cartan subalgebra. Therefore, a highest weight vector is characterised by the L_0-eigenvalue h and W_0-eigenvalue w, and it should satisfy

$$L_n|h,w\rangle = 0 \quad \text{and} \quad W_n|h,w\rangle = 0 \quad \text{for all} \quad n > 0 \ . \tag{2.112}$$

For Ramond sectors of fermionic $\mathcal{W}(2,\delta)$ algebras, highest weight vectors are characterised similarly while odd generators of a superalgebra cannot be included in its Cartan subalgebra in general[14].

In the following, we summarise a few examples of W-algebras that are relevant to the discussion of this thesis.

• $\mathcal{W}(2,2)$

In $\mathcal{W}(2,2)$, there is the weight 2 chiral primary field $W(z)$ in addition to the stress-energy tensor $T(z)$. Their modes satisfy

$$[L_n, W_m] = (n-m)W_{n+m} \quad \text{and}$$
$$[W_n, W_m] = \frac{1}{2}C_{WW}^{W}(n-m)W_{n+m} + (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.113}$$

as well as the usual Virasoro relations (2.46). In (2.113), C_{WW}^{W} is a free parameter, and $\mathcal{W}(2,2)$ exists for generic values of c.

It is possible to express $\mathcal{W}(2,2)$ as $\mathcal{V}\mathrm{ir} \oplus \mathcal{V}\mathrm{ir}$. Let

$$T(z) = T^{(1)}(z) + T^{(2)}(z) \quad \text{and} \quad W(z) = \alpha\left(\sqrt{\frac{c_2}{c_1}}\,T^{(1)}(z) - \sqrt{\frac{c_1}{c_2}}\,T^{(2)}(z)\right) , \tag{2.114}$$

where $\alpha = \pm 1$ is arbitrary. Straightforward calculations show that $W(z)$ is primary with respect to $T(z)$, and that they are normalised properly; $d_T = d_W = \frac{c}{2}$ where $c = c_1 + c_2$. Using the definition given in (2.114), one can calculate that the modes L_n and W_n satisfy the commutators (2.113) with

$$C_{WW}^{W} = 2\alpha\,\frac{c_2 - c_1}{\sqrt{c_1 c_2}} . \tag{2.115}$$

When $c_1 = c_2$, this self-coupling constant vanishes and there is an outer automorphism given by (2.109), which gives rise to the twisted sector of $\mathcal{W}(2,2)$.

• $N = 1$ **Super-Virasoro Algebra**

The $N = 1$ super-Virasoro algebra, which we denote by $\mathcal{SV}\mathrm{ir}$, is the extension of the Virasoro algebra by a chiral primary field $G(z)$ of weight $\frac{3}{2}$. Together with the usual bosonic stress-energy tensor $T(z)$, $G(z)$ forms the super-stress-energy tensor[9]

$$\mathbf{T}(Z) = \frac{1}{2}G(z) + \theta T(z) , \tag{2.116}$$

where $Z := (z,\theta)$ is a superspace coordinate with θ being a Grassmann variable. The commutators and anticommutators involving the modes G_n are

$$[L_n, G_m] = \left(\frac{n}{2} - m\right) G_{n+m} \quad \text{and}$$
$$\{G_n, G_m\} = 2L_{n+m} + \frac{c}{3}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} . \tag{2.117}$$

14. On a $\mathbb{Z}_2$-graded representation space $\mathcal{H} = \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}}$, an odd operator can be regarded as an off-diagonal matrix $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$, where $A \in \mathrm{Hom}(\mathcal{H}^{\bar{1}}, \mathcal{H}^{\bar{0}})$ and $B \in \mathrm{Hom}(\mathcal{H}^{\bar{0}}, \mathcal{H}^{\bar{1}})$. Therefore, if an odd operator acts diagonally on some representation space, then this space is not $\mathbb{Z}_2$-graded by fermion parity in general.

The untwisted sector is also called the Neveu–Schwarz sector in which the modes G_n take $n \in \mathbb{Z} + \frac{1}{2}$. In the Ramond sector, G_n has integer n. In the Neveu–Schwarz sector, L_0, $L_{\pm 1}$, and $G_{\pm\frac{1}{2}}$ form the Lie superalgebra $\mathrm{osp}(1|2)$ which is a graded extension of $\mathrm{sl}(2)$ and corresponds to the global superconformal group[9] on a supermanifold $\mathbb{CP}^{1|1}$.

A representation space $\mathcal{H}$ of $\mathcal{SV}\mathrm{ir}$ is $\mathbb{Z}_2$-graded, and it can be written as $\mathcal{H} = \mathcal{H}^{\bar{0}} \oplus \mathcal{H}^{\bar{1}}$, where $\mathcal{H}^{\bar{0}}$ is the subspace formed by bosonic states and $\mathcal{H}^{\bar{1}}$ corresponds to fermionic states. Note that the vacuum state $|0\rangle$ is defined to be bosonic. So-called the fermion parity operator $(-1)^F$ is defined to act as $\mathbf{1}$ on $\mathcal{H}^{\bar{0}}$ and $-\mathbf{1}$ on $\mathcal{H}^{\bar{1}}$, from which we can deduce that $(-1)^F$ commutes with bosonic fields and anticommutes with fermionic fields. Using the fermion parity operator, actions of the outer automorphism Ω_F of $\mathcal{SV}\mathrm{ir}$, under which $G(z) \mapsto -G(z)$, can be written as

$$\rho(\Omega_F(x))|0\rangle = (-1)^F \rho(x)|0\rangle \ , \tag{2.118}$$

where $x \in \mathcal{SV}\mathrm{ir}$ and $\rho : \mathcal{SV}\mathrm{ir} \to \mathrm{End}(\mathcal{H})$ is the representation map.

Similar to the Virasoro case, reducible Verma modules of the $N = 1$ super-Virasoro algebra occur at values of c and h that can be parametrised[15] as

$$c(p,q) = \frac{3}{2}\left(1 - \frac{2(q-p)^2}{pq}\right) \quad \text{and} \quad h_{r,s} = \frac{(qr-ps)^2 - (q-p)^2}{8pq} + \frac{1}{32}\left(1 - (-1)^{r+s}\right) \ , \tag{2.119}$$

where $1 < p < q$ should satisfy

$$\begin{aligned} &p, q \in \mathbb{Z} \ , \quad p \text{ and } q \text{ coprime} \ , \quad p+q \in 2\mathbb{Z} \ , \quad \text{or} \\ &p, q \in 2\mathbb{Z} \ , \ \frac{p}{2} \text{ and } \frac{q}{2} \text{ coprime} \ , \ \frac{p}{2} + \frac{q}{2} \notin 2\mathbb{Z} \ , \end{aligned} \tag{2.120}$$

and we have $1 \le r \le p-1$ and $1 \le s \le q-1$. Note that p and q must be either both odd or both even. For these ranges of r and s, we can form a table containing the values of $h_{r,s}$ using (2.119) which is called the Kac table. Representations labelled by $h_{r,s}$ with $r+s \in 2\mathbb{Z}$ are in the Neveu–Schwarz sector, and those with $r+s \in 2\mathbb{Z}+1$ are in the Ramond sector. An $N = 1$ super-Virasoro minimal model $SM(p,q)$ is constructed from the representations with $c = c(p,q)$ and $h = h_{r,s}$ given in (2.119). When $q = p+2$, the irreducible quotients of the highest weight representations with $c(p,q)$ and $h_{r,s}$ are unitary. As in the Virasoro cases, we have $h_{r,s} = h_{p-r,q-s}$ which leads to the identification of Kac labels $(r,s) \sim (p-r, q-s)$. Note that, unlike the Virasoro cases, it is possible to have $p, q \in 2\mathbb{Z}$ in which case there is a representation labelled by the Kac label $(\frac{p}{2}, \frac{q}{2})$ which is the fixed point of this identification. Moreover, this fixed point is in the Ramond sector and has $h = \frac{c}{24}$, which is the lowest in this sector.

Highest weight representations of $\mathcal{SV}\mathrm{ir}$ in the Neveu–Schwarz sector are very similar to those of the Virasoro algebra. A highest weight vector $|h\rangle$ is characterised by its L_0-eigenvalue h and the fermion parity $\epsilon = \pm 1$, that is $(-1)^F|h\rangle = \epsilon|h\rangle$. For the values of

15. The formula for unitary cases can be found in [12]. The more general expression quoted here which includes non-unitary cases can be found, for example, in [50].

c and h parametrised by (2.119), Verma modules are reducible, and irreducible highest weight modules can be obtained by taking their quotients. Their Virasoro characters are given in Appendix A.1.

On the other hand, representations of $\mathcal{SV}\mathrm{ir}$ in the Ramond sector are more involved. Since there is the zero mode G_0, a highest weight vector can be taken as an eigenvector of G_0. From the $\mathcal{SV}\mathrm{ir}$ relations (2.117), we can write

$$(G_0)^2 = L_0 - \frac{c}{24} \; . \tag{2.121}$$

Therefore, if a highest weight vector has the G_0-eigenvalue λ, then its L_0-eigenvalue h is given by

$$h = \lambda^2 + \frac{c}{24} \; , \tag{2.122}$$

and for a given value of h, there are two highest weight vectors $|\pm\lambda\rangle$ unless $h = \frac{c}{24}$. In addition, these highest weight vectors do not have definite fermion parities; since G_0 and $(-1)^F$ anticommute, $(-1)^F|\lambda\rangle$ has the G_0-eigenvalue $-\lambda$, and we can identify this vector as

$$(-1)^F|\lambda\rangle = |-\lambda\rangle \; . \tag{2.123}$$

In a L_0-eigensubspace of the Ramond Verma module generated from $|\lambda\rangle$, each of the basis vectors of the form

$$L_{-m_l} \cdots L_{-m_1} G_{-n_k} \cdots G_{-n_1} |\lambda\rangle \; , \tag{2.124}$$

where $0 < m_1 \leq \cdots \leq m_l$ and $0 < n_1 < \cdots < n_k$, is not necessarily an eigenvector of G_0, and one needs to change bases to obtain G_0 eigenvectors. From the parametrisation (2.119), we define

$$(\lambda_{r,s})^2 = h_{r,s} - \frac{1}{24} c(p,q) \; . \tag{2.125}$$

Then, the highest weight state $|\lambda_{r,s}\rangle$ has the L_0- and G_0-eigenvalues $h_{r,s}$ and $\lambda_{r,s}$ respectively. For a given value of c, we may write M_λ to denote the Verma module constructed from a highest weight state $|\lambda\rangle$.

So far, we are treating the fermion parity operator $(-1)^F$ as an operator defined on a representation space, but we may treat this operator as a part of the algebra, and consider the so-called extended Ramond algebra[21]. Then, L_0 and $(-1)^F$ form the Cartan subalgebra of the extended Ramond algebra, and a highest weight vector $|h\rangle$ has definite fermion parity $\epsilon = \pm 1$ and clearly cannot be an eigenvector of G_0. Then, there is another vector $G_0|h\rangle$, which has the opposite fermion parity $-\epsilon$ and thus orthogonal to $|h\rangle$, at the level zero subspace of the Verma module. For the Verma module M_h of the extended Ramond algebra constructed from $|h\rangle$, there are pairs of vectors of the form

$$L_{-m_l} \cdots L_{-m_1} G_{-n_k} \cdots G_{-n_1} |h\rangle \quad \text{and} \quad L_{-m_l} \cdots L_{-m_1} G_{-n_k} \cdots G_{-n_1} G_0 |h\rangle \; , \tag{2.126}$$

where $0 < m_1 \leq \cdots \leq m_l$ and $0 < n_1 < \cdots < n_k$, that have the same conformal dimensions but opposite fermion parities. If $h \neq \frac{c}{24}$, a Verma module M_h can be written as the direct sum

$$\mathrm{M}_h = \mathrm{M}_\lambda \oplus \mathrm{M}_{-\lambda} \; , \tag{2.127}$$

where h and λ are related by (2.122). If M_λ and $\mathrm{M}_{-\lambda}$ are irreducible, M_h is also irreducible as a representation of the extended Ramond algebra. When $h \neq \frac{c}{24}$, the vectors $|h\rangle$, $G_0|h\rangle$, and $|\pm\lambda\rangle$ are related by

$$\begin{cases} \quad |h\rangle = \quad a\left(|\lambda\rangle + \epsilon|-\lambda\rangle\right) \\ G_0|h\rangle = a\lambda\left(|\lambda\rangle - \epsilon|-\lambda\rangle\right) \end{cases} \quad \leftrightarrow \quad \begin{cases} \quad |\lambda\rangle = \dfrac{1}{2a}\left(|h\rangle + \dfrac{1}{\lambda}G_0|h\rangle\right) \\ |-\lambda\rangle = \dfrac{\epsilon}{2a}\left(|h\rangle - \dfrac{1}{\lambda}G_0|h\rangle\right) \end{cases} . \tag{2.128}$$

where $a \in \mathbb{C}$ satisfies

$$d_h = |a|^2\, 2d_\lambda \tag{2.129}$$

in which $d_h \coloneqq \langle h|h\rangle$ and $d_\lambda \coloneqq \langle \pm\lambda|\pm\lambda\rangle$. Since the fermion parity operator acts diagonally on a Verma module M_h, it is preferable to use M_h, rather than $\mathrm{M}_{\pm\lambda}$, as it is a $\mathbb{Z}_2$-graded representation of a superalgebra.

When $h = \frac{c}{24}$, we have $(G_0)^2|h\rangle = 0$ from (2.121), and we can see that $G_0|h\rangle$ is a null vector which is also an eigenvector of G_0 with the zero eigenvalue. Since it is a null vector, we can set $G_0|h\rangle = 0$, then $|h\rangle$ becomes an eigenvector of G_0 as well. If we write $|\lambda_0\rangle$ as an eigenvector of G_0 with the eigenvalue $\lambda_0 = 0$, then we cannot determine $(-1)^F|\lambda_0\rangle$ from the action of G_0 alone. Therefore, the level zero subspace of the irreducible module is one dimensional when $h = \frac{c}{24}$.

The Virasoro characters of Ramond Verma modules and irreducible modules are given in Appendix A.1.

2.2.3 Super W-Algebras

- $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$

A super W-algebra $\mathcal{SW}(\frac{3}{2}, h_1, \ldots, h_N)$ is the extension of the $N = 1$ super-Virasoro algebra by chiral primary superfields $\mathbf{W}^{(i)}(Z) = W^{(i)}(z) + \theta U^{(i)}(z)$ of integer or half-integer conformal weights h_i. The component fields $W^{(i)}(z)$ and $U^{(i)}(z)$ are Virasoro primary fields. In order for $\mathbf{W}^{(i)}(Z)$ to be a primary superfield, the following supercommutators involving its component fields have to hold[40]

$$[G_n, W_m^{(i)}] = C_{GW_i}^{U_i}\, U_{n+m}^{(i)} \quad \text{and} \quad [G_n, U_m^{(i)}] = \frac{C_{GU_i}^{W_i}}{2h_i}(n(2h_i - 1) - m)W_{n+m}^{(i)} . \tag{2.130}$$

By taking the normalisation

$$d_{W^{(i)}} = \langle 0|W_{h_i}^{(i)} W_{-h_i}^{(i)}|0\rangle = \frac{c}{h_i} \quad \text{and} \quad d_{U^{(i)}} = \langle 0|U_{h_i'}^{(i)} U_{-h_i'}^{(i)}|0\rangle = \frac{c}{h_i'} , \tag{2.131}$$

where $h_i' \coloneqq h_i + \frac{1}{2}$, structure constants in (2.130) are given by

$$(C_{GW_i}^{U_i})^2 = (-1)^{2h_i+1}(2h_i + 1) \quad \text{and} \quad C_{GW_i}^{U_i} = \frac{2h_i + 1}{2h_i}(-1)^{2h_i+1} C_{GU_i}^{W_i} . \tag{2.132}$$

Since $U^{(i)}(z)$ is the superdescendant of $W^{(i)}(z)$, their state vectors are related by

$$U_{-h_i'}^{(i)}|0\rangle = \frac{1}{C_{GW_i}^{U_i}}\, G_{-\frac{1}{2}} W_{-h_i}^{(i)}|0\rangle . \tag{2.133}$$

The remaining supercommutators can be obtained by the same methods as for the W-algebra cases.

In $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$, there are four chiral fields $T(z)$, $G(z)$, $W(z)$, and $U(z)$. The Virasoro primary field $W(z)$ has the conformal dimension $3/2$, and its superdescendant $U(z)$, which is also a Virasoro primary field, has the conformal dimension 2. In addition to the $\mathcal{SV}\mathrm{ir}$ relations, the modes of the generators satisfy

$$\begin{aligned} [L_n, W_m] &= \left(\frac{n}{2} - m\right) W_{n+m} \;, \qquad & [L_n, U_m] &= (n-m) U_{n+m} \;, \\ \{G_n, W_m\} &= 2U_{n+m} \;, & [G_n, U_m] &= \left(n - \frac{m}{2}\right) W_{n+m} \;, \\ \{W_n, W_m\} &= C^U_{WW}\, U_{n+m} + 2L_{n+m} + \frac{c}{3}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} \;, \\ [W_n, U_m] &= \left(n - \frac{m}{2}\right)\left(\frac{2}{3} C^W_{WU}\, W_{n+m} + G_{n+m}\right) \;, \quad \text{and} \\ [U_n, U_m] &= \frac{1}{2} C^U_{UU} (n-m) U_{n+m} + (n-m) L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \;, \end{aligned} \tag{2.134}$$

where

$$C^U_{WW} = C^U_{UU} \quad \text{and} \quad C^W_{WU} = \frac{3}{4} C^U_{UU} \;, \tag{2.135}$$

and C^U_{UU} is a free parameter. $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ exists for generic values of c.

Similar to the non-supersymmetric case, it is possible to embed $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ in $\mathcal{SV}\mathrm{ir} \oplus \mathcal{SV}\mathrm{ir}$. In (2.134), L_n and U_n form the $\mathcal{W}(2,2)$ subalgebra which suggests that we can take

$$\begin{aligned} G(z) &= \beta_1 G^{(1)}(z) + \beta_2 G^{(2)}(z) \;, \quad T(z) = T^{(1)}(z) + T^{(2)}(z) \\ W(z) &= \alpha\left(\sqrt{\frac{c_2}{c_1}}\,\beta_1 G^{(1)}(z) - \sqrt{\frac{c_1}{c_2}}\,\beta_2 G^{(2)}(z)\right) \;, \quad \text{and} \\ U(z) &= \alpha\left(\sqrt{\frac{c_2}{c_1}}\,T^{(1)}(z) - \sqrt{\frac{c_1}{c_2}}\,T^{(2)}(z)\right) \;, \end{aligned} \tag{2.136}$$

where $\alpha = \pm 1$ and $\beta_i = \pm 1$ are arbitrary constants. The chiral fields in (2.136) are normalised according to (2.131); $G(z)$, $W(z)$, and $U(z)$ are Virasoro primary with respect to $T(z)$; and $W(z)$ is superprimary with respect to $G(z)$. Modes of the chiral fields in (2.136) satisfy the $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ relations (2.134), and the free parameter is given by

$$C^U_{UU} = 2\alpha\,\frac{c_2 - c_1}{\sqrt{c_1 c_2}} \;, \tag{2.137}$$

where the relations (2.135) hold regardless of the values of β_i. Therefore, different values of β_i are related by automorphisms of $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$.

Each $\mathcal{SV}\mathrm{ir}$ in $\mathcal{SV}\mathrm{ir} \oplus \mathcal{SV}\mathrm{ir}$ has an outer automorphism $\Omega_{F_i} : G^{(i)}(z) \mapsto -G^{(i)}(z)$, under which the other fields are invariant; at generic values of c, $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group coming from the $\mathbb{Z}_2$ automorphisms of the two copies of $\mathcal{SV}\mathrm{ir}$ whose actions are equivalent to taking different values of β_i. Considering $\mathcal{SV}\mathrm{ir} \subset \mathcal{SW}(\frac{3}{2},\frac{3}{2})$, the overall fermionic automorphism of $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ is given by

$$(\Omega_{F_1}, \Omega_{F_2}) : G(z) \mapsto -G(z) \;, \quad W(z) \mapsto -W(z) \;, \tag{2.138}$$

which leaves the bosonic fields invariant. For brevity, denote $\Omega_F := (\Omega_{F_1}, \Omega_{F_2})$. This automorphism can be used for the usual GSO projection, and is related to the usual Ramond sector of $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$. If we define

$$\tilde{G}(z) = \beta_1 G^{(1)}(z) - \beta_2 G^{(2)}(z) \quad \text{and} \quad \tilde{W}(z) = \alpha \left(\sqrt{\frac{c_2}{c_1}}\, \beta_1 G^{(1)}(z) + \sqrt{\frac{c_1}{c_2}}\, \beta_2 G^{(2)}(z) \right) , \tag{2.139}$$

then, the other elements of this $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group can be written as

$$(\mathrm{id}_1, \Omega_{F_2}) : G(z) \mapsto \tilde{G}(z) , \quad W(z) \mapsto \tilde{W}(z) , \quad \text{and} \tag{2.140}$$

$$(\Omega_{F_1}, \mathrm{id}_2) : G(z) \mapsto -\tilde{G}(z) , \quad W(z) \mapsto -\tilde{W}(z) . \tag{2.141}$$

They leave the bosonic fields invariant. We may also denote them as $\Omega_{F_1} := (\Omega_{F_1}, \mathrm{id}_2)$ and $\Omega_{F_2} := (\mathrm{id}_1, \Omega_{F_2})$.

When $c_1 = c_2$, the self-coupling of $U(z)$ vanishes, and there will be another $\mathbb{Z}_2$ outer automorphism given by

$$\Omega_U : W(z) \mapsto -W(z) , \quad U(z) \mapsto -U(z) , \tag{2.142}$$

which leaves $G(z)$ and $T(z)$ invariant. In this case, we have

$$\tilde{G}(z) = \alpha W(z) \quad \text{and} \quad \tilde{W}(z) = \alpha G(z) . \tag{2.143}$$

We emphasis that $\alpha = \pm 1$ is a constant determined by an embedding of $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ in $\mathcal{SV}\mathrm{ir} \oplus \mathcal{SV}\mathrm{ir}$. Note that changing the sign of α swaps the actions of Ω_{F_1} and Ω_{F_2}. Combining the fermionic automorphisms and Ω_U, we get three more automorphisms

$$\Omega_F \circ \Omega_U = \Omega_U \circ \Omega_F \; : G(z) \mapsto -G(z) , \quad W(z) \mapsto W(z) , \quad U(z) \mapsto -U(z) , \tag{2.144}$$

$$\Omega_{F_1} \circ \Omega_U = \Omega_U \circ \Omega_{F_2} : G(z) \mapsto -\alpha W(z) , \quad W(z) \mapsto \alpha G(z) , \quad U(z) \mapsto -U(z) , \tag{2.145}$$

$$\Omega_{F_2} \circ \Omega_U = \Omega_U \circ \Omega_{F_1} : G(z) \mapsto \alpha W(z) , \quad W(z) \mapsto -\alpha G(z) , \quad U(z) \mapsto -U(z) . \tag{2.146}$$

We denote them by $\tilde{\Omega}_F := \Omega_F \circ \Omega_U$, $\tilde{\Omega}_{F_1} := \Omega_{F_1} \circ \Omega_U$, and $\tilde{\Omega}_{F_2} := \Omega_{F_2} \circ \Omega_U$. While $\tilde{\Omega}_F$ is an involution, that is, it is the inverse of itself, $\tilde{\Omega}_{F_1}$ and $\tilde{\Omega}_{F_2}$ square to Ω_F. From this information, we find that the set of outer automorphisms

$$\{\mathrm{id},\, \Omega_{F_1},\, \Omega_{F_2},\, \Omega_F,\, \Omega_U,\, \tilde{\Omega}_{F_1},\, \tilde{\Omega}_{F_2},\, \tilde{\Omega}_F\} \tag{2.147}$$

forms a group which is isomorphic[16] to the dihedral group D_4. Conjugacy classes and corresponding centralisers of this automorphism group are summarised in Table 2.1, in which $\mathrm{V}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ denotes the Klein four group.

2.2.4 Intertwiners

In Subsection 2.2.2, we have encountered the vertex operator, which is defined as a map $V : \mathcal{H}_0^{\mathcal{A}} \times \mathbb{C} \to \mathrm{End}(\mathcal{H}_0^{\mathcal{A}})$ where $\mathcal{H}_0^{\mathcal{A}}$ is the vacuum irreducible module of a chiral algebra $\mathcal{A}$

16. One way to identify with $\mathrm{D}_4 = \langle x, a \mid a^4 = x^2 = e,\ xax^{-1} = a^{-1} \rangle$ is to let $a = \tilde{\Omega}_{F_1}$, $a^2 = \Omega_F$, $a^3 = \tilde{\Omega}_{F_2}$, $x = \Omega_U$, $ax = \Omega_{F_1}$, $a^2 x = \tilde{\Omega}_F$, and $a^3 x = \Omega_{F_2}$.

Conjugacy classes	Centralisers
$\{\text{id}\}$	All elements in D_4
$\{\Omega_F\}$	All elements in D_4
$\{\Omega_U,\, \tilde{\Omega}_F\}$	$\{\text{id},\, \Omega_F,\, \Omega_U,\, \tilde{\Omega}_F\} \cong \mathrm{V}_4$
$\{\Omega_{F_1},\, \Omega_{F_2}\}$	$\{\text{id},\, \Omega_F,\, \Omega_{F_1},\, \Omega_{F_2}\} \cong \mathrm{V}_4$
$\{\tilde{\Omega}_{F_1},\, \tilde{\Omega}_{F_2}\}$	$\{\text{id},\, \Omega_F,\, \tilde{\Omega}_{F_1},\, \tilde{\Omega}_{F_2}\} \cong \mathbb{Z}_4$

Table 2.1: Conjugacy classes and corresponding centralisers of the D_4 outer automorphism group of $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ when $c_1 = c_2$.

with a given value of the central charge c. We can generalise this concept to an intertwining operator, which is defined as a map

$$V^k_{ij;\alpha} : \mathcal{H}_i \times \mathbb{C} \to \mathrm{Hom}(\mathcal{H}_j, \mathcal{H}_k) \ , \tag{2.148}$$

where $\mathcal{H}_i$, $\mathcal{H}_j$, and $\mathcal{H}_k$ are irreducible $\mathcal{A}$-modules with the central charge c, and this map is an intertwiner for $\mathcal{A}$-representations. The dimension of the space of intertwiners $V^k_{ij;\alpha}$ is given by N^k_{ij}, and the multiplicity labels run $\alpha = 1, \ldots, N^k_{ij}$. An intertwining operator exists when the fusion coefficient N^k_{ij} is non-zero. We can understand this operator in several ways: for a given $z \in \mathbb{C}$, we can view this as

$$V^k_{ij;\alpha}(\,\cdot\,, z) : \mathcal{H}_i \times \mathcal{H}_j \to \mathcal{H}_k \tag{2.149}$$

in the sense that

$$V^k_{ij;\alpha}(|\psi_i\rangle, z)|\psi_j\rangle \in \mathcal{H}_k \ , \tag{2.150}$$

where $|\psi_i\rangle \in \mathcal{H}_i$ and $|\psi_j\rangle \in \mathcal{H}_j$; equivalently, we can think of this as

$$V^k_{ij;\alpha}(\,\cdot\,, z) : (\mathcal{H}_k)^* \times \mathcal{H}_i \times \mathcal{H}_j \to \mathbb{C} \tag{2.151}$$

given by

$$\langle\psi_k|V^k_{ij;\alpha}(|\psi_i\rangle, z)|\psi_j\rangle \in \mathbb{C} \ , \tag{2.152}$$

where $\langle\psi_k| \in (\mathcal{H}_k)^*$.

In order to see how $V^k_{ij;\alpha}$ intertwines $\mathcal{A}$-actions, we need a comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. For $|\psi_i\rangle \otimes |\psi_j\rangle \in \mathcal{H}_i \otimes \mathcal{H}_j$ and $W_n \in \mathcal{A}$, where $n \in \mathbb{Z} - h_W$, the comultiplication formula is given by

$$\Delta_{z,0}(W_n)(|\psi_i\rangle \otimes |\psi_j\rangle) = \oint_C \frac{\mathrm{d}w}{2\pi i} w^{n+h_W-1}\, W(w) V^k_{ij;\alpha}(|\psi_i\rangle, z)|\psi_j\rangle \ , \tag{2.153}$$

where the contour C encircles z and 0, and we have assumed $\mathcal{H}_i$ and $\mathcal{H}_j$ are untwisted representations. By deforming the contour C to the contours around z and 0, we can write this as

$$\begin{aligned}
&\oint_z \frac{\mathrm{d}w}{2\pi i} w^{n+h_W-1}\, W(w) V^k_{ij;\alpha}(|\psi_i\rangle, z)|\psi_j\rangle + \epsilon_{W\psi_i} \oint_0 \frac{\mathrm{d}w}{2\pi i} w^{n+h_W-1}\, V^k_{ij;\alpha}(|\psi_i\rangle, z) W(w)|\psi_j\rangle \\
&= \oint_z \frac{\mathrm{d}w}{2\pi i} w^{n+h_W-1} \sum_{m \le h_{\psi_i}} (w-z)^{-m-h_W}\, V^k_{ij;\alpha}(W_m|\psi_i\rangle, z)|\psi_j\rangle + \epsilon_{W\psi_i} V^k_{ij;\alpha}(|\psi_i\rangle, z) W_n |\psi_j\rangle \ ,
\end{aligned} \tag{2.154}$$

where $m \in \mathbb{Z} - h_W$. The phase factor $\epsilon_{W\psi_i} \in \mathbb{C}$ is defined by

$$W(w)V^k_{ij;\alpha}(|\psi_i\rangle, z) = \epsilon_{W\psi_i}\, V^k_{ij;\alpha}(|\psi_i\rangle, z)W(w)\ , \tag{2.155}$$

where $|w| > |z|$ on the left hand side, and the right hand side is the clockwise analytic continuation[57] from $|w| < |z|$. For untwisted representations, $\epsilon_{W\psi_i} = \pm 1$ is determined by the fermion parities of $W(w)$ and $|\psi_i\rangle$. By shifting $w \mapsto w + z$, we can evaluate the remaining integral, and obtain

$$\begin{aligned}&\Delta_{z,0}(W_n)(|\psi_i\rangle \otimes |\psi_j\rangle)\\ &= \sum_{k=0}^{\infty} \binom{n + h_W - 1}{k} z^{n+h_W-1-k}\, V^k_{ij;\alpha}(W_{1+k-h_W}|\psi_i\rangle, z)|\psi_j\rangle + \epsilon_{W\psi_i}\, V^k_{ij;\alpha}(|\psi_i\rangle, z)W_n|\psi_j\rangle\ .\end{aligned} \tag{2.156}$$

Therefore, the comultiplication gives $\Delta_{z,0}(W_n) \in \mathcal{A} \otimes \mathcal{A}$ as

$$\Delta_{z,0}(W_n) = \sum_{k=0}^{\infty} \binom{n + h_W - 1}{k} z^{n+h_W-1-k}\, W_{1+k-h_W} \otimes \mathbf{1} + \epsilon_{W\psi_i}\, \mathbf{1} \otimes W_n\ . \tag{2.157}$$

For given representation maps $\rho_i : \mathcal{A} \to \mathrm{End}(\mathcal{H}_i)$ and $\rho_j : \mathcal{A} \to \mathrm{End}(\mathcal{H}_j)$, the fusion product

$$(\rho_i \otimes \rho_j) \circ \Delta : \mathcal{A} \to \mathrm{End}(\mathcal{H}_i \otimes \mathcal{H}_j) \tag{2.158}$$

defines an $\mathcal{A}$-representation on $\mathcal{H}_i \otimes \mathcal{H}_j$. In general, the fusion product of two irreducible representations is not irreducible, and it can be decomposed into irreducible representations as

$$\mathcal{H}_i \otimes \mathcal{H}_j = \bigoplus_k N^k_{ij}\, \mathcal{H}_k \tag{2.159}$$

with maps $\rho_{k;\alpha} : \mathcal{A} \to \mathrm{End}(\mathcal{H}_k)$ that are the restrictions of (2.158). Using the fusion product, the operator $V^k_{ij;\alpha}$ can be understood as an intertwiner for $\mathcal{A}$-representation

$$V^k_{ij;\alpha} : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_k \tag{2.160}$$

given by

$$\rho_{k;\alpha}(W_n)\, V^k_{ij;\alpha}(|\psi_i\rangle \otimes |\psi_j\rangle) = V^k_{ij;\alpha}(\, (\rho_i \otimes \rho_j) \circ \Delta(W_n)(|\psi_i\rangle \otimes |\psi_j\rangle)\,)\ . \tag{2.161}$$

This means that $W_n V^k_{ij;\alpha}(|\psi_i\rangle, z)|\psi_j\rangle$ is given by the right hand side of (2.156). In order to calculate fusion products involving twisted representations, one needs to modify the integral for $\Delta_{z,0}(\widetilde{W}_n)$ where $\widetilde{W}_n$ is a series in W_m. The comultiplication formula and fusion products of twisted representations are given by Gaberdiel[57].

For the highest weight state $|i\rangle \in \mathcal{H}_i$ with the conformal weight h_i, the operator $V^k_{ij;\alpha}(|i\rangle, z)$ acts as a chiral primary field, that is

$$[L_n, V^k_{ij;\alpha}(|i\rangle, z)] = \big(h_i(n+1)z^n + z^{n+1}\partial\big)\, V^k_{ij;\alpha}(|i\rangle, z)\ . \tag{2.162}$$

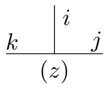
Intertwiners $V_{ij;\alpha}^{k}$ are also called chiral vertex operators (CVOs), and they can be used to construct chiral parts of correlation functions. For simplicity, consider $\mathcal{A} = \mathcal{V}\mathrm{ir}$, in which case $N_{ij}^{k} \in \{0,1\}$ and the multiplicity labels α are suppressed. Given the highest weight vectors $|i\rangle$, $|j\rangle$, $|k\rangle$, and $|l\rangle$, a Virasoro four-point block (2.64) can be written as

$$\mathcal{F}_{ijkl}^{p}(z) = \langle i|V_{jp}^{i}(|j\rangle, 1)V_{kl}^{p}(|k\rangle, z)|l\rangle \ . \tag{2.163}$$

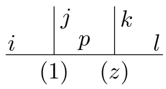
By comparing (2.98) and (2.68), we see that the duality relation for CVOs gives rise to the fusing matrix as

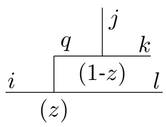
$$V_{jp}^{i}(|j\rangle, 1)V_{kl}^{p}(|k\rangle, z) = \sum_{q} \mathsf{F}_{pq}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] V_{ql}^{i}(V_{jk}^{q}(|j\rangle, 1-z)|k\rangle, z) \ . \tag{2.164}$$

If we introduce a graphical notation[17]

$V_{ij}^{k}(\,\cdot\,, z) =$  (2.166)

then, the relation (2.164) can be understood as



$= \sum_{q} \mathsf{F}_{pq}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right]$  (2.167)

For intertwiners, Condition (4) of (2.97) becomes a relation

$$V_{jp}^{i}(|j\rangle, z)V_{kl}^{p}(|k\rangle, w) = \sum_{q} \mathsf{B}_{pq}^{(\varepsilon)}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] V_{kq}^{i}(|k\rangle, w)V_{jl}^{q}(|j\rangle, z) \ , \tag{2.168}$$

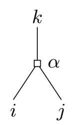
where B is called the braiding matrix, and $\varepsilon = \pm$ specifies the direction of analytic continuation. Note that $\mathsf{B}^{(-\varepsilon)}$ is the inverse of $\mathsf{B}^{(\varepsilon)}$, that is

$$\sum_{t} \mathsf{B}_{pt}^{(\varepsilon)}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \mathsf{B}_{tq}^{(-\varepsilon)}\left[\begin{smallmatrix} k & j \\ i & l \end{smallmatrix}\right] = \delta_{p,q} \ , \tag{2.169}$$

but $\mathsf{B}^2 := \mathsf{B}^{(\varepsilon)}\mathsf{B}^{(\varepsilon)}$ is not necessarily the identity; B^2 is the monodromy matrix for the analytic continuation of z around w.

The matrices F and B satisfy various relations and identities—among them, the hexagon identity for B resembles the Yang–Baxter equation, and the pentagon identity for F can be used to prove the Verlinde formula—their details can be found, for example, in [30], [31], [32], and [76]. For Virasoro minimal models, elements of the fusing matrices are given in Appendix A.3.

17. Since $V_{ij,\alpha}^{k}(\,\cdot\,, z) \in \mathrm{Hom}(\mathcal{H}_i \otimes \mathcal{H}_j, \mathcal{H}_k)$, one has to take care of the directions of lines. Using the notation of [76], we can write

$V_{ij,\alpha}^{k}(\,\cdot\,, z) =$  (2.165)

where the diagram is read from bottom to top.

From the point of view of Moore and Seiberg[30, 31], we can picture $V_{ij}^{k}(\,\cdot\,, z)$ as a Riemann sphere with three punctures that are located at ∞, z, and 0 in some local coordinates, and the representations $\mathcal{H}_k$, $\mathcal{H}_i$, and $\mathcal{H}_j$ labelling the punctures. In order to distinguish the puncture labelled by k from the other two, we may introduce "orientations" to the punctures[18]; they are represented by arrows around punctures in Figure 2.1. In this picture, compositions of CVOs are understood as sewings of Riemann spheres. Conversely, the relations (2.164) and (2.168) can be understood as the relations between the three distinct ways of decomposing a Riemann sphere with four punctures into two Riemann spheres with three punctures; they are called pants decompositions.

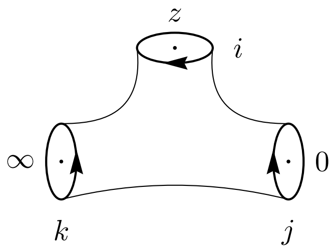
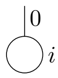


Figure 2.1: $V_{ij}^{k}(\,\cdot\,, z)$ as a Riemann sphere with three punctures

Writing the identity operator as a CVO, the Virasoro character (A.1) of a representation $\mathcal{H}_i$ can be written as

$$\chi_i(q) = \mathrm{Tr}_i\left(V_{0i}^{i}(|0\rangle, 1)\, q^{L_0 - \frac{c}{24}}\right) = \tag{2.170}$$



where 0 denotes the vacuum representation. In this sense, Virasoro characters can be viewed as zero-point functions on a torus. Then, the modular S transformation relates the two distinct ways of sewing to obtain the torus as depicted in Figure 2.2. For Virasoro and $N = 1$ super-Virasoro minimal models, elements of the modular S matrices are given in Appendix A.2.

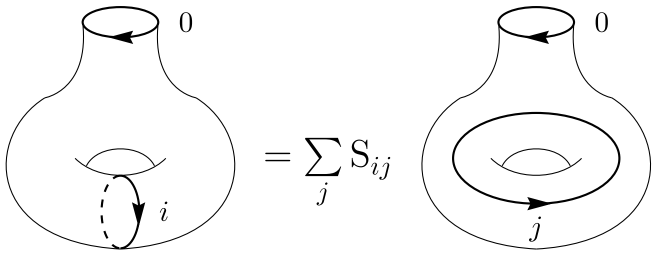


Figure 2.2: The modular S transformation and the pants decompositions of a torus.

On an $N = 1$ super-Riemann surface, which is locally isomorphic to $\mathbb{C}^{1|1}$, there are two kinds of punctures: Neveu–Schwarz (NS) punctures and Ramond (R) punctures. While NS punctures can be inserted at any point on a surface, a Ramond puncture is a singularity in the super-Riemann surface structure. As a consequence, we cannot assign fermionic coordinates to R punctures. Moreover, it is possible to take local coordinates such that a

18. For example, in [88].

Ramond puncture becomes a square-root branch point of fermions, that is $\theta \to -\theta$ for the fermionic coordinate when the bosonic coordinate z is rotated around the R puncture by 2π. More details can be found, for example, in Witten's notes[107].

On a compact super-Riemann surface, the number of Ramond punctures is always even. Therefore, a super-Riemann sphere with three punctures can either have three NS punctures or two R punctures and one NS puncture. This determines sectors of representations appearing in the labels of a CVO. In addition, it is only possible to glue punctures in the same sector. We draw a line between two Ramond punctures representing a branch cut of fermions[17, 43]. Then, the modular S transformation of a Ramond character into NS supercharacters can be understood geometrically as shown in Figure 2.3. Here, the numbered arrows correspond to: (1) sewing; (2) modular S transformation; and (3) pants decomposition. On the resulting sphere, the branch cut encircles one of the NS punctures; considering radial quantisation around this puncture and recalling the definition of a Virasoro supercharacter (A.7), we can associate the fermion parity operator $(-1)^F$ to the branch cut of fermions.

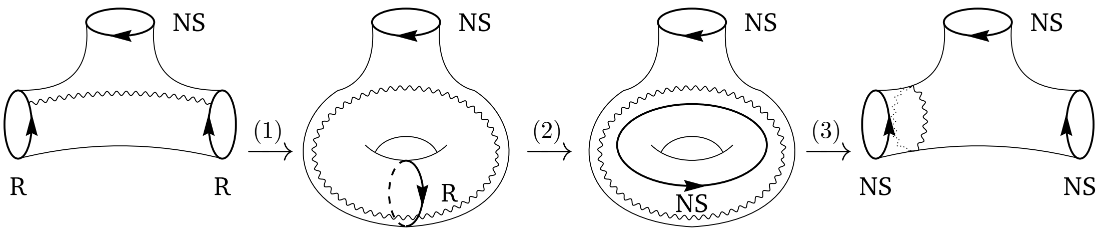


Figure 2.3: The modular S transformation of a Ramond character

One of the remarkable features of CFT is that there is a relation between the modular S matrix and the fusion rules, which is given by the Verlinde formula. From the fusion coefficients N_{ij}^k, one can form a fusion matrix N_i which is defined by

$$(\mathrm{N}_i)_{jk} := N_{ij}^k \ . \tag{2.171}$$

Some of their properties are apparent from those of the fusion coefficients. Since $N_{0i}^j = \delta_i^j$, the fusion matrix N_0 is the identity. Using the fusion coefficient identities

$$N_{ij}^k = N_{ji}^k \ , \quad N_{ij}^k = N_{ik^+}^{j+} \ , \quad N_{ij}^k = N_{i^+j^+}^{k^+} \ , \quad \text{and} \quad \sum_{p\in\mathcal{I}} N_{jp}^i N_{kl}^p = \sum_{q\in\mathcal{I}} N_{ql}^i N_{jk}^q \ , \tag{2.172}$$

one can show $\mathrm{N}_i^{\mathrm{T}} = \mathrm{N}_{i^+}$ and so on. The fusion matrices form the adjoint representation of the fusion rule algebra, that is

$$\mathrm{N}_i \, \mathrm{N}_j = \sum_{k\in\mathcal{I}} N_{ij}^k \, \mathrm{N}_k \ . \tag{2.173}$$

Since elements of N_i are all non-negative integers, the set of fusion matrices is also called a non-negative integer matrix representation (NIM-rep). Since the fusion algebra is commutative, fusion matrices commute as well, and consequently they can be simultaneously diagonalised. The Verlinde formula[25] (2.176) shows the modular S matrix diagonalises the fusion matrices, that is

$$\mathrm{N}_i = \mathrm{S} \, \mathrm{D}_i \, \mathrm{S}^{-1} \ , \tag{2.174}$$

where D_i is a diagonal matrix given by

$$(\mathrm{D}_i)_{jk} = \delta_{jk} \, \frac{\mathrm{S}_{ij}}{\mathrm{S}_{0j}} \; . \tag{2.175}$$

In terms of the components, the Verlinde formula can be written as

$$N_{ij}^{k} = \sum_{l \in \mathcal{I}} \frac{\mathrm{S}_{il} \, \mathrm{S}_{jl} \, \mathrm{S}_{lk}^{-1}}{\mathrm{S}_{0l}} \; . \tag{2.176}$$

Since the matrices D_i are diagonal, their elements form one-dimensional representations of the fusion algebra

$$(\mathrm{D}_i)_{nn} (\mathrm{D}_j)_{nn} = \sum_{k \in \mathcal{I}} N_{ij}^{k} \, (\mathrm{D}_k)_{nn} \; , \tag{2.177}$$

In particular, the quantity

$$\mathcal{D}_i = (D_i)_{00} = \frac{\mathrm{S}_{i0}}{\mathrm{S}_{00}} \tag{2.178}$$

is called the quantum dimension of a representation $\mathcal{H}_i$, which is also the eigenvalue associated with the Perron–Frobenius eigenvector of N_i. Quantum dimensions must satisfy[51]

$$\mathcal{D}_i \in \{2 \cos \left(\frac{\pi}{n} \right) : n \in \mathbb{Z}_{\geq 3}\} \cup [2, \infty) \tag{2.179}$$

in which 1 is the smallest possible number. Note that $\mathcal{D}_0 = 1$, and if another representation has $\mathcal{D}_i = 1$ then the fusion of i with some other representation contains only one representation. Chiral primary fields associated to representations with quantum dimension $\mathcal{D}_i = 1$ are called simple currents. Given a fusion rule, one can relate this to a polynomial equation. For example, if we have a Lee-Yang type fusion rule $\varphi \otimes \varphi = 0 \oplus \varphi$, then by writing $\mathcal{D}_\varphi = x$, we get $x^2 = 1 + x$ whose solutions are the golden ratio $x = \frac{1}{2}(1 \pm \sqrt{5})$. Only the positive solution can be interpreted as the quantum dimension.

The Verlinde formula relates the fusion coefficients, that are algebraic in its nature, to the modular S matrix which is of geometric origin, and it has far-reaching consequences. The proofs of this formula are given in [23], [33], and [28].

2.2.5 Modular Invariant Partition Functions

A torus T_τ can be regarded as the quotient $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ of the complex plane by a lattice $\mathbb{Z} + \tau\mathbb{Z}$ where $\tau \in \mathbb{C}$. This means that $z \in \mathbb{C}$ with the identifications

$$z \sim z + n + m\tau \quad \forall n, m \in \mathbb{Z} \tag{2.180}$$

describes a coordinate on T_τ. As Riemann surfaces, T_τ with $\tau \in \mathbb{H}$, where $\mathbb{H}$ is the upper half plane $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}\, \tau > 0\}$, are conformally equivalent if they are related by the modular group. The modular group $\mathrm{PSL}(2, \mathbb{Z})$ is the group of transformations of τ that leave $\tau \in \mathbb{H}$, and this group is generated by the two elements

$$\mathrm{S} : \tau \mapsto -\frac{1}{\tau} \quad \text{and} \quad \mathrm{T} : \tau \mapsto \tau + 1 \; . \tag{2.181}$$

They are called the modular S and T transformations.

In the basis of characters, modular S and T transformations are represented by the matrices given in (A.16). For a bosonic chiral algebra, they should satisfy

$$\mathrm{S}^2 = \mathrm{C} = (\mathrm{ST})^3 \,, \tag{2.182}$$

where C is the charge conjugation matrix given by $\mathrm{C}_{ij} = \delta_{j,i^+}$. The equation above is nothing but the defining relation of the modular group except for C is not necessarily the identity matrix. This is due to the fact that S transformation exchanges two fundamental cycles of a torus by $\mathrm{S} : (\mathsf{a}, \mathsf{b}) \to (-\mathsf{b}, \mathsf{a})$, and consequently $\mathrm{S}^2 : (\mathsf{a}, \mathsf{b}) \to (-\mathsf{a}, -\mathsf{b})$ corresponds to space and time reversal, which should result in charge conjugation. In addition, S and T are unitary matrices and S is symmetric as well, that is

$$\mathrm{S}^{\mathrm{T}} = \mathrm{S} \,, \quad \mathrm{S}^{\dagger} = \mathrm{S}^{-1} \,, \quad \text{and} \quad \mathrm{T}^{\dagger} = \mathrm{T}^{-1} \,. \tag{2.183}$$

Complex conjugate of a S matrix element gives $\overline{\mathrm{S}}_{ij} = \mathrm{S}_{i^+j}$. For fermionic theories, especially for $N = 1$ super-Virasoro minimal models, there are some subtleties involved in defining the modular S and T matrices to obey the conditions above. They are discussed in Appendix A.2.

In a CFT, we usually start from a complex plane with coordinate z and map it to an infinite cylinder by $z = e^{-2\pi i w}$. On the cylinder, $w \sim w + n$ for all $n \in \mathbb{Z}$. Due to radial quantisation on the plane, $\operatorname{Im} w$ and $\operatorname{Re} w$ correspond to 'time' and 'space' directions, respectively. By introducing $w = x + iy$, we can write the Hamiltonian on this cylinder as

$$H_{\text{cyl.}} = \int_0^1 T_{yy}^{\text{cyl.}} \, \mathrm{d}x \,. \tag{2.184}$$

Using the transformation law (2.51), this can be written in terms of the Virasoro generators on the plane as

$$H_{\text{cyl.}} = 2\pi \left(L_0 + \bar{L}_0 - \frac{c}{12} \right) \,. \tag{2.185}$$

If we make the time direction periodic by choosing purely imaginary $\tau > 0$ and imposing $w \sim w + m\tau$ for all $m \in \mathbb{Z}$, then we obtain a theory defined on the torus T_τ. It is sometimes useful to think this torus as the parallelogram with vertices at 0, 1, τ, and $\tau + 1$ on the w-plane. If we let $\tau = it$ where $t \in \mathbb{R}$, the torus partition function is given by

$$Z = \operatorname{Tr}_{\mathcal{H}} e^{-tH_{\text{cyl.}}} = \operatorname{Tr}_{\mathcal{H}} e^{2\pi i \tau \left(L_0 + \bar{L}_0 - \frac{c}{12} \right)} \,, \tag{2.186}$$

where $\mathcal{H}$ is the space of states for the bulk fields. If $\mathcal{H}$ is given by (2.54), then the partition function becomes

$$Z = \sum_{I \in \mathcal{S}} \chi_i(q) \, \chi_{\bar{\imath}}(\bar{q}) \,, \tag{2.187}$$

where $I = (i, \bar{\imath})$ labels a bulk field, $\chi_i(q)$ is the Virasoro character (A.1) of an irreducible module $\mathcal{H}_i$ with $q := e^{2\pi i \tau}$, and $\bar{q}$ is the complex conjugate of q, that is $\bar{q} = e^{-2\pi i \bar{\tau}}$ with the complex conjugate $\bar{\tau}$ of τ. If we assume this theory has the same holomorphic and antiholomorphic chiral algebras $\mathcal{A} \cong \bar{\mathcal{A}}$, then we can write

$$Z = \sum_{i,j \in \mathcal{I}} M_{ij} \, \chi_i(q) \, \chi_j(\bar{q}) \,, \tag{2.188}$$

where $\mathcal{I}$ is the indexing set for $\mathcal{A}$ irreducible representations, and the multiplicity matrix M encodes the bulk spectrum $\mathcal{S}$ of the theory. In order for this theory to be defined on a torus, its partition function must be invariant under modular transformations, in particular, the modular S and T transformations. This means that the multiplicity matrix M has to commute with the modular S and T matrices, that is

$$M = \mathrm{S}M\mathrm{S}^{-1} \quad \text{and} \quad M = \mathrm{T}M\mathrm{T}^{-1} \,. \tag{2.189}$$

In addition, the vacuum state has to be unique, which imposes $M_{00} = 1$. The invariance under T transformation restricts bulk fields to have integer spins, that is $h - \bar{h} \in \mathbb{Z}$. For fermionic theories, this condition is weakened to the invariance under T^2 only. There are always two types of modular invariant partition functions possible: a diagonal invariant $M_{ij} = \delta_{ij}$ and a charge conjugation invariant $M_{ij} = \delta_{ij^+}$.

For $\widehat{\mathrm{sl}}(2)_k$-WZW[19], Virasoro[19] and super-Virasoro minimal models[18], modular invariant partition functions obeys the so-called A-D-E classification. In a $\widehat{\mathrm{sl}}(2)_k$-WZW model at level $k \in \mathbb{Z}_{>0}$, primary fields correspond to the integrable highest weight representations of $\widehat{\mathrm{sl}}(2)_k$, and they can be labelled by integers $1 \le i \le k+1$. The A-D-E classification associates a simply-laced Dynkin diagram G with the (dual) Coxeter number[19] $g = k+2$ to a $\widehat{\mathrm{sl}}(2)_k$-WZW model. When G is A_n, D_{even}, E_6, or E_8, the corresponding modular invariant partition function consists of the characters χ_i with the representation labels i appearing in the exponents of G. Properties of simply-laced Dynkin diagrams are summarised in Table 2.2, and the modular invariant partition functions of $\widehat{\mathrm{sl}}(2)_k$-WZW models are given in Table 2.3. Note that D_{even} invariants are related to the simple current extension[36] by the primary field with label $i = k+1$, and they have extended conformal symmetries; the two primary fields with degenerate conformal dimensions are distinguished by their W_0 eigenvalues. In addition D_{even} invariants can be written as diagonal invariants for the extended chiral algebras. D_{odd} invariants are due to the action of a non-trivial automorphism of the fusion algebra on the chiral half of the representations. Among the exceptional invariants, E_6 and E_8 invariants can be understood in terms of the conformal embeddings $\widehat{\mathrm{sl}}(2)_{10} \subset \widehat{\mathrm{sp}}(4)_1$ and $\widehat{\mathrm{sl}}(2)_{28} \subset (\widehat{G}_2)_1$, respectively. Note that $\widehat{\mathrm{sp}}(4)_1 \cong \widehat{\mathrm{so}}(5)_1$.

Modular invariant partition functions of Virasoro minimal models are constructed similarly. For $M(p,q)$, one associates a pair of simply-laced Dynkin diagrams (G, G') where they have the Coxeter numbers p and q respectively. Since p and q are coprime, one of them should be an odd number. Then, one of the diagrams in (G, G') must be A_n as they are the only simply-laced Dynkin diagrams that can have odd Coxeter numbers. In our notation, a pair (G, G') is related to Kac labels (r, s) in this order. Note that (A_{p-1}, A_{q-1}) corresponds to the diagonal modular invariant. If one of the diagrams in (G, G'), say G, is not A_n, then the corresponding modular invariant is obtained by taking the summations over r in the same form as the corresponding $\widehat{\mathrm{sl}}(2)_k$-WZW invariant. For example, (A_4, D_4)

19. For simply-laced Dynkin diagrams, they are the same.

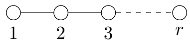
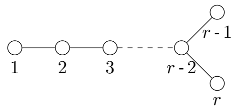
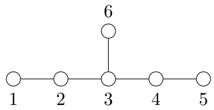
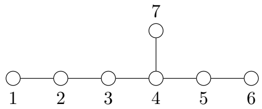
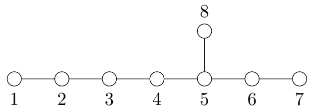
Name	Diagram	g	Exponents
$A_{r\geq 2}$		$r+1$	$1,2,\ldots,r$
$D_{r\geq 4}$		$2r-2$	$1,3,\ldots,2r-3,r-1$
E_6		12	$1,4,5,7,8,11$
E_7		18	$1,5,7,9,11,13,17$
E_8		30	$1,7,11,13,17,19,23,29$

Table 2.2: Properties of simply-laced Dynkin diagrams with ranks r and Coxeter numbers g.

$g=k+2$	G	Partition function
$g\geq 2$	A_{g-1}	$\sum_{i=1}^{g-1}\lvert\chi_i\rvert^2$
$g=4\rho+2,\ \rho\geq 1$	$D_{2\rho+2}$	$\sum_{\substack{i=1\\ i\in 2\mathbb{Z}+1}}^{2\rho-1}\lvert\chi_i+\chi_{4\rho+2-i}\rvert^2+2\lvert\chi_{2\rho+1}\rvert^2$
$g=4\rho,\ \rho\geq 2$	$D_{2\rho+1}$	$\sum_{\substack{i=1\\ i\in 2\mathbb{Z}+1}}^{4\rho-1}\lvert\chi_i\rvert^2+\lvert\chi_{2\rho}\rvert^2+\sum_{\substack{i=2\\ i\in 2\mathbb{Z}}}^{2\rho-2}(\chi_i\bar{\chi}_{4\rho-i}+\chi_{4\rho-i}\bar{\chi}_i)$
$g=12$	E_6	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_4+\chi_8\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$
$g=18$	E_7	$\lvert\chi_1+\chi_{17}\rvert^2+\lvert\chi_5+\chi_{13}\rvert^2+\lvert\chi_7+\chi_{11}\rvert^2+\lvert\chi_9\rvert^2$ $+(\chi_3+\chi_{15})\bar{\chi}_9+\chi_9(\bar{\chi}_3+\bar{\chi}_{15})$
$g=30$	E_8	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2+\lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$

Table 2.3: Modular invariant partition functions of $\widehat{\mathrm{sl}}(2)_k$-WZW models associated to simply-laced Dynkin diagrams G.

invariant is given by

$$Z_{(A_4,D_4)}=\frac{1}{2}\sum_{r=1}^{4}\left(|\chi_{r,1}+\chi_{r,5}|^2+2|\chi_{r,3}|^2\right)\ , \tag{2.190}$$

where the factor of half is needed to compensate the overcounting due to the identification $(r,s)\sim(p-r,q-s)$. The non-diagonal modular invariants of Virasoro minimal models can be analysed similarly to the $\widehat{\mathrm{sl}}(2)_k$ case. For example, in $M(5,6)$, the primary field with $h_{1,5}=3$ is a simple current, and (A_4,D_4) invariant corresponds to the W_3 minimal model with $c=\frac{4}{5}$.

The same principle applies to super-Virasoro minimal models while the possible types of modular invariants are slightly different from the Virasoro case. Recall that p and q in $SM(p,q)$ are either both even or both odd; this results in (A_{p-1},A_{q-1}) being the only possibility for $p\in 2\mathbb{Z}+1$, and there are not only invariants of the form (A,A), (A,D),

(D, A), (A, E), and (E, A) but also (D, E) and (E, D) are possible for $p \in 2\mathbb{Z}$. Note that (D, D) is equivalent to either of (A, D) or (D, A). More details can be found in [18].

It is important to note that fully modular invariant partition functions of fermionic theories are necessarily GSO projected. The GSO projection is a projection of both Neveu–Schwarz and Ramond sectors to the states that are invariant under the action of the fermion parity operator $(-1)^{F+\bar{F}}$. In a fermionic theory, the Neveu–Schwarz sector is only invariant under S and T^2, the Ramond sector is invariant under S^2 and T, and TST intertwines these two sectors. This kind of behaviour is not uncommon in theories with extended conformal algebras: twisted sectors of bosonic $\mathcal{W}(2, \delta)$ algebras are only invariant under T^2 and TST while untwisted sectors are fully modular invariant; modular invariant partition functions are obtained by orbifolding with respect to outer automorphisms of W-algebras[46].

2.3 Conformal Boundaries

So far, we have been considering CFTs defined on closed surfaces but it is possible to construct CFTs on surfaces with boundaries.

For concreteness, let us consider the closure of upper half plane $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ and construct a CFT with the boundary at the real axis $z = \bar{z}$. Away from the boundary, we can treat infinitesimal conformal transformations for z and $\bar{z}$ independently, and thus there are holomorphic and antiholomorphic copies of the Virasoro algebra, or its extension, as usual. But the situation changes at the boundary; from the physical perspective, we would like to have conformal transformations that preserve the boundary, and therefore the transformations for z and $\bar{z}$ are no longer independent. If the condition

$$T(z) = \overline{T}(\bar{z}) \quad \text{at} \quad z = \bar{z} \in \mathbb{R}\,, \tag{2.191}$$

holds, then this boundary is said to be conformal. This condition can be viewed as the statement that there is no energy flow across the boundary. In addition, this restricts the conformal group to $\mathrm{PSL}(2, \mathbb{R})$ which maps $\overline{\mathbb{H}}$ to itself and, in particular, leaves the real axis invariant. Note that the condition (2.191) implies we need $c = \bar{c}$ in order for conformal boundaries to exist.

If we perform radial quantization about the origin, this results in the single set of $\mathcal{V}\mathrm{ir}$ generators

$$L_n^{(\mathrm{UHP})} = \oint_{C_+} \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z) - \oint_{C_+} \frac{\mathrm{d}\bar{z}}{2\pi i} \bar{z}^{n+1} \overline{T}(\bar{z})\,, \tag{2.192}$$

where the contour C_+ is a semicircle on the upper half plane, which includes the real axis and the origin. Since $T(z)$ and $\overline{T}(\bar{z})$ are independent away from the boundary, we may "unfold" the antiholomorphic part of the theory about the real axis to the lower half plane, and define

$$\mathsf{T}(z) := \begin{cases} T(z) & \text{for} \quad \operatorname{Im} z \geq 0 \\ \overline{T}(\bar{z}) & \text{for} \quad \operatorname{Im} z < 0 \end{cases}\,, \tag{2.193}$$

which can be regarded as the stress-tensor of the chiral theory which is defined on the entire complex plane. Then, $L_n^{(\mathrm{UHP})}$ corresponds to the usual Laurent modes of $\mathsf{T}(z)$.

For theories with extended conformal symmetries, we may impose boundary conditions that preserve extra symmetries as well. Consider a theory with chiral algebras $\mathcal{A} \cong \overline{\mathcal{A}}$ and corresponding generators $W^{(i)}(z)$ and $\overline{W}^{(i)}(\bar{z})$. If the boundary at $z = \bar{z}$ preserves $\mathcal{A}$, then we should also have

$$W^{(i)}(z) = \Omega \overline{W}^{(i)}(\bar{z}) \quad \text{at} \quad z = \bar{z} \in \mathbb{R} , \tag{2.194}$$

where Ω is a local automorphism of $\mathcal{A}$ which leaves $\overline{T}(\bar{z})$ invariant. An automorphism Ω induces a permutation ω of the labelling set for $\mathcal{A}$ representations given by the isomorphism of representations

$$(\rho_i \circ \Omega \, , \, \mathcal{H}_i) \cong (\rho_{\omega(i)} \, , \, \mathcal{H}_{\omega(i)}) \, , \tag{2.195}$$

where $\rho_i : \mathcal{A} \to \mathcal{H}_i$ is a representation map. If Ω is an outer automorphism, then i and $\omega(i)$ are inequivalent[109].

2.3.1 Boundary Fields

Using the definitions (2.193) and (2.192) of $\mathsf{T}(w)$ and $L_n^{(\mathrm{UHP})}$, one can calculate its OPE with a bulk primary field $\varphi_I(z, \bar{z})$, and equivalently the commutator

$$[L_n^{(\mathrm{UHP})}, \varphi_I(z, \bar{z})] = \left(h_I(n+1)z^n + z^{n+1}\partial\right)\varphi_I(z, \bar{z}) + \left(\bar{h}_I(n+1)\bar{z}^n + \bar{z}^{n+1}\bar{\partial}\right)\varphi_I(z, \bar{z}) \, . \tag{2.196}$$

We take $I = (i, \bar{i})$ for some $\mathcal{A} \cong \overline{\mathcal{A}}$ representation labels i and $\bar{i}$. From the "unfolded" point of view, the bulk field $\varphi_I(z, \bar{z})$ behaves as two chiral fields $\phi_i(z)$ and $\phi_{\omega(\bar{i})}(\bar{z})$ in the presence of a conformal boundary labelled by $a = (\alpha, \Omega)$ at $z = \bar{z}$. In this case, $\bar{z}$ explicitly means the complex conjugate of z, and the two chiral fields are located on the opposite halves of the plane as depicted in Figure 2.4. Then, the one-point function of $\varphi_I(z, \bar{z})$ can be considered as the two-point function of $\phi_i(z)$ and $\phi_{\omega(\bar{i})}$ which should vanish unless $i^+ = \omega(\bar{i})$. Therefore, we can write

$$\langle \varphi_I(z, \bar{z}) \rangle_a = \frac{A_I^a}{|z - \bar{z}|^{h_I + \bar{h}_I}} \, , \tag{2.197}$$

where A_I^a is some constant, which depends on the boundary condition a and vanishes unless $i^+ = \omega(\bar{i})$. Non-vanishing one-point functions depend on their distances from the boundary; this is not surprising as translations in the imaginary direction are no longer symmetries due to the boundary along the real axis.

We can view the gluing conditions for W generators (2.194) and the upper half plane one-point functions (2.197) in another way. Instead of taking the usual[109] analytic continuation $\mathsf{W}^{(i)}(z) = \Omega \overline{W}^{(i)}(\bar{z})$ for $\operatorname{Im} z < 0$, we can simply set $\mathsf{W}^{(i)}(z) = \overline{W}^{(i)}(\bar{z})$ on the lower half plane but take the whole plane with a chiral topological defect D_a at $z = \bar{z}$ whose action implements the automorphism Ω. Then, the one-point function of $\varphi_I(z, \bar{z})$ is given by the two-point function of $\phi_i(z)$ and $\phi_{\bar{i}}(\bar{z})$ with D_a which corresponds to the picture on the right side of Figure 2.4.

One-point functions (2.197) can become singular as they approach the boundary. Since OPEs can be considered as a way of expressing singular behaviours of two-point functions, we can view the singularity of (2.197) as an OPE of the two chiral fields resulting in fields

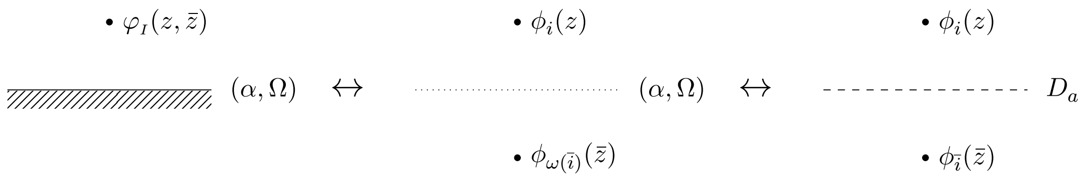


Figure 2.4: Boundary CFT on the upper half plane and its analytic continuations to the whole plane.

on the boundary. By introducing boundary fields $\psi_i^{(a)}(x)$ that only live on the boundary, a bulk–boundary OPE is given by

$$\varphi_I(z,\bar{z}) = \sum_j B_{Ij}^{(a)} |z-\bar{z}|^{h_j - h_I - \bar{h}_I} \left(\psi_j^{(a)}(x) + \cdots \right) , \tag{2.198}$$

where $x = \mathrm{Re}\, z$ and $B_{Ij}^{(a)}$ are called the bulk–boundary couplings. On the right hand side of (2.198), omitted terms are descendants of the primary field $\psi_j^{(a)}$. If a bulk field is close to the boundary labelled by a, we can view it as boundary fields $\psi_i^{(a)}(x)$. By computing the one-point functions of the both sides of (2.198), we obtain

$$B_{I0}^{(a)} = \frac{A_I^a}{A_{\mathbf{1}}^a} , \tag{2.199}$$

where $\psi_0^{(a)}$ is the identity field on the boundary, and $A_{\mathbf{1}}^a = \langle \mathbf{1} \rangle_a$ is the one-point function of the identity field which is not usually free to normalise it to one. Using (2.196) and (2.198), one can derive

$$[L_n^{(\mathrm{UHP})}, \psi_i^{(a)}(x)] = \left(h_i(n+1)x^n + x^{n+1}\frac{\mathrm{d}}{\mathrm{d}x} \right) \psi_i^{(a)}(x) , \tag{2.200}$$

which implies that we can take i as a label for an irreducible representation of one copy of $\mathcal{V}\mathrm{ir}$ or its extension. For minimal theories, the state space of boundary fields living on a boundary labelled by a can be written as

$$\mathcal{H}_a = \bigoplus_{i \in \mathcal{I}} n_i^a \, \mathcal{H}_i , \tag{2.201}$$

where $n_i^a \in \mathbb{Z}_{\geq 0}$ counts multiplicities, and $\mathcal{I}$ is the indexing set for irreducible representations. If there is more than one boundary field carrying the same representation of the chiral algebra, that is when $n_i^a > 1$, these boundary fields must be distinguished by introducing the multiplicity label $\alpha \in [1, n_i^a]$ and writing $\psi_{i;\alpha}^{(a)}$. We make multiplicity labels implicit as long as they are not important, in order to avoid unnecessary cluttering of notations. In addition, if the vacuum state is non-degenerate, that is $n_0^a = 1$, the boundary condition a is said to be elementary.

As we can rewrite bulk two-point functions as boundary two-point functions using the bulk–boundary OPEs (2.198), we can expect boundary OPEs of the form

$$\psi_i^{(a)}(x)\psi_j^{(a)}(y) = \sum_k C_{ij}^{(a)k} (x-y)^{h_k - h_i - h_j} \left(\psi_k^{(a)}(y) + \cdots \right) , \tag{2.202}$$

where $C_{ij}^{(a)k}$ are called boundary structure constants. This equation is understood to hold for $x > y$. Since boundary fields are only defined on a boundary, analytic continuations of (2.202) to $y > x$ are not unique in general. Therefore, $C_{ij}^{(a)k}$ and $C_{ji}^{(a)k}$ are not always equal up to signs unlike bulk fields.

We may allow boundary conditions to change along boundaries. If the boundary condition changes from a for $x < 0$ to b for $x > 0$, this gives rise to a boundary field $\psi_i^{(ab)}(x)$ at the origin which is also called a boundary condition changing field. Since boundary fields $\psi_i^{(a)}(x)$ can be considered as the boundary condition changing fields changing the a-boundary to itself, we simply call both of them boundary fields. In this case, boundary OPEs (2.202) generalise to

$$\psi_i^{(ab)}(x)\psi_j^{(bc)}(y) = \sum_k C_{ij}^{(abc)k}(x-y)^{h_k-h_i-h_j}\left(\psi_k^{(ac)}(y)+\cdots\right) \tag{2.203}$$

for $x > y$. In this case, the state space of boundary fields on a a–b boundary junction is denoted by

$$\mathcal{H}_{ab} = \bigoplus_{i\in\mathcal{I}} n_{ab}^i \mathcal{H}_i \ , \tag{2.204}$$

where $n_{ab}^i \in \mathbb{Z}_{\geq 0}$ as usual.

2.3.2 Boundary States

Consider mapping a boundary CFT on the upper half plane considered in the previous subsection to the whole plane with the unit disk removed. If we use a contour encircling the unit disk for radial quantisation, we can consider the boundary as a certain state in the completion of bulk state space, which is called a boundary state.

On the upper half plane with coordinates $z = x + iy$, radial quantisation about the origin gives $|z| =$ constant as the equal-time surfaces. Then, the real axis can be regarded as two portions of the spatial boundary with boundary conditions a for $x < 0$ and b for $x > 0$. We can map the upper half plane to an infinite strip of width $L \in \mathbb{R}_{>0}$ by

$$w = \frac{L}{\pi}\ln z \ . \tag{2.205}$$

The spatial boundaries correspond to the parallel edges of the strip. If we make this strip periodic in time by imposing

$$w \sim w + nR \quad \forall n \in \mathbb{Z} \ , \tag{2.206}$$

where $R \in \mathbb{R}_{>0}$ can be regarded as the length of a finite strip. Then, we can map this strip to an annulus by

$$\zeta = e^{-\frac{2\pi i}{R}w} \ . \tag{2.207}$$

The two circles of the annulus correspond to the boundaries as depicted in Figure 2.5. If we swap the roles of time and space, and consider radial quantization about the origin of the ζ-plane with contours defined on the annulus, the inner circle can be regarded as a boundary state $\|b\rangle\rangle$ and the outer circle corresponds to a “dual”[20] boundary state $\langle\langle a\|$.

20. As we shall see later, the usual inner product $\langle\langle a\|a\rangle\rangle$ diverges. We will elaborate on in which sense $\langle\langle a\|$ is considered as a dual of $\|a\rangle\rangle$.

Using (2.51), the gluing condition (2.191) becomes

$$\left(L_n - \bar{L}_{-n}\right) \|b\rangle\rangle = 0 \; . \tag{2.208}$$

Since $W^{(i)}(z)$ is a chiral primary field with conformal weight h_i, we can use (2.17) to rewrite (2.194) as

$$\left(W_n^{(i)} - (-1)^{h_i}\,\Omega\overline{W}_{-n}^{(i)}\right) \|b\rangle\rangle = 0 \; . \tag{2.209}$$

As we shall see later, boundary states can be used to determine the spectra of boundary fields as well as a consistent set of boundary conditions.

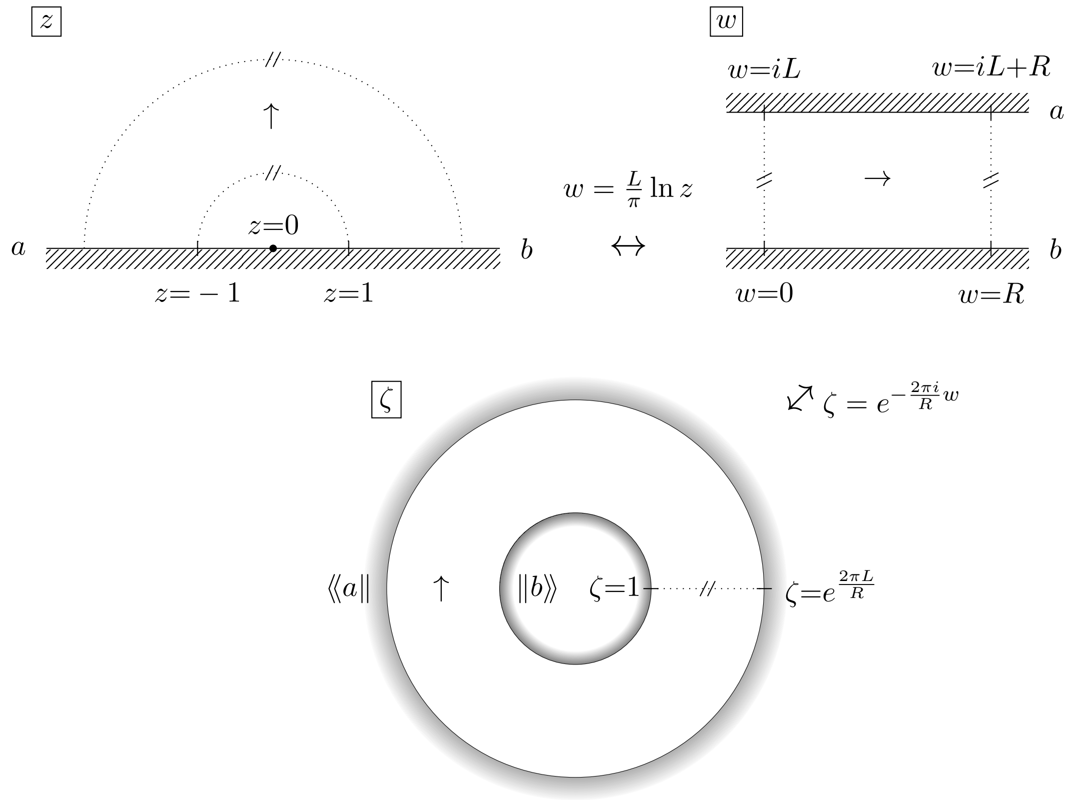


Figure 2.5: Mapping boundary CFT on the upper half plane to a cylinder, and then to an annulus.

From the condition (2.208) for $n = 0$, we see that a boundary state lives in a subspace of the bulk state space with $h = \bar{h}$. Moreover, (2.208) and (2.209) implies that a boundary state $\|b\rangle\rangle$ acts as an intertwiner for $\mathcal{A}$ irreducible representations, and from Schur's lemma, $\|b\rangle\rangle$ must be composed of zero maps and isomorphisms. For each irreducible representation $\mathcal{H}_i$ of the chiral algebra $\mathcal{A}$, the Ishibashi state[29], 21 is given by

$$|i\rangle\rangle = \sum_{n=0}^{\infty} |i;n\rangle \otimes U\overline{|i;n\rangle} \; , \tag{2.210}$$

where $|i;n\rangle$ is an orthonormal basis of $\mathcal{H}_i$ with $|i;0\rangle$ being the highest weight state, and the antiunitary[22] operator U satisfies

$$U\bar{L}_n = \bar{L}_n U \quad \text{and} \quad U|i;0\rangle = |i^+;0\rangle \; . \tag{2.211}$$

21. We use $|i\rangle\rangle$ to denote an Ishibashi state, and $\|b\rangle\rangle$ means a generic boundary state.
22. One has to be careful when using the Dirac notation with antiunitary operators. For some Hilbert space $\mathcal{H}$

For other generators of $\mathcal{A}$, U satisfies

$$U\overline{W}_n^{(i)} = (-1)^{h_i}\overline{W}_n^{(i)}U \qquad \text{for} \quad h_i \in \mathbb{Z}\ , \quad \text{and}$$
$$U\overline{W}_n^{(i)} = (-1)^{h_i}\overline{W}_n^{(i)}U(-1)^{\bar{F}} \quad \text{for} \quad h_i \in \mathbb{Z} + \frac{1}{2}\ . \tag{2.212}$$

Then, the Ishibashi state $|i\rangle\!\rangle$ is a solution of (2.208) and (2.209) with $\Omega = \mathrm{id}$. The antiunitary operator maps $U : \overline{\mathcal{H}}_i \to \overline{\mathcal{H}}_{i^+}$, therefore $|i\rangle\!\rangle \in \mathcal{H}_i \otimes \overline{\mathcal{H}}_{i^+}$. Since $\overline{\mathcal{H}}_{i^+}$ is isomorphic to the dual space of $\overline{\mathcal{H}}_i$, we can view $|i\rangle\!\rangle$ as an intertwiner of the chiral algebra representations $\overline{\mathcal{H}}_i$ and $\mathcal{H}_i$. Ishibashi states are unique up to overall normalisations.

Since a bulk one-point function (2.197) vanishes unless $i^+ = \omega(\bar{i})$, we expect an Ishibashi state $|i\rangle\!\rangle_\Omega$, which is a solution of (2.209) for non-trivial Ω, to be an element of $\mathcal{H}_i \otimes \overline{\mathcal{H}}_{\omega^{-1}(i^+)}$. If we define a unitary operator

$$V_\Omega : \mathcal{H}_{i^+} \to \mathcal{H}_{\omega^{-1}(i^+)} \tag{2.213}$$

such that

$$\rho_{\omega^{-1}(i^+)}(\Omega x) = V_\Omega\, \rho_{i^+}(x)\, V_\Omega^{-1} \quad \forall x \in \mathcal{A}\ , \tag{2.214}$$

then a twisted Ishibashi state $|i\rangle\!\rangle_\Omega$ can be written as[109]

$$|i\rangle\!\rangle_\Omega = (\mathbf{1} \otimes V_\Omega)|i\rangle\!\rangle \tag{2.215}$$

where $|i\rangle\!\rangle \in \mathcal{H}_i \otimes \overline{\mathcal{H}}_{i^+}$ is given by (2.210).

We define a dual Ishibashi state as

$$\langle\!\langle i| = \sum_{n=0}^{\infty} \langle i;n| \otimes \overline{\langle U(i;n)|}\ , \tag{2.216}$$

where $\langle i;n|$ is the dual vector of $|i;n\rangle$ defined by $\langle i;n|j;m\rangle = \delta_{i,j}\delta_{n,m}$. Since U is an antiunitary operator, we can calculate

$$\overline{\langle U(i;n)|}\, U\overline{|j;m\rangle} = \overline{\langle j;m|i;n\rangle} = \delta_{j,i}\delta_{m,n}\ . \tag{2.217}$$

In addition, we define

$${}_\Omega\langle\!\langle i| = \langle\!\langle i|(\mathbf{1} \otimes V_\Omega^\dagger)\ . \tag{2.218}$$

Recall that V_Ω is a unitary operator.

As it is clear from the definitions (2.210) and (2.216), the norm of an Ishibashi state $|i\rangle\!\rangle$ diverges if we use the usual inner product $\langle\!\langle i|i\rangle\!\rangle$. It is not too surprising since boundary states do not correspond to any local fields in the theory. Instead of $\langle\!\langle i|i\rangle\!\rangle$, we may consider

$$\langle\!\langle i|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{c}{12}\right)}|j\rangle\!\rangle = {}_\Omega\langle\!\langle i|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{c}{12}\right)}|j\rangle\!\rangle_\Omega = \delta_{i,j}\,\chi_i(\tilde{q})\ , \tag{2.219}$$

with an inner product $\langle\cdot,\cdot\rangle$, the Dirac notation is understood as $\langle x|A|y\rangle = \langle x, Ay\rangle$, where $x, y \in \mathcal{H}$ and A is an operator acting on $\mathcal{H}$. If A is a linear operator, we can use the definition of adjoint $\langle Ax, y\rangle = \langle x, A^\dagger y\rangle$ and write the "dual vector" as $\langle Ax| = \langle x|A^\dagger$. The problem is that antiunitary operators are antilinear; an antiunitary operator U on $\mathcal{H}$ satisfies $\langle Ux, Uy\rangle = \langle x, y\rangle^* = \langle y, x\rangle$ and its adjoint is defined by $\langle Ux, y\rangle = \langle x, U^\dagger y\rangle^*$ where * means complex conjugation. Therefore, we cannot write a dual vector $\langle Ux|$ using $\langle x|$ and $U^\dagger$.

where $\chi_i(\tilde{q})$ is the Virasoro character of $\mathcal{H}_i$, as an "inner product" for Ishibashi states. The overlaps of Ishibashi states (2.219) converge for $|\tilde{q}| < 1$. As we shall see explicitly for the $N = 1$ super-Virasoro case, overlaps of untwisted Ishibashi states with twisted ones may produce certain "twisted" characters.

So far, Ishibashi states are given in terms of orthonormal bases of irreducible representations. While they are convenient for deriving properties of Ishibashi states, we often prefer them to be written in terms of basis vectors of the form

$$W^{(N)}_{-l_c} \cdots W^{(N)}_{-l_1} \cdots\cdots W^{(1)}_{-m_b} \cdots W^{(1)}_{-m_1} L_{-n_a} \cdots L_{-n_1} |i\rangle \ , \tag{2.220}$$

where $|i\rangle$ is the highest weight vector of $\mathcal{H}_i$, and the modes are ordered similar to (2.96) but $n_1 > 0$, $m_1 > 0$, etc. in this case. It is simpler to implement Ishibashi states in a computer program using vectors of the form (2.220) as we can start from the corresponding Verma module and discard some basis vectors level by level to obtain a basis of the irreducible module[23]. For an irreducible representation $\mathcal{H}_i$ of a given chiral algebra, let $(\mathcal{H}_i)_N$ denote the level N subspace with the L_0-eigenvalue $h_i + N$, and let $|i; N, l\rangle$ be a basis vector of this subspace, which is given in the form (2.220) and labelled by $1 \leq l \leq \dim(\mathcal{H}_i)_N$. Since it is an irreducible representation, the Gram matrix for this subspace is invertible. In this basis, the Gram matrix is given by

$$(\Gamma_N)_{lm} = \langle i; N, l | i; N, m\rangle \ . \tag{2.221}$$

Then, the Ishibashi state corresponding to $\mathcal{H}_i$ can be written as

$$|i\rangle\!\rangle = \sum_{N=0} \sum_{l,m=1}^{\dim(\mathcal{H}_i)_N} (\Gamma_N^{-1})_{lm} \, |i; N, l\rangle \otimes U\overline{|i; N, m\rangle} \ . \tag{2.222}$$

Using (2.211) and (2.212), we can move U all the way to the right and act it on the highest weight vector $|i\rangle$. Therefore, we can write

$$U\overline{|i; N, m\rangle} = u_{N,m}\overline{|i^+; N, m\rangle} \ , \tag{2.223}$$

where $u_{N,m} \in \mathbb{C}$ is a factor which depends on $\overline{|i; N, m\rangle}$. Once we have obtained an Ishibashi state $|i\rangle\!\rangle$, the corresponding dual Ishibashi state $\langle\!\langle i|$ can be constructed using (2.75) and taking the complex conjugates of the coefficients.

Any boundary state can be written as a linear combination of Ishibashi states

$$\begin{aligned} \|b\rangle\!\rangle &= \sum_{i \in \mathcal{I}_\mathrm{B}} g_b^i \, |i\rangle\!\rangle \quad \text{if} \quad b = (\beta, \mathrm{id}) \ , \quad \text{or} \\ \|b\rangle\!\rangle &= \sum_{i \in \mathcal{I}_\mathrm{B}^\Omega} g_b^i \, |i\rangle\!\rangle_\Omega \quad \text{if} \quad b = (\beta, \Omega) \ , \end{aligned} \tag{2.224}$$

where $g_b^i \in \mathbb{C}$. The indexing sets $i \in \mathcal{I}_\mathrm{B}$ and $i \in \mathcal{I}_\mathrm{B}^\Omega$ are given by

$$\mathcal{I}_\mathrm{B} = \{i \in \mathcal{I} : (i, i^+) \in \mathcal{S}\} \quad \text{and} \quad i \in \mathcal{I}_\mathrm{B}^\Omega = \{i \in \mathcal{I} : (i, \omega^{-1}(i^+)) \in \mathcal{S}\} \ , \tag{2.225}$$

23. This procedure may not be unique in general but it will give a basis of the irreducible module since vectors of the form (2.220) are linearly independent.

where $\mathcal{I}$ is the indexing set for $\mathcal{A}$ irreducible representations, and $\mathcal{S}$ is the spectrum of bulk fields. Similarly, dual boundary states can be written as

$$\langle\!\langle b \| = \sum_{i\in\mathcal{I}_{\mathrm{B}}} \langle\!\langle i \| \, \bar{g}_b^i \quad \text{if} \quad b = (\beta, \mathrm{id}) \ , \quad \text{or}$$
$$\langle\!\langle b \| = \sum_{i\in\mathcal{I}_{\mathrm{B}}^{\Omega}} {}_{\Omega}\langle\!\langle i \| \, \bar{g}_b^i \quad \text{if} \quad b = (\beta, \Omega) \ , \tag{2.226}$$

where $\bar{g}_b^i$ is the complex conjugate of g_b^i. In addition, we have[67, 109] $\bar{g}_b^i = g_b^{i^+} = g_{b^+}^i$, where b^+ is understood as a "charge conjugation", or a certain involution on the labelling set for boundary conditions, which is defined by this relation.

2.3.3 Cardy Constraints

Let us go back to the situation depicted in Figure 2.5. As before, we take the w-plane to be periodic in the time direction with the condition (2.206). On this strip, we can write the Hamiltonian as

$$H_{\text{str.}} = \frac{2\pi}{L}\left(L_0^{(\mathrm{UHP})} - \frac{c}{24}\right) \ . \tag{2.227}$$

Using this Hamiltonian, the partition function on this strip is given by

$$Z_{ab} = \mathrm{Tr}_{\mathcal{H}_{ab}} \, e^{-RH_{\text{str.}}} = \mathrm{Tr}_{\mathcal{H}_{ab}} \, q^{L_0^{(\mathrm{UHP})} - \frac{c}{24}} \ , \tag{2.228}$$

where $q = e^{2\pi i\tau}$ with $\tau = \frac{iR}{2L}$. Since the state space $\mathcal{H}_{ab}$ can be written as (2.204), we obtain

$$Z_{ab} = \sum_{i\in\mathcal{I}} n_{ab}^i \, \chi_i(q) \ . \tag{2.229}$$

If we swap the roles of space and time, we can view this strip as a cylinder with the circumference R and the length L, which is equivalent to the annulus in Figure 2.5. Using the Hamiltonian on this cylinder

$$H_{\text{cyl.}} = \frac{2\pi}{R}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \ , \tag{2.230}$$

the amplitude between the two boundary states can be written as

$$\langle\!\langle b \| e^{-LH_{\text{cyl.}}} \| a \rangle\!\rangle = \langle\!\langle b \| \tilde{q}^{\frac{1}{2}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} \| a \rangle\!\rangle = \sum_{i\in\mathcal{I}_{\mathrm{B}}} \bar{g}_b^i \, g_a^i \, \chi_i(\tilde{q}) \ , \tag{2.231}$$

where $\tilde{q} = e^{-2\pi i/\tau}$, and we have used (2.219), (2.224), and (2.226) assuming the both boundary states are untwisted. Since the two quantities (2.229) and (2.231) are related by the modular S transformation, we obtain

$$\sum_{i\in\mathcal{I}_{\mathrm{B}}} \sum_{j\in\mathcal{I}} \bar{g}_b^i \, g_a^i \, \mathrm{S}_{ij} \chi_j(q) = \sum_{i\in\mathcal{I}} n_{ab}^i \, \chi_i(q) \ , \tag{2.232}$$

which is called the Cardy constraint[28]. Since $n_{ab}^i \in \mathbb{Z}_{\geq 0}$, if we calculate the overlap of two boundary states, its modular S transformation should be interpretable as a sum of Virasoro characters of $\mathcal{A}$ irreducible representations with non-negative integer coefficients. In addition, if the boundary state coefficients g_a^i are know for a given boundary, then the spectrum of boundary fields n_{aa}^i can be obtained from (2.232). In Boundary CFT, one of the important tasks is to determine the maximal set of elementary boundary conditions that satisfy the Cardy constraint from a given bulk partition function.

2.3.4 A - D - E for Boundaries

For theories with charge conjugate modular invariant bulk partition functions, Cardy[28] gave solutions to the constraint (2.232). For untwisted Ishibashi states, their indexing set (2.225) is the same as that of the $\mathcal{A}$ representation, that is $\mathcal{I}_{\rm B} = \mathcal{I}$. In addition, the labelling set $\mathcal{B}$ for boundaries is the same as $\mathcal{I}$. Cardy's solutions are

$$g_a^i = \frac{\mathrm{S}_{ai}}{\sqrt{\mathrm{S}_{0i}}} \,, \tag{2.233}$$

where S_{ai} are the elements of modular S matrix. Then, the Cardy equation (2.232) gives

$$n_{ab}^i = \sum_{j\in\mathcal{I}} \bar{g}_b^j \, g_a^j \, \mathrm{S}_{ji} = \sum_{j\in\mathcal{I}} g_a^j \, g_{b^+}^j \, \bar{\mathrm{S}}_{ji^+} = \sum_{j\in\mathcal{I}} \frac{\mathrm{S}_{aj} \, \mathrm{S}_{b^+j} \, \mathrm{S}^{-1}_{ji^+}}{\mathrm{S}_{0j}} = N_{ab^+}^{i^+} = N_{a^+b}^{i} \,, \tag{2.234}$$

therefore the Cardy constraint is satisfied as the fusion coefficients are non-negative integers. In addition, these boundary states are elementary as we always have $N_{a^+a}^0 = 1$. Note that we have used properties of the modular S matrix (2.183) to show the Cardy constraint. Boundary states with coefficients of the form (2.233) are called the Cardy boundaries.

In [67], boundary states satisfying the Cardy constraint are systematically constructed for $\widehat{\mathrm{sl}}(2)_k$-WZW and Virasoro minimal models. As it will turn out to be convenient later, we rewrite the boundary state coefficients as

$$g_a^i = \frac{\psi_a^i}{\sqrt{\mathrm{S}_{0i}}} \,. \tag{2.235}$$

Then, the Cardy equation (2.232) becomes

$$n_{ab}^i = \sum_{j\in\mathcal{I}_{\rm B}} \psi_a^j \, \bar{\psi}_b^j \, \frac{\mathrm{S}_{ij}}{\mathrm{S}_{0j}} \,. \tag{2.236}$$

where $\bar{\psi}_a^i$ is the complex conjugate of ψ_a^i, and they satisfy $\bar{\psi}_a^i = \psi_a^{i^+} = \psi_{a^+}^i$. A labelling set for boundary conditions $\mathcal{B}$ is called complete[55, 67] if

$$\sum_{a\in\mathcal{B}} \psi_a^i \, \bar{\psi}_a^j = \delta_{i,j} \,, \tag{2.237}$$

which implies that the number of boundary states $|\mathcal{B}|$ is the same as that of Ishibashi states $|\mathcal{I}_{\rm B}|$ or $|\mathcal{I}_{\rm B}^{\Omega}|$. If boundary states are complete, we can calculate

$$\begin{aligned}
\sum_{b\in\mathcal{B}} n_{ab}^i \, n_{bc}^j &= \sum_{b\in\mathcal{B}} \sum_{l,m\in\mathcal{I}_{\rm B}} \psi_a^l \, \bar{\psi}_b^l \, \frac{\mathrm{S}_{il}}{\mathrm{S}_{0l}} \, \psi_b^m \, \bar{\psi}_c^m \, \frac{\mathrm{S}_{jm}}{\mathrm{S}_{0m}} \\
&= \sum_{l\in\mathcal{I}_{\rm B}} \psi_a^l \, \bar{\psi}_c^l \, \frac{\mathrm{S}_{il} \, \mathrm{S}_{jl}}{(\mathrm{S}_{0l})^2} \\
&= \sum_{l\in\mathcal{I}_{\rm B}} \psi_a^l \, \bar{\psi}_c^l \, \frac{1}{\mathrm{S}_{0l}} \sum_{m\in\mathcal{I}_{\rm B}} \delta_{l,m} \frac{\mathrm{S}_{im} \, \mathrm{S}_{jm}}{\mathrm{S}_{0m}} \\
&= \sum_{l,m\in\mathcal{I}_{\rm B}} \sum_{k\in\mathcal{I}} \psi_a^l \, \bar{\psi}_c^l \, \frac{\mathrm{S}_{kl}}{\mathrm{S}_{0l}} \, \frac{\mathrm{S}_{im} \, \mathrm{S}_{jm} \mathrm{S}^{-1}_{km}}{\mathrm{S}_{0m}} = \sum_{k\in\mathcal{I}} N_{ij}^k \, n_{ac}^k \,.
\end{aligned} \tag{2.238}$$

By defining $|\mathcal{B}| \times |\mathcal{B}|$ matrices $(\mathrm{n}_i)_{ab} = n^i_{ab}$, we see that they form a NIM-rep of the fusion algebra

$$\mathrm{n}_i\,\mathrm{n}_j = \sum_{k\in\mathcal{I}} N^k_{ij}\,\mathrm{n}_k\ . \tag{2.239}$$

From (2.236), we have a relation $n^i_{ab} = n^{i^+}_{ba} = n^i_{b^+a^+}$, and therefore the matrices satisfy $\mathrm{n}_{i^+} = \mathrm{n}_i^{\mathrm{T}}$. Furthermore, if the boundary states are orthonormal, that is, they satisfy

$$\sum_{i\in\mathcal{I}_{\mathrm{B}}} \psi^i_a\,\bar{\psi}^i_b = \delta_{a,b}\ , \tag{2.240}$$

then $\mathrm{n}_0 = \mathbf{1}$ is the identity matrix. As a representation of the fusion algebra, the matrices n_i commute each other, and they can be simultaneously diagonalised. Using the diagonal matrix D_i defined in (2.175) and writing the boundary state coefficients as a matrix $(\psi)_{ai} = \psi^i_a$, the Cardy equation (2.236) can be written as[77]

$$\mathrm{n}_i = \psi\,\mathrm{D}_i\,\psi^\dagger\ , \tag{2.241}$$

from which we see that ψ diagonalises n_i, and therefore ψ can be constructed as a matrix of eigenvectors of n_i.

For $\widehat{\mathrm{sl}}(2)_k$-WZW theories, fusion rules are given by

$$(i)\otimes(j) = \bigoplus_{\substack{l=|i-j|+1\\ i+j+l=1 \mod 2}}^{\min\{i+j-1,2k-1-i-j\}} (l)\ , \tag{2.242}$$

where $i,j,l \in [1,k+1]$ label integrable highest weight representations of $\widehat{\mathrm{sl}}(2)_k$. Since $(2)\otimes(2) = (1)\oplus(3)$, it is possible to obtain fusion matrices N_i from the repeated actions of N_2 using the recursion relation

$$\mathrm{N}_{i+1} = \mathrm{N}_2\,\mathrm{N}_i - \mathrm{N}_{i-1}\ . \tag{2.243}$$

This applies to the matrices n_i as well, which means that we can obtain a set of boundary states if we could find n_2. As shown in [67], it turns out that n_2 is an adjacency matrix of A - D - E (and tadpole) diagrams by considering the eigenvalue matrix D_2. In addition, the indexing set $\mathcal{I}_{\mathrm{B}}$ for Ishibashi states is given by the set of exponents of the corresponding diagram. For a given bulk partition function associated with a diagram G of A - D - E type, boundary states satisfying the Cardy constraint can be obtained from an eigenvector matrix of the adjacency matrix of G.

For a Virasoro minimal model $M(p,q)$, one can use the fact[67] that the fusion matrices can be written in terms of tensor (Kronecker) products of the $\widehat{\mathrm{sl}}(2)_k$ fusion matrices at level $k=p-2$ and level $k=q-2$. Denoting a $\widehat{\mathrm{sl}}(2)_k$ fusion matrix by $\mathrm{N}^{(k)}_i$, fusion matrices of $M(p,q)$ are given by

$$\mathrm{N}_{r,s} = \mathrm{N}^{(p-2)}_r \otimes \mathrm{N}^{(q-2)}_s + \mathrm{N}^{(p-2)}_{p-r} \otimes \mathrm{N}^{(q-2)}_{q-s}\ . \tag{2.244}$$

Since one of the diagrams denoting a bulk partition function is always of A_n type, it suffices to consider an invariant of the form (A_{p-1}, G) for $M(p,q)$. In this case, the matrices $\mathrm{n}_{r,s}$ can be written as

$$\mathrm{n}_{r,s} = \mathrm{N}^{(p-2)}_r \otimes \mathrm{n}_s + \mathrm{N}^{(p-2)}_{p-r} \otimes \mathrm{n}_{q-s}\ , \tag{2.245}$$

where n_s, which is often referred to as a fused adjacency matrix, is constructed from n_2, the adjacency matrix of G. Boundary states are labelled by (r, a) where r and a are nodes of A_{p-1} and G Dynkin diagrams, respectively. When G is one of A_n, D_{odd}, or E_6, there is an identification of boundary states given by

$$(r, a) \sim (p - r, \gamma(a)) \,, \tag{2.246}$$

where γ is the $\mathbb{Z}_2$ automorphism of the Dynkin diagram. For other cases, we have

$$(r, a) \sim (p - r, a) \,. \tag{2.247}$$

The set of Ishibashi states is given by

$$\mathcal{I}_{\mathrm{B}} = \{(r, s) \sim (p - r, q - s) : 1 \le r \le p - 1 \quad \text{and} \quad s \in \mathcal{E}(G)\} \,, \tag{2.248}$$

where $\mathcal{E}(G)$ denotes the set of exponents of the G Dynkin diagram which can be found in Table 2.2. The boundary state coefficients are given by

$$\Psi^{(r,s)}_{(r',a)} = \sqrt{2}\, \mathrm{S}^{(p-2)}_{r'r}\ {}^{(G)}\psi^s_a \,, \tag{2.249}$$

where $\mathrm{S}^{(p-2)}_{r'r}$ is an element of the $\widehat{\mathrm{sl}}(2)_k$ modular S matrix

$$\mathrm{S}^{(k)}_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi i j}{k+2}\right) \tag{2.250}$$

with $k = p - 2$, and ${}^{(G)}\psi^s_a$ is an eigenvector of the adjacency matrix of G whose explicit expression can be found in [67]. Then, the boundary states corresponding to the (A_{p-1}, G) bulk modular invariant can be written as

$$\|r', a\rangle\!\rangle = \sum_{(r,s)\in\mathcal{I}_{\mathrm{B}}} \frac{\Psi^{(r,s)}_{(r',a)}}{\sqrt{\mathrm{S}_{(1,1)(r,s)}}} \, |r, s\rangle\!\rangle \,, \tag{2.251}$$

where $\mathrm{S}_{(1,1)(r,s)}$ is an element of the modular S matrix of $M(p, q)$ which is given in (A.18). If we consider a unitary Virasoro minimal model, the modular S matrices of $M(p, p+1)$ and $\widehat{\mathrm{sl}}(2)_k$ are related by

$$\mathrm{S}_{(r_1,s_1)(r_2,s_2)} = \sqrt{2}(-1)^{(r_1+s_1)(r_2+s_2)}\, \mathrm{S}^{(p-2)}_{r_1 r_2}\, \mathrm{S}^{(p-1)}_{s_1 s_2} \,, \tag{2.252}$$

from which we can understand the factor of $\sqrt{2}$ in (2.249) since ${}^{(A_p)}\psi^s_a = \mathrm{S}^{(p-1)}_{as}$.

2.4 Conformal Defects

Defects in two-dimensional CFTs are one-dimensional objects that can be considered as inhomogeneities in a theory or interfaces between two CFTs.

As an example, let us consider a situation where the upper and lower half planes correspond to, possibly different, CFTs separated by a defect along the real axis. We denote the stress-energy tensors on the upper half plane by $T^{(1)}(z)$ and $\overline{T}^{(1)}(\bar{z})$, and those on the lower half plane by $T^{(2)}(z)$ and $\overline{T}^{(2)}(\bar{z})$. We may call these theories CFT_1 and CFT_2. For the moment, we consider $\bar{z}$ to be independent from z. If the condition

$$T^{(1)}(z) - \overline{T}^{(1)}(\bar{z}) = T^{(2)}(z) - \overline{T}^{(2)}(\bar{z}) \quad \text{at} \quad z = \bar{z} \in \mathbb{R} \tag{2.253}$$

holds, then this defects is said to be conformal. In the Cartesian coordinates $z = x + iy$, this means $T^{(1)}_{xy} = T^{(2)}_{xy}$, that is, the total momentum is conserved across the defect. If we use transformations

$$w = \frac{1 - iz}{1 + iz} \quad \text{and} \quad \bar{w} = \frac{1 - i\bar{z}}{1 + i\bar{z}} , \tag{2.254}$$

then the real axis on the z-plane is mapped to the unit circle on the w-plane, and the upper half plane is mapped to the exterior of the unit circle. Using (2.51), and considering radial quantisation about the origin of the w-plane, the condition (2.253) becomes

$$(L^{(1)}_n - \bar{L}^{(1)}_{-n})D = D(L^{(2)}_n - \bar{L}^{(2)}_{-n}) , \tag{2.255}$$

where the defect operator $D : \mathcal{H}_2 \to \mathcal{H}_1$ is a map from the bulk state space of CFT_2 to that of CFT_1. By considering a map similar to (2.254), which sends the upper half plane to the interior of the unit circle, we can obtain the operator of the orientation reversed defect $D^\dagger : \mathcal{H}_1 \to \mathcal{H}_2$ which satisfies the conformal condition (2.255) with the labels (1) and (2) exchanged.

Going back to the z-plane, we can consider "folding"[54, 59] the whole theory about the real axis, and obtain the product theory $\mathrm{CFT}_1 \times \overline{\mathrm{CFT}}_2$ which is defined only on the upper half plane. By $\overline{\mathrm{CFT}}_2$, we mean the holomorphic sector of CFT_2 becomes a part of the antiholomorphic sector of the product theory, and vice versa. For z and $\bar{z}$ on the upper half plane, the stress-energy tensors of the product theory are defined by

$$T(z) = T^{(1)}(z) + \overline{T}^{(2)}(z^*) \quad \text{and} \quad \overline{T}(\bar{z}) = \overline{T}^{(1)}(\bar{z}) + T^{(2)}(\bar{z}^*) , \tag{2.256}$$

where * means complex conjugation, and the quantities on the right hand sides are defined on the whole plane as z and z^* will correspond to the same point after folding. Then, the condition (2.253) becomes

$$T(z) = \overline{T}(\bar{z}) \quad \text{at} \quad z = \bar{z} \in \mathbb{R} , \tag{2.257}$$

and we see that conformal defects correspond to conformal boundaries in the folded theories. In this sense, defects satisfying the condition (2.253) are called conformal. From the condition (2.257), the central charges have to satisfy $c_1 + \bar{c}_2 = \bar{c}_1 + c_2$ in order to have conformal boundary conditions corresponding to conformal defects between the two theories.

There are two obvious solutions to the equation (2.255). The first one is given by

$$(L^{(1)}_n - \bar{L}^{(1)}_{-n})D = 0 \quad \text{and} \quad D(L^{(2)}_n - \bar{L}^{(2)}_{-n}) = 0 , \tag{2.258}$$

and D acts as boundary states of CFT_1 and CFT_2. Such conformal defects are called factorising defects, and their operators can be expressed as a sum of

$$D = \|a\rangle\!\rangle\langle\!\langle b\| , \tag{2.259}$$

where $\|a\rangle\rangle$ and $\langle\langle b\|$ are boundary states of CFT_1 and CFT_2 respectively. Since the two theories need to have conformal boundaries, the requirement for factorising defects is $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$. The second solution is given by

$$L_n^{(1)} D = D L_n^{(2)} \quad \text{and} \quad \bar{L}_n^{(1)} D = D \bar{L}_n^{(2)} \,, \tag{2.260}$$

and conformal defects satisfying this condition are called topological defects. They act as intertwiners for each of holomorphic and antiholomorphic representations of the Virasoro algebra, and therefore they can be moved freely without changing the values of correlators as long as they do not cross field insertions or other defects. Topological defects only exist for the cases when $c_1 = c_2$ and $\bar{c}_1 = \bar{c}_2$.

2.4.1 Topological Defects

From (2.260), we can view topological defects as interfaces relating two CFTs at the same central charges. If they are described by the same modular invariant partition function, topological defects can be considered as something internal to the theory. In the context of two-dimensional CFT, topological defects were first studied in [74] and [75] from this perspective. On the other hand, topological defects can relate theories with different modular invariants. In [89], such topological defects were formulated in terms of the topological field theory (TFT) approach to RCFT[76, 84, 85, 86, 87], where the topological defects between the $c = \frac{4}{5}$ CFTs, the tetra-critical Ising model and the three-states Potts model, were constructed as an example. Topological defects between the free boson CFTs with different compactification radii are studied in [90]. For rational CFTs, all the correlators involving topological defects can be obtained by the TFT approach. In particular, it gives the classifying algebra for topological defects[102], which is similar to the sewing constraints for conformal boundaries. In the context of TFT approach, topological defects were studied, for example, in [83] and [89].

Topological defects can exist within a CFT. In this case, the condition (2.260) becomes

$$[L_n, D] = 0 \quad \text{and} \quad [\bar{L}_n, D] = 0 \,. \tag{2.261}$$

If the chiral algebras are larger than the Virasoro algebra, it is possible to take D to commute or anticommute with the extra generators but, for now, let us consider Virasoro minimal models. Then, the bulk state space can be written as

$$\mathcal{H} = \bigoplus_{i,\bar{i} \in \mathcal{I}} M_{i\bar{i}} \, \mathcal{H}_i \otimes \overline{\mathcal{H}}_{\bar{i}} \,, \tag{2.262}$$

where $\mathcal{I}$ is the indexing set for the Virasoro irreducible representations $\mathcal{H}_i$, and non-negative integers $M_{i\bar{i}}$ are the multiplicities. For brevity, let us denote

$$\mathcal{H}_i \otimes \overline{\mathcal{H}}_{\bar{i}} = \mathcal{H}_{i,\bar{i}}^{(\alpha)} \tag{2.263}$$

if $M_{i\bar{i}} \neq 0$, and $\alpha \in [1, M_{i\bar{i}}]$ distinguishes degenerate representations. Since topological defects act as intertwiners for the irreducible representations (2.263), the operators D

should consist of isomorphisms and zero maps for these subspaces of $\mathcal{H}$. Degenerate representations are isomorphic as vector spaces, and therefore we can introduce projectors

$$P_{i,\bar{i};\alpha,\alpha'} : \mathcal{H}_{i,\bar{i}}^{(\alpha')} \to \mathcal{H}_{i,\bar{i}}^{(\alpha)} \tag{2.264}$$

that act as the identity maps of the vector spaces. We define

$$(P_{i,\bar{i};\alpha,\alpha'})^{\dagger} = P_{i,\bar{i};\alpha',\alpha} , \tag{2.265}$$

which can be regarded as a part of the orientation reversed topological defect. As projectors, they satisfy

$$P_{i,\bar{i};\alpha,\alpha'} P_{j,\bar{j};\beta,\beta'} = \delta_{i,j}\, \delta_{\bar{i},\bar{j}}\, \delta_{\alpha',\beta}\, P_{i,\bar{i};\alpha,\beta'} . \tag{2.266}$$

Then, we may write an operator satisfying (2.261) as

$$D_a = \sum_{i,\bar{i}\in\mathcal{I}} \sum_{\alpha,\alpha'=1}^{M_{i\bar{i}}} g_a^{i,\bar{i};\alpha,\alpha'} P_{i,\bar{i};\alpha,\alpha'} , \tag{2.267}$$

where $g_a^{i,\bar{i};\alpha,\alpha'} \in \mathbb{C}$ in general, and a labels distinct topological defects.

Similar to the conformal boundary case, we need to impose certain consistency conditions on (2.267) in order to obtain the operators describing "sensible" topological defects. One of the consistency conditions, which is discussed in [74], is similar to the Cardy constraints for conformal boundaries, and we will describe it in the next section. At least, there is one operator which can be written down immediately: the identity operator on the bulk state space $\mathcal{H}$ corresponds to the identity defect.

2.4.2 Cardy-type Constraint for Topological Defects

The Cardy-type constraint for topological defects can be obtained by considering compatibility with the modular S transformation of a torus.

Let us consider a torus specified by the modular parameter τ with two topological defects a and b in the opposite directions along non-contractible circles. By mapping this torus to an annulus in such a way that the defects are mapped to two concentric circles about the origin, the torus partition function can be expressed as the trace over the bulk state space $\mathcal{H}$

$$\mathrm{Tr}_{\mathcal{H}} \left(D_a^{\dagger}\, D_b\, \tilde{q}^{L_0-\frac{c}{24}}\, \tilde{\bar{q}}^{\bar{L}_0-\frac{c}{24}} \right) = \sum_{i,\bar{i}\in\mathcal{I}} \sum_{\alpha,\alpha'=1}^{M_{i\bar{i}}} \bar{g}_a^{i,\bar{i};\alpha,\alpha'}\, g_b^{i,\bar{i};\alpha,\alpha'}\, \chi_i(\tilde{q})\chi_{\bar{i}}(\tilde{\bar{q}}) , \tag{2.268}$$

where $\tilde{q} = e^{-2\pi i/\tau}$ and $\tilde{\bar{q}} = e^{2\pi i/\bar{\tau}}$. Here, we used

$$\mathrm{Tr}_{\mathcal{H}} \left(P_{i,\bar{i};\alpha,\alpha'}\, \tilde{q}^{L_0-\frac{c}{24}}\, \tilde{\bar{q}}^{\bar{L}_0-\frac{c}{24}} \right) = \delta_{\alpha,\alpha'}\, \chi_i(\tilde{q})\chi_{\bar{i}}(\tilde{\bar{q}}) . \tag{2.269}$$

This annulus is equivalent to the complex plane with punctures at the origin and the point at infinity, which is depicted in Figure 2.7. On the other hand, it is possible to use another map from the torus to an annulus, on which the defects are placed in the radial directions as in Figure 2.6. As a consequence, the space of states in radial quantisation is different from the bulk state space $\mathcal{H}$, and this state space is given by

$$\mathcal{H}_{a|b} = \bigoplus_{i,\bar{i}\in\mathcal{I}} \mathcal{N}_{ab}^{i,\bar{i}}\, \mathcal{H}_i \otimes \mathcal{H}_{\bar{i}} \,, \tag{2.270}$$

where $\mathcal{N}_{ab}^{i,\bar{i}} \in \mathbb{Z}_{\geq 0}$ are multiplicities. Then, the torus partition function is equivalently written as

$$Z_{a|b}(q,\bar{q}) = \mathrm{Tr}_{\mathcal{H}_{a|b}}\left(q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) = \sum_{i,\bar{i}\in\mathcal{I}} \mathcal{N}_{ab}^{i,\bar{i}}\,\chi_i(q)\chi_{\bar{i}}(\bar{q}) \,. \tag{2.271}$$

where $q = e^{2\pi i\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$. Since (2.268) and (2.271) are related by the modular S transformation, we obtain the consistency equation for topological defects[74]

$$\sum_{i,\bar{i}\in\mathcal{I}}\sum_{j,\bar{j}\in\mathcal{I}}\sum_{\alpha,\alpha'=1}^{M_{i\bar{i}}} \bar{g}_a^{i,\bar{i};\alpha,\alpha'}\, g_b^{i,\bar{i};\alpha,\alpha'}\, \mathrm{S}_{ij}\mathrm{S}_{\bar{i}\bar{j}}\,\chi_j(q)\chi_{\bar{j}}(\bar{q}) = \sum_{j,\bar{j}\in\mathcal{I}} \mathcal{N}_{ab}^{j,\bar{j}}\,\chi_j(q)\chi_{\bar{j}}(\bar{q}) \,. \tag{2.272}$$

Compare this with the Cardy constraint (2.232). Since $\mathcal{N}_{ab}^{j,\bar{j}}$ must be non-negative integers, the equation (2.272) constrains possible values of $\bar{g}_a^{i,\bar{i};\alpha,\alpha'}$ and $g_b^{i,\bar{i};\alpha,\alpha'}$.

As in the boundary case, we rewrite the coefficients of topological defect operators as

$$g_a^{i,\bar{i};\alpha,\alpha'} = \frac{\psi_a^{i,\bar{i};\alpha,\alpha'}}{\sqrt{\mathrm{S}_{0i}\mathrm{S}_{0\bar{i}}}} \,, \tag{2.273}$$

then the Cardy-type constraint (2.272) becomes

$$\mathcal{N}_{ab}^{i,\bar{i}} = \sum_{j,\bar{j}\in\mathcal{I}}\sum_{\alpha,\alpha'=1}^{M_{j\bar{j}}} \bar{\psi}_a^{j,\bar{j};\alpha,\alpha'}\,\psi_b^{j,\bar{j};\alpha,\alpha'}\,\frac{\mathrm{S}_{ij}\mathrm{S}_{\bar{i}\bar{j}}}{\mathrm{S}_{0j}\mathrm{S}_{0\bar{j}}} \,. \tag{2.274}$$

If the set $\mathcal{T}$ of topological defects is complete, in the sense that

$$\sum_{a\in\mathcal{T}} \psi_a^{i,\bar{i};\alpha,\alpha'}\,\bar{\psi}_a^{j,\bar{j};\beta,\beta'} = \delta_{i,j}\,\delta_{\bar{i},\bar{j}}\,\delta_{\alpha,\beta}\,\delta_{\alpha',\beta'} \tag{2.275}$$

holds, using (2.274), we can obtain

$$\sum_{b\in\mathcal{T}} \mathcal{N}_{ab}^{i,\bar{i}}\,\mathcal{N}_{bc}^{j,\bar{j}} = \sum_{k,\bar{k}\in\mathcal{I}} N_{ij}^{k} N_{\bar{i}\bar{j}}^{\bar{k}}\,\mathcal{N}_{ac}^{k,\bar{k}} \,. \tag{2.276}$$

By introducing $|\mathcal{T}|\times|\mathcal{T}|$ matrices $(\mathcal{N}_{i,\bar{i}})_{ab} = \mathcal{N}_{ab}^{i,\bar{i}}$, the above equation becomes

$$\mathcal{N}_{i,\bar{i}}\,\mathcal{N}_{j,\bar{j}} = \sum_{k,\bar{k}\in\mathcal{I}} N_{ij}^{k} N_{\bar{i}\bar{j}}^{\bar{k}}\,\mathcal{N}_{k,\bar{k}} \,, \tag{2.277}$$

from which we see that the matrices $\mathcal{N}_{i,\bar{i}}$ form a NIM-rep of the "double" fusion algebra. Their relation to Ocneanu's double triangle algebra is studied in [75]. In the context of the topological field theory (TFT) approach to RCFT, these "double NIM-reps" are discussed in [76].

If the bulk partition function is a charge conjugate modular invariant, a solution of (2.272) is given by[74, 79]

$$g_a^{i,i^+} = \frac{\mathrm{S}_{ai}}{\mathrm{S}_{0i}} \tag{2.278}$$

and the indexing set $\mathcal{T}$ for topological defects is the same as that for the irreducible representations $\mathcal{I}$. In this case, elements of the matrices $\mathcal{N}_{i,\bar{\imath}}$ are

$$\mathcal{N}_{ab}^{i,\bar{\imath}} = \sum_{k\in\mathcal{I}} N_{ai}{}^{k}\, N_{k\bar{\imath}}{}^{b}\ . \tag{2.279}$$

These topological defects are elementary in the sense that the vacuum representation is unique, that is $\mathcal{N}_{aa}^{0,0} = 1$. In general, the number of elementary topological defects is given by[76] $\sum_{i,\bar{\imath}\in\mathcal{I}} (M_{i\bar{\imath}})^2$, which is $|\mathcal{I}|$ in this case.

2.4.3 Defect Fields and Disorder Fields

Let us consider the space of states in (2.270). The state space $\mathcal{H}_{a|b}$ is obtained by mapping the torus to a complex plane, and considering radial quantisation about the origin. This map is depicted in Figure 2.6. Since the topological defects labelled by a and b join at the origin of the plane, this can be viewed as a defect changing field $\psi^{(ab)}_{i,\bar{\imath};\alpha}(z,\bar{z})$ inserted at the origin. There may be more than one defect changing fields carrying the representations $(i,\bar{\imath})$, and they are distinguished by multiplicity labels $\alpha \in [1, \mathcal{N}_{ab}^{i,\bar{\imath}}]$. The state space $\mathcal{H}_{a|b}$ corresponds to the defect changing fields $\psi^{(ab)}_{i,\bar{\imath};\alpha}(z,\bar{z})$. Similarly, if we consider a state space $\mathcal{H}_{a|a}$, this gives topological defect fields $\psi^{(a)}_{i,\bar{\imath};\alpha}(z,\bar{z})$ living on the topological defect labelled by a. For brevity, we often call them defect fields. An important class of defect fields is obtained by considering a state space $\mathcal{H}_{a|0}$, where the label 0 corresponds to the identity defect. The defect fields $\psi^{(a0)}_{i,\bar{\imath};\alpha}(z,\bar{z})$ join the defect a and the identity defect. In other words, the topological defect a can end on a defect field $\psi^{(a0)}_{i,\bar{\imath};\alpha}(z,\bar{z})$. Such defect fields are called disorder fields. From this perspective, bulk fields can be considered as the defect fields living on the identity defect. Therefore, $\mathcal{H}_{0|0} = \mathcal{H}$.

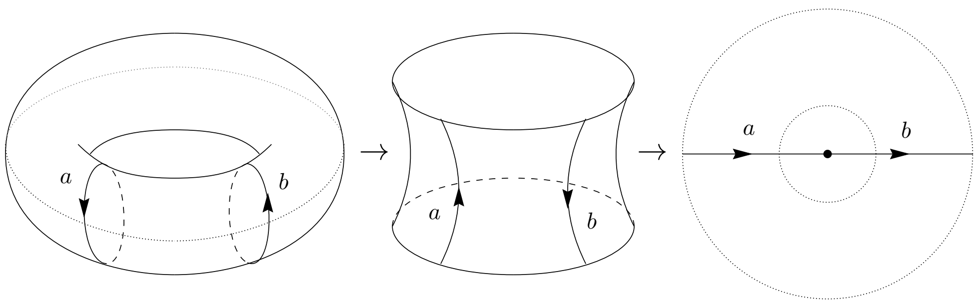


Figure 2.6: Mapping the torus to the complex plane with defect fields.

Since the consistency equation (2.272) relates the two ways of mapping the torus to the complex plane depicted in Figure 2.6 and Figure 2.7, topological defect fields are also related to the bulk fields with loops of topological defects around them. Given a bulk field $\varphi_I(z,\bar{z})$, the action of a topological defect loop a surrounding $\varphi_I(z,\bar{z})$ clockwise is denoted by $D_a\varphi_I(z,\bar{z})$, and the action of the loop surrounding $\varphi_I(z,\bar{z})$ anticlockwise is denoted by $D_a^\dagger\varphi_I(z,\bar{z})$.

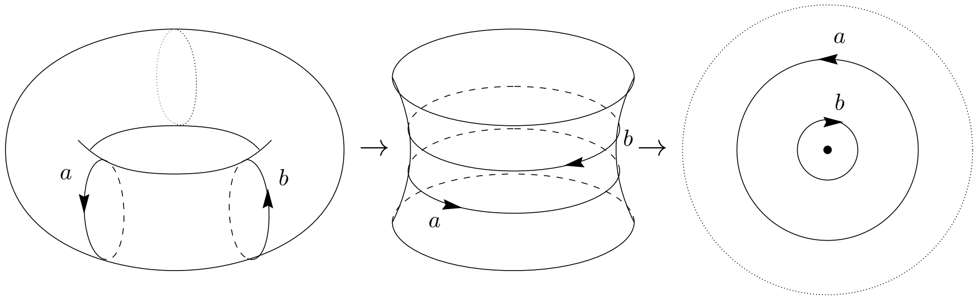


Figure 2.7: Mapping the torus to the complex plane with defect loops.

Since topological defects can be moved freely without changing the values of correlators as long as they do not cross field insertions, we can consider a situation where a bulk field approaching a topological defect giving rise to a defect field, which is illustrated in Figure 2.8. In general, the results of two limits depicted in Figure 2.8 do not have to be the same. In addition, not all of defect fields arise in this way as they may carry combinations of holomorphic and antiholomorphic representation labels that are not available in the bulk spectrum.

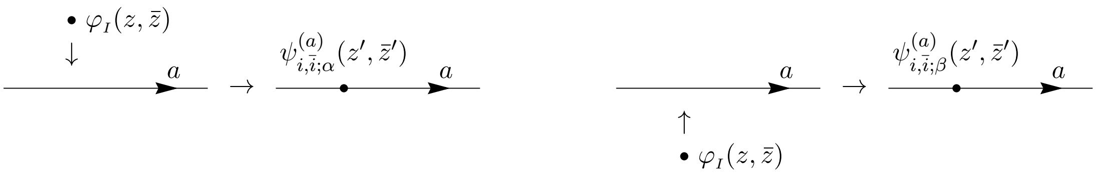


Figure 2.8: A bulk field $\varphi_I(z,\bar{z})$ with $I=(i,\bar{i})$ approaching a topological defect labelled by a.

For a defect a along the real axis, it is possible to pick a basis of defect fields, and define the bulk–defect OPE[110]

$$\varphi_I(z,\bar{z}) = \sum_{(j,\bar{j};\alpha)} \begin{cases} {}^{(a)}C_I^{j,\bar{j};\alpha}\ |y|^{h_I-h_j}|y|^{\bar{h}_I-h_{\bar{j}}}\ \psi^{(a)}_{j,\bar{j};\alpha}(x) & \text{for} \quad y>0 \\ {}^{(a)}\tilde{C}_I^{j,\bar{j};\alpha}\ |y|^{h_I-h_j}|y|^{\bar{h}_I-h_{\bar{j}}}\ \psi^{(a)}_{j,\bar{j};\alpha}(x) & \text{for} \quad y<0 \end{cases}, \tag{2.280}$$

where $z=x+iy$. For topological defects, bulk–defect OPEs must be non-singular, and therefore ${}^{(a)}C_I^{j,\bar{j};\alpha} = {}^{(a)}\tilde{C}_I^{j,\bar{j};\alpha} = 0$ unless $h_I = h_j$ and $\bar{h}_I = h_{\bar{j}}$.

2.4.4 Fusion of Topological Defects

As long as they do not cross field insertions, we can bring two topological defects arbitrarily close, and consider them as another topological defect. This procedure is know as fusion of topological defects. Given two topological defect operators[24] D_a and D_b, their fusion is

24. So far we are assuming that topological defect operators characterise topological defects uniquely but this may not be true in certain cases. See, for example, Section 4.5 and Appendix A of [90]. In general, one has to calculate correlators involving defect fields in order to distinguish topological defects. We only consider the cases where defect operators are unique for topological defects.

given by the composition of operators $D_a D_b$. A priori, fusions of topological defects do not have to be commutative.

If the bulk partition function is given by a charge conjugate modular invariant, we can use the operators describing elementary topological defects given in (2.278), and obtain the fusion rules for such defects

$$D_a D_b = \sum_{c \in \mathcal{I}} N_{ab}^c \, D_c \; , \tag{2.281}$$

where N_{ab}^c are the usual fusion coefficients of the chiral algebra $\mathcal{A}$. In this case, topological defects are not only labelled by the irreducible representations of $\mathcal{A}$ but they also obey the same fusion rules. As it is clear from (2.281), fusion products of elementary topological defects are not necessarily elementary but decompose into elementary ones. In addition, this implies that if D_c appears in the fusion rule of D_a and D_b, then we can form a junction of such topological defects as depicted in Figure 2.9. Since the positions of such junctions can be moved freely, corresponding junction fields necessarily have zero conformal weights.

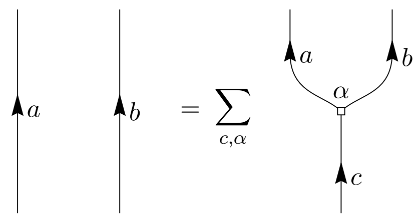


Figure 2.9: Fusion of topological defects and junction fields.

Moreover, topological defects can act on conformal boundaries and change their boundary conditions[79]. As before, let us consider a theory with a charge conjugate modular invariant bulk partition function. Since elementary boundary conditions are given by (2.233), we can use (2.278) and obtain

$$D_a \| b \rangle\rangle = \sum_{c \in \mathcal{I}} N_{ab}^c \| c \rangle\rangle \; , \tag{2.282}$$

where $\|b\rangle\rangle$ and $\|c\rangle\rangle$ are Cardy boundary states. In addition, we can consider a situation where D_a acts only on a part of the boundary. Then, this can be viewed as the topological defect D_a ending on the boundary at the point where the boundary conditions change form b to c. Thus, if b appears in the fusion rule of a and b, then a topological defect D_a can end on a boundary with the boundary condition b.

By sweeping a topological defect D_a across a bulk field insertion, we obtain a disorder field with the topological defect $D_a D_a^\dagger$ as illustrated in Figure 2.10. The disorder field and $D_a D_a^\dagger$ may decompose into the sum of elementary topological defects with certain disorder fields. In Figure 2.10, $D_{ba\alpha}$ denotes a linear map which assigns a disorder field $D_{ba\alpha}(\varphi_I(z,\bar{z}))$ at the end of D_b for a given bulk field $\varphi_I(z,\bar{z})$ as a result of surrounding it clockwise by a loop of D_a with a D_b tether. The junction between two D_a and one D_b is labelled by a suitably normalised intertwiner[83, 89] α. Some of $D_{ba\alpha}(\varphi_I(z,\bar{z}))$ in the expansion of Figure 2.10 may vanish; such 'diagrams' do not contribute to the expansion.

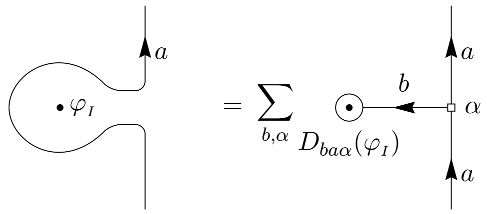


Figure 2.10: Sweeping a topological defect across a bulk field.

If the bulk partition function is given by a charge conjugate modular invariant, the spectrum of disorder fields associated with a topological defect D_a is given by

$$\mathcal{N}_{a0}^{i,\bar{i}} = \sum_{k \in \mathcal{I}} N_{ai}^{k} N_{k\bar{i}}^{0} = N_{i\bar{i}}^{a^+} \ . \tag{2.283}$$

That is, if a^+ appears in the fusion rule of i and $\bar{i}$, then D_a can end on a disorder field $\psi_{i,\bar{i};\alpha}^{(a0)}(z,\bar{z})$. In particular, we always have the disorder fields $\psi_{a^+,0}^{(a0)}(z)$ and $\psi_{0,a^+}^{(a0)}(\bar{z})$, which do not appear in the bulk spectrum.

In certain cases, it is possible to deduce the results of Figure 2.10 by a simple method. If the bulk partition function is a charge conjugate modular invariant, we can use (2.278) and (2.279). Let us consider sweeping a topological defect D_a across a bulk field $\varphi_I(z,\bar{z})$ with $I = (i,\bar{i})$ assuming that we know the fusion rule $D_a D_a^\dagger = \sum_b D_b$. If $b = 0$ is allowed by the fusion rule and $D_a \varphi_I(z,\bar{z})$ does not vanish, then this term appears in the expansion of Figure 2.10. For other D_b with $b \neq 0$, if the representation $(i,\bar{i})$ appears in $\mathcal{H}_{0|b}$, then $\psi_{i,\bar{i};\alpha}^{(b0)}(z,\bar{z})$ with topological defect D_b appears in Figure 2.10. For this calculation, we can use (2.283). For example, we can use these rules to understand the actions of topological defects on the bulk fields in the Ising model given in Figure 3 of [83].

There are two special classes of topological defects that are of great importance. A topological defect D_a is called group-like if and only if $D_a^\dagger D_a = D_0$, where D_0 denotes the identity defect. If we sweep a group-like defect across bulk fields as in Figure 2.10, the resulting disorder fields are again bulk fields. Therefore, the set of group-like defects forms a group and it describes internal symmetries of this CFT. A topological defect D_a is called a duality defect if and only if $D_a^\dagger D_a$ is a sum of group-like defects. Duality defects describe order-disorder dualities of the CFT[83, 89].

2.4.5 Defect Flows

Similar to bulk and boundary perturbations, it is possible to consider perturbations of conformal defects by defect fields. In particular, we focus on perturbations of topological defects.

Let us take a theory defined on the complex plane with a topological defect D_a along the real line $x \in \mathbb{R}$. Then, a perturbation of D_a by a defect field $\psi_{i,\bar{i};\alpha}^{(a)}(x)$ is given by adding the action a term

$$S_{\text{pert.}} = \lambda \int_{\mathbb{R}} \psi_{i,\bar{i};\alpha}^{(a)}(x) \, \mathrm{d}x \ , \tag{2.284}$$

where $\lambda \in \mathbb{C}$ is a coupling constant. This modifies correlation functions and the perturbed defect is denoted by $D_a(\lambda \psi_{i,\bar{i};\alpha}^{(a)})$. If the perturbing defect field has the scaling dimension

$h_i + h_{\bar{i}} < 1$, then it is relevant. We are interested in the defect flows whose end points correspond to non-topological and non-factorising conformal defects.

Consider perturbations of a topological defect D_a by a chiral defect field $\psi_{i,0}^{(a)}(x)$. In reality, existence of such defect fields depends on a model and D_a, however let us assume that the topological defect D_a has chiral defect fields. Since $\lambda\psi_{i,0}^{(a)}(x)$ commutes with all $\bar{L}_n$, the end point of this flow corresponds a topological defect[81]. Furthermore, if $h_i < 1/2$, perturbations do not require UV regularisations[92]. Relevant perturbations by chiral defect fields are studied in, for example, [65] and [92]. Truly marginal deformations by chiral defect fields are studied in [90].

In certain cases, the end points of flows correspond to conformal but non-topological defects. In particular, in a unitary Virasoro minimal model $M(p, p+1)$ with the diagonal modular invariant bulk partition function, perturbations of the topological defect $D_{(1,2)}$, where the subscript denotes its Kac label, by a linear combination of chiral defect fields $\lambda_l\psi_{(1,3),0} + \lambda_r\psi_{0,(1,3)}$ living on this defect give flows leading to four possible endpoints[98]: another topological defect $D_{(2,1)}$, the identity defect, a factorising defect, and a non-topological and non-factorising conformal defect.

Chapter 3

Conformal Defects and W-Algebras

In this short chapter, we discuss the $\mathcal{W}(2,2)$ symmetry of the product theory and their relations to the boundary states associated with topological and factorising defects.

3.1 Product Theory and $\mathcal{W}(2,2)$

In the product theory $\mathrm{CFT}_1 \times \overline{\mathrm{CFT}}_2$ defined on the upper half plane with the stress-energy tensors (2.256), we can introduce the chiral primary fields

$$W(z) = \sqrt{\frac{\bar{c}_2}{c_1}} T^{(1)}(z) - \sqrt{\frac{c_1}{\bar{c}_2}} \overline{T}^{(2)}(z^*) \quad \text{and} \quad \overline{W}(\bar{z}) = \sqrt{\frac{c_2}{\bar{c}_1}} \overline{T}^{(1)}(\bar{z}) - \sqrt{\frac{\bar{c}_1}{c_2}} T^{(2)}(\bar{z}^*) \ . \tag{3.1}$$

Therefore, the chiral algebras of $\mathrm{CFT}_1 \times \overline{\mathrm{CFT}}_2$ can be regarded as two copies of $\mathcal{W}(2,2)$. In addition to (2.257), if we have

$$W(z) = \overline{W}(\bar{z}) \quad \text{at} \quad z = \bar{z} \in \mathbb{R} \tag{3.2}$$

as well as $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$, then this gives

$$T^{(1)}(z) = \overline{T}^{(1)}(\bar{z}) \quad \text{and} \quad T^{(2)}(z) = \overline{T}^{(2)}(\bar{z}) \ , \tag{3.3}$$

which means this boundary condition corresponds to a factorising defect. If the central charges satisfy $c_1 = c_2$ and $\bar{c}_1 = \bar{c}_2$, the two copies of $\mathcal{W}(2,2)$ have the $\mathbb{Z}_2$ automorphisms given by $W(z) \mapsto -W(z)$ and $\overline{W}(\bar{z}) \mapsto -\overline{W}(\bar{z})$. Therefore, in addition to the untwisted boundary condition (3.2), we can also impose the twisted boundary condition

$$W(z) = -\overline{W}(\bar{z}) \quad \text{at} \quad z = \bar{z} \in \mathbb{R} \ , \tag{3.4}$$

then this yields the conditions for a topological defect

$$T^{(1)}(z) = T^{(2)}(z) \quad \text{and} \quad \overline{T}^{(1)}(\bar{z}) = \overline{T}^{(2)}(\bar{z}) \ . \tag{3.5}$$

Therefore we see that the factorising and topological defects correspond to $\mathcal{W}(2,2)$ boundary conditions in the folded theory.

If there is a conformal boundary state in the product theory which corresponds to a non-topological and non-factorising conformal defect, it has to break the $\mathcal{W}(2,2)$ symmetry of the theory.

3.2 Reflection and Transmission Coefficients

The reflection and transmission coefficient of a conformal defect D was defined in [91] using the matrix

$$R_{ij} := \frac{\langle 0 | L_2^{(i)} \bar{L}_2^{(j)} \| D \rangle\!\rangle}{\langle 0 \| D \rangle\!\rangle} \ , \tag{3.6}$$

where $\|D\rangle\rangle$ is the boundary state in the folded model corresponding to the defect D. The reflection coefficient $\mathcal{R}$ and the transmission coefficient $\mathcal{T}$ of D are given by

$$\mathcal{R} = \frac{2}{c_1+c_2}(R_{11}+R_{22}) = \frac{(c_1)^2 + 2c_1c_2\omega_D + (c_2)^2}{(c_1+c_2)^2} \quad \text{and} \tag{3.7}$$

$$\mathcal{T} = \frac{2}{c_1+c_2}(R_{12}+R_{21}) = \frac{2c_1c_2(1-\omega_D)}{(c_1+c_2)^2} \tag{3.8}$$

where the quantity ω_D can be expressed as

$$\omega_D = \frac{2}{c_1+c_2}\frac{\langle 0|W_2\overline{W}_2\|D\rangle\rangle}{\langle 0\|D\rangle\rangle} \tag{3.9}$$

using our definition of $\mathcal{W}(2,2)$ generators given in (3.1).

In the folded theory, there is a diagonal $\mathcal{V}\mathrm{ir}$ primary state defined by

$$|W\rangle := \frac{2}{c_1+c_2}W_{-2}\overline{W}_{-2}|0\rangle \ . \tag{3.10}$$

Since the $\mathcal{W}(2,2)$ generators are normalised as $d_W = \frac{c}{2}$, where $c = c_1 + c_2$, we have $\langle W|W\rangle = 1$. Using $|W\rangle$, we can simplify (3.9) as

$$\omega_D = \frac{\langle W\|D\rangle\rangle}{\langle 0\|D\rangle\rangle} = \frac{g_D^W}{g_D^0} \ , \tag{3.11}$$

where g_D^0 and g_D^W are given by the expansion of $\|D\rangle\rangle$ in terms of the Virasoro Ishibashi states

$$\|D\rangle\rangle = g_D^0\,|0\rangle\rangle + g_D^W\,|W\rangle\rangle + \cdots \ . \tag{3.12}$$

This means that it is straightforward to calculate the reflection and transmission coefficients once we obtain the explicit expression for the boundary state corresponding to a conformal defect.

Consider the $\mathcal{W}(2,2)$ Ishibashi condition

$$\left(W_n - \epsilon\,\overline{W}_{-n}\right)|h,\epsilon\rangle\rangle = 0 \ , \tag{3.13}$$

where $\epsilon = 1$ gives the untwisted sector, and $\epsilon = -1$ corresponds to the twisted sector which only appears when $c_1 = c_2$. If we focus on the Vacuum sector of $\mathcal{W}(2,2)$, we can write the corresponding Ishibashi state as

$$|0,\epsilon\rangle\rangle = |0\rangle\rangle + \epsilon\,|W\rangle\rangle + \cdots \ . \tag{3.14}$$

From (3.2) and (3.4), we know the factorising and topological defects correspond to untwisted and twisted $\mathcal{W}(2,2)$ boundary conditions, respectively. Therefore, using the relation (3.11), we obtain

$$\omega_D = \ \ 1 \quad \text{for factorising defects, and} \tag{3.15}$$

$$\omega_D = -1 \quad \text{for topological defects,} \tag{3.16}$$

which is in agreement with [91].

Chapter 4

Perturbative Calculation of Reflection Coefficients

In this chapter, we calculate the reflection and transmission coefficients of the conformal defect C in diagonal Virasoro minimal models using the leading-order perturbation calculation. The defect C is the endpoint of a defect flow of the topological defect $D_{(1,2)}$ by the combination of chiral defect fields, which was considered in [98]. One characteristic of a conformal defect is its transmission coefficient $\mathcal{T}$, or equivalently its reflection coefficient $\mathcal{R} = 1 - \mathcal{T}$, which was defined in [91]. These take the values $\mathcal{R} = 0$ for a topological defect and $\mathcal{R} = 1$ for a factorised defect, and $0 < \mathcal{R} < 1$ for a general conformal defect in a unitary theory [115].

The aim of this chapter is to calculate the reflection coefficient of C perturbatively, and compare the value for the tri-critical Ising model with the result obtained from our construction in Chapter 6.

4.1 $D_{(r,2)}$ Defect and Its Perturbations

In this chapter, we focus on diagonal Virasoro minimal models $M(p, q)$, also know as the (A_{p-1}, A_{q-1}) invariant. Recall that the coprime integers satisfy $1 < p < q$, and we shall take $p \geq 2$ and $q \geq 5$. For $M(p, q)$, there are $(p-1)(q-1)/2$ primary fields corresponding to the Virasoro highest weight representations, and we use the Kac labels $(r, s) \sim (p-r, q-s)$ for both of them. For the defect perturbations we consider in this chapter, we are going to be especially interested in the representation $(r, s) = (1, 3)$, and we denote the corresponding conformal weight as

$$\mathrm{h} := h_{1,3} = \frac{2p}{q} - 1 \ . \tag{4.1}$$

As we discussed in Chapter 2, the elementary conformal boundaries and topological defects in this model are also labelled by the Kac labels, and their properties are well-known. In particular, we know the spectra of defects fields which can be calculated from the multiplicity formula (2.279) and the fusion numbers N_{ij}^k.

From the formula (2.279), a general topological defect $D_{(r,s)}$ has (for $s > 2$ and q large enough) one chiral defect field of weights $(\mathrm{h}, 0)$, another chiral defect field of weights $(0, \mathrm{h})$, and three defect fields of weights (h, h). A topological defect of type $D_{(r,2)}$ is special in that it has one chiral defect field ϕ of conformal weights $(\mathrm{h}, 0)$, another chiral defect field $\bar{\phi}$ of weights $(0, \mathrm{h})$, but only a two dimensional space of defect fields $\{\varphi_\alpha\}$ of weights (h, h).

Furthermore, the $D_{(r,2)}$ topological defect can be constructed as the fusion of $D_{(r,1)}$ and $D_{(1,2)}$, and the operator product algebra of defect fields of type $(h_{1,s}, h_{1,s'})$ is unaffected by this fusion, in exactly the same way that the action of topological defects on boundaries leaves operator algebras invariant[79]. This means that when considering the algebra of defect fields generated by the set $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_\alpha\}$, we can restrict attention to just the $D_{(1,2)}$ topological defect.

The fact that there is a two-dimensional space of fields $\{\varphi_a\}$ on the $D_{(r,2)}$ topological defects allows one to choose a canonical basis of these fields with special properties so that the analysis of the sewing constraints is correspondingly simpler. These sewing constraints have been solved in [110] for the $D_{(1,2)}$ topological defect in the non-unitary Lee-Yang model, the (A_1, A_4) theory, in which $D_{(1,2)}$ is the only non-trivial defect and $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_\alpha\}$ are the only non-trivial primary defect fields. In this chapter, we extend this analysis to the fields $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_\alpha\}$ on defects of type $D_{(r,2)}$ in all the (A_p, A_q) models.

We are interested in the perturbations of the topological defect $D_{(r,2)}$ by a combination of the fields ϕ and $\bar{\phi}$,

$$S = \int \left(\lambda\phi(x) + \bar{\lambda}\bar{\phi}(x)\right) \, \mathrm{d}x \,. \tag{4.2}$$

where the parameters λ and $\bar{\lambda}$ are independent, and x is a coordinate on the defect. This is a relevant perturbation if $\mathrm{h} < 1$ which is the case if $p < q$.

One important question is that of the transformation properties of fields on a defect under a conformal transformation. We will use the conventions of [100] which imply that defect fields always transform with the absolute value of the derivative of the conformal map, even if they are "chiral" defect fields. This is possible because the defect defines a direction through the insertion point of the field (the tangent vector along the defect), and so a defect field can pick up an extra phase under a conformal transformation: this is chosen so that all defect fields transform with the absolute value of the derivative of the conformal map. This has the advantage of making the perturbation well-defined on defects that are closed loops and making the correlation function independent of the orientation of the defect at the location of the defect field (as one would expect if the defect is genuinely topological). The question remains whether this choice for the transformation law of "chiral" defect fields is unique: the corresponding situation for a boundary and boundary fields was considered by Runkel [71], and there seems no way to fix it a priori; we stick to the conventions of [100] here for the good reasons cited above.

The expectation values in the perturbed defect $D_{(r,2)}(\lambda, \bar{\lambda})$ are formally given by

$$\langle\, \mathcal{O} \,\rangle_{D_{(r,2)}(\lambda,\bar{\lambda})} = \langle\, \mathcal{O}\, \exp(-S) \,\rangle_{D_{(r,2)}} \,. \tag{4.3}$$

This is only formal since there may be UV divergences in the integrals when the insertion points of two fields ϕ or two fields $\bar{\phi}$ meet, and IR divergences from integration along the whole real axis. This means that the general procedure of regularisation and renormalisation may be needed to give meaning to the expression (4.3). This is explained in Affleck and Ludwig [37], and applied by Recknagel et al. in [70] to the case of boundary perturbations of the unitary minimal models where $q = p + 1$.

As explained in [98], when $y := 1 - \mathrm{h}$ is small and positive, the results of [70] can immediately be applied to the case of defects with the perturbation (4.2), and we obtain the prediction (from third order perturbation theory) of three conformal defects at the fixed points

$$\text{(i)}\ \ \lambda = \lambda^*,\, \bar{\lambda} = 0 \tag{4.4}$$

$$\text{(ii)}\ \ \lambda = 0,\ \ \bar{\lambda} = \lambda^* \tag{4.5}$$

$$\text{(iii)}\ \ \lambda = \bar{\lambda} = \lambda^* \tag{4.6}$$

The fixed points (i) and (ii) can be identified as the topological defect $D_{(2,1)}$ (if $r = 2$) and (more generally) the superposition $D_{(r-1,1)} \oplus D_{(r+1,1)}$; the fixed point (iii) is a potential new conformal defect, denoted by C in [98], in the case of the perturbation of the defect $D_{(1,2)}$. The value of λ^* is given (to first order in y) by

$$\lambda^* = \frac{y}{C_{\phi\phi\phi}} = \frac{y}{C^{\phi}_{\phi\phi} d_{\phi\phi}}\ , \tag{4.7}$$

where $C^{\phi}_{\phi\phi}$ is the three point coupling of the fields ϕ. Note that λ^* depends on the normalisation of ϕ, but this will cancel in any physical quantities.

4.1.1 Perturbative Calculation of Reflection and Transmission Coefficients

Previously, the reflection and transmission coefficients, (3.7) and (3.8), of a conformal defect along the real axis were defined in terms of the matrix (3.6) but, for perturbative calculations, it is more convenient to use the equivalent definition which was also given in [91]:

$$\mathcal{R} = \frac{\langle T^1\overline{T}^1 + T^2\overline{T}^2\rangle}{\langle (T^1 + \overline{T}^2)(\overline{T}^1 + T^2)\rangle} \quad \text{and} \quad \mathcal{T} = 1 - \mathcal{R}\ , \tag{4.8}$$

where T^1 and $\overline{T}^1$ are inserted at the point iY on the upper half-plane, while T^2 and $\overline{T}^2$ are inserted at the point $-iY$. For the unperturbed topological defect,

$$\langle\, T^1\overline{T}^1\,\rangle = \langle\, T^2\overline{T}^2\,\rangle = 0 \quad \text{and} \quad \langle\, T^1 T^2\,\rangle = \langle\, \overline{T}^1\overline{T}^2\,\rangle = \frac{c}{32Y^4}\ , \tag{4.9}$$

and so $\mathcal{R} = 0$ and $\mathcal{T} = 1$.

For the defect with perturbation (4.2), the expansion of the perturbed quantities using (4.3) gives

$$\begin{aligned}\langle\, T^1\overline{T}^1\,\rangle = {}& \frac{1}{4}\lambda^2\bar{\lambda}^2 \int \mathrm{d}x\ \mathrm{d}x'\ \mathrm{d}y\ \mathrm{d}y' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\,\rangle \\ & - \frac{1}{24}\lambda^3\bar{\lambda}^2 \int \mathrm{d}x\ \mathrm{d}x'\ \mathrm{d}x''\ \mathrm{d}y\ \mathrm{d}y' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\phi(x'')\bar{\phi}(y)\bar{\phi}(y')\,\rangle \\ & - \frac{1}{24}\lambda^2\bar{\lambda}^3 \int \mathrm{d}x\ \mathrm{d}x'\ \mathrm{d}y\ \mathrm{d}y'\ \mathrm{d}y'' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\bar{\phi}(y'')\,\rangle \\ & + O(\lambda^6)\ ,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\langle\, T^1 T^2\,\rangle = {}& \frac{c}{32Y^4} \\ & + \frac{1}{2}\lambda^2 \int \mathrm{d}x\ \mathrm{d}x' \langle\, T(iY)\,T(-iY)\,\phi(x)\phi(x')\,\rangle \\ & + \frac{1}{2}\bar{\lambda}^2 \int \mathrm{d}y\ \mathrm{d}y' \langle\, T(iY)\,T(-iY)\,\bar{\phi}(y)\bar{\phi}(y')\,\rangle + O(\lambda^3)\ ,\end{aligned} \tag{4.11}$$

and in order to find the leading order term in $\mathcal{R}$, we only need to calculate the first term in $\langle\, T^1\overline{T}^1 \,\rangle$ and $\langle\, T^2\overline{T}^2 \,\rangle$. It turns out there are neither UV nor IR divergences in these integrals; their dependence on Y is simply Y^{-4} and the reflection coefficient $\mathcal{R}$ (to leading order) is indeed independent of Y as expected. We shall take $Y = 1$ from now on.

The consequence is that the only correlation function we need to evaluate is

$$\langle\, T(i)\overline{T}(i)\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y') \,\rangle \,, \tag{4.12}$$

where the insertion points can be in any order. This is equal to

$$\langle\, T(-i)\overline{T}(-i)\phi(-x)\phi(-x')\bar{\phi}(-y)\bar{\phi}(-y') \,\rangle \,, \tag{4.13}$$

by rotation through π. The analytic structure is simple,

$$\Big\langle\, T(i)\,\overline{T}(i)\,\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y') \,\Big\rangle = \mathcal{C}\,\frac{(x'-x)^{2-2h}\,(y'-y)^{2-2h}}{(i-x)^2(i-x')^2(i+y)^2(i+y')^2} \,, \tag{4.14}$$

but the constant $\mathcal{C}$ depends on the order of the insertion points $\{x, x', y, y'\}$ and is determined by the operator algebra structure constants. Therefore, we now turn to the calculation of some of the structure constants of the local fields on the topological defect $D_{(r,2)}$.

4.2 Structure Constants

In this section we will calculate some structure constants for the $(r,2)$ topological defect in the diagonal Virasoro minimal models. These structure constants can be found in terms of topological field theory data [86, 100] which is a general method allowing one to find all the structure constants in the defect theory, but we will not use it here and instead only use elementary properties of the conformal field theory to find the particular structure constants we need for the perturbative calculation of the reflection coefficient in the minimal models.

We note here that we will use the conventions of [100] so that the structure constant $C_{\alpha\beta}^{\gamma}$ is the coefficient of the field ϕ_γ appearing in the OPE of the fields $\phi_\alpha(x)$ with $\phi_\beta(y)$ on the defect oriented opposite to the real line with $x > y$, which means that this coefficient appears in the OPE of the fields ϕ_α with ϕ_β as they appear along the defect. Rotating by π, we obtain the picture in Figure 4.1.

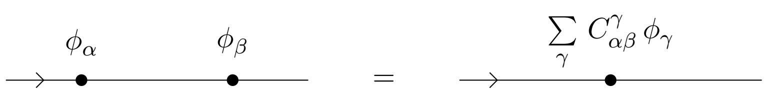


Figure 4.1: The OPE of defect fields

4.2.1 Bulk Theory

The (A_{p-1}, A_{q-1}) Virasoro minimal model has $(p-1)(q-1)/2$ bulk primary fields, of which we are especially interested in the bulk field φ of type $(1,3)$. If we set $t := p/q$, then

$\mathrm{h} := h_{1,3} = 2t - 1$ and $\mathrm{h} < 1$ if $t < 1$, that is $p < q$. The fusion rules for this field are

$$[\varphi] \otimes [\varphi] = [1] \oplus [\varphi] \oplus [\chi] \,, \tag{4.15}$$

where χ is of type (1,5) and has conformal weights $(\mathrm{h}', \mathrm{h}')$ with $\mathrm{h}' := h_{1,5} = 6t - 2$. Hence, the OPE of φ with itself is

$$\varphi(z, \bar{z})\varphi(w, \bar{w}) = \frac{d_{\varphi\varphi}}{|z - w|^{4h}} + \frac{C_{\varphi\varphi}^{\varphi}\, \varphi(w, \bar{w})}{|z - w|^{2h}} + \frac{C_{\varphi\varphi}^{\chi}\, \chi(w, \bar{w})}{|z - w|^{4h - 2h'}} + \ldots \,. \tag{4.16}$$

The structure constant $C_{\varphi\varphi}^{\varphi}$ clearly depends on the choice of $d_{\varphi\varphi}$ (see eg [7, 66] for different conventions) but the combination

$$\frac{(C_{\varphi\varphi}^{\varphi})^2}{d_{\varphi\varphi}} = -(1 - 2t)^2 \frac{\Gamma(2 - 3t)}{\Gamma(3t - 1)} \frac{\Gamma(4t - 1)^2}{\Gamma(2 - 4t)^2} \frac{\Gamma(t)^3}{\Gamma(1 - t)^3} \frac{\Gamma(1 - 2t)^4}{\Gamma(2t)^4} \,, \tag{4.17}$$

is independent of the normalisation. If $\mathrm{h} = 1 - y$ then

$$\frac{(C_{\varphi\varphi}^{\varphi})^2}{d_{\varphi\varphi}} = \frac{16}{3} - 16y + O(y)^2 \,. \tag{4.18}$$

4.2.2 Defect Theory

The defects of the (A_{p-1}, A_{q-1}) Virasoro models are not intrinsically oriented, but the operator product of fields along the defect depends on the ordering of the fields, we shall assume that we can define an orientation for the defects but that all results will be independent of this orientation.

Since the space of defect fields $\{\varphi_\alpha\}$ of weights (h, h) is only two-dimensional for a defect of type $(r, 2)$, we can take as a basis the fields φ_L and φ_R which are the limits of the bulk field φ as it approaches the defect from the left or the right respectively as one looks along the defects — see Figure 4.2.

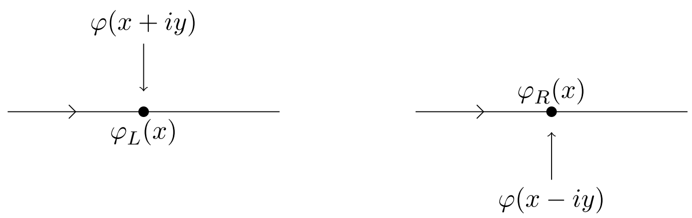


Figure 4.2: The fields φ_L and φ_R defined as limits of the bulk field

Note that the operator product algebra of the defect fields $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_L, \varphi_R\}$ does not close on these fields, other fields can arise as well, namely fields with weights $(\mathrm{h}, \mathrm{h}')$, $(\mathrm{h}', \mathrm{h})$ and $(\mathrm{h}', \mathrm{h}')$ which we denote by ψ, $\bar{\psi}$ and $\{\chi_L, \chi_R\}$ (which again are the limits of the bulk field $\chi(z, \bar{z})$ as it approaches the defect from the left and the right). Although we should mention the existence of these fields and their occurrence in the operator products of some of the fields $\{\phi, \bar{\phi}, \varphi_\alpha\}$, we will not need any of the structure constants including these fields as they will not contribute to any of the sewing constraints considered later on.

Φ_a	h_a	$\bar{h}_a$
$\mathbf{1}$	0	0
ϕ	h	0
$\bar{\phi}$	0	h
φ_α	h	h
ψ	h′	h
$\bar{\psi}$	h	h′
χ_α	h′	h′

Table 4.1: Some of the primary fields occurring on the defect $(r, 2)$

We use the generic labels $\{a, b, \dots\}$ for all of these fields and the labels $\{\alpha, \beta, \dots\}$ for the set $\{L, R\}$. The conformal weights of the field Φ_a are $(h_a, \bar{h}_a)$ as in table 4.1.

We now define the structure constants between these defect fields from their operator product expansions (we show the possibility of fields $\{\psi, \bar{\psi}, \chi_\alpha\}$ appearing in an OPE by placing the fields in square brackets [ ]).

If both fields are chiral, there are 8 structure constants $\{d_{\phi\phi}, d_{\bar{\phi}\bar{\phi}}, C^{\phi}_{\phi\phi}, C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}}, C^{\alpha}_{\phi\bar{\phi}}, C^{\alpha}_{\bar{\phi}\phi}\}$ appearing in the OPEs (recall here that x and y are ordered along the defect):

$$\phi(x)\phi(y) = \frac{d_{\phi\phi}}{|x-y|^{2\mathrm{h}}} + \frac{C^{\phi}_{\phi\phi}\,\phi(y)}{|x-y|^{\mathrm{h}}} + \dots\,, \tag{4.19}$$

$$\bar{\phi}(x)\bar{\phi}(y) = \frac{d_{\bar{\phi}\bar{\phi}}}{|x-y|^{2\mathrm{h}}} + \frac{C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}}\,\bar{\phi}(y)}{|x-y|^{\mathrm{h}}} + \dots\,, \tag{4.20}$$

$$\phi(x)\bar{\phi}(y) = C^{L}_{\phi\bar{\phi}}\,\varphi_L(x,y) + C^{R}_{\phi\bar{\phi}}\,\varphi_R(x,y) + \dots\,, \tag{4.21}$$

$$\bar{\phi}(x)\phi(y) = C^{L}_{\bar{\phi}\phi}\,\varphi_L(y,x) + C^{R}_{\bar{\phi}\phi}\,\varphi_R(y,x) + \dots\,. \tag{4.22}$$

With one chiral field on the left, there are 12 structure constants $\{C^{\bar{\phi}}_{\phi\alpha}, C^{\phi}_{\bar{\phi}\alpha}, C^{\beta}_{\phi\alpha}, C^{\beta}_{\bar{\phi}\alpha}\}$ in the OPEs

$$\phi(x)\varphi_\alpha(z,\bar{z}) = \frac{C^{\bar{\phi}}_{\phi\alpha}\,\bar{\phi}(\bar{z})}{|x-z|^{2\mathrm{h}}} + \frac{C^{L}_{\phi\alpha}\,\varphi_L(z,\bar{z})}{|x-z|^{\mathrm{h}}} + \frac{C^{R}_{\phi\alpha}\,\varphi_R(z,\bar{z})}{|x-z|^{\mathrm{h}}} + [\psi] + \dots\,, \tag{4.23}$$

$$\bar{\phi}(x)\varphi_\alpha(z,\bar{z}) = \frac{C^{\phi}_{\bar{\phi}\alpha}\,\phi(z)}{|x-\bar{z}|^{2\mathrm{h}}} + \frac{C^{L}_{\bar{\phi}\alpha}\,\varphi_L(z,\bar{z})}{|x-\bar{z}|^{\mathrm{h}}} + \frac{C^{R}_{\bar{\phi}\alpha}\,\varphi_R(z,\bar{z})}{|x-\bar{z}|^{\mathrm{h}}} + [\bar{\psi}] + \dots\,. \tag{4.24}$$

Likewise there are 12 structure constants $\{C^{\bar{\phi}}_{\alpha\phi}, C^{\phi}_{\alpha\bar{\phi}}, C^{\beta}_{\alpha\phi}, C^{\beta}_{\alpha\bar{\phi}}\}$ in the OPEs with one chiral field on the right:

$$\varphi_\alpha(z,\bar{z})\phi(x) = \frac{C^{\bar{\phi}}_{\alpha\phi}\,\bar{\phi}(\bar{z})}{|z-x|^{2\mathrm{h}}} + \frac{C^{L}_{\alpha\phi}\,\varphi_L(z,\bar{z})}{|z-x|^{\mathrm{h}}} + \frac{C^{R}_{\alpha\phi}\,\varphi_R(z,\bar{z})}{|z-x|^{\mathrm{h}}} + [\psi] + \dots\,, \tag{4.25}$$

$$\varphi_\alpha(z,\bar{z})\bar{\phi}(x) = \frac{C^{\phi}_{\alpha\bar{\phi}}\,\phi(z)}{|\bar{z}-x|^{2\mathrm{h}}} + \frac{C^{L}_{\alpha\bar{\phi}}\,\varphi_L(z,\bar{z})}{|\bar{z}-x|^{\mathrm{h}}} + \frac{C^{R}_{\alpha\bar{\phi}}\,\varphi_R(z,\bar{z})}{|\bar{z}-x|^{\mathrm{h}}} + [\bar{\psi}] + \dots\,. \tag{4.26}$$

Finally there are 20 structure constants $\{d_{\alpha\beta}, C^{\phi}_{\alpha\beta}, C^{\bar{\phi}}_{\alpha\beta}, C^{\gamma}_{\alpha\beta}\}$ in the OPEs involving no chiral fields:

$$\varphi_\alpha(z,\bar z)\varphi_\beta(w,\bar w) = \frac{d_{\alpha\beta}}{|z-w|^{4\mathrm{h}}} + \frac{C^{\phi}_{\alpha\beta}\phi(w)}{|z-w|^{\mathrm{h}}|\bar z-\bar w|^{2\mathrm{h}}} + \frac{C^{\bar\phi}_{\alpha\beta}\bar\phi(\bar w)}{|\bar z-\bar w|^{\mathrm{h}}|z-w|^{2\mathrm{h}}} + \frac{C^{L}_{\alpha\beta}\varphi_L(w,\bar w)}{|z-w|^{2\mathrm{h}}} + \frac{C^{R}_{\alpha\beta}\varphi_R(w,\bar w)}{|z-w|^{2\mathrm{h}}} + [\psi,\bar\psi,\chi_\alpha] + \dots \tag{4.27}$$

Having defined the fifty-two structure constants we need to calculate, we now set about finding relations. The simplest come from the fact that the orientation of the defect is in fact not physical.

4.2.3 Symmetry Relations

Since the defect is not intrinsically oriented, our labelling over-counts the structure constants: sixteen constants are related by changing the orientation of the defect, as follows:

$$C^L_{\phi\bar\phi} = C^R_{\bar\phi\phi}\ , \quad C^R_{\phi\bar\phi} = C^L_{\bar\phi\phi}\ , \quad d_{LL} = d_{RR}\ , \quad d_{LR} = d_{RL}\ , \tag{4.28}$$

$$C^L_{LL} = C^R_{RR}\ , \quad C^R_{LL} = C^L_{RR}\ , \quad C^L_{LR} = C^R_{RL}\ , \quad C^L_{RL} = C^R_{LR}\ , \tag{4.29}$$

$$C^R_{\phi R} = C^L_{L\phi}\ , \quad C^L_{\phi R} = C^R_{L\phi}\ , \quad C^R_{\phi L} = C^L_{R\phi}\ , \quad C^L_{\phi L} = C^R_{R\phi}\ , \tag{4.30}$$

$$C^R_{\bar\phi R} = C^L_{L\bar\phi}\ , \quad C^L_{\bar\phi R} = C^R_{L\bar\phi}\ , \quad C^R_{\bar\phi L} = C^L_{R\bar\phi}\ , \quad C^L_{\bar\phi L} = C^R_{R\bar\phi}\ . \tag{4.31}$$

Bulk Field Relations

We can use the fact that φ_L and φ_R are the limits of bulk fields to find d_{LL}, d_{LR}, d_{RL}, and d_{RR}, as well as C^L_{LL}, C^R_{LL}, C^L_{RR}, and C^R_{RR}.

In the bulk, we have (4.16). Bringing this OPE towards a defect from the left, we obtain

$$d_{LL} = d_{\varphi\varphi}\ , \quad C^L_{LL} = C^{\varphi}_{\varphi\varphi}\ , \quad C^R_{LL} = C^{\phi}_{LL} = C^{\bar\phi}_{LL} = 0\ . \tag{4.32}$$

We have also found that

$$C^{\chi_L}_{LL} = C^{\chi}_{\varphi\varphi}\ , \quad C^{\chi_R}_{LL} = C^{\psi}_{LL} = C^{\bar\psi}_{LL} = 0\ , \tag{4.33}$$

but these four constants are not of interest to us. Likewise, bringing the bulk OPE (4.16) towards a defect from the right, we obtain

$$d_{RR} = d_{\varphi\varphi}\ , \quad C^R_{RR} = C^{\varphi}_{\varphi\varphi}\ , \quad C^L_{RR} = C^{\phi}_{RR} = C^{\bar\phi}_{RR} = 0\ . \tag{4.34}$$

Finally, using the expression (2.267) for the topological defect operator in terms of projectors, and the coefficients (2.278) for the Virasoro minimal model cases, we can write

$$D_{(r,2)} = \sum_{r',s} \frac{\mathrm{S}_{(r,2)(r',s)}}{\mathrm{S}_{(1,1)(r',s)}} P_{r',s}\ , \tag{4.35}$$

where $\mathrm{S}_{(rs)(r's')}$ is the modular S-matrix given in (A.18). From this, we can obtain

$$\begin{aligned} d_{LR} = \frac{\langle\varphi|\, D_{(r,2)}\,|\varphi\rangle}{\langle 0|\, D_{(r,2)}\,|0\rangle} &= \frac{\mathrm{S}_{(r,2)(1,3)}/\mathrm{S}_{(1,1)(1,3)}}{\mathrm{S}_{(r,2)(1,1)}/\mathrm{S}_{(1,1)(1,1)}}\, \frac{\langle\varphi|\varphi\rangle}{\langle 0|0\rangle} \\ &= (2\cos(2\pi t) - 1)\, d_{LL} \\ &= \gamma\, d_{LL}\ , \end{aligned} \tag{4.36}$$

where we define

$$\gamma = 2\cos(2\pi t) - 1 , \tag{4.37}$$

which is independent of r, as expected.

4.2.4 Defect – Boundary Identification

We next use the fact that the OPE algebra of ϕ along the real axis is the same as that of the boundary field on the $(r, 2)$ boundary — we obtain this identification by bringing the $(r, 2)$ topological defect next to the identity boundary as considered in [79]. Likewise, the algebra of $\bar{\phi}$ is also the same as the boundary algebra.

This means that

$$d_{\phi\phi} = d_{\bar{\phi}\bar{\phi}} , \quad C^{\phi}_{\phi\phi} = C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}} , \tag{4.38}$$

and these values are are given by Runkel's solution to the boundary algebra [66],

$$\frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} = \frac{\Gamma(2-3t)\Gamma(t)\Gamma(1-2t)^3}{\Gamma(2-4t)^2\Gamma(-1+2t)\Gamma(1-t)^2} . \tag{4.39}$$

If $\mathrm{h} = 1 - y$ then

$$\frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} = \frac{8}{3} - 4y + O(y^2) . \tag{4.40}$$

Note that the structure constant again does not depend on r.

Three-Point Function Constraints

We can express the three point function,

$$\langle \Phi_a(u)\Phi_b(v)\Phi_c(w)\rangle , \tag{4.41}$$

in two different ways—using the OPE of Φ_a with Φ_b first, or instead using the OPE of Φ_b with Φ_c first—leading to the constraint

$$\sum_e C^e_{ab} d_{ec} = \sum_f d_{af} C^f_{bc} . \tag{4.42}$$

Taking a and c chiral, this gives the simple relations

$$C^{\bar{\phi}}_{\phi R} d_{\bar{\phi}\bar{\phi}} = C^{\phi}_{R\bar{\phi}} d_{\phi\phi} , \quad C^{\bar{\phi}}_{\phi L} d_{\bar{\phi}\bar{\phi}} = C^{\phi}_{L\bar{\phi}} d_{\phi\phi} , \tag{4.43}$$

$$C^{\phi}_{\bar{\phi} R} d_{\phi\phi} = C^{\bar{\phi}}_{R\phi} d_{\bar{\phi}\bar{\phi}} , \quad C^{\phi}_{\bar{\phi} L} d_{\phi\phi} = C^{\bar{\phi}}_{L\phi} d_{\bar{\phi}\bar{\phi}} , \tag{4.44}$$

which, using (4.38) become

$$C^{\bar{\phi}}_{\phi R} = C^{\phi}_{R\bar{\phi}} , \quad C^{\bar{\phi}}_{\phi L} = C^{\phi}_{L\bar{\phi}} , \quad C^{\phi}_{\bar{\phi} R} = C^{\bar{\phi}}_{R\phi} , \quad C^{\phi}_{\bar{\phi} L} = C^{\bar{\phi}}_{L\phi} . \tag{4.45}$$

Taking only a chiral and the two non-chiral fields equal, this gives the slightly more complicated

$$C^{R}_{\phi R} d_{RR} + C^{L}_{\phi R} d_{LR} = C^{\phi}_{RR} d_{\phi\phi} = 0 , \quad C^{R}_{\bar{\phi} R} d_{RR} + C^{L}_{\bar{\phi} R} d_{LR} = C^{\bar{\phi}}_{RR} d_{\bar{\phi}\bar{\phi}} = 0 , \tag{4.46}$$

$$C^{R}_{\phi L} d_{RL} + C^{L}_{\phi L} d_{LL} = C^{\phi}_{LL} d_{\phi\phi} = 0 , \quad C^{R}_{\bar{\phi} L} d_{RL} + C^{L}_{\bar{\phi} L} d_{LL} = C^{\bar{\phi}}_{LL} d_{\bar{\phi}\bar{\phi}} = 0 , \tag{4.47}$$

which using (4.36) become

$$C^R_{\phi R} = -\gamma C^L_{\phi R}\ , \quad C^R_{\bar\phi R} = -\gamma C^L_{\bar\phi R}\ , \quad C^L_{\phi L} = -\gamma C^R_{\phi L}\ , \quad C^L_{\bar\phi L} = -\gamma C^R_{\bar\phi L}\ . \tag{4.48}$$

Taking a chiral and the other two fields different, we get

$$C^{\phi}_{LR} d_{\phi\phi} = d_{LL} C^L_{R\phi} + d_{LR} C^R_{R\phi}\ , \quad C^{\bar\phi}_{LR} d_{\bar\phi\bar\phi} = d_{LL} C^L_{R\bar\phi} + d_{LR} C^R_{R\bar\phi}\ , \tag{4.49}$$

$$C^{\phi}_{RL} d_{\phi\phi} = d_{RR} C^R_{R\phi} + d_{RL} C^L_{R\phi}\ , \quad C^{\bar\phi}_{RL} d_{\bar\phi\bar\phi} = d_{RR} C^R_{R\bar\phi} + d_{RL} C^L_{R\bar\phi}\ . \tag{4.50}$$

Using $d_{LR} = \gamma d_{\varphi\varphi}$, these become

$$C^{\phi}_{LR} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^L_{R\phi} + \gamma C^R_{R\phi})\ , \quad C^{\bar\phi}_{LR} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^L_{R\bar\phi} + \gamma C^R_{R\bar\phi})\ , \tag{4.51}$$

$$C^{\phi}_{RL} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^R_{L\phi} + \gamma C^L_{L\phi})\ , \quad C^{\bar\phi}_{RL} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^R_{L\bar\phi} + \gamma C^L_{L\bar\phi})\ . \tag{4.52}$$

Finally, taking only b chiral, we get

$$C^{\bar\phi}_{R\phi} d_{\bar\phi\bar\phi} = d_{RR} C^R_{\phi\bar\phi} + d_{RL} C^L_{\phi\bar\phi}\ , \quad C^{\bar\phi}_{L\phi} d_{\bar\phi\bar\phi} = d_{LR} C^R_{\phi\bar\phi} + d_{LL} C^L_{\phi\bar\phi}\ , \tag{4.53}$$

$$C^{\phi}_{R\bar\phi} d_{\phi\phi} = d_{RR} C^R_{\bar\phi\phi} + d_{RL} C^L_{\bar\phi\phi}\ , \quad C^{\phi}_{L\bar\phi} d_{\phi\phi} = d_{LR} C^R_{\bar\phi\phi} + d_{LL} C^L_{\bar\phi\phi}\ . \tag{4.54}$$

Looking at the first of these, it becomes

$$\begin{aligned} C^{\bar\phi}_{R\phi} &= \frac{1}{d_{\bar\phi\bar\phi}}(d_{RR} C^R_{\phi\bar\phi} + d_{RL} C^L_{\phi\bar\phi}) \\ &= \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^R_{\phi\bar\phi} + \gamma C^L_{\phi\bar\phi}) \\ &= \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^R_{\phi\bar\phi} + \gamma C^R_{\bar\phi\phi})\ . \end{aligned} \tag{4.55}$$

Likewise we get

$$C^{\bar\phi}_{L\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^R_{\phi\bar\phi} + C^R_{\bar\phi\phi})\ , \quad C^{\phi}_{R\bar\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^R_{\bar\phi\phi} + \gamma C^R_{\phi\bar\phi})\ , \quad C^{\phi}_{L\bar\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^R_{\bar\phi\phi} + C^R_{\phi\bar\phi})\ , \tag{4.56}$$

which also imply

$$C^{\phi}_{R\bar\phi} = C^{\bar\phi}_{L\phi}\ , \quad C^{\phi}_{L\bar\phi} = C^{\bar\phi}_{R\phi}\ . \tag{4.57}$$

Bulk Field Expectation Operator Product

To find C^R_{LR} we use the inner product matrix $d_{\alpha\beta}$ of defect fields φ_L and φ_R and cyclicity of the three point constant $C_{\alpha\beta\gamma}$ defined by

$$\langle \varphi_\alpha(u,\bar u)\varphi_\beta(v,\bar v)\varphi_\gamma(w,\bar w)\rangle = C_{\alpha\beta\gamma}\left(|u-v||v-w||v-w|\right)^{-2\mathrm{h}}\ . \tag{4.58}$$

Using $C^{\gamma}_{\alpha\beta} = d^{\gamma\epsilon} C_{\alpha\beta\epsilon}$ and $C_{\alpha\beta\gamma} = C_{\gamma\beta\alpha}$ and the relations (4.32) and (4.34), we get

$$\begin{aligned} C^R_{LR} &= d^{RR} C_{LRR} + d^{RL} C_{LRL} \\ &= d^{RR} C_{RRL} + d^{RL} C_{LLR} \\ &= d^{RR}(d_{LL} C^L_{RR} + d_{LR} C^R_{RR}) + d^{RL}(d_{RL} C^L_{LL} + d_{RR} C^R_{LL}) \\ &= (d^{RR} d_{LR} + d^{RL} d_{RL}) C^{\varphi}_{\varphi\varphi} \\ &= (d^{RR} + d^{RL}) d_{RL} C^{\varphi}_{\varphi\varphi}\ . \end{aligned} \tag{4.59}$$

With the inner-product matrix $d_{\alpha\beta} = \langle\varphi_\alpha|\varphi_\beta\rangle$,

$$d_{\alpha\beta} = \begin{pmatrix} d_{LL} & d_{LR} \\ d_{RL} & d_{RR} \end{pmatrix} = d_{\varphi\varphi} \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} , \tag{4.60}$$

and its inverse

$$d^{\alpha\beta} = \begin{pmatrix} d^{LL} & d^{LR} \\ d^{RL} & d^{RR} \end{pmatrix} = \frac{1}{d_{\varphi\varphi}(1-\gamma^2)} \begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix} , \tag{4.61}$$

we obtain

$$C_{LR}^{R} = \frac{\gamma}{1+\gamma} C_{\varphi\varphi}^{\varphi} . \tag{4.62}$$

Likewise, we find all four of these structure constants are equal,

$$C_{RL}^{R} = C_{LR}^{L} = C_{RL}^{L} = C_{LR}^{R} = \frac{\gamma}{1+\gamma} C_{\varphi\varphi}^{\varphi} . \tag{4.63}$$

Continuity of Bulk Fields

We can relate the structure constants C_{aL}^{b} and $C_{L,a}^{b}$ by moving the insertion point of the field φ_L from the right of the field a to the left through the bulk. If the defect is oriented along the x axis in the plane, then the field φ_L can be moved through the upper half plane, as in Figure 4.3.

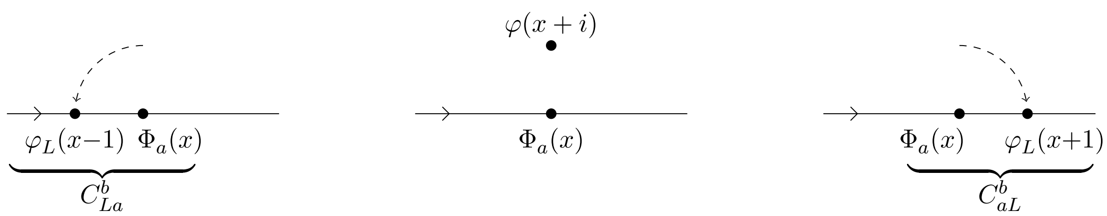


Figure 4.3: The relation between C_{La}^{b} and C_{aL}^{b} from continuity in the bulk.

Likewise, we can relate C_{aR}^{b} and $C_{R,a}^{b}$ by moving the field φ_R through the lower half plane.

Since the OPEs of the bulk field φ and the defect field φ_L with Φ_a are

$$\phi_a(u,\bar{u})\varphi(z,\bar{z}) = C_{a\varphi}^{b}\Phi_b(u,\bar{u})(u-z)^{h_b-h_a-\mathrm{h}}(\bar{u}-\bar{z})^{\bar{h}_b-\bar{h}_a-\mathrm{h}} + \dots , \tag{4.64}$$

$$\phi_a(u,\bar{u})\varphi_L(z,\bar{z}) = C_{aL}^{b}\Phi_b(u,\bar{u})|u-z|^{h_b-h_a-\mathrm{h}}|\bar{u}-\bar{z}|^{\bar{h}_b-\bar{h}_a-\mathrm{h}} + \dots , \tag{4.65}$$

$$\varphi_L(z,\bar{z})\phi_a(u,\bar{u}) = C_{La}^{b}\Phi_b(u,\bar{u})|z-u|^{h_b-h_a-\mathrm{h}}|\bar{z}-\bar{u}|^{\bar{h}_b-\bar{h}_a-\mathrm{h}} + \dots , \tag{4.66}$$

we get the relations

$$C_{La}^{b} = \exp(i\pi(h_b - \bar{h}_b - h_a + \bar{h}_a))C_{aL}^{b} , \tag{4.67}$$

$$C_{Ra}^{b} = \exp(-i\pi(h_b - \bar{h}_b - h_a + \bar{h}_a))C_{aR}^{b} . \tag{4.68}$$

We again list the cases according to the number of chiral fields involved:

- No chiral fields: we find identities consistent with Equation (4.63)

$$C_{LR}^{R} = C_{RL}^{R} , \quad C_{LR}^{L} = C_{RL}^{L} . \tag{4.69}$$

- If Φ_b is chiral and Φ_a is not; with $\zeta = \exp(i\pi \mathrm{h})$:

$$C^{\phi}_{L\alpha} = \zeta C^{\phi}_{\alpha L} \ , \quad C^{\bar\phi}_{L\alpha} = \zeta^{-1} C^{\bar\phi}_{\alpha L} \ , \quad C^{\phi}_{R\alpha} = \zeta^{-1} C^{\phi}_{\alpha R} \ , \quad C^{\bar\phi}_{R\alpha} = \zeta C^{\bar\phi}_{\alpha R} \ , \tag{4.70}$$

and hence

$$C^{\phi}_{LL} = C^{\bar\phi}_{LL} = C^{\phi}_{RR} = C^{\bar\phi}_{RR} = 0 \ , \quad C^{\phi}_{LR} = \zeta C^{\phi}_{RL} \ , \quad C^{\bar\phi}_{LR} = \zeta^{-1} C^{\bar\phi}_{RL} \ . \tag{4.71}$$

where the first four structure constants were already found to be zero in Equations (4.32) and (4.34).

- If Φ_a is chiral and Φ_b is not:

$$C^{L}_{L\phi} = \zeta^{-1} C^{L}_{\phi L} \ , \quad C^{R}_{L\phi} = \zeta^{-1} C^{R}_{\phi L} \ , \quad C^{L}_{L\bar\phi} = \zeta C^{L}_{\bar\phi L} \ , \quad C^{R}_{L\bar\phi} = \zeta C^{R}_{\bar\phi L} \ , \tag{4.72}$$

$$C^{L}_{R\phi} = \zeta C^{L}_{\phi R} \ , \quad C^{R}_{R\phi} = \zeta C^{R}_{\phi R} \ , \quad C^{L}_{R\bar\phi} = \zeta^{-1} C^{L}_{\bar\phi R} \ , \quad C^{R}_{R\bar\phi} = \zeta^{-1} C^{R}_{\bar\phi R} \ , \tag{4.73}$$

$$C^{\phi}_{LR} = \zeta C^{\phi}_{RL} \ , \quad C^{\bar\phi}_{LR} = \zeta^{-1} C^{\bar\phi}_{RL} \ . \tag{4.74}$$

- If both Φ_a and Φ_b are chiral:

$$C^{\bar\phi}_{L\phi} = \zeta^{-2} C^{\bar\phi}_{\phi L} \ , \quad C^{\phi}_{L\bar\phi} = \zeta^{2} C^{\phi}_{\bar\phi L} \ , \quad C^{\bar\phi}_{R\phi} = \zeta^{2} C^{\bar\phi}_{\phi R} \ , \quad C^{\phi}_{R\bar\phi} = \zeta^{-2} C^{\phi}_{\bar\phi R} \ , \tag{4.75}$$

4.2.5 Unknown Constants

We summarise the results so far, distinguishing the structure constants by the number of chiral fields they involve.

No Chiral Fields

These are all known in terms of the bulk field data:

$$d_{RR} = d_{LL} = d_{\varphi\varphi} \ , \quad d_{LR} = d_{RL} = \gamma \, d_{\varphi\varphi} \ , \tag{4.76}$$

$$C^{L}_{LL} = C^{R}_{RR} = C^{\varphi}_{\varphi\varphi} \ , \quad C^{R}_{LL} = C^{L}_{RR} = 0 \ , \tag{4.77}$$

$$C^{R}_{LR} = C^{L}_{LR} = C^{R}_{RL} = C^{L}_{RL} = \tfrac{\gamma}{1+\gamma} \, C^{\varphi}_{\varphi\varphi} \ . \tag{4.78}$$

Three Chiral Fields

These are also all known in terms of the boundary field theory data [66]:

$$C^{\bar\phi}_{\bar\phi\bar\phi} = C^{\phi}_{\phi\phi} \ , \quad d_{\bar\phi\bar\phi} = d_{\phi\phi} \ , \tag{4.79}$$

$$C^{\phi}_{\bar\phi\bar\phi} = C^{\bar\phi}_{\phi\phi} = C^{\bar\phi}_{\phi\bar\phi} = C^{\bar\phi}_{\bar\phi\phi} = C^{\phi}_{\phi\bar\phi} = C^{\phi}_{\bar\phi\phi} = 0 \ . \tag{4.80}$$

Two Chiral Fields

The 24 structure constants involving two chiral fields can be written in terms of just two of these, which we can take to be

$$C^{L}_{\bar{\phi}\phi} \quad \text{and} \quad C^{L}_{\phi\bar{\phi}} \,. \tag{4.81}$$

Listing the remaining 22 structure constants:

$$C^{R}_{\phi\bar{\phi}} = C^{L}_{\bar{\phi}\phi} \,, \qquad C^{R}_{\bar{\phi}\phi} = C^{L}_{\phi\bar{\phi}} \,, \tag{4.82}$$

$$C^{\bar{\phi}}_{R\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\bar{\phi}\phi} + \gamma C^{L}_{\phi\bar{\phi}}) \,, \qquad C^{\bar{\phi}}_{L\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\bar{\phi}\phi} + C^{L}_{\phi\bar{\phi}}) \,, \tag{4.83}$$

$$C^{\phi}_{R\bar{\phi}} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\phi\bar{\phi}} + \gamma C^{L}_{\bar{\phi}\phi}) \,, \qquad C^{\phi}_{L\bar{\phi}} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\phi\bar{\phi}} + C^{L}_{\bar{\phi}\phi}) \,, \tag{4.84}$$

$$C^{\bar{\phi}}_{\phi R} = \zeta^{-2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\bar{\phi}\phi} + \gamma C^{L}_{\phi\bar{\phi}}) \,, \quad C^{\bar{\phi}}_{\phi L} = \zeta^{2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\bar{\phi}\phi} + C^{L}_{\phi\bar{\phi}}) \,, \tag{4.85}$$

$$C^{\phi}_{\bar{\phi} R} = \zeta^{2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\phi\bar{\phi}} + \gamma C^{L}_{\bar{\phi}\phi}) \,, \qquad C^{\phi}_{\bar{\phi} L} = \zeta^{-2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\phi\bar{\phi}} + C^{L}_{\bar{\phi}\phi}) \,, \tag{4.86}$$

$$C^{R}_{\phi\phi} = C^{L}_{\phi\phi} = C^{R}_{\bar{\phi}\bar{\phi}} = C^{L}_{\bar{\phi}\bar{\phi}} = 0 \,, \tag{4.87}$$

$$C^{\phi}_{R\phi} = C^{\phi}_{L\phi} = C^{\bar{\phi}}_{R\bar{\phi}} = C^{\bar{\phi}}_{L\bar{\phi}} = 0 \,, \tag{4.88}$$

$$C^{\phi}_{\phi R} = C^{\phi}_{\phi L} = C^{\bar{\phi}}_{\bar{\phi} R} = C^{\bar{\phi}}_{\bar{\phi} L} = 0 \,. \tag{4.89}$$

It will be convenient to introduce κ and Γ to parametrise $C^{L}_{\phi\bar{\phi}}$ and $C^{L}_{\bar{\phi}\phi}$ as

$$C^{L}_{\phi\bar{\phi}} = \kappa\Gamma \,, \quad C^{L}_{\bar{\phi}\phi} = \kappa^{-1}\Gamma \,, \quad C^{L}_{\phi\bar{\phi}} = \kappa^{2} C^{L}_{\bar{\phi}\phi} \,. \tag{4.90}$$

It will turn out that Γ is real and non-negative, and κ is a pure phase. We note that these two structure constants can be found from the results in [100]—they are related to C_s defined in [100]: Equation (2.19).

One Chiral Field

The twenty-four structure constants involving just one chiral field can, using the previous identities, be written in terms of just four:

$$C^{R}_{\phi L} \,, \quad C^{R}_{\bar{\phi} L} \,, \quad C^{L}_{\phi R} \,, \quad C^{L}_{\bar{\phi} R} \,. \tag{4.91}$$

We list the remaining twenty constants here for convenience:

$$C^R_{L\phi} = \zeta^{-1} C^R_{\phi L}\,, \qquad C^R_{L\bar\phi} = \zeta C^R_{\bar\phi L}\,, \tag{4.92}$$

$$C^L_{R\phi} = \zeta^{-1} C^L_{\phi R}\,, \qquad C^L_{R\bar\phi} = \zeta C^L_{\bar\phi R}\,, \tag{4.93}$$

$$C^L_{\phi L} = -\gamma C^R_{\phi L}\,, \qquad C^R_{\phi R} = -\gamma C^L_{\phi R}\,, \tag{4.94}$$

$$C^L_{\bar\phi L} = -\gamma C^R_{\bar\phi L}\,, \qquad C^R_{\bar\phi R} = -\gamma C^L_{\bar\phi R}\,, \tag{4.95}$$

$$C^L_{L\phi} = \zeta^{-1} C^L_{\phi L} = -\gamma\zeta^{-1}\, C^R_{\phi L}\,, \qquad C^L_{L\bar\phi} = \zeta C^L_{\bar\phi L} = -\gamma\zeta\, C^R_{\bar\phi L}\,, \tag{4.96}$$

$$C^R_{R\phi} = \zeta^{-1} C^R_{\phi R} = -\gamma\zeta^{-1}\, C^L_{\phi R}\,, \qquad C^R_{R\bar\phi} = \zeta C^R_{\bar\phi R} = -\gamma\zeta\, C^L_{\bar\phi R}\,, \tag{4.97}$$

$$C^\phi_{LR} = \frac{1-\gamma^2}{\zeta}\frac{d_{\varphi\varphi}}{d_{\phi\phi}} C^L_{\phi R}\,, \qquad C^{\bar\phi}_{LR} = (1-\gamma^2)\frac{d_{\varphi\varphi}}{d_{\phi\phi}} C^R_{\bar\phi L}\,, \tag{4.98}$$

$$C^\phi_{RL} = \frac{1-\gamma^2}{\zeta^2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}} C^L_{\phi R}\,, \qquad C^{\bar\phi}_{RL} = \zeta\,(1-\gamma^2)\frac{d_{\varphi\varphi}}{d_{\phi\phi}} C^R_{\bar\phi L}\,, \tag{4.99}$$

$$C^\phi_{LL} = C^{\bar\phi}_{LL} = C^\phi_{RR} = C^{\bar\phi}_{RR} = 0\,. \tag{4.100}$$

4.2.6 Four-Point Function Sewing Constraints

We will use crossing relations for four point correlation functions to find sewing constraints that will enable us to determined the remaining six structure constants $\{C^L_{\bar\phi\phi}, C^L_{\phi\bar\phi}, C^R_{\phi L}, C^R_{\bar\phi L}, C^L_{\phi R}, C^L_{\bar\phi R}\}$.

The four-point function $\langle \Phi_a \Phi_b \Phi_c \Phi_d \rangle$ of fields on a defect can be expressed in terms of conformal blocks in two different ways, as illustrated in Figure 4.4.

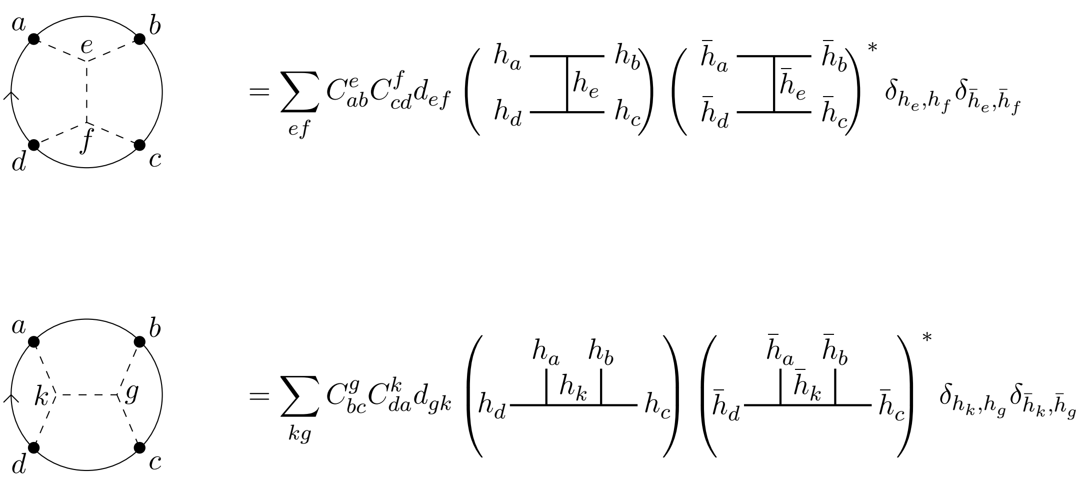
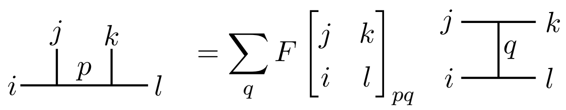


Figure 4.4: Two ways of calculating a four-point defect field correlation function

The conformal blocks are functions which satisfy the crossing relations [66]



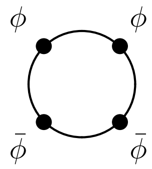
(4.101)

where the F-matrices are known constants, again given explicitly in [66]. Substituting (4.101) into the expressions in Figure 4.4 leads to further sewing constraints that the structure constants must satisfy.

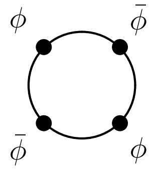
The simplest relations arise when there is only a single channel in both diagrams, i.e. the sum is over a single pair of weights $(h_e, \bar{h}_e)$ and a single pair of weights $(h_g, \bar{h}_g)$. Note that since the space of fields with weights (h, h) is two-dimensional, this does not mean that the OPE has to include only a single field. In all the cases where there is only a single channel, the F-matrix is just the number 1 and so the sewing constraints become just

$$\sum_{e,f} C^e_{ab} C^f_{cd} d_{ef} = \sum_{g,k} C^g_{bc} C^k_{da} d_{gk} \ . \tag{4.102}$$

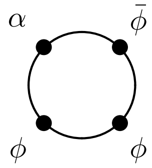
We now list all the non-zero cases in which the fields a, b, c and d are taken from $\{\phi, \bar{\phi}, \varphi_\alpha\}$ and for which there is only a single intermediate channel in both diagrams, and state the corresponding equations. We will in fact only use the first eight of these, where there is at most one field of weights (h, h) but we list them all for completeness. The eight we use are:



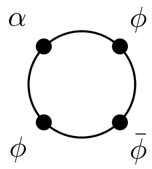
$$d_{\phi\phi}\, d_{\bar{\phi}\bar{\phi}} = \sum_{\alpha,\beta} C^\alpha_{\phi\bar{\phi}} C^\beta_{\bar{\phi}\phi} d_{\alpha\beta} \tag{4.103}$$



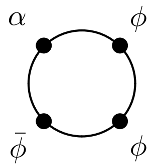
$$\sum_{\alpha,\beta} C^\alpha_{\phi\bar{\phi}} C^\beta_{\phi\bar{\phi}} d_{\alpha\beta} = \sum_{\alpha,\beta} C^\alpha_{\bar{\phi}\phi} C^\beta_{\bar{\phi}\phi} d_{\alpha\beta} \tag{4.104}$$



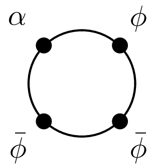
$$C^\phi_{\alpha\bar{\phi}} C^\phi_{\phi\phi} d_{\phi\phi} = \sum_{\beta,\gamma} C^\beta_{\bar{\phi}\phi} C^\gamma_{\phi\alpha} d_{\beta\gamma} \tag{4.105}$$



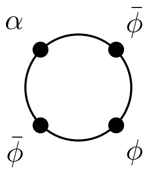
$$\sum_{\beta\gamma} C^\beta_{\alpha\phi} C^\gamma_{\bar{\phi}\phi} d_{\beta\gamma} = \sum_{\beta,\gamma} C^\beta_{\phi\bar{\phi}} C^\gamma_{\phi\alpha} d_{\beta\gamma} \tag{4.106}$$



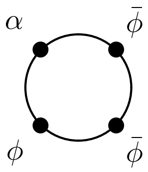
$$\sum_{\beta\gamma} C^\beta_{\alpha\phi} C^\gamma_{\phi\bar{\phi}} d_{\beta\gamma} = C^\phi_{\phi\phi} C^\phi_{\bar{\phi}\alpha} d_{\phi\phi} \tag{4.107}$$



$$C^{\bar{\phi}}_{\alpha\phi} C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}} d_{\bar{\phi}\bar{\phi}} = \sum_{\beta,\gamma} C^\beta_{\phi\bar{\phi}} C^\gamma_{\bar{\phi}\alpha} d_{\beta\gamma} \tag{4.108}$$

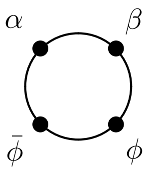


$$\sum_{\beta\gamma} C^{\beta}_{\alpha\bar\phi} C^{\gamma}_{\phi\bar\phi} d_{\beta\gamma} = \sum_{\beta,\gamma} C^{\beta}_{\bar\phi\phi} C^{\gamma}_{\bar\phi\alpha} d_{\beta\gamma} \tag{4.109}$$

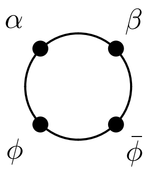


$$\sum_{\beta\gamma} C^{\beta}_{\alpha\bar\phi} C^{\gamma}_{\bar\phi\phi} d_{\beta\gamma} = C^{\bar\phi}_{\bar\phi\bar\phi} C^{\bar\phi}_{\phi\alpha} d_{\bar\phi\bar\phi} \tag{4.110}$$

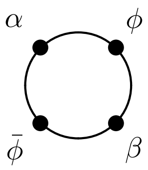
The remaining three which include two fields of type φ_α but still only have a single intermediate channel are:



$$\sum_{\gamma\epsilon} C^{\gamma}_{\alpha\beta} C^{\epsilon}_{\phi\bar\phi} d_{\gamma\epsilon} = \sum_{\gamma\epsilon} C^{\gamma}_{\beta\phi} C^{\epsilon}_{\bar\phi\alpha} d_{\gamma\epsilon} \tag{4.111}$$



$$\sum_{\gamma\epsilon} C^{\gamma}_{\alpha\beta} C^{\epsilon}_{\bar\phi\phi} d_{\gamma\epsilon} = \sum_{\gamma\epsilon} C^{\gamma}_{\beta\bar\phi} C^{\epsilon}_{\phi\alpha} d_{\gamma\epsilon} \tag{4.112}$$



$$\sum_{\gamma\epsilon} C^{\gamma}_{\alpha\phi} C^{\epsilon}_{\beta\bar\phi} d_{\gamma\epsilon} = \sum_{\gamma\epsilon} C^{\gamma}_{\phi\beta} C^{\epsilon}_{\bar\phi\alpha} d_{\gamma\epsilon} \tag{4.113}$$

4.2.7 Analysis of Sewing Constraints

We need to use only the first eight relations. We consider these in turn:

- Equation (4.103)

  Written out in full, this is

$$d_{\phi\phi} d_{\bar\phi\bar\phi} = C^{L}_{\phi\bar\phi} C^{L}_{\bar\phi\phi} d_{LL} + C^{L}_{\phi\bar\phi} C^{R}_{\bar\phi\phi} d_{LR} + C^{R}_{\phi\bar\phi} C^{L}_{\bar\phi\phi} d_{RL} + C^{R}_{\phi\bar\phi} C^{R}_{\bar\phi\phi} d_{RR} \; . \tag{4.114}$$

  Using $C^{L}_{\phi\bar\phi} = C^{R}_{\bar\phi\phi} = \kappa\Gamma$ and $C^{R}_{\phi\bar\phi} = C^{L}_{\bar\phi\phi} = \kappa^{-1}\Gamma$, together with $d_{LR} = d_{RL} = \gamma d_{\varphi\varphi}$, and $d_{\phi\phi} = d_{\bar\phi\bar\phi}$, this becomes

$$\frac{d^2_{\phi\phi}}{d_{\varphi\varphi}} = \Gamma^2 (2 + \gamma\kappa^2 + \gamma\kappa^{-2}) \; , \tag{4.115}$$

  or

$$\Gamma = \sqrt{\frac{d^2_{\phi\phi}}{d_{\varphi\varphi} \, (2 + \gamma\kappa^2 + \gamma\kappa^{-2})}} \; . \tag{4.116}$$

- Equation (4.104)

  This is

$$C^L_{\phi\bar\phi}C^L_{\phi\bar\phi}d_{LL} + C^L_{\phi\bar\phi}C^R_{\phi\bar\phi}d_{LR} + C^R_{\phi\bar\phi}C^L_{\phi\bar\phi}d_{RL} + C^R_{\phi\bar\phi}C^R_{\phi\bar\phi}d_{RR} = C^L_{\bar\phi\phi}C^L_{\bar\phi\phi}d_{LL} + C^L_{\bar\phi\phi}C^R_{\bar\phi\phi}d_{LR} + C^R_{\bar\phi\phi}C^L_{\bar\phi\phi}d_{RL} + C^R_{\bar\phi\phi}C^R_{\bar\phi\phi}d_{RR} \ , \quad (4.117)$$

  which is satisfied identically.

- Equation (4.105)

  This leads to two equations: for $\alpha = L$:

$$C^\phi_{L\bar\phi}C^\phi_{\phi\phi}d_{\phi\phi} = C^L_{\bar\phi\phi}C^L_{\phi L}d_{LL} + C^L_{\bar\phi\phi}C^R_{\phi L}d_{LR} + C^R_{\bar\phi\phi}C^L_{\phi L}d_{RL} + C^R_{\bar\phi\phi}C^R_{\phi L}d_{RR} \ , \quad (4.118)$$

  and for $\alpha = R$:

$$C^\phi_{R\bar\phi}C^\phi_{\phi\phi}d_{\phi\phi} = C^L_{\bar\phi\phi}C^L_{\phi R}d_{LL} + C^L_{\bar\phi\phi}C^R_{\phi R}d_{LR} + C^R_{\bar\phi\phi}C^L_{\phi R}d_{RL} + C^R_{\bar\phi\phi}C^R_{\phi R}d_{RR} \ . \quad (4.119)$$

  The first equation becomes:

$$(\gamma C^L_{\phi\bar\phi} + C^L_{\bar\phi\phi})C^\phi_{\phi\phi} = C^R_{\phi L}C^L_{\phi\bar\phi}\left(1-\gamma^2\right) \ , \quad (4.120)$$

  or

$$C^R_{\phi L} = \frac{1+\kappa^2\gamma}{\kappa^2(1-\gamma^2)}C^\phi_{\phi\phi} \ . \quad (4.121)$$

  The second equation implies

$$C^L_{\phi R} = \frac{\kappa^2+\gamma}{(1-\gamma^2)}C^\phi_{\phi\phi} \ . \quad (4.122)$$

- Equation (4.106)

  These two equations imply

$$\kappa^2 = \zeta = \exp(i\pi\mathrm{h}) \ . \quad (4.123)$$

  (We will not need to fix the sign of κ as only κ^2 appears in our final answers.)

- Equation (4.107)

  These equations imply (for $\alpha = L$)

$$C^R_{\phi L} = \frac{1+\kappa^2\gamma}{\zeta(1-\gamma^2)}C^\phi_{\phi\phi} \ , \quad (4.124)$$

  and (for $\alpha = R$)

$$C^L_{\phi R} = \frac{\zeta^2}{\kappa}\frac{\gamma+\kappa^2}{(1-\gamma^2)}C^\phi_{\phi\phi} \ , \quad (4.125)$$

  which are consistent with the results so far.

- Equation (4.108)

  These two equations lead to ($\alpha = L$):

$$C^R_{\bar\phi L} = \frac{\gamma + \kappa^2}{1-\gamma^2} C^\phi_{\phi\phi} \,, \tag{4.126}$$

  and (with $\alpha = R$)

$$C^L_{\bar\phi R} = \frac{1 + \gamma\kappa^2}{\kappa^2(1-\gamma^2)} C^\phi_{\phi\phi} \,. \tag{4.127}$$

  Together, these imply

$$C^R_{\bar\phi L} = C^L_{\phi R} \quad \text{and} \quad C^L_{\bar\phi R} = C^R_{\phi L} \,. \tag{4.128}$$

This completes the derivation of the structure constants. They agree with the specific case in [110] (apart from a typo in [110], where it should $\rho = \exp(i\pi/10)$). The remaining crossing relations (4.111) – (4.113) are not needed for the derivation of the structure constants but we have checked that they hold.

4.3 Integrals

We want to calculate the leading term in the expansion (4.10), that is

$$I = \frac{1}{4}\lambda^2\bar\lambda^2 \int \mathrm{d}x\ \mathrm{d}x'\ \mathrm{d}y\ \mathrm{d}y' \langle\, T(iY)\, \overline{T}(iY)\, \phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\,\rangle \,. \tag{4.129}$$

The correlation function has the same functional form whatever the order of the fields, but a different constant depending on the order of the insertions. We can restrict to $x < x'$ and $y < y'$ to get

$$I = (\lambda\bar\lambda)^2 \left\langle\, T(i)\bar T(i) \int_{x<x'\,,\,y<y'} \mathrm{d}x\ \mathrm{d}x'\ \mathrm{d}y\ \mathrm{d}y'\ \phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\, \right\rangle_{D_{r2}} \,. \tag{4.130}$$

This correlation function is

$$\langle\, T(i)\,\bar T(i)\,\phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\,\rangle = \Delta\,\mathrm{h}^2\, \frac{(x'-x)^{2-2\mathrm{h}}\,(y'-y)^{2-2\mathrm{h}}}{(i-x)^2(i-x')^2(i+y)^2(i+y')^2} \,, \tag{4.131}$$

where the constant Δ depends on the order of the field insertions as in table 4.2. The values Δ_i are

$$\Delta_1 = d_{\phi\phi}\, d_{\bar\phi\bar\phi} = (d_{\phi\phi})^2 \,, \tag{4.132}$$

$$\Delta_2 = d_{\alpha\beta}\, C^\alpha_{\phi\bar\phi} C^\beta_{\phi\bar\phi} = (d_{\phi\phi})^2 \frac{2\gamma + \kappa^2 + \kappa^{-2}}{2 + \gamma\kappa^2 + \gamma\kappa^{-2}} \,. \tag{4.133}$$

We only need to evaluate three of these integrations, the other three being given by complex conjugation. Furthermore, we only need the leading order term in y in the correlation function,

$$\langle\, T(i)\bar T(i)\phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\,\rangle_{D_{r2}} = \frac{\Delta}{(i-x)^2(i-x')^2(i+y)^2(i+y')^2} + O(y) \,. \tag{4.134}$$

Integration region	Order of fields	Value of Δ
$x < x' < y < y'$	$\phi\phi\bar\phi\bar\phi$	Δ_1
$x < y < x' < y'$	$\phi\bar\phi\phi\bar\phi$	Δ_2
$x < y < y' < x'$	$\phi\bar\phi\bar\phi\phi$	Δ_1
$y < x < x' < y'$	$\bar\phi\phi\phi\bar\phi$	Δ_1
$y < x < y' < x'$	$\bar\phi\phi\bar\phi\phi$	Δ_2
$y < y' < x < x'$	$\bar\phi\bar\phi\phi\phi$	Δ_1

Table 4.2: The coefficient in the four-point function (4.131)

Integration region	Order of fields	Value of the integral
$x < x' < y < y'$	$\phi\phi\bar\phi\bar\phi$	$-\frac{3\pi i}{16}\Delta_1$
$x < y < x' < y'$	$\phi\bar\phi\phi\bar\phi$	$-\frac{\pi^2+3\pi i}{8}\Delta_2$
$x < y < y' < x'$	$\phi\bar\phi\bar\phi\phi$	$\frac{\pi^2}{8}\Delta_1$
$y < x < x' < y'$	$\bar\phi\phi\phi\bar\phi$	$\frac{\pi^2}{8}\Delta_1$
$y < x < y' < x'$	$\bar\phi\phi\bar\phi\phi$	$-\frac{\pi^2-3\pi i}{8}\Delta_2$
$y < y' < x < x'$	$\bar\phi\bar\phi\phi\phi$	$\frac{3\pi i}{16}\Delta_1$

Table 4.3: The integrals

The results are given in table 4.3.

Adding all six together, we get

$$\begin{aligned} I &= (\lambda\bar\lambda)^2 \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{d}x'\ \mathrm{d}y\ \mathrm{d}y'\ \left\langle\ T(i)\bar T(i)\phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\ \right\rangle_{D_{r2}} \\ &= (\lambda\bar\lambda)^2 \left[\frac{\pi^2}{4}(\Delta_1 - \Delta_2) + O(y)\right] \\ &= \frac{\pi^2}{4}\ (\lambda\bar\lambda)^2\ (d_{\phi\phi})^2 \left[1 - \left[\frac{2\gamma + \kappa^2 + \kappa^{-2}}{2 + \gamma\kappa^2 + \gamma\kappa^{-2}}\right] + O(y)\right]\ . \end{aligned} \tag{4.135}$$

4.4 Value of Reflection Coefficient for Defect C

We now put the various terms together to find the value of $\mathcal{R}$ at the fixed point (λ^*, λ^*),

$$\mathcal{R} = \frac{\langle T^1\overline{T}^1 + T^2\overline{T}^2\rangle}{\langle (T^1 + \overline{T}^2)(\overline{T}^1 + T^2)\rangle}\ . \tag{4.136}$$

The leading term in the numerator is $2I$ and leading term in the denominator is $c/16$.

We first give the expansion in $y = 1 - \mathrm{h}$ of the various constants. With $\mathrm{h} = 2t - 1$ we get $t = 1 - y/2$ and so

$$\kappa^2 = \zeta = \exp(i\pi\mathrm{h}) = -1 + O(y)\ , \tag{4.137}$$

$$\gamma = 2\cos(2\pi t) - 1 = 1 + O(y^2)\ , \tag{4.138}$$

$$\frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} = \frac{8}{3} + O(y)\ . \tag{4.139}$$

At the fixed point,

$$I = \frac{\pi^2}{4} \, (\lambda^*)^4 \, (d_{\phi\phi})^2 \, \left[1 - \left[\frac{2\gamma + \kappa^2 + \kappa^{-2}}{2 + \gamma\kappa^2 + \gamma\kappa^{-2}} \right] + O(y) \right]$$
$$= \frac{9\pi^2 y^4}{256} + O(y^5) \; , \qquad (4.140)$$

and with $c = 1 + O(y)$, we find

$$\mathcal{R} = \frac{2\frac{9\pi^2 y^4}{256} + O(y^5)}{1/16 + O(y)} = \frac{9\pi^2 y^4}{8} + O(y^5) \; . \qquad (4.141)$$

We can now calculate this for the tri-critical Ising model. In this case, $\mathrm{h} = 3/5$, $y = 2/5$ and we are far from the small y regime, but we calculate the leading correction and get

$$\mathcal{R} \sim \frac{18\pi^2}{625} = 0.284\ldots \; . \qquad (4.142)$$

Note that the corresponding transmission coefficients is $\mathcal{T} \sim 0.715\ldots$. This can be compared with the values[117] we find in (6.115) and (6.116) from Chapter 6, which are

$$\frac{\sqrt{3} - 1}{2} = 0.366\ldots \quad \text{and} \quad \frac{3 - \sqrt{3}}{2} = 0.633\ldots \; . \qquad (4.143)$$

The latter value is close enough not to rule out that the conformal defect we found in Chapter 6 is related to the one found by perturbation theory.

Chapter 5

Interfaces Between Conformal Field Theories

In this chapter, we construct topological interfaces between the Ising model and the free fermion theory at $c = \frac{1}{2}$, and the tri-critical Ising model considered as a bosonic theory and as a $N = 1$ super-Virasoro minimal model at $c = \frac{7}{10}$. The aim of this chapter is to use the latter interface and the superconformal defects we construct in the next chapter to obtain conformal defects in the bosonic tri-critical Ising model. These also provide new examples of topological interfaces between conformal field theories.

5.1 Ising Model and Free Fermion Theory

As a warm-up, we consider topological interfaces between the Ising model and the free fermion theory. While the free fermion theory is not supersymmetric, the construction of interfaces between the bosonic and fermionic theories can be generalised to the tri-critical Ising model case.

5.1.1 Bulk Fields in Free Fermion Theory

A single free massless Majorana fermion theory on the complex plane has the action

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right) \,, \tag{5.1}$$

where $\psi(z)$ and $\bar{\psi}(\bar{z})$ are the holomorphic and antiholomorphic components of the free fermion. The equations of motion imply ψ and $\bar{\psi}$ are independent on the full complex plane, and the chiral components of the free fermion have conformal dimensions $h_\psi = \frac{1}{2}$ and $\bar{h}_{\bar{\psi}} = \frac{1}{2}$. This theory has the holomorphic and antiholomorphic stress-energy tensors given by the normal ordered products

$$T(z) = -\frac{1}{2}(\psi \partial \psi)(z) \quad \text{and} \quad \bar{T}(\bar{z}) = -\frac{1}{2}(\bar{\psi} \bar{\partial} \bar{\psi})(\bar{z}) \,. \tag{5.2}$$

Mode expansions of $\psi(z)$ and $\bar{\psi}(\bar{z})$ are given by

$$\psi(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \psi_n \, z^{-n-\frac{1}{2}} \quad \text{and} \quad \bar{\psi}(\bar{z}) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \bar{\psi}_n \, \bar{z}^{-n-\frac{1}{2}} \,, \tag{5.3}$$

and they satisfy the anticommutation relations

$$\{\psi_n, \psi_m\} = \delta_{n+m,0} \,, \quad \{\bar{\psi}_n, \bar{\psi}_m\} = \delta_{n+m,0} \,, \quad \text{and} \quad \{\psi_n, \bar{\psi}_m\} = 0 \,. \tag{5.4}$$

These modes define the untwisted representation, which is called the Neveu–Schwarz sector, of the free fermion algebra. The chiral representation $\mathcal{H}_{\text{NS}}$ is the Fock space spanned by vectors of the form

$$\psi_{n_1} \psi_{n_2} \cdots \psi_{n_k} |0\rangle \quad \text{with} \quad n_1 < n_2 < \cdots < n_k < 0 \,, \tag{5.5}$$

where $|0\rangle$ is the unique vacuum state. Since $\psi(z)$ and $\bar{\psi}(\bar{z})$ are real, we have $\psi_n^\dagger = \psi_{-n}$ and $\bar{\psi}_n^\dagger = \bar{\psi}_{-n}$.

From the anticommutation relations and the form of stress-energy tensors, we can see there is an automorphism of the algebra given by $\psi(z) \mapsto -\psi(z)$. Therefore, we can impose a twisted boundary condition

$$\psi(e^{2\pi i} z) = -\psi(z) \ . \tag{5.6}$$

As a consequence, mode expansions of $\psi(z)$ becomes

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n \, z^{-n-\frac{1}{2}} \ . \tag{5.7}$$

They define the twisted representation of the free fermion algebra, which is called the Ramond sector. As we have seen in (2.111), such representations correspond to non-local fields. The ground states of the bulk theory in the Ramond sector corresponds to the spin fields $\sigma(z,\bar{z})$ and $\mu(z,\bar{z})$ that are discussed in Appendix C.

Roughly speaking, spin structures on a Riemann surface are determined by the periodicities of fermions around homotopically inequivalent uncontractible loops. For a Riemann surface of genus g, there are 2^{2g} spin structures: the circle S^1 has two inequivalent spin structures that are called Neveu–Schwarz and Ramond, and for genus 1, four spin structures are called NS–NS, NS–R, R–NS, and R–R. For a more general definition of spin structures on Riemannian manifolds in terms of principle fibre bundles, see, for example, [1].

We take the bulk sector to be the theory of local fermion fields $\psi(z)$ and $\bar{\psi}(\bar{z})$ in the Neveu–Schwarz sector. Therefore, the bulk state space is given by

$$\mathcal{H}_{\text{f.f.}} = \mathcal{H}_{\text{NS}} \otimes \overline{\mathcal{H}}_{\text{NS}} = (\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}}) \otimes (\overline{\mathcal{H}}_0 \oplus \overline{\mathcal{H}}_{\frac{1}{2}}) \ , \tag{5.8}$$

where $\mathcal{H}_h$ on the right hand side is the irreducible Virasoro representation with $c = \frac{1}{2}$ and the conformal weight h. The corresponding torus partition function can be written as

$$Z_{\text{f.f.}} = \text{Tr}_{\mathcal{H}_{\text{f.f.}}} \left(q^{L_0 - \frac{1}{48}} \bar{q}^{\bar{L}_0 - \frac{1}{48}} \right) = |\chi_{\text{NS}}(q)|^2 \ , \tag{5.9}$$

where $q = e^{2\pi i \tau}$ and $\chi_{\text{NS}}(q)$ is given by (C.4). This partition function is invariant under $\text{S} : \tau \mapsto -1/\tau$ and $\text{T}^2 : \tau \mapsto \tau + 2$ but under $\text{T} : \tau \mapsto \tau + 1$, it becomes

$$\text{T} : |\chi_{\text{NS}}(q)|^2 \mapsto |\tilde{\chi}_{\text{NS}}(q)|^2 \ , \tag{5.10}$$

where the supercharacter $\tilde{\chi}_{\text{NS}}(q)$ is given in (C.5).

5.1.2 Topological Defects in Free Fermion Theory

We consider topological defects $D_{\epsilon,\bar{\epsilon}}$ that preserve the free fermion algebra up to automorphisms. Then the defect operators must satisfy

$$\psi_n \, D_{\epsilon,\bar{\epsilon}} = \epsilon \, D_{\epsilon,\bar{\epsilon}} \, \psi_n \quad \text{and} \quad \bar{\psi}_n \, D_{\epsilon,\bar{\epsilon}} = \bar{\epsilon} \, D_{\epsilon,\bar{\epsilon}} \, \bar{\psi}_n \ , \tag{5.11}$$

where $\epsilon = \pm$ and $\bar{\epsilon} = \pm$. As we can write the Virasoro generators as (C.1), if a defect operator satisfies (5.11), then it satisfies the topological condition (2.261) as well. There are operators $\mathbf{1}$, $(-1)^F$, $(-1)^{\bar{F}}$, and $(-1)^{F+\bar{F}}$ acting on the space $\mathcal{H}_{\text{f.f.}}$, and the condition (5.11) determines the topological defect operators as

$$D_{++} = a_{++}\mathbf{1} \ , \quad D_{-+} = a_{-+}(-1)^F \ , \quad D_{+-} = a_{+-}(-1)^{\bar{F}} \ , \quad D_{--} = a_{--}(-1)^{F+\bar{F}} \tag{5.12}$$

up to some normalisation $a_{\epsilon,\bar{\epsilon}}$. We determine $a_{\epsilon,\bar{\epsilon}}$ from the Cardy-type constraint (2.272) and the topological defect fusion rules (2.281).

The torus partition functions with one defect inserted are

$$\mathrm{Tr}_{\mathcal{H}_{\text{f.f.}}}\left(D_{++}\, \tilde{q}^{L_0-\frac{1}{48}}\tilde{\bar{q}}^{\bar{L}_0-\frac{1}{48}}\right) = a_{++}|\chi_{\text{NS}}(\tilde{q})|^2 \qquad = a_{++}|\chi_{\text{NS}}(q)|^2 \ , \tag{5.13}$$

$$\mathrm{Tr}_{\mathcal{H}_{\text{f.f.}}}\left(D_{-+}\, \tilde{q}^{L_0-\frac{1}{48}}\tilde{\bar{q}}^{\bar{L}_0-\frac{1}{48}}\right) = a_{-+}\tilde{\chi}_{\text{NS}}(\tilde{q})\chi_{\text{NS}}(\tilde{\bar{q}}) = \sqrt{2}a_{-+}\chi_{\text{R}}(q)\chi_{\text{NS}}(\bar{q}) \ , \tag{5.14}$$

$$\mathrm{Tr}_{\mathcal{H}_{\text{f.f.}}}\left(D_{+-}\, \tilde{q}^{L_0-\frac{1}{48}}\tilde{\bar{q}}^{\bar{L}_0-\frac{1}{48}}\right) = a_{+-}\chi_{\text{NS}}(\tilde{q})\tilde{\chi}_{\text{NS}}(\tilde{\bar{q}}) = \sqrt{2}a_{+-}\chi_{\text{NS}}(q)\chi_{\text{R}}(\bar{q}) \ , \tag{5.15}$$

$$\mathrm{Tr}_{\mathcal{H}_{\text{f.f.}}}\left(D_{--}\, \tilde{q}^{L_0-\frac{1}{48}}\tilde{\bar{q}}^{\bar{L}_0-\frac{1}{48}}\right) = a_{--}|\tilde{\chi}_{\text{NS}}(\tilde{q})|^2 \qquad = 2a_{--}|\chi_{\text{R}}(q)|^2 \ , \tag{5.16}$$

where $\tilde{q} = e^{-2\pi i/\tau}$. From these we see $a_{++},\ a_{--} \in \mathbb{Z}_{>0}$, and a_{+-} and a_{-+} must be positive integer multiples of $\sqrt{2}$ or $1/\sqrt{2}$. Compositions of topological defect operators give

$$D_{\epsilon_1,\bar{\epsilon}_1}D_{\epsilon_2,\bar{\epsilon}_2} = \frac{a_{\epsilon_1,\bar{\epsilon}_1}a_{\epsilon_2,\bar{\epsilon}_2}}{a_{\epsilon_1\epsilon_2,\bar{\epsilon}_1\bar{\epsilon}_2}}D_{\epsilon_1\epsilon_2,\bar{\epsilon}_1\bar{\epsilon}_2} \ , \tag{5.17}$$

which rules out a_{-+} and a_{+-} being multiples of $1/\sqrt{2}$. Therefore, the smallest solutions are

$$a_{++} = a_{--} = 1 \quad \text{and} \quad a_{-+} = a_{+-} = \sqrt{2} \ . \tag{5.18}$$

The space of disorder fields for these topological defects are summarised in Table 5.1. Notations and conventions for various fields and state spaces used in this table are summarised in Appendix C. Note that i in $i\psi\bar{\psi}(z,\bar{z})$ is introduced to make its two-point function positive.

D	$Z_{D\|0}(q,\bar{q})$	$\mathcal{H}_{D\|0}$	$\mathcal{V}$ir primary disorder fields	
			bosons	fermions
$D_{++} = \mathbf{1}$	$Z_{\text{f.f.}} = \|\chi_{\text{NS}}(q)\|^2$	$\mathcal{H}_{\text{f.f}}$	$\mathbf{1},\ i\psi\bar{\psi}(z,\bar{z})$	$\psi(z),\ \bar{\psi}(\bar{z})$
$D_{-+} = \sqrt{2}(-1)^F$	$2\chi_{\text{R}}(q)\chi_{\text{NS}}(\bar{q})$	$\mathcal{H}_{\text{R}} \otimes \overline{\mathcal{H}}_{\text{NS}}$	$\sigma(z),\ \mu\bar{\psi}(z,\bar{z})$	$\mu(z),\ \sigma\bar{\psi}(z,\bar{z})$
$D_{+-} = \sqrt{2}(-1)^{\bar{F}}$	$2\chi_{\text{NS}}(q)\chi_{\text{R}}(\bar{q})$	$\mathcal{H}_{\text{NS}} \otimes \overline{\mathcal{H}}_{\text{R}}$	$\bar{\sigma}(\bar{z}),\ \psi\bar{\mu}(z,\bar{z})$	$\bar{\mu}(\bar{z}),\ \psi\bar{\sigma}(z,\bar{z})$
$D_{--} = (-1)^{F+\bar{F}}$	$2\|\chi_{\text{R}}(q)\|^2$	$\mathcal{H}_{\text{R}}$	$\sigma(z,\bar{z})$	$\mu(z,\bar{z})$

Table 5.1: Disorder fields of free fermion topological defects.

In this picture of local free fermions with topological defects, the spin fields $\sigma(z,\bar{z})$ and $\mu(z,\bar{z})$ corresponding to the non-chiral Ramond ground states arise as the disorder fields of the topological defect D_{--} which gives the branch cut for all fermions $\psi(z)$ and $\bar{\psi}(\bar{z})$.

5.1.3 Conformal Boundaries in Free Fermion Theory

From the condition (2.209), if a boundary state $\|b\rangle\rangle$ preserves the free fermion algebra, it must satisfy

$$\left(\psi_n - i\epsilon\bar{\psi}_{-n}\right)\|b\rangle\rangle = 0 \ , \tag{5.19}$$

where $\epsilon = 1$ corresponds to untwisted boundary states and boundary states with $\epsilon = -1$ are twisted by the automorphism of the free fermion algebra $\bar{\psi}(\bar{z}) \mapsto -\bar{\psi}(\bar{z})$. Again, from (C.1), if a boundary state satisfies (5.19), then it satisfies the conformal boundary condition (2.208) as well.

In fermionic theories, sectors of boundary conditions are determined by whether the gluing conditions for the fermionic generators change at $z = 0$ on the upper half plane or not[109]. In this case, if a boundary along the real axis on the upper half plane satisfies

$$\psi(z) = \epsilon\bar{\psi}(\bar{z}) \quad \text{for} \quad z = \bar{z} \in \mathbb{R} \,, \tag{5.20}$$

then this boundary condition is in the Neveu–Schwarz sector. If the gluing condition for $z = \bar{z}$ is specified by

$$\psi(x) = \begin{cases} \epsilon\bar{\psi}(\bar{x}) & \text{for} \quad x < 0 \\ -\epsilon\bar{\psi}(\bar{x}) & \text{for} \quad x > 0 \end{cases} , \tag{5.21}$$

where $x \in \mathbb{R}$, then this boundary condition is in the Ramond sector. For Ramond boundary conditions, change in the gluing conditions imply existence of boundary fields carrying Ramond representations inserted at the origin.

Since Ishibashi states live in the completion of the bulk state space, we only consider boundary states in the Neveu–Schwarz sector. Nevertheless, we will see that the Ramond boundary states arise in the modular S transformed picture of the boundary theory.

The Ishibashi states in the NS sector can be written as[105]

$$|\text{NS}, \epsilon\rangle\rangle = \prod_{n=0}^{\infty} e^{i\epsilon\psi_{-n-\frac{1}{2}}\bar{\psi}_{-n-\frac{1}{2}}}|0\rangle \tag{5.22}$$

that are solutions of (5.19) with corresponding ϵ. Their overlaps can be calculated as

$$\langle\langle \text{NS}, \pm|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{1}{24}\right)}|\text{NS}, \pm\rangle\rangle = \chi_{\text{NS}}(\tilde{q}) = \chi_{\text{NS}}(q) \qquad \text{and} \tag{5.23}$$

$$\langle\langle \text{NS}, \pm|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{1}{24}\right)}|\text{NS}, \mp\rangle\rangle = \tilde{\chi}_{\text{NS}}(\tilde{q}) = \sqrt{2}\chi_{\text{R}}(q) \,. \tag{5.24}$$

We define the boundary state of the free fermion theory as

$$\|\text{free}\rangle\rangle = |\text{NS}, +\rangle\rangle \quad \text{and} \quad \|\text{fixed}\rangle\rangle = \sqrt{2}|\text{NS}, -\rangle\rangle \,. \tag{5.25}$$

These "free" and "fixed" boundary states are different from the usual Ising boundary states with the same names; we have defined the free and fixed boundary conditions for free fermions in the sense of [52] and [49] where the boundary conditions associated to the upper half plane gluing condition (5.20) with $\epsilon = 1$ is called "free" and $\epsilon = -1$ is called "fixed". The overlaps of these boundary states are given by

$$\begin{aligned} \langle\langle \text{free}\|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{1}{24}\right)}\|\text{free}\rangle\rangle &= \chi_{\text{NS}}(\tilde{q}) &&= \chi_{\text{NS}}(q) \,, \\ \langle\langle \text{free}\|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{1}{24}\right)}\|\text{fixed}\rangle\rangle &= \sqrt{2}\tilde{\chi}_{\text{NS}}(\tilde{q}) &&= 2\chi_{\text{R}}(q) \,, \quad \text{and} \\ \langle\langle \text{fixed}\|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{1}{24}\right)}\|\text{fixed}\rangle\rangle &= 2\chi_{\text{NS}}(\tilde{q}) &&= 2\chi_{\text{NS}}(q) \,. \end{aligned} \tag{5.26}$$

Recalling the Cardy constraint (2.232) and its geometric meaning illustrated in Figure 2.5, we can analyse the spectra of boundary fields from the overlaps (5.26).

5.1.4 Topological Defects in Ising Model

For $c = \frac{1}{2}$, there are three irreducible unitary representations of the Virasoro algebra. We denote their Kac labels as $(1,1) = (2,3) = \mathbf{1}$, $(1,2) = (2,2) = \sigma$, and $(1,3) = (2,1) = \varepsilon$. In addition, we introduce the indexing set, which is denoted by $\mathcal{I} = \{\mathbf{1}, \sigma, \varepsilon\}$. The conformal weights of the representations are $h_{\mathbf{1}} = 0$, $h_\sigma = \frac{1}{16}$, and $h_\varepsilon = \frac{1}{2}$. For the Ising model, the fusion rules (2.91) are given by

$$\sigma \otimes \sigma = \mathbf{1} \oplus \varepsilon \ , \quad \sigma \otimes \varepsilon = \sigma \ , \quad \varepsilon \otimes \varepsilon = \mathbf{1} \ . \tag{5.27}$$

In addition, the modular S-matrix of the Ising model is given by

$$\mathrm{S} = \begin{pmatrix} \mathrm{S}_{\mathbf{11}} & \mathrm{S}_{\mathbf{1}\varepsilon} & \mathrm{S}_{\mathbf{1}\sigma} \\ \mathrm{S}_{\varepsilon\mathbf{1}} & \mathrm{S}_{\varepsilon\varepsilon} & \mathrm{S}_{\varepsilon\sigma} \\ \mathrm{S}_{\sigma\mathbf{1}} & \mathrm{S}_{\sigma\varepsilon} & \mathrm{S}_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} . \tag{5.28}$$

For the Ising model, there are three elementary topological defects $D_{\mathbf{1}}$, D_σ, and D_ε. From (2.278) and using (A.18), the corresponding defect operators are given by

$$\begin{aligned} D_{\mathbf{1}} &= P_0\overline{P}_0 + P_{\frac{1}{2}}\overline{P}_{\frac{1}{2}} + P_{\frac{1}{16}}\overline{P}_{\frac{1}{16}} \ , \\ D_\varepsilon &= P_0\overline{P}_0 + P_{\frac{1}{2}}\overline{P}_{\frac{1}{2}} - P_{\frac{1}{16}}\overline{P}_{\frac{1}{16}} \ , \\ D_\sigma &= \sqrt{2}(P_0\overline{P}_0 - P_{\frac{1}{2}}\overline{P}_{\frac{1}{2}}) \ . \end{aligned} \tag{5.29}$$

From (2.270) and (2.279), the state spaces of defect fields living on these topological defects are

$$\begin{aligned} \mathcal{H}_{\mathbf{1}|\mathbf{1}} = \mathcal{H}_{\text{bulk}} &= (\mathbf{1},\mathbf{1}) \oplus (\sigma,\sigma) \oplus (\varepsilon,\varepsilon) \ , \\ \mathcal{H}_{\sigma|\sigma} &= (\mathbf{1},\mathbf{1}) \oplus 2(\sigma,\sigma) \oplus (\varepsilon,\varepsilon) \oplus (\mathbf{1},\varepsilon) \oplus (\varepsilon,\mathbf{1}) \ , \\ \mathcal{H}_{\varepsilon|\varepsilon} &= (\mathbf{1},\mathbf{1}) \oplus (\sigma,\sigma) \oplus (\varepsilon,\varepsilon) \ , \end{aligned} \tag{5.30}$$

where (i,j) is a short-hand for $\mathcal{H}_i \otimes \overline{\mathcal{H}}_j$ for $i,j \in \mathcal{I}$.

As it is clear from the fusion rules (5.27), D_ε is a group-like defect. From (5.29), we see that this defect flips the sign of the bulk field with label σ, which is often called the spin field. As a consequence, the defect fields on D_ε are again the bulk fields, which can be seen in (5.30). Also, recall that the partition function of defect fields on D_ε is the same as the torus partition function with two D_ε inserted with the opposite orientations as in Figure 2.6. In this case, two D_ε on the torus fuse to give the identity defect, therefore the partition function is the same as the bulk one.

Using (2.283), we can calculated the state spaces of disorder fields

$$\begin{aligned} \mathcal{H}_{\sigma|\mathbf{1}} &= (\sigma,\mathbf{1}) \oplus (\mathbf{1},\sigma) \oplus (\sigma,\varepsilon) \oplus (\varepsilon,\sigma) \ , \\ \mathcal{H}_{\varepsilon|\mathbf{1}} &= (\varepsilon,\mathbf{1}) \oplus (\mathbf{1},\varepsilon) \oplus (\sigma,\sigma) \ . \end{aligned} \tag{5.31}$$

The disorder field of D_ε with the representation (σ,σ) is not the same as the bulk spin field σ. This is what we usually call the disorder field $\mu(z,\bar{z})$ of the Ising model. In addition, the disorder fields of D_ε with the representations $(\varepsilon, \mathbf{1})$ and $(\mathbf{1}, \varepsilon)$ are related to the free fermions $\psi(z)$ and $\bar{\psi}(\bar{z})$.

Since D_ε and $D_\mathbf{1}$ are group-like defects, the fusion rules (5.27) indicate that D_σ is a duality defect. As discussed in [83], D_σ implements the order-disorder duality, which is know as the Kramers-Wannier duality, of the Ising model.

In addition, the state space of defect changing fields is

$$\mathcal{H}_{\sigma|\varepsilon} = (\sigma, \mathbf{1}) \oplus (\mathbf{1}, \sigma) \oplus (\sigma, \varepsilon) \oplus (\varepsilon, \sigma) \ . \tag{5.32}$$

This is the same as $\mathcal{H}_{\sigma|\mathbf{1}}$ since, by folding this defect about a defect changing field, the defect field becomes a disorder field at the end of the topological defect $D_\sigma D_\varepsilon = D_\sigma$. Using the same argument, we can understand $\mathcal{H}_{\sigma|\sigma} \cong \mathcal{H}_{\mathbf{1}|\mathbf{1}} \oplus \mathcal{H}_{\varepsilon|\mathbf{1}}$ as a result of $D_\sigma D_\sigma = D_\mathbf{1} + D_\varepsilon$.

5.1.5 Factorising Defects in Ising Model

Since a factorising defect acts as conformal boundaries to the both sides of the defect, the defect operator $F_{ab} : \mathcal{H}_{\text{bulk}} \to \mathcal{H}_{\text{bulk}}$ of a factorising defect with the left boundary condition a and the right boundary condition b can be written as

$$F_{ab} = \|a\rangle\!\rangle \langle\!\langle b\| \ , \tag{5.33}$$

where $\|a\rangle\!\rangle$ and $\|b\rangle\!\rangle$ are the corresponding boundary states. For the Ising model, there are three Cardy boundary states

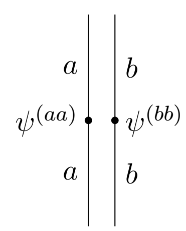
$$\begin{aligned} \|+\rangle\!\rangle &= \frac{1}{\sqrt{2}}|0\rangle\!\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\!\rangle + \frac{1}{\sqrt[4]{2}}|\tfrac{1}{16}\rangle\!\rangle \ , \\ \|-\rangle\!\rangle &= \frac{1}{\sqrt{2}}|0\rangle\!\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle\!\rangle - \frac{1}{\sqrt[4]{2}}|\tfrac{1}{16}\rangle\!\rangle \ , \\ \|f\rangle\!\rangle &= |0\rangle\!\rangle - |\tfrac{1}{2}\rangle\!\rangle \ . \end{aligned} \tag{5.34}$$

In this notation $+ = \mathbf{1}$, $- = \varepsilon$, and $f = \sigma$. The space of defect fields living on a factorising defect F_{ab} is

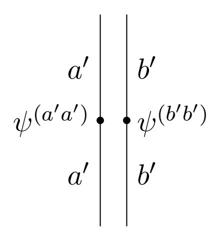
$$\mathcal{H}_{F_{ab}} = \mathcal{H}_{aa} \otimes \overline{\mathcal{H}}_{bb} \ , \tag{5.35}$$

where $\mathcal{H}_{aa}$ is the space of boundary fields $\psi_i^{(\alpha\alpha)}$ living on a boundary (2.201). Here, the bar indicates the boundary has the opposite orientation.

As we shall see later, we often need to consider the space of defect fields on $F_{ab} \cup F_{a'b'}$. In this case, we have the following boundary configurations:

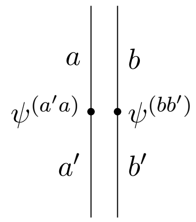


and

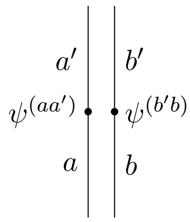


,

in addition to



and



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Therefore, the space of defect fields is

$$\mathcal{H}_{F_{ab}\cup F_{a'b'}} = \left(\mathcal{H}_{aa}\otimes\overline{\mathcal{H}}_{bb}\right)\oplus\left(\mathcal{H}_{a'a'}\otimes\overline{\mathcal{H}}_{b'b'}\right)\oplus 2\left(\mathcal{H}_{aa'}\otimes\overline{\mathcal{H}}_{bb'}\right)\ . \tag{5.36}$$

For the Ising model, we can calculate (2.204) and obtain

$$\begin{aligned} \mathcal{H}_{++} = \mathcal{H}_{--} &= (\mathbf{1})\ , \\ \mathcal{H}_{+f} = \mathcal{H}_{-f} &= (\sigma)\ , \\ \mathcal{H}_{+-} &= (\varepsilon)\ , \\ \mathcal{H}_{ff} &= (\mathbf{1})\oplus(\varepsilon)\ . \end{aligned}$$

For the purpose of the next section, we note the following state spaces of factorising defect fields:

$$\begin{aligned} \mathcal{H}_{F_{++}\cup F_{--}} &= 2(\mathbf{1},\mathbf{1})\oplus 2(\varepsilon,\varepsilon)\ , \\ \mathcal{H}_{F_{ff}} &= (\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\varepsilon)\oplus(\varepsilon,\mathbf{1})\oplus(\varepsilon,\varepsilon)\ , \\ \mathcal{H}_{F_{-+}\cup F_{+-}} &= 2(\mathbf{1},\mathbf{1})\oplus 2(\varepsilon,\varepsilon)\ , \\ \mathcal{H}_{F_{f+}\cup F_{f-}} &= 2(\mathbf{1},\mathbf{1})\oplus 2(\mathbf{1},\varepsilon)\oplus 2(\varepsilon,\mathbf{1})\oplus 2(\varepsilon,\varepsilon)\ , \\ \mathcal{H}_{F_{+f}\cup F_{-f}} &= 2(\mathbf{1},\mathbf{1})\oplus 2(\mathbf{1},\varepsilon)\oplus 2(\varepsilon,\mathbf{1})\oplus 2(\varepsilon,\varepsilon)\ . \end{aligned} \tag{5.37}$$

5.1.6 Interfaces Between Ising Model and Free Fermion Theory

If there exists a topological interface between the Ising model and the free fermion theory constructed in the previous sections, the corresponding operator I should be a map from the bulk state space $\mathcal{H}_{\text{f.f.}}$ of the free fermion theory to that of the Ising model $\mathcal{H}_{\text{Is.}}$. The operator $I^\dagger$ corresponds to the orientation reversed interface, and it is a map from $\mathcal{H}_{\text{Is.}}$ to $\mathcal{H}_{\text{f.f.}}$. In order for this interface to be topological, the operator I has to satisfy

$$I\,L_n^{\text{NS}} = L_n\,I \quad\text{and}\quad I\,\bar{L}_n^{\text{NS}} = \bar{L}_n\,I\ , \tag{5.38}$$

where L_n^{NS} is given by (C.1) and L_n acts on $\mathcal{H}_{\text{Is.}}$. Then, the operator I should consist of projectors on the Virasoro representations. In terms of the $c=\frac{1}{2}$ irreducible Virasoro representations $\mathcal{H}_h$ with conformal weights $h=0,\ \frac{1}{2},\ \frac{1}{16}$, the bulk state spaces can be expressed as

$$\mathcal{H}_{\text{Is.}} = (\mathcal{H}_0\otimes\overline{\mathcal{H}}_0)\oplus(\mathcal{H}_{\frac{1}{2}}\otimes\overline{\mathcal{H}}_{\frac{1}{2}})\oplus(\mathcal{H}_{\frac{1}{16}}\otimes\overline{\mathcal{H}}_{\frac{1}{16}}) \quad\text{and} \tag{5.39}$$

$$\mathcal{H}_{\text{f.f.}} = (\mathcal{H}_0\oplus\mathcal{H}_{\frac{1}{2}})\otimes(\overline{\mathcal{H}}_0\oplus\overline{\mathcal{H}}_{\frac{1}{2}})\ . \tag{5.40}$$

By writing the projector onto $\mathcal{H}_h \otimes \overline{\mathcal{H}}_{h'}$ as $P_h \overline{P}_{h'}$, the interface operator I should project $\mathcal{H}_{\text{f.f.}}$ onto the subspace which is common with $\mathcal{H}_{\text{Is.}}$

$$I_{\alpha,\beta} = \alpha\, P_0 \overline{P}_0 + \beta\, P_{\frac{1}{2}} \overline{P}_{\frac{1}{2}} \ . \tag{5.41}$$

We are going to determine the allowed coefficients α and β for the interfaces $I_{\alpha,\beta}$ by evaluating their fusion rules with topological defects, boundary states, and interfaces.

By considering the fusion of an interface $I_{\alpha,\beta}$ with a topological defect D_a, which is parallel to $I_{\alpha,\beta}$, in either theory, we demand the compositions of operators

$$D_a^{\text{Is.}}\, I_{\alpha,\beta} = \sum_{(\alpha',\beta')} \widetilde{N}^{(\alpha',\beta')}_{a\,(\alpha,\beta)}\, I_{\alpha',\beta'} \quad \text{and} \quad I_{\alpha,\beta}\, D_a^{\text{f.f.}} = \sum_{(\alpha',\beta')} N^{(\alpha',\beta')}_{(\alpha,\beta)\,a}\, I_{\alpha',\beta'} \tag{5.42}$$

have non-negative integer coefficients $\widetilde{N}^{(\alpha',\beta')}_{a\,(\alpha,\beta)}$ and $N^{(\alpha',\beta')}_{(\alpha,\beta)\,a}$. Conformal defects in the Ising model have been studied before[54, 59], and they are summarised in Appendix B. In particular, topological defect operators in the Ising model can be written as

$$\begin{aligned}
D_{\mathbf{1}} &= P_0 \overline{P}_0 + P_{\frac{1}{2}} \overline{P}_{\frac{1}{2}} + P_{\frac{1}{16}} \overline{P}_{\frac{1}{16}} \ , \\
D_{\varepsilon} &= P_0 \overline{P}_0 + P_{\frac{1}{2}} \overline{P}_{\frac{1}{2}} - P_{\frac{1}{16}} \overline{P}_{\frac{1}{16}} \ , \\
D_{\sigma} &= \sqrt{2}(P_0 \overline{P}_0 - P_{\frac{1}{2}} \overline{P}_{\frac{1}{2}}) \ .
\end{aligned} \tag{5.43}$$

Similarly, topological defect operators in the free fermion theory summarised in Table 5.1 can be expressed as

$$\begin{aligned}
D_{++} &= (P_0 + P_{\frac{1}{2}})(\overline{P}_0 + \overline{P}_{\frac{1}{2}}) \ , \\
D_{-+} &= \sqrt{2}(P_0 - P_{\frac{1}{2}})(\overline{P}_0 + \overline{P}_{\frac{1}{2}}) \ , \\
D_{+-} &= \sqrt{2}(P_0 + P_{\frac{1}{2}})(\overline{P}_0 - \overline{P}_{\frac{1}{2}}) \ , \\
D_{--} &= (P_0 - P_{\frac{1}{2}})(\overline{P}_0 - \overline{P}_{\frac{1}{2}}) \ .
\end{aligned} \tag{5.44}$$

Using these expressions, we can calculate

$$\begin{gathered}
D_{\mathbf{1}}\, I_{\alpha,\beta} = I_{\alpha,\beta} \ , \quad D_{\varepsilon}\, I_{\alpha,\beta} = I_{\alpha,\beta} \ , \quad D_{\sigma}\, I_{\alpha,\beta} = \sqrt{2} I_{\alpha,-\beta} \ , \\
I_{\alpha,\beta}\, D_{++} = I_{\alpha,\beta} \ , \quad I_{\alpha,\beta}\, D_{-+} = \sqrt{2} I_{\alpha,-\beta} \ , \quad I_{\alpha,\beta}\, D_{+-} = \sqrt{2} I_{\alpha,-\beta} \ , \quad I_{\alpha,\beta}\, D_{--} = I_{\alpha,\beta} \ .
\end{gathered} \tag{5.45}$$

Next, we consider the action of topological interfaces on the boundary states. We demand interfaces to map boundary states of one theory to those of the other theory as

$$I_{\alpha,\beta} \| a \rangle\!\rangle_{\text{f.f.}} = \sum_{b \in \mathcal{B}_{\text{Is.}}} N^{b}_{(\alpha,\beta)\,a} \| b \rangle\!\rangle_{\text{Is.}} \quad \text{and} \quad I^{\dagger}_{\alpha,\beta} \| a \rangle\!\rangle_{\text{Is.}} = \sum_{b \in \mathcal{B}_{\text{f.f.}}} \widetilde{N}^{b}_{(\alpha,\beta)\,a} \| b \rangle\!\rangle_{\text{f.f.}} \tag{5.46}$$

with non-negative integer coefficients $N^{b}_{(\alpha,\beta)\,a}$ and $\widetilde{N}^{b}_{(\alpha,\beta)\,a}$. Using the Ising boundary states (5.34) and

$$|\text{NS}, \pm\rangle\!\rangle = |0\rangle\!\rangle \pm |\tfrac{1}{2}\rangle\!\rangle \ , \tag{5.47}$$

we obtain the action of interfaces on the free fermion boundary states as

$$I_{\alpha,\beta}\|\text{free}\rangle\!\rangle = \frac{\alpha+\beta}{2\sqrt{2}}(\|+\rangle\!\rangle + \|-\rangle\!\rangle) + \frac{\alpha-\beta}{2}\|f\rangle\!\rangle \quad \text{and}$$
$$I_{\alpha,\beta}\|\text{fixed}\rangle\!\rangle = \frac{\alpha-\beta}{2}(\|+\rangle\!\rangle + \|-\rangle\!\rangle) + \frac{\alpha+\beta}{\sqrt{2}}\|f\rangle\!\rangle \; . \tag{5.48}$$

The action of orientation reversed interfaces on the Ising boundary states are

$$I^\dagger_{\alpha,\beta}\|+\rangle\!\rangle = \frac{\alpha+\beta}{2\sqrt{2}}\|\text{free}\rangle\!\rangle + \frac{\alpha-\beta}{4}\|\text{fixed}\rangle\!\rangle \; ,$$
$$I^\dagger_{\alpha,\beta}\|-\rangle\!\rangle = \frac{\alpha+\beta}{2\sqrt{2}}\|\text{free}\rangle\!\rangle + \frac{\alpha-\beta}{4}\|\text{fixed}\rangle\!\rangle \; ,$$
$$I^\dagger_{\alpha,\beta}\|f\rangle\!\rangle = \frac{\alpha-\beta}{2}\|\text{free}\rangle\!\rangle + \frac{\alpha+\beta}{2\sqrt{2}}\|\text{fixed}\rangle\!\rangle \; . \tag{5.49}$$

From these, we obtain the condition $\alpha = \sqrt{2}m + 2n$ and $\beta = \sqrt{2}m - 2n$ for $m, n \in \mathbb{Z}$.

Finally, we consider fusions of interfaces with orientation reversed interfaces. We require the compositions of interface operators

$$I_{\alpha,\beta}\, I^\dagger_{\alpha',\beta'} = \sum_{a\in\mathcal{T}_{\text{Is.}}} \widetilde{N}^a_{(\alpha,\beta)(\alpha',\beta')}\, D^{\text{Is.}}_a \quad \text{and} \quad I^\dagger_{\alpha,\beta}\, I_{\alpha',\beta'} = \sum_{a\in\mathcal{T}_{\text{f.f.}}} N^a_{(\alpha,\beta)(\alpha',\beta')}\, D^{\text{f.f.}}_a \tag{5.50}$$

to have non-negative integer coefficients $\widetilde{N}^a_{(\alpha,\beta)(\alpha',\beta')}$ and $N^a_{(\alpha,\beta)(\alpha',\beta')}$. Using (5.43) and (5.44), we can calculate

$$I_{\alpha,\beta}\, I^\dagger_{\alpha',\beta'} = \frac{\alpha\alpha'+\beta\beta'}{4}(D_{\mathbf{1}} + D_\varepsilon) + \frac{\alpha\alpha'-\beta\beta'}{2\sqrt{2}}D_\sigma \quad \text{and}$$
$$I^\dagger_{\alpha,\beta}\, I_{\alpha',\beta'} = \frac{\alpha\alpha'+\beta\beta'}{4}(D_{++} + D_{--}) + \frac{\alpha\alpha'-\beta\beta'}{4\sqrt{2}}(D_{-+} + D_{+-}) \; . \tag{5.51}$$

These observations above suggest that there are two elementary interfaces I and I' given by

$$I := I_{\sqrt{2},\sqrt{2}} = \sqrt{2}(P_0\overline{P}_0 + P_{\frac{1}{2}}\overline{P}_{\frac{1}{2}}) \quad \text{and}$$
$$I' := I_{2,-2} = 2(P_0\overline{P}_0 - P_{\frac{1}{2}}\overline{P}_{\frac{1}{2}}) \; . \tag{5.52}$$

Then, the non-trivial fusions of interfaces and topological defects in (5.45) become

$$D_\sigma\, I = I' \; , \quad I\, D_{+-} = I\, D_{-+} = I' \; ,$$
$$D_\sigma\, I' = 2I \; , \quad I'\, D_{+-} = I'\, D_{-+} = 2I \; . \tag{5.53}$$

The action of interfaces on the free fermion boundary states (5.48) and on the Ising boundary states (5.49) becomes

$$I\|\text{free}\rangle\!\rangle = \|+\rangle\!\rangle + \|-\rangle\!\rangle \; , \qquad I^\dagger\|+\rangle\!\rangle = I^\dagger\|-\rangle\!\rangle = \|\text{free}\rangle\!\rangle \; ,$$
$$I\|\text{fixed}\rangle\!\rangle = 2\|f\rangle\!\rangle \; , \qquad I^\dagger\|f\rangle\!\rangle = \|\text{fixed}\rangle\!\rangle \; ,$$
$$I'\|\text{free}\rangle\!\rangle = 2\|f\rangle\!\rangle \; , \qquad I'^\dagger\|+\rangle\!\rangle = I'^\dagger\|-\rangle\!\rangle = \|\text{fixed}\rangle\!\rangle \; ,$$
$$I'\|\text{fixed}\rangle\!\rangle = 2(\|+\rangle\!\rangle + \|-\rangle\!\rangle) \; , \qquad I'^\dagger\|f\rangle\!\rangle = 2\|\text{free}\rangle\!\rangle \; . \tag{5.54}$$

Finally, the fusions of interfaces with orientation reversed ones (5.51) become

$$I\, I^\dagger = D_{\mathbf{1}} + D_\varepsilon \; , \qquad I\, I'^\dagger = 2D_\sigma \; ,$$
$$I^\dagger\, I = D_{++} + D_{--} \; , \quad I'^\dagger\, I = D_{-+} + D_{+-} \; . \tag{5.55}$$

5.1.7 Consistency of Interfaces

Since they are topological, we can consider sweeping an interface across bulk fields in one theory to obtain corresponding disorder fields in the other theory as in Figure 2.10. The action of topological interfaces does not change conformal weights of fields, and therefore we can identify the resulting disorder fields by their conformal weights in this case.

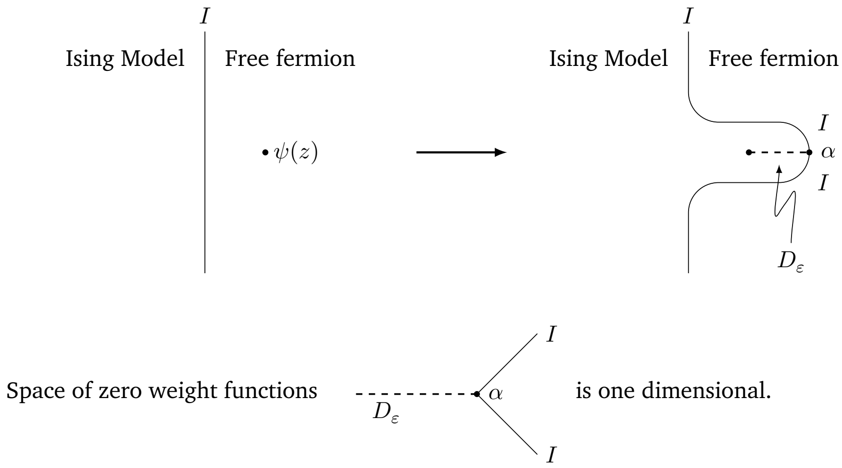

Figure 5.1: Moving the interface I across the field $\psi(z)$ in the free fermion theory, and the junction between I and D_ε.

Let us consider sweeping the interface I across the bulk fields of the free fermion theory. From (5.55), the resulting fields must be either bulk fields of the Ising model or disorder fields of the topological defect D_ε. We also need to check the existence of a weight zero field at the three-legged junction between the interface and a topological defect as depicted in Figure 5.1. Denoting the space of fields at the junction between the interface I and a topological defect D_i as $\mathcal{H}_{II^\dagger D_i}$, we can calculate

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{II^\dagger D_\varepsilon}}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) &= \mathrm{Tr}_{\mathcal{H}_{\text{Is.}}}\left(II^\dagger D_\varepsilon\, \tilde{q}^{L_0-\frac{c}{24}}\tilde{\bar{q}}^{\bar{L}_0-\frac{c}{24}}\right) \\ &= \mathrm{Tr}_{\mathcal{H}_{\text{Is.}}}\left((D_{\mathbf{1}}+D_\varepsilon)\,\tilde{q}^{L_0-\frac{c}{24}}\tilde{\bar{q}}^{\bar{L}_0-\frac{c}{24}}\right) \\ &= 2|\chi_0(\tilde{q})|^2+2|\chi_{\frac{1}{2}}(\tilde{q})|^2 = |\chi_0(q)+\chi_{\frac{1}{2}}(q)|^2+2|\chi_{\frac{1}{16}}(q)|^2\,,\end{aligned} \tag{5.56}$$

where $\chi_h(q)$ is the character of the Virasoro representation with a highest weight h and $c=\frac{1}{2}$. Therefore, we see that there is a one-dimensional space of weight zero field at the three-legged junction between I and D_ε. Since $II^\dagger D_\varepsilon = II^\dagger = II^\dagger D_{\mathbf{1}}$, the same is true for the three-legged junction between I and $D_{\mathbf{1}}$. As there are no chiral bulk fields in the Ising model, $\psi(z)$ and $\bar{\psi}(\bar{z})$ become disorder fields of D_ε while $i\psi\bar{\psi}(z,\bar{z})$ corresponds to the bulk field $\varepsilon(z,\bar{z})$. In addition, we assume the weight zero field α at the three-legged junction between I and D_ε only couples to fermions. In terms of Figure 2.10, this means that the linear map $D_{\varepsilon I\alpha}$ is zero for bosons in the free fermion theory.

We can also consider sweeping an interface across disorder fields attached to topological defects. For example, we can fuse the bottom half of the interface I with the topological defect D_{--}. From the fusion rules (5.45), a disorder field of D_{--} becomes a defect field on I. Then, by moving the interface, these fields become disorder fields of $D_{\mathbf{1}}$ or D_{ε} in the Ising model. As in Table 5.1, the disorder fields $\sigma(z,\bar{z})$ and $\mu(z,\bar{z})$ of D_{--} have the same conformal weights but opposite fermion parities. As we have assumed the junction field α between I and D_{ε} only couples to fermions, $\sigma(z,\bar{z})$ becomes a bulk field of the Ising model and $\mu(z,\bar{z})$ becomes a disorder field of D_{ε}.

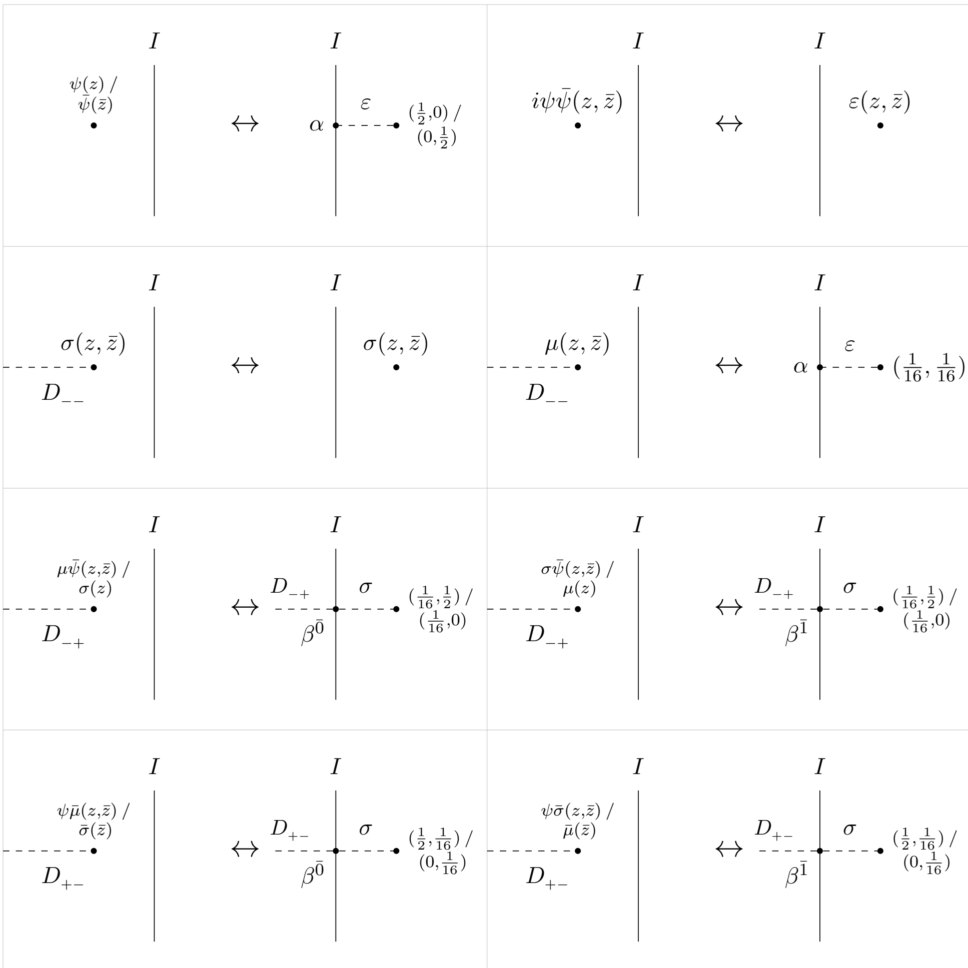

Figure 5.2: Correspondence between bulk and disorder fields of the free fermion theory and the Ising model.

Since $ID_{-+} = I'$ and $II'^{\dagger} = 2D_{\sigma}$, if we move the interface I across a disorder field at the end of D_{-+} defect, it will become a disorder field of D_{σ} in two possible ways. We can check this by calculating the number of weight zero fields at the four-legged junction between I, D_{-+} and D_{σ}. Denoting the space of fields at this junction by $\mathcal{H}_{ID_{-+}I^{\dagger}D_{\sigma}}$, we can calculate

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{ID_{-+}I^\dagger D_\sigma}}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) &= \mathrm{Tr}_{\mathcal{H}_{\mathrm{Is.}}}\left(ID_{-+}I^\dagger D_\sigma\, \tilde{q}^{L_0-\frac{c}{24}}\tilde{\bar{q}}^{\bar{L}_0-\frac{c}{24}}\right)\\ &= \mathrm{Tr}_{\mathcal{H}_{\mathrm{Is.}}}\left(2(D_{\mathbf{1}}+D_\varepsilon)\, \tilde{q}^{L_0-\frac{c}{24}}\tilde{\bar{q}}^{\bar{L}_0-\frac{c}{24}}\right)\\ &= 4|\chi_0(\tilde{q})|^2 + 4|\chi_{\frac{1}{2}}(\tilde{q})|^2 = 2|\chi_0(q)+\chi_{\frac{1}{2}}(q)|^2 + 4|\chi_{\frac{1}{16}}(q)|^2\ ,\end{aligned} \tag{5.57}$$

and therefore there is a two-dimensional space of weight zero fields at the junction. We label these two weight zero fields by $\beta^{\bar{0}}$ and $\beta^{\bar{1}}$ and assume the former only couples to bosons and the latter only couples to fermions in the free fermion theory. As in Table 5.1, disorder fields of D_{-+} come in pairs in which the fields have the same conformal weights but with opposite fermion parities; by specifying $\beta^{\bar{0}}$ and $\beta^{\bar{1}}$ in this way, we can have unique maps from the space of D_{-+} disorder fields to that of D_σ disorder fields. By looking at (5.31), one can see that the half of D_σ disorder fields correspond to those of D_{-+}. Using the same argument, we find the other half of D_σ disorder fields come from D_{+-} disorder fields.

We have obtained the linear maps from the spaces of bulk fields and disorder fields of the free fermion theory to those of the Ising model. These results are summarised in Figure 5.2. As we can invert these linear maps, we see that the bulk and disorder fields of the two theories have unique correspondence. Together with the fusion rules (5.53), (5.54), and (5.55), we conclude the topological interfaces (5.52) are consistent.

5.2 $N = 1$ Superconformal Field Theory

In $N = 1$ superconformal field theories (SCFTs), we often employ the superspace formalism in which the theories are defined on super-Riemann surfaces. Coordinates on super-Riemann surfaces are denoted by $Z = (z, \theta)$ where θ is a Grassmann variable which satisfies $\theta_1\theta_2 = -\theta_2\theta_1$ and $\theta^2 = 0$. In the superspace formalism, bulk fields of an SCFT are given by Neveu–Schwarz superfields[15]

$$\Phi_I(Z,\overline{Z}) = \varphi_I(z,\bar{z}) + \theta\psi_I(z,\bar{z}) + \bar{\theta}\bar{\psi}_I(z,\bar{z}) + \theta\bar{\theta}\tilde{\varphi}_I(z,\bar{z})\ . \tag{5.58}$$

The component $\varphi_I(z,\bar{z})$ of a superfield is called a superprimary field if the corresponding state

$$|\varphi_I\rangle = \lim_{z,\bar{z}\to 0}\varphi_I(z,\bar{z})|0\rangle \tag{5.59}$$

is a highest weight state for holomorphic and antiholomorphic copies of the $N = 1$ super-Virasoro algebra, that is

$$G_n|\varphi_I\rangle = 0\ ,\quad L_n|\varphi_I\rangle = 0\ ,\quad \bar{G}_n|\varphi_I\rangle = 0\ ,\quad \bar{L}_n|\varphi_I\rangle = 0 \quad \text{for} \quad n > 0\ , \tag{5.60}$$

and

$$L_0|\varphi_I\rangle = h_I|\varphi_I\rangle \quad \text{and} \quad \bar{L}_0|\varphi_I\rangle = \bar{h}_I|\varphi_I\rangle\ . \tag{5.61}$$

Then, the other components are obtained as the superdescendant states

$$|\psi_I\rangle = G_{-\frac{1}{2}}|\varphi_I\rangle\ , \quad |\bar{\psi}_I\rangle = \bar{G}_{-\frac{1}{2}}|\varphi_I\rangle\ , \quad \text{and} \quad |\tilde{\varphi}_I\rangle = G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|\varphi_I\rangle\ . \tag{5.62}$$

Using the $\mathcal{SV}\mathrm{ir}$ relations (2.117), one can show that these superdescendant fields are Virasoro primary fields.

5.2.1 Bosonic Tri-Critical Ising Model

The unitary Virasoro minimal model $M(4,5)$ has $c=\frac{7}{10}$, and there are six representations that are summarised in Table 5.2. Our choice of labels comes from the fusion rules of $M(4,5)$, which can be considered as (Lee-Yang) $\times$ (Ising). The fields labelled by $\mathbf{1}$, ε, and σ obey the Ising fusion rules

$$\varepsilon\otimes\varepsilon=\mathbf{1}\ , \quad \varepsilon\otimes\sigma=\sigma\ , \quad \text{and} \quad \sigma\otimes\sigma=\mathbf{1}\oplus\varepsilon\ , \tag{5.63}$$

and $\mathbf{1}$ and $\hat{\mathbf{1}}$ satisfy the Lee-Yang fusion rule

$$\hat{\mathbf{1}}\otimes\hat{\mathbf{1}}=\mathbf{1}\oplus\hat{\mathbf{1}}\ . \tag{5.64}$$

In general, we can write

$$x\otimes\hat{y}=(\widehat{x\otimes y}) \quad \text{and} \quad \hat{x}\otimes\hat{y}=(x\otimes y)\oplus(\widehat{x\otimes y})\ , \tag{5.65}$$

where $x,\ y\in\{\mathbf{1},\ \varepsilon,\ \sigma\}$.

Label i	$\mathbf{1}$	ε	σ	$\hat{\mathbf{1}}$	$\hat{\varepsilon}$	$\hat{\sigma}$
Kac label (r,s)	$(1,1)$ $=(3,4)$	$(3,1)$ $=(1,4)$	$(2,1)$ $=(2,4)$	$(1,3)$ $=(3,2)$	$(1,2)$ $=(3,3)$	$(2,2)$ $=(2,3)$
Weight h_i	0	$\frac{3}{2}$	$\frac{7}{16}$	$\frac{3}{5}$	$\frac{1}{10}$	$\frac{3}{80}$
Quantum dimension $\mathcal{D}_i$	1	1	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{\sqrt{2}}$
Boundary entropy g_i^0	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{5}\right)^{\frac{1}{4}}$

Table 5.2: Virasoro representations of $M(4,5)$.

Since there is only one modular invariant bulk partition function at $c=\frac{7}{10}$, the theory defined by the diagonal modular invariant for the Virasoro algebra is also denoted by $M(4,5)$. As a diagonal theory, $M(4,5)$ has six elementary boundary conditions and six elementary topological defects that are labelled by the Virasoro representations. Their explicit expressions are summarised in Table 5.3 and 5.4. One of the important quantities characterising conformal boundaries and topological defects are their entropies. They are defined as the coefficients of the vacuum Ishibashi state or those of the vacuum projector. The entropy of the conformal defect labelled by a representation i is the same as the quantum dimension (2.178) of i in the diagonal theories. They are also summarised in Table 5.2.

In terms of the $\mathcal{V}\mathrm{ir}$ representation labels at $c=\frac{7}{10}$

$$\mathcal{I}=\{\mathbf{1},\ \varepsilon,\ \sigma,\ \hat{\mathbf{1}},\ \hat{\varepsilon},\ \hat{\sigma}\}\ , \tag{5.66}$$

the modular S matrix of $M(4,5)$ is given by

$$\mathrm{S} = \frac{1}{\sqrt{5}} \begin{pmatrix} s_2 & s_2 & \sqrt{2}s_2 & s_1 & s_1 & \sqrt{2}s_1 \\ s_2 & s_2 & -\sqrt{2}s_2 & s_1 & s_1 & -\sqrt{2}s_1 \\ \sqrt{2}s_2 & -\sqrt{2}s_2 & 0 & \sqrt{2}s_1 & -\sqrt{2}s_1 & 0 \\ s_1 & s_1 & \sqrt{2}s_1 & -s_2 & -s_2 & -\sqrt{2}s_2 \\ s_1 & s_1 & -\sqrt{2}s_1 & -s_2 & -s_2 & \sqrt{2}s_2 \\ \sqrt{2}s_1 & -\sqrt{2}s_1 & 0 & -\sqrt{2}s_2 & \sqrt{2}s_2 & 0 \end{pmatrix} \tag{5.67}$$

where $s_1 = \sin(\frac{2\pi}{5})$ and $s_2 = \sin(\frac{4\pi}{5})$. Therefore the Cardy boundary states of $M(4,5)$ are given by

$$\|a\rangle\!\rangle = \sum_{i \in \mathcal{I}} g_a^i \, |i\rangle\!\rangle \ , \quad \text{where} \quad g_a^i = \frac{\mathrm{S}_{ai}}{\sqrt{\mathrm{S}_{0i}}} \ . \tag{5.68}$$

These boundary state coefficients are given in Table 5.3.

$\|a\rangle\!\rangle \setminus \|h_i\rangle\!\rangle$	$\|0\rangle\!\rangle$	$\|\frac{3}{2}\rangle\!\rangle$	$\|\frac{7}{16}\rangle\!\rangle$	$\|\frac{3}{5}\rangle\!\rangle$	$\|\frac{1}{10}\rangle\!\rangle$	$\|\frac{3}{80}\rangle\!\rangle$
$\|\mathbf{1}\rangle\!\rangle$	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5+\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5+\sqrt{5}}{20}\right)^{\frac{1}{4}}$
$\|\varepsilon\rangle\!\rangle$	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$-\left(\frac{5-\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5+\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$-\left(\frac{5+\sqrt{5}}{20}\right)^{\frac{1}{4}}$
$\|\sigma\rangle\!\rangle$	$\left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}}$	$-\left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}}$	0	$\left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{4}}$	$-\left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{4}}$	0
$\|\hat{\mathbf{1}}\rangle\!\rangle$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{10}\right)^{\frac{1}{4}}$	$-\left(\frac{5-2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$-\left(\frac{5-2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$-\left(\frac{5-2\sqrt{5}}{10}\right)^{\frac{1}{4}}$
$\|\hat{\varepsilon}\rangle\!\rangle$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$-\left(\frac{5+2\sqrt{5}}{10}\right)^{\frac{1}{4}}$	$-\left(\frac{5-2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$-\left(\frac{5-2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5-2\sqrt{5}}{10}\right)^{\frac{1}{4}}$
$\|\hat{\sigma}\rangle\!\rangle$	$\left(\frac{5+2\sqrt{5}}{5}\right)^{\frac{1}{4}}$	$-\left(\frac{5+2\sqrt{5}}{5}\right)^{\frac{1}{4}}$	0	$-\left(\frac{5-2\sqrt{5}}{5}\right)^{\frac{1}{4}}$	$\left(\frac{5-2\sqrt{5}}{5}\right)^{\frac{1}{4}}$	0

Table 5.3: Boundary state coefficients g_a^i for $M(4,5)$.

The elementary topological defect operators of $M(4,5)$ are given by

$$D_a = \sum_{i \in \mathcal{I}} g_a^{i,i} \, P_i \overline{P}_i \ , \quad \text{where} \quad g_a^{i,i} = \frac{\mathrm{S}_{ai}}{\mathrm{S}_{0i}} \ . \tag{5.69}$$

These topological defect operator coefficients are summarised in Table 5.4.

$D_a \setminus P_{h_i}\overline{P}_{h_i}$	$P_0\overline{P}_0$	$P_{\frac{3}{2}}\overline{P}_{\frac{3}{2}}$	$P_{\frac{7}{16}}\overline{P}_{\frac{7}{16}}$	$P_{\frac{3}{5}}\overline{P}_{\frac{3}{5}}$	$P_{\frac{1}{10}}\overline{P}_{\frac{1}{10}}$	$P_{\frac{3}{80}}\overline{P}_{\frac{3}{80}}$
$D_{\mathbf{1}}$	1	1	1	1	1	1
D_ε	1	1	-1	1	1	-1
D_σ	$\sqrt{2}$	$-\sqrt{2}$	0	$\sqrt{2}$	$-\sqrt{2}$	0
$D_{\hat{\mathbf{1}}}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$D_{\hat{\varepsilon}}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$-\frac{1-\sqrt{5}}{2}$
$D_{\hat{\sigma}}$	$\frac{1+\sqrt{5}}{\sqrt{2}}$	$-\frac{1+\sqrt{5}}{\sqrt{2}}$	0	$\frac{1-\sqrt{5}}{\sqrt{2}}$	$-\frac{1-\sqrt{5}}{\sqrt{2}}$	0

Table 5.4: Topological defect operator coefficients $g_a^{i,i}$ for $M(4,5)$.

5.2.2 Supersymmetric Tri-Critical Ising Model

In $M(4,5)$, the Virasoro representation labelled by ε is a simple current with the conformal weight $\frac{3}{2}$. The unitary super-Virasoro minimal model $SM(3,5)$, which also has $c = \frac{7}{10}$, can be regarded as the fermionic extension of $M(4, 5)$ by the simple current ε. In $SM(3, 5)$, there are four representations of the super-Virasoro algebra: two of which are in the Neveu–Schwarz sector and the other two are in the Ramond sector. They are summarised in Table 5.5. The fusion rules of $SM(3, 5)$ are given by[8, 6, 16, 24, 47, 50, 57]

$$\mathbf{1}_{\text{A}} \otimes \mathbf{1}_{\text{B}} = a\, \mathbf{1}_{\text{C}} \quad \text{and} \quad \varphi_{\text{A}} \otimes \varphi_{\text{B}} = a(\mathbf{1}_{\text{C}} \oplus \varphi_{\text{C}}) \,, \tag{5.70}$$

where the subscripts and coefficients obey

A	B	C	a
NS	NS	NS	1
NS	R	R	1
R	NS	R	1
R	R	NS	2

(5.71)

In the fusion rules of two Ramond representations, NS representations appear twice due to the fact that the even and odd fusion rules are the same for these cases[24].

Label i	$\mathbf{1}_{\text{NS}}$	φ_{R}	φ_{NS}	$\mathbf{1}_{\text{R}}$
Kac label (r, s)	$(1,1)$ $= (2,4)$	$(1,2)$ $= (2,3)$	$(1,3)$ $= (2,2)$	$(1,4)$ $= (2,1)$
Weight h_i	0	$\frac{3}{80}$	$\frac{1}{10}$	$\frac{7}{16}$

Table 5.5: Super-Virasoro representations of $SM(4, 5)$.

As in the free fermion theory, we take the bulk sector to be the theory of local NS superfields. Therefore, the bulk state space is given by

$$\begin{aligned} \mathcal{H}_{\text{NS}} &= (\mathcal{H}_0^{\text{NS}} \otimes \overline{\mathcal{H}}_0^{\text{NS}}) \oplus (\mathcal{H}_{\frac{1}{10}}^{\text{NS}} \otimes \overline{\mathcal{H}}_{\frac{1}{10}}^{\text{NS}}) \\ &= (\mathcal{H}_0 \oplus \mathcal{H}_{\frac{3}{2}}) \otimes (\overline{\mathcal{H}}_0 \oplus \overline{\mathcal{H}}_{\frac{3}{2}}) \oplus (\mathcal{H}_{\frac{1}{10}} \oplus \mathcal{H}_{\frac{3}{5}}) \otimes (\overline{\mathcal{H}}_{\frac{1}{10}} \oplus \overline{\mathcal{H}}_{\frac{3}{5}}) \,, \end{aligned} \tag{5.72}$$

where $\mathcal{H}_h^{\text{NS}}$ and $\mathcal{H}_h$ are the super-Virasoro and Virasoro representations with $c = \frac{7}{10}$ the conformal weight h respectively. We take the highest weight states of $\mathcal{H}_{\frac{1}{10}}^{\text{NS}}$ and $\overline{\mathcal{H}}_{\frac{1}{10}}^{\text{NS}}$ to be fermionic, which means

$$(-1)^F |\tfrac{1}{10}\rangle = -|\tfrac{1}{10}\rangle \quad \text{and} \quad (-1)^{\bar{F}} |\overline{\tfrac{1}{10}}\rangle = -|\overline{\tfrac{1}{10}}\rangle \tag{5.73}$$

for the highest weight states $|\frac{1}{10}\rangle \in \mathcal{H}_{\frac{1}{10}}^{\text{NS}}$ and $|\overline{\frac{1}{10}}\rangle \in \overline{\mathcal{H}}_{\frac{1}{10}}^{\text{NS}}$. The corresponding bulk superprimary field is still bosonic as it commutes with $(-1)^{F+\bar{F}}$. The bulk partition function can be expressed in terms of the characters $\chi_h^{\text{NS}}(q)$ of $\mathcal{SV}\text{ir}$, which is given by (A.9), and $\chi_h(q)$ of $\mathcal{V}\text{ir}$ as

$$\begin{aligned} Z_{\text{NS}} = \text{Tr}_{\mathcal{H}_{\text{NS}}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right) &= |\chi_0^{\text{NS}}(q)|^2 + |\chi_{\frac{1}{10}}^{\text{NS}}(q)|^2 \\ &= |\chi_0(q) + \chi_{\frac{3}{2}}(q)|^2 + |\chi_{\frac{1}{10}}(q) + \chi_{\frac{3}{5}}(q)|^2 \,. \end{aligned} \tag{5.74}$$

This partition function is invariant under modular S and T^2 transformations but the T transformation changes $|\chi_h^{\text{NS}}(q)|^2$ to $|\tilde{\chi}_h^{\text{NS}}(q)|^2$.

The theory defined by the diagonal bulk partition function (5.74) is also denoted by $SM(3, 5)$. As we saw in the free fermion case, the Ramond fields arise as disorder fields of topological defects.

5.2.3 Topological Defects in $SM(3,5)$

We call a topological defect superconformal if it also preserves the $N = 1$ super-Virasoro algebra up to automorphisms. Therefore, the operator D of a topological defect in supersymmetric theories satisfies

$$G_n\, D_{\epsilon,\bar\epsilon} = \epsilon\, D_{\epsilon,\bar\epsilon}\, G_n \quad \text{and} \quad \bar G_n\, D_{\epsilon,\bar\epsilon} = \bar\epsilon\, D_{\epsilon,\bar\epsilon}\, \bar G_n \ , \tag{5.75}$$

where $\epsilon = \pm 1$ and $\bar\epsilon = \pm 1$. Using the anticommutation relation in (2.117), one can show that the condition above implies the Virasoro condition (2.261) as well. Recalling (5.73), a topological defect operator satisfying (5.75) can be written as

$$D_{\epsilon,\bar\epsilon} = a(P_0 + \epsilon\, P_{\frac{3}{2}})(\overline{P}_0 + \bar\epsilon\, \overline{P}_{\frac{3}{2}}) + b(\epsilon\, P_{\frac{1}{10}} + P_{\frac{3}{5}})(\bar\epsilon\, \overline{P}_{\frac{1}{10}} + \overline{P}_{\frac{3}{5}}) \ , \tag{5.76}$$

where a and b are constants, and $P_h \overline{P}_{h'}$ is the projector onto the Virasoro representations $\mathcal{H}_h \otimes \overline{\mathcal{H}}_{h'}$.

The identity defect operator has $a,\ b = 1$ and $\epsilon,\ \bar\epsilon = 1$ in (5.76). Using the same argument as the free fermion case, the other topological defect operators with the same a and b that are related by the automorphisms of $\mathcal{SV}\mathrm{ir}$ can be obtained by composing it with $\sqrt{2}(-1)^F$, $\sqrt{2}(-1)^{\bar F}$, and $(-1)^{F+\bar F}$. Since the NS sector is closed under fusion and the bulk partition function is diagonal, we can use (2.278) to obtain the other solution for the coefficients a and b. Then, the topological defect operators with $\epsilon = 1$ and $\bar\epsilon = 1$ are given by

$$D_{\mathbf{1}} = (P_0 + P_{\frac{3}{2}})(\overline{P}_0 + \overline{P}_{\frac{3}{2}}) + (P_{\frac{1}{10}} + P_{\frac{3}{5}})(\overline{P}_{\frac{1}{10}} + \overline{P}_{\frac{3}{5}}) \qquad \text{and} \tag{5.77}$$

$$\begin{aligned} D_\varphi &= \frac{\mathrm{S}^{[\mathrm{NS,NS}]}_{\varphi\mathbf{1}}}{\mathrm{S}^{[\mathrm{NS,NS}]}_{\mathbf{11}}} P_0^{\mathrm{NS}} \overline{P}_0^{\mathrm{NS}} + \frac{\mathrm{S}^{[\mathrm{NS,NS}]}_{\varphi\varphi}}{\mathrm{S}^{[\mathrm{NS,NS}]}_{\mathbf{1}\varphi}} P_{\frac{1}{10}}^{\mathrm{NS}} \overline{P}_{\frac{1}{10}}^{\mathrm{NS}} \\ &= \frac{1+\sqrt{5}}{2}(P_0 + P_{\frac{3}{2}})(\overline{P}_0 + \overline{P}_{\frac{3}{2}}) + \frac{1-\sqrt{5}}{2}(P_{\frac{1}{10}} + P_{\frac{3}{5}})(\overline{P}_{\frac{1}{10}} + \overline{P}_{\frac{3}{5}}) \ , \end{aligned} \tag{5.78}$$

where $\mathrm{S}^{[\mathrm{NS,NS}]}_{ij}$ is an element of the $\mathcal{SV}\mathrm{ir}$ modular S matrix given by (A.42), and $P_h^{\mathrm{NS}} \overline{P}_{h'}^{\mathrm{NS}}$ is the projector onto the super-Virasoro representations $\mathcal{H}_h^{\mathrm{NS}} \otimes \overline{\mathcal{H}}_{h'}^{\mathrm{NS}}$. In this way, we find a complete set of eight elementary topological defects

$$\begin{aligned} \mathcal{T} = \{D_{\mathbf{1}},\ D_\varphi,\ \sqrt{2}(-1)^F D_{\mathbf{1}},\ \sqrt{2}(-1)^F D_\varphi,\ \sqrt{2}(-1)^{\bar F} D_{\mathbf{1}}, \\ \sqrt{2}(-1)^{\bar F} D_\varphi,\ (-1)^{F+\bar F} D_{\mathbf{1}},\ (-1)^{F+\bar F} D_\varphi\} \ . \end{aligned} \tag{5.79}$$

The identity defect $D_{\mathbf{1}}$ in the supersymmetric theory is also denoted by $D_{\mathbf{1}}^{\mathrm{NS}}$ in order to distinguish it from the identity defect in the non-supersymmetric theory. Compositions of these topological defect operators satisfy an algebra with non-negative integer structure constants. In addition to the usual compositions of fermion parity operators, we have

$$D_\varphi\, D_\varphi = D_{\mathbf{1}} + D_\varphi \ . \tag{5.80}$$

The torus partition function with one defect inserted are

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(D_{\mathbf{1}}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=|\chi_0^{\mathrm{NS}}(q)|^2+|\chi_{\frac{1}{10}}^{\mathrm{NS}}(q)|^2\,, \tag{5.81}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(\sqrt{2}(-1)^F D_{\mathbf{1}}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=2\chi_{\frac{7}{16}}^{\mathrm{R}}(q)\chi_0^{\mathrm{NS}}(\bar{q})+2\chi_{\frac{3}{80}}^{\mathrm{R}}(q)\chi_{\frac{1}{10}}^{\mathrm{NS}}(\bar{q})\,, \tag{5.82}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(\sqrt{2}(-1)^{\bar{F}} D_{\mathbf{1}}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=2\chi_0^{\mathrm{NS}}(q)\chi_{\frac{7}{16}}^{\mathrm{R}}(\bar{q})+2\chi_{\frac{1}{10}}^{\mathrm{NS}}(q)\chi_{\frac{3}{80}}^{\mathrm{R}}(\bar{q})\,, \tag{5.83}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left((-1)^{F+\bar{F}} D_{\mathbf{1}}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=2|\chi_{\frac{3}{80}}^{\mathrm{R}}(q)|^2+2|\chi_{\frac{7}{16}}^{\mathrm{R}}(q)|^2\,, \tag{5.84}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(D_{\varphi}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=|\chi_{\frac{1}{10}}^{\mathrm{NS}}(q)|^2+\chi_0^{\mathrm{NS}}(q)\chi_{\frac{1}{10}}^{\mathrm{NS}}(\bar{q})+\chi_{\frac{1}{10}}^{\mathrm{NS}}(q)\chi_0^{\mathrm{NS}}(\bar{q})\,, \tag{5.85}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(\sqrt{2}(-1)^F D_{\varphi}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=2\chi_{\frac{3}{80}}^{\mathrm{R}}(q)\left(\chi_0^{\mathrm{NS}}(\bar{q})+\chi_{\frac{1}{10}}^{\mathrm{NS}}(\bar{q})\right)+2\chi_{\frac{7}{16}}^{\mathrm{R}}(q)\chi_{\frac{1}{10}}^{\mathrm{NS}}(\bar{q})\,, \tag{5.86}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(\sqrt{2}(-1)^{\bar{F}} D_{\varphi}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=2\left(\chi_0^{\mathrm{NS}}(q)+\chi_{\frac{1}{10}}^{\mathrm{NS}}(q)\right)\chi_{\frac{3}{80}}^{\mathrm{R}}(\bar{q})+2\chi_{\frac{1}{10}}^{\mathrm{NS}}(q)\chi_{\frac{7}{16}}^{\mathrm{R}}(\bar{q})\,, \tag{5.87}$$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left((-1)^{F+\bar{F}} D_{\varphi}\,\tilde{q}^{L_0-\frac{7}{240}}\bar{\tilde{q}}^{\bar{L}_0-\frac{7}{240}}\right)=2|\chi_{\frac{3}{80}}^{\mathrm{R}}(q)|^2+2\chi_{\frac{3}{80}}^{\mathrm{R}}(q)\chi_{\frac{7}{16}}^{\mathrm{R}}(\bar{q})+2\chi_{\frac{7}{16}}^{\mathrm{R}}(q)\chi_{\frac{3}{80}}^{\mathrm{R}}(\bar{q})\,, \tag{5.88}$$

where $\chi_h^{\mathrm{R}}(q)$ is the character of the unextended Ramond algebra representation[1] with $c=\frac{7}{10}$ and the conformal weight h whose explicit expression is given by (A.13).

D	$\mathcal{H}_{D\|0}$	$\mathcal{V}$ir primary disorder fields	
		bosons	fermions
$D_{\mathbf{1}}$	$\mathcal{H}_{\mathrm{NS}}$	$\mathbf{1}$, $G\bar{G}(z,\bar{z})$, $\varphi_\varphi(z,\bar{z})$, $\tilde{\varphi}_\varphi(z,\bar{z})$	$G(z)$, $\bar{G}(\bar{z})$, $\psi^b_\varphi(z,\bar{z})$, $\bar{\psi}^b_\varphi(z,\bar{z})$
$\sqrt{2}(-1)^F D_{\mathbf{1}}$	$(\mathcal{H}^{\mathrm{R}}_{\frac{7}{16}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_0)\oplus(\mathcal{H}^{\mathrm{R}}_{\frac{3}{80}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_{\frac{1}{10}})$	$\sigma_{\mathbf{1}}(z)$, $\mu_{\mathbf{1}}\bar{G}(z,\bar{z})$, $\sigma_\varphi\bar{\psi}_\varphi(z,\bar{z})$, $\mu_\varphi\bar{\phi}_\varphi(z,\bar{z})$	$\mu_{\mathbf{1}}(z)$, $\sigma_{\mathbf{1}}\bar{G}(z,\bar{z})$ $\mu_\varphi\bar{\psi}_\varphi(z,\bar{z})$, $\sigma_\varphi\bar{\phi}_\varphi(z,\bar{z})$
$\sqrt{2}(-1)^{\bar{F}} D_{\mathbf{1}}$	$(\mathcal{H}^{\mathrm{NS}}_0\otimes\overline{\mathcal{H}}^{\mathrm{R}}_{\frac{7}{16}})\oplus(\mathcal{H}^{\mathrm{NS}}_{\frac{1}{10}}\otimes\overline{\mathcal{H}}^{\mathrm{R}}_{\frac{3}{80}})$	$\sigma_{\mathbf{1}}(z)$, $\mu_{\mathbf{1}}\bar{G}(z,\bar{z})$, $\sigma_\varphi\bar{\psi}_\varphi(z,\bar{z})$, $\mu_\varphi\bar{\phi}_\varphi(z,\bar{z})$	$\mu_{\mathbf{1}}(z)$, $\sigma_{\mathbf{1}}\bar{G}(z,\bar{z})$ $\mu_\varphi\bar{\psi}_\varphi(z,\bar{z})$, $\sigma_\varphi\bar{\phi}_\varphi(z,\bar{z})$
$(-1)^{F+\bar{F}}D_{\mathbf{1}}$	$\mathcal{H}_{\mathrm{R}}=\mathcal{H}^{\mathrm{R}}_{\frac{3}{80}}\oplus\mathcal{H}^{\mathrm{R}}_{\frac{7}{16}}$	$\sigma^b_{\mathbf{1}}(z,\bar{z})$, $\sigma^b_\varphi(z,\bar{z})$	$\mu^b_{\mathbf{1}}(z,\bar{z})$, $\mu^b_\varphi(z,\bar{z})$
D_φ	$(\mathcal{H}^{\mathrm{NS}}_{\frac{1}{10}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_{\frac{1}{10}})\oplus(\mathcal{H}^{\mathrm{NS}}_0\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_{\frac{1}{10}})$ $\oplus(\mathcal{H}^{\mathrm{NS}}_{\frac{1}{10}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_0)$	$\vdots$	$\vdots$
$\sqrt{2}(-1)^F D_\varphi$	$(\mathcal{H}^{\mathrm{R}}_{\frac{3}{80}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_0)\oplus(\mathcal{H}^{\mathrm{R}}_{\frac{3}{80}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_{\frac{1}{10}})$ $\oplus(\mathcal{H}^{\mathrm{R}}_{\frac{7}{16}}\otimes\overline{\mathcal{H}}^{\mathrm{NS}}_{\frac{1}{10}})$		
$\sqrt{2}(-1)^{\bar{F}} D_\varphi$	$(\mathcal{H}^{\mathrm{NS}}_0\otimes\overline{\mathcal{H}}^{\mathrm{R}}_{\frac{3}{80}})\oplus(\mathcal{H}^{\mathrm{NS}}_{\frac{1}{10}}\otimes\overline{\mathcal{H}}^{\mathrm{R}}_{\frac{3}{80}})$ $\oplus(\mathcal{H}^{\mathrm{NS}}_{\frac{1}{10}}\otimes\overline{\mathcal{H}}^{\mathrm{R}}_{\frac{7}{16}})$		
$(-1)^{F+\bar{F}}D_\varphi$	$\mathcal{H}^{\mathrm{R}}_{\frac{3}{80}}\oplus\mathcal{H}^{\mathrm{R}}_{\frac{3}{80},\frac{7}{16}}\oplus\mathcal{H}^{\mathrm{R}}_{\frac{7}{16},\frac{3}{80}}$		

Table 5.6: Disorder fields of $SM(3,5)$ topological defects.

1. Since two unextended Ramond modules with λ and $-\lambda$ have the same characters, we use $h=\lambda^2+\frac{c}{24}$ to label characters.

5.2.4 Conformal Boundaries in $SM(3,5)$

For the $N = 1$ super-Virasoro algebra, the Ishibashi condition (2.209) becomes

$$(G_n + i\epsilon \bar{G}_{-n})\|b\rangle\!\rangle = 0 , \tag{5.89}$$

where $\epsilon = \pm 1$ is the gluing condition with $\epsilon = 1$ corresponding to the identity automorphism of $\mathcal{SV}\mathrm{ir}$. As in the free fermion case, we only consider Ishibashi states in the NS sector since the bulk state space entirely consists of the NS representations. There are two NS representations at $c = \frac{7}{10}$, and two possible gluing conditions give four Ishibashi states $|h, \epsilon\rangle\!\rangle$ that are given by

$$\begin{aligned} |0, \pm\rangle\!\rangle &= |0\rangle \mp \frac{i}{2c/3} G_{-\frac{3}{2}} \bar{G}_{-\frac{3}{2}} |0\rangle + \frac{1}{c/2} L_{-2} \bar{L}_{-2} |0\rangle + \cdots , \\ |\tfrac{1}{10}, \pm\rangle\!\rangle &= |\tfrac{1}{10}\rangle \mp \frac{i}{1/5} G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} |\tfrac{1}{10}\rangle + \frac{1}{1/5} L_{-1} \bar{L}_{-1} |\tfrac{1}{10}\rangle + \cdots . \end{aligned} \tag{5.90}$$

They are solutions of (5.89) with corresponding ϵ. Their normalisation is given by

$$\langle\!\langle h, \pm| \tilde{q}^{\frac{1}{2}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} |h', \pm\rangle\!\rangle = \delta_{h,h'}\, \chi_h^{\mathrm{NS}}(\tilde{q}) , \tag{5.91}$$

$$\langle\!\langle h, \pm| (-1)^F \tilde{q}^{\frac{1}{2}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} |h', \pm\rangle\!\rangle = \delta_{h,h'}\, \tilde{\chi}_h^{\mathrm{NS}}(\tilde{q}) . \tag{5.92}$$

In addition, the fermion parity operators act on the Ishibashi states as

$$(-1)^F |h, \pm\rangle\!\rangle = (-1)^{\bar{F}} |h, \pm\rangle\!\rangle = \epsilon(h) |h, \mp\rangle\!\rangle \quad \text{and} \quad (-1)^{F+\bar{F}} |h, \pm\rangle\!\rangle = |h, \pm\rangle\!\rangle , \tag{5.93}$$

where $\epsilon(h) = \pm 1$ is the fermion parity of the highest weight state $|h\rangle$ as specified by (5.73).

Similar to the topological defect case, we can apply the Cardy's solution (2.233) to obtain boundary states labelled by $a \in \{\mathbf{1}_{\mathrm{NS}},\ \varphi_{\mathrm{NS}}\}$

$$\|a\rangle\!\rangle = \frac{\mathrm{S}_{a\mathbf{1}}^{[\mathrm{NS,NS}]}}{\sqrt{\mathrm{S}_{\mathbf{11}}^{[\mathrm{NS,NS}]}}} |0, +\rangle\!\rangle + \frac{\mathrm{S}_{a\varphi}^{[\mathrm{NS,NS}]}}{\sqrt{\mathrm{S}_{\mathbf{1}\varphi}^{[\mathrm{NS,NS}]}}} |\tfrac{1}{10}, +\rangle\!\rangle , \tag{5.94}$$

where $\mathrm{S}_{ij}^{[\mathrm{NS,NS}]}$ are the modular S matrix elements of $SM(3,5)$ given by (A.42). We expect there are two more boundary states carrying the labels in the twisted sector. Since topological defects can change boundary conditions as in (2.282), we can use (5.79) and (5.93) to obtain

$$\|\mathbf{1}_{\mathrm{R}}\rangle\!\rangle \coloneqq \sqrt{2}(-1)^F \|\mathbf{1}_{\mathrm{NS}}\rangle\!\rangle = \sqrt{2}(-1)^{\bar{F}} \|\mathbf{1}_{\mathrm{NS}}\rangle\!\rangle \quad \text{and} \tag{5.95}$$

$$\|\varphi_{\mathrm{R}}\rangle\!\rangle \coloneqq \sqrt{2}(-1)^F \|\varphi_{\mathrm{NS}}\rangle\!\rangle = \sqrt{2}(-1)^{\bar{F}} \|\varphi_{\mathrm{NS}}\rangle\!\rangle . \tag{5.96}$$

In addition, we can check

$$\|\varphi_{\mathrm{NS}}\rangle\!\rangle = D_\varphi\, \|\mathbf{1}_{\mathrm{NS}}\rangle\!\rangle . \tag{5.97}$$

Explicitly, these four boundary states can be written as

$$\|\mathbf{1}_{\rm NS}\rangle\!\rangle = \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}} |0,+\rangle\!\rangle + \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{4}} |\tfrac{1}{10},+\rangle\!\rangle \ ,$$
$$\|\varphi_{\rm NS}\rangle\!\rangle = \left(\frac{\sqrt{5}+2}{\sqrt{5}}\right)^{\frac{1}{4}} |0,+\rangle\!\rangle - \left(\frac{\sqrt{5}-2}{\sqrt{5}}\right)^{\frac{1}{4}} |\tfrac{1}{10},+\rangle\!\rangle \ ,$$
$$\|\mathbf{1}_{\rm R}\rangle\!\rangle = \left(\frac{2(5-\sqrt{5})}{5}\right)^{\frac{1}{4}} |0,-\rangle\!\rangle - \left(\frac{2(5+\sqrt{5})}{5}\right)^{\frac{1}{4}} |\tfrac{1}{10},-\rangle\!\rangle \ ,$$
$$\|\varphi_{\rm R}\rangle\!\rangle = \left(\frac{4(\sqrt{5}+2)}{\sqrt{5}}\right)^{\frac{1}{4}} |0,-\rangle\!\rangle + \left(\frac{4(\sqrt{5}-2)}{\sqrt{5}}\right)^{\frac{1}{4}} |\tfrac{1}{10},-\rangle\!\rangle \ . \tag{5.98}$$

There is another set of consistent boundary states obtained by exchanging $|h,+\rangle\!\rangle$ and $|h,-\rangle\!\rangle$ above. The overlaps (2.231) of these boundary states are summarised in Table 5.7. They obey the fusion rules (5.70), and we may call them "Cardy boundary states" of $SM(3,5)$. In the overlaps of two boundary states with Ramond labels, NS characters appear twice in the results in agreement with the fusion rules. This suggests we may have to weaken the condition for elementary boundary states in the supersymmetric theories.

	$\|\mathbf{1}_{\rm NS}\rangle\!\rangle$	$\|\varphi_{\rm NS}\rangle\!\rangle$	$\|\mathbf{1}_{\rm R}\rangle\!\rangle$	$\|\varphi_{\rm R}\rangle\!\rangle$
$\langle\!\langle\mathbf{1}_{\rm NS}\|$	$\chi_0^{\rm NS}(q)$	$\chi_{\frac{1}{10}}^{\rm NS}(q)$	$2\chi_{\frac{7}{16}}^{\rm R}(q)$	$2\chi_{\frac{3}{80}}^{\rm R}(q)$
$\langle\!\langle\varphi_{\rm NS}\|$		$\chi_0^{\rm NS}(q)+\chi_{\frac{1}{10}}^{\rm NS}(q)$	$2\chi_{\frac{3}{80}}^{\rm R}(q)$	$2\chi_{\frac{7}{16}}^{\rm R}(q)+2\chi_{\frac{3}{80}}^{\rm R}(q)$
$\langle\!\langle\mathbf{1}_{\rm R}\|$			$2\chi_0^{\rm NS}(q)$	$2\chi_{\frac{1}{10}}^{\rm NS}(q)$
$\langle\!\langle\varphi_{\rm R}\|$				$2\chi_0^{\rm NS}(q)+2\chi_{\frac{1}{10}}^{\rm NS}(q)$

Table 5.7: Overlaps of $SM(3,5)$ boundary states.

5.2.5 Interfaces Between $M(4,5)$ and $SM(3,5)$

As in the $c=\frac{1}{2}$ case, we need to consider the common sectors of the $M(4,5)$ bulk state space and that of $SM(3,5)$ in order to obtain the topological interfaces. In terms of the Virasoro representations, the common sectors are given by

$$(\mathcal{H}_0\otimes\overline{\mathcal{H}}_0)\oplus(\mathcal{H}_{\frac{3}{2}}\otimes\overline{\mathcal{H}}_{\frac{3}{2}})\oplus(\mathcal{H}_{\frac{1}{10}}\otimes\overline{\mathcal{H}}_{\frac{1}{10}})\oplus(\mathcal{H}_{\frac{3}{5}}\otimes\overline{\mathcal{H}}_{\frac{3}{5}})\ . \tag{5.99}$$

Therefore, a topological interface operator I satisfying

$$L_n\, I = I\, L_n \quad \text{and} \quad \bar{L}_n\, I = I\, \bar{L}_n \tag{5.100}$$

can be written in terms of the projectors onto Virasoro representations as

$$I(a,b,c,d) = a\, P_0\overline{P}_0 + b\, P_{\frac{3}{2}}\overline{P}_{\frac{3}{2}} + c\, P_{\frac{1}{10}}\overline{P}_{\frac{1}{10}} + d\, P_{\frac{3}{5}}\overline{P}_{\frac{3}{5}}\ , \tag{5.101}$$

where a, b, c, and d are constants. In addition, we need a map identifying the Virasoro highest weight states of weights $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{3}{5}, \frac{3}{5})$, which we take to be

$$
\begin{aligned}
|\tfrac{3}{2}, \tfrac{3}{2}\rangle_{M(4,5)} &= \frac{i\xi_{\frac{3}{2}}}{2c/3} G_{-\frac{3}{2}} \bar{G}_{-\frac{3}{2}} |0\rangle_{SM(3,5)} \ , \\
|\tfrac{3}{5}, \tfrac{3}{5}\rangle_{M(4,5)} &= \frac{i\xi_{\frac{3}{5}}}{2h} G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} |\tfrac{1}{10}, \tfrac{1}{10}\rangle_{SM(3,5)} \ ,
\end{aligned} \tag{5.102}
$$

where $c = \frac{7}{10}$, $h = \frac{1}{10}$ in the second line, and $\xi_{\frac{3}{2}} = \pm 1$ and $\xi_{\frac{3}{5}} = \pm 1$ are free. These highest weight states are normalised to have unit norm.

Requiring the condition (5.46) for the $c = \frac{7}{10}$ case allows us to solve for (a, b, c, d). In particular, from the $SM(3,5)$ boundary states (5.98) and the Ishibashi states (5.90), and the identifications (5.102), we get

$$
I(a, b, c, d)^\dagger \|\mathbf{1}\rangle\!\rangle = m_1 \|\mathbf{1}_{\rm NS}\rangle\!\rangle + m_2 \|\varphi_{\rm NS}\rangle\!\rangle + m_3 \|\mathbf{1}_{\rm R}\rangle\!\rangle + m_4 \|\varphi_{\rm R}\rangle\!\rangle \tag{5.103}
$$

with

$$
a = \sqrt{2} m_1 + \frac{1+\sqrt{5}}{\sqrt{2}} m_2 + 2m_3 + (1+\sqrt{5}) m_4 \ , \tag{5.104}
$$

$$
-\xi_{\frac{3}{2}}\, b = \sqrt{2} m_1 + \frac{1+\sqrt{5}}{\sqrt{2}} m_2 - 2m_3 - (1+\sqrt{5}) m_4 \ , \tag{5.105}
$$

$$
c = \sqrt{2} m_1 + \frac{1-\sqrt{5}}{\sqrt{2}} m_2 - 2m_3 - (1-\sqrt{5}) m_4 \ , \tag{5.106}
$$

$$
-\xi_{\frac{3}{5}}\, d = \sqrt{2} m_1 + \frac{1-\sqrt{5}}{\sqrt{2}} m_2 + 2m_3 + (1-\sqrt{5}) m_4 \ . \tag{5.107}
$$

We choose $\xi_{\frac{3}{2}} = \xi_{\frac{3}{5}} = -1$ so that any interface can be expressed as a combination

$$
I(a, b, c, d) = m_1\, I + m_2\, I_2 + m_3\, I_3 + m_4\, I_4 \ , \tag{5.108}
$$

where these interface operators are given by

$$
\begin{aligned}
I &= I(\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}) \ , \\
I_2 &= I(\tfrac{1+\sqrt{5}}{\sqrt{2}}, \tfrac{1+\sqrt{5}}{\sqrt{2}}, \tfrac{1-\sqrt{5}}{\sqrt{2}}, \tfrac{1-\sqrt{5}}{\sqrt{2}}) \ , \\
I_3 &= I(2, -2, -2, 2) \ , \\
I_4 &= I(1+\sqrt{5}, -1-\sqrt{5}, -1+\sqrt{5}, 1-\sqrt{5}) \ .
\end{aligned} \tag{5.109}
$$

They can be viewed as being created from the fundamental interface I by the action of topological defects since they satisfy the relations

$$
I = D_{\mathbf{1}}\, I = D_\varepsilon\, I = I\, D_{\mathbf{1}}^{\rm NS} = I\, (-1)^{F+\bar{F}} D_{\mathbf{1}}^{\rm NS} \ , \tag{5.110}
$$

$$
I_2 = D_{\hat{\mathbf{1}}}\, I = D_{\hat{\varepsilon}}\, I = I\, D_\varphi = I\, (-1)^{F+\bar{F}} D_\varphi \ , \tag{5.111}
$$

$$
I_3 = D_\sigma\, I = I\, \sqrt{2}(-1)^F D_{\mathbf{1}}^{\rm NS} = I\, \sqrt{2}(-1)^{\bar{F}} D_{\mathbf{1}}^{\rm NS} \ , \tag{5.112}
$$

$$
I_4 = D_{\hat{\sigma}}\, I = I\, \sqrt{2}(-1)^F D_\varphi = I\, \sqrt{2}(-1)^{\bar{F}} D_\varphi \ . \tag{5.113}
$$

The action of the fundamental interface I on the boundary states is as follows

$$I\|\mathbf{1}_{\rm NS}\rangle\!\rangle = \|\mathbf{1}\rangle\!\rangle + \|\varepsilon\rangle\!\rangle\ , \quad I^\dagger\|\mathbf{1}\rangle\!\rangle = I^\dagger\|\varepsilon\rangle\!\rangle = \|\mathbf{1}_{\rm NS}\rangle\!\rangle\ , \tag{5.114}$$

$$I\|\varphi_{\rm NS}\rangle\!\rangle = \|\hat{\mathbf{1}}\rangle\!\rangle + \|\hat\varepsilon\rangle\!\rangle\ , \quad I^\dagger\|\hat{\mathbf{1}}\rangle\!\rangle = I^\dagger\|\hat\varepsilon\rangle\!\rangle = \|\varphi_{\rm NS}\rangle\!\rangle\ , \tag{5.115}$$

$$I\|\mathbf{1}_{\rm R}\rangle\!\rangle = 2\|\sigma\rangle\!\rangle\ , \qquad I^\dagger\|\sigma\rangle\!\rangle = \|\mathbf{1}_{\rm R}\rangle\!\rangle\ , \tag{5.116}$$

$$I\|\varphi_{\rm R}\rangle\!\rangle = 2\|\hat\sigma\rangle\!\rangle\ , \qquad I^\dagger\|\hat\sigma\rangle\!\rangle = \|\varphi_{\rm R}\rangle\!\rangle\ . \tag{5.117}$$

Comparing these with the free fermion results (5.53) and (5.54), we may view the $SM(3,5)$ boundary conditions with Neveu–Schwarz labels as “free” boundary conditions and those with Ramond labels as “fixed” ones.

As it did in the $c = \frac{1}{2}$ case, requiring $II^\dagger$ and $I^\dagger I$ to be expressible as sums of topological defects in $M(4,5)$ and $SM(3,5)$, respectively, provides a strong constraint. For example, by taking $a,b,c,d \in \mathbb{R}$, we can calculate

$$\begin{aligned} &I(a,b,c,d)\; I(a,b,c,d)^\dagger \\ &= \frac{(5-\sqrt{5})(a^2+b^2)+(5+\sqrt{5})(c^2+d^2)}{40}\,(D_{\mathbf{1}}+D_\varepsilon) + \frac{a^2+b^2-c^2-d^2}{4\sqrt{5}}\,\big(D_{\hat{\mathbf{1}}}+D_{\hat\varepsilon}\big) \\ &\qquad + \frac{(5-\sqrt{5})(a^2-b^2)-(5+\sqrt{5})(c^2-d^2)}{20\sqrt{2}}D_\sigma + \frac{a^2-b^2+c^2-d^2}{2\sqrt{10}}D_{\hat\sigma}\ . \end{aligned} \tag{5.118}$$

We find the interfaces given by (5.108) indeed give integer coefficients, and the fundamental interface I given by (5.109) satisfies

$$I\,I^\dagger = D_{\mathbf{1}} + D_\varepsilon \quad \text{and} \quad I^\dagger\, I = D^{\rm NS}_{\mathbf{1}} + (-1)^{F+\bar F} D^{\rm NS}_{\mathbf{1}}\ . \tag{5.119}$$

To summarise, we have found a set of four elementary boundary conditions and a set of eight elementary topological defects in the supersymmetric theory $SM(3,5)$, and a set of four fundamental interfaces between $M(4,5)$ and $SM(3,5)$. Our next task is to find non-factorising and non-topological superconformal defects in $SM(3,5)$ and generate the corresponding defects in $M(4,5)$ by using the interface operators.

Chapter 6

Conformal Defects in Tri-Critical Ising Model

In this chapter we consider the superconformal boundary conditions in the folded theory of $SM(3,5)$ at $c = \frac{7}{5}$, and identify them with superconformal defects in $SM(3,5)$. Then we use the topological interfaces constructed in the previous chapter to obtain conformal defects in the tri-critical Ising model $M(4,5)$. We calculate their reflection and transmission coefficients, and comment on differences between our results and those given in [94].

6.1 Folding $SM(3,5)$ and (D_6, E_6) Theory

If we consider "folding" the diagonal theory $SM(3,5)$, which has $c = \frac{7}{10}$, it will result in a theory with $c = \frac{7}{5}$. It turns out that $c = \frac{7}{5}$ appears in the unitary series of (2.119), which is the super-Virasoro minimal model $SM(10,12)$. As we are folding a theory which is diagonal with respect to $\mathcal{SV}\mathrm{ir}$, the resulting theory should admit an interpretation as an $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ diagonal theory. In [94], the partition function of the folded theory is identified as the (D_6, E_6) modular invariant of $\mathcal{SV}\mathrm{ir}$. The NS sector of (D_6, E_6) partition function is given[18] by

$$\begin{aligned} Z_{\mathrm{NS}}^{(D_6,E_6)} = &|\chi_0^{\mathrm{NS}}(q) + \chi_{\frac{3}{2}}^{\mathrm{NS}}(q) + \chi_{\frac{7}{2}}^{\mathrm{NS}}(q) + \chi_{10}^{\mathrm{NS}}(q)|^2 \\ &+ |\chi_{\frac{1}{5}}^{\mathrm{NS}}(q) + \chi_{\frac{7}{10}}^{\mathrm{NS}}(q) + \chi_{\frac{6}{5}}^{\mathrm{NS}}(q) + \chi_{\frac{57}{10}}^{\mathrm{NS}}(q)|^2 + 2|\chi_{\frac{1}{10}}^{\mathrm{NS}}(q) + \chi_{\frac{13}{5}}^{\mathrm{NS}}(q)|^2 \\ &= (Z_{\mathrm{NS}})^2 \ , \end{aligned} \tag{6.1}$$

which can be identified with the square of the $SM(3,5)$ bulk partition function Z_{NS} given by (5.74) using the character identities in Appendix D.1. We take (6.1) as the bulk partition function of the folded theory.

The Ishibashi states are constructed from the diagonal terms in the partition function (5.74). There are 12 terms with $h = \bar{h}$ in the partition function, and together with two gluing conditions $\epsilon = \pm 1$, there will be 24 Ishibashi states. These diagonal terms can be identified by Kac labels (r,s) with r and s taking values in the exponents of D_6 diagram and a subset of E_6 exponents respectively. From Table 2.2, these exponents are

$$\mathcal{E}(D_6) = \{1,\ 3,\ 5,\ 5',\ 7,\ 9\} \quad \text{and} \quad \mathcal{E}(E_6) = \{1,\ 4,\ 5,\ 7,\ 8,\ 11\} \ . \tag{6.2}$$

Since D_6 exponents are all odd, we take the subset of E_6 exponents that are odd numbers in order to obtain the Kac labels in the NS sector. The result is that the Ishibashi states are labelled by (r,s) with $r \in \{1,\ 3,\ 5,\ 5',\ 7,\ 9\}$ and $s \in \{1,\ 5,\ 7,\ 11\}$ modulo $(r,s) \sim (10-r, 12-s)$.

It is instructive to rewrite the partition function (6.1) with the characters labelled by (r,s). Keeping in mind the D_6 and E_6 partition functions of $\widehat{\mathrm{sl}}(2)_k$-WZW models given in

Table 2.3, we can write

$$\begin{aligned} &Z_{\rm NS}^{(D_6,E_6)} \\ &\quad = \frac{1}{2}\left(|\chi_{1,1}+\chi_{1,7}+\chi_{9,1}+\chi_{9,7}|^2+|\chi_{3,1}+\chi_{3,7}+\chi_{7,1}+\chi_{7,7}|^2+2|\chi_{5,1}+\chi_{5,7}|^2\right) \\ &\qquad + \frac{1}{2}\left(|\chi_{1,5}+\chi_{1,11}+\chi_{9,5}+\chi_{9,11}|^2+|\chi_{3,5}+\chi_{3,11}+\chi_{7,5}+\chi_{7,11}|^2+2|\chi_{5,5}+\chi_{5,11}|^2\right) \ , \end{aligned} \tag{6.3}$$

where the first line and the second line are identical in terms of the characters due to the identification $(r,s)\sim(10-r,12-s)$. This suggests that a natural way to restrict the Kac labels to obtain unique representations is to take either $s\in\{1,\ 7\}$ or $s\in\{5,\ 11\}$ while r still takes value in $\{1,\ 3,\ 5,\ 5',\ 7,\ 9\}$. As we prefer to use $(1,1)$, rather than $(9,11)$, to label the vacuum representation of $\mathcal{SV}\mathrm{ir}$ at $c=\frac{7}{5}$, we take $s\in\{1,\ 7\}$. Our choice of representatives is different from that of [94], and this will make differences later on.

From the partition function (6.1) and the character identity (D.1), the chiral vacuum character of the (D_6,E_6) theory can be written as

$$ {}^{(10)}\chi_0^{\rm NS}(q) + {}^{(10)}\chi_{\frac{3}{2}}^{\rm NS}(q) + {}^{(10)}\chi_{\frac{7}{2}}^{\rm NS}(q) + {}^{(10)}\chi_{10}^{\rm NS}(q) = ({}^{(3)}\chi_0^{\rm NS}(q))^2 \ . \tag{6.4}$$

This shows that the full chiral algebra $\mathcal{SV}\mathrm{ir}\oplus\mathcal{SV}\mathrm{ir}$ at $c=\frac{7}{5}$ has three chiral superprimary fields of conformal weights $\frac{3}{2}$, $\frac{7}{2}$, and 10. Various extensions of $\mathcal{SV}\mathrm{ir}$ at $c=\frac{7}{5}$ have been considered in [35], [39], [40], and [42]. We will return to these super W-algebras later.

Finally, we note that it is also possible to express a single copy of the $SM(3,5)$ partition function in terms of characters of $\mathcal{SV}\mathrm{ir}$ at $c=\frac{7}{5}$ as

$$\begin{aligned} Z_{\rm NS} &= ({}^{(3)}\chi_0^{\rm NS}(q))^2 + ({}^{(3)}\chi_{\frac{1}{10}}^{\rm NS}(q))^2 \\ &= {}^{(10)}\chi_0^{\rm NS}(q) + {}^{(10)}\chi_{\frac{3}{2}}^{\rm NS}(q) + {}^{(10)}\chi_{\frac{7}{2}}^{\rm NS}(q) + {}^{(10)}\chi_{10}^{\rm NS}(q) \\ &\qquad + {}^{(10)}\chi_{\frac{1}{5}}^{\rm NS}(q) + {}^{(10)}\chi_{\frac{7}{10}}^{\rm NS}(q) + {}^{(10)}\chi_{\frac{6}{5}}^{\rm NS}(q) + {}^{(10)}\chi_{\frac{57}{10}}^{\rm NS}(q) \ , \end{aligned} \tag{6.5}$$

where q is real. The reason is that one can embed $\mathcal{SV}\mathrm{ir}$ at $c=\frac{7}{5}$ into the two, holomorphic and antiholomorphic, copies of $\mathcal{SV}\mathrm{ir}$ at $c=\frac{7}{10}$.

6.1.1 Boundary States Corresponding to Topological and Factorising Defects

As usual, we start from the diagonal theory $SM(3,5)$ defined on the whole complex plane with a conformal defect running along the real axis, and fold this theory along the defect to obtain the (D_6,E_6) theory defined on the upper half plane with the corresponding boundary condition at $z=\bar{z}$. For each field $\varphi_I(z,\bar{z})$, there are two copies $\varphi_I^{(a)}(z,\bar{z})$ in the folded theory: for $(z,\bar{z})$ on the upper half plane, the $a=1$ copy is the original field $\varphi_I^{(1)}(z,\bar{z})=\varphi_I(z,\bar{z})$, and the $a=2$ copy coming from the the lower half plane which is being folded $\varphi_I^{(2)}(z,\bar{z})=\varphi_I(z^*,\bar{z}^*)$.

As we did in Section 2.4, it is more useful to map the boundary/defect along the real axis onto the unit circle, and consider boundary states/defect operators. For that we employ a family of Möbius maps $w\mapsto z(w)$ such that the image of the real axis changes smoothly from the real axis to the unit circle. In terms of the parameter $R \in \mathbb{R}$, they are defined by

$$w = 2iR \left(\frac{z + \frac{1}{iR}}{z + 2iR + \frac{1}{iR}} \right) , \tag{6.6}$$

where $R = \infty$ gives the identity map, and $R = 1$ maps the real axis to the unit circle. This map has the property that the derivative at the origin is 1,

$$\left. \frac{\partial z}{\partial w} \right|_{w=0} = 1 . \tag{6.7}$$

On the upper half plane, the generator $G^{(2)}(w)$ is the image of $\bar{G}(\bar{w})$ on the lower half plane

$$G^{(2)}(w)\big|_{w=a} = \bar{G}(\bar{w})\big|_{\bar{w}=a^*} . \tag{6.8}$$

We would like to relate the modes $G^{(2)}_n$ and $\bar{G}_n$ that are defined by the contour integrals along the unit circle on the z-plane. They can also be expressed as the expansions

$$G^{(2)}(z) = \sum_n G^{(2)}_n \, z^{-n-\frac{3}{2}} \quad \text{and} \quad \bar{G}(\bar{z}) = \sum_n \bar{G}_n \, \bar{z}^{-n-\frac{3}{2}} , \tag{6.9}$$

when $R = 1$. Using the Möbius map (6.6), the relation (6.8) becomes

$$z^{3/2} \, G^{(2)}(z)\big|_{w=a} = \left(\frac{(2iR - a)(2aR^2 + 2iR - a)}{(2iR + a^*)(2a^*R^2 - 2iR - a^*)} \right)^{\frac{3}{2}} \bar{z}^{3/2} \, \bar{G}(\bar{z})\big|_{\bar{w}=a^*} . \tag{6.10}$$

By taking $R = 1$ and $z = u$ on the unit circle, we have $a = a^*$ and $\bar{z} = u^{-1}$. Therefore,

$$u^{3/2} G^{(2)}(u) = -iu^{-3/2} \bar{G}(u^{-1}) , \tag{6.11}$$

which yields

$$G^{(2)}_n = -i\bar{G}_{-n} . \tag{6.12}$$

Likewise, we find

$$\bar{G}^{(2)}_n = iG_{-n} . \tag{6.13}$$

In the folded theory on the z-plane, the generators $G^{(1)}(z)$, $\bar{G}^{(1)}(\bar{z})$, $G^{(2)}(z)$, and $\bar{G}^{(2)}(\bar{z})$ exist only on the exterior of the unit circle; upon folding, the generators $G(z)$ and $\bar{G}(\bar{z})$ with $|z| < 1$ and $|\bar{z}| < 1$ are mapped to $\bar{G}^{(2)}(\bar{z}')$ and $G^{(2)}(z')$ with appropriate coefficients.

We are going to construct a map ρ which takes a boundary state in the folded theory and maps it to the corresponding defect operator in the unfolded theory. If it relates a boundary state and a defect operator by

$$\rho(\|b\rangle\rangle) = D_b , \tag{6.14}$$

we define

$$\begin{aligned} \rho(G^{(1)}_n \|b\rangle\rangle) &= G_n D_b , \\ \rho(\bar{G}^{(1)}_n \|b\rangle\rangle) &= \bar{G}_n D_b , \\ \rho(G^{(2)}_n \|b\rangle\rangle) &= -i(-1)^{F+\bar{F}} D_b (-1)^{F+\bar{F}} \bar{G}_{-n} , \\ \rho(\bar{G}^{(2)}_n \|b\rangle\rangle) &= i(-1)^{F+\bar{F}} D_b (-1)^{F+\bar{F}} G_{-n} , \end{aligned} \tag{6.15}$$

where the coefficients are due to the relations (6.12) and (6.13), and the bulk fermion parity operators $(-1)^{F+\bar{F}}$ are introduced to make all the fermionic generators in the folded theory anticommute.

In order to complete the definition of the "unfolding map" ρ, we need to consider the image of the highest weight states in the (D_6, E_6) theory that are tensor products of $SM(3,5)$ highest weight states. They are the lowest weight components of each "block" appearing in the partition function (6.1). For brevity, when there is no danger of confusion, we denote the diagonal bulk highest weight states by $|h\rangle$ as they have $h = \bar{h}$. The simplest choice is

$$\rho(|h_1\rangle \otimes |h_2\rangle) = |h_1\rangle\langle h_2| \tag{6.16}$$

for bulk highest weight states $|h_1\rangle$ and $|h_2\rangle$ of $SM(3,5)$. As we shall see later, it will be helpful to define in addition the map

$$\rho'(|h_1\rangle \otimes |h_2\rangle) = |h_1\rangle\langle h_2|(-1)^F \ . \tag{6.17}$$

In the bulk (D_6, E_6) theory, there are four such highest weight states, and their images under ρ and ρ' are summarised in Table 6.1.

(r,s)	$(1,1)$	$(7,7)$	$(5,7)$	$(5',7)$
$h_{r,s}$	0	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$
$\rho\,(\|r,s\rangle)$	$\|0\rangle\langle 0\|$	$\|\frac{1}{10}\rangle\langle\frac{1}{10}\|$	$\|0\rangle\langle\frac{1}{10}\|$	$\|\frac{1}{10}\rangle\langle 0\|$
$\rho'(\|r,s\rangle)$	$\|0\rangle\langle 0\|$	$-\|\frac{1}{10}\rangle\langle\frac{1}{10}\|$	$-\|0\rangle\langle\frac{1}{10}\|$	$\|\frac{1}{10}\rangle\langle 0\|$

Table 6.1: Images of bulk highest weight states of (D_6, E_6) theory under ρ and ρ'.

For a topological defect in $SM(3,5)$, there are two signs corresponding to automorphisms of two copies of $\mathcal{SV}$ir. From (5.75), the boundary state corresponding to a topological defect should satisfy

$$\begin{cases} G_n D_b = \eta\, D_b G_n \\ \bar{G}_n D_b = \eta' D_b \bar{G}_n \end{cases} \rightarrow \begin{cases} \left(G_n^{(1)} + i\eta\, \bar{G}_{-n}^{(2)}\right) \|b\rangle\rangle \\ \left(G_n^{(2)} + i\eta' \bar{G}_{-n}^{(1)}\right) \|b\rangle\rangle \end{cases} \tag{6.18}$$

For a factorising defect in $SM(3,5)$, we have two signs coming from the gluing conditions of the two boundary states. For the corresponding boundary state, these conditions become

$$\begin{cases} \left(G_n + i\eta \bar{G}_{-n}\right) \|a,\eta\rangle\rangle\langle\langle b,\eta'\| = 0 \\ 0 = \|a,\eta\rangle\rangle\langle\langle b,\eta'\| \left(G_n - i\eta' \bar{G}_{-n}\right) \end{cases} \rightarrow \begin{cases} \left(G_n^{(1)} + i\eta\, \bar{G}_{-n}^{(1)}\right) \|a,b\rangle\rangle \\ \left(G_n^{(2)} - i\eta' \bar{G}_{-n}^{(2)}\right) \|a,b\rangle\rangle \end{cases} \tag{6.19}$$

From these equations, we see that it is not possible to express all the gluing conditions by only using a single set of combinations of the form $G_n^{(1)} \pm G_n^{(2)}$ and $\bar{G}_n^{(1)} \pm \bar{G}_n^{(2)}$. In the next section, we consider exactly how we can organise the boundary states corresponding to the know defects into boundary states of the (D_6, E_6) theory. There are 24 known conformal defects in the diagonal $SM(3,5)$ theory of which 8 are topological and 16 are factorising.

6.1.2 Embeddings of $\mathcal{SV}\mathrm{ir}$ into $\mathcal{SV}\mathrm{ir} \oplus \mathcal{SV}\mathrm{ir}$

In order to view the folded theory as a boundary SCFT at $c = \frac{7}{5}$, we need to define an embedding of $\mathcal{SV}\mathrm{ir}$ at $c = \frac{7}{5}$ into the algebra $\mathcal{SV}\mathrm{ir} \oplus \mathcal{SV}\mathrm{ir}$, where each of $\mathcal{SV}\mathrm{ir}$ has $c = \frac{7}{10}$. Considering the holomorphic and antiholomorphic copies of the chiral algebra at the same time, we write an embedding as

$$\iota_{\alpha\beta\gamma\delta}(G_n^{\mathrm{tot}}) = \alpha G_n^{(1)} + \beta G_n^{(2)} \quad \text{and} \quad \iota_{\alpha\beta\gamma\delta}(\bar{G}_n^{\mathrm{tot}}) = \gamma \bar{G}_n^{(1)} + \delta \bar{G}_n^{(2)} \;, \tag{6.20}$$

where α, β, γ, and δ are signs, and G_n^{tot} and $\bar{G}_n^{\mathrm{tot}}$ are the generators of the two copies of $\mathcal{SV}\mathrm{ir}$ at $c = \frac{7}{5}$. We will denote the combined mat $\rho \circ \iota_{\alpha\beta\gamma\delta}$ by $\rho_{\alpha\beta\gamma\delta}$.

We also need to define a map from the Ishibashi states of the (D_6, E_6) theory to the tensor products of the $SM(3,5)$ Ishibashi states. As Ishibashi states are determined by highest weight states and gluing conditions, we need to express the bulk highest weight states of the (D_6, E_6) theory in terms of vectors in the folded theory, and find relations between gluing conditions of the (D_6, E_6) Ishibashi states and those of the tensor product of the $SM(3,5)$ Ishibashi states.

Gluing conditions of (D_6, E_6) boundary states are related to the signs η and η' in (6.18) and (6.19) via the embedding maps $\iota_{\alpha\beta\gamma\delta}$. Consider an Ishibashi state $|h, \epsilon\rangle\rangle$ of the (D_6, E_6) theory, which satisfies

$$(G_n^{\mathrm{tot}} + i\epsilon \bar{G}_{-n}^{\mathrm{tot}})\, |h, \epsilon\rangle\rangle = 0 \;. \tag{6.21}$$

Using the embedding $\iota_{\alpha\beta\gamma\delta}$, this condition becomes

$$\left(\alpha G_n^{(1)} + \beta G_n^{(2)} + i\epsilon\gamma \bar{G}_{-n}^{(1)} + i\epsilon\delta \bar{G}_{-n}^{(2)}\right) |h, \epsilon\rangle\rangle = 0 \;. \tag{6.22}$$

Therefore, a topological defect with the gluing conditions η and η' in (6.18) corresponds to $\eta = \alpha\delta\epsilon$ and $\eta' = \beta\gamma\epsilon$, which gives $\alpha\beta\gamma\delta = \eta\eta'$. Similarly, a factorising defect with η and η' in (6.19) corresponds to $\eta = \alpha\gamma\epsilon$ and $\eta' = -\beta\delta\epsilon$, which yields $\alpha\beta\gamma\delta = -\eta\eta'$. From these observations, we find that there are two equivalence classes of embeddings given by $\alpha\beta\gamma\delta = \pm 1$. If an embedding satisfies $\alpha\beta\gamma\delta = 1$, topological defects will satisfy $\eta\eta' = 1$ and factorising defects will satisfy $\eta\eta' = -1$. An embedding with $\alpha\beta\gamma\delta = -1$ corresponds to topological defects with $\eta\eta' = -1$ and factorising defects with $\eta\eta' = 1$.

From the above result, we expect there are two sets of boundary states in the (D_6, E_6) theory arising from the two equivalence classes of embeddings; one set should correspond to half the defects of $SM(3,5)$ and the other set giving the other half. In order to obtain the exact correspondence, we need to consider embeddings of the diagonal bulk highest weight states of the (D_6, E_6) theory into bulk states of the folded theory. As we already know some of the relations that are given in Table 6.1, we now need to determine embeddings of the remaining 8 states corresponding to the diagonal terms in the partition function (6.1). Namely, they are the states with Kac labels $(1,7)$, $(9,1)$, $(9,7)$, $(3,1)$, $(3,7)$, $(7,1)$, $(5,1)$, and $(5',1)$ that correspond to $h = \bar{h}$ given by $\frac{7}{2}$, 10, $\frac{3}{2}$, $\frac{7}{10}$, $\frac{6}{5}$, $\frac{57}{10}$, $\frac{13}{5}$, and $\frac{13}{5}'$ respectively. Definitions of these states depend on the embedding $\iota_{\alpha\beta\gamma\delta}$. For example, the state with $h = \bar{h} = \frac{3}{2}$ in the (D_6, E_6) theory is given by

$$\iota_{\alpha\beta\gamma\delta}(|\tfrac{3}{2}\rangle) = \frac{i\eta_{\frac{3}{2}}}{4c/3}(\alpha G_{-\frac{3}{2}}^{(1)} - \beta G_{-\frac{3}{2}}^{(2)})(\gamma \bar{G}_{-\frac{3}{2}}^{(1)} - \delta \bar{G}_{-\frac{3}{2}}^{(2)})|0\rangle \;, \tag{6.23}$$

where $c = \frac{7}{10}$ and $\eta_{\frac{3}{2}} = \pm 1$ is arbitrary. Another example is the state with $h = \bar{h} = \frac{7}{10}$, which can be written as

$$\iota_{\alpha\beta\gamma\delta}(|\tfrac{7}{10}\rangle) = \frac{i\eta_{\frac{7}{10}}}{2h}(\alpha G^{(1)}_{-\frac{1}{2}} - \beta G^{(2)}_{-\frac{1}{2}})(\gamma \bar{G}^{(1)}_{-\frac{1}{2}} - \delta \bar{G}^{(2)}_{-\frac{1}{2}})|\tfrac{1}{5}\rangle \ , \tag{6.24}$$

where $h = \frac{1}{10}$ and $\eta_{\frac{7}{10}} = \pm 1$ is arbitrary. Here, $|\frac{1}{5}\rangle$ is a highest weight state of the (D_6,E_6) theory which can be written as the tensor product $|\frac{1}{10}\rangle \otimes |\frac{1}{10}\rangle$ of the highest weight states of the diagonal $SM(3,5)$ theory. We have a free sign η_h for each of these eight (D_6,E_6) diagonal highest weight states. The image of these states under ι and ρ as well as the corresponding results for the Ishibashi states are summarised in Appendix D.2.

Using these facts, it is possible to construct the boundary states corresponding to all the know topological and factorising defects in the diagonal theory $SM(3,5)$. We find that these boundary states can all be written in terms of $\|(a,b)_{\text{NS}}\rangle\rangle$ and $\|(a,b)_{\widetilde{\text{NS}}}\rangle\rangle$ defined in [94] in at least two ways. We illustrate this in the next two subsections with the case of the identity defect and the factorising defect $\|\mathbf{1}_{\text{NS}}\rangle\rangle\langle\langle\mathbf{1}_{\text{NS}}\|$ in $SM(3,5)$.

6.1.3 Identity Defect in $SM(3,5)$

In terms of an orthonormal basis of the bulk state space $\mathcal{H}_{\text{NS}}$ of $SM(3,5)$, the identity defect can be expressed as

$$D_{\mathbf{1}} = \sum_{\psi} |\psi\rangle\langle\psi| \ . \tag{6.25}$$

Expanding this as

$$\begin{aligned} D_{\mathbf{1}} = |0\rangle\langle 0| &+ \frac{1}{2c/3}\Big(G_{-\frac{3}{2}}|0\rangle\langle 0|G_{\frac{3}{2}} + \bar{G}_{-\frac{3}{2}}|0\rangle\langle 0|\bar{G}_{\frac{3}{2}} + \cdots\Big) \\ &+ |\tfrac{1}{10}\rangle\langle\tfrac{1}{10}| + \frac{1}{1/5}\Big(G_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|G_{\frac{1}{2}} + \bar{G}_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{G}_{\frac{1}{2}} + \cdots\Big) + \cdots \ , \end{aligned} \tag{6.26}$$

we see that this must arise from a combination of the (D_6,E_6) Ishibashi states $|0,\epsilon\rangle\rangle$, $|\frac{3}{2},\epsilon\rangle\rangle$, $|\frac{7}{2},\epsilon\rangle\rangle$, $|10,\epsilon\rangle\rangle$, $|\frac{1}{5},\epsilon\rangle\rangle$, $|\frac{7}{10},\epsilon\rangle\rangle$, $|\frac{6}{5},\epsilon\rangle\rangle$, and $|\frac{57}{10},\epsilon\rangle\rangle$. Since the identity defect satisfies (6.18) with $\eta = \eta' = 1$, the gluing condition ϵ and embedding $\iota_{\alpha\beta\gamma\delta}$ satisfy $\epsilon = \alpha\delta = \beta\gamma$ and $\alpha\beta\gamma\delta = 1$.

The simplest choice is $\alpha = \beta = \gamma = \delta = 1$. This still leaves the signs η_h free for the $\mathcal{SV}\text{ir}$ primary states that can be considered as super W-algebra descendants. Given the freedom to choose these signs, the boundary state $\|D_{\mathbf{1}}\rangle\rangle$ can be expressed as

$$\begin{aligned} \|D_{\mathbf{1}}\rangle\rangle = |0,+\rangle\rangle + |\tfrac{3}{2},+\rangle\rangle + |\tfrac{7}{2},+\rangle\rangle + |10,+\rangle\rangle \\ + |\tfrac{1}{5},+\rangle\rangle + |\tfrac{7}{10},+\rangle\rangle + |\tfrac{6}{5},+\rangle\rangle + |\tfrac{57}{10},+\rangle\rangle \end{aligned} \tag{6.27}$$

such that

$$D_{\mathbf{1}} = \rho_{++++}(\|D_{\mathbf{1}}\rangle\rangle) \ , \tag{6.28}$$

where ρ acts as in Table 6.1. Using the explicit expressions of the Ishibashi states given in Appendix D.2, we can see that this fixes the signs η_h. In particular, we have

$$\eta_{\frac{3}{2}} = 1 \ , \quad \eta_{\frac{7}{10}} = 1 \ , \quad \text{and} \quad \eta_{\frac{6}{5}} = -1 \ . \tag{6.29}$$

By choosing the signs η_h in a different fashion, we can equally express $\|D_{\mathbf{1}}\rangle\!\rangle$ as

$$\|D_{\mathbf{1}}\rangle\!\rangle = |0,+\rangle\!\rangle - |\tfrac{3}{2},+\rangle\!\rangle + |\tfrac{7}{2},+\rangle\!\rangle - |10,+\rangle\!\rangle \\ - |\tfrac{1}{5},+\rangle\!\rangle + |\tfrac{7}{10},+\rangle\!\rangle + |\tfrac{6}{5},+\rangle\!\rangle - |\tfrac{57}{10},+\rangle\!\rangle \tag{6.30}$$

with

$$D_{\mathbf{1}} = \rho'_{++++}(\|D_{\mathbf{1}}\rangle\!\rangle) \ . \tag{6.31}$$

In this case, we have the opposite choices of η_h

$$\eta_{\frac{3}{2}} = -1 \ , \quad \eta_{\frac{7}{10}} = -1 \ , \quad \text{and} \quad \eta_{\frac{6}{5}} = 1 \ . \tag{6.32}$$

Note that in the expressions like (6.28) and (6.31), information regarding the choices of signs η_h is suppressed, and only information regarding the choice of the signs for the highest weight states given in Table 6.1 is kept.

6.1.4 Factorising Defect $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ in $SM(3,5)$

We take the set of consistent boundary states in $SM(3,5)$ as given in (5.98). Then, the factorising defect $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ can be written as

$$\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\| = \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{2}} |0,+\rangle\!\rangle\langle\!\langle 0,+| \ + \ \left(\frac{1}{5}\right)^{\frac{1}{4}} |0,+\rangle\!\rangle\langle\!\langle \tfrac{1}{10},+| \\ + \ \left(\frac{1}{5}\right)^{\frac{1}{4}} |\tfrac{1}{10},+\rangle\!\rangle\langle\!\langle 0,+| \ + \ \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{2}} |\tfrac{1}{10},+\rangle\!\rangle\langle\!\langle \tfrac{1}{10},+| \ . \tag{6.33}$$

Since this factorising defect satisfies (6.19) with $\eta = \eta' = 1$, the gluing condition ϵ and embedding $\iota_{\alpha\beta\gamma\delta}$ satisfy $\epsilon = \alpha\gamma = -\beta\delta$ and $\alpha\beta\gamma\delta = -1$.

The simplest choice is $\alpha = \beta = \gamma = 1$ and $\delta = -1$, which we take for this case. As before, we have the freedom to choose the signs η_h, and we can express this factorising defect in many ways. For later use, we make a specific choice

$$\|\mathbf{1}_{\rm NS},\mathbf{1}_{\rm NS}\rangle\!\rangle = \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{2}} \big(|0,+\rangle\!\rangle + |\tfrac{3}{2},+\rangle\!\rangle + |\tfrac{7}{2},+\rangle\!\rangle + |10,+\rangle\!\rangle\big) \\ + \left(\frac{1}{5}\right)^{\frac{1}{4}} \Big(|\tfrac{1}{10},+\rangle\!\rangle + |\tfrac{13}{5},+\rangle\!\rangle + |\tfrac{1}{10}{}',+\rangle\!\rangle + |\tfrac{13}{5}{}',+\rangle\!\rangle\Big) \\ + \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{2}} \big(|\tfrac{1}{5},+\rangle\!\rangle + |\tfrac{7}{10},+\rangle\!\rangle + |\tfrac{6}{5},+\rangle\!\rangle + |\tfrac{57}{10},+\rangle\!\rangle\big) \tag{6.34}$$

such that

$$\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\| = \rho_{+++-}(\|\mathbf{1}_{\rm NS},\mathbf{1}_{\rm NS}\rangle\!\rangle) \ , \tag{6.35}$$

where ρ acts as in Table 6.1. In addition, we have

$$\eta_{\frac{3}{2}} = -1 \ , \quad \eta_{\frac{7}{10}} = -1 \ , \quad \text{and} \quad \eta_{\frac{6}{5}} = 1 \tag{6.36}$$

in this case. Similarly, the other η_h are fixed.

6.2 Boundary Conditions in (D_6, E_6) Theory

A set of boundary states for the (D_6, E_6) theory was proposed by Gang and Yamaguchi in [94]. They considered GSO projected boundary states by including both Neveu–Schwarz and Ramond contributions whereas our construction only needs boundary states in the NS sector. In addition, there are some difficulties with the boundary states in [94] as we will explain later.

The boundary conditions of the (D_6, E_6) theory are labelled by pairs of nodes on the D_6 and E_6 Dynkin diagrams together with a choice of gluing condition. As we have seen for the Virasoro boundary conditions in Subsection 2.3.4, it over-counts the number of boundary conditions if we include all the pairs of nodes. The situation is quite different from the Virasoro case in which one of the diagrams is always of the type A_n. In this case, it turns out that the nodes of the E_6 diagram related by the diagram symmetry,

$$r : 1 \leftrightarrow 5 \quad \text{and} \quad r : 2 \leftrightarrow 4\,, \tag{6.37}$$

lead to the same Neveu–Schwarz contribution while the Ramond contributions are different. We may think of this as replacing the E_6 Dynkin diagram by the F_4 diagram with the nodes related by the $\mathbb{Z}_2$ symmetry corresponding the short simple roots of F_4. In order to label distinct boundary conditions, we take the E_6 diagram nodes 1, 2, 3, and 6, which gives 24 pairs of nodes. Furthermore, we can bi-colour the Dynkin diagrams as in Figure 6.1, and split the boundary conditions into two sets with 12 elements each: one in which the pairs consist of nodes of the same colour, and the other set with pairs of nodes of opposite colour. A pair of nodes is denoted by (a, b) where a is a node of the D_6 diagram and b belongs to the E_6 diagram. The set of nodes with the same colouration is given by

$$\mathcal{B}_e = \{(1,1),\ (3,1),\ (5,1),\ (6,1),\ (2,2),\ (4,2),\ (1,3),\ (3,3),\ (5,3),\ (6,3),\ (2,6),\ (4,6)\}\,, \tag{6.38}$$

and the set of nodes with opposite colouration is

$$\mathcal{B}_o = \{(2,1),\ (4,1),\ (1,2),\ (3,2),\ (5,2),\ (6,2),\ (2,3),\ (4,3),\ (1,6),\ (3,6),\ (5,6),\ (6,6)\}\,. \tag{6.39}$$

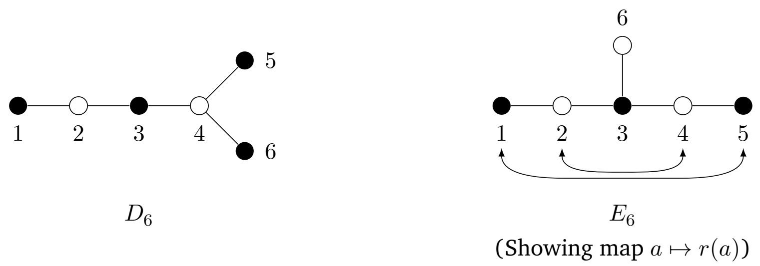

Figure 6.1: Dynkin diagrams of D_6 and E_6 showing bi-colouration and map r.

In [94], boundary states were constructed by applying the method of [67], which was discussed in Subsection 2.3.4, directly to the (D_6, E_6) theory. Based on the coefficients

(2.249) of the boundary states (2.251) for Virasoro minimal cases, Gang and Yamaguchi introduce matrices

$$\Psi^{(r,s)}_{(a,b)} = \frac{{}^{(D_6)}\psi^{r}_{a}\; {}^{(E_6)}\psi^{s}_{b}}{\sqrt{\mathrm{S}^{(8)}_{1r}\mathrm{S}^{(10)}_{1s}}} \,, \tag{6.40}$$

where ${}^{(G)}\psi^{r}_{a}$ are eigenvectors of the adjacency matrix of the Dynkin diagram of G, and $\mathrm{S}^{(k)}_{ij}$ are elements of the modular S matrix of $\widehat{\mathrm{sl}}(2)_k$-WZW model, which is given by (2.250). Explicit expressions for ${}^{(D_6)}\psi^{r}_{a}$, ${}^{(E_6)}\psi^{s}_{b}$, and $\Psi^{(r,s)}_{(a,b)}$ can be found in Appendix D.3. These matrices have the property that under the Kac-symmetry

$$\Psi^{(r,s)}_{(a,b)} = \begin{cases} \Psi^{(10-r,12-s)}_{(a,b)} & \text{if} \quad (a,b) \in \mathcal{B}_e \quad \text{(same colouration)} \\ -\Psi^{(10-r,12-s)}_{(a,b)} & \text{if} \quad (a,b) \in \mathcal{B}_o \quad \text{(opposite colouration)} \end{cases} \tag{6.41}$$

Therefore, some care is needed when defining the boundary states.

Following [94], we define the boundary states $\|(a,b)_{\mathrm{NS}}\rangle\!\rangle$ using the matrices (6.40), but we take a slightly different choice

$$\|(a,b)_{\mathrm{NS}}\rangle\!\rangle = \sum_{\substack{r\in\{1,3,5,5',7,9\} \\ s\in\{1,7\}}} \Psi^{(r,s)}_{(a,b)} \, |h_{r,s}, +\rangle\!\rangle \,, \tag{6.42}$$

whereas in [94], the sum over s is taken $s \in \{1,\ 5\}$. The sums are over exactly the same representations but the choice of different representatives result in expressions which differ by a sign for $s = 7$ when the nodes are of opposite colour. For example, $(r,s) = (1,7)$ and $(9,5)$ denote the same representation but $\Psi^{(1,7)}_{(1,2)} = -\Psi^{(9,5)}_{(1,2)}$.

Our choice of representatives was motivated by the form of the (D_6, E_6) bulk partition function (6.3) and the fact that the E_6 invariant of the $\widehat{\mathrm{sl}}(2)_{10}$-WZW model has an extended symmetry algebra consisting of the representations $1 \oplus 7$ as in Table (2.3). Our choice seems natural when considering fusion rules of $\mathcal{SV}\mathrm{ir}$ at $c = \frac{7}{5}$, and we think it results in more natural expression for the final boundary states.

One consequence is that, unlike the situation in [94], our choice of representatives results in sets of boundary states that only differ by factors of $\sqrt{2}$,

$$\|(a,6)_{\mathrm{NS}}\rangle\!\rangle = \sqrt{2}\|(a,1)_{\mathrm{NS}}\rangle\!\rangle \quad \text{and} \quad \|(a,3)_{\mathrm{NS}}\rangle\!\rangle = \sqrt{2}\|(a,2)_{\mathrm{NS}}\rangle\!\rangle \,. \tag{6.43}$$

These may seem redundant but it will turn out to be helpful when we consider consistent descriptions of all the possible boundary states for the (D_6, E_6) theory.

In addition, we define the boundary states $\|(a,b)_{\widetilde{\mathrm{NS}}}\rangle\!\rangle$ in a slightly different way to [94]

$$\|(a,b)_{\widetilde{\mathrm{NS}}}\rangle\!\rangle = (-1)^F \|(a,b)_{\mathrm{NS}}\rangle\!\rangle \,. \tag{6.44}$$

These differ from the ones in [94] by an extra sign for each of the Ishibashi states corresponding to a fermionic chiral highest weight states $|h_{r,s}\rangle$ with

$$(r,s) \in \{(1,7),\ (3,1),\ (5,7),\ (5',7),\ (7,1),\ (9,7)\} \,. \tag{6.45}$$

In terms of the conformal weights, fermionic highest weights are

$$h = \frac{3}{2},\ \frac{7}{2},\ \frac{7}{10},\ \frac{57}{10},\ \frac{1}{10},\ \frac{1}{10}' \,. \tag{6.46}$$

This simplifies the identification of the known boundary states. Our choice of fermion parity assignment is explained in Appendix D.4.

6.2.1 Identifying Known Defects

Using the relations (6.43) and the definition (6.44), we can identify the boundary states corresponding to all the know defects in $SM(3,5)$. As shown in Table 6.2, these split into two sets: those with $\alpha\beta\gamma\delta = 1$ for which we need the highest weight state map ρ' supplemented by suitable descendant maps η_h, and those with $\alpha\beta\gamma\delta = -1$ for which we use ρ. These two sets cannot be defined at the same time as they use different embeddings. For example, we cannot describe the defects $D_{\mathbf{1}}$ and $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ as supersymmetric boundary conditions for the (D_6, E_6) theory at the same time.

	$\alpha\beta\gamma\delta{=}{-1},\ \alpha{=}\beta{=}\gamma{=}1,\ \delta{=}{-1},\ \text{map} = \rho_{+++-}$		$\alpha\beta\gamma\delta{=}1,\ \alpha{=}\beta{=}\gamma{=}\delta{=}1,\ \text{map} = \rho'_{++++}$		
	Defect	Boundary states	Defect	Boundary states	
a	$\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$	$\sqrt{2}\,\|(1,1)_{\rm NS}\rangle\!\rangle\,,\quad \|(1,6)_{\rm NS}\rangle\!\rangle$	$\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm R}\|$	$\sqrt{2}\,\|(1,6)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(1,1)_{\rm NS}\rangle\!\rangle$	a'
	$\|\varphi_{\rm NS}\rangle\!\rangle\langle\!\langle\varphi_{\rm NS}\|$	$\sqrt{2}\,\|(3,1)_{\rm NS}\rangle\!\rangle\,,\quad \|(3,6)_{\rm NS}\rangle\!\rangle$	$\|\varphi_{\rm NS}\rangle\!\rangle\langle\!\langle\varphi_{\rm R}\|$	$\sqrt{2}\,\|(3,6)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(3,1)_{\rm NS}\rangle\!\rangle$	
	$\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\varphi_{\rm NS}\|$	$\sqrt{2}\,\|(5,1)_{\rm NS}\rangle\!\rangle\,,\quad \|(5,6)_{\rm NS}\rangle\!\rangle$	$\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\varphi_{\rm R}\|$	$\sqrt{2}\,\|(5,6)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(5,1)_{\rm NS}\rangle\!\rangle$	
	$\|\varphi_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$	$\sqrt{2}\,\|(6,1)_{\rm NS}\rangle\!\rangle\,,\quad \|(6,6)_{\rm NS}\rangle\!\rangle$	$\|\varphi_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm R}\|$	$\sqrt{2}\,\|(6,6)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(6,1)_{\rm NS}\rangle\!\rangle$	
$\tilde{a}$	$(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F$	$\sqrt{2}\,\|(1,1)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle$	$\|\mathbf{1}_{\rm R}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$	$\sqrt{2}\,\|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(1,1)_{\widetilde{\rm NS}}\rangle\!\rangle$	$\tilde{a}'$
	$(-1)^F\|\varphi_{\rm NS}\rangle\!\rangle\langle\!\langle\varphi_{\rm NS}\|(-1)^F$	$\sqrt{2}\,\|(3,1)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(3,6)_{\widetilde{\rm NS}}\rangle\!\rangle$	$\|\varphi_{\rm R}\rangle\!\rangle\langle\!\langle\varphi_{\rm NS}\|$	$\sqrt{2}\,\|(3,6)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(3,1)_{\widetilde{\rm NS}}\rangle\!\rangle$	
	$(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\varphi_{\rm NS}\|(-1)^F$	$\sqrt{2}\,\|(5,1)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(5,6)_{\widetilde{\rm NS}}\rangle\!\rangle$	$\|\mathbf{1}_{\rm R}\rangle\!\rangle\langle\!\langle\varphi_{\rm NS}\|$	$\sqrt{2}\,\|(5,6)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(5,1)_{\widetilde{\rm NS}}\rangle\!\rangle$	
	$(-1)^F\|\varphi_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F$	$\sqrt{2}\,\|(6,1)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(6,6)_{\widetilde{\rm NS}}\rangle\!\rangle$	$\|\varphi_{\rm R}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$	$\sqrt{2}\,\|(6,6)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(6,1)_{\widetilde{\rm NS}}\rangle\!\rangle$	
b	$\sqrt{2}(-1)^F D_{\mathbf{1}}$	$\sqrt{2}\,\|(2,6)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(2,1)_{\rm NS}\rangle\!\rangle$	$D_{\mathbf{1}}$	$\sqrt{2}\,\|(2,1)_{\rm NS}\rangle\!\rangle\,,\quad \|(2,6)_{\rm NS}\rangle\!\rangle$	b'
	$\sqrt{2}(-1)^{\bar{F}} D_{\mathbf{1}}$	$\sqrt{2}\,\|(2,6)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(2,1)_{\widetilde{\rm NS}}\rangle\!\rangle$	$(-1)^{F+\bar{F}} D_{\mathbf{1}}$	$\sqrt{2}\,\|(2,1)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(2,6)_{\widetilde{\rm NS}}\rangle\!\rangle$	
	$\sqrt{2}(-1)^F D_{\varphi}$	$\sqrt{2}\,\|(4,6)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(4,1)_{\rm NS}\rangle\!\rangle$	D_{φ}	$\sqrt{2}\,\|(4,1)_{\rm NS}\rangle\!\rangle\,,\quad \|(4,6)_{\rm NS}\rangle\!\rangle$	
	$\sqrt{2}(-1)^{\bar{F}} D_{\varphi}$	$\sqrt{2}\,\|(4,6)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(4,1)_{\widetilde{\rm NS}}\rangle\!\rangle$	$(-1)^{F+\bar{F}} D_{\varphi}$	$\sqrt{2}\,\|(4,1)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(4,6)_{\widetilde{\rm NS}}\rangle\!\rangle$	

Table 6.2: Identifications of the boundary states corresponding to the known defects.

As an example, we present the overlaps of the boundary state corresponding to the identity defect,

$$\sqrt{2}\,\|(2,1)_{\rm NS}\rangle\!\rangle = \|(2,6)_{\rm NS}\rangle\!\rangle\ , \tag{6.47}$$

with the one representing the topological defect $(-1)^{F+\bar{F}}D_{\mathbf{1}}$,

$$\sqrt{2}\,\|(2,1)_{\widetilde{\rm NS}}\rangle\!\rangle = \|(2,6)_{\widetilde{\rm NS}}\rangle\!\rangle\ , \tag{6.48}$$

the one representing the factorising defect $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm R}\|$,

$$2\,\|(1,1)_{\rm NS}\rangle\!\rangle = \sqrt{2}\,\|(1,6)_{\rm NS}\rangle\!\rangle\ , \tag{6.49}$$

and the one representing another factorising defect $\|\mathbf{1}_{\rm R}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$,

$$2\,\|(1,1)_{\widetilde{\rm NS}}\rangle\!\rangle = \sqrt{2}\,\|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle \tag{6.50}$$

all of which have $\alpha\beta\gamma\delta = 1$. We have exactly the expected results:

$$\begin{aligned}\langle\!\langle(2,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(2,6)_{\rm NS}\rangle\!\rangle = {}& {}^{(10)}\chi^{\rm NS}_{0}(q)+{}^{(10)}\chi^{\rm NS}_{\frac{3}{2}}(q)+{}^{(10)}\chi^{\rm NS}_{\frac{7}{2}}(q)+{}^{(10)}\chi^{\rm NS}_{10}(q)\\ &+{}^{(10)}\chi^{\rm NS}_{\frac{1}{5}}(q)+{}^{(10)}\chi^{\rm NS}_{\frac{7}{10}}(q)+{}^{(10)}\chi^{\rm NS}_{\frac{6}{5}}(q)+{}^{(10)}\chi^{\rm NS}_{\frac{57}{10}}(q)\end{aligned}$$

$$= \left({}^{(3)}\chi_0(q)\right)^2 + \left({}^{(3)}\chi_{\frac{1}{10}}(q)\right)^2 , \tag{6.51}$$

$$\langle\!\langle (2,6)_{\rm NS} \| \tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)} \| (2,6)_{\widetilde{\rm NS}} \rangle\!\rangle = 2\left({}^{(10)}\chi^{\rm R}_{\frac{7}{8}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{39}{8}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{3}{40}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{83}{40}}(q)\right)$$
$$= 2\left({}^{(3)}\chi^{\rm R}_{\frac{3}{80}}(q)\right)^2 + 2\left({}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(q)\right)^2 , \tag{6.52}$$

$$2\,\langle\!\langle (2,6)_{\rm NS} \| \tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)} \| (1,1)_{\rm NS} \rangle\!\rangle = 2\left({}^{(10)}\chi^{\rm NS}_{\frac{21}{80}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{261}{80}}(q)\right)$$
$$= 2\;{}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(\sqrt{q}) , \tag{6.53}$$

$$2\,\langle\!\langle (2,6)_{\rm NS} \| \tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)} \| (1,1)_{\widetilde{\rm NS}} \rangle\!\rangle = 2\left({}^{(10)}\chi^{\rm R}_{\frac{21}{80}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{61}{80}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{181}{80}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{621}{80}}(q)\right)$$
$$= 2\;{}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(\sqrt{q}) , \tag{6.54}$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$ with $\tau = \frac{i}{L}$, and $L_n^{\rm tot}$ and $\bar{L}_n^{\rm tot}$ are the Virasoro generators for $c = \frac{7}{5}$. Note that the overlaps of $\|(2,6)_{\rm NS}\rangle\!\rangle$ with $2\,\|(1,1)_{\rm NS}\rangle\!\rangle$ and $2\,\|(1,1)_{\widetilde{\rm NS}}\rangle\!\rangle$ are the same, $2\;{}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(\sqrt{q})$, thanks to two different identities relating the characters of $SM(10,12)$ and $SM(3,5)$ that are given in Appendix D.1. This is a function of $\sqrt{q}$ since geometrically it corresponds to a strip of width $2L$, as shown in figure 6.2.

However, if we consider defects with different values of $\alpha\beta\gamma\delta$ we do not get sensible results. The overlap of the boundary state in the (D_6, E_6) theory corresponding to the identity defect with the boundary state corresponding to the factorising defect $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ will give the partition function on the strip of width $2L$ and boundary conditions $\mathbf{1}_{\rm NS}$ on both sides, that is

$$\mathrm{Tr}_{\mathcal{H}_{\rm NS}}\left(\tilde{q}^{L_0-\frac{7}{240}}\,\tilde{\bar{q}}^{\bar{L}_0-\frac{7}{240}}\;D_{\mathbf{1}}\;\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|\right) = {}^{(3)}\chi_0^{\rm NS}(\sqrt{q}) . \tag{6.55}$$

But ${}^{(3)}\chi_0^{\rm NS}(\sqrt{q}) = q^{-\frac{7}{480}}(1+q^{\frac{3}{4}}+q+q^{\frac{5}{4}}+q^{\frac{3}{2}}+\cdots)$ cannot be expressed as a sum of characters of the $c = \frac{7}{5}$ algebra, and therefore, it is not possible for the two defects $D_{\mathbf{1}}$ and $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ to be represented as boundary states for the (D_6, E_6) theory at the same time. If we look at Table 6.2, we see that $D_{\mathbf{1}}$ corresponds to $\|(2,6)_{\rm NS}\rangle\!\rangle$ defined with embedding ι_{++++} but $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ corresponds to $\|(1,6)_{\rm NS}\rangle\!\rangle$ with embedding ι_{+++-}, and so their overlap being calculated as

$$\langle\!\langle (2,6)_{\rm NS} \| \tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)} \| (1,6)_{\rm NS} \rangle\!\rangle = \sqrt{2}\;{}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(\sqrt{q}) \tag{6.56}$$

has nothing to do with the required quantity.

6.3 Identifying New Defects

In the preceding subsection, we have see that all the know defects in $SM(3,5)$ correspond to the boundary conditions in the (D_6, E_6) theory that are all labelled by the E_6 diagram nodes 1 $(= 5)$ and 6. If we instead use the nodes 2 $(= 4)$ and 3 of the E_6 diagram, we find new conformal defects that are neither topological nor factorising. This can be compared with the boundary conditions in the $\widehat{\rm sl}(2)_{10}$-WZW model with the E_6 invariant; it is known[67, 68] that the boundary conditions labelled by the nodes 2, 4, 6 breaks the $\widehat{\rm sp}(4)_1 \cong \widehat{\rm so}(5)_1$ symmetry. In [91], non-factorising and non-topological conformal defects between the Lee-Yang model and the Ising model are labelled by the E_6 diagram nodes 2, 4, and 6.

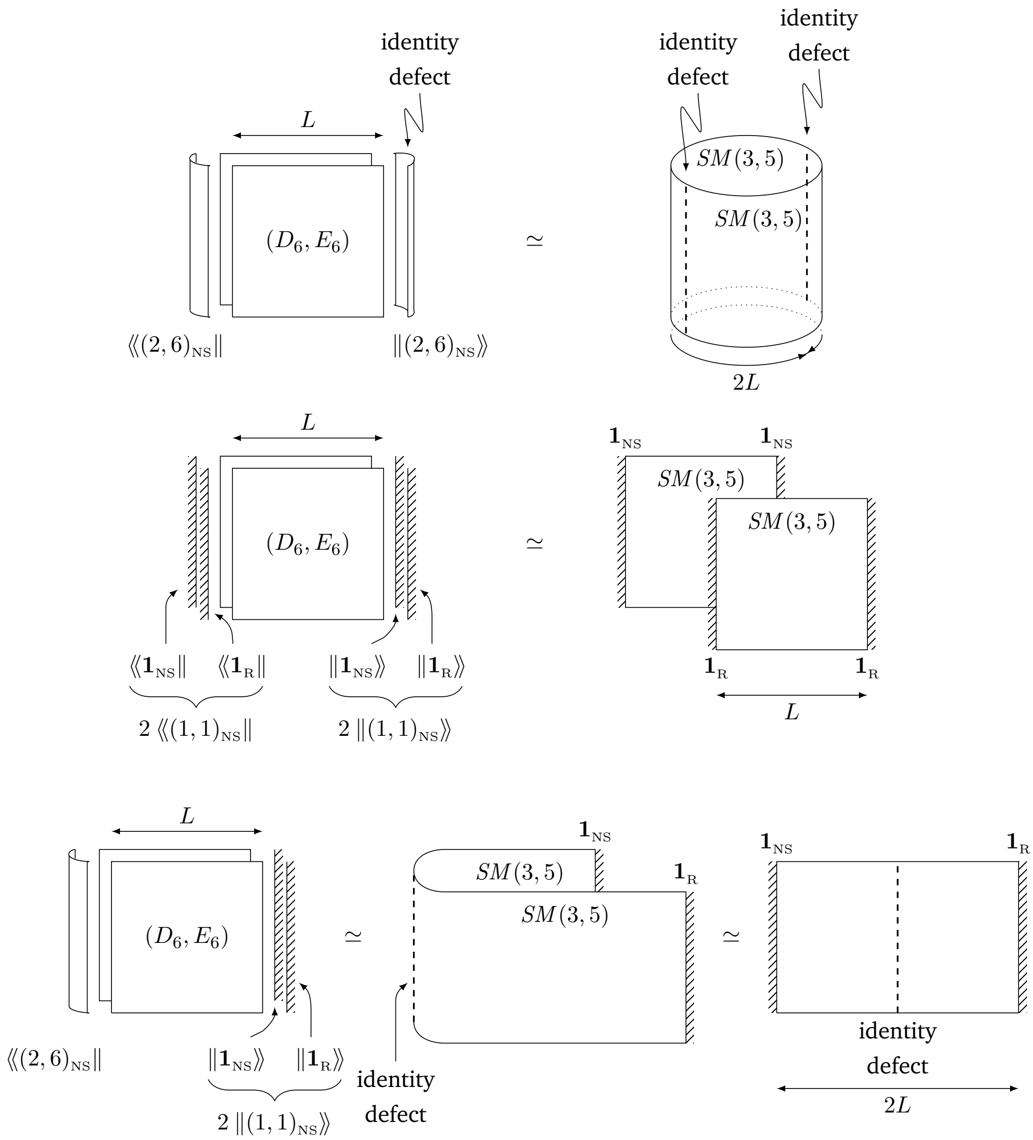

Figure 6.2: Different boundary conditions on (D_6, E_6) result in different geometrical set-ups for $SM(3,5)$.

Furthermore, in Table 6.2, all the factorising defects defects correspond to the D_6 diagram nodes 1, 3, 5, and 6 while all the topological defects are labelled by the nodes 2 and 4. This suggests that the boundary conditions labelled by the D_6 diagram nodes 1, 3, 5, and 6 are in the untwisted sector of $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$, and those with the nodes 2 and 4 are in the twisted sector. This observation agrees with a $\mathcal{W}(2,3)$ case (the three-states Potts model[62]) and with a $\mathcal{W}(2,2)$ case (the doubled theory of Lee-Yang[91]).

As we know the explicit expressions for the (D_6, E_6) boundary states, we can calculate their reflection and transmission coefficients as well as their entropies (defect g-values). For the $\mathcal{R}$ and $\mathcal{T}$ values of the defects, we can use the formula give in (3.11). In this case, the Virasoro highest weight state $|W\rangle$ can be written as

$$|W\rangle = \frac{1}{c}(L_{-2}^{(1)} - \bar{L}_{-2}^{(2)})(\bar{L}_{-2}^{(1)} - L_{-2}^{(2)})|0\rangle \ , \tag{6.57}$$

where $c = \frac{7}{10}$. Using the expansion of $|\frac{3}{2}, \epsilon\rangle\!\rangle$ given in (D.6), if the boundary state $\|D\rangle\!\rangle$ corresponding to a conformal defect D can be expressed as

$$\|D\rangle\!\rangle = A\,|0, \epsilon\rangle\!\rangle + B\,|\tfrac{3}{2}, \epsilon\rangle\!\rangle + \cdots \ , \tag{6.58}$$

then ω_D given in (3.11) can be written as

$$\omega_D = -\epsilon\, \eta_{\frac{3}{2}}\, \frac{B}{A}\ , \tag{6.59}$$

and the reflection and transmission coefficients of D are given by

$$\mathcal{R} = \frac{1}{2}(1 + \omega_D) \quad \text{and} \quad \mathcal{T} = \frac{1}{2}(1 - \omega_D)\ . \tag{6.60}$$

They do not depend on the highest weight embedding maps ρ_{+++-} nor ρ'_{++++} but depending on the signs of the boundary state gluing condition ϵ and embedding of the descendant state $\eta_{\frac{3}{2}}$, the values of $\mathcal{R}$ and $\mathcal{T}$ may be swapped.

The g-values of conformal defects can be obtained from corresponding boundary state coefficients; they are the coefficients of the Ishibashi state $|0, \epsilon\rangle\!\rangle$. From (6.40) and Appendix D.3, the g-values of (D_6, E_6) boundary states are given by

$$g\big(\|(a,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big) = \begin{cases} \dfrac{\mathrm{S}^{(8)}_{a1}}{\sqrt{\mathrm{S}^{(8)}_{1,1}}} = \sqrt{1 + \frac{1}{\sqrt{5}}}\,\sin\left(\frac{a\pi}{10}\right) & \text{for } a = 1,2,3,4 \\ \dfrac{\mathrm{S}^{(8)}_{5,1}}{2\sqrt{\mathrm{S}^{(8)}_{1,1}}} = \frac{1}{2}\sqrt{1 + \frac{1}{\sqrt{5}}} & \text{for } a = 5,6 \end{cases} \tag{6.61}$$

$$g\big(\|(a,2)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big) = \sqrt{2 + \sqrt{3}}\quad g\big(\|(a,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big)\ , \tag{6.62}$$

$$g\big(\|(a,3)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big) = (1 + \sqrt{3})\quad g\big(\|(a,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big)\ , \tag{6.63}$$

$$g\big(\|(a,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big) = \sqrt{2}\quad g\big(\|(a,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle\big)\ , \tag{6.64}$$

where the last two relations follow from (6.43). These values are independent of the embedding and choice of signs η_h.

Note that $\|D\rangle\!\rangle$ and $(-1)^F\|D\rangle\!\rangle$ have the same value of g and $\mathcal{T}$ as we have

$$(-1)^F\|D\rangle\!\rangle = A\,|0, -\epsilon\rangle\!\rangle - B\,|\tfrac{3}{2}, -\epsilon\rangle\!\rangle + \cdots \tag{6.65}$$

due to fermion parity assignment (6.46), and this results in the same value of ω_D. We find that the boundary states $\|(a,b)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ only take four different values for $\mathcal{T}$ as in Table 6.3, but a large range of g-values. We also list the g-values for the known topological and factorising defects in the (D_6, E_6) theory in the same table.

If the g-value of a boundary state cannot be expressed as a sum of the g-values of known topological and factorising defects, this boundary state must correspond to a "new" defect.

Again, these new defects fall into two sets: those defined from the boundary state using embedding ι_{++++} and map ρ', and those defined with embedding ι_{+++-} and map ρ. With each set, the boundary states satisfy Cardy's condition, that is, the overlaps of any two boundary states corresponding to the same embedding ρ, or ρ', are non-negative integer

$\mathcal{T}$	g		boundary states defined with ι_{+++-} and map ρ
1	$\sqrt{2}$	1.414..	$2\,\Vert(2,1)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(2,1)_{\widetilde{\rm NS}}\rangle\!\rangle$
	$\frac{1+\sqrt{5}}{\sqrt{2}}$	2.288...	$2\,\Vert(4,1)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(4,1)_{\widetilde{\rm NS}}\rangle\!\rangle$
0	$\left(\frac{5-\sqrt{5}}{10}\right)^{1/2}$	0.5257...	$\Vert(1,6)_{\rm NS}\rangle\!\rangle,\ \Vert(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle$
	$\left(\frac{5+\sqrt{5}}{10}\right)^{1/2}$	0.8506...	$\Vert(5,6)_{\rm NS}\rangle\!\rangle,\ \Vert(6,6)_{\rm NS}\rangle\!\rangle,\ \Vert(5,6)_{\widetilde{\rm NS}}\rangle\!\rangle,\ \Vert(6,6)_{\widetilde{\rm NS}}\rangle\!\rangle$
	$\left(\frac{5+2\sqrt{5}}{5}\right)^{1/2}$	1.3763...	$\Vert(3,6)_{\rm NS}\rangle\!\rangle,\ \Vert(3,6)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\frac{\sqrt{3}-1}{2}$	$\sqrt{2+\sqrt{3}}$	1.9318...	$\Vert(2,3)_{\rm NS}\rangle\!\rangle,\ \Vert(2,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
		3.1258...	$\Vert(4,3)_{\rm NS}\rangle\!\rangle,\ \Vert(4,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\frac{3-\sqrt{3}}{2}$		1.4363...	$2\,\Vert(1,2)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(1,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
		3.7603...	$2\,\Vert(3,2)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(3,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
		2.3240...	$2\,\Vert(5,2)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(6,2)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(5,2)_{\widetilde{\rm NS}}\rangle\!\rangle,\ 2\,\Vert(6,2)_{\widetilde{\rm NS}}\rangle\!\rangle$

$\mathcal{T}$	g		boundary states defined with ι_{++++} and map ρ'
1	1	1	$\Vert(2,6)_{\rm NS}\rangle\!\rangle,\ \Vert(2,6)_{\widetilde{\rm NS}}\rangle\!\rangle$
	$\frac{1+\sqrt{5}}{2}$	1.618...	$\Vert(4,6)_{\rm NS}\rangle\!\rangle,\ \Vert(4,6)_{\widetilde{\rm NS}}\rangle\!\rangle$
0	$\left(\frac{5-\sqrt{5}}{5}\right)^{1/2}$	0.7434...	$2\,\Vert(1,1)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(1,1)_{\widetilde{\rm NS}}\rangle\!\rangle$
	$\left(\frac{5+\sqrt{5}}{5}\right)^{1/2}$	1.2030...	$2\,\Vert(5,1)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(6,1)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(5,1)_{\widetilde{\rm NS}}\rangle\!\rangle,\ 2\,|(6,1)_{\widetilde{\rm NS}}\rangle\!\rangle$
	$\left(\frac{10-2\sqrt{5}}{5}\right)^{1/2}$	1.9465...	$2\,\Vert(3,1)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(3,1)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\frac{\sqrt{3}-1}{2}$	$1+\sqrt{3}$	2.732..	$2\,\Vert(2,2)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(2,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
		4.4205...	$2\,\Vert(4,2)_{\rm NS}\rangle\!\rangle,\ 2\,\Vert(4,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\frac{3-\sqrt{3}}{2}$		1.0156...	$\Vert(1,3)_{\rm NS}\rangle\!\rangle,\ \Vert(1,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
		2.6589...	$\Vert(3,3)_{\rm NS}\rangle\!\rangle,\ \Vert(3,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
		1.6433...	$\Vert(5,3)_{\rm NS}\rangle\!\rangle,\ \Vert(6,3)_{\widetilde{\rm NS}}\rangle\!\rangle$

Table 6.3: $\mathcal{T}$ and g-values for the (D_6, E_6) boundary states

combinations of characters of $SM(10,12)$. The overlaps of states corresponding to different maps do not satisfy Cardy's condition.

Further, the overlaps involving the known topological and factorising defects can be expressed in terms of the characters of $SM(3,5)$, but those involving the new defects can not.

As an example, we consider the overlaps of the boundary states representing the identity defect,

$$\sqrt{2}\,\Vert(2,1)_{\rm NS}\rangle\!\rangle = \Vert(2,6)_{\rm NS}\rangle\!\rangle\ , \tag{6.66}$$

with a new defect,

$$\sqrt{2}\,\Vert(1,2)_{\rm NS}\rangle\!\rangle = \Vert(1,3)_{\rm NS}\rangle\!\rangle\ , \tag{6.67}$$

and one representing $\Vert I_{\rm NS}\rangle\!\rangle\langle\!\langle I_{\rm NS}\Vert$,

$$\sqrt{2}\,\Vert(1,1)_{\rm NS}\rangle\!\rangle = \Vert(1,6)_{\rm NS}\rangle\!\rangle\ , \tag{6.68}$$

with another new defect

$$\sqrt{2}\,\Vert(2,2)_{\rm NS}\rangle\!\rangle = \Vert(2,3)_{\rm NS}\rangle\!\rangle\ . \tag{6.69}$$

We have

$$\begin{aligned}\langle\!\langle(2,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(2,6)_{\rm NS}\rangle\!\rangle &= {}^{(10)}\chi_0^{\rm NS}(q)+{}^{(10)}\chi_{\frac{3}{2}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{7}{2}}^{\rm NS}(q)+{}^{(10)}\chi_{10}^{\rm NS}(q)\\ &\quad+{}^{(10)}\chi_{\frac{1}{5}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{7}{10}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{6}{5}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{57}{10}}^{\rm NS}(q)\\ &= \left({}^{(3)}\chi_0^{\rm NS}(q)\right)^2+\left({}^{(3)}\chi_{\frac{1}{10}}^{\rm NS}(q)\right)^2 , \qquad (6.70)\\ \langle\!\langle(2,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(1,3)_{\rm NS}\rangle\!\rangle &= {}^{(10)}\chi_{\frac{1}{80}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{21}{80}}^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{323}{240}}^{\rm NS}(q)\\ &\quad+{}^{(10)}\chi_{\frac{261}{80}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{481}{80}}^{\rm NS}(q) , \qquad (6.71)\\ \langle\!\langle(1,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(1,6)_{\rm NS}\rangle\!\rangle &= {}^{(10)}\chi_0^{\rm NS}(q)+{}^{(10)}\chi_{\frac{3}{2}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{7}{2}}^{\rm NS}(q)+{}^{(10)}\chi_{10}^{\rm NS}(q)\\ &= \left({}^{(3)}\chi_0^{\rm NS}(q)\right)^2 , \qquad (6.72)\\ \langle\!\langle(1,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(2,3)_{\rm NS}\rangle\!\rangle &= {}^{(10)}\chi_{\frac{1}{80}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{21}{80}}^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{323}{240}}^{\rm NS}(q)\\ &\quad+{}^{(10)}\chi_{\frac{261}{80}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{481}{80}}^{\rm NS}(q) , \qquad (6.73)\end{aligned}$$

where $q=e^{2\pi i\tau}$ and $\tilde{q}=e^{-2\pi i/\tau}$ with $\tau=\frac{i}{L}$, and $L_n^{\rm tot}$ and $\bar{L}_n^{\rm tot}$ are the Virasoro generators for $c=\frac{7}{5}$.

Since $h_{2,2}^{(10)}=\frac{1}{80}\neq h_{r,s}^{(3)}+h_{r',s'}^{(3)}$ for any pair of Kac labels (r,s) and (r',s') in $SM(3,5)$, the overlap

$$\langle\!\langle(2,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(1,3)_{\rm NS}\rangle\!\rangle \qquad (6.74)$$

cannot be expressed as a sum of products of characters ${}^{(3)}\chi_{r,s}(q)\;{}^{(3)}\chi_{r',s'}(q)$. In addition, since $h_{2,2}^{(10)}-\frac{7}{120}=-\frac{11}{240}\neq\frac{1}{2}(h_{r,s}^{(3)}-\frac{7}{240})$ for any (r,s) in $SM(3,5)$, it cannot be expressed as a sum of characters ${}^{(3)}\chi_{r,s}(\sqrt{q})$.

Note that

$$\langle\!\langle(2,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(1,3)_{\rm NS}\rangle\!\rangle=\langle\!\langle(1,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(2,3)_{\rm NS}\rangle\!\rangle , \qquad (6.75)$$

which suggest that these overlaps are related by the insertion of a topological defect in the doubled model labelled by the Dynkin nodes $(2,1)$, which we will return to later.

For reference, we give the overlaps of the new boundary states with themselves to show that they satisfy Cardy's condition, but also cannot be expressed in terms of characters of $SM(3,5)$:

$$\begin{aligned}&\langle\!\langle(1,3)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(1,3)_{\rm NS}\rangle\!\rangle\\ &\quad= {}^{(10)}\chi_0^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{1}{3}}^{\rm NS}(q)+3\,{}^{(10)}\chi_{\frac{3}{2}}^{\rm NS}(q)+3\,{}^{(10)}\chi_{\frac{7}{2}}^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{19}{3}}^{\rm NS}(q)+{}^{(10)}\chi_{10}^{\rm NS}(q) , \qquad (6.76)\end{aligned}$$

$$\begin{aligned}&\langle\!\langle(2,3)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(2,3)_{\rm NS}\rangle\!\rangle\\ &\quad= {}^{(10)}\chi_0^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{1}{3}}^{\rm NS}(q)+3\,{}^{(10)}\chi_{\frac{1}{5}}^{\rm NS}(q)+3\,{}^{(10)}\chi_{\frac{1}{7}}^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{19}{3}}^{\rm NS}(q)+{}^{(10)}\chi_{10}^{\rm NS}(q)\\ &\qquad+{}^{(10)}\chi_{\frac{7}{10}}^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{1}{30}}^{\rm NS}(q)+3\,{}^{(10)}\chi_{\frac{1}{5}}^{\rm NS}(q)+3\,{}^{(10)}\chi_{\frac{6}{5}}^{\rm NS}(q)+2\,{}^{(10)}\chi_{\frac{91}{30}}^{\rm NS}(q)+{}^{(10)}\chi_{\frac{57}{10}}^{\rm NS}(q) . \qquad (6.77)\end{aligned}$$

6.3.1 New Factorising Defects in $SM(3,5)$

While the boundary state $\|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle$ can be identified as the defect

$$(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F\ , \tag{6.78}$$

this is not actually the product of two boundary states in $SM(3,5)$. The state $(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle$ does not satisfy Cardy's constraint. For example, its overlap with $\|\mathbf{1}_{\rm NS}\rangle\!\rangle$ is not an integer combination of characters in the crossed channel:

$$\langle\!\langle\mathbf{1}_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{7}{120}\right)}(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle=\sqrt{2}\ {}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(q)\ . \tag{6.79}$$

The defect $(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F$ does however satisfy the constraint. For example

$$\begin{aligned}&\langle\!\langle(1,6)_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60}\right)}\|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle\\&=\langle\!\langle\mathbf{1}_{\rm NS}\|\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{7}{120}\right)}(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F\tilde{q}^{\frac{1}{2}\left(L_0+\bar{L}_0-\frac{7}{120}\right)}\|\mathbf{1}_{\rm NS}\rangle\!\rangle=2\left({}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(q)\right)^2\ .\end{aligned} \tag{6.80}$$

Conversely, the factorising defect $\|\mathbf{1}_{\rm R}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm R}\|$ does not arise in the tables 6.2. The resolution seems to be that these factorising defects are not fundamental and instead we have

$$\|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle\simeq\ (-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F \tag{6.81}$$

$$2\|(1,6)_{\widetilde{\rm NS}}\rangle\!\rangle\simeq 2(-1)^F\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|(-1)^F=\|\mathbf{1}_{\rm R}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm R}\|\ . \tag{6.82}$$

This illustrates the possibility that each known factorising and topological defect in $SM(3,5)$ gives rise to a superconformal boundary state in the (D_6,E_6) theory, but the converse need not to be true.

6.3.2 Fundamental Defects

All the $SM(3,5)$ boundary states given in (5.98) can be considered as being generated from the fundamental boundary state $\|\mathbf{1}_{\rm NS}\rangle\!\rangle$ by the action of topological defects. In this regard, we can view the factorising defect $\|\mathbf{1}_{\rm NS}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\rm NS}\|$ and the topological defect $D_{\mathbf{1}}$ as fundamental, and the other known defects and the corresponding boundary states in Table 6.2 can be regarded as the result of left and/or right action of topological defects on them.

We can conjecture the same structure holds for the sector corresponding to new defects. We take

$$\mathcal{D}_f:=\rho'_{++++}(|(1,3)_{\rm NS}\rangle\!\rangle)=\rho'_{++++}(\sqrt{2}\|(1,2)_{\rm NS}\rangle\!\rangle) \tag{6.83}$$

and denote the action of $SM(3,5)$ topological defects on this defect as

$$\mathcal{D}_{\varphi f\varphi}:=D_\varphi\,\mathcal{D}_f\,D_\varphi\ ,\quad \mathcal{D}_{f\varphi}:=\mathcal{D}_f\,D_\varphi\ ,\quad \mathcal{D}_{\varphi f}:=D_\varphi\,\mathcal{D}_f\ . \tag{6.84}$$

In addition, we define

$$\mathcal{D}_t:=\rho_{+++-}(\|(2,3)_{\rm NS}\rangle\!\rangle)=\rho_{+++-}(\sqrt{2}\|(2,2)_{\rm NS}\rangle\!\rangle)\ . \tag{6.85}$$

The other non-factorising and non-topological defects can be viewed as the result of their fusion with topological defects in $SM(3,5)$ as summarised in Table 6.4.

$\alpha\beta\gamma\delta{=}1,\ \alpha{=}\beta{=}\gamma{=}\delta{=}1,$ map = ρ'_{++++}	
Defect	Boundary states
$\mathcal{D}_f$	$\sqrt{2}\,\|(1,2)_{\rm NS}\rangle\!\rangle\,,\quad \|(1,3)_{\rm NS}\rangle\!\rangle$
$\mathcal{D}_{\varphi f\varphi}$	$\sqrt{2}\,\|(3,2)_{\rm NS}\rangle\!\rangle\,,\quad \|(3,3)_{\rm NS}\rangle\!\rangle$
$\mathcal{D}_{f\varphi}$	$\sqrt{2}\,\|(5,2)_{\rm NS}\rangle\!\rangle\,,\quad \|(5,3)_{\rm NS}\rangle\!\rangle$
$\mathcal{D}_{\varphi f}$	$\sqrt{2}\,\|(6,2)_{\rm NS}\rangle\!\rangle\,,\quad \|(6,3)_{\rm NS}\rangle\!\rangle$
$(-1)^F\mathcal{D}_f(-1)^F$	$\sqrt{2}\,\|(1,2)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(1,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
$(-1)^F\mathcal{D}_{\varphi f\varphi}(-1)^F$	$\sqrt{2}\,\|(3,2)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(3,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
$(-1)^F\mathcal{D}_{f\varphi}(-1)^F$	$\sqrt{2}\,\|(5,2)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(5,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
$(-1)^F\mathcal{D}_{\varphi f}(-1)^F$	$\sqrt{2}\,\|(6,2)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(6,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\sqrt{2}(-1)^F\mathcal{D}_t=\mathcal{D}_t\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(2,3)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(2,2)_{\rm NS}\rangle\!\rangle$
$\sqrt{2}(-1)^{\bar F}\mathcal{D}_t=\mathcal{D}_t\sqrt{2}(-1)^{\bar F}$	$\sqrt{2}\,\|(2,3)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(2,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\sqrt{2}(-1)^F D_\varphi\mathcal{D}_t=\mathcal{D}_t D_\varphi\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(4,3)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(4,2)_{\rm NS}\rangle\!\rangle$
$\sqrt{2}(-1)^{\bar F} D_\varphi\mathcal{D}_t=\mathcal{D}_t D_\varphi\sqrt{2}(-1)^{\bar F}$	$\sqrt{2}\,\|(4,3)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(4,2)_{\widetilde{\rm NS}}\rangle\!\rangle$

$\alpha\beta\gamma\delta{=}{-1},\ \alpha{=}\beta{=}\gamma{=}1,\ \delta{=}{-1},$ map = ρ_{+++-}	
Defect	Boundary states
$\mathcal{D}_f\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(1,3)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(1,2)_{\rm NS}\rangle\!\rangle$
$\mathcal{D}_{\varphi f\varphi}\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(3,3)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(3,2)_{\rm NS}\rangle\!\rangle$
$\mathcal{D}_{f\varphi}\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(5,3)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(5,2)_{\rm NS}\rangle\!\rangle$
$\mathcal{D}_{\varphi f}\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(6,3)_{\rm NS}\rangle\!\rangle\,,\,2\,\|(6,2)_{\rm NS}\rangle\!\rangle$
$\sqrt{2}(-1)^F\mathcal{D}_f$	$\sqrt{2}\,\|(1,3)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(1,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\sqrt{2}(-1)^F\mathcal{D}_{\varphi f\varphi}$	$\sqrt{2}\,\|(3,3)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(3,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\sqrt{2}(-1)^F\mathcal{D}_{f\varphi}$	$\sqrt{2}\,\|(5,3)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(5,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\sqrt{2}(-1)^F\mathcal{D}_{\varphi f}$	$\sqrt{2}\,\|(6,3)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\,2\,\|(6,2)_{\widetilde{\rm NS}}\rangle\!\rangle$
$\mathcal{D}_t$	$\sqrt{2}\,\|(2,2)_{\rm NS}\rangle\!\rangle\,,\quad \|(2,3)_{\rm NS}\rangle\!\rangle$
$(-1)^{F+\bar F}\mathcal{D}_t=\mathcal{D}_t(-1)^{F+\bar F}$	$\sqrt{2}\,\|(2,2)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(2,3)_{\widetilde{\rm NS}}\rangle\!\rangle$
$D_\varphi\mathcal{D}_t=\mathcal{D}_t D_\varphi$	$\sqrt{2}\,\|(4,2)_{\rm NS}\rangle\!\rangle\,,\quad \|(4,3)_{\rm NS}\rangle\!\rangle$
$(-1)^{F+\bar F}D_\varphi\mathcal{D}_t=\mathcal{D}_t D_\varphi(-1)^{F+\bar F}$	$\sqrt{2}\,\|(4,2)_{\widetilde{\rm NS}}\rangle\!\rangle\,,\quad \|(4,3)_{\widetilde{\rm NS}}\rangle\!\rangle$

Table 6.4: Structure of new defects.

For one of the fundamental defect $\mathcal{D}_t$, the left and right action of a topological defect D_a coincides

$$D_a\,\mathcal{D}_t=\mathcal{D}_t\,D_a\,. \tag{6.86}$$

This does not, however, imply that the defect $\mathcal{D}_t$ is topological.

One of the simplest checks is to compare the ratios of boundary state coefficients since the action of topological defects is multiplicative in terms of the coefficients. Indeed we have

$$\Psi^{(r,s)}_{(3,6)}/\Psi^{(r,s)}_{(1,6)}=\Psi^{(r,s)}_{(3,3)}/\Psi^{(r,s)}_{(1,3)}\,,\quad \Psi^{(r,s)}_{(4,6)}/\Psi^{(r,s)}_{(2,6)}=\Psi^{(r,s)}_{(4,3)}/\Psi^{(r,s)}_{(2,3)}\,,$$
$$\Psi^{(r,s)}_{(5,6)}/\Psi^{(r,s)}_{(1,6)}=\Psi^{(r,s)}_{(5,3)}/\Psi^{(r,s)}_{(1,3)}\,,\quad \Psi^{(r,s)}_{(1,1)}/\Psi^{(r,s)}_{(2,6)}=\Psi^{(r,s)}_{(1,2)}/\Psi^{(r,s)}_{(2,3)}\,,$$

$$\begin{aligned} &\Psi^{(r,s)}_{(6,6)}/\Psi^{(r,s)}_{(1,6)} = \Psi^{(r,s)}_{(6,3)}/\Psi^{(r,s)}_{(1,3)} \ , \quad \Psi^{(r,s)}_{(3,1)}/\Psi^{(r,s)}_{(2,6)} = \Psi^{(r,s)}_{(3,2)}/\Psi^{(r,s)}_{(2,3)} \ , \\ &\Psi^{(r,s)}_{(2,1)}/\Psi^{(r,s)}_{(1,6)} = \Psi^{(r,s)}_{(2,2)}/\Psi^{(r,s)}_{(1,3)} \ , \quad \Psi^{(r,s)}_{(5,1)}/\Psi^{(r,s)}_{(2,6)} = \Psi^{(r,s)}_{(5,2)}/\Psi^{(r,s)}_{(2,3)} \ , \\ &\Psi^{(r,s)}_{(4,1)}/\Psi^{(r,s)}_{(1,6)} = \Psi^{(r,s)}_{(4,2)}/\Psi^{(r,s)}_{(1,3)} \ , \quad \Psi^{(r,s)}_{(6,1)}/\Psi^{(r,s)}_{(2,6)} = \Psi^{(r,s)}_{(6,2)}/\Psi^{(r,s)}_{(2,3)} \ . \end{aligned} \tag{6.87}$$

6.4 Super W-Algebra Boundary States

From the bulk partition function (6.1), we see that the $c = \frac{7}{5}$ $\mathcal{SV}\mathrm{ir}$ representations with $h = 0,\ \frac{3}{2},\ \frac{7}{2},\ 10$ form the vacuum sector of the extended superconformal algebra for which the (D_6, E_6) partition function of $\mathcal{SV}\mathrm{ir}$ can be regarded as diagonal. At first, this extended superconformal algebra seems to have three extra generators of weight $\frac{3}{2}$, $\frac{7}{2}$, and 10 but we will argue this is just $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$.

At $c = \frac{7}{5}$, there are four possible modular invariant partition functions[18] for $\mathcal{SV}\mathrm{ir}$. Besides the diagonal one, (A_9, A_{11}), there are (D_6, A_{11}), (A_9, E_6) and (D_6, E_6). As the $h = 10$ superprimary field is a simple current, the corresponding extension yields[40] the (D_6, A_{11}) invariant and $\mathcal{SW}(\frac{3}{2}, 10)$. As in Table 6.5, the exceptional invariant (A_9, E_6) contains the field of conformal dimension $\frac{7}{2}$, and it is associated[40] with $\mathcal{SW}(\frac{3}{2}, \frac{7}{2})$.

Invariant	Algebra	Vacuum sector
(A_9, A_{11})	$\mathcal{SV}\mathrm{ir}$	χ_0^{NS}
(D_6, A_{11})	$\mathcal{SW}(\frac{3}{2}, 10)$	$\chi_0^{\mathrm{NS}} + \chi_{10}^{\mathrm{NS}}$
(A_9, E_6)	$\mathcal{SW}(\frac{3}{2}, \frac{7}{2})$	$\chi_0^{\mathrm{NS}} + \chi_{\frac{7}{2}}^{\mathrm{NS}}$
(D_6, E_6)	$\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$	$\chi_0^{\mathrm{NS}} + \chi_{\frac{3}{2}}^{\mathrm{NS}} + \chi_{\frac{7}{2}}^{\mathrm{NS}} + \chi_{10}^{\mathrm{NS}}$

Table 6.5: Modular invariant partition functions and extended superconformal algebras at $c = \frac{7}{5}$.

As we have discussed in Subsection 2.2.2, $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ at $c = \frac{7}{5}$ can be expressed as the direct sum of two copies of $\mathcal{SV}\mathrm{ir}$ at $c = \frac{7}{10}$. We define

$$\begin{aligned} G^{\mathrm{tot}}(z) &= \alpha G^{(1)}(z) + \beta G^{(2)}(z) \ , \quad W(z) = \alpha G^{(1)}(z) - \beta G^{(2)}(z) \\ T^{\mathrm{tot}}(z) &= T^{(1)}(z) + T^{(2)}(z) \ , \qquad U(z) = T^{(1)}(z) - T^{(2)}(z) \ , \end{aligned} \tag{6.88}$$

where the generators with superscripts are those of $c = \frac{7}{10}$. The chiral highest weight states $|\frac{3}{2}\rangle$ and $|\frac{7}{2}\rangle$ of $SM(10, 12)$ can be decomposed as

$$|\tfrac{3}{2}\rangle = \sqrt{\tfrac{1}{4c/3}}\left(\alpha G^{(1)}_{-\frac{3}{2}} - \beta G^{(2)}_{-\frac{3}{2}}\right)|0\rangle = \sqrt{\tfrac{15}{14}}\, W_{-\frac{3}{2}}|0\rangle \quad \text{and} \tag{6.89}$$

$$\begin{aligned} &|\tfrac{7}{2}\rangle \\ &= \tfrac{10}{7}\sqrt{\tfrac{6}{323}}\left(\alpha(L^{(1)}_{-2}G^{(1)}_{-\frac{3}{2}} - \tfrac{3}{4}G^{(1)}_{-\frac{7}{2}}) + \beta(L^{(2)}_{-2}G^{(2)}_{-\frac{3}{2}} - \tfrac{3}{4}G^{(2)}_{-\frac{7}{2}}) - \tfrac{17}{2}(\beta L^{(1)}_{-2}G^{(2)}_{-\frac{3}{2}} + \alpha L^{(2)}_{-2}G^{(1)}_{-\frac{3}{2}})\right)|0\rangle \\ &\qquad = \tfrac{5}{7}\sqrt{\tfrac{57}{34}}\left(U_{-2}W_{-\frac{3}{2}} - \tfrac{15}{19}L^{\mathrm{tot}}_{-2}G^{\mathrm{tot}}_{-\frac{3}{2}} - \tfrac{3}{19}G^{\mathrm{tot}}_{-\frac{7}{2}}\right)|0\rangle \ , \end{aligned} \tag{6.90}$$

where $c = \frac{7}{10}$. The expansion of $|10\rangle$ is too lengthy to present, and it is not unique due to null states. We find that the states $|\frac{3}{2}\rangle$ and $|10\rangle$ change signs if we interchange $G^{(1)}$ and

$G^{(2)}$ while $|\frac{7}{2}\rangle$ does not. This implies the automorphism of $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ given by $W\mapsto -W$ and $U\mapsto -U$ induces $|\frac{3}{2}\rangle\mapsto -|\frac{3}{2}\rangle$ and $|10\rangle\mapsto -|10\rangle$ on the vacuum representation. In [39], it is shown that $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ contains $\mathcal{SW}(\frac{3}{2},\frac{7}{2})$ as a subalgebra at $c=\frac{7}{5}$. Furthermore, the character identities between $SM(10,12)$ and $SM(3,5)$ suggest the (D_6,E_6) invariant corresponds to $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$.

We define the untwisted Ishibashi states corresponding to $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ as

$$|0,\epsilon\rangle\!\rangle^{\mathcal{W}} = |0,\epsilon\rangle\!\rangle + |\tfrac{3}{2},\epsilon\rangle\!\rangle + |\tfrac{7}{2},\epsilon\rangle\!\rangle + |10,\epsilon\rangle\!\rangle \ , \tag{6.91}$$

$$|\tfrac{1}{5},\epsilon\rangle\!\rangle^{\mathcal{W}} = |\tfrac{1}{5},\epsilon\rangle\!\rangle + |\tfrac{7}{10},\epsilon\rangle\!\rangle + |\tfrac{6}{5},\epsilon\rangle\!\rangle + |\tfrac{57}{10},\epsilon\rangle\!\rangle \ , \tag{6.92}$$

$$|\tfrac{1}{10},\epsilon\rangle\!\rangle^{\mathcal{W}} = |\tfrac{1}{10},\epsilon\rangle\!\rangle + |\tfrac{13}{5},\epsilon\rangle\!\rangle \ , \tag{6.93}$$

$$|\tfrac{1}{10}',\epsilon\rangle\!\rangle^{\mathcal{W}} = |\tfrac{1}{10}',\epsilon\rangle\!\rangle + |\tfrac{13}{5}',\epsilon\rangle\!\rangle \ . \tag{6.94}$$

We can also define the twisted Ishibashi states corresponding to the automorphism $W\mapsto -W$ and $U\mapsto -U$ as

$$|0,\epsilon\rangle\!\rangle^{\mathcal{W}}_T = \ |0,\epsilon\rangle\!\rangle - |\tfrac{3}{2},\epsilon\rangle\!\rangle + |\tfrac{7}{2},\epsilon\rangle\!\rangle - |10,\epsilon\rangle\!\rangle \quad \text{and} \tag{6.95}$$

$$|\tfrac{1}{5},\epsilon\rangle\!\rangle^{\mathcal{W}}_T = -|\tfrac{1}{5},\epsilon\rangle\!\rangle + |\tfrac{7}{10},\epsilon\rangle\!\rangle + |\tfrac{6}{5},\epsilon\rangle\!\rangle - |\tfrac{57}{10},\epsilon\rangle\!\rangle \ . \tag{6.96}$$

They satisfy the Ishibashi conditions

$$(G^{\rm tot}_n + i\epsilon\bar{G}^{\rm tot}_{-n})\,|h,\epsilon\rangle\!\rangle^{\mathcal{W}}_{\Omega} = 0 \quad \text{and} \quad \big(W_n + i\epsilon\xi_\Omega\overline{W}_{-n}\big)\,|h,\epsilon\rangle\!\rangle^{\mathcal{W}}_{\Omega} = 0 \ , \tag{6.97}$$

where $\xi_\Omega = 1$ for $\Omega = \mathrm{id}$ and $\xi_\Omega=-1$ for $\Omega = T$.

Using the extended modular S matrix of $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ at $c=\frac{7}{5}$ defined in Appendix D.6, we define the untwisted $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ boundary states as

$$\|a,\epsilon\rangle\!\rangle = \sum_{a'\in\mathcal{I}^{\rm ext}_{\rm NS}} \frac{\mathcal{S}^{[\rm NS,NS]}_{aa'}}{\sqrt{\mathcal{S}^{[\rm NS,NS]}_{1a'}}}\,|h_{a'},\epsilon\rangle\!\rangle^{\mathcal{W}} \ , \tag{6.98}$$

where $a\in\mathcal{I}^{\rm ext}_{\rm NS}$ and $\mathcal{I}^{\rm ext}_{\rm NS} = \{1,\,3,\,5,\,5'\}$. The conformal weights of the $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ representations are given by $h_1=0$, $h_3=\frac{1}{5}$, $h_5=\frac{1}{10}$, and $h_{5'}=\frac{1}{10}'$. For the Neveu–Schwarz twisted sector labels[1] $a=2$ and $a=4$, we can write the twisted boundary states as

$$\|a,\epsilon\rangle\!\rangle_T = \sum_{a'\in\{1,3\}} \frac{\mathcal{S}^{[\rm NS,NS]}_{aa'}}{\sqrt{\mathcal{S}^{[\rm NS,NS]}_{1a'}}}\,|h_{a'},\epsilon\rangle\!\rangle^{\mathcal{W}}_T \ . \tag{6.99}$$

Note that $\mathcal{S}^{[\rm NS,NS]}_{aa'}$ vanishes when $a\in\{2,\,4\}$ and $a'\in\{5,\,5'\}$.

We can also use the Ramond labels $a\in\mathcal{I}^{\rm ext}_{\rm R} = \{1\pm,\,3\pm,\,5\pm,\,5'\pm\}$ and define

$$\|a,\epsilon\rangle\!\rangle = \sum_{a'\in\mathcal{I}^{\rm ext}_{\rm NS}} \frac{\mathcal{S}^{[\rm R,NS]}_{aa'}}{\sqrt{\mathcal{S}^{[\rm NS,NS]}_{1a'}}}\,|h_{a'},\epsilon\rangle\!\rangle^{\mathcal{W}} \ . \tag{6.100}$$

By introducing the Ramond twisted sector labels $a\in\{2\pm,\,4\pm\}$, the twisted boundary states are given by

$$\|a,\epsilon\rangle\!\rangle_T = \sum_{a'\in\{1,3\}} \frac{\mathcal{S}^{[\rm R,NS]}_{aa'}}{\sqrt{\mathcal{S}^{[\rm NS,NS]}_{1a'}}}\,|h_{a'},\epsilon\rangle\!\rangle^{\mathcal{W}}_T \ . \tag{6.101}$$

1. c.f. Virasoro D-type boundary states discussed e.g. in [62] and [67].

If we ignore the signs in the Ramond labels, these modular S matrices are related by

$$\frac{1}{\sqrt{2}}\mathcal{S}^{[\mathrm{NS},\mathrm{NS}]}_{aa'} = \mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{aa'} \ . \tag{6.102}$$

In addition, two boundary states $\|a+,\epsilon\rangle\!\rangle$ and $\|a-,\epsilon\rangle\!\rangle$ with the Ramond labels $a\pm$ are the same. We shall see this degeneracy has nice interpretation in terms of the (D_6, E_6) boundary states discussed previously.

By considering $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$, we have obtained 6 boundary states labelled by the extended algebra representations in the Neveu–Schwarz sector, and 12 boundary states labelled by $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ representations in the Ramond sector of which 6 are distinct. By comparing the coefficients of Ishibashi states, we can identify them as the (D_6, E_6) boundary conditions labelled by the nodes of the Dynkin diagrams. We summarise the relation in Table 6.6. We can compare this with the identification of the (D_6, E_6) boundary states given in Table 6.2. As in the Virasoro case, all the boundary states that preserve $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ correspond to the factorising and topological defects. We also see that the identification of boundary states $\|(a,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle = \|(a,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ comes from degeneracy of $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ Ramond representations $a\pm$ due to the way the (D_6, E_6) partition function is constructed. Note that the extended S matrix $\mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{aa'}$ is constructed using the modified Ramond characters, and this explains why we had to normalise the (D_6, E_6) boundary states as in Table 6.2.

$\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ states		(D_6, E_6) states
$\|1,\epsilon\rangle\!\rangle$	=	$\|(1,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|2,\epsilon\rangle\!\rangle_T$	=	$\|(2,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|3,\epsilon\rangle\!\rangle$	=	$\|(3,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|4,\epsilon\rangle\!\rangle_T$	=	$\|(4,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|5,\epsilon\rangle\!\rangle$	=	$\|(5,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|5',\epsilon\rangle\!\rangle$	=	$\|(6,6)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$

$\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ states		(D_6, E_6) states
$\|1\pm,\epsilon\rangle\!\rangle$	=	$\|(1,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ or $\|(1,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|2\pm,\epsilon\rangle\!\rangle_T$	=	$\|(2,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ or $\|(2,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|3\pm,\epsilon\rangle\!\rangle$	=	$\|(3,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ or $\|(3,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|4\pm,\epsilon\rangle\!\rangle_T$	=	$\|(4,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ or $\|(4,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|5\pm,\epsilon\rangle\!\rangle$	=	$\|(5,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ or $\|(5,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$
$\|5'\pm,\epsilon\rangle\!\rangle$	=	$\|(6,1)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$ or $\|(6,5)_{\mathrm{NS}/\widetilde{\mathrm{NS}}}\rangle\!\rangle$

Table 6.6: Relation between $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ and (D_6, E_6) boundary states.

6.5 (D_6, E_6) Fusion Rules

As in the Virasoro and $\widehat{\mathrm{sl}}(2)_k$-WZW model cases, it would be nice to have certain fusion rules from which we can obtain the characters appearing in the overlaps of the (D_6, E_6) boundary states. The obvious starting point is the graph fusion algebras of the D_6 and E_6 diagrams that are summarised in Appendix D.5.

The first obstacle is that, as we saw in Table 6.2, specifying a pair of D_6 and E_6 diagram nodes together with a choice of gluing condition does not determine the corresponding defect uniquely but we also need a normalisation of the boundary state which in turn specifies the embedding. In Table 6.2, there are 12 distinct pairs of diagram nodes with various normalisations and gluing conditions; the same set of nodes appear in the sectors labelled by a or b, and dashes and tildes represent different embeddings and gluing conditions. It is possible to pick representatives such that the E_6 diagram nodes 1 and 6 specify the gluing conditions. As discussed in Appendix D.5, the E_6 diagram node 1 is associated to the representation $1 \oplus 7$ and the node 6 carries $4 \oplus 8$ of $\widehat{\mathrm{sl}}(2)_{10}$. Looking at Figure D.1, which is the D_6 and E_6 diagrams with the corresponding $\widehat{\mathrm{sl}}(2)_8$ and $\widehat{\mathrm{sl}}(2)_{10}$ representations on each node, we can associate[2] $SM(10,12)$ Kac labels to each pair of the diagram nodes. In fact, the even nodes specified by (6.38) are associated with the NS representations, and the odd nodes (6.39) correspond to the Ramond representations. From this observation, we take

$$\begin{aligned} &\|(a,b)_{\mathrm{NS}}\rangle\!\rangle \quad \text{if} \quad (a,b) \in \mathcal{B}_e \ , \quad \text{and} \\ &\|(a,b)_{\widetilde{\mathrm{NS}}}\rangle\!\rangle \quad \text{if} \quad (a,b) \in \mathcal{B}_o \end{aligned} \tag{6.104}$$

as representatives.

Now, we can use the graph fusion algebras of D_6 and E_6 to calculate overlaps. As we have the identifications $1 \sim 5$ and $2 \sim 4$ of the E_6 diagram nodes, we modified the E_6 graph fusion algebra

$$\begin{aligned} &(2) \otimes (2) = (1) \oplus (3) \ , \quad && (3) \otimes (3) = (1) \oplus 2(3) \\ &(2) \otimes (3) = (2) \oplus (6) \ , \quad && (3) \otimes (6) = (2) \\ &(2) \otimes (6) = (3) \ , && (6) \otimes (6) = (1) \ . \end{aligned} \tag{6.105}$$

In addition, for odd nodes of the form $(a,1)$ and $(a,2)$ appearing inside a fusion rule, their contributions have to be doubled.

For example, we can re-calculate (6.51), (6.52), (6.53), and (6.54) as

$$\begin{aligned} (2,6) \otimes (2,6) &= (1,1) \oplus (3,1) \\ &\to \chi^{\mathrm{NS}}_{1,1}(q) + \chi^{\mathrm{NS}}_{1,7}(q) + \chi^{\mathrm{NS}}_{9,1}(q) + \chi^{\mathrm{NS}}_{9,7}(q) \\ &\qquad + \chi^{\mathrm{NS}}_{3,1}(q) + \chi^{\mathrm{NS}}_{3,7}(q) + \chi^{\mathrm{NS}}_{7,1}(q) + \chi^{\mathrm{NS}}_{7,7}(q) \\ &= \chi^{\mathrm{NS}}_{0}(q) + \chi^{\mathrm{NS}}_{\frac{3}{2}}(q) + \chi^{\mathrm{NS}}_{\frac{7}{2}}(q) + \chi^{\mathrm{NS}}_{10}(q) \\ &\qquad + \chi^{\mathrm{NS}}_{\frac{1}{5}}(q) + \chi^{\mathrm{NS}}_{\frac{7}{10}}(q) + \chi^{\mathrm{NS}}_{\frac{6}{5}}(q) + \chi^{\mathrm{NS}}_{\frac{57}{10}}(q) \ , \end{aligned} \tag{6.106}$$

$$\begin{aligned} (2,6) \otimes (2,1) &= (1,6) \oplus (3,6) \\ &\to \chi^{\mathrm{R}}_{1,4}(q) + \chi^{\mathrm{R}}_{1,8}(q) + \chi^{\mathrm{R}}_{9,4}(q) + \chi^{\mathrm{R}}_{9,8}(q) \\ &\qquad + \chi^{\mathrm{R}}_{3,4}(q) + \chi^{\mathrm{R}}_{3,8}(q) + \chi^{\mathrm{R}}_{7,4}(q) + \chi^{\mathrm{R}}_{7,8}(q) \\ &= 2\left(\chi^{\mathrm{R}}_{\frac{7}{8}}(q) + \chi^{\mathrm{R}}_{\frac{39}{8}}(q) + \chi^{\mathrm{R}}_{\frac{3}{40}}(q) + \chi^{\mathrm{R}}_{\frac{83}{40}}(q)\right) \ , \end{aligned} \tag{6.107}$$

$$\begin{aligned} (2,6) \otimes (1,1) &= (2,6) \\ &\to \chi^{\mathrm{NS}}_{2,4}(q) + \chi^{\mathrm{NS}}_{2,8}(q) + \chi^{\mathrm{NS}}_{8,4}(q) + \chi^{\mathrm{NS}}_{8,8}(q) \\ &= 2\big(\chi^{\mathrm{NS}}_{\frac{21}{80}}(q) + \chi^{\mathrm{NS}}_{\frac{261}{80}}(q)\big) \ , \end{aligned} \tag{6.108}$$

2. It is due to the fact that the $\mathcal{SV}\mathrm{ir}$ representations in $SM(10,12)$ can be constructed from a coset

$$\frac{\widehat{\mathrm{sl}}(2)_8 \oplus \widehat{\mathrm{sl}}(2)_2}{\widehat{\mathrm{sl}}(2)_{10}} \ , \tag{6.103}$$

where the $\widehat{\mathrm{sl}}(2)_2$ factor gives the sector structure[12].

$$\begin{aligned}(2,6)\otimes(1,6) = (2,1)&\\ &\to 2\left(\chi^{\mathrm{R}}_{2,1}(q)+\chi^{\mathrm{R}}_{2,7}(q)+\chi^{\mathrm{R}}_{8,1}(q)+\chi^{\mathrm{R}}_{8,7}(q)\right)\\ &= 2\left(\chi^{\mathrm{R}}_{\frac{21}{80}}(q)+\chi^{\mathrm{R}}_{\frac{61}{80}}(q)+\chi^{\mathrm{R}}_{\frac{181}{80}}(q)+\chi^{\mathrm{R}}_{\frac{621}{80}}(q)\right)\ ,\end{aligned} \tag{6.109}$$

where, in the last line, we doubled the contribution of $(2,1)$.

In this way, we can calculate the characters appearing in (D_6, E_6) boundary overlaps, but some care is needed to obtain the correct overall normalisation. For overlaps of two boundary states corresponding to topological defects, the results obtained from the fusion rules have to be doubled. In addition, these fusion rules yield overlaps for the boundary states that are normalised according to the embedding ρ_{+++-}. For the boundary states defined with ρ'_{+++-}, we have to double all the results from the fusion rules.

6.6 Projection of Defects

As we have found the boundary conditions in the (D_6, E_6) theory corresponding to non-topological and non-factorising defects in $SM(3,5)$, we would now like to use the topological interface and obtain the corresponding defects in the tri-critical Ising model $M(4,5)$. For a defect $D_{SM(3,5)}$ in the supersymmetric theory, we can obtain the corresponding defect $D_{M(4,5)}$ in $M(4,5)$ by

$$D_{M(4,5)} = I\, D_{SM(3,5)}\, I^\dagger\ , \tag{6.110}$$

where I is the fundamental topological interface operator defined in (5.109). The defect transmission coefficient $\mathcal{T}$ and the g value of $D_{M(4,5)}$ can be obtained straightforwardly,

$$\mathcal{T}(D_{M(4,5)}) = \mathcal{T}(D_{SM(3,5)}) \quad \text{and} \quad g(D_{M(4,5)}) = 2g(D_{SM(3,5)})\ . \tag{6.111}$$

As we know from the Ising–free fermion case, it is unlikely that the image of a $SM(3,5)$ defect is elementary in $M(4,5)$. For example, we can map topological and factorising defects of $SM(3,5)$ as

$$\begin{aligned} &I\, D^{\mathrm{NS}}_{\mathbf{1}}\, I^\dagger = D_{\mathbf{1}} + D_\varepsilon\ , \quad I\,\sqrt{2}(-1)^F D^{\mathrm{NS}}_{\mathbf{1}}\, I^\dagger = 2D_\sigma\ ,\\ &I\, D^{\mathrm{NS}}_{\varphi}\, I^\dagger = D_{\hat{\mathbf{1}}} + D_{\hat\varepsilon}\ , \quad I\,\sqrt{2}(-1)^F D^{\mathrm{NS}}_{\varphi}\, I^\dagger = 2D_{\hat\sigma}\ ,\\ &I\, \|\mathbf{1}_{\mathrm{NS}}\rangle\!\rangle\langle\!\langle\mathbf{1}_{\mathrm{NS}}\|\, I^\dagger = (\|\mathbf{1}\rangle\!\rangle + \|\varepsilon\rangle\!\rangle)\,(\langle\!\langle\mathbf{1}\| + \langle\!\langle\varepsilon\|)\ . \end{aligned} \tag{6.112}$$

We can see that these images are the orbits of the action of D_ε. As we know the defect $\mathcal{D}_t$ commutes with topological defect operators in $SM(3,5)$ while $\mathcal{D}_f$ does not, thus we can make an ansatz

$$I\,\mathcal{D}_t\, I^\dagger = \mathcal{D}^{M(4,5)}_t + D_\varepsilon\, \mathcal{D}^{M(4,5)}_t \quad \text{and} \tag{6.113}$$

$$I\,\mathcal{D}_f\, I^\dagger = (D_{\mathbf{1}} + D_\varepsilon)\mathcal{D}^{M(4,5)}_f (D_{\mathbf{1}} + D_\varepsilon)\ , \tag{6.114}$$

where $\mathcal{D}^{M(4,5)}_t$ and $\mathcal{D}^{M(4,5)}_f$ are defects in $M(4,5)$. From the definitions (6.83) and (6.85), and from Table 6.3, these defects have

$$\mathcal{D}_t : \mathcal{T} = \frac{\sqrt{3}-1}{2} = 0.366025... \quad \text{and} \quad g = \sqrt{2+\sqrt{3}}\,, \tag{6.115}$$

$$\mathcal{D}_f : \mathcal{T} = \frac{3-\sqrt{3}}{2} = 0.633975... \quad \text{and} \quad g = \frac{1}{4}(\sqrt{3}+1)(\sqrt{5}-1)\sqrt{1+\frac{1}{\sqrt{5}}}\,. \tag{6.116}$$

Since D_ε has the g-value of 1, $\mathcal{D}_t^{M(4,5)}$ should have the same g-value as $\mathcal{D}_t$, and $\mathcal{D}_f^{M(4,5)}$ has half the g-value of $\mathcal{D}_f$,

$$g(\mathcal{D}_t^{M(4,5)}) = \sqrt{2+\sqrt{3}} \qquad = 1.93185... \quad \text{and} \tag{6.117}$$

$$g(\mathcal{D}_f^{M(4,5)}) = \frac{1}{8}(\sqrt{3}+1)(\sqrt{5}-1)\sqrt{1+\frac{1}{\sqrt{5}}} = 0.507817...\,. \tag{6.118}$$

The conjectures (6.113) and (6.114) may be too optimistic, but it is quite likely that they satisfy

$$g(\mathcal{D}_a) > g(\mathcal{D}_a^{M(4,5)}) > \frac{1}{2}\,g(\mathcal{D}_a)\,, \tag{6.119}$$

where a stands t or f.

In [98], Kormos, Runkel, and Watts studied defect perturbations of the topological defect $D_{(1,2)}$ by the linear combination of chiral defect fields

$$\lambda\psi_{(1,3),(1,1)}(z) + \bar{\lambda}\psi_{(1,1),(1,3)}(\bar{z})\,. \tag{6.120}$$

For $M(4,5)$, these Kac labels corresponds to the representation labels given in Table 5.2 as $(1,2) = \hat{\varepsilon}$ and $(1,3) = \hat{\mathbf{1}}$. In [98], perturbative analysis and truncated conformal space approach calculations show that, for $\lambda \neq 0$ and $\bar{\lambda} \neq 0$, the endpoint of these flows is a new non-topological and non-factorising defect which is denoted by C. Since $D_{\hat{\varepsilon}}$ has $g = \frac{1+\sqrt{5}}{2} = 1.61803...$, only $\mathcal{D}_f^{M(4,5)}$ can be a candidate for an endpoint of this defect flow as the g-theorem states that the g-value decreases along the flow. In addition, if we set $\lambda = 0$ or $\bar{\lambda} = 0$ in (6.120), these purely chiral perturbations lead to the topological defect D_σ which has $g = \sqrt{2} = 1.41421...$, and there are defect flows from D_σ to the defect C as depicted in Figure 6.3. Therefore, the g-value of C has to be smaller than $\sqrt{2}$. The defect $\mathcal{D}_f^{M(4,5)}$ still satisfies the requirement.

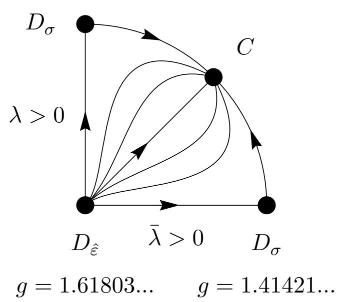

Figure 6.3: Defect flows of $D_{\hat{\varepsilon}}$ with positive λ and $\bar{\lambda}$ given in [98].

6.6.1 Projection of Ishibashi States

The interface projector (5.101) acts on the state space of $SM(3,5)$ as

$$I = \frac{1}{\sqrt{2}}\left(1 + (-1)^{F+\bar{F}}\right) \,, \tag{6.121}$$

and we can rewrite (6.110) as

$$D_{M(4,5)} = \frac{1}{2}\left(1 + (-1)^{F+\bar{F}}\right) D_{SM(3,5)} \left(1 + (-1)^{F+\bar{F}}\right) \,. \tag{6.122}$$

If we consider the folded theory, this becomes

$$\|D\rangle\rangle_{M(4,5)} = \frac{1}{2}\left(1 + (-1)^{F_1+\bar{F}_1} + (-1)^{F_2+\bar{F}_2} + (-1)^{F_{\text{tot}}+\bar{F}_{\text{tot}}}\right) \|D\rangle\rangle_{SM(3,5)} \tag{6.123}$$

where $(-1)^{F_{\text{tot}}+\bar{F}_{\text{tot}}} = (-1)^{F_1+\bar{F}_1+F_2+\bar{F}_2}$. Note that the Neveu–Schwarz Ishibashi states are invariant under $(-1)^{F_{\text{tot}}+\bar{F}_{\text{tot}}}$. For brevity, we write

$$\mathcal{P} := \frac{1}{2}\left(1 + (-1)^{F_1+\bar{F}_1} + (-1)^{F_2+\bar{F}_2} + (-1)^{F_{\text{tot}}+\bar{F}_{\text{tot}}}\right) \,. \tag{6.124}$$

In addition, for Ishibashi states of $SM(10,12)$ and $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ at $c=\frac{7}{5}$, we introduce

$$|h,\epsilon\rangle\rangle_{\mathcal{P}} := \mathcal{P}\,|h,\epsilon\rangle\rangle \quad \text{and} \quad |h,\epsilon\rangle\rangle^{\mathcal{W}}_{\Omega,\mathcal{P}} := \mathcal{P}\,|h,\epsilon\rangle\rangle^{\mathcal{W}}_{\Omega} \,. \tag{6.125}$$

The exact definition of these states depend on the embedding $\iota_{\alpha\beta\gamma\delta}$ which was defined in (6.20).

Using the expansion (2.223) and the corresponding Gram matrices, it is straightforward to write algorithms that generate $SM(10,12)$ and $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ Ishibashi states. For $SM(10,12)$ Ishibashi states, we use $\iota_{\alpha\beta\gamma\delta}$ to express $\mathcal{SV}\text{ir}$ generators in terms of those of the folded theory, and apply the projector $\mathcal{P}$. For $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ Ishibashi states, we can apply $\mathcal{P}$ directly since the action of fermion parity operators are given in (2.138), (2.140), and (2.141). In addition, we can use the automorphism (2.142) to construct twisted Ishibashi states. For the calculations we pick ι_{++++}.

For example, we can calculate the overlap

$$\begin{aligned} {}^{\mathcal{W}}_{\mathcal{P}}\langle\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}\left(L_0^{\text{tot}}+\bar{L}_0^{\text{tot}}-\frac{7}{60}\right)}|0,\epsilon\rangle\rangle^{\mathcal{W}}_{\mathcal{P}} &= {}^{\mathcal{W}}_{\mathcal{P},T}\langle\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}\left(L_0^{\text{tot}}+\bar{L}_0^{\text{tot}}-\frac{7}{60}\right)}|0,\epsilon\rangle\rangle^{\mathcal{W}}_{T,\mathcal{P}} \\ &= 4\tilde{q}^{-\frac{7}{60}}\left(1+2\tilde{q}^{\frac{3}{2}}+2\tilde{q}^2+2\tilde{q}^{\frac{5}{2}}+3\tilde{q}^3+6\tilde{q}^{\frac{7}{2}}+7\tilde{q}^4+8\tilde{q}^{\frac{9}{2}}+11\tilde{q}^5+\cdots\right) \\ &= 4\left(\chi^{\mathcal{V}}_0(\tilde{q})+\chi^{\mathcal{V}}_{\frac{3}{2}}(\tilde{q})\right)^2 \,, \end{aligned} \tag{6.126}$$

where $\chi^{\mathcal{V}}_h(q)$ denotes the character of a Virasoro representation at $c=\frac{7}{10}$. Similarly, we obtain

$$\begin{aligned} {}^{\mathcal{W}}_{\mathcal{P}}\langle\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}\left(L_0^{\text{tot}}+\bar{L}_0^{\text{tot}}-\frac{7}{60}\right)}|0,-\epsilon\rangle\rangle^{\mathcal{W}}_{\mathcal{P}} &= {}^{\mathcal{W}}_{\mathcal{P},T}\langle\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}\left(L_0^{\text{tot}}+\bar{L}_0^{\text{tot}}-\frac{7}{60}\right)}|0,-\epsilon\rangle\rangle^{\mathcal{W}}_{T,\mathcal{P}} \\ &= 4\left(\chi^{\mathcal{V}}_0(\tilde{q})-\chi^{\mathcal{V}}_{\frac{3}{2}}(\tilde{q})\right)^2 \,. \end{aligned} \tag{6.127}$$

However, the overlaps of untwisted and twisted projected Ishibashi states yield

$$\begin{aligned} {}^{\mathcal{W}}_{\mathcal{P}}\langle\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}\left(L_0^{\text{tot}}+\bar{L}_0^{\text{tot}}-\frac{7}{60}\right)}|0,\epsilon\rangle\rangle^{\mathcal{W}}_{T,\mathcal{P}} &= {}^{\mathcal{W}}_{\mathcal{P}}\langle\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}\left(L_0^{\text{tot}}+\bar{L}_0^{\text{tot}}-\frac{7}{60}\right)}|0,-\epsilon\rangle\rangle^{\mathcal{W}}_{T,\mathcal{P}} \\ &= 4\tilde{q}^{-\frac{7}{60}}\left(1-\tilde{q}^3+\tilde{q}^4-\tilde{q}^5+\tilde{q}^6-2\tilde{q}^7+2\tilde{q}^8-\cdots\right) \,, \end{aligned} \tag{6.128}$$

which does not admit a straightforward interpretation in terms of the $M(4,5)$ Virasoro characters.

As the $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ Ishibashi state $|0,\epsilon\rangle\!\rangle_{\Omega}^{\mathcal{W}}$ decomposes into $SM(10,12)$ Ishibashi states, we can expect the overlaps of projected $SM(10,12)$ Ishibashi states are more involved. For the vacuum Ishibashi state, we get

$$\begin{aligned} {}_{\mathcal{P}}\langle\!\langle 0,\epsilon|\tilde{q}^{\frac{1}{2}(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60})}|0,\epsilon\rangle\!\rangle_{\mathcal{P}} &= 4\tilde{q}^{-\frac{7}{60}}\Big(1+\tfrac{1}{2}\tilde{q}^{\frac{3}{2}}+\tilde{q}^2+\tfrac{1}{2}\tilde{q}^{\frac{5}{2}}+\tilde{q}^3+\tilde{q}^{\frac{7}{2}}+\tfrac{49123}{17689}\tilde{q}^4+\tfrac{3}{2}\tilde{q}^{\frac{9}{2}} \\ &+\tfrac{49123}{17689}\tilde{q}^5+\tfrac{5}{2}\tilde{q}^{\frac{11}{2}}+\tfrac{102941115}{16999129}\tilde{q}^6+\tfrac{7}{2}\tilde{q}^{\frac{13}{2}}+\tfrac{116150060}{16999129}\tilde{q}^7+6\tilde{q}^{\frac{15}{2}}+\tfrac{390058397535}{31431389521}\tilde{q}^8 \\ &+8\tilde{q}^{\frac{17}{2}}+\tfrac{486102350607}{31431389521}\tilde{q}^9+\tfrac{25}{2}\tilde{q}^{\frac{19}{2}}+\tfrac{781645765079}{31431389521}\tilde{q}^{10}+\cdots\Big)\,. \end{aligned} \tag{6.129}$$

The coefficient of $\tilde{q}^4$ can be written as

$$\frac{49123}{17689}=\left(\frac{116}{133}\right)^2+\left(\frac{17}{133}\right)^2+2\,, \tag{6.130}$$

so we may interpret this result as

$$|0,\epsilon\rangle\!\rangle_{\mathcal{P}}\sim 2\left(|0\rangle\!\rangle+\frac{1}{2}\,|\tfrac{3}{2}^+\rangle\!\rangle+\frac{1}{2}\,|\tfrac{3}{2}^-\rangle\!\rangle+\frac{17}{133}\,|4^+\rangle\!\rangle+\frac{116}{133}\,|4^-\rangle\!\rangle+\cdots\right)\,, \tag{6.131}$$

where $|h^\pm\rangle\!\rangle$ are Virasoro Ishibashi states. As a bosonic projection of $\mathcal{SV}\mathrm{ir}$, we have expected $\mathcal{W}(2,4,6)$ but the normalisations of highest weight states seem to be incompatible.

For reference, some other overlaps are

$$\begin{aligned} &{}_{\mathcal{P}}\langle\!\langle \tfrac{1}{5},\epsilon|\tilde{q}^{\frac{1}{2}(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60})}|\tfrac{1}{5},\epsilon\rangle\!\rangle_{\mathcal{P}} \\ &\quad=4\tilde{q}^{-\frac{7}{60}}\Big(\tfrac{1}{2}\tilde{q}^{\frac{1}{2}}+\tilde{q}+\tilde{q}^{\frac{3}{2}}+3\tilde{q}^2+2\tilde{q}^{\frac{5}{2}}+5\tilde{q}^3+\tfrac{7}{2}\tilde{q}^{\frac{7}{2}}+10\tilde{q}^4+\cdots\Big)\,, \end{aligned} \tag{6.132}$$

$$\begin{aligned} &{}_{\mathcal{P}}\langle\!\langle \tfrac{1}{10},\epsilon|\tilde{q}^{\frac{1}{2}(L_0^{\rm tot}+\bar{L}_0^{\rm tot}-\frac{7}{60})}|\tfrac{1}{10},\epsilon\rangle\!\rangle_{\mathcal{P}} \\ &\quad=4\tilde{q}^{-\frac{7}{60}}\Big(\tilde{q}^{\frac{1}{2}}+\tilde{q}+2\tilde{q}^{\frac{3}{2}}+3q^2+\tfrac{107732}{30625}\tilde{q}^{\frac{5}{2}}+\tfrac{5881}{1250}\tilde{q}^3+\tfrac{199607}{30625}\tilde{q}^{\frac{7}{2}}+\tfrac{17054233}{1805000}\tilde{q}^4+\cdots\Big)\,. \end{aligned} \tag{6.133}$$

6.7 Comparisons with Gang and Yamaguchi's Results

We can now attempt to compare our results for non-topological, non-factorising defects with those of Gang and Yamaguchi. The simplest such defects we have found are $\mathcal{D}_f$ defined in (6.83) with $g=2.031..$ and $\mathcal{T}=(3-\sqrt{3})/2$. Looking at the list of proposed defects in section 3.2 of [94] the only candidates to which we can hope to relate $\mathcal{D}_f$ are those from the boundary state $|(1,3)\rangle_{A_\pm}$ which have the same value of $\mathcal{T}$ and half the g-value.

From the definitions in equation (3.9) of [94], the states $|(1,3)\rangle_{A_\pm}$ have equal and opposite components in the Ramond sector. Since our defects have no components in the Ramond sector, we must consider the sum $|(1,3)\rangle_{A_+}+|(1,3)\rangle_{A_-}$ which has the same $\mathcal{T}$ and g values as each of $\mathcal{D}_f$.

There are no precise definitions given in [94] on how to obtain a defect from a boundary state, but we can see that $|(1,3)\rangle_{A_+}+|(1,3)\rangle_{A_-}$ has zero overlap with the states $|(5,3,5)_{10}\rangle$

and $|(5',3,5)_{10}\rangle$ (in the notation of [94]) which are equivalent to (in our notation) $|\frac{1}{10}\rangle$ and $|\frac{1}{10}{}'\rangle$ of the (D_6,E_6) theory. This means that whatever map $\tilde{\rho}$ is required to obtain a defect from a boundary state in the formalism of [94], the corresponding defect has zero matrix elements between $\langle 0|$ and $|\frac{1}{10}\rangle$

$$\langle 0|\tilde{\rho}\Big(|(1,3)\rangle_{A_+} + |(1,3)\rangle_{A_-}\Big)|\tfrac{1}{10}\rangle = 0\ , \tag{6.134}$$

and so cannot be equal to $\mathcal{D}_f$.

Gang and Yamaguchi do not give details on the the precise map $\tilde{\rho}$ required to obtain a defect from a boundary state in their formalism. We can be sure that the method we use cannot work, as this will result in defects which are not GSO projected, that is defects which are not maps from $SM(3,5)$ to $SM(3,5)$. To illustrate this, we consider the states used in [94] in the representation

$$\Big[\mathcal{H}^3_{1,3}\otimes\mathcal{H}^3_{1,3}\Big]^{\otimes 2} = \Big[\mathcal{H}^{10}_{3,1}\oplus\mathcal{H}^{10}_{3,5}\oplus\mathcal{H}^{10}_{3,7}\oplus\mathcal{H}^{10}_{3,11}\Big]^{\otimes 2} \tag{6.135}$$

The paper [94] uses coset representations, and each highest weight representation $\mathcal{H}^{10}_{r,s}\equiv\mathcal{H}^{10}_{10-r,12-s}$ of $SM(10)$ splits into two coset representations,

$$\begin{aligned}\mathcal{H}^{10}_{3,1} &= \mathcal{H}_{(3,1,1)_{10}}\oplus\mathcal{H}_{(3,3,1)_{10}}\ , \quad \mathcal{H}^{10}_{3,5} = \mathcal{H}_{(3,1,5)_{10}}\oplus\mathcal{H}_{(3,3,5)_{10}}\ ,\\ \mathcal{H}^{10}_{3,7} &= \mathcal{H}_{(7,1,5)_{10}}\oplus\mathcal{H}_{(7,3,5)_{10}}\ , \quad \mathcal{H}^{10}_{3,11} = \mathcal{H}_{(7,1,1)_{10}}\oplus\mathcal{H}_{(7,3,1)_{10}}\ .\end{aligned} \tag{6.136}$$

Only four of these coset representations appear in the boundary states of [94], with conformal weights as follows

Representation	Weight
$(3,3,5)_{10}$	$\frac{1}{5}$
$(3,1,1)_{10}$	$\frac{6}{5}$
$(7,3,5)_{10}$	$\frac{6}{5}$
$(7,1,1)_{10}$	$\frac{31}{5}$

(6.137)

In our terms, these can be identified with $\mathcal{SW}(\frac{3}{2},\frac{3}{2})$ descendants of the $SM(10,12)$ highest weight states,

$$\begin{aligned}|(3,3,5)_{10}\rangle &= |\tfrac{1}{5}\rangle\ , & |(7,3,5)_{10}\rangle &= |\tfrac{6}{5}\rangle\ ,\\ |(3,1,1)_{10}\rangle &= \frac{i\eta}{7/5}G^{\rm tot}_{-\frac{1}{2}}\bar{G}^{\rm tot}_{-\frac{1}{2}}|\tfrac{7}{10}\rangle\ , & |(7,1,1)_{10}\rangle &= \frac{i\eta'}{57/5}G^{\rm tot}_{-\frac{1}{2}}\bar{G}^{\rm tot}_{-\frac{1}{2}}|\tfrac{57}{10}\rangle\ ,\end{aligned} \tag{6.138}$$

where η and η' are undetermined signs. Further, given an embedding $\iota_{\alpha\beta\gamma\delta}$, the states $|\frac{6}{5}\rangle$ and $|\frac{7}{10}\rangle$ can be identified from Appendix D.2 as

$$|\tfrac{7}{10}\rangle = \frac{i\eta_{\frac{7}{10}}}{2/5}(\alpha G^{(1)}_{-\frac{1}{2}} - \beta G^{(2)}_{-\frac{1}{2}})(\gamma\bar{G}^{(1)}_{-\frac{1}{2}} - \delta\bar{G}^{(2)}_{-\frac{1}{2}})|\tfrac{1}{5}\rangle\ , \tag{6.139}$$

$$|\tfrac{6}{5}\rangle = \frac{\eta_{\frac{6}{5}}}{7/5}(L^{(1)}_{-1} - L^{(2)}_{-1} + \frac{\alpha\beta}{1/5}G^{(1)}_{-\frac{1}{2}}G^{(2)}_{-\frac{1}{2}})(\bar{L}^{(1)}_{-1} - \bar{L}^{(2)}_{-1} + \frac{\gamma\delta}{1/5}\bar{G}^{(1)}_{-\frac{1}{2}}\bar{G}^{(2)}_{-\frac{1}{2}})|\tfrac{1}{5}\rangle\ . \tag{6.140}$$

This means that the state $|(3,1,1)_{10}\rangle$ is

$$|(3,1,1)_{10}\rangle = \frac{-\eta\,\eta_{\frac{7}{10}}}{(2/5)(7/5)}(L^{(1)}_{-1} - L^{(2)}_{-1} - 2\alpha\beta G^{(1)}_{-\frac{1}{2}}G^{(2)}_{-\frac{1}{2}})(\bar{L}^{(1)}_{-1} - \bar{L}^{(2)}_{-1} - 2\gamma\delta\bar{G}^{(1)}_{-\frac{1}{2}}\bar{G}^{(2)}_{-\frac{1}{2}})|\tfrac{1}{5}\rangle\ . \tag{6.141}$$

Putting these together with the results in Appendix D.2, and the fact that the boundary state

$$|(3,3,5)_{10}\rangle\!\rangle = |\tfrac{1}{5}\rangle + \frac{1}{2/5} L_{-1}\bar{L}_{-1}|\tfrac{1}{5}\rangle + \cdots \ , \tag{6.142}$$

we can find the expansion up to level one of defect given by a combination of boundary states constructed from the four states (6.7):

$$|\Psi\rangle\!\rangle = A|(3,3,5)_{10}\rangle\!\rangle + B|(3,1,1)_{10}\rangle\!\rangle + C|(7,3,5)_{10}\rangle\!\rangle + D|(7,1,1)_{10}\rangle\!\rangle \ , \tag{6.143}$$

$$\begin{aligned}
\rho_{\alpha\beta\gamma\delta}(|\Psi\rangle\!\rangle) = {}& A|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}| \\
&+ \left(\frac{A}{2/5} - \frac{B\eta\eta_{\frac{7}{10}}}{(2/5)(7/5)} + \frac{C\eta_{\frac{6}{5}}}{7/5}\right)\left(L_{-1}\bar{L}_{-1}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}| + |\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|\bar{L}_1 L_1\right) \\
&+ \left(\frac{A}{2/5} + \frac{B\eta\eta_{\frac{7}{10}}}{(2/5)(7/5)} - \frac{C\eta_{3,7}}{7/5}\right)\left(L_{-1}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|L_1 + \bar{L}_{-1}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|\bar{L}_1\right) \\
&+ i\alpha\beta\frac{(B\eta\eta_{\frac{7}{10}} + C\eta_{\frac{6}{5}})}{7/25}\left(G_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|\bar{G}_{\frac{1}{2}}L_1 - \bar{L}_{-1}G_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|\bar{G}_{\frac{1}{2}}\right) \\
&+ i\gamma\delta\frac{(B\eta\eta_{\frac{7}{10}} + C\eta_{\frac{6}{5}})}{7/25}\left(L_{-1}\bar{G}_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|G_{\frac{1}{2}} - \bar{G}_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|\bar{G}_{\frac{1}{2}}\bar{L}_1\right) \\
&+ \alpha\beta\gamma\delta\left(\frac{2B\eta\eta_{\frac{7}{10}}}{7/25} - \frac{C\eta_{\frac{6}{5}}}{7/125}\right)\left(G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|\tfrac{1}{10}\rangle\!\langle\tfrac{1}{10}|\bar{G}_{\frac{1}{2}}G_{\frac{1}{2}}\right) + \cdots
\end{aligned} \tag{6.144}$$

The expression (6.144) is only GSO projected if $B\eta\eta_{\frac{7}{10}} + C\eta_{\frac{6}{5}} = 0$, otherwise it is not. We can fix $\eta\eta_{\frac{7}{10}}$ and $\eta_{\frac{6}{5}}$ by comparing (6.144) with equation (3.20) of [94]. Equation (3.20) says that the expression (6.144) should be purely transmitting for $A = B = 1, C = -1$ and purely reflecting for $B = -1, A = C = 1$, from which we deduce that $\eta\eta_{\frac{7}{10}} = \eta_{\frac{6}{5}} = 1$. We can now decide if the defects arising from the boundary states of [94] are GSO projected or not by looking at the ratio of the coefficients B and C of the states $|(3,1,1)_{10}\rangle\!\rangle$ and $|(7,3,5)_{10}\rangle\!\rangle$. If this ratio is -1, the resulting defect can be GSO projected, if it is not -1 then it is not GSO projected:

$B = -C$, GSO projected	$B \neq -C$, not GSO projected
$|(2,6)\rangle_{A_\pm},\ |(4,6)\rangle_{A_\pm}$	$|(1,3)\rangle_{A_\pm},\ |(3,3)\rangle_{A_\pm}, |(5,3)\rangle_{A_\pm},\ |(6,3)\rangle_{A_\pm}$
$|(1,1)\rangle_B,\ |(3,1)\rangle_B,\ |(5,1)\rangle_B,\ |(6,1)\rangle_B$	$|(2,2)\rangle_B,\ |(4,2)\rangle_B$

(6.145)

Those which are GSO projected correspond to topological or factorising defects; none of the "new" defects proposed in [94] lead to GSO projected defects in our formalism, and so it is difficult for us to make a stronger comparison with the proposals of [94].

Chapter 7

Summary and Outlook

In this thesis, we have developed a systematic method to obtain conformal defects in the tri-critical Ising model from the $N = 1$ super-Virasoro symmetry of the folded theory by constructing topological interfaces between the supersymmetric and bosonic theories. We found that the extended conformal symmetry of the folded theory can be an useful guide to construct conformal defects. In the folded theory, $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ boundary conditions correspond to topological and factorising defects, and those break the extended superconformal symmetry but preserve $\mathcal{SV}\mathrm{ir}$ correspond to non-topological and non-factorising superconformal defects.

We constructed consistent supersymmetric theory with topological defects and boundaries at $c = \frac{7}{10}$. We also found that Ramond fields can be considered as disorder fields—fields on which defects can terminate—associated with the topological defects that act as fermion parity operators, and even if we restrict the bulk theory to the NS sector only, it is possible to construct consistent interfaces to obtain the various quantities in the bosonic theory.

Our construction uses many elements from the paper of Gang and Yamaguchi [94] but the defects we propose are not the same as theirs. While the expressions for topological and factorising defects are the same, it may be possible that the new defects proposed in [94] are not properly GSO projected.

As part of our construction, we found evidence for two non-commensurate sets of boundary states in the doubled theory of $SM(3, 5)$ corresponding to two inequivalent embeddings of $c = \frac{7}{5}$ algebra into two copies of $c = \frac{7}{10}$. We have identified half of these boundary states as known objects, the remaining half are new and lead to non-topological and non-factorising defects in the tri-critical Ising model. By considering fusions with topological defects in $SM(3, 5)$, we conjecture that there are two fundamental non-topological and non-factorising defects $\mathcal{D}_f$ and $\mathcal{D}_t$.

We think it should be possible to derive the boundary states we have proposed for the (D_6, E_6) theory using topological field theory methods. It would be nice to compare our method with the construction of fermionic models of Novak and Runkel [113] using topological field theory methods incorporating spin structure.

In order to gain more insight into the structure of conformal defects, we have also calculated the leading term in the perturbative expansion of the reflection coefficient for the defect of type $(r, 2)$ in a diagonal Virasoro minimal model.

It is possible that at least one of the conformal defects, $\mathcal{D}_f^{M(4,5)}$, we discovered in Chapter 6 is related to the conformal defect C found by perturbation theory; the value of $\mathcal{R}$ is close enough not to rule this out. It would be good to extend this calculation to next-to-leading order where there are UV divergences to be regulated, but so far we have not yet managed this.

We have also calculated defect structure constants for various fields on defects of type

$(r, 2)$ extending the results of [110]. These results are not complete—they do not include all fields, and use special properties of the $(r, 2)$ defect, but it would be good to check that these constants in fact agree with the general results of [86] where the same constants were constructed using topological field theory methods.

For further research, there are several possible directions. So far we have only obtained defect entropies, reflection and transmission coefficients, and various partition functions involving defects. It would be nice to obtain more quantities, for example, various correlation functions involving bulk, boundary, defect, and interface fields. Also, it would be interesting to see if similar constructions are possible for other extended conformal algebras—non-topological and non-factorising defects should correspond to symmetry breaking boundaries in the folded theory.

There are several other situations in which it may be possible to construct topological interfaces, for example, the $N = 1$ and $N = 2$ super-Virasoro minimal models at $c = 1$.

Appendix A

Properties of Virasoro and Super-Virasoro Representations

In this appendix, we summarise Virasoro characters, and modular S and T matrices for Virasoro and super-Virasoro minimal representations. In addition, we quote the expression for fusing matrix elements for Virasoro minimal models.

A.1 Virasoro Characters

Given a highest weight representation $\mathcal{H}$ of the chiral algebra $\mathcal{A}$, its Virasoro character[1] is defined by

$$\chi_{\mathcal{H}}(q) := \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-\frac{c}{24}} = q^{-\frac{c}{24}} \sum_{N=0}^{\infty} (\dim \mathcal{H}_N)\, q^{h+N} \ , \tag{A.1}$$

where $q \in \mathbb{C}$ is a formal variable. If all the generators of $\mathcal{A}$ take integer modes, then $N \in \mathbb{Z}$ in the summation; or $N \in \frac{1}{2}\mathbb{Z}$ if there are generators with half-integer modes. Since L_0-eigensubspaces of $\mathcal{H}$ are mutually orthogonal, this is nothing but a generating function for the dimensions of these subspaces.

A.1.1 Virasoro Representations

For a Verma module M of the Virasoro algebra, we know the dimension of the level N subspace is given by $p(N)$, the number of integer partitions of N, and the corresponding Virasoro character is given by the famous generating function due to Euler

$$\chi_{\mathrm{M}}(q) = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = q^{h-\frac{c}{24}} (1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots) \ . \tag{A.2}$$

Note that even when $h = 0$, this formula gives the dimension of the level 1 subspace to be one; the condition $L_{-1}|0\rangle = 0$ is due to the fact that $L_{-1}|0\rangle$ is a null vector.

In order to obtain the Virasoro character of an irreducible module, we need to subtract contributions from null vectors from (A.2). One way to obtain the dimension of an irreducible subspace is to explicitly calculate the maximal number of basis vectors of the form (2.74) which makes the determinant of corresponding Gram matrix positive. However, this method is rather impractical as $p(N)$ increases rapidly[2].

1. We use the term "Virasoro character" to distinguish (A.1) from other specialised characters, for example, W characters defined by $\mathrm{Tr}\, e^{2\pi i \alpha W_0} q^{L_0-\frac{c}{24}}$.

2. For example, $p(10) = 42$, $p(20) = 627$, and $p(30) = 5604$. In addition, the Hardy-Ramanujan asymptotic approximation gives

$$p(N) \sim \frac{1}{4N\sqrt{3}} \exp\left(\pi\sqrt{\frac{2N}{3}}\right) \ . \tag{A.3}$$

Fortunately, there is a useful structure of null vectors in a Verma module with $c = c(p,q)$ and $h = h_{r,s}$ where the parametrisations are given by (2.85). From the Kac determinant formula (2.83), we know the first null vector $|\chi_{r,s}\rangle$ appears at level $N = rs$, which is not degenerate in the sense that there is no other linearly independent null vectors at this level. By acting L_n with $n < 0$ on $|\chi_{r,s}\rangle$, we can construct a sub-Verma module whose elements are all null vectors. The null vector with the lowest conformal weight in a sub-Verma module is called a singular vector. Since $h_{r,s} = h_{p-r,q-s}$, there is another singular vector at level $N = (p-r)(q-s)$. In each of the sub-Verma modules corresponding to these singular vectors, there will be two singular vectors obtained by the same argument. However, it turns out that these two pairs of singular vectors are linearly dependent[3]. Taking this embedding pattern of the sub-Verma modules into account, the Virasoro character of an irreducible module $\mathcal{H}$ with $c = c(p,q)$ and $h = h_{r,s}$ is given by the Rocha-Caridi formula[10]

$$\chi_{\mathcal{H}}(q) = q^{-\frac{c}{24}} \sum_{n\in\mathbb{Z}} \left(q^{h_{2pn+r,s}} - q^{h_{2pn-r,s}} \right) \prod_{m=1}^{\infty} \frac{1}{1-q^m} \, . \tag{A.4}$$

One of the important properties of a Verma module of the Virasoro algebra is that the maximal submodule is generated by the singular vectors. This may not be true for Verma modules of other W-algebras; even after taking the quotient of a Verma module by dividing the submodule generated by the singular vectors, there may be new singular vectors called subsingular vectors.

A.1.2 $N = 1$ Super-Virasoro Representations

For the Neveu–Schwarz sector of the $N = 1$ super-Virasoro algebra, Virasoro characters are obtained similarly. Consider the level N subspace of a Verma module which is spanned by vectors of the form

$$L_{-m_l} \cdots L_{-m_1} G_{-n_k} \cdots G_{-n_1} |h\rangle \, , \tag{A.5}$$

where $0 < m_1 \leq \cdots \leq m_l$ and $0 < n_1 < \cdots < n_k$. In addition, $\sum_i m_i + \sum_j n_j = N$ and n_i are half-integers. Then, the dimension of this subspace is given by the number of integer partitions of $2N$ into distinct odd parts while there is no restrictions on even parts. A generating function for this quantity is known[3], and the Virasoro character of a Verma module M can be written as

$$\chi_{\mathrm{M}}(q) = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n} = q^{h-\frac{c}{24}} (1 + q^{\frac{1}{2}} + q + 2q^{\frac{3}{2}} + 3q^2 + 4q^{\frac{5}{2}} + 5q^3 + \cdots) \, . \tag{A.6}$$

For the Neveu–Schwarz representations, integer levels and half-integer levels have the opposite fermion parities. If we define the Virasoro supercharacter[4]

$$\tilde{\chi}_{\mathrm{M}}(q) := \mathrm{Tr}_{\mathrm{M}} (-1)^F q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_{\substack{N\in\frac{1}{2}\mathbb{Z} \\ N\geq 0}} (\mathrm{sdim}\, \mathrm{M}_N)\, q^{h+N} \, , \tag{A.7}$$

3. OEIS A006950.
4. Note that this definition is slightly different from that of the “odd supercharacters” given in [21].

where the superdimension $\mathrm{sdim}\, \mathrm{M}_N$ of a subspace M_N is defined as the dimension of its bosonic subspace minus that of the fermionic subspace. Then,

$$\tilde{\chi}_{\mathrm{M}}(q) = \epsilon\, q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1-q^{n-\frac{1}{2}}}{1-q^n} = \epsilon\, q^{h-\frac{c}{24}}(1 - q^{\frac{1}{2}} + q - 2q^{\frac{3}{2}} + 3q^2 - 4q^{\frac{5}{2}} + 5q^3 - \cdots)\ , \quad \text{(A.8)}$$

where $\epsilon = \pm 1$ is the fermion parity of $|h\rangle$.

The Kac determinant formula for the $\mathcal{SV}\mathrm{ir}$ highest weight representations is first given by Kac[2] for the Neveu–Schwarz sector and by Friedan, Qiu, and Shenker[9] for the Ramond sector, and later proven by Meurman and Rocha-Caridi[14] for both cases. With the parametrisation given in (2.119), the Verma module with $c = c(p,q)$ and $h = h_{r,s}$ has the first null vector at level $N = rs/2$, which also applies to the Ramond representations. For the Neveu–Schwarz sector, the structure of singular vectors in a Verma module is very similar to the Virasoro case, and the Virasoro character of an irreducible module $\mathcal{H}$ with $c = c(p,q)$ and $h = h_{r,s}$ is given by

$$\chi_{\mathcal{H}}(q) = q^{-\frac{c}{24}} \sum_{n\in\mathbb{Z}} \left(q^{h_{2pn+r,s}} - q^{h_{2pn-r,s}}\right) \prod_{m=1}^{\infty} \frac{1+q^{m-\frac{1}{2}}}{1-q^m}\ . \quad \text{(A.9)}$$

In addition, the Virasoro supercharacter of this module is given[5] by

$$\tilde{\chi}_{\mathcal{H}}(q) = \epsilon\, q^{-\frac{c}{24}} \sum_{n\in\mathbb{Z}} (-1)^{np} \left(q^{h_{2pn+r,s}} - (-1)^{rs} q^{h_{2pn-r,s}}\right) \prod_{m=1}^{\infty} \frac{1-q^{m-\frac{1}{2}}}{1-q^m}\ . \quad \text{(A.10)}$$

We do not assume a highest weight state to be always bosonic, and keep the factor ϵ coming from its fermion parity explicit in our character formulae.

For the Ramond sector of the $N = 1$ super-Virasoro algebra, there are several kinds of embedding diagrams for Verma modules (see, for example [72] and [80]) but the Virasoro characters of irreducible modules with c and h parametrised by (2.119) are similar to the previous cases.

Consider the Verma module M_λ constructed from a highest weight state $|\lambda\rangle$, which is a G_0-eigenvector. Since it is spanned by vectors of the form (2.124), the Virasoro character is given by the convolution of the generating functions of $p(N)$ and $q(N)$, which is the number of integer partitions of N into distinct parts. The generating function for $q(n)$ is given by

$$\prod_{n=1}^{\infty}(1+x^n) = \sum_{n=0}^{\infty} q(n) x^n\ . \quad \text{(A.11)}$$

Therefore,

$$\chi_{\mathrm{M}_\lambda}(q) = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = q^{h-\frac{c}{24}}(1+2q+4q^2+8q^3+14q^4+24q^5+40q^6+\cdots)\ , \quad \text{(A.12)}$$

where h is related to λ by (2.122). If we take $c = c(p,q)$ and $\lambda = \lambda_{r,s}$ given by (2.119) and (2.125), the first null vector appears at level $N = rs/2$, which is not degenerate unlike

5. The formula for unitary cases can be found in [13]. It is not difficult to generalise this by noticing $r+s \in 2\mathbb{Z}$ and $p+q \in 2\mathbb{Z}$.

in the extended Ramond algebra module M_h. In this case, the embedding diagram is the same as the Neveu–Schwarz case, and the Virasoro character of the corresponding irreducible module $\mathcal{H}_\lambda$ is given by

$$\chi_{\mathcal{H}_\lambda}(q) = q^{-\frac{c}{24}} \sum_{n\in\mathbb{Z}} \left(q^{h_{2pn+r,s}} - q^{h_{2pn-r,s}} \right) \prod_{m=1}^{\infty} \frac{1+q^m}{1-q^m} \ . \tag{A.13}$$

Since $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$ are isomorphic as Virasoro representations, we have $\chi_{\mathcal{H}_\lambda}(q) = \chi_{\mathcal{H}_{-\lambda}}(q)$. When $h \neq c/24$, that is $(r,s) \neq (p/2, q/2)$, the irreducible $\mathbb{Z}_2$-graded modules is given by $\mathcal{H}_h = \mathcal{H}_\lambda \oplus \mathcal{H}_{-\lambda}$, where $h = h_{r,s}$. For $h = c/24$, $\mathcal{H}_h$ and $\mathcal{H}_{\lambda_0}$ are isomorphic as Virasoro representations. Therefore, we obtain[50, 80]

$$\chi_{\mathcal{H}_h}(q) = (2 - \delta_{r,\frac{p}{2}} \delta_{s,\frac{q}{2}})\, \chi_{\mathcal{H}_\lambda}(q) \ , \tag{A.14}$$

where $c = c(p,q)$, $h = h_{r,s}$, and $\lambda = \lambda_{r,s}$.

Virasoro supercharacters are trivial in the Ramond sector. If $h \neq c/24$, each subspace of $\mathcal{H}_h$ has $\mathrm{sdim}(\mathcal{H}_h)_N = 0$ since the even and odd subspaces of $(\mathcal{H}_h)_N$ are isomorphic as vector spaces from (2.126). When $h = c/24$, we have $\mathrm{sdim}(\mathcal{H}_h)_N = 0$ for $N \geq 1$, and $\mathrm{sdim}(\mathcal{H}_h)_0 = \epsilon$ since it is one-dimensional and $|h\rangle$ has the fermion parity ϵ. Therefore,

$$\tilde{\chi}_{\mathcal{H}_h}(q) = \epsilon_{\frac{p}{2},\frac{q}{2}}\, \delta_{r,\frac{p}{2}} \delta_{s,\frac{q}{2}} \ , \tag{A.15}$$

where $c = c(p,q)$, $h = h_{r,s}$, $\lambda = \lambda_{r,s}$, and $\epsilon_{r,s} = \pm 1$ is the fermion parity of a Ramond highest weight state $|h_{r,s}\rangle$.

A.2 Elements of Modular S and T Matrices

In a rational conformal field theory, modular transformations of Virasoro characters can be written as finite sums

$$\chi_i(-1/\tau) = \sum_{j\in\mathcal{I}} \mathrm{S}_{ij}\, \chi_j(\tau) \quad \text{and} \quad \chi_i(\tau+1) = \sum_{j\in\mathcal{I}} \mathrm{T}_{ij}\, \chi_j(\tau) \ , \tag{A.16}$$

where $\chi_i(\tau) := \chi_i(q)$ is the Virasoro character (A.1) of an irreducible representation $\mathcal{H}_i$ with $q = e^{2\pi i\tau}$, and $\mathcal{I}$ is the indexing set for the irreducible representations of the chiral algebra $\mathcal{A}$ at the given value of the central charge c.

A.2.1 Virasoro Minimal Models

Since detailed derivation of elements of the modular S and T matrices for Virasoro minimal models can be found, for example, in [56], we only quote the results.

For a Virasoro minimal model $M(p,q)$, the modular T matrix is given by

$$\mathrm{T}_{ij} = e^{2\pi i\left(h_i - \frac{c}{24}\right)} \delta_{i,j} \ , \tag{A.17}$$

where i and j denote Kac labels. The modular S matrix is given by

$$\mathrm{S}_{(r_1,s_1)(r_2,s_2)} = \sqrt{\frac{8}{pq}} (-1)^{1+r_1 s_2 + s_1 r_2} \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) \ . \tag{A.18}$$

Since Virasoro representations are all self-conjugate, the charge conjugation matrix is trivial, and we have $\mathrm{S}^2 = \mathbf{1}$. In addition, S is an orthogonal matrix as all the elements (A.18) are real. Note that the summations (A.17) are over the indexing set $\mathcal{I}$ for the Virasoro irreducible representations, thus we should not forget the identification $(r,s) \sim (p-r, q-s)$ and sum over distinct Kac labels.

A.2.2 $N = 1$ Super-Virasoro Minimal Models

For $N = 1$ Super-Virasoro minimal models, derivation of the modular S and T matrices is similar to the Virasoro case while S and T transformations change the sectors of characters. We present a derivation of the elements of S and T based on [18].

Consider a $N = 1$ super-Virasoro minimal model $SM(p,q)$. For an irreducible highest weight module $\mathcal{H}_{r,s} := \mathcal{H}_h$ with $c = c(p,q)$ and $h = h_{r,s}$ in the Neveu–Schwarz sector, we denote

$$\chi^{\mathrm{NS}}_{r,s}(\tau) := \chi_{\mathcal{H}_{r,s}}(q) \quad \text{and} \quad \tilde{\chi}^{\mathrm{NS}}_{r,s}(\tau) := \tilde{\chi}_{\mathcal{H}_{r,s}}(q) \,, \tag{A.19}$$

where $q = e^{2\pi i \tau}$. Similarly, for an irreducible module $\mathcal{H}^{(\pm)}_{r,s} := \mathcal{H}_{\pm\lambda}$ with $c = c(p,q)$ and $\lambda = \lambda_{r,s}$ in the Ramond sector[6], we write

$$\chi^{\mathrm{R}(\pm)}_{r,s}(\tau) := \chi_{\mathcal{H}^{(\pm)}_{r,s}}(q) \,. \tag{A.20}$$

When $\lambda = 0$, we denote the corresponding character by $\chi^{\mathrm{R}(0)}_{r,s}(\tau)$. Note that these signs have nothing to do with fermion parities as $\mathcal{H}^{(\pm)}_{r,s}$ are ungraded representations. If we introduce the Dedekind eta-function and the Jacobi theta-functions

$$\begin{aligned}
\eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \,, \\
\vartheta_2(\tau) := \vartheta_2(0,\tau) &= 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2 \,, \\
\vartheta_3(\tau) := \vartheta_3(0,\tau) &= \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-\frac{1}{2}})^2 \,, \quad \text{and} \\
\vartheta_4(\tau) := \vartheta_4(0,\tau) &= \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-\frac{1}{2}})^2 \,,
\end{aligned}$$

where $q = e^{2\pi i \tau}$ as usual, then the infinite products in the Verma characters (A.6), (A.8), and (A.12) can be written as

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n} &= q^{\frac{1}{16}} \sqrt{\frac{\vartheta_3(\tau)}{\eta^3(\tau)}} = q^{\frac{1}{16}} \sqrt{\frac{2}{\vartheta_2(\tau)\vartheta_4(\tau)}} \,, \\
\prod_{n=1}^{\infty} \frac{1-q^{n-\frac{1}{2}}}{1-q^n} &= q^{\frac{1}{16}} \sqrt{\frac{\vartheta_4(\tau)}{\eta^3(\tau)}} = q^{\frac{1}{16}} \sqrt{\frac{2}{\vartheta_2(\tau)\vartheta_3(\tau)}} \,, \quad \text{and} \\
\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} &= \sqrt{\frac{\vartheta_2(\tau)}{2\eta^3(\tau)}} = \sqrt{\frac{1}{\vartheta_3(\tau)\vartheta_4(\tau)}} \,,
\end{aligned}$$

6. Note that $\lambda_{r,s}$ and $-\lambda_{r,s}$ give the same character, and thus $\chi^{\mathrm{R}(+)}_{r,s}(\tau) = \chi^{\mathrm{R}(-)}_{r,s}(\tau)$.

where we have used the identity

$$\eta^3(\tau) = \frac{1}{2}\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)\ . \tag{A.21}$$

We can write

$$h_{r,s} - \frac{1}{24}c(p,q) = \frac{(qr-ps)^2}{8pq} - \frac{1}{32}\left(1+(-1)^{r+s}\right)\ , \tag{A.22}$$

which motivate us to define

$$K_\lambda(\tau) \coloneqq \sum_{n\in\mathbb{Z}} q^{\frac{(Nn+\lambda)^2}{4N}} \quad \text{and} \quad \tilde{K}_\lambda(\tau) \coloneqq \sum_{n\in\mathbb{Z}} (-1)^{np} q^{\frac{(Nn+\lambda)^2}{4N}}\ , \tag{A.23}$$

where $N = 2pq$. Note that $K_\lambda = K_{-\lambda} = K_{\lambda+N}$, so we only need to consider $\lambda \mod N$. Since $\gcd(p,q) = 1$ or $\gcd(\frac{p}{2},\frac{q}{2}) = 1$, there exist unique $r_0, s_0 \in \mathbb{Z}$ such that

$$\begin{cases} qr_0 - ps_0 = 1 & \text{if} \quad \gcd(p,q) = 1 \\ qr_0 - ps_0 = 2 & \text{if} \quad \gcd(\frac{p}{2},\frac{q}{2}) = 1 \end{cases}\ . \tag{A.24}$$

Then, we define $\omega_0,\ \tilde{\omega}_0 \in \mathbb{Z}$ to be

$$\begin{cases} \omega_0 \coloneqq qr_0 + ps_0 \mod N & \text{if} \quad \gcd(p,q) = 1 \\ \tilde{\omega}_0 \coloneqq \frac{q}{2}r_0 + \frac{p}{2}s_0 \mod N & \text{if} \quad \gcd(\frac{p}{2},\frac{q}{2}) = 1 \end{cases}\ . \tag{A.25}$$

If we let $\lambda = qr - ps$, we obtain

$$\begin{cases} \omega_0\lambda = qr + ps \mod N & \text{if} \quad \gcd(p,q) = 1 \\ \tilde{\omega}_0\lambda = qr + ps \mod N & \text{if} \quad \gcd(\frac{p}{2},\frac{q}{2}) = 1 \end{cases}\ . \tag{A.26}$$

Therefore, we can write the Virasoro characters as

$$\begin{aligned} \chi^{\rm NS}_{r,s}(\tau) &= \left(K_\lambda(\tau) - K_{\lambda'}(\tau)\right)\sqrt{\frac{2}{\vartheta_2(\tau)\vartheta_4(\tau)}}\ , \\ \tilde{\chi}^{\rm NS}_{r,s}(\tau) &= \epsilon_{r,s}\left(\tilde{K}_\lambda(\tau) - (-1)^{rs}\tilde{K}_{\lambda'}(\tau)\right)\sqrt{\frac{2}{\vartheta_2(\tau)\vartheta_3(\tau)}}\ , \\ \chi^{\rm R(\pm)}_{r,s}(\tau) &= \left(K_\lambda(\tau) - K_{\lambda'}(\tau)\right)\sqrt{\frac{1}{\vartheta_3(\tau)\vartheta_4(\tau)}}\ , \end{aligned} \tag{A.27}$$

where $\lambda = qr - ps$ and

$$\lambda' = \begin{cases} \omega_0\lambda & \text{if} \quad \gcd(p,q) = 1 \\ \tilde{\omega}_0\lambda & \text{if} \quad \gcd(\frac{p}{2},\frac{q}{2}) = 1 \end{cases}\ . \tag{A.28}$$

We need to determine the modular transformation properties of the functions appearing in (A.27).

Modular transformations of the Jacobi theta-functions are well-known. They are given by

$$\begin{aligned} &\vartheta_2(\tau+1) = e^{\frac{i\pi}{4}}\vartheta_2(\tau)\ , && \vartheta_3(\tau+1) = \vartheta_4(\tau)\ , && \vartheta_4(\tau+1) = \vartheta_3(\tau)\ , \\ &\vartheta_2(-1/\tau) = \sqrt{-i\tau}\vartheta_4(\tau)\ , && \vartheta_3(-1/\tau) = \sqrt{-i\tau}\vartheta_3(\tau)\ , && \vartheta_4(-1/\tau) = \sqrt{-i\tau}\vartheta_2(\tau)\ . \end{aligned} \tag{A.29}$$

For T transformation, let us write $\lambda_{r,s} = qr - ps$, and consider

$$\frac{(Nn + \lambda_{r,s})^2}{4N} = \frac{1}{2}\left(pqn^2 + \lambda_{r,s}\, n + \frac{\lambda_{r,s}^2}{2N}\right) , \tag{A.30}$$

in which

$$pqn^2 = np \mod 2 \quad \text{and} \quad \lambda_{r,s}\, n = np(r-s) \mod 2 , \tag{A.31}$$

since p and q are either both odd or both even. Also notice that

$$\begin{aligned} \frac{\lambda_{r,s}^2}{2N} - \frac{\lambda_{-r,s}^2}{2N} &= -rs \quad \text{and} \\ \frac{\lambda_{r,s}^2}{4N} &= h_{r,s} - \frac{1}{24}c(p,q) + \frac{1}{32}\left(1 + (-1)^{r+s}\right) . \end{aligned}$$

Thus, under $\mathrm{T} : q \mapsto e^{2\pi i}q$, we have

$$\begin{aligned} & q^{\frac{(Nn+\lambda_{r,s})^2}{4N}} - q^{\frac{(Nn+\lambda_{-r,s})^2}{4N}} \\ & \overset{\mathrm{T}}{\mapsto} \begin{cases} e^{\frac{i\pi}{8}}\, e^{2\pi i\left(h_{r,s} - \frac{c}{24}\right)} (-1)^{np} \left(q^{\frac{(Nn+\lambda_{r,s})^2}{4N}} - (-1)^{rs} q^{\frac{(Nn+\lambda_{-r,s})^2}{4N}}\right) & \text{for} \quad r+s \in 2\mathbb{Z} \\ e^{2\pi i\left(h_{r,s} - \frac{c}{24}\right)} \left(q^{\frac{(Nn+\lambda_{r,s})^2}{4N}} - q^{\frac{(Nn+\lambda_{-r,s})^2}{4N}}\right) & \text{for} \quad r+s \in 2\mathbb{Z}+1 \end{cases} . \end{aligned}$$

Taking T transformation of the Jacobi theta-functions (A.29) into account, we find the the action of T on the Virasoro characters as

$$\begin{aligned} \chi_{r,s}^{\mathrm{NS}}(\tau+1) &= \epsilon_{r,s}\, e^{2\pi i\left(h_{r,s} - \frac{c}{24}\right)}\, \tilde{\chi}_{r,s}^{\mathrm{NS}}(\tau) , \\ \tilde{\chi}_{r,s}^{\mathrm{NS}}(\tau+1) &= \epsilon_{r,s}\, e^{2\pi i\left(h_{r,s} - \frac{c}{24}\right)}\, \chi_{r,s}^{\mathrm{NS}}(\tau) , \\ \chi_{r,s}^{\mathrm{R}(\pm)}(\tau+1) &= \qquad e^{2\pi i\left(h_{r,s} - \frac{c}{24}\right)}\, \chi_{r,s}^{\mathrm{R}(\pm)}(\tau) . \end{aligned}$$

The matrix T is diagonal except it exchanges the Virasoro characters and supercharacters in the Neveu–Schwarz sector.

For S transformation, using the Poisson resummation formula, we can write

$$\begin{aligned} K_\lambda(-1/\tau) &= \sqrt{\frac{2\tau}{iN}} \sum_{k\in\mathbb{Z}} q^{\frac{k^2}{N}}\, e^{-2\pi i \frac{k\lambda}{N}} \quad \text{and} \\ \tilde{K}_\lambda(-1/\tau) &= \sqrt{\frac{2\tau}{iN}} \sum_{k} q^{\frac{k^2}{N}}\, e^{-2\pi i \frac{k\lambda}{N}} \quad \text{where} \quad \begin{cases} k \in \mathbb{Z} & \text{if} \quad p \in 2\mathbb{Z} \\ k \in \mathbb{Z} + \frac{1}{2} & \text{if} \quad p \in 2\mathbb{Z}+1 \end{cases} . \end{aligned}$$

Let us rewrite the summations by defining

$$k = \frac{1}{2}(Nn + \mu) \quad \text{where} \quad n \in \mathbb{Z} \quad \text{and} \quad \begin{cases} \mu \in [0, N-2] \cap 2\mathbb{Z} & \text{for} \quad k \in \mathbb{Z} \\ \mu \in [1, N-1] \cap 2\mathbb{Z}+1 & \text{for} \quad k \in \mathbb{Z} + \frac{1}{2} \end{cases} , \tag{A.32}$$

then

$$\sqrt{\frac{2\tau}{iN}} \sum_{k} q^{\frac{k^2}{N}}\, e^{-2\pi i \frac{k\lambda}{N}} = \sqrt{\frac{2\tau}{iN}} \sum_{\mu} e^{-\frac{i\pi\mu\lambda}{N}} \sum_{n\in\mathbb{Z}} e^{-i\pi n\lambda} q^{\frac{(Nn+\mu)^2}{4N}} . \tag{A.33}$$

Note that

$$e^{-i\pi n\lambda_{r,s}} = \begin{cases} 1 & \text{for} \quad r+s \in 2\mathbb{Z} \\ (-1)^{np} & \text{for} \quad r+s \in 2\mathbb{Z}+1 \end{cases} . \tag{A.34}$$

In addition, $\lambda_{\pm r,s} = \pm qr - ps = p(r+s) \mod 2$, which means $\lambda_{r,s}$ is a even number except for $p \in 2\mathbb{Z}+1$ and $r+s \in 2\mathbb{Z}+1$. Finally, we obtain, for $\lambda = \lambda_{r,s}$, with $r+s \in 2\mathbb{Z}$,

$$K_\lambda(-1/\tau) = \sqrt{\frac{2\tau}{iN}} \sum_{\substack{\mu=0 \\ \mu\in 2\mathbb{Z}}}^{N-2} e^{-\frac{i\pi\mu\lambda}{N}} K_\mu(\tau) \quad \text{and} \quad \tilde{K}_\lambda(-1/\tau) = \sqrt{\frac{2\tau}{iN}} \sum_\mu e^{-\frac{i\pi\mu\lambda}{N}} K_\mu(\tau) , \tag{A.35}$$

where the range of the second summation depends on the condition given in (A.32), and for $\lambda = \lambda_{r,s}$ with $r+s \in \mathbb{Z}+1$,

$$K_\lambda(-1/\tau) = \sqrt{\frac{2\tau}{iN}} \sum_{\substack{\mu=0 \\ \mu\in 2\mathbb{Z}}}^{N-2} e^{-\frac{i\pi\mu\lambda}{N}} \tilde{K}_\mu(\tau) . \tag{A.36}$$

We still need to combine the contributions from $K_{\lambda'}(-1/\tau)$ and $\tilde{K}_{\lambda'}(-1/\tau)$. Since $\omega_0^2 = 1 \mod 2N$ and $\tilde{\omega}_0^2 = 1 \mod 2N$, we can write

$$K_{\lambda'}(-1/\tau) = \sqrt{\frac{2\tau}{iN}} \sum_\mu e^{-\frac{i\pi\mu\lambda'}{N}} K_\mu(\tau) = \sqrt{\frac{2\tau}{iN}} \sum_\nu e^{-\frac{i\pi\nu\lambda}{N}} K_{\nu'}(\tau) , \tag{A.37}$$

where $\nu = \omega_0\mu$ or $\nu = \tilde{\omega}_0\mu$. If we define

$$X_\lambda(\tau) := K_\lambda(\tau) - K_{\lambda'}(\tau) \quad \text{and} \quad \tilde{X}_\lambda(\tau) := \tilde{K}_\lambda(\tau) - (-1)^{rs}\tilde{K}_{\lambda'}(\tau) , \tag{A.38}$$

where $\lambda = \lambda_{r,s}$, they satisfy $X_\lambda = X_{\lambda+N} = X_{-\lambda} = -X_{\lambda'}$. Using the fact that

$$\begin{aligned} \lambda_{r,s} + 2 &= \begin{cases} \lambda_{r+2r_0,s+2s_0} & \text{for} \quad p \in 2\mathbb{Z}+1 \\ \lambda_{r+r_0,s+s_0} & \text{for} \quad p \in 2\mathbb{Z} \end{cases} , \\ N - \lambda'_{r,s} &= \lambda'_{p-r,q-s} \mod N , \\ \lambda'_{r,s} &= \lambda_{r,-s} \mod N , \end{aligned}$$

and so on, if we denote the set of distinct Kac labels in the NS and R sectors by $\mathcal{I}_{\mathrm{NS}}$ and $\mathcal{I}_{\mathrm{R}}$, we obtain

$$\begin{aligned} X_\lambda(-1/\tau) &= \sqrt{\frac{2\tau}{iN}} \sum_{\substack{\mu=\mu_{r_2,s_2} \\ (r_2,s_2)\in\mathcal{I}_{\mathrm{NS}}}} \left(e^{-\frac{i\pi\mu\lambda}{N}} + e^{\frac{i\pi\mu\lambda}{N}} - e^{-\frac{i\pi\mu'\lambda}{N}} - e^{\frac{i\pi\mu'\lambda}{N}} \right) X_\mu(\tau) \\ &= \sqrt{\frac{2\tau}{iN}} \sum_\mu 2 \left(\cos\left(\frac{\pi\mu\lambda}{N}\right) - \cos\left(\frac{\pi\mu'\lambda}{N}\right) \right) X_\mu(\tau) \\ &= \sqrt{\frac{-i\tau}{pq}} \sum_{(r_2,s_2)\in\mathcal{I}_{\mathrm{NS}}} 4(-1)^{\frac{1}{2}(r_1-s_1)(r_2+s_2)} \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) X_{\mu_{r_2,s_2}}(\tau) \end{aligned}$$

for $\lambda = \lambda_{r_1,s_2}$ with $(r_1, s_1) \in \mathcal{I}_{\rm NS}$. Note that there is a factor of $(-1)^{\frac{1}{2}(r_1-s_1)(r_2+s_2)}$ in the final line, which is always one for this range of summation. Similarly, we can calculate

$$\begin{aligned}
&\tilde{X}_\lambda(-1/\tau) \\
&= \sqrt{\frac{-i\tau}{pq}} \sum_{(r_2,s_2)\in\mathcal{I}_{\rm R}} (-1)^{\frac{1}{2}(r_1-s_1)} 2(2-\delta_{r_2,\frac{p}{2}}\delta_{s_2,\frac{q}{2}}) \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) X_{\mu_{r_2,s_2}}(\tau)
\end{aligned}$$

for $\lambda = \lambda_{r_1,s_2}$ with $(r_1, s_1) \in \mathcal{I}_{\rm NS}$. Note that the coefficient is different at the Ramond fixed point $(r_2, s_2) = (\frac{p}{2}, \frac{q}{2})$ as the identification of Kac labels is trivial. The remaining case is

$$X_\lambda(-1/\tau) = \sqrt{\frac{-i\tau}{pq}} \sum_{(r_2,s_2)\in\mathcal{I}_{\rm NS}} (-1)^{\frac{1}{2}(r_2-s_2)} 4 \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) \tilde{X}_{\mu_{r_2,s_2}}(\tau) \tag{A.39}$$

for $\lambda = \lambda_{r_1,s_2}$ with $(r_1, s_1) \in \mathcal{I}_{\rm R}$.

Finally, taking S transformation of the Jacobi theta-functions (A.29) into account, we obtain the modular S transformations of the Virasoro characters as

$$\begin{aligned}
\chi^{\rm NS}_{r_1,s_1}(-1/\tau) &= \frac{4}{\sqrt{pq}} \sum_{(r_2,s_2)\in\mathcal{I}_{\rm NS}} \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) \chi^{\rm NS}_{r_2,s_2}(\tau)\ , \\
\tilde{\chi}^{\rm NS}_{r_1,s_1}(-1/\tau) &= \frac{2}{\sqrt{pq}} \sum_{(r_2,s_2)\in\mathcal{I}_{\rm R}} \epsilon_{r_1,s_1}(2-\delta_{r_2,\frac{p}{2}}\delta_{s_2,\frac{q}{2}})(-1)^{\frac{1}{2}(r_1-s_1)} \\
&\qquad \times \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) \sqrt{2}\chi^{\rm R(\pm)}_{r_2,s_2}(\tau)\ , \\
\sqrt{2}\chi^{\rm R(\pm)}_{r_1,s_1}(-1/\tau) &= \frac{4}{\sqrt{pq}} \sum_{(r_2,s_2)\in\mathcal{I}_{\rm NS}} \epsilon_{r_2,s_2}(-1)^{\frac{1}{2}(r_2-s_2)} \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right) \tilde{\chi}^{\rm NS}_{r_2,s_2}(\tau)\ .
\end{aligned}$$

Not only the presence of Ramond fixed point but also the factors of $\sqrt{2}$ make the S matrix non-symmetric. In addition, if we recall the definition of $\chi^{\rm R(\pm)}_{r,s}(\tau)$ given in (A.20), they correspond to ungraded representations, and it is more desirable to write the S matrix in terms of the characters of graded modules given by

$$\chi^{\rm R}_{r,s}(\tau) := \begin{cases} \chi^{\rm R(+)}_{r,s}(\tau) + \chi^{\rm R(-)}_{r,s}(\tau) & \text{if} \quad (r,s) \neq (\frac{p}{2}, \frac{q}{2}) \\ \chi^{\rm R(0)}_{\frac{p}{2},\frac{q}{2}}(\tau) \end{cases} . \tag{A.40}$$

In the literature, there are two ways to define the S matrix: the first method, which is more or less common, introduces so-called modified Ramond characters and yields a symmetric and orthogonal S matrix[13, 18, 73, 78]; the second method defines the S matrix of a fermionic theory, which is not necessarily symmetric or unitary, together with some additional matrices[50]. Nevertheless, we can always write the modular S transformation of $\chi^{\rm NS}_{r,s}(\tau)$ as

$$\chi^{\rm NS}_{r_1,s_1}(-1/\tau) = \sum_{(r_2,s_2)\in\mathcal{I}_{\rm NS}} {\rm S}^{\rm [NS,NS]}_{(r_1,s_1)(r_2,s_2)}\, \chi^{\rm NS}_{r_2,s_2}(\tau) \tag{A.41}$$

with

$${\rm S}^{\rm [NS,NS]}_{(r_1,s_1)(r_2,s_2)} = \frac{4}{\sqrt{pq}} \sin\left(\pi r_1 r_2 \frac{q}{p}\right) \sin\left(\pi s_1 s_2 \frac{p}{q}\right)\ , \tag{A.42}$$

and the corresponding submatrix of S is symmetric and orthogonal.

• Symmetric Modular S Matrix

If we introduce modified Ramond characters that are given by

$$\hat{\chi}^{\mathrm{R}}_{r,s}(\tau) := \begin{cases} \frac{1}{\sqrt{2}}\chi^{\mathrm{R}}_{r,s}(\tau) = \sqrt{2}\chi^{\mathrm{R}(\pm)}_{r,s}(\tau) & \text{if} \quad (r,s) \neq (\frac{p}{2}, \frac{q}{2}) \\ \chi^{\mathrm{R}}_{\frac{p}{2},\frac{q}{2}}(\tau) & \end{cases} , \tag{A.43}$$

then we can write

$$\begin{aligned} \tilde{\chi}^{\mathrm{NS}}_{r_1,s_1}(-1/\tau) &= \sum_{(r_2,s_2)\in\mathcal{I}_{\mathrm{R}}} \hat{\mathrm{S}}^{[\mathrm{NS,R}]}_{(r_1,s_1)(r_2,s_2)}\,\hat{\chi}^{\mathrm{R}}_{r_2,s_2}(\tau) \quad \text{and} \\ \hat{\chi}^{\mathrm{R}}_{r_1,s_1}(-1/\tau) &= \sum_{(r_2,s_2)\in\mathcal{I}_{\mathrm{NS}}} \hat{\mathrm{S}}^{[\mathrm{R,NS}]}_{(r_1,s_1)(r_2,s_2)}\,\tilde{\chi}^{\mathrm{NS}}_{r_2,s_2}(\tau) \end{aligned}$$

with

$$\begin{aligned} \hat{\mathrm{S}}^{[\mathrm{NS,R}]}_{(r_1,s_1)(r_2,s_2)} &= \epsilon_{r_1,s_1}\frac{4\,g_{r_2,s_2}}{\sqrt{pq}}(-1)^{\frac{1}{2}(r_1-s_1)}\sin\left(\pi r_1 r_2\frac{q}{p}\right)\sin\left(\pi s_1 s_2\frac{p}{q}\right) \quad \text{and} \\ \hat{\mathrm{S}}^{[\mathrm{R,NS}]}_{(r_1,s_1)(r_2,s_2)} &= \epsilon_{r_2,s_2}\frac{4\,g_{r_1,s_1}}{\sqrt{pq}}(-1)^{\frac{1}{2}(r_2-s_2)}\sin\left(\pi r_1 r_2\frac{q}{p}\right)\sin\left(\pi s_1 s_2\frac{p}{q}\right) , \end{aligned} \tag{A.44}$$

where $g_{r,s}$ is defined as

$$g_{r,s} := \begin{cases} 1 & \text{if} \quad (r,s) \neq (\frac{p}{2}, \frac{q}{2}) \\ \frac{1}{\sqrt{2}} & \text{if} \quad (r,s) = (\frac{p}{2}, \frac{q}{2}) \end{cases} . \tag{A.45}$$

Then, the modular S matrix

$$\hat{\mathrm{S}} = \begin{pmatrix} \mathrm{S}^{[\mathrm{NS,NS}]} & 0 & 0 \\ 0 & 0 & \hat{\mathrm{S}}^{[\mathrm{NS,R}]} \\ 0 & \hat{\mathrm{S}}^{[\mathrm{R,NS}]} & 0 \end{pmatrix} \tag{A.46}$$

is symmetric and orthogonal.

At the chiral level, it seems strange to have characters of the form $\sqrt{2}q^{h-\frac{c}{24}}(1+\cdots)$ but it turns out to be rather convenient from the bulk point of view. Since the bulk Ramond ground state is two-dimensional rather than four-dimensional, the Ramond sector of bulk partition function can be written in terms of $\hat{\chi}^{\mathrm{R}}_i(\tau)\hat{\chi}^{\mathrm{R}}_i(\bar{\tau})$.

• Non-Symmetric Modular S Matrix

It is possible to define a modular S matrix using the Ramond characters given in (A.40) but some care is needed as it results in a non-symmetric S matrix. In this case, we can write

$$\begin{aligned} \tilde{\chi}^{\mathrm{NS}}_{r_1,s_1}(-1/\tau) &= \sum_{(r_2,s_2)\in\mathcal{I}_{\mathrm{R}}} \mathrm{S}^{[\mathrm{NS,R}]}_{(r_1,s_1)(r_2,s_2)}\,\chi^{\mathrm{R}}_{r_2,s_2}(\tau) \quad \text{and} \\ \chi^{\mathrm{R}}_{r_1,s_1}(-1/\tau) &= \sum_{(r_2,s_2)\in\mathcal{I}_{\mathrm{NS}}} \mathrm{S}^{[\mathrm{R,NS}]}_{(r_1,s_1)(r_2,s_2)}\,\tilde{\chi}^{\mathrm{NS}}_{r_2,s_2}(\tau) \end{aligned}$$

with

$$\begin{aligned} S^{[\mathrm{NS,R}]}_{(r_1,s_1)(r_2,s_2)} &= \epsilon_{r_1,s_1}\frac{2\sqrt{2}}{\sqrt{pq}}(-1)^{\frac{1}{2}(r_1-s_1)}\sin\left(\pi r_1 r_2\frac{q}{p}\right)\sin\left(\pi s_1 s_2\frac{p}{q}\right) \quad \text{and} \\ S^{[\mathrm{R,NS}]}_{(r_1,s_1)(r_2,s_2)} &= \epsilon_{r_2,s_2}\frac{4\sqrt{2}}{\sqrt{pq}}\,g^2_{r_1,s_1}(-1)^{\frac{1}{2}(r_2-s_2)}\sin\left(\pi r_1 r_2\frac{q}{p}\right)\sin\left(\pi s_1 s_2\frac{p}{q}\right)\ , \end{aligned} \tag{A.47}$$

where $g_{r,s}$ is the same as before. They give the modular S matrix

$$\mathrm{S} = \begin{pmatrix} \mathrm{S}^{[\mathrm{NS,NS}]} & 0 & 0 \\ 0 & 0 & \mathrm{S}^{[\mathrm{NS,R}]} \\ 0 & \mathrm{S}^{[\mathrm{R,NS}]} & 0 \end{pmatrix}\ , \tag{A.48}$$

which is clearly not symmetric. Since (A.44) and (A.47) are related by

$$\mathrm{S}^{[\mathrm{NS,R}]}_{ij} = \frac{1}{\sqrt{2}g_j}\hat{\mathrm{S}}^{[\mathrm{NS,R}]}_{ij} \quad \text{and} \quad \mathrm{S}^{[\mathrm{R,NS}]}_{ij} = \sqrt{2}g_i\,\hat{\mathrm{S}}^{[\mathrm{R,NS}]}_{ij}\ , \tag{A.49}$$

we introduce a $|\mathcal{I}_{\mathrm{R}}| \times |\mathcal{I}_{\mathrm{R}}|$ diagonal matrix g defined as

$$\mathrm{g}_{ij} := \sqrt{2}\,g_i\,\delta_{i,j}\ . \tag{A.50}$$

Then, the submatrices are related by $\mathrm{S}^{[\mathrm{NS,R}]} = \hat{\mathrm{S}}^{[\mathrm{NS,R}]}\mathrm{g}^{-1}$ and $\mathrm{S}^{[\mathrm{R,NS}]} = \mathrm{g}\hat{\mathrm{S}}^{[\mathrm{R,NS}]}$. Furthermore, if we define

$$\mathrm{G} := \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathrm{g} \end{pmatrix}\ , \tag{A.51}$$

these relations can be written as

$$\mathrm{S} = \mathrm{G}\,\hat{\mathrm{S}}\,\mathrm{G}^{-1}\ . \tag{A.52}$$

Note that G is related to the matrix D, which was give in [50], by $\mathrm{G}^2 = D$. As described in [50], S obeys the equation $\mathrm{S}^{\mathrm{T}}\,D^{-1}\,\mathrm{S} = D^{-1}$.

A.3 Fusing Matrices for Virasoro Minimal Models

We reproduce the explicit formula for fusing matrix elements given in Appendix A.4 of Runkel's PhD thesis[71] which is based on the results given in [7] and [34].

For a Virasoro minimal model $M(p,q)$, define the quantities $t := p/q$ and $d_i := r_i - s_i\,t$, where $i = (r_i, s_i)$ denotes a Kac label. Note that $1 < p < q$, and $p, q \in \mathbb{Z}$ are coprime. In addition, define the functions

$$\begin{aligned} b_{xy}(\alpha,\beta;\rho) &:= \prod_{g=1}^{y}\frac{\Gamma(g\rho)\Gamma(\alpha+g\rho)\Gamma(\beta+g\rho)}{\Gamma(\rho)\Gamma(\alpha+\beta-2x+(y+g)\rho)} \quad \text{and} \\ m_{xy}&(\alpha,\beta) \\ &:= t^{2xy}\prod_{g=1}^{x}\prod_{h=1}^{y}\big((ht-g)(\alpha+ht-g)(\beta+ht-g)(\alpha+\beta+(y+h)t-(x+g))\big)^{-1}\ , \end{aligned}$$

from which we let

$$J(x,y;\alpha,\beta) \coloneqq m_{xy}(\alpha,\beta)\, b_{yx}(-\tfrac{1}{t}\alpha,-\tfrac{1}{t}\beta;\tfrac{1}{t})\, b_{xy}(\alpha,\beta;t)\ . \tag{A.53}$$

We need one more function

$$\begin{aligned}
&A(s;x,y;\alpha,\beta,\gamma,\delta;\rho)\\
&\quad\coloneqq \sum_{h=\max(x,y)}^{\min(s,x+y-1)} \frac{\prod_{g=1}^{s-h}\sin\pi(\delta+(x-1+g)\rho)\prod_{g=1}^{h-y}\sin\pi(-\alpha+(s-x+g)\rho)}{\prod_{g=1}^{s-y}\sin\pi(-\alpha+\delta+(s-y+g)\rho)}\\
&\qquad\times\frac{\prod_{g=1}^{y-1-(h-x)}\sin\pi(\beta+(s-x+g)\rho)\prod_{g=1}^{h-x}\sin\pi(\gamma+(x-1+g)\rho)}{\prod_{g=1}^{y-1}\sin\pi(\beta+\gamma+(y-1+g)\rho)}\\
&\qquad\qquad\times\prod_{g=1}^{h-x}\frac{\sin\pi((x+y-h-1+g)\rho)}{\sin\pi(g\rho)}\prod_{g=1}^{s-h}\frac{\sin\pi((h-y+g)\rho)}{\sin\pi(g\rho)}\ .
\end{aligned}$$

In terms of the functions J and A, fusing matrix elements are given by

$$\begin{aligned}
\mathsf{F}_{pq}\big[\begin{smallmatrix} j & k\\ i & l\end{smallmatrix}\big] &= \frac{J\big(\frac{1}{2}(r_l-r_i-1+r_q),\frac{1}{2}(s_l-s_i-1+s_q);-d_i,d_l\big)}{J\big(\frac{1}{2}(r_j-r_i-1+r_p),\frac{1}{2}(s_j-s_i-1+s_p);-d_i,d_j\big)}\\
&\qquad\times\frac{J\big(\frac{1}{2}(r_j+r_k-1-r_q),\frac{1}{2}(s_j+s_k-1-s_q);d_j,d_k\big)}{J\big(\frac{1}{2}(r_k+r_l-1-r_p),\frac{1}{2}(s_k+s_l-1-s_p);d_k,d_l\big)}\\
&\times A\big(\tfrac{1}{2}(-r_i+r_j+r_k+r_l);\tfrac{1}{2}(r_k+r_l+1-r_p),\tfrac{1}{2}(r_j+r_k+1-r_q);-\tfrac{1}{t}d_i,-\tfrac{1}{t}d_j,-\tfrac{1}{t}d_k,-\tfrac{1}{t}d_l;\tfrac{1}{t}\big)\\
&\quad\times A\big(\tfrac{1}{2}(-s_i+s_j+s_k+s_l);\tfrac{1}{2}(s_k+s_l+1-s_p),\tfrac{1}{2}(s_j+s_k+1-s_q);d_i,d_j,d_k,d_l;t\big)\ .
\end{aligned} \tag{A.54}$$

While this equation (A.54) looks rather complicated, it is straightforward to implement it in a computer program.

Appendix B

Conformal Defects in Ising Model

In [54] and [59], conformal defects in the Ising model are obtained by identifying the folded model, which has $c = 1$, as a special case of the Ashkin-Teller model and the $\mathbb{Z}_2$-orbifold of a free boson compactified on a circle of radius $r = 1$. As in [59], we use the normalisation in which the self-dual radius of the free boson theory is $r = 1/\sqrt{2}$. In addition, we always take the Ising model as the unitary Virasoro minimal model $M(3,4)$ with the diagonal modular invariant bulk partition function.

B.1 Conformal Defects from Orbifolded Free Boson Theory

In the orbifolded free boson theory, there are two classes of conformal boundary conditions: continuous families of Dirichlet and Neumann boundary conditions. In [59], they are constructed from the boundary states of the unorbifolded theory preserving the U(1) symmetry. As mentioned in [91], there may be extra defects associated with the boundary states of the unorbifolded theory braking the U(1) symmetry, however we do not consider this possibility in this report.

The Dirichlet boundary conditions are denoted by $D_O(\varphi_0)$, where $\varphi_0 \in \mathbb{R}$ are the eigenvalues of the free boson at the boundaries, and the Neumann boundary conditions are denoted by $N_O(\tilde{\varphi}_0)$, where $\tilde{\varphi}$ is the dual field. In terms of the chiral components of the free boson $\varphi = \varphi_L + \varphi_R$, the dual field is given by $\tilde{\varphi} = \varphi_L - \varphi_R$. In this section, we follow the notations of [59]; the subscripts distinguish $D_O(\varphi_0)$ and $N_O(\tilde{\varphi}_0)$ from the boundary states $D(\varphi_0)$ and $N(\tilde{\varphi}_0)$ in the unorbifolded theory. The boundary conditions satisfy

$$D_O(\varphi_0) = D_O(-\varphi_0) = D_O(\varphi_0 + 2\pi) \quad \text{and} \quad N_O(\tilde{\varphi}_0) = N_O(-\tilde{\varphi}_0) = N_O(\tilde{\varphi}_0 + \pi) \ , \quad \text{(B.1)}$$

thus we take the fundamental domains $\varphi_0 \in [0, \pi]$ and $\tilde{\varphi}_0 \in [0, \pi/2]$. These boundary conditions are elementary except at the endpoints of the fundamental domains, where they split into two elementary boundary conditions due to the presence of twisted sectors. The elementary boundary states are

$$\begin{gathered} \| D_O(\varphi_0) \rangle\!\rangle \quad \text{with} \quad \varphi_0 \in (0, \pi) \ , \quad \| D_O(0) \rangle\!\rangle_\pm \ , \quad \| D_O(\pi) \rangle\!\rangle_\pm \ , \\ \| N_O(\tilde{\varphi}_0) \rangle\!\rangle \quad \text{with} \quad \tilde{\varphi}_0 \in (0, \pi/2) \ , \quad \| N_O(0) \rangle\!\rangle_\pm \ , \quad \text{and} \quad \| N_O(\pi/2) \rangle\!\rangle_\pm \ . \end{gathered}$$

Consider a torus of circumferences β and $2l$ with two defect lines along non-contractible circles at the diametrically opposite locations separated by l. After folding, this torus becomes a cylinder with two boundaries separated by l. Then, from [59], the torus partition function (2.268) for two continuous Dirichlet defects can be written in terms of the boundary states of the orbifolded theory as

$$Z_{D_O(\varphi_0)|D_O(\varphi_0')} = \langle\!\langle D_O(\varphi_0) \| \tilde{q}^{\frac{1}{2}(L_0^{(c=1)} + \bar{L}_0^{(c=1)} - 1/12)} \| D_O(\varphi_0') \rangle\!\rangle = Z_r(\varphi_0 - \varphi_0') + Z_r(\varphi_0 + \varphi_0') \ , \quad \text{(B.2)}$$

where

$$Z_r(\varphi_0 \pm \varphi_0') = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{2r^2(n+(\varphi_0\pm\varphi_0')/2\pi)^2} \tag{B.3}$$

with the Dedekind eta function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \ , \tag{B.4}$$

$\tilde{q} = e^{-2\pi i/\tau}$, $q = e^{2\pi i\tau}$, and $\tau = i\beta/2l$. Here, $L_0^{(c=1)}$ and $\bar{L}_0^{(c=1)}$ are the Virasoro generators of the compactified free boson theory on the plane, and $\varphi \sim \varphi + 2\pi r$. We should keep in mind that the boundary states carry momentum and winding modes. Since the Neumann boundary conditions become the Dirichlet boundary conditions for the dual fields $\tilde{\varphi}$, the torus partition functions for Neumann defects can be calculated by substituting $r \to \tilde{r} = 1/2r$, $\varphi_0 \to 2\tilde{\varphi}_0$, and $\varphi_0' \to 2\tilde{\varphi}_0'$ in (B.2). By computing the spectra of defect fields $Z_{D_O(\varphi_0)|D_O(\varphi_0)}$ with $r=1$, we can identify the factorising and topological defects in the family of Dirichlet defects and Neumann defects. This has been done in [59] and [91].

B.1.1 Topological and Factorising Points of Dirichlet and Neumann Defects

As in [59] and [91], by comparing the torus partition functions of Dirichlet and Neumann defects (B.2), and the partition functions of (5.30) and (5.37), we can identified the factorising and topological defects in the family of Dirichlet and Neumann defects. In summary, we have

$$\begin{gathered} D_O(0) = F_{++} \cup F_{--} \ , \quad D_O(\pi/4) = D_{\mathbf{1}} \ , \quad D_O(\pi/2) = F_{ff} \ , \\ D_O(3\pi/4) = D_{\varepsilon} \ , \quad \text{and} \quad D_O(\pi) = F_{-+} \cup F_{+-} \ . \end{gathered} \tag{B.5}$$

Here, $D_O(0) = D_O(0)_+ + D_O(0)_-$ and $D_O(\pi) = D_O(\pi)_+ + D_O(\pi)_-$, where they are identified as $D_O(0)_+ = F_{++}$, $D_O(0)_- = F_{--}$, $D_O(\pi)_+ = F_{-+}$, and $D_O(\pi)_- = F_{+-}$. For the family of Neumann defects, we have

$$N_O(0) = F_{f+} \cup F_{f-}, \quad N_O(\pi/4) = D_{\sigma}, \quad \text{and} \quad N_O(\pi/2) = F_{+f} \cup F_{-F} \ . \tag{B.6}$$

We take $N_O(0) = N_O(0)_+ + N_O(0)_-$ and $N_O(\pi/2) = N_O(\pi/2)_+ + N_O(\pi/2)_-$, where $N_O(0)_+ = F_{f+}$, $N_O(0)_- = F_{f-}$, $N_O(\pi/2)_+ = F_{+f}$, and $N_O(\pi/2)_- = F_{-f}$.

As discussed in [59], the family of Dirichlet defects can be understood in terms of the underlying classical system. Let us consider a square-lattice Ising model on a cylinder with a defect line along the axial direction. The classical Hamiltonian is given by

$$\mathcal{E} = -\sum_{i=1}^{M-1}\sum_{j=1}^{N} \sigma_{i,j}\Big(J_1\sigma_{i,j+1} + J_2\sigma_{i+1,j} + h_{i,j}\Big) - \sum_{j=1}^{N} \sigma_{M,j}\Big(J_1\sigma_{M,j+1} + J_D\sigma_{1,j} + h_{M,j}\Big) \ , \tag{B.7}$$

where $\sigma_{i,j} \in \{-1, 1\}$ are classical spin variables, J_1 is the vertical coupling, J_2 is the horizontal coupling, J_D is the defect coupling, and $h_{i,j}$ are the couplings to the external magnetic field at each site, which we set to zero for the moment. The defect line is given by the altered horizontal couplings between the sites at $i = M$ and $i = 1$. Figure B.1 depicts lattice sites near the defect line. Let us introduce various quantities of our interest. The defect strength is defined by $b = J_D/J_2$. We define $K_1 = \beta J_1$, $K_2 = \beta J_2$, and $K_D = \beta J_D$, where β is the inverse temperature. In terms of the dual coupling K_1^* defined by

$$\sinh 2K_1^* \sinh 2K_1 = 1, \tag{B.8}$$

the critical values of the bulk couplings must satisfy $K_2 = K_1^*$. This is a consequence of the Kramers-Wannier duality. Defect lines in a square-lattice Ising model are discussed in [26]. In [59], the lattice parameters corresponding to the Dirichlet defects in (B.5) are give by

$$\begin{aligned}
D_O(0) = F_{++} \cup F_{--} \quad &\leftrightarrow \quad b \to \infty \quad \text{with} \quad K_2 \to 0 \quad \text{(Infinitely ferromagnetic)},\\
D_O(\pi/4) = D_{\mathbf{1}} \quad &\leftrightarrow \quad b = 1 \quad \text{(No defect)},\\
D_O(\pi/2) = F_{ff} \quad &\leftrightarrow \quad b = 0 \quad \text{(Free boundary)},\\
D_O(3\pi/4) = D_\varepsilon \quad &\leftrightarrow \quad b = -1 \quad \text{(Antiferromagnetic defect)}, \quad \text{and}\\
D_O(\pi) = F_{-+} \cup F_{+-} \quad &\leftrightarrow \quad b \to -\infty \quad \text{with} \quad K_2 \to 0\\
&\qquad \text{(Infinitely antiferromagnetic)} .
\end{aligned}$$

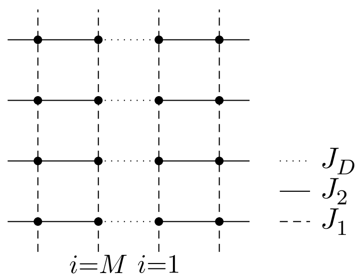

Figure B.1: The square-lattice Ising model with a defect line.

On the other hand, the family of Neumann defects does not have obvious classical descriptions. In the anisotropic limit $K_2 \to 0$, the square-lattice Ising model considered before is equivalent to the one-dimensional quantum transverse field Ising model, whose Hamiltonian is given by

$$H = -\sum_{n\in\mathbb{Z}} \hat{\sigma}^x(n) - \sum_{n\neq 0} \hat{\sigma}^z(n-1)\hat{\sigma}^z(n) - b\hat{\sigma}^z(-1)\hat{\sigma}^z(0) \ , \tag{B.9}$$

where $\hat{\sigma}^x(n)$ and $\hat{\sigma}^z(n)$ are Pauli spin operators. In [59], the Hamiltonian of the one-dimensional quantum system corresponding to the continuous Neumann defects is given by

$$H = -\sum_{n\neq 0} \hat{\sigma}^x(n) - \sum_{n\neq 0} \hat{\sigma}^z(n-1)\hat{\sigma}^z(n) - b\hat{\sigma}^z(-1)\hat{\sigma}^x(0) \ , \tag{B.10}$$

which has the defective link between $n = -1$ and $n = 0$ different from (B.9). This Hamiltonian is the same as (B.9) with the spin operators replaced by the disorder operators for the half of the chain $n \geq 0$. The disorder operators are given by

$$\hat{\mu}^z(n) = \prod_{0\leq m\leq n} \hat{\sigma}^x(m) \quad \text{and} \quad \hat{\mu}^x(n) = \hat{\sigma}^z(n)\hat{\sigma}^z(n+1) \ . \tag{B.11}$$

In addition, continuous Neumann defects can be realised in a certain quantum Hall state with a single vortex in the bulk [97].

In [91], the defect fusion rules of the Ising model are given by

$$\begin{gathered} D_\varepsilon\, D_O(\varphi_0) = D_O(\varphi_0)\, D_\varepsilon = D_O(\pi - \varphi_0)\ , \quad D_\varepsilon\, N_O(\tilde{\varphi}_0) = N_O(\tilde{\varphi}_0)\, D_\varepsilon = N_O(\tilde{\varphi}_0)\ , \\ D_\sigma\, D_O(\varphi_0) = N_O(\varphi_0)\ , \quad D_O(\varphi_0)\, D_\sigma = N_O(\pi/2 - \varphi_0)\ , \\ D_\sigma\, N_O(\tilde{\varphi}_0) = D_O(\tilde{\varphi}_0) + D_O(\pi - \tilde{\varphi}_0)\ , \quad \text{and} \\ N_O(\tilde{\varphi}_0)\, D_\sigma = D_O(\pi/2 - \tilde{\varphi}_0) + D_O(\pi/2 + \tilde{\varphi}_0)\ . \end{gathered} \tag{B.12}$$

From these fusion rules, it is clear that the topological defect D_ε commutes with all the other defects. The topological defect D_σ changes a Dirichlet defect to a Neumann defect and vice versa. Furthermore, the Neumann defects behave like the representation (σ) in the bulk fusion rules (5.27) due to the second line in (B.12); for example, the last two equations of (B.12) can be written as

$$\begin{gathered} D_\sigma\, N_O(\tilde{\varphi}_0) = D_\sigma\, D_\sigma\, D_O(\tilde{\varphi}_0) = (D_{\mathbf{1}} + D_\varepsilon) D_O(\tilde{\varphi}_0) \quad \text{and} \\ N_O(\tilde{\varphi}_0)\, D_\sigma = D_O(\pi/2 - \tilde{\varphi}_0)\, D_\sigma\, D_\sigma = D_O(\pi/2 - \tilde{\varphi}_0)(D_{\mathbf{1}} + D_\varepsilon)\ . \end{gathered}$$

The general fusion rules of conformal defects in the Ising model are given in [108].

B.2 Defect Flows in Ising Model

In this section, we summarise the defect flows in the Ising model investigated in [97] and [98].

One of the important quantities characterising a conformal defect is its g-value, which is also called the entropy or the universal ground state degeneracy. The defect g-value is defined as that of the corresponding conformal boundary in the folded theory. The boundary g-value is the coefficient of $|0\rangle\!\rangle$ in (2.224). Similar to the c-theorem for bulk perturbations, there is the g-theorem, which states that the g-value decreases along the boundary RG flows.

For the Ising model, the g-values of conformal defects are given in [59]: the continuous Neumann defects have $g = \sqrt{2}$ and the continuous Dirichlet defects have $g = 1$. Furthermore, the two elementary defects at the endpoints of the fundamental domains of φ_0 and $\tilde{\varphi}_0$ have $g = 1/\sqrt{2}$ for the Neumann case and $g = 1/2$ for the Dirichlet case, therefore they add up to the same g-values as the continuous cases.

Before we analyse defect flows, let us consider the spectra of defect fields in details. For the topological and factorising defects, we know the spectra from (5.30) and (5.37). For the other Dirichlet and Neumann defects, we can calculate (B.2).

For the Dirichlet defects, the spectra of defect fields are given by

$$Z_{D_O(\varphi_0)|D_O(\varphi_0)} = Z_{r=1}(0) + Z_{r=1}(2\varphi_0)\ , \tag{B.13}$$

where

$$Z_{r=1}(0) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{2n^2} \quad \text{and} \quad Z_{r=1}(2\varphi_0) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{2(n+\varphi_0/\pi)^2}\ . \tag{B.14}$$

They should be considered as $c=1$ Virasoro characters and the scaling dimensions of the defect fields can be read off from these partition functions. Since $c=1$ Virasoro representations are degenerate when $h=(n/2)^2$, where $n\in\mathbb{Z}$, it is useful to rewrite $Z_{r=1}(0)$ as

$$Z_{r=1}(0)=\frac{1}{\eta(q)}\sum_{n=1}^{\infty}2q^{2n^2}+\frac{1}{\eta(q)}\sum_{m=0}^{\infty}\left(q^{m^2}-q^{(m+1)^2}\right)\ . \tag{B.15}$$

Then, it is clear that there are single defect fields with scaling dimensions $\Delta=m^2$, where $m\in\mathbb{Z}_{\geq 0}$, and pairs of defect fields with $\Delta=2n^2$, where $n\in\mathbb{Z}_{>0}$. They appear in the spectra regardless of the values of φ_0. The scaling dimensions of the other defect fields depend on φ_0 and these are given by

$$\Delta_D^n(\varphi_0)=2\left(n+\frac{\varphi_0}{\pi}\right)^2 \quad \text{with} \quad n\in\mathbb{Z} \quad \text{and} \quad \varphi_0\neq 0,\pi\ . \tag{B.16}$$

When $\varphi_0=0,\pi$, the partition function (B.13) reduces to $2Z_{r=1}(0)$. These defect fields with scaling dimensions less than one are plotted in Figure B.2. At $\varphi_0=\pi/4$, the curve Δ_D^{-1} corresponds to the bulk spin field σ on the defect line, which couples to the defect magnetic field. This field becomes dimension zero at $\varphi_0=0$, which means that even an infinitesimal defect magnetic field will split $F_{++}\cup F_{--}$ into either F_{++} or F_{--} [97]. The same argument holds for the curve Δ_D^0 and the factorising defect $F_{-+}\cup F_{+-}$.

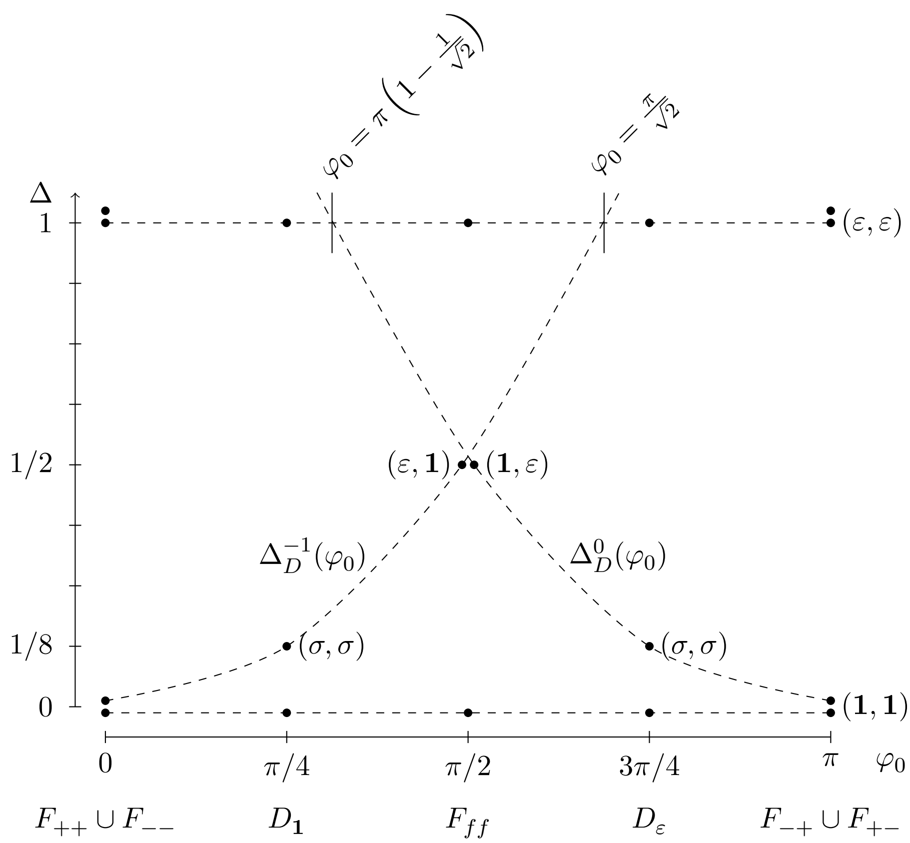

Figure B.2: The spectra of defect fields with $\Delta\leq 1$ for Dirichlet defects.

For the Neumann defects, the spectra of defect fields are given by

$$Z_{N_O(\tilde{\varphi}_0)|N_O(\tilde{\varphi}_0)}=Z_{r=1/2}(0)+Z_{r=1/2}(4\tilde{\varphi}_0)\ . \tag{B.17}$$

Similar to the Dirichlet case, we write

$$Z_{r=1/2}(0) = \frac{1}{\eta(q)} \sum_{n=1}^{\infty} 2q^{n^2/2} + \frac{1}{\eta(q)} \sum_{m=0}^{\infty} \left(q^{m^2} - q^{(m+1)^2} \right) , \tag{B.18}$$

which gives single defect fields with scaling dimensions $\Delta = m^2$, where $m \in \mathbb{Z}_{\geq 0}$, and pairs of defect fields with $\Delta = n^2/2$, where $n \in \mathbb{Z}_{>0}$. The other term

$$Z_{r=1/2}(4\tilde{\varphi}_0) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{(n+2\tilde{\varphi}_0/\pi)^2/2} \tag{B.19}$$

gives defect fields with scaling dimensions

$$\Delta_N^n(\tilde{\varphi}_0) = \frac{1}{2}\left(n + \frac{2\tilde{\varphi}_0}{\pi}\right)^2 \quad \text{with} \quad n \in \mathbb{Z} \quad \text{and} \quad \tilde{\varphi}_0 \neq 0, \pi/2 \, . \tag{B.20}$$

These defect fields with scaling dimensions less than one are plotted in Figure B.3. For the Neumann case, there are always two chiral defect fields with $\Delta = 1/2$, therefore it is possible to perturb the Neumann defects by these chiral defect fields.

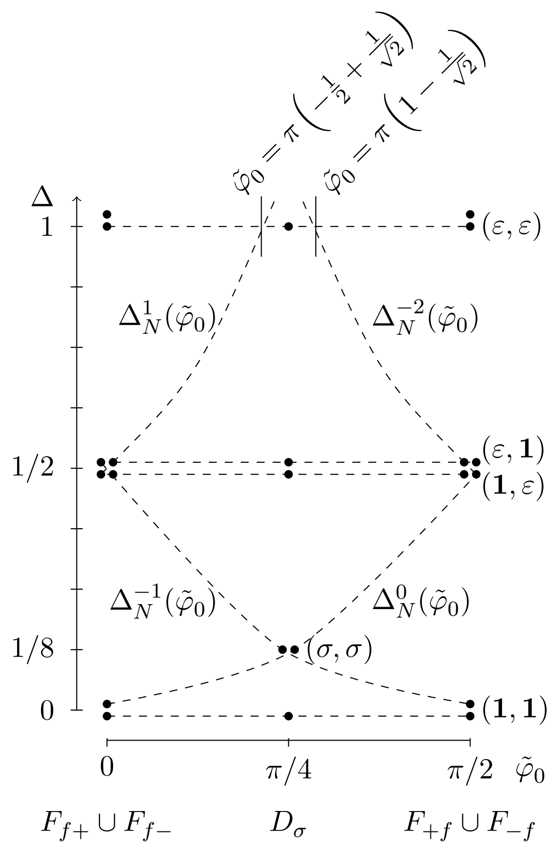

Figure B.3: The spectra of defect fields with $\Delta \leq 1$ for Neumann defects.

B.2.1 Marginal Defect Perturbations

Perturbations of a conformal defect by a defect field with scaling dimension one are marginal and this defect remains conformal along such RG flows. Since a defect field with $\Delta = 1$ is always present in the Dirichlet and Neumann defects, we can consider the two continuous families of conformal defects are generated by the marginal defect perturbations.

In [97], the continuous family of Dirichlet defects is obtained by considering the perturbations of the identity defect by

$$S_{\text{pert}} = \lambda_D \int_{\mathbb{R}} \left(\psi_{\varepsilon,\varepsilon}^{(\mathbf{11})1}(x) + \psi_{\varepsilon,\varepsilon}^{(\mathbf{11})2}(x) \right) \mathrm{d}x \,, \tag{B.21}$$

where $\psi_{\varepsilon,\varepsilon}^{(\mathbf{11})1}(x)$ and $\psi_{\varepsilon,\varepsilon}^{(\mathbf{11})2}(x)$ are the bulk energy fields $\varphi_{\varepsilon,\varepsilon}(z,\bar{z})$ brought to the defect from the opposite sides. At the identity defect, these two defect fields are the same, thus the perturbation term is the integral of $2\lambda_D \psi_{\varepsilon,\varepsilon}^{(\mathbf{11})}$. By parametrising the coupling λ_D as

$$\tan(\delta/2) = \frac{\lambda_D}{2} \,, \tag{B.22}$$

where we have set $v_n = 1$, it is related to φ_0 by

$$\varphi_0 = \frac{\delta}{2} + \frac{\pi}{4} \,. \tag{B.23}$$

For the continuous family of Neumann defects, [97] considers the perturbations of the topological defect D_σ by

$$S_{\text{pert}} = \lambda_N \int_{\mathbb{R}} \left(\psi_{\varepsilon,\varepsilon}^{(\sigma\sigma)1}(x) - \psi_{\varepsilon,\varepsilon}^{(\sigma\sigma)2}(x) \right) \mathrm{d}x \,. \tag{B.24}$$

Since the action of D_σ on a bulk energy field is $\varphi_{\varepsilon,\varepsilon} \to -\varphi_{\varepsilon,\varepsilon}$, the perturbing term is the integral of $2\lambda_N \psi_{\varepsilon,\varepsilon}^{(\sigma\sigma)}$. Again, the coupling λ_N is parametrised by

$$\tan(\tilde{\delta}/2) = \frac{\lambda_N}{2} \tag{B.25}$$

and related to $\tilde{\varphi}_0$ by

$$\tilde{\varphi}_0 = \frac{\tilde{\delta}}{2} + \frac{\pi}{4} \,. \tag{B.26}$$

B.2.2 Perturbation by Chiral Defect Fields

Since the Neumann defects have the highest g-values of $\sqrt{2}$, there should be RG flows to more stable defects, for example, to the Dirichlet defects, whose g-values are one. In [97], the Dirichlet defects are obtained as the endpoints of the perturbations of a Neumann defect by the two chiral defect fields with the scaling dimensions $\Delta = 1/2$. Let us denote these two chiral defect fields by $\psi_{\varepsilon,\mathbf{1}}^{N_O(\tilde{\varphi}_0)}$ and $\psi_{\mathbf{1},\varepsilon}^{N_O(\tilde{\varphi}_0)}$. Then the perturbation term in the action is given by

$$S_{\text{pert}} = \int_{\mathbb{R}} \left(\lambda_l \psi_{\varepsilon,\mathbf{1}}^{N_O(\tilde{\varphi}_0)} + \lambda_r \psi_{\mathbf{1},\varepsilon}^{N_O(\tilde{\varphi}_0)} \right) \mathrm{d}x \,. \tag{B.27}$$

As in [98], by introducing

$$\tan\alpha = \frac{\lambda_r}{\lambda_l} \quad \text{with} \quad \begin{cases} \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & \text{for} \quad \lambda_l > 0 \,, \\ \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) & \text{for} \quad \lambda_l < 0 \end{cases} \tag{B.28}$$

and $\theta = \pi - \alpha$, these flows give

$$N_O(\tilde{\varphi}_0) \to D_O(\tilde{\varphi}_0 + \theta) \ . \tag{B.29}$$

In particular, perturbations of D_σ by (B.27) give

$$D_\sigma \to D_O(5\pi/4 - \alpha) \ . \tag{B.30}$$

These flows are verified by the truncated conformal space approach (TCSA) in [98]. For certain values of α, the endpoints correspond to the factorising and topological defects in the family of Dirichlet defects. These flows are depicted in Figure 13 (a) of [98].

Appendix C

Free Fermion Conventions

Let us briefly summarise the representation theory of the $c = 1/2$ free fermion and its relation to the irreducible highest weight modules of the Virasoro algebra with $c = 1/2$.

From the stress-energy tensors (5.2) and the anticommutation relations (5.4), if we define the operators[109]

$$
\begin{aligned}
L_n^{\mathrm{NS}} &= \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \left(r+\frac{n}{2}\right)(\psi_{-r}\psi_{n+r}) \quad \text{and} \\
L_n^{\mathrm{R}} &= \frac{1}{2}\sum_{r\in\mathbb{Z}} \left(r+\frac{n}{2}\right)(\psi_{-r}\psi_{n+r}) + \frac{1}{16}\delta_{n,0}
\end{aligned}
\tag{C.1}
$$

with $n \in \mathbb{Z}$, then we see that they satisfy the Virasoro algebra relations (2.46) with $c = 1/2$.

C.1 Neveu–Schwarz Sector

For the NS sector, a highest weight representation is constructed from the vacuum state $|0\rangle$ which satisfies

$$
\psi_n|0\rangle = 0 \quad \text{for all} \quad n \in \mathbb{Z}+\frac{1}{2} \quad \text{and} \quad n > 0 \, . \tag{C.2}
$$

Let us denote the highest weight module constructed from $|0\rangle$ by $\mathcal{H}_{\mathrm{NS}}$. Then, $\mathcal{H}_{\mathrm{NS}}$ is spanned by vectors of the form

$$
\psi_{n_1}\psi_{n_2}\cdots\psi_{n_k}|0\rangle \quad \text{with} \quad n_1 < n_2 < \cdots < n_k < 0 \quad \text{and} \quad n_1, n_2, \ldots, n_k \in \mathbb{Z}+\frac{1}{2} \, . \tag{C.3}
$$

Note that the anticommutation relations (5.4) imply the Verma module constructed from $|0\rangle$ is unitary, and therefore irreducible. In addition, one can show that $\psi_{-\frac{1}{2}}|0\rangle$ is a Virasoro primary state using (C.1).

The Virasoro character of $\mathcal{H}_{\mathrm{NS}}$ is given by[56]

$$
\chi_{\mathrm{NS}}(q) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\, q^{L_0^{\mathrm{NS}}-\frac{1}{48}} = q^{-\frac{1}{48}}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right) \, . \tag{C.4}
$$

Since $|0\rangle$ is bosonic, the corresponding supercharacter is given by

$$
\tilde{\chi}_{\mathrm{NS}}(q) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}(-1)^F q^{L_0^{\mathrm{NS}}-\frac{1}{48}} = q^{-\frac{1}{48}}\prod_{n=1}^{\infty}\left(1-q^{n-\frac{1}{2}}\right) \, . \tag{C.5}
$$

If we denote the Virasoro character of the irreducible representation with conformal weight h in the Virasoro minimal model $M(3,4)$ as $\chi_h(q)$, we obtain the relations

$$
\chi_0(q) = \frac{1}{2}\left(\chi_{\mathrm{NS}}(q)+\tilde{\chi}_{\mathrm{NS}}(q)\right) \quad \text{and} \quad \chi_{\frac{1}{2}}(q) = \frac{1}{2}\left(\chi_{\mathrm{NS}}(q)-\tilde{\chi}_{\mathrm{NS}}(q)\right) \, . \tag{C.6}
$$

C.2 Ramond Ground States

For the Ramond sector, some care is needed due to the fermionic zero modes. Since $(-1)^F$ and $\sqrt{2}\psi_0$ form the two-dimensional real Clifford algebra, the Ramond ground states are degenerate. As we have seen in the $N=1$ super-Virasoro case, there are two choices for a basis: one in which the basis vectors are $(-1)^F$ eigenstates, and the other in which they are ψ_0 eigenstates. We take the $(-1)^F$ eigenstates, and they are denoted by $|\frac{1}{16}\rangle_\pm$ which satisfy

$$\psi_n|\tfrac{1}{16}\rangle_\pm = 0 \quad \text{for all} \quad n \in \mathbb{Z} \quad \text{and} \quad n > 0 \tag{C.7}$$

in addition to

$$\psi_0|\tfrac{1}{16}\rangle_\pm = \frac{1}{\sqrt{2}}|\tfrac{1}{16}\rangle_\pm \quad \text{and} \quad (-1)^F|\tfrac{1}{16}\rangle_\pm = \pm|\tfrac{1}{16}\rangle_\pm \,. \tag{C.8}$$

By $\mathcal{H}_{\mathrm{R}}$ let us denote the module of the extended Ramond free fermion algebra spanned by vectors of the form

$$\psi_{n_1}\psi_{n_2}\cdots\psi_{n_k}|\tfrac{1}{16}\rangle_\pm \quad \text{with} \quad n_1 < n_2 < \cdots < n_k < 0 \quad \text{and} \quad n_1, n_2, \ldots, n_k \in \mathbb{Z} \,. \tag{C.9}$$

By introducing the Virasoro character of the unextended Ramond algebra module $\mathcal{H}_{\mathrm{R}}^{(\pm)}$

$$\chi_{\mathrm{R}}(q) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}^{(\pm)}}\, q^{L_0^{\mathrm{R}} - \frac{1}{48}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n) \ , \tag{C.10}$$

the character of $\mathcal{H}_{\mathrm{R}}$ is given by $2\chi_{\mathrm{R}}(q)$.

In the Ramond sector, chiral free fermions $\psi(z)$ are non-local fields, and their two-point functions have square root branch cuts. As in [22], these branch cuts are described by the chiral spin fields $\sigma(z)$ and $\mu(z)$ that satisfy the OPE

$$\psi(z)\sigma(w) = \frac{1}{\sqrt{z-w}}\,\mu(w) + \text{reg.} \,. \tag{C.11}$$

These spin fields have $h = \frac{1}{16}$, and they are inserted at the endpoints of branch cuts. As we usually require fields to have definite fermion parities, the above OPE gives $\sigma(z)$ and $\mu(z)$ to have the opposite fermion parities. Then, the Ramond ground states in the $(-1)^F$ eigenbasis can be written as

$$\lim_{z\to 0} \sigma(z)|0\rangle = |\tfrac{1}{16}\rangle_+ \quad \text{and} \quad \lim_{z\to 0} \mu(z)|0\rangle = |\tfrac{1}{16}\rangle_- \ , \tag{C.12}$$

where we chose $\sigma(z)$ to be bosonic.

Unlike in the NS sector, simply taking the tensor product $\mathcal{H}_{\mathrm{R}} \otimes \overline{\mathcal{H}}_{\mathrm{R}}$ does not yield the bulk irreducible representation of the Ramond free fermion. In the bulk case, $\sqrt{2}\psi_0$ and $\sqrt{2}\bar{\psi}_0$ form the two-dimensional real Clifford algebra even without the chiral fermion parity operators. Therefore, the bulk Ramond ground states are two-fold degenerate and denoted by $|\pm\rangle_{\mathrm{R}}$ with the conformal dimensions $(h, \bar{h}) = (\frac{1}{16}, \frac{1}{16})$. We take them to be $(-1)^{F+\bar{F}}$ eigenstates

$$(-1)^{F+\bar{F}}|\pm\rangle_{\mathrm{R}} = \pm|\pm\rangle_{\mathrm{R}} \ , \tag{C.13}$$

and they correspond to a rotated basis of the Clifford algebra[22, 105], which gives

$$\psi_0|\pm\rangle_{\mathrm{R}} = \frac{1}{\sqrt{2}} e^{\pm i\frac{\pi}{4}} |\mp\rangle_{\mathrm{R}} \quad \text{and} \quad \bar{\psi}_0|\pm\rangle_{\mathrm{R}} = \frac{1}{\sqrt{2}} e^{\mp i\frac{\pi}{4}} |\mp\rangle_{\mathrm{R}} \,. \tag{C.14}$$

They are related to the chiral Ramond ground states by[22]

$$|\pm\rangle_{\pm} \sim \left(|\tfrac{1}{16}\rangle_{+} \otimes |\overline{\tfrac{1}{16}}\rangle_{\pm}\right) + \left(|\tfrac{1}{16}\rangle_{-} \otimes |\overline{\tfrac{1}{16}}\rangle_{\mp}\right) \,, \tag{C.15}$$

and the action of $(-1)^F$ or $(-1)^{\bar{F}}$ connects them to the orthogonal space spanned by

$$|\pm\rangle_{\pm} \sim \left(|\tfrac{1}{16}\rangle_{+} \otimes |\overline{\tfrac{1}{16}}\rangle_{\pm}\right) - \left(|\tfrac{1}{16}\rangle_{-} \otimes |\overline{\tfrac{1}{16}}\rangle_{\mp}\right) \,. \tag{C.16}$$

In this case, non-local behaviour of Ramond fermions is captured by the bulk spin fields $\sigma(z,\bar{z})$ and $\mu(z,\bar{z})$ that correspond to

$$\lim_{z,\bar{z}\to 0} \sigma(z,\bar{z})|0\rangle = |+\rangle_{\mathrm{R}} \quad \text{and} \quad \lim_{z,\bar{z}\to 0} \mu(z,\bar{z})|0\rangle = |-\rangle_{\mathrm{R}} \,. \tag{C.17}$$

Their OPEs can be found, for example in [22] and [56].

Let $\mathcal{H}_{\mathrm{R}}$ denote the Fock space generated by the action of negative modes of $\psi(z)$ and $\bar{\psi}(\bar{z})$ on the states $|\pm\rangle_{\mathrm{R}}$. Then, the corresponding torus partition function can be written as

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}} \left(q^{L_0^{\mathrm{R}} - \frac{1}{48}} \bar{q}^{\bar{L}_0^{\mathrm{R}} - \frac{1}{48}} \right) = 2|\chi_{\mathrm{R}}(q)|^2 \,, \tag{C.18}$$

where $\chi_{\mathrm{R}}(q)$ is given by (C.10).

C.3 GSO Projection of Boundary States

Boundary states

The boundary states of the free fermion theory are

$$|\mathrm{NS},\epsilon\rangle\!\rangle = \prod_{r\in\mathbb{Z}_{>0}} e^{i\epsilon\psi_{-r+1/2}\bar{\psi}_{-r+1/2}} |0\rangle_{\mathrm{NS}} \otimes |0\rangle_{\overline{\mathrm{NS}}}, \tag{C.19}$$

where $\epsilon = \pm$. They are the unique (up to normalisation) solutions of the gluing conditions

$$\left(\psi_r - i\epsilon\bar{\psi}_{-r}\right)|\mathrm{NS},\epsilon\rangle\!\rangle = 0 \quad \text{for all} \quad r \in \mathbb{Z}+1/2. \tag{C.20}$$

Therefore, there are only two boundary states in the free fermion theory, which consists of a single Neveu–Schwarz free fermion.

If we include R-sector as well, then the boundary states become linear combinations of (C.19) and

$$|\mathrm{R},\epsilon\rangle\!\rangle = \sqrt[4]{2} \prod_{r\in\mathbb{Z}_{>0}} e^{i\epsilon\psi_{-r}\bar{\psi}_{-r}} |\epsilon\rangle_{\mathrm{R}}. \tag{C.21}$$

Here, we follow normalisation of [105]—this choice will be apparent when we project these boundary states to obtain the Ising boundary states. In R-sector, the gluing conditions are

$$\left(\psi_r - i\epsilon\bar{\psi}_{-r}\right)|\mathrm{R},\epsilon\rangle\!\rangle = 0 \quad \text{for all} \quad r \in \mathbb{Z} \tag{C.22}$$

and (C.21) solve these.

Let us consider the cylinder partition functions with the boundary states (C.19) and (C.21) placed at the ends. Straightforward calculations show

$$\begin{aligned}
\langle\!\langle \mathrm{NS}, \epsilon \mid \tilde{q}^{\frac{1}{2}(L_0^{\mathrm{NS}}+\bar{L}_0^{\mathrm{NS}}-1/24)} \mid \mathrm{NS}, \epsilon \rangle\!\rangle &= \tilde{q}^{-1/48} \prod_{n=1}^{\infty} \left(1+\tilde{q}^{n-1/2}\right) = \chi_0(\tilde{q}) + \chi_{1/2}(\tilde{q}) \\
\langle\!\langle \mathrm{NS}, \epsilon \mid \tilde{q}^{\frac{1}{2}(L_0^{\mathrm{NS}}+\bar{L}_0^{\mathrm{NS}}-1/24)} \mid \mathrm{NS}, -\epsilon \rangle\!\rangle &= \tilde{q}^{-1/48} \prod_{n=1}^{\infty} \left(1-\tilde{q}^{n-1/2}\right) = \chi_0(\tilde{q}) - \chi_{1/2}(\tilde{q}) \\
\langle\!\langle \mathrm{R}, \epsilon \mid \tilde{q}^{\frac{1}{2}(L_0^{\mathrm{R}}+\bar{L}_0^{\mathrm{R}}-1/24)} \mid \mathrm{R}, \epsilon \rangle\!\rangle &= \sqrt{2}\tilde{q}^{1/24} \prod_{n=1}^{\infty} \left(1+\tilde{q}^{n}\right) = \sqrt{2}\chi_{1/16}(\tilde{q}) \quad \text{and} \\
\langle\!\langle \mathrm{R}, \epsilon \mid \tilde{q}^{\frac{1}{2}(L_0^{\mathrm{R}}+\bar{L}_0^{\mathrm{R}}-1/24)} \mid \mathrm{R}, -\epsilon \rangle\!\rangle &= 0,
\end{aligned}$$

where χ_0, $\chi_{1/2}$, and $\chi_{1/16}$ are the $c = 1/2$ Virasoro characters. In the equations above, the cylinder partition functions are identified with the torus partition of a chiral fermion associated with the given spin structures by comparing the characters. This can be understood as 'unfolding' the boundary theory on the cylinder and considering it as a theory of chiral fermion on the torus.

The GSO projection of the free fermion boundary states are discussed, for example, in [105]. The GSO projected boundary states are obtained as the boundary states in the $\mathbb{Z}_2$-orbifold of the free fermion theory generated by $(-1)^{F+\bar{F}}$. We start from the boundary states of the free fermion theory $|\mathrm{NS}, \epsilon\rangle\!\rangle$. Since each of them is invariant under $(-1)^{F+\bar{F}}$, they should be resolved by adding the R-sector boundary conditions, which yields

$$\|\epsilon, \pm\rangle\!\rangle_{\text{ferm}} = \frac{1}{\sqrt{2}} \left(|\mathrm{NS}, \epsilon\rangle\!\rangle \pm |\mathrm{R}, \epsilon\rangle\right), \tag{C.23}$$

where the factor of $|\mathbb{Z}_2|^{-1/2} = 1/\sqrt{2}$ is chosen as the normalisation. Then, we need to consider the orbits of the boundary states (C.23) under the action of the orbifold group $G \cong \mathbb{Z}_2$. For the moment, let us take the type 0B projection. Since $(-1)^{F+\bar{F}}$ acts non-trivially only on $|\mathrm{R}, -\rangle\!\rangle$ in this case, the boundary states $\|+, \pm\rangle\!\rangle_{\text{ferm}}$ are invariant under G while $\|-, +\rangle\!\rangle_{\text{ferm}}$ and $\|-, -\rangle\!\rangle_{\text{ferm}}$ are related by $(-1)^{F+\bar{F}}$. Thus, there are three orbits which partition the set of boundary states (C.23). We sum the boundary states lying in an orbit and normalise it by $(|\mathrm{Stab}_G|/|G|)^{1/2}$, where Stab_G is the stabiliser subgroup associated with this orbit—for the two fixed points, $\mathrm{Stab}_G = G$, and for the other orbit, it is trivial. This gives the projected boundary states.

Let us summarise the GSO projected boundary states and their relations to the Cardy boundary states of the Ising model (5.34) given in [105]. For the type 0A projection, we have

$$\begin{aligned}
\|+\rangle\!\rangle_{\text{Ising}} &= \|\text{charged}, +\rangle\!\rangle_{\text{ferm}}^{0\text{A}} = \frac{1}{\sqrt{2}} \left(|\mathrm{NS}, -\rangle\!\rangle + |\mathrm{R}, -\rangle\!\rangle\right), \\
\|-\rangle\!\rangle_{\text{Ising}} &= \|\text{charged}, -\rangle\!\rangle_{\text{ferm}}^{0\text{A}} = \frac{1}{\sqrt{2}} \left(|\mathrm{NS}, -\rangle\!\rangle - |\mathrm{R}, -\rangle\!\rangle\right), \quad \text{and} \\
\|f\rangle\!\rangle_{\text{Ising}} &= \|\text{neutral}\rangle\!\rangle_{\text{ferm}}^{0\text{A}} = |\mathrm{NS}, +\rangle\!\rangle.
\end{aligned}$$

For the type 0B projection, we have

$$\begin{aligned} \|+\rangle\rangle_{\text{Ising}} &= \|\text{charged}, +\rangle\rangle^{0\text{B}}_{\text{ferm}} = \frac{1}{\sqrt{2}}\left(|\text{NS}, +\rangle\rangle + |\text{R}, +\rangle\rangle\right), \\ \|-\rangle\rangle_{\text{Ising}} &= \|\text{charged}, -\rangle\rangle^{0\text{B}}_{\text{ferm}} = \frac{1}{\sqrt{2}}\left(|\text{NS}, +\rangle\rangle - |\text{R}, +\rangle\rangle\right), \quad \text{and} \\ \|f\rangle\rangle_{\text{Ising}} &= \|\text{neutral}\rangle\rangle^{0\text{B}}_{\text{ferm}} \qquad = |\text{NS}, -\rangle\rangle. \end{aligned}$$

From these, we can write the free fermion boundary states in terms of the Ishibashi states of the Ising model. If we take the type 0B projection, we can identify

$$\frac{1}{2}\left(|\text{NS}, +\rangle\rangle + |\text{NS}, -\rangle\rangle\right) = |\mathbf{1}\rangle\rangle, \quad \frac{1}{2}\left(|\text{NS}, +\rangle\rangle - |\text{NS}, -\rangle\rangle\right) = |\varepsilon\rangle\rangle, \quad \text{and} \quad \frac{1}{\sqrt[4]{2}}|\text{R}, +\rangle\rangle = |\sigma\rangle\rangle. \tag{C.24}$$

Similarly, for the type 0A case, we have

$$\frac{1}{2}\left(|\text{NS}, -\rangle\rangle + |\text{NS}, +\rangle\rangle\right) = |\mathbf{1}\rangle\rangle, \quad \frac{1}{2}\left(|\text{NS}, -\rangle\rangle - |\text{NS}, +\rangle\rangle\right) = |\varepsilon\rangle\rangle, \quad \text{and} \quad \frac{1}{\sqrt[4]{2}}|\text{R}, -\rangle\rangle = |\sigma\rangle\rangle. \tag{C.25}$$

Expanding (C.19) and (C.21), we can write

$$\begin{aligned} \frac{1}{2}\left(|\text{NS}, \epsilon\rangle\rangle + |\text{NS}, -\epsilon\rangle\rangle\right) &= \big(1 + \psi_{-1/2}\psi_{-3/2}\bar{\psi}_{-1/2}\bar{\psi}_{-3/2} + \psi_{-1/2}\psi_{-5/2}\bar{\psi}_{-1/2}\bar{\psi}_{-5/2} + \cdots \\ &\qquad + \psi_{-3/2}\psi_{-5/2}\bar{\psi}_{-3/2}\bar{\psi}_{-5/2} + \cdots\big)|0\rangle_{\text{NS}} \otimes |0\rangle_{\overline{\text{NS}}}, \\ \frac{1}{2}\left(|\text{NS}, \epsilon\rangle\rangle - |\text{NS}, -\epsilon\rangle\rangle\right) &= i\epsilon\big(\psi_{-1/2}\bar{\psi}_{-1/2} + \psi_{-3/2}\bar{\psi}_{-3/2} + \psi_{-5/2}\bar{\psi}_{-5/2} + \cdots \\ &\qquad + \psi_{-1/2}\psi_{-3/2}\psi_{-5/2}\bar{\psi}_{-1/2}\bar{\psi}_{-3/2}\bar{\psi}_{-5/2} + \cdots\big)|0\rangle_{\text{NS}} \otimes |0\rangle_{\overline{\text{NS}}}, \quad \text{and} \\ |\text{R}, \epsilon\rangle\rangle = \sqrt[4]{2}\big(1 &+ i\epsilon\psi_{-1}\bar{\psi}_{-1} + i\epsilon\psi_{-2}\bar{\psi}_{-2} + i\epsilon\psi_{-3}\bar{\psi}_{-3} + \cdots \\ &+ \psi_{-1}\psi_{-2}\bar{\psi}_{-1}\bar{\psi}_{-2} + \psi_{-1}\psi_{-3}\bar{\psi}_{-1}\bar{\psi}_{-3} + \cdots + \psi_{-2}\psi_{-3}\bar{\psi}_{-2}\bar{\psi}_{-3} + \cdots \\ &\qquad + i\epsilon\psi_{-1}\psi_{-2}\psi_{-3}\bar{\psi}_{-1}\bar{\psi}_{-2}\bar{\psi}_{-3} + \cdots\big)|\epsilon\rangle_{\text{R}}. \end{aligned}$$

Therefore, we see that the type 0B projection relates the highest weight vectors as

$$\begin{aligned} |0\rangle_{\text{NS}} \otimes |0\rangle_{\overline{\text{NS}}} \leftrightarrow |0\rangle \otimes |0\rangle, \quad i\psi_{-1/2}\bar{\psi}_{-1/2}|0\rangle_{\text{NS}} \otimes |0\rangle_{\overline{\text{NS}}} &\leftrightarrow |1/2\rangle \otimes |1/2\rangle, \quad \text{and} \\ |+\rangle_{\text{R}} \leftrightarrow |1/16\rangle \otimes |1/16\rangle&, \end{aligned} \tag{C.26}$$

while for the type 0A projection, we have

$$\begin{aligned} |0\rangle_{\text{NS}} \otimes |0\rangle_{\overline{\text{NS}}} \leftrightarrow |0\rangle \otimes |0\rangle, \quad -i\psi_{-1/2}\bar{\psi}_{-1/2}|0\rangle_{\text{NS}} \otimes |0\rangle_{\overline{\text{NS}}} &\leftrightarrow |1/2\rangle \otimes |1/2\rangle, \quad \text{and} \\ |-\rangle_{\text{R}} \leftrightarrow |1/16\rangle \otimes |1/16\rangle&. \end{aligned} \tag{C.27}$$

Since the factor of i in front of bilinears of the form $\psi_{-r}\bar{\psi}_{-r}$ is introduced to ensure that the left and right fermion modes obey the usual anti-commutation rules, there is no i appearing in (C.24) and (C.25).

Appendix D

Various Quantities in (D_6, E_6) Theory

9	10	$\frac{65}{8}$	$\frac{19}{3}$	$\frac{39}{8}$	$\frac{7}{2}$	$\frac{59}{24}$	$\frac{3}{2}$	$\frac{7}{8}$	$\frac{1}{3}$	$\frac{1}{8}$	0
8	$\frac{621}{80}$	$\frac{481}{80}$	$\frac{1103}{240}$	$\frac{261}{80}$	$\frac{181}{80}$	$\frac{323}{240}$	$\frac{61}{80}$	$\frac{21}{80}$	$\frac{23}{240}$	$\frac{1}{80}$	$\frac{21}{80}$
7	$\frac{57}{10}$	$\frac{173}{40}$	$\frac{91}{30}$	$\frac{83}{40}$	$\frac{6}{5}$	$\frac{79}{120}$	$\frac{1}{5}$	$\frac{3}{40}$	$\frac{1}{30}$	$\frac{13}{40}$	$\frac{7}{10}$
6	$\frac{65}{16}$	$\frac{45}{16}$	$\frac{91}{48}$	$\frac{17}{16}$	$\frac{9}{16}$	$\frac{7}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{19}{48}$	$\frac{13}{16}$	$\frac{25}{16}$
5	$\frac{13}{5}$	$\frac{69}{40}$	$\frac{14}{15}$	$\frac{19}{40}$	$\frac{1}{10}$	$\frac{7}{120}$	$\frac{1}{10}$	$\frac{19}{40}$	$\frac{14}{15}$	$\frac{69}{40}$	$\frac{13}{5}$
4	$\frac{25}{16}$	$\frac{13}{16}$	$\frac{19}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{7}{48}$	$\frac{9}{16}$	$\frac{17}{16}$	$\frac{91}{48}$	$\frac{45}{16}$	$\frac{65}{16}$
3	$\frac{7}{10}$	$\frac{13}{40}$	$\frac{1}{30}$	$\frac{3}{40}$	$\frac{1}{5}$	$\frac{79}{120}$	$\frac{6}{5}$	$\frac{83}{40}$	$\frac{91}{30}$	$\frac{173}{40}$	$\frac{57}{10}$
2	$\frac{21}{80}$	$\frac{1}{80}$	$\frac{23}{240}$	$\frac{21}{80}$	$\frac{61}{80}$	$\frac{323}{240}$	$\frac{181}{80}$	$\frac{261}{80}$	$\frac{1103}{240}$	$\frac{481}{80}$	$\frac{621}{80}$
1	0	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{7}{8}$	$\frac{3}{2}$	$\frac{59}{24}$	$\frac{7}{2}$	$\frac{39}{8}$	$\frac{19}{3}$	$\frac{65}{8}$	10
r / s	1	2	3	4	5	6	7	8	9	10	11

Table D.1: Kac table for $SM(10, 12)$.

9	$\frac{21}{2}$		$\frac{41}{6}$		4		2		$\frac{5}{6}$		$\frac{3}{2}$
8		$\frac{521}{80}$		$\frac{301}{80}$		$\frac{443}{240}$		$\frac{61}{80}$		$\frac{41}{80}$	
7	$\frac{31}{5}$		$\frac{53}{15}$		$\frac{17}{10}$		$\frac{7}{10}$		$\frac{8}{15}$		$\frac{6}{5}$
6		$\frac{53}{16}$		$\frac{25}{16}$		$\frac{31}{48}$		$\frac{9}{16}$		$\frac{21}{16}$	
5	$\frac{31}{10}$		$\frac{43}{30}$		$\frac{3}{5}$		$\frac{3}{5}$		$\frac{43}{30}$		$\frac{31}{10}$
4		$\frac{21}{16}$		$\frac{9}{16}$		$\frac{31}{48}$		$\frac{25}{16}$		$\frac{53}{16}$	
3	$\frac{6}{5}$		$\frac{8}{15}$		$\frac{7}{10}$		$\frac{17}{10}$		$\frac{53}{15}$		$\frac{31}{5}$
2		$\frac{41}{80}$		$\frac{61}{80}$		$\frac{443}{240}$		$\frac{301}{80}$		$\frac{521}{80}$	
1	$\frac{3}{2}$		$\frac{5}{6}$		2		4		$\frac{41}{6}$		$\frac{21}{2}$
r / s	1	2	3	4	5	6	7	8	9	10	11

Table D.2: Conformal weights of superdescendants for $SM(10, 12)$.

D.1 Character Identities

We label characters by either Kac labels or conformal weights. ${}^{(3)}\chi_h^{\mathrm{NS}}(q)$ and ${}^{(3)}\chi_h^{\mathrm{R}}(q)$ denote characters of $SM(3, 5)$, and ${}^{(10)}\chi_h^{\mathrm{NS}}(q)$ and ${}^{(10)}\chi_h^{\mathrm{R}}(q)$ denote characters of $SM(10, 12)$. Ramond characters correspond to the unextended algebra representations.

Relations expressing products of NS characters of $SM(3, 5)$ as sums of NS characters of $SM(10, 12)$ are[94]

$$\begin{aligned}
{}^{(10)}\chi_{1,1}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{1,7}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{9,1}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{9,7}^{\mathrm{NS}}(q) &= ({}^{(3)}\chi_{1,1}^{\mathrm{NS}}(q))^2 \ , \\
{}^{(10)}\chi_{3,1}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{3,7}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{7,1}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{7,7}^{\mathrm{NS}}(q) &= ({}^{(3)}\chi_{1,3}^{\mathrm{NS}}(q))^2 \ , \\
{}^{(10)}\chi_{5,1}^{\mathrm{NS}}(q) + {}^{(10)}\chi_{5,7}^{\mathrm{NS}}(q) &= {}^{(3)}\chi_{1,1}^{\mathrm{NS}}(q)\,{}^{(3)}\chi_{1,3}^{\mathrm{NS}}(q) \ .
\end{aligned} \tag{D.1}$$

The same relations with characters labelled by conformal weights are

$$
\begin{aligned}
{}^{(10)}\chi^{\rm NS}_{0}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{3}{2}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{7}{2}}(q) + {}^{(10)}\chi^{\rm NS}_{10}(q) &= ({}^{(3)}\chi^{\rm NS}_{0}(q))^2 \, , \\
{}^{(10)}\chi^{\rm NS}_{\frac{1}{5}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{7}{10}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{6}{5}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{57}{10}}(q) &= ({}^{(3)}\chi^{\rm NS}_{\frac{1}{10}}(q))^2 \, , \\
{}^{(10)}\chi^{\rm NS}_{\frac{1}{10}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{13}{5}}(q) &= {}^{(3)}\chi^{\rm NS}_{0}(q)\, {}^{(3)}\chi^{\rm NS}_{\frac{1}{10}}(q) \, .
\end{aligned}
$$

There are similar relations for Ramond characters[94]

$$
\begin{aligned}
{}^{(10)}\chi^{\rm R}_{1,4}(q) + {}^{(10)}\chi^{\rm R}_{1,8}(q) &= ({}^{(3)}\chi^{\rm R}_{1,4}(q))^2 \, , & {}^{(10)}\chi^{\rm R}_{\frac{7}{8}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{39}{8}}(q) &= ({}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(q))^2 \, , \\
{}^{(10)}\chi^{\rm R}_{3,4}(q) + {}^{(10)}\chi^{\rm R}_{3,8}(q) &= ({}^{(3)}\chi^{\rm R}_{1,2}(q))^2 \, , & {}^{(10)}\chi^{\rm R}_{\frac{3}{40}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{83}{40}}(q) &= ({}^{(3)}\chi^{\rm R}_{\frac{3}{80}}(q))^2 \, , \\
{}^{(10)}\chi^{\rm R}_{5,4}(q) &= {}^{(3)}\chi^{\rm R}_{1,2}(q)\, {}^{(3)}\chi^{\rm R}_{1,4}(q) \, , & {}^{(10)}\chi^{\rm R}_{\frac{19}{40}}(q) &= {}^{(3)}\chi^{\rm R}_{\frac{3}{80}}(q)\, {}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(q) \, .
\end{aligned}
$$

Moreover, we found that there are relations expressing Ramond characters of $SM(3,5)$ with $\sqrt{q}$ as sums of $SM(10,12)$ characters

$$
\begin{aligned}
{}^{(10)}\chi^{\rm NS}_{2,4}(q) + {}^{(10)}\chi^{\rm NS}_{2,8}(q) &= {}^{(3)}\chi^{\rm R}_{1,4}(\sqrt{q}) \, , & {}^{(10)}\chi^{\rm NS}_{\frac{21}{80}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{261}{80}}(q) &= {}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(\sqrt{q}) \, , \\
{}^{(10)}\chi^{\rm NS}_{4,4}(q) + {}^{(10)}\chi^{\rm NS}_{4,8}(q) &= {}^{(3)}\chi^{\rm R}_{1,2}(\sqrt{q}) \, , & {}^{(10)}\chi^{\rm NS}_{\frac{1}{16}}(q) + {}^{(10)}\chi^{\rm NS}_{\frac{17}{16}}(q) &= {}^{(3)}\chi^{\rm R}_{\frac{3}{80}}(\sqrt{q}) \, .
\end{aligned}
$$

These $\sqrt{q}$ characters of $SM(3,5)$ can be expressed as sums of Ramond characters of $SM(10,12)$ as well

$$
\begin{aligned}
{}^{(10)}\chi^{\rm R}_{2,1}(q) + {}^{(10)}\chi^{\rm R}_{2,7}(q) + {}^{(10)}\chi^{\rm R}_{8,1}(q) + {}^{(10)}\chi^{\rm R}_{8,7}(q) &= {}^{(3)}\chi^{\rm R}_{1,4}(\sqrt{q}) \, , \\
{}^{(10)}\chi^{\rm R}_{4,1}(q) + {}^{(10)}\chi^{\rm R}_{4,7}(q) + {}^{(10)}\chi^{\rm R}_{6,1}(q) + {}^{(10)}\chi^{\rm R}_{6,7}(q) &= {}^{(3)}\chi^{\rm R}_{1,2}(\sqrt{q}) \, .
\end{aligned}
$$

The same relations with characters labelled by conformal weights are

$$
\begin{aligned}
{}^{(10)}\chi^{\rm R}_{\frac{21}{80}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{61}{80}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{181}{80}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{621}{80}}(q) &= {}^{(3)}\chi^{\rm R}_{\frac{7}{16}}(\sqrt{q}) \, , \\
{}^{(10)}\chi^{\rm R}_{\frac{1}{16}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{9}{16}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{25}{16}}(q) + {}^{(10)}\chi^{\rm R}_{\frac{65}{16}}(q) &= {}^{(3)}\chi^{\rm R}_{\frac{3}{80}}(\sqrt{q}) \, .
\end{aligned}
$$

D.2 Expansions of (D_6, E_6) Ishibashi States

In this appendix, we expand the Ishibashi state of the (D_6, E_6) theory as states of the folded theory using the map $\iota_{\alpha\beta\gamma\delta}$ defined in (6.20). In addition we further expand them as defect operators using the map ρ given by (6.15).

In order to do this, we first need to find the image of the highest weight states corresponding to the diagonal terms in the bulk partition function (6.1) under the map $\iota_{\alpha\beta\gamma\delta}$. Among these states

$$
|0\rangle \, , \quad |\tfrac{1}{5}\rangle \, , \quad |\tfrac{1}{10}\rangle \, , \quad \text{and} \quad |\tfrac{1}{10}{}'\rangle \tag{D.2}
$$

can be considered as the super W-algebra highest weight states, therefore the map $\iota_{\alpha\beta\gamma\delta}$ acts trivially. The remaining 8 states

$$
|\tfrac{3}{2}\rangle \, , \quad |\tfrac{7}{2}\rangle \, , \quad |10\rangle \, , \quad |\tfrac{7}{10}\rangle \, , \quad |\tfrac{6}{5}\rangle \, , \quad |\tfrac{57}{10}\rangle \, , \quad |\tfrac{13}{5}\rangle \, , \quad \text{and} \quad |\tfrac{13}{5}{}'\rangle \tag{D.3}
$$

are the super W-algebra descendants, and we need to consider the action of $\iota_{\alpha\beta\gamma\delta}$ explicitly. Throughout this appendix, we shall use $c' = 2c = \frac{7}{5}$.

For the vacuum sector, the Ishibashi states are given by

$$|0,\epsilon\rangle\!\rangle = |0\rangle - \frac{i\epsilon}{2c'/3}G_{-\frac{3}{2}}\bar{G}_{-\frac{3}{2}}|0\rangle + \frac{1}{c'/2}L_{-2}\bar{L}_{-2}|0\rangle - \frac{i\epsilon}{2c'}G_{-\frac{5}{2}}\bar{G}_{-\frac{5}{2}}|0\rangle$$
$$+ \frac{1}{2c'}L_{-3}\bar{L}_{-3}|0\rangle - \frac{3i\epsilon}{c'(c'+12)}L_{-2}G_{-\frac{3}{2}}\bar{L}_{-2}\bar{G}_{-\frac{3}{2}}|0\rangle$$
$$- \frac{81i\epsilon}{c'(c'+12)(21+4c')}\left(L_{-2}G_{-\frac{3}{2}} - \tfrac{c'+12}{9}G_{-\frac{7}{2}}\right)\left(\bar{L}_{-2}\bar{G}_{-\frac{3}{2}} - \tfrac{c'+12}{9}\bar{G}_{-\frac{7}{2}}\right)|0\rangle + \cdots\ ,$$

which can be expanded as

$$\iota_{\alpha\beta\gamma\delta}(|0,\epsilon\rangle\!\rangle) = |0\rangle - \frac{i\epsilon}{4c/3}(\alpha G^{(1)}_{-\frac{3}{2}} + \beta G^{(2)}_{-\frac{3}{2}})(\gamma\bar{G}^{(1)}_{-\frac{3}{2}} + \delta\bar{G}^{(2)}_{-\frac{3}{2}})|0\rangle + \frac{1}{c}(L^{(1)}_{-2} + L^{(2)}_{-2})(\bar{L}^{(1)}_{-2} + \bar{L}^{(2)}_{-2})|0\rangle$$
$$- \frac{i\epsilon}{4c}(\gamma\bar{G}^{(1)}_{-\frac{5}{2}} + \delta\bar{G}^{(2)}_{-\frac{5}{2}})(\gamma\bar{G}^{(1)}_{-\frac{5}{2}} + \delta\bar{G}^{(2)}_{-\frac{5}{2}})|0\rangle + \frac{1}{4c}(L^{(1)}_{-3} + L^{(2)}_{-3})(L^{(1)}_{-3} + L^{(2)}_{-3})|0\rangle$$
$$- \frac{3i\epsilon}{4c(c+6)}(L^{(1)}_{-2} + L^{(2)}_{-2})(\alpha G^{(1)}_{-\frac{3}{2}} + \beta G^{(2)}_{-\frac{3}{2}})(\bar{L}^{(1)}_{-2} + \bar{L}^{(2)}_{-2})(\gamma\bar{G}^{(1)}_{-\frac{3}{2}} + \delta\bar{G}^{(2)}_{-\frac{3}{2}})|0\rangle$$
$$- \frac{81i\epsilon}{4c(c+6)(21+8c)}\left((L^{(1)}_{-2} + L^{(2)}_{-2})(\alpha G^{(1)}_{-\frac{3}{2}} + \beta G^{(2)}_{-\frac{3}{2}}) - \tfrac{2(c+6)}{9}(\alpha G^{(1)}_{-\frac{7}{2}} + \beta G^{(2)}_{-\frac{7}{2}})\right)$$
$$\times\left((\bar{L}^{(1)}_{-2} + \bar{L}^{(2)}_{-2})(\gamma\bar{G}^{(1)}_{-\frac{3}{2}} + \delta\bar{G}^{(2)}_{-\frac{3}{2}}) - \tfrac{2(c+6)}{9}(\gamma\bar{G}^{(1)}_{-\frac{7}{2}} + \delta\bar{G}^{(2)}_{-\frac{7}{2}})\right)|0\rangle + \cdots$$

and

$$\rho_{\alpha\beta\gamma\delta}(|0,\epsilon\rangle\!\rangle) = |0\rangle\langle 0|$$
$$- \frac{i\epsilon}{4c/3}\left(\alpha\gamma G_{-\frac{3}{2}}\bar{G}_{-\frac{3}{2}}|0\rangle\langle 0| + i\alpha\delta G_{-\frac{3}{2}}|0\rangle\langle 0|G_{\frac{3}{2}} + i\beta\gamma\bar{G}_{-\frac{3}{2}}|0\rangle\langle 0|\bar{G}_{\frac{3}{2}} + \beta\delta|0\rangle\langle 0|\bar{G}_{\frac{3}{2}}G_{\frac{3}{2}}\right)$$
$$+ \frac{1}{c}\left(L_{-2}\bar{L}_{-2}|0\rangle\langle 0| + L_{-2}|0\rangle\langle 0|L_2 + \bar{L}_{-2}|0\rangle\langle 0|\bar{L}_2 + |0\rangle\langle 0|\bar{L}_2L_2\right)$$
$$- \frac{i\epsilon}{4c}\left(\alpha\gamma G_{-\frac{5}{2}}\bar{G}_{-\frac{5}{2}}|0\rangle\langle 0| + i\alpha\delta G_{-\frac{5}{2}}|0\rangle\langle 0|G_{\frac{3}{2}} + i\beta\gamma\bar{G}_{-\frac{5}{2}}|0\rangle\langle 0|\bar{G}_{\frac{3}{2}} + \beta\delta|0\rangle\langle 0|\bar{G}_{\frac{3}{2}}G_{\frac{3}{2}}\right)$$
$$+ \frac{1}{4c}\left(L_{-3}\bar{L}_{-3}|0\rangle\langle 0| + L_{-3}|0\rangle\langle 0|L_3 + \bar{L}_{-3}|0\rangle\langle 0|\bar{L}_3 + |0\rangle\langle 0|\bar{L}_3L_3\right) + \cdots\ .$$

For the highest weight state with $h = \bar{h} = \frac{3}{2}$, the Ishibashi state is given by

$$|\tfrac{3}{2},\epsilon\rangle\!\rangle = |\tfrac{3}{2}\rangle - \frac{i\epsilon}{3}G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|\tfrac{3}{2}\rangle + \frac{1}{3}L_{-1}\bar{L}_{-1}|\tfrac{3}{2}\rangle + \cdots\ . \tag{D.4}$$

Using the expansion of the highest weight state

$$\iota_{\alpha\beta\gamma\delta}(|\tfrac{3}{2}\rangle) = \frac{i\eta_{\frac{3}{2}}}{4c/3}(\alpha G^{(1)}_{-\frac{3}{2}} - \beta G^{(2)}_{-\frac{3}{2}})(\gamma\bar{G}^{(1)}_{-\frac{3}{2}} - \delta\bar{G}^{(2)}_{-\frac{3}{2}})|0\rangle\ , \tag{D.5}$$

the Ishibashi state can also be expanded as

$$\iota_{\alpha\beta\gamma\delta}(|\tfrac{3}{2},\epsilon\rangle\!\rangle)$$
$$= \frac{i\eta_{\frac{3}{2}}}{4c/3}(\alpha G^{(1)}_{-\frac{3}{2}} - \beta G^{(2)}_{-\frac{3}{2}})(\gamma\bar{G}^{(1)}_{-\frac{3}{2}} - \delta\bar{G}^{(2)}_{-\frac{3}{2}})|0\rangle - \frac{\epsilon\eta_{\frac{3}{2}}}{c}(L^{(1)}_{-2} - L^{(2)}_{-2})(\bar{L}^{(1)}_{-2} - \bar{L}^{(2)}_{-2})|0\rangle + \cdots \tag{D.6}$$

and

$$\rho_{\alpha\beta\gamma\delta}(|\tfrac{3}{2},\epsilon\rangle\rangle) = \frac{i\eta_{\frac{3}{2}}}{4c/3}\Big(\alpha\gamma G_{-\frac{3}{2}}\bar{G}_{-\frac{3}{2}}|0\rangle\langle 0| + \beta\delta|0\rangle\langle 0|\bar{G}_{\frac{3}{2}}G_{\frac{3}{2}} - i\alpha\delta G_{-\frac{3}{2}}|0\rangle\langle 0|G_{\frac{3}{2}} - i\beta\gamma\bar{G}_{-\frac{3}{2}}|0\rangle\langle 0|\bar{G}_{\frac{3}{2}}\Big) - \frac{\epsilon\eta_{\frac{3}{2}}}{c}\big(L_{-2}\bar{L}_{-2}|0\rangle\langle 0| - L_{-2}|0\rangle\langle 0|L_2 - \bar{L}_{-2}|0\rangle\langle 0|\bar{L}_2 + |0\rangle\langle 0|\bar{L}_2L_2\big) + \cdots .$$

We now consider the sector corresponding to $\mathcal{H}^{\rm NS}_{\frac{1}{10}}\otimes\mathcal{H}^{\rm NS}_{\frac{1}{10}}$. We give the results in terms of a state of weight $2h$, but of course, in this particular case $h = \frac{1}{10}$. The states are identified as

$$|2h\rangle = |\tfrac{1}{5}\rangle\ , \quad |2h{+}\tfrac{1}{2}\rangle = |\tfrac{7}{10}\rangle\ , \quad \text{and} \quad |2h{+}1\rangle = |\tfrac{6}{5}\rangle\ , \tag{D.7}$$

and the constants are $\eta := \eta_{\frac{7}{10}}$ and $\eta' := \eta_{\frac{6}{5}}$. For the Ishibashi state corresponding to $2h = 2\bar{h} = \frac{1}{5}$

$$|2h,\epsilon\rangle\rangle = |2h\rangle - \frac{i\epsilon}{4h}G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|2h\rangle + \frac{1}{4h}L_{-1}\bar{L}_{-1}|2h\rangle + \cdots\ , \tag{D.8}$$

the expansions are

$$\iota_{\alpha\beta\gamma\delta}(|2h,\epsilon\rangle\rangle) = |2h\rangle - \frac{i\epsilon}{4h}(\alpha G^{(1)}_{-\frac{1}{2}} + \beta G^{(2)}_{-\frac{1}{2}})(\gamma\bar{G}^{(1)}_{-\frac{1}{2}} + \delta\bar{G}^{(2)}_{-\frac{1}{2}})|2h\rangle + \frac{1}{4h}(L^{(1)}_{-1} + L^{(2)}_{-1})(\bar{L}^{(1)}_{-1} + \bar{L}^{(2)}_{-1})|2h\rangle + \cdots$$

and

$$\rho_{\alpha\beta\gamma\delta}(|2h,\epsilon\rangle\rangle) = |h\rangle\langle h| - \frac{i\epsilon}{4h}\Big(\alpha\gamma G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|h\rangle\langle h| + i\alpha\delta G_{-\frac{1}{2}}|h\rangle\langle h|G_{\frac{1}{2}} + i\beta\gamma\bar{G}_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}} + \beta\delta|h\rangle\langle h|\bar{G}_{\frac{1}{2}}G_{\frac{1}{2}}\Big) + \frac{1}{4h}\big(L_{-1}\bar{L}_{-1}|h\rangle\langle h| + L_{-1}|h\rangle\langle h|L_1 + \bar{L}_{-1}|h\rangle\langle h|\bar{L}_1 + |h\rangle\langle h|\bar{L}_1L_1\big) + \cdots .$$

For the Ishibashi state with $2h + \frac{1}{2} = 2\bar{h} + \frac{1}{2} = \frac{7}{10}$

$$|2h{+}\tfrac{1}{2},\epsilon\rangle\rangle = |2h{+}\tfrac{1}{2}\rangle - \frac{i\epsilon}{4h{+}1}G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|2h{+}\tfrac{1}{2}\rangle + \cdots\ , \tag{D.9}$$

we can use the expression for the highest weight state

$$\iota_{\alpha\beta\gamma\delta}(|2h{+}\tfrac{1}{2}\rangle) = \frac{i\eta}{4h}(\alpha G^{(1)}_{-\frac{1}{2}} - \beta G^{(2)}_{-\frac{1}{2}})(\gamma\bar{G}^{(1)}_{-\frac{1}{2}} - \delta\bar{G}^{(2)}_{-\frac{1}{2}})|2h\rangle\ , \tag{D.10}$$

and obtain the expansions

$$\iota_{\alpha\beta\gamma\delta}(|2h{+}\tfrac{1}{2},\epsilon\rangle\rangle) = \frac{i\eta}{4h}(\alpha G^{(1)}_{-\frac{1}{2}} - \beta G^{(2)}_{-\frac{1}{2}})(\gamma\bar{G}^{(1)}_{-\frac{1}{2}} - \delta\bar{G}^{(2)}_{-\frac{1}{2}})|2h\rangle - \frac{\epsilon\eta}{4h(4h+1)}(L^{(1)}_{-1} - L^{(2)}_{-1} - 2\alpha\beta G^{(1)}_{-\frac{1}{2}}G^{(2)}_{-\frac{1}{2}})(\bar{L}^{(1)}_{-1} - \bar{L}^{(2)}_{-1} - 2\gamma\delta\bar{G}^{(1)}_{-\frac{1}{2}}\bar{G}^{(2)}_{-\frac{1}{2}})|2h\rangle + \cdots$$

and

$$\rho_{\alpha\beta\gamma\delta}(|2h+\tfrac{1}{2},\epsilon\rangle\rangle) = \frac{\eta}{4h}\Big(i\alpha\gamma G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|h\rangle\langle h| + i\beta\delta|h\rangle\langle h|\bar{G}_{\frac{1}{2}}G_{\frac{1}{2}} + \alpha\delta G_{-\frac{1}{2}}|h\rangle\langle h|G_{\frac{1}{2}} + \beta\gamma\bar{G}_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}\Big) - \frac{\epsilon\eta}{4h(4h+1)}\left(L_{-1}\bar{L}_{-1}|h\rangle\langle h| + L_{-1}|h\rangle\langle h|L_1 + \bar{L}_{-1}|h\rangle\langle h|\bar{L}_1 + |h\rangle\langle h|\bar{L}_1L_1\right) + \frac{2i\epsilon\eta\alpha\beta}{4h(4h+1)}\Big(G_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}L_1 - \bar{L}_{-1}G_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}\Big) + \frac{2i\epsilon\eta\gamma\delta}{4h(4h+1)}\Big(L_{-1}\bar{G}_{-\frac{1}{2}}|h\rangle\langle h|G_{\frac{1}{2}} - \bar{G}_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}\bar{L}_1\Big) + \frac{4\epsilon\eta\alpha\beta\gamma\delta}{4h(4h+1)}\Big(G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}G_{\frac{1}{2}}\Big) + \cdots .$$

For the Ishibashi state with $2h+1 = 2\bar{h}+1 = \frac{6}{5}$

$$|2h{+}1,\epsilon\rangle\rangle = |2h{+}1\rangle + \cdots , \tag{D.11}$$

we can expand

$$\iota_{\alpha\beta\gamma\delta}(|2h{+}1,\epsilon\rangle\rangle) = \frac{\eta'}{4h{+}1}(L_{-1}^{(1)} - L_{-1}^{(2)} + \tfrac{\alpha\beta}{2h}G_{-\frac{1}{2}}^{(1)}G_{-\frac{1}{2}}^{(2)})(\bar{L}_{-1}^{(1)} - \bar{L}_{-1}^{(2)} + \tfrac{\gamma\delta}{2h}\bar{G}_{-\frac{1}{2}}^{(1)}\bar{G}_{-\frac{1}{2}}^{(2)})|2h\rangle + \cdots$$

and

$$\rho_{\alpha\beta\gamma\delta}(|2h{+}1,\epsilon\rangle\rangle) = \frac{\eta'}{4h{+}1}\left(L_{-1}\bar{L}_{-1}|h\rangle\langle h| - L_{-1}|h\rangle\langle h|L_1 - \bar{L}_{-1}|h\rangle\langle h|\bar{L}_1 + |h\rangle\langle h|\bar{L}_1L_1\right) + \frac{i\eta'\alpha\beta}{2h(4h+1)}\Big(G_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}L_1 - \bar{L}_{-1}G_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}\Big) + \frac{i\eta'\gamma\delta}{2h(4h+1)}\Big(L_{-1}\bar{G}_{-\frac{1}{2}}|h\rangle\langle h|G_{\frac{1}{2}} - \bar{G}_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}\bar{L}_1\Big) - \frac{\eta'\alpha\beta\gamma\delta}{4h^2(4h+1)}\Big(G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}|h\rangle\langle h|\bar{G}_{\frac{1}{2}}G_{\frac{1}{2}}\Big) + \cdots .$$

D.3 Boundary States Coefficients of (D_6, E_6) Theory

The matrices $\Psi_{(r,s)}^{(a,b)}$ are given in terms of the eigenvectors of adjacency matrices of the Dynkin diagrams of D_6 and E_6 in equation (6.40):

$$\Psi_{(r,s)}^{(a,b)} = \frac{\psi_a^r(D_6)\,\psi_b^s(E_6)}{\sqrt{S_{1r}^{(8)}S_{1s}^{(10)}}} , \tag{D.12}$$

We repeat here for convenience the vectors $\psi_a^r(G)$ given in [94]:

The eigenvectors of the D_6 adjacency matrix $\psi_a^r(D_6)$ are given by

$$\psi_a^r(D_6) = \sqrt{2}S_{ar}^{(8)} \quad \text{for } a,r \neq 5 \qquad\qquad \psi_a^{5^\pm}(D_6) = S_{a5}^{(8)} \quad \text{for } a \neq 5$$

$$\psi_{5^\pm}^r(D_6) = \frac{1}{\sqrt{2}}S_{5r}^{(8)} \quad \text{for } r \neq 5 \qquad\qquad \psi_{5^\epsilon}^{5^{\epsilon'}}(D_6) = \frac{1}{2}\left(S_{55}^{(8)} - \epsilon\epsilon'\right)$$

where $a = 1, 2, 3, 4, 5^+, 5^-$ ($a = 5^\pm$ correspond to 5 and 6 nodes on the D_6 Dynkin diagram), $r \in \mathcal{E}(D_6) = \{1, 3, 5, 5', 7, 9\}$ ($r = 5^\pm$ above correspond to 5 and $5'$), and $S^{(8)}_{ij}$ is the $\hat{\mathrm{su}}(2)_8$ modular S matrix elements,

$$S^{(k)}_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi i j}{k+2}\right)$$

Explicitly, the entries in $\psi^r_a(D_6)$ are

$a \setminus r$	1	3	$5^+ (=5)$	$5^- (=5')$	7	9
1	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$
2	$\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$	$\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$	0	0	$-\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$	$-\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$
3	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$	$-\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$
4	$\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$	$-\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$	0	0	$\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$	$-\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$
$5^+ (=5)$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{10}\left(-5+\sqrt{5}\right)$	$\frac{1}{10}\left(5+\sqrt{5}\right)$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
$5^- (=6)$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{10}\left(5+\sqrt{5}\right)$	$\frac{1}{10}\left(-5+\sqrt{5}\right)$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$

The eigenvectors of the E_6 adjacency matrix $\psi^s_b(E_6)$ are given by

$b \setminus s$	1	4	5	7	8	11
1	a	$\frac{1}{2}$	b	b	$\frac{1}{2}$	a
2	b	$\frac{1}{2}$	a	$-a$	$-\frac{1}{2}$	$-b$
3	c	0	$-d$	$-d$	0	c
4	b	$-\frac{1}{2}$	a	$-a$	$\frac{1}{2}$	$-b$
5	a	$-\frac{1}{2}$	b	b	$-\frac{1}{2}$	a
6	d	0	$-c$	c	0	$-d$

where

$$a = \frac{1}{2}\sqrt{\frac{3-\sqrt{3}}{6}} \quad b = \frac{1}{2}\sqrt{\frac{3+\sqrt{3}}{6}}$$
$$c = \frac{1}{2}\sqrt{\frac{3+\sqrt{3}}{3}} \quad d = \frac{1}{2}\sqrt{\frac{3-\sqrt{3}}{3}}$$

Putting these together, we can calculate the entries of Ψ. Since it is helpful to have an overview of the properties of Ψ when discussing the boundary states from the extended algebra point of view, we include a table of the approximate numerical values in table D.3.

(D.13)

(a,b)	$(1,1)$	$(1,5)$	$(1,7)$	$(1,11)$	$(3,1)$	$(3,5)$	$(3,7)$	$(3,11)$	$(5,1)$	$(5,5)$	$(5',1)$	$(5',5)$
$(1,1)$	0.3717	0.3717	0.3717	0.3717	0.6015	0.6015	0.6015	0.6015	0.4729	0.4729	0.4729	0.4729
$(1,2)$	0.7182	−0.1924	−0.1924	0.7182	1.162	−0.3114	−0.3114	1.162	0.9135	−0.2448	0.9135	−0.2448
$(1,3)$	1.016	−0.2721	−0.2721	1.016	1.643	−0.4403	−0.4403	1.643	1.292	−0.3462	1.292	−0.3462
$(1,6)$	0.5257	0.5257	0.5257	0.5257	0.8507	0.8507	0.8507	0.8507	0.6687	0.6687	0.6687	0.6687
$(2,1)$	0.7071	−0.7071	0.7071	−0.7071	0.7071	−0.7071	0.7071	−0.7071	0	0	0	0
$(2,2)$	1.366	0.3660	−0.3660	−1.366	1.366	0.3660	−0.3660	−1.366	0	0	0	0
$(2,3)$	1.932	0.5176	−0.5176	−1.932	1.932	0.5176	−0.5176	−1.932	0	0	0	0
$(2,6)$	1.000	−1.000	1.000	−1.000	1.000	−1.000	1.000	−1.000	0	0	0	0
$(3,1)$	0.9732	0.9732	0.9732	0.9732	0.2298	0.2298	0.2298	0.2298	−0.4729	−0.4729	−0.4729	−0.4729
$(3,2)$	1.880	−0.5038	−0.5038	1.880	0.4438	−0.1189	−0.1189	0.4438	−0.9135	0.2448	−0.9135	0.2448
$(3,3)$	2.659	−0.7125	−0.7125	2.659	0.6277	−0.1682	−0.1682	0.6277	−1.292	0.3462	−1.292	0.3462
$(3,6)$	1.376	1.376	1.376	1.376	0.3249	0.3249	0.3249	0.3249	−0.6687	−0.6687	−0.6687	−0.6687
$(4,1)$	1.144	−1.144	1.144	−1.144	−0.4370	0.4370	−0.4370	0.4370	0	0	0	0
$(4,2)$	2.210	0.5922	−0.5922	−2.210	−0.8443	−0.2262	0.2262	0.8443	0	0	0	0
$(4,3)$	3.126	0.8376	−0.8376	−3.126	−1.194	−0.3199	0.3199	1.194	0	0	0	0
$(4,6)$	1.618	−1.618	1.618	−1.618	−0.6180	0.6180	−0.6180	0.6180	0	0	0	0
$(5,1)$	0.6015	0.6015	0.6015	0.6015	−0.3717	−0.3717	−0.3717	−0.3717	−0.2923	−0.2923	0.7651	0.7651
$(5,2)$	1.162	−0.3114	−0.3114	1.162	−0.7182	0.1924	0.1924	−0.7182	−0.5646	0.1513	1.478	−0.3961
$(5,3)$	1.643	−0.4403	−0.4403	1.643	−1.016	0.2721	0.2721	−1.016	−0.7984	0.2139	2.090	−0.5601
$(5,6)$	0.8507	0.8507	0.8507	0.8507	−0.5257	−0.5257	−0.5257	−0.5257	−0.4133	−0.4133	1.082	1.082
$(6,1)$	0.6015	0.6015	0.6015	0.6015	−0.3717	−0.3717	−0.3717	−0.3717	0.7651	0.7651	−0.2923	−0.2923
$(6,2)$	1.162	−0.3114	−0.3114	1.162	−0.7182	0.1924	0.1924	−0.7182	1.478	−0.3961	−0.5646	0.1513
$(6,3)$	1.643	−0.4403	−0.4403	1.643	−1.016	0.2721	0.2721	−1.016	2.090	−0.5601	−0.7984	0.2139
$(6,6)$	0.8507	0.8507	0.8507	0.8507	−0.5257	−0.5257	−0.5257	−0.5257	1.082	1.082	−0.4133	−0.4133

The column heads $(1,1)$ to $(5',5)$ are the values of (r,s).

Table D.3: Numerical values of the boundary state coefficients $\Psi^{(a,b)}_{(r,s)}$

D.4 Fermion Parity Assignment of NS Highest Weight Vectors

In most cases, a choice of $\varepsilon(r,s)$ for NS highest weight vectors is irrelevant. Usually, NS highest weight vectors $|r,s\rangle$ are taken to be bosonic (i.e. $G_{-1/2}|r,s\rangle$ and $G_{-3/2}|0\rangle$ are fermionic). However, we take the following convention:

- For m odd,

$$r+s \in 4\mathbb{Z}+2 \to |r,s\rangle \text{ bosonic i.e. } \varepsilon(r,s)=1$$
$$r+s \in 4\mathbb{Z} \to |r,s\rangle \text{ fermionic i.e. } \varepsilon(r,s)=-1$$

  (In particular, $|1,3\rangle = |2,2\rangle$ with $h=\frac{1}{10}$ is fermionic in $m=3$.)

- For $m=10$ with the (D_6,E_6) bulk partition function,

$$(r,s) = (1,5),(1,7),(3,1),(3,11),(5,5),(5,7),(7,1),(7,11),(9,5),(9,7) \to \text{fermionic}$$
$$\text{others} \to \text{bosonic}$$

The first choice for m odd cases makes all the fusion coefficients $\left(N_{\widetilde{\rm NS}\,\widetilde{\rm NS}}{}^{\widetilde{\rm NS}}\right)_{ij}^{\;k}$ non-negative. However, there is no obvious procedure to make all these coefficients non-negative for m even cases. The second choice for $m=10$ comes from two observations: modular transformations of the bulk partition function and character identities between $m=3$ and $m=10$.

- Consider the (D_6,E_6) bulk partition function,

$$\begin{aligned}
Z &= \frac{1}{2}\left(Z_{\rm NS} + Z_{\widetilde{\rm NS}}\right) + Z_{\rm R} \\
Z_{\rm NS} &= \left|{}^{(10)}\chi_{1,1} + {}^{(10)}\chi_{1,5} + {}^{(10)}\chi_{1,7} + {}^{(10)}\chi_{1,11}\right|^2 \\
&\qquad + \left|{}^{(10)}\chi_{3,1} + {}^{(10)}\chi_{3,5} + {}^{(10)}\chi_{3,7} + {}^{(10)}\chi_{3,11}\right|^2 \\
&\qquad\qquad + 2\left|{}^{(10)}\chi_{5,1} + {}^{(10)}\chi_{5,5}\right|^2 \\
Z_{\rm R} &= 2\left|{}^{(10)}\chi_{1,4} + {}^{(10)}\chi_{1,8}\right|^2 + 2\left|{}^{(10)}\chi_{3,4} + {}^{(10)}\chi_{3,8}\right|^2 + 4\left|{}^{(10)}\chi_{5,4}\right|^2
\end{aligned}$$

  If we demand $Z_{\widetilde{\rm NS}}$ to have the same form as $Z_{\rm NS}$, we need

$$\begin{aligned}
\varepsilon(1,1) = \varepsilon(1,11) &= -\varepsilon(1,5) = -\varepsilon(1,7) \\
\varepsilon(3,1) = \varepsilon(3,11) &= -\varepsilon(3,5) = -\varepsilon(3,7) \\
\varepsilon(5,1) &= -\varepsilon(5,5)
\end{aligned}$$

  to ensure modular S transformation $\frac{1}{2}Z_{\widetilde{\rm NS}} \leftrightarrow Z_{\rm R}$.

- From the NS character identities between $m=3$ and $m=10$, if we want something similar for $\widetilde{\rm NS}$ characters, that is (again with q real)

$$\begin{aligned}
\left({}^{(3)}\tilde{\chi}_{1,1}\right)^2 &= {}^{(10)}\tilde{\chi}_{1,1} + {}^{(10)}\tilde{\chi}_{1,5} + {}^{(10)}\tilde{\chi}_{1,7} + {}^{(10)}\tilde{\chi}_{1,11} \\
\left({}^{(3)}\tilde{\chi}_{1,3}\right)^2 &= {}^{(10)}\tilde{\chi}_{3,1} + {}^{(10)}\tilde{\chi}_{3,5} + {}^{(10)}\tilde{\chi}_{3,7} + {}^{(10)}\tilde{\chi}_{3,11}
\end{aligned}$$

then they fix $\varepsilon(1,1) = 1$, $\varepsilon(3,1) = -1$, etc. Furthermore, if we take $\varepsilon(1,3) = -1$ for $m = 3$,

$$^{(3)}\tilde{\chi}_{1,1} \cdot {}^{(3)}\tilde{\chi}_{1,3} = {}^{(10)}\tilde{\chi}_{5,1} + {}^{(10)}\tilde{\chi}_{5,5}$$

fixes $\varepsilon(5,1) = 1$ and $\varepsilon(5,5) = -1$.

The above arguments fix $\varepsilon(r,s)$ of the NS representations with (r,s) appearing in the (D_6, E_6) bulk partition function. For the other NS representations, we simply pick $\varepsilon(r,s) = 1$.

D.5 Graph Fusion Algebras and Induced Modules for D_6 and E_6

Using the graph fusion algebras that were discussed in [67], and the α induced modules the were introduced in the context of subfactor theory[61, 63, 64] but also appears in the TFT construction of RCFTs[76, 91], we can calculate the boundary overlaps (2.232) for $\widehat{\mathrm{sl}}(2)_k$-WZW models efficiently.

We only consider the D_6 invariant of $\widehat{\mathrm{sl}}(2)_8$ and the E_6 invariant of $\widehat{\mathrm{sl}}(2)_{10}$ here. Using the method summarised in Appendix C of [91], we can associate simple induced modules to each node of the D_6 and E_6 diagrams as in Figure D.1. Using the graph fusion coefficients defined by[67]

$$\hat{N}_{ab}^{c} = \sum_{i \in \mathcal{E}(G)} \frac{{}^{(G)}\psi_a^j \, {}^{(G)}\psi_b^j \, {}^{(G)}\bar{\psi}_c^j}{{}^{(G)}\psi_1^j} , \tag{D.14}$$

we can calculate the graph fusion algebra of D_6 as

$$\begin{aligned}
&(2) \otimes (2) = (1) \oplus (3) , && (3) \otimes (3) = (1) \oplus (3) \oplus (5) \oplus (6) , \\
&(2) \otimes (3) = (2) \oplus (4) , && (3) \otimes (4) = (2) \oplus 2(4) , \\
&(2) \otimes (4) = (3) \oplus (5) \oplus (6) , && (3) \otimes (5) = (3) \oplus (6) , \\
&(2) \otimes (5) = (4) , && (3) \otimes (6) = (3) \oplus (5) , \\
&(2) \otimes (6) = (4) , && \\
&(4) \otimes (4) = (1) \oplus 2(3) \oplus (5) \oplus (6) , && (5) \otimes (5) = (1) \oplus (5) , \\
&(4) \otimes (5) = (2) \oplus (4) , && (5) \otimes (6) = (3) , \\
&(4) \otimes (6) = (2) \oplus (4) , && (6) \otimes (6) = (1) \oplus (6) ,
\end{aligned}$$

and that of E_6 as

$$
\begin{aligned}
&(2)\otimes(2) = (1)\oplus(3)\ , && (3)\otimes(3) = (1)\oplus 2(3)\oplus(5)\ ,\\
&(2)\otimes(3) = (2)\oplus(4)\oplus(6)\ , && (3)\otimes(4) = (2)\oplus(4)\oplus(6)\ ,\\
&(2)\otimes(4) = (3)\oplus(5)\ , && (3)\otimes(5) = (3)\ ,\\
&(2)\otimes(5) = (4)\ , && (3)\otimes(6) = (2)\oplus(4)\ ,\\
&(2)\otimes(6) = (3)\ , && \\
&(4)\otimes(4) = (1)\oplus(3)\ , && (5)\otimes(5) = (1)\ ,\\
&(4)\otimes(5) = (2)\ , && (5)\otimes(6) = (6)\ ,\\
&(4)\otimes(6) = (3)\ , && (6)\otimes(6) = (1)\oplus(5)\ .
\end{aligned}
$$

Note that the nodes (1), (5), and (6) of the E_6 diagram satisfy the Ising fusion rules.

One can calculate boundary overlaps using these algebras. For example, the overlap of the E_6 boundary states labelled by (4) and (5) can be written as the character of the simple induced module associated with the node (2).

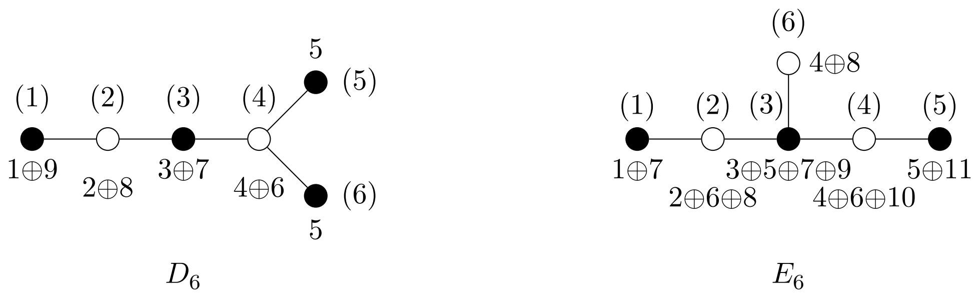

Figure D.1: Simple induced modules for D_6 and E_6.

D.6 Extended Modular S Matrix For $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$ at $c = \frac{7}{5}$

We define the extended characters

$$
\begin{aligned}
&\mathrm{ch}^{\mathrm{NS}}_{1}(q) = \chi^{\mathrm{NS}}_{0}(q) + \chi^{\mathrm{NS}}_{\frac{3}{2}}(q) + \chi^{\mathrm{NS}}_{\frac{7}{2}}(q) + \chi^{\mathrm{NS}}_{10}(q)\ , && \mathrm{ch}^{\mathrm{R}}_{1\pm}(q) = \hat{\chi}^{\mathrm{R}}_{\frac{7}{8}}(q) + \hat{\chi}^{\mathrm{R}}_{\frac{39}{8}}(q)\ ,\\
&\mathrm{ch}^{\mathrm{NS}}_{3}(q) = \chi^{\mathrm{NS}}_{\frac{1}{5}}(q) + \chi^{\mathrm{NS}}_{\frac{7}{10}}(q) + \chi^{\mathrm{NS}}_{\frac{6}{5}}(q) + \chi^{\mathrm{NS}}_{\frac{57}{10}}(q)\ , && \mathrm{ch}^{\mathrm{R}}_{3\pm}(q) = \hat{\chi}^{\mathrm{R}}_{\frac{3}{40}}(q) + \hat{\chi}^{\mathrm{R}}_{\frac{83}{40}}(q)\ ,\\
&\mathrm{ch}^{\mathrm{NS}}_{5}(q) = \chi^{\mathrm{NS}}_{\frac{1}{10}}(q) + \chi^{\mathrm{NS}}_{\frac{13}{5}}(q)\ , && \mathrm{ch}^{\mathrm{R}}_{5\pm}(q) = \hat{\chi}^{\mathrm{R}}_{\frac{19}{40}}(q)\ ,\\
&\mathrm{ch}^{\mathrm{NS}}_{5'}(q) = \chi^{\mathrm{NS}}_{\frac{1}{10}}(q) + \chi^{\mathrm{NS}}_{\frac{13}{5}}(q)\ , && \mathrm{ch}^{\mathrm{R}}_{5'\pm}(q) = \hat{\chi}^{\mathrm{R}}_{\frac{19}{40}}(q)\ ,
\end{aligned}
$$

so that the (D_6, E_6) bulk partition function can be written as

$$
Z^{(D_6,E_6)}_{\mathrm{NS}} = \sum_{a\in\mathcal{I}^{\mathrm{ext}}_{\mathrm{NS}}} |\mathrm{ch}^{\mathrm{NS}}_{a}(q)|^2\ , \qquad Z^{(D_6,E_6)}_{\mathrm{R}} = \sum_{a\in\mathcal{I}^{\mathrm{ext}}_{\mathrm{R}}} |\mathrm{ch}^{\mathrm{R}}_{a}(q)|^2\ , \tag{D.15}
$$

where the indexing sets are

$$
\mathcal{I}^{\mathrm{ext}}_{\mathrm{NS}} = \{1,\ 3,\ 5,\ 5'\} \quad \text{and} \quad \mathcal{I}^{\mathrm{ext}}_{\mathrm{R}} = \{1\pm,\ 3\pm,\ 5\pm,\ 5'\pm\}\ . \tag{D.16}
$$

Note that the Ramond characters are modified ones.

We can calculate extended S matrix elements. For $a = 1,\ 3$

$$\begin{aligned}\mathrm{ch}_a^{\mathrm{NS}}(\tilde{q}) &= \chi_{a,1}^{\mathrm{NS}}(\tilde{q}) + \chi_{a,5}^{\mathrm{NS}}(\tilde{q}) + \chi_{a,7}^{\mathrm{NS}}(\tilde{q}) + \chi_{a,11}^{\mathrm{NS}}(\tilde{q}) \\ &= \sum_{(r,s)\in\mathcal{I}_{\mathrm{NS}}} \tilde{\mathrm{S}}_{a\,(r,s)}^{[\mathrm{NS,NS}]}\, \chi_{r,s}^{\mathrm{NS}}(q)\ ,\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathrm{S}}_{a\,(r,s)}^{[\mathrm{NS,NS}]} &:= \mathrm{S}_{(a,1)(r,s)}^{[\mathrm{NS,NS}]} + \mathrm{S}_{(a,5)(r,s)}^{[\mathrm{NS,NS}]} + \mathrm{S}_{(a,7)(r,s)}^{[\mathrm{NS,NS}]} + \mathrm{S}_{(a,11)(r,s)}^{[\mathrm{NS,NS}]} \\ &= 2\mathrm{S}_{ar}^{(8)}\left(\mathrm{S}_{1,s}^{(10)} + \mathrm{S}_{5,s}^{(10)} + \mathrm{S}_{7,s}^{(10)} + \mathrm{S}_{11,s}^{(10)}\right)\ .\end{aligned}$$

For $a = 5,\ 5'$

$$\begin{aligned}\mathrm{ch}_a^{\mathrm{NS}}(\tilde{q}) &= \chi_{5,1}^{\mathrm{NS}}(\tilde{q}) + \chi_{5,5}^{\mathrm{NS}}(\tilde{q}) \\ &= \frac{1}{2}\left(\chi_{5,1}^{\mathrm{NS}}(\tilde{q}) + \chi_{5,5}^{\mathrm{NS}}(\tilde{q}) + \chi_{5,7}^{\mathrm{NS}}(\tilde{q}) + \chi_{5,11}^{\mathrm{NS}}(\tilde{q})\right) \\ &= \sum_{(r,s)\in\mathcal{I}_{\mathrm{NS}}} \frac{1}{2}\tilde{\mathrm{S}}_{a\,(r,s)}^{[\mathrm{NS,NS}]}\, \chi_{r,s}^{\mathrm{NS}}(q)\ .\end{aligned}$$

In order to simplify the S matrix further, consider the E_6 invariant of the $\widehat{\mathrm{sl}}(2)_{10}$-WZW model

$$Z = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2\ , \tag{D.17}$$

and calculate

$$\mathrm{S}_{1,s}^{(10)} + \mathrm{S}_{7,s}^{(10)} = \begin{cases}\frac{1}{2} & \text{for } s = 1,7 \\ \frac{1}{\sqrt{2}} & \text{for } s = 4,8 \\ \frac{1}{2} & \text{for } s = 5,11 \\ 0 & \text{for } s = 2,3,6,9,10\end{cases}$$

$$\mathrm{S}_{4,s}^{(10)} + \mathrm{S}_{8,s}^{(10)} = \begin{cases}\frac{1}{\sqrt{2}} & \text{for } s = 1,7 \\ 0 & \text{for } s = 4,8 \\ -\frac{1}{\sqrt{2}} & \text{for } s = 5,11 \\ 0 & \text{for } s = 2,3,6,9,10\end{cases}$$

$$\mathrm{S}_{5,s}^{(10)} + \mathrm{S}_{11,s}^{(10)} = \begin{cases}\frac{1}{2} & \text{for } s = 1,7 \\ -\frac{1}{\sqrt{2}} & \text{for } s = 4,8 \\ \frac{1}{2} & \text{for } s = 5,11 \\ 0 & \text{for } s = 2,3,6,9,10\end{cases}$$

In fact, by mapping $\{1,7\} \ni i \mapsto 1,\ \{4,8\} \ni i \mapsto 2,\ \{5,11\} \ni i \mapsto 3$ to $\widehat{\mathrm{sl}}(2)_2$ labels, we see these numbers are just $\mathrm{S}_{ij}^{(2)}$. Denote this map by ρ.

Therefore, $\tilde{\mathrm{S}}_{a\,(r,s)}^{[\mathrm{NS,NS}]}$ is non-vanishing for $s = 1,\ 5,\ 7,\ 11$ and

$$\tilde{\mathrm{S}}_{a\,(r,s)}^{[\mathrm{NS,NS}]} = 2\mathrm{S}_{ar}^{(8)}\left(\mathrm{S}_{1,\rho(s)}^{(2)} + \mathrm{S}_{3,\rho(s)}^{(2)}\right) = 2\mathrm{S}_{ar}^{(8)}\ . \tag{D.18}$$

This shows $\tilde{\mathrm{S}}^{[\mathrm{NS,NS}]}_{a\,(r,s)}$ does not depend on s and we can write, for example, for $a' = 1,\ 3$

$$\tilde{\mathrm{S}}^{[\mathrm{NS,NS}]}_{a\,(a',1)}\,\chi^{\mathrm{NS}}_{a',1}(q) + \tilde{\mathrm{S}}^{[\mathrm{NS,NS}]}_{a\,(a',5)}\,\chi^{\mathrm{NS}}_{a',5}(q) + \tilde{\mathrm{S}}^{[\mathrm{NS,NS}]}_{a\,(a',7)}\,\chi^{\mathrm{NS}}_{a',7}(q) + \tilde{\mathrm{S}}^{[\mathrm{NS,NS}]}_{a\,(a',11)}\,\chi^{\mathrm{NS}}_{a',11}(q) = \mathcal{S}^{[\mathrm{NS,NS}]}_{aa'}\,\mathrm{ch}^{\mathrm{NS}}_{a'}(q)\ . \tag{D.19}$$

Similar expression holds for $a' = 5, 5'$.

After resolving the fixed point $a = 5,\ 5'$ (by guess), we get the extended S matrix elements

$$\mathcal{S}^{[\mathrm{NS,NS}]}_{aa'} = \begin{cases} 2\mathrm{S}^{(8)}_{aa'} = \frac{2}{\sqrt{5}}\sin\left(\frac{aa'\pi}{10}\right) & \text{for} \quad a, a' \neq 5, 5' \\ \mathrm{S}^{(8)}_{aa'} = \frac{1}{\sqrt{5}}\sin\left(\frac{aa'\pi}{10}\right) & \text{for} \quad \begin{array}{l}(a{=}5{,}5' \text{ and } a'{\neq}5{,}5') \text{ or} \\ (a{\neq}5{,}5' \text{ and } a'{=}5{,}5')\end{array} \end{cases},$$

$$\mathcal{S}^{[\mathrm{NS,NS}]}_{5,5} = \mathcal{S}^{[\mathrm{NS,NS}]}_{5',5'} = \frac{1}{2}\left(\mathrm{S}^{(8)}_{5,5} - 1\right) = \frac{1}{2}\left(\frac{1}{\sqrt{5}} - 1\right)\ ,$$

$$\mathcal{S}^{[\mathrm{NS,NS}]}_{5,5'} = \mathcal{S}^{[\mathrm{NS,NS}]}_{5',5} = \frac{1}{2}\left(\mathrm{S}^{(8)}_{5,5} + 1\right) = \frac{1}{2}\left(\frac{1}{\sqrt{5}} + 1\right)\ .$$

These give the correct character transformation

$$\mathrm{ch}^{\mathrm{NS}}_{a}(\tilde{q}) = \sum_{a' \in \mathcal{I}^{\mathrm{ext}}_{\mathrm{NS}}} \mathcal{S}^{[\mathrm{NS,NS}]}_{aa'}\,\mathrm{ch}^{\mathrm{NS}}_{a'}(q)\ . \tag{D.20}$$

For extended $\mathcal{S}^{[\mathrm{R,NS}]}_{aa'}$, some care is needed regarding the various signs. For $a = 1\pm,\ 3\pm$

$$\mathrm{ch}^{\mathrm{R}}_{a}(\tilde{q}) = \hat{\chi}^{\mathrm{R}}_{a,4}(\tilde{q}) + \hat{\chi}^{\mathrm{R}}_{a,8}(\tilde{q}) = \sum_{(r,s)\in\mathcal{I}_{\mathrm{NS}}} \tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,s)}\,\tilde{\chi}^{\mathrm{NS}}_{r,s}(q)\ , \tag{D.21}$$

where

$$\tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,s)} := \hat{\mathrm{S}}^{[\mathrm{R,NS}]}_{(a,4)(r,s)} + \hat{\mathrm{S}}^{[\mathrm{R,NS}]}_{(a,8)(r,s)} = \varepsilon(r,s)\,(-1)^{\frac{r-s}{2}}\,2\mathrm{S}^{(8)}_{ar}\left(\mathrm{S}^{(10)}_{4s} + \mathrm{S}^{(10)}_{8s}\right)\ , \tag{D.22}$$

and the symmetric S matrix $\hat{\mathrm{S}}^{[\mathrm{R,NS}]}$ is given in (A.44). For $a = 5\pm,\ 5'\pm$

$$\mathrm{ch}^{\mathrm{R}}_{a}(\tilde{q}) = \hat{\chi}^{\mathrm{R}}_{5,4}(\tilde{q}) = \frac{1}{2}\left(\hat{\chi}^{\mathrm{R}}_{5,4}(\tilde{q}) + \hat{\chi}^{\mathrm{R}}_{5,8}(\tilde{q})\right) = \sum_{(r,s)\in\mathcal{I}_{\mathrm{NS}}} \frac{1}{2}\tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,s)}\,\tilde{\chi}^{\mathrm{NS}}_{r,s}(q)\ . \tag{D.23}$$

$\tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,s)}$ is non-vanishing for $s = 1, 5, 7, 11$ and

$$\tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,s)} = \varepsilon(r,s)\,(-1)^{\frac{r-s}{2}}\,2\mathrm{S}^{(8)}_{ar}\,\mathrm{S}^{(2)}_{2\rho(s)}\ . \tag{D.24}$$

Explicitly, these coefficients are

$$\begin{aligned} \tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,1)} &= \varepsilon(r,1)\,(-1)^{\frac{r-1}{2}}\,\sqrt{2}\mathrm{S}^{(8)}_{ar}\ , & \tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,5)} &= -\varepsilon(r,5)\,(-1)^{\frac{r-1}{2}}\,\sqrt{2}\mathrm{S}^{(8)}_{ar}\ , \\ \tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,7)} &= -\varepsilon(r,7)\,(-1)^{\frac{r-1}{2}}\,\sqrt{2}\mathrm{S}^{(8)}_{ar}\ , & \tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,11)} &= \varepsilon(r,11)\,(-1)^{\frac{r-1}{2}}\,\sqrt{2}\mathrm{S}^{(8)}_{ar}\ . \end{aligned}$$

However, our choice of $\varepsilon(r,s)$ exactly cancels signs in front of $\mathrm{S}^{(8)}_{ar}$, and we obtain

$$\tilde{\mathrm{S}}^{[\mathrm{R,NS}]}_{a\,(r,s)} = \sqrt{2}\mathrm{S}^{(8)}_{ar}\ . \tag{D.25}$$

After resolving the fixed point $a = 5,\ 5'$, we get the extended S matrix elements

$$\mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{aa'} = \begin{cases} \sqrt{2}\mathrm{S}^{(8)}_{aa'} = \sqrt{\frac{2}{5}}\sin\left(\frac{aa'\pi}{10}\right) & \text{for} \quad a, a' \neq 5, 5' \\ \frac{1}{\sqrt{2}}\mathrm{S}^{(8)}_{aa'} = \frac{1}{\sqrt{10}}\sin\left(\frac{aa'\pi}{10}\right) & \text{for} \quad {(a{=}5{,}5' \text{ and } a'{\neq}5{,}5') \text{ or} \atop (a{\neq}5{,}5' \text{ and } a'{=}5{,}5')} \end{cases} ,$$

$$\mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{5,5} = \mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{5',5'} = \frac{1}{2\sqrt{2}}\left(\mathrm{S}^{(8)}_{5,5} - 1\right) = \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{5}} - 1\right) ,$$

$$\mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{5,5'} = \mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{5',5} = \frac{1}{2\sqrt{2}}\left(\mathrm{S}^{(8)}_{5,5} + 1\right) = \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{5}} + 1\right) .$$

These give the correct character transformation

$$\mathrm{ch}^{\mathrm{R}}_a(\tilde{q}) = \sum_{a' \in \mathcal{I}^{\mathrm{ext}}_{\mathrm{NS}}} \mathcal{S}^{[\mathrm{R},\mathrm{NS}]}_{aa'} \, \mathrm{ch}^{\widetilde{\mathrm{NS}}}_{a'}(q) \ . \tag{D.26}$$

References

[1] J. Milnor. "Spin structures on manifolds". In: *L'Enseignement Mathématique* 9.2 (1963), pp. 198–203.

[2] V. G. Kac. "Contravariant form for infinite-dimensional Lie algebras and superalgebras". In: *Group Theoretical Methods in Physics*. Ed. by W. Beiglböck, A. Böhm, and E. Takasugi. Vol. 94. Lecture Notes in Physics. Springer, 1979, pp. 441–445.

[3] B. L. Feigin and D. B. Fuks. "Invariant skew symmetric differential operators on the line and Verma modules over the Virasoro algebra". In: *Functional Analysis and Its Applications* 16 (1982). [Funkt. Anal. Pril.16,47(1982)], pp. 114–126.

[4] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory". In: *Nuclear Physics* B241 (1984), pp. 333–380.

[5] J. L. Cardy. "Conformal Invariance and Surface Critical Behavior". In: *Nuclear Physics* B240 (1984), pp. 514–532.

[6] M. A. Bershadsky, V. G. Knizhnik, and M. G. Teitelman. "Superconformal symmetry in two dimensions". In: *Physics Letters* B151 (1985), pp. 31–36.

[7] V. S. Dotsenko and V. A. Fateev. "Four-Point Correlation Functions and the Operator Algebra in 2D Conformal Invariant Theories with Central Charge $c \leq 1$". In: *Nuclear Physics* B251 (1985), pp. 691–734.

[8] H. Eichenherr. "Minimal operator algebras in superconformal quantum field theory". In: *Physics Letters* B151 (1985).

[9] D. Friedan, Z. Qiu, and S. Shenker. "Superconformal invariance in two dimensions and the tricritical Ising model". In: *Physics Letters* B151.1 (1985), pp. 37–43.

[10] A. Rocha-Caridi. "Vacuum Vector Representations of the Virasoro Algebra". In: *Vertex Operators in Mathematics and Physics*. Ed. by J. Lepowsky, S. Mandelstam, and I. Singer. Vol. 3. Mathematical Sciences Research Institute Publications. Springer, 1985, pp. 451–473.

[11] A. B. Zamolodchikov. "Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory". In: *Theoretical and Mathematical Physics* 65 (1985). [Teor. Mat. Fiz.65,347(1985)], pp. 1205–1213.

[12] P. Goddard, A. Kent, and D. I. Olive. "Unitary Representations of the Virasoro and Super-Virasoro Algebras". In: *Communications in Mathematical Physics* 103 (1986), pp. 105–119.

[13] Y. Matsuo and S. Yahikozawa. "Superconformal field theory with modular invariance on a torus". In: *Physics Letters* B178.2-3 (1986), pp. 211–218.

[14] A. Meurman and A. Rocha-Caridi. "Highest weight representations of the Neveu–Schwarz and Ramond algebras". In: *Communications in Mathematical Physics* 107 (1986), pp. 263–294.

[15] Z. Qiu. “Supersymmetry, two-dimensional critical phenomena and the tricritical Ising model”. In: *Nuclear Physics* B270 (1986), pp. 205–234.

[16] G. M. Sotkov and M. S. Stanishkov. “N=1 superconformal operator product expansions and superfield fusion rules”. In: *Physics Letters* B177.3–4 (1986), pp. 361–367.

[17] J. J. Atick and A. Sen. “Correlation Functions of Spin Operators on a Torus”. In: *Nuclear Physics* B286 (1987), pp. 189–210.

[18] A. Cappelli. “Modular Invariant Partition Functions of Superconformal Theories”. In: *Physics Letters* B185 (1987), pp. 82–88.

[19] A. Cappelli, C. Itzykson, and J. B. Zuber. “The ADE Classification of Minimal and A1(1) Conformal Invariant Theories”. In: *Communications in Mathematical Physics* 113 (1987), p. 1.

[20] V. G. Kac and A. K. Raina. *Bombay Lectures On Highest Weight Representations of Infinite Dimensional Lie algebras*. 1st. World Scientific, 1987.

[21] J. D. Cohn and D. Friedan. “Super characters and chiral asymmetry in superconformal field theory”. In: *Nuclear Physics* B296 (1988), pp. 779–799.

[22] P. H. Ginsparg. “Applied Conformal Field Theory”. In: *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28–August 5, 1988*. 1988, pp. 1–168. arXiv: `hep-th/9108028`.

[23] G. W. Moore and N. Seiberg. “Polynomial Equations for Rational Conformal Field Theories”. In: *Physics Letters* B212 (1988), pp. 451–460.

[24] G. Mussardo, G. Sotkov, and M. Stanishkov. “Fine structure of the supersymmetric operator product expansion algebras”. In: *Nuclear Physics* B305 (1988), pp. 69–108.

[25] E. P. Verlinde. “Fusion Rules and Modular Transformations in 2D Conformal Field Theory”. In: *Nuclear Physics* B300 (1988), pp. 360–376.

[26] D. B. Abraham, L. F. Ko, and N. M. Švrakić. “Transfer matrix spectrum for the finite-width Ising model with adjustable boundary conditions: exact solution”. In: *Journal of statistical physics* 56.4 (1989), pp. 563–587.

[27] L. Alvarez-Gaumé, G. Sierra, and C. Gomez. “Topics in Conformal Field Theory”. In: *Physics and mathematics of strings*. Ed. by L. Brink et al. 1989, pp. 16–184.

[28] J. L. Cardy. “Boundary Conditions, Fusion Rules and the Verlinde Formula”. In: *Nuclear Physics* B324 (1989), pp. 581–596.

[29] N. Ishibashi. “The Boundary and Crosscap States in Conformal Field Theories”. In: *Modern Physics Letters* A4 (1989), p. 251.

[30] G. W. Moore and N. Seiberg. “Classical and Quantum Conformal Field Theory”. In: *Communications in Mathematical Physics* 123 (1989), p. 177.

[31] G. W. Moore and N. Seiberg. “Lectures On RCFT”. In: *1989 Banff NATO ASI: Physics, Geometry and Topology, Banff, Canada, August 14-25, 1989*. 1989, pp. 1–129.

[32] G. W. Moore and N. Seiberg. “Naturality in Conformal Field Theory”. In: *Nuclear Physics* B313 (1989), pp. 16–40.

[33] E. Witten. “Quantum Field Theory and the Jones Polynomial”. In: *Communications in Mathematical Physics* 121 (1989), pp. 351–399.

[34] P. Furlan, A. C. Ganchev, and V. B. Petkova. “Fusion Matrices and $c < 1$ (Quasi)local Conformal Theories”. In: *International Journal of Modern Physics* A5 (1990). [Erratum: Int. J. Mod. Phys. A5,3641(1990)], pp. 2721–2736.

[35] K. Hornfeck. “Supersymmetrizing the W(4) algebra”. In: *Physics Letters* B252 (1990), pp. 357–360.

[36] A. N. Schellekens and S. Yankielowicz. “Simple Currents, Modular Invariants and Fixed Points”. In: *International Journal of Modern Physics* A5 (1990), pp. 2903–2952.

[37] I. Affleck and A. W. W. Ludwig. “Universal noninteger ’ground state degeneracy’ in critical quantum systems”. In: *Physics Review Letters* 67 (1991), pp. 161–164.

[38] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel, and R. Varnhagen. “W-Algebras with Two and Three Generators”. In: *Nuclear Physics* B361 (1991), pp. 255–289.

[39] K. Hornfeck. *Realizations for the SW(7/2)-Algebra and the Minimal Supersymmetric Extension of WA(3)*. PRINT-91-0341 (KCL). 1991.

[40] R. Blumenhagen, W. Eholzer, A. Honecker, and R. Hübel. “New N=1 Extended Superconformal Algebras with Two and Three Generators”. In: *International Journal of Modern Physics* A7 (1992), pp. 7841–7871. arXiv: `hep-th/9207072`.

[41] W. Eholzer, M. Flohr, A. Honecker, R. Hübel, W. Nahm, and R. Varnhagen. “Representations of W-algebras with two generators and new rational models”. In: *Nuclear Physics* B383 (1992), pp. 249–288.

[42] J. M. Figueroa-O’Farrill and S. Schrans. “The Conformal Bootstrap and Super W Algebras”. In: *International Journal of Modern Physics* A7 (1992), pp. 591–617.

[43] S. B. Giddings. “Punctures on super Riemann surfaces”. In: *Communications in Mathematical Physics* 143 (1992), pp. 355–370.

[44] P. Bouwknegt and K. Schoutens. “W symmetry in conformal field theory”. In: *Physics Reprots* 223 (1993), pp. 183–276. arXiv: `hep-th/9210010`.

[45] W. Eholzer, A. Honecker, and R. Hübel. “Representations of N=1 Extended Superconformal Algebras with Two Generators”. In: *Modern Physics Letters* A8 (1993), pp. 725–738. arXiv: `hep-th/9209030`.

[46] A. Honecker. “Automorphisms of W algebras and extended rational conformal field theories”. In: *Nuclear Physics* B400 (1993), pp. 574–596. arXiv: `hep-th/9211130`.

[47] G. M. T. Watts. “Null vectors of the superconformal algebra: The Ramond sector”. In: *Nuclear Physics* B407 (1993), pp. 213–236. arXiv: hep-th/9306034.

[48] P. Bowcock and G. M. T. Watts. “Null vectors, three point and four point functions in conformal field theory”. In: *Theoretical and Mathematical Physics* 98 (1994). [Teor. Mat. Fiz.98,500(1994)], pp. 350–356. arXiv: hep-th/9309146.

[49] R. Chatterjee and A. B. Zamolodchikov. “Local magnetization in critical Ising model with boundary magnetic field”. In: *Modern Physics Letters* A9 (1994), p. 2227. arXiv: hep-th/9311165.

[50] W. Eholzer and R. Hubel. “Fusion Algebras of Fermionic Rational Conformal Field Theories via a Generalized Verlinde Formula”. In: *Nuclear Physics* B414 (1994), pp. 348–378. arXiv: hep-th/9307031.

[51] J. Fuchs. “Fusion rules in conformal field theory”. In: *Fortschritte der Physik* 42 (1994), pp. 1–48. arXiv: hep-th/9306162.

[52] S. Ghoshal and A. B. Zamolodchikov. “Boundary S matrix and boundary state in two-dimensional integrable quantum field theory”. In: *International Journal of Modern Physics* A9 (1994). [Erratum: Int. J. Mod. Phys.A9,4353(1994)], pp. 3841–3886. arXiv: hep-th/9306002.

[53] J. Polchinski. “Dirichlet Branes and Ramond-Ramond charges”. In: *Physics Review Letters* 75 (1995), pp. 4724–4727. arXiv: hep-th/9510017.

[54] M. Oshikawa and I. Affleck. “Defect lines in the Ising model and boundary states on orbifolds”. In: *Physics Review Letters* 77 (1996), pp. 2604–2607. arXiv: hep-th/9606177.

[55] G. Pradisi, A. Sagnotti, and Ya. S. Stanev. “Completeness conditions for boundary operators in 2-D conformal field theory”. In: *Physics Letters* B381 (1996), pp. 97–104. arXiv: hep-th/9603097.

[56] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. 1st. Springer, 1997.

[57] M. R. Gaberdiel. “Fusion of twisted representations”. In: *International Journal of Modern Physics* A12 (1997), pp. 5183–5208. arXiv: hep-th/9607036.

[58] V. G. Kac. *Vertex algebras for beginners*. University lecture series. American Mathematical Society, 1997.

[59] M. Oshikawa and I. Affleck. “Boundary conformal field theory approach to the critical two-dimensional Ising model with a defect line”. In: *Nuclear Physics* B495 (1997), pp. 533–582. arXiv: cond-mat/9612187.

[60] G. M. T. Watts. “W-algebras and their representations”. In: *Conformal Field Theories and Integrable Models: Lectures Held at the Eötvös Graduate Course, Budapest, Hungary, 13–18 August 1996*. Ed. by Z. Horváth and L. Palla. Springer, 1997, pp. 55–84.

[61] J. Bockenhauer and D. E. Evans. "Modular Invariants, Graphs and α Induction for Nets of Subfactors I". In: *Communications in Mathematical Physics* 197 (1998), pp. 361–386. arXiv: hep-th/9801171.

[62] J. Fuchs and C. Schweigert. "Completeness of boundary conditions for the critical three state Potts model". In: *Physics Letters* B441 (1998), pp. 141–146. arXiv: hep-th/9806121.

[63] J. Bockenhauer and D. E. Evans. "Modular Invariants, Graphs and α Induction for Nets of Subfactors II". In: *Communications in Mathematical Physics* 200 (1999), pp. 57–103. arXiv: hep-th/9805023.

[64] J. Bockenhauer and D. E. Evans. "Modular Invariants, Graphs and α Induction for Nets of Subfactors III". In: *Communications in Mathematical Physics* 205 (1999), pp. 183–228. arXiv: hep-th/9812110.

[65] R. Konik and A. LeClair. "Purely Transmitting Defect Field Theories". In: *Nuclear Physics B* 538.3 (1999), pp. 587–611. arXiv: hep-th/9703085.

[66] I. Runkel. "Boundary structure constants for the A series Virasoro minimal models". In: *Nuclear Physics* B549 (1999), pp. 563–578. arXiv: hep-th/9811178.

[67] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J. B. Zuber. "Boundary conditions in rational conformal field theories". In: *Nuclear Physics* B570 (2000), pp. 525–589. arXiv: hep-th/9908036.

[68] J. Fuchs and C. Schweigert. "Solitonic sectors, α-induction and symmetry breaking boundaries". In: *Physics Letters* B490 (2000), pp. 163–172. arXiv: hep-th/0006181.

[69] M. R. Gaberdiel. "An Introduction to Conformal Field Theory". In: *Reports on Progress in Physics* 63 (2000), pp. 607–667. arXiv: hep-th/9910156.

[70] A. Recknagel, D. Roggenkamp, and V. Schomerus. "On relevant boundary perturbations of unitary minimal models". In: *Nuclear Physics* B588 (2000), pp. 552–564. arXiv: hep-th/0003110 [hep-th].

[71] I. Runkel. "Boundary Problems in Conformal Field Theory". PhD. King's College London, 2000.

[72] M. Dörrzapf. "Highest weight representations of the N=1 Ramond algebra". In: *Nuclear Physics* B595.3 (2001), pp. 605–653. arXiv: hep-th/9905150.

[73] R. I. Nepomechie. "Consistent superconformal boundary states". In: *Journal of Physics* A34 (2001), pp. 6509–6524. arXiv: hep-th/0102010.

[74] V. B. Petkova and J. B. Zuber. "Generalised twisted partition functions". In: *Physics Letters B* 504.1-2 (2001), pp. 157–164. arXiv: hep-th/0011021.

[75] V. B. Petkova and J. B. Zuber. "The many faces of Ocneanu cells". In: *Nuclear Physics B* 603.3 (2001), pp. 449–496. arXiv: hep-th/0101151.

[76] J. Fuchs, I. Runkel, and C. Schweigert. “TFT construction of RCFT correlators I. Partition functions”. In: *Nuclear Physics* B646 (2002), pp. 353–497. arXiv: hep-th/0204148.

[77] H. Ishikawa. “Boundary states in coset conformal field theories”. In: *Nuclear Physics* B629 (2002), pp. 209–232. arXiv: hep-th/0111230.

[78] C. Richard and P. A. Pearce. “Integrable lattice realizations of N=1 superconformal boundary conditions”. In: *Nuclear Physics* B631 (2002), pp. 447–470. arXiv: hep-th/0109083.

[79] K. Graham and G. M. T. Watts. “Defect Lines and Boundary Flows”. In: *Journal of High Energy Physics* 2004.04 (2003), p. 1. arXiv: hep-th/0306167.

[80] K. Iohara and Y. Koga. “Representation theory of Neveu–Schwarz and Ramond algebras I: Verma modules”. In: *Advances in Mathematics* 178 (2003), pp. 1–65.

[81] C. Bachas and M. Gaberdiel. “Loop Operators and the Kondo Problem”. In: *Journal of High Energy Physics* 2004.11 (2004). arXiv: hep-th/0411067.

[82] E. Frenkel and D. Ben-Zvi. *Vertex Algebras and Algebraic Curves*. Mathematical surveys and monographs. American Mathematical Society, 2004.

[83] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert. “Kramers-Wannier duality from conformal defects”. In: *Physical Review Letters* 93.7 (2004). arXiv: cond-mat/0404051.

[84] J. Fuchs, I. Runkel, and C. Schweigert. “TFT construction of RCFT correlators. II. Unoriented world sheets”. In: *Nuclear Physics* B678 (2004), pp. 511–637. arXiv: hep-th/0306164.

[85] J. Fuchs, I. Runkel, and C. Schweigert. “TFT construction of RCFT correlators. III. Simple currents”. In: *Nuclear Physics* B694 (2004), pp. 277–353. arXiv: hep-th/0403157.

[86] J. Fuchs, I. Runkel, and C. Schweigert. “TFT construction of RCFT correlators IV: Structure constants and correlation functions”. In: *Nuclear Physics* B715 (2005), pp. 539–638. arXiv: hep-th/0412290.

[87] J. Fjelstad, J. Fuchs, I. Runkel, and C. Schweigert. “TFT construction of RCFT correlators. V. Proof of modular invariance and factorisation”. In: *Theory and Applications of Categories* 16 (2006), pp. 342–433. arXiv: hep-th/0503194.

[88] T. Gannon. *Moonshine Beyond the Monster*. 1st. Cambridge University Press, 2006.

[89] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert. “Duality and defects in rational conformal field theory”. In: *Nuclear Physics B* 763.3 (2007), pp. 354–430. arXiv: hep-th/0607247.

[90] J. Fuchs, M. R. Gaberdiel, I. Runkel, and C. Schweigert. “Topological defects for the free boson CFT”. In: *Journal of Physics A: Mathematical and Theoretical* 40.37 (2007). arXiv: 0705.3129 [hep-th].

[91] T. Quella, I. Runkel, and G. M. T. Watts. "Reflection and transmission for conformal defects". In: *Journal of High Energy Physics* 2007.04 (2007). arXiv: `hep-th/0611296`.

[92] I. Runkel. "Perturbed Defects and T-Systems in Conformal Field Theory". In: *Journal of Physics A: Mathematical and Theoretical* 41.10 (2007). arXiv: `0711.0102 [hep-th]`.

[93] I. Brunner and D. Roggenkamp. "Defects and bulk perturbations of boundary Landau-Ginzburg orbifolds". In: *Journal of High Energy Physics* 2008.04 (2008). eprint: `0712.0188v2`.

[94] D. Gang and S. Yamaguchi. "Superconformal defects in the tricritical Ising model". In: *Journal of High Energy Physics* 12 (2008), p. 076. arXiv: `0809.0175 [hep-th]`.

[95] P. Giokas and G. M. T. Watts. "Differential equations from null vectors of the Ramond algebra". In: (2008). arXiv: `0812.3108 [hep-th]`.

[96] R. Blumenhagen and E. Plauschinn. *Introduction to Conformal Field Theory*. 1st. Springer, 2009.

[97] P. Fendley, M. Fisher, and C. Nayak. "Boundary conformal field theory and tunneling of edge quasiparticles in non-Abelian topological states". In: *Annals of Physics* 324.7 (2009), pp. 1547–1572. arXiv: `0902.0998 [cond-mat]`.

[98] M. Kormos, I. Runkel, and G. M. T. Watts. "Defect flows in minimal models". In: *Journal of High Energy Physics* 2009.11 (2009). arXiv: `0907.1497 [hep-th]`.

[99] L. Alday, D. Gaiotto, and Y. Tachikawa. "Liouville correlation functions from four-dimensional gauge theories". In: *Letters in Mathematical Physics* 91.2 (2010), pp. 167–197. eprint: `0906.3219v2`.

[100] I. Runkel. "Non-local conserved charges from defects in perturbed conformal field theory". In: *Journal of Physics* A43.36 (2010), p. 365206. arXiv: `1004.1909 [hep-th]`.

[101] N. Drukker, D. Gaiotto, and J. Gomis. "The Virtue of Defects in 4D Gauge Theories and 2D CFTs". In: *Journal of High Energy Physics* 2011.25 (2011). eprint: `1003.1112v2`.

[102] J. Fuchs, C. Schweigert, and C. Stigner. "The classifying algebra for defects". In: *Nuclear Physics B* 843.3 (2011), pp. 673–723. arXiv: `1007.0401 [hep-th]`.

[103] M. R. Gaberdiel and R. Gopakumar. "An AdS3 Dual for Minimal Model CFTs". In: *Physics Reviews* D83 (2011), p. 066007. arXiv: `1011.2986 [hep-th]`.

[104] P. Giokas. "Mean field theory for boundary Ising and tricritical Ising models". In: *arXiv preprint* (2011). arXiv: `1106.1843 [cond-mat.stat-mech]`.

[105] C. Bachas, I. Brunner, and D. Roggenkamp. "A worldsheet extension of O(d,d;Z)". In: *Journal of High Energy Physics* 2012.10 (2012). arXiv: `1205.4647 [hep-th]`.

[106] D. Gaiotto. "Domain Walls for Two-Dimensional Renormalization Group Flows". In: *Journal of High Energy Physics* 2012.12 (Jan. 2012), p. 29. eprint: `1201.0767`.

[107] E. Witten. “Notes On Super Riemann Surfaces And Their Moduli”. In: (2012). arXiv: 1209.2459 [hep-th].

[108] C. Bachas, I. Brunner, and D. Roggenkamp. “Fusion of critical defect lines in the 2D Ising model”. In: *Journal of Statistical Mechanics: Theory and Experiment* 2013.08 (2013), P08008. arXiv: 1303.3616 [cond-mat].

[109] A. Recknagel and V. Schomerus. *Boundary Conformal Field Theory and the World-sheet Approach to D-Branes*. 1st. Cambridge University Press, 2013.

[110] Z. Bajnok, L. Holló, and G. M. T. Watts. “Defect scaling Lee–Yang model from the perturbed DCFT point of view”. In: *Nuclear Physics* B886 (2014), pp. 93–124. arXiv: 1307.4536 [hep-th].

[111] A. Konechny and C. Schmidt-Colinet. “Entropy of conformal perturbation defects”. In: *Journal of Physics* A47.48 (2014), p. 485401. arXiv: 1407.6444 [hep-th].

[112] O. Foda and J.-F. Wu. “From topological strings to minimal models”. In: *arXiv preprint* (2015). eprint: 1504.01925v1.

[113] S. Novak and I. Runkel. “Spin from defects in two-dimensional quantum field theory”. In: (2015). arXiv: 1506.07547 [hep-th].

[114] G. Poghosyan and H. Poghosyan. “RG domain wall for the N = 1 minimal super-conformal models”. In: *Journal of High Energy Physics* 1505 (2015), p. 043. eprint: 1412.6710v1.

[115] M. Billò, V. Gonçalves, E. Lauria, and M. Meineri. “Defects in conformal field theory”. In: *Journal of High Energy Physics* 04 (2016), p. 091. arXiv: 1601.02883 [hep-th].

[116] I. Brunner and C. Schmidt-Colinet. “Reflection and transmission of conformal perturbation defects”. In: *Journal of Physics* A49.19 (2016), p. 195401. arXiv: 1508.04350 [hep-th].

[117] I. Makabe and G. M. T. Watts. “Defects in the Tri-critical Ising model”. In: *Journal of High Energy Physics* 09 (2017), p. 013. arXiv: 1703.09148 [hep-th].

[118] I. Makabe and G. M. T. Watts. “The reflection coefficient for minimal model conformal defects from perturbation theory”. In: (2017). arXiv: 1712.07234 [hep-th].