SCHATTEN CLASS CONDITIONS FOR FUNCTIONS OF SCHRÖDINGER OPERATORS

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Abstract. We consider the difference $f(H_1) - f(H_0)$, where $H_0 = -\Delta$ and $H_1 = -\Delta + V$ are the free and the perturbed Schrödinger operators in $L^2(\mathbb{R}^d)$, and V is a real-valued short range potential. We give a sufficient condition for this difference to belong to a given Schatten class $\mathbf{S}_p$, depending on the rate of decay of the potential and on the smoothness of f (stated in terms of the membership in a Besov class). In particular, for $p > 1$ we allow for some unbounded functions f.

1. Introduction and main results

1.1. **Overview.**

Let H_0 and H_1 be the free and the perturbed (self-adjoint) Schrödinger operators,

$$H_0 = -\Delta, \quad H_1 = -\Delta + V \quad \text{in } L^2(\mathbb{R}^d),\ d \geq 1, \tag{1.1}$$

where the real-valued potential V satisfies the bound

$$|V(x)| \leq C(1 + |x|)^{-\rho}, \quad \rho > 1. \tag{1.2}$$

The purpose of this paper is to give new sufficient conditions for the boundedness and the Schatten class membership of the difference

$$D(f) := f(H_1) - f(H_0)$$

where f is a complex-valued function on $\mathbb{R}$ of an appropriate class. These conditions are given in terms of the smoothness of f and the exponent ρ in (1.2). This paper is a continuation of [7], where this problem was considered in the general operator theoretic context. It is also a further development of [5], where the trace class membership of $D(f)$ was considered. As explained in [5] and briefly recalled in Subsection 1.6 below, this problem is in part motivated by applications to mathematical physics.

As it is well known, the continuous spectrum of both H_0 and H_1 consists of the closed positive half-line $[0, \infty)$. We focus on the local behaviour of f on $(0, \infty)$. The questions of the behaviour of f at $+\infty$ and near zero are of a very different nature, so in what follows we assume that f is compactly supported on $(0, \infty)$. As explained in Subsection 1.6, this is not a severe restriction in the applications that we have in mind.

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If f is sufficiently smooth, say, $f \in C_0^\infty(0,\infty)$, and the exponent ρ is sufficiently large, then it is not difficult to show, by a variety of standard methods, that the difference $D(f)$ is trace class. On the other hand, as shown in [16], if f has a jump discontinuity at a point $\lambda > 0$, then $D(f)$ is never compact, unless scattering at energy λ is trivial. Thus, a question arises how the transition from the non-compact to the compact difference $D(f)$ occurs when the smoothness of f increases. The "degree of compactness" of $D(f)$ will be measured by its Schatten class membership, and the "degree of smoothness" of f — by its Besov class membership.

Our key example is of f having an isolated cusp-like singularity (see (1.3), (1.4) below) on the positive half-line, smooth elsewhere and compactly supported.

1.2. **Boundedness and compactness of** $D(f)$**.** Below $\mathrm{BMO}(\mathbb{R})$ is the class of functions of bounded mean oscillation on $\mathbb{R}$, and $\mathrm{VMO}(\mathbb{R})$ (vanishing mean oscillation) is the closure of $C(\mathbb{R}) \cap \mathrm{BMO}(\mathbb{R})$ in BMO. Further, $\mathcal{B}$ and $\mathbf{S}_\infty$ are the classes of bounded and compact operators on $L^2(\mathbb{R}^d)$. Precise definitions are given in Section 2.

Theorem 1.1. *Let* H_0*,* H_1 *be as in* (1.1)*,* (1.2) *with* $\rho > 1$*.*

(i) *For any* $f \in \mathrm{BMO}(\mathbb{R})$ *with compact support in* $(0,\infty)$*, we have* $D(f) \in \mathcal{B}$*.*
(ii) *For any* $f \in \mathrm{VMO}(\mathbb{R})$ *with compact support in* $(0,\infty)$*, we have* $D(f) \in \mathbf{S}_\infty$*.*

To illustrate the type of admissible singularities for the function f in the above theorem, let us consider the following example. Let $\chi_0 \in C_0^\infty(\mathbb{R})$ be a function which equals 1 in a neighbourhood of the origin and vanishes outside the interval $(-c, c)$ with some $0 < c < 1$. Then the function

$$f(x) = \chi_0(x)|\log|x|| \tag{1.3}$$

is in $\mathrm{BMO}(\mathbb{R})$, and the function

$$f_\gamma(x) = \chi_0(x)|\log|x||^\gamma$$

is in $\mathrm{VMO}(\mathbb{R})$ if $\gamma < 1$. Of course, the same applies to all shifted functions $f(x-\lambda)$, $f_\gamma(x-\lambda)$ for $\lambda \in \mathbb{R}$. Observe that these functions are unbounded for $\gamma > 0$; this is perhaps the most striking feature of Theorem 1.1. Observe also that functions with a jump discontinuity are in BMO, but not in VMO.

1.3. **Schatten class membership of** $D(f)$**.** For $0 < p < \infty$, $B^{1/p}_{p,p}(\mathbb{R})$ is the Besov class of functions on $\mathbb{R}$ and $\mathbf{S}_p$ is the Schatten class of all compact operators in $L^2(\mathbb{R}^d)$; see Section 2.

Theorem 1.2. *Let* H_0*,* H_1 *be as in* (1.1)*,* (1.2)*.*

(i) *Assume* $1 < \rho \le d$*. Then for any* $p > \dfrac{d-1}{\rho-1}$ *and for any* $f \in B^{1/p}_{p,p}(\mathbb{R})$ *with compact support in* $(0,\infty)$*, we have* $D(f) \in \mathbf{S}_p$*.*

(ii) *Assume $\rho > d$. Then for any $p > d/\rho$ and for any $f \in B^{1/p}_{p,p}(\mathbb{R})$ with compact support in $(0,\infty)$, we have $D(f) \in \mathbf{S}_p$.*

For $p = 1$, this is the main result of [5].

To illustrate the type of local singularities allowed for the functions $f \in B^{1/p}_{p,p}(\mathbb{R})$, consider the following example. Let $\chi_0 \in C_0^\infty(\mathbb{R})$ be as above; fix $\alpha > -1$, $a_+, a_- \in \mathbb{C}$, and consider the function

$$F_\alpha(x) = \begin{cases} a_+\chi_0(x)|\log|x||^{-\alpha}, & x > 0, \\ a_-\chi_0(x)|\log|x||^{-\alpha}, & x < 0. \end{cases} \tag{1.4}$$

It can be shown that (see [15] or [7, Proposition 1.3])

(i) If $a_+ \neq a_-$ and $\alpha > 0$, then $F_\alpha \in B^{1/p}_{p,p}(\mathbb{R})$ if and only if $p > 1/\alpha$.
(ii) If $a_+ = a_- \neq 0$ and $\alpha > -1$, then $F_\alpha \in B^{1/p}_{p,p}(\mathbb{R})$ if and only if $p > 1/(\alpha+1)$.

We see that for $p > 1$, the functions F_α may be unbounded. On the other hand, for $0 < p \leq 1$, the functions in $B^{1/p}_{p,p}(\mathbb{R})$ are always bounded and continuous.

1.4. **Discussion.** Prior to our work [5], the sharpest sufficient conditions for Schatten class inclusions for $D(f)$ were obtained through general operator theoretic estimates of the form [13]

$$\|f(H_1) - f(H_0)\|_p \leq C(p)\|f\|_{\mathrm{Lip}(\mathbb{R})}\|H_1 - H_0\|_p, \quad 1 < p < \infty, \tag{1.5}$$

with appropriate modifications for $p = 1$ and $p = \infty$; see [12]. Here Lip($\mathbb{R}$) is the Lipschitz class and $\|\cdot\|_p$ is the norm in $\mathbf{S}_p$. Of course, for the Schrödinger operator, the difference $V = H_1 - H_0$ is never in $\mathbf{S}_p$, but one can apply (1.5) to the resolvents of H_0, H_1 or their powers.

Observe that none of the functions (1.3), (1.4) is in Lip($\mathbb{R}$) (unless $\alpha = 0$); they are not even in any Hölder class. So one cannot hope to deduce Theorem 1.2 from (1.5).

In [5], we have used an ad hoc calculation, combining Kato smoothness with an integral representation for $B^1_{1,1}$ functions to prove Theorem 1.2 for $p = 1$. In [7] we approach the problem in a more systematic fashion; working in a general operator theoretic framework, we introduce the notion of $\mathbf{S}_p$-valued Kato smoothness and combine it with the double operator integral technique of Birman and Solomyak to treat all cases $0 < p < \infty$; see Sections 2.4 and 2.5 below. Here we apply and adapt the general results of [7] to the Schrödinger operators H_0, H_1.

We emphasize that while the arguments in the present paper are much more special than the theory developed in [7], they are by no means restricted to the case where the unperturbed operator is the Laplacian. Rather, the basic underlying assumption is that the unperturbed operator has a 'nice' diagonalization in an interval containing the support of the function f and that its resolvent, or powers thereof, satisfy some trace ideal properties when multiplied by decaying functions. For instance, our results should remain valid when $-\Delta$ is replaced by $-\Delta + V_0(x)$

where V_0 is periodic and the function f is supported away from band edges. Other examples are the three dimensional Landau Hamiltonian (with f supported away from the Landau levels) or the Stark operator. In these cases the function $(1+|x|)^{-\rho}$ in (1.2) needs to be modified appropriately. Yet another example is the discrete Laplacian. We omit the details, but refer to Section 11 of [14] for some of the necessary ingredients for these extensions in some cases.

Another generalization that we do not pursue here is to replace the pointwise assumption (1.2) on V by an integral assumption. In [5] we showed that this was possible for $p = 1$.

1.5. **Some ideas of the proof.** To prove our main results we proceed as follows. Let Λ be an open bounded interval in $\mathbb{R}$, such that $\operatorname{supp} f \subset \Lambda$ and the closure of Λ is included in $(0, \infty)$. We denote by $\mathbb{1}_\Lambda$ (resp. by $\mathbb{1}_{\Lambda^c}$) the characteristic function of Λ (resp. of the complement Λ^c) in $\mathbb{R}$. We write

$$\begin{aligned} D(f) &= (\mathbb{1}_\Lambda(H_1) + \mathbb{1}_{\Lambda^c}(H_1))D(f)(\mathbb{1}_\Lambda(H_0) + \mathbb{1}_{\Lambda^c}(H_0)) \\ &= \mathbb{1}_\Lambda(H_1)D(f)\mathbb{1}_\Lambda(H_0) - \mathbb{1}_{\Lambda^c}(H_1)f(H_0) + f(H_1)\mathbb{1}_{\Lambda^c}(H_0); \end{aligned} \tag{1.6}$$

here several terms vanish because of the assumption $\operatorname{supp} f \subset \Lambda$. We estimate the "diagonal term" $\mathbb{1}_\Lambda(H_1)D(f)\mathbb{1}_\Lambda(H_0)$ by directly applying the results of [7] and some variants of the limiting absorption principle. We estimate the "off-diagonal terms" (the second and third terms in the right side of (1.6)) by using rather standard Schatten class bounds for Schrödinger operators.

Following the proofs, it is not difficult to obtain estimates for the relevant norms of $D(f)$ in terms of the exponents p, ρ, d, and the geometry of the support of f. However, these estimates are clearly very far from being optimal (perhaps with the exception of the ones for the diagonal term in (1.6) above), and so we have not attempted to work them out explicitly.

1.6. **Motivations from mathematical physics.** In a number of problems from mathematical physics one encounters differences $f(H_1) - f(H_0)$ where H_1 and H_0 are Schrödinger operators as in (1.1) (or their generalizations mentioned in Subsection 1.4) and where either the function f is non-smooth at a certain $\mu > 0$ or where the function f belongs to a family of functions whose smoothness at a point $\mu > 0$ degenerates in an asymptotic regime. While in these applications bounds on $f(H_1) - f(H_0)$ are needed most frequently in trace class norm, bounds in other Schatten norms or in operator norm are often a useful tool in the proofs.

We believe that our theorems and the methods we use to prove them are relevant in several such problems. The fact that our theorems are only stated for functions with compact support in $(0, \infty)$ is not a severe restriction since in many applications one can decompose $f = f_1 + f_2$ where f_1 has compact support in $(0, \infty)$ and where f_2 is smooth. The contribution of f_2 to the difference can be controlled by (1.6) or other standard bounds, while our theorems apply to f_1, which in the situations we have in mind gives the main contribution.

To be more specific, the function $f(x) = -\min\{x-\mu, 0\}$ with $\mu > 0$ appears in the problem of estimating the energy cost of making a hole in the Fermi sea. This cost was quantified through a version of the Lieb–Thirring inequality at positive density [3, 4]. In order to convert the 'density version' of this inequality into its 'potential version', one needs the a priori information that $f(H_1) - f(H_0)$ is trace class. This was shown in [5] and is one of the basic motivations of this and our previous work [7]. We emphasize that the above function f does not satisfy the sufficient conditions from [12] which guarantee membership in the trace class.

The case where a family of smooth functions f approaches a discontinuous function is relevant in the study of what is known as the Anderson orthogonality catastrophe; see [8, 6] and references therein. The discontinuous limiting function is $f(x) = \chi_{\{x<\mu\}}$, while the functions approximating this function can be chosen smooth; see Section 3 in [8]. To be more precise, in this problem the product of $f(H_1)$ and $f(H_0)$ rather than their difference appears, but a mathematically closely related problem for the difference was studied by one of us in [14]. In fact, in view of the latter work we believe that for both the operator norm and the Schatten norm with any fixed $0 < p < \infty$ the assumptions on ρ and f in Theorems 1.1 and 1.2 are best possible. Investigating this optimality, however, is beyond the scope of the present paper.

Different, but not unrelated bounds are relevant in the study of the entanglement entropy in quantum systems. We refer to [11, 19] and references therein.

1.7. **The structure of the paper.** The paper can be divided into two parts: in Sections 2–3, we work in a general operator theoretic framework, and in Sections 4–6 we specialise to the case of the Schrödinger operator.

In Section 2 we recall definitions of relevant function and operator classes, discuss the notions of Kato smoothness and $\mathbf{S}_p$-valued Kato smoothness and recall the main results of [7], which apply to estimates for the diagonal terms in (1.6). In Section 3, we prove preliminary estimates for the off-diagonal terms in (1.6).

In Section 4 we give sufficient conditions for $\mathbf{S}_p$-valued smoothness in the context of the Schrödinger operator. In Section 5 we prove that certain auxiliary operators belong to relevant $\mathbf{S}_p$ classes; these facts are needed to treat the off-diagonal terms in (1.6). Finally, in Section 6 we put everything together and prove Theorems 1.1 and 1.2.

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2. Preliminaries

2.1. **The classes** BMO **and** VMO. The space BMO($\mathbb{R}$) (bounded mean oscillation) consists of all locally integrable functions f on $\mathbb{R}$ such that the following supremum over all bounded intervals $I \subset \mathbb{R}$ is finite:

$$\sup_I \langle |f - \langle f \rangle_I| \rangle_I < \infty, \quad \langle f \rangle_I = |I|^{-1} \int_I f(x)dx. \tag{2.1}$$

Observe that this supremum vanishes on constant functions. Strictly speaking, the elements of $\mathrm{BMO}(\mathbb{R})$ should be regarded not as functions but as equivalence classes $\{f + \mathrm{const}\}$. However, since here we are interested in compactly supported functions f, this issue is not important to us. Functions in $\mathrm{BMO}(\mathbb{R})$ belong to $L^p(-R, R)$ for any $R > 0$ and any $p < \infty$, but not for $p = \infty$: they may have logarithmic singularities, see (1.3).

Many explicit equivalent norms on $\mathrm{BMO}(\mathbb{R})$ are known (see e.g. [9]). The easiest one to define is the supremum in (2.1). In [7] we use the norm related to Fefferman's duality theorem, which identifies $\mathrm{BMO}(\mathbb{R})$ with the dual to the Hardy class H^1. This choice of the norm allowed us to explicitly determine the optimal constant appearing in the right hand side of (2.8). However, in this paper we do not attempt to keep track of all constants appearing in estimates, and so the choice of the norm in $\mathrm{BMO}(\mathbb{R})$ is not important here.

The subspace $\mathrm{VMO}(\mathbb{R}) \subset \mathrm{BMO}(\mathbb{R})$ is characterised by the condition

$$\lim_{\epsilon \to 0} \sup_{|I| \leq \epsilon} \langle |f - \langle f \rangle_I| \rangle_I = 0.$$

Alternatively, $\mathrm{VMO}(\mathbb{R})$ is the closure of $C(\mathbb{R}) \cap \mathrm{BMO}(\mathbb{R})$ in $\mathrm{BMO}(\mathbb{R})$.

In [7], we also use the space $\mathrm{CMO}(\mathbb{R})$ (continuous mean oscillation) which can be characterised as the closure of $C_{\mathrm{comp}}(\mathbb{R}) \cap \mathrm{BMO}(\mathbb{R})$ in $\mathrm{BMO}(\mathbb{R})$. However, for a compactly supported function f, conditions $f \in \mathrm{VMO}$ and $f \in \mathrm{CMO}$ coincide.

2.2. **The Besov class $B^{1/p}_{p,p}$.** Let $w \in C_0^\infty(\mathbb{R})$, $w \geq 0$, be a function with $\operatorname{supp} w \subset [1/2, 2]$ and such that

$$\sum_{j \in \mathbb{Z}} w_j(x) = 1, \quad x > 0, \qquad \text{where } w_j(x) = w(x/2^j).$$

The (homogeneous) Besov class $B^{1/p}_{p,p}(\mathbb{R})$ is defined as the space of tempered distributions f on $\mathbb{R}$ such that

$$\|f\|^p_{B^{1/p}_{p,p}} := \sum_{j \in \mathbb{Z}} 2^j \big(\|f * \widehat{w}_j\|^p_{L^p(\mathbb{R})} + \|f * \overline{\widehat{w}_j}\|^p_{L^p(\mathbb{R})} \big) < \infty. \tag{2.2}$$

Here $\widehat{w}_j$ is the Fourier transform of w_j, and $*$ is the convolution.

We will only be interested in compactly supported elements in $B^{1/p}_{p,p}(\mathbb{R})$. For compactly supported functions f, sufficient conditions for Besov class membership can be given in terms of the usual Sobolev spaces:

$$f \in W^s_p(\mathbb{R}) \Rightarrow f \in B^{1/p}_{p,p}(\mathbb{R}), \quad s > 1/p.$$

(For $p \geq 2$, this follows from [1, Theorem 6.4.4], even with $s = 1/p$. For $0 < p < 2$, this follows from a slight modification of [1, Lemma 6.2.1(1)].) On the other hand, it may be useful to note that

$$f \in B^{1/p}_{p,p}(\mathbb{R}) \Rightarrow f \in W^{1/p}_p(\mathbb{R}), \quad 0 < p \leq 1.$$

(Again, this follows from an adaptation of [1, Lemma 6.2.1(1)] to $0 < p \leq 1$.) In particular, $B^1_{1,1}(\mathbb{R}) \subset C(\mathbb{R})$.

2.3. **Schatten classes.** For $0 < p < \infty$, the Schatten class $\mathbf{S}_p$ is the class of all compact operators A in a given Hilbert space such that

$$\|A\|_p = \Big(\sum_{n=1}^{\infty} s_n(A)^p\Big)^{1/p} < \infty,$$

where $\{s_n(A)\}_{n=1}^{\infty}$ is the sequence of all singular values of A, enumerated with multiplicities taken into account. The expression $\|\cdot\|_p$ is a norm for $p \geq 1$ and a quasinorm for $0 < p < 1$. For $0 < p \leq 1$ we have the following modified triangle inequality in $\mathbf{S}_p$:

$$\|A + B\|_p^p \leq \|A\|_p^p + \|B\|_p^p, \quad A, B \in \mathbf{S}_p, \quad 0 < p \leq 1. \tag{2.3}$$

We will also need the following Hölder inequality in Schatten classes:

$$\|AB\|_p \leq \|A\|_q \|B\|_r, \quad \tfrac{1}{p} = \tfrac{1}{q} + \tfrac{1}{r}. \tag{2.4}$$

2.4. **Kato smoothness.** Here we briefly recall (with minor simplifications) the relevant definitions and main results of [7].

To motivate what comes next, we should explain that we will factorise the potential V in the form

$$V = (\operatorname{sign} V)|V|^{1-\theta}|V|^{\theta}$$

with an appropriate exponent $\theta \in (0, 1)$. This corresponds to the "abstract" factorisation

$$V = G_1^* G_0$$

of [7]. In [7], we consider the general case, where G_0, G_1 are possibly unbounded operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, such that G_0 is H_0-bounded and G_1 is H_1-bounded. In this paper, since V is assumed to be bounded, we will only consider the case of bounded operators G_0, G_1; this simplifies the exposition. We shall also assume $\mathcal{H} = \mathcal{K}$.

Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let G be a bounded operator in $\mathcal{H}$. One says that G is *Kato smooth with respect to* H (we will write $G \in \mathrm{Smooth}(H)$), if

$$\|G\|_{\mathrm{Smooth}(H)} := \sup_{\|\varphi\|_{L^2(\mathbb{R})}=1} \|G\varphi(H)\| < \infty. \tag{2.5}$$

As shown in [7], this definition coincides with the standard definition (see [10]) of Kato smoothness. The advantage of the definition (2.5) is that it extends naturally to Schatten classes. Generalising (2.5), we will say that $G \in \mathrm{Smooth}_p(H)$ for some $0 < p < \infty$, if

$$\|G\|_{\mathrm{Smooth}_p(H)} := \sup_{\|\varphi\|_{L^2(\mathbb{R})}=1} \|G\varphi(H)\|_p < \infty.$$

Finally, we shall write $G \in \mathrm{Smooth}_\infty(H)$, if $G \in \mathrm{Smooth}(H)$ and if

$$G\mathbb{1}_{(-R,R)}(H) \in \mathbf{S}_\infty \quad \forall R > 0.$$

It is very easy to prove [7, Lemma 2.3] that for $G \in \mathrm{Smooth}_\infty(H)$, one has

$$G\varphi(H) \in \mathbf{S}_\infty, \quad \forall \varphi \in L^2(\mathbb{R}). \tag{2.6}$$

2.5. **Main results from** [7]**.** In the following theorem, H_0 and H_1 are self-adjoint operators in a Hilbert space $\mathcal{H}$ such that the perturbation $H_1 - H_0$ factorises as

$$H_1 - H_0 = G_1^* G_0,$$

where G_0, G_1 are bounded operators in $\mathcal{H}$. Let $\Lambda \subset \mathbb{R}$ be a measurable set; the case $\Lambda = \mathbb{R}$ is not excluded. (In fact, during the first reading of this subsection, the reader is encouraged to think of the simplest case $\Lambda = \mathbb{R}$.) Here we are interested in the "diagonal term" in (1.6),

$$D_\Lambda(f) := \mathbb{1}_\Lambda(H_1) D(f) \mathbb{1}_\Lambda(H_0).$$

Since functions $f \in \mathrm{BMO}(\mathbb{R})$ in general need not be bounded, we need to take some care in defining the operator $D_\Lambda(f)$. We define the corresponding sesquilinear form

$$d_{\Lambda,f}[u,v] := (\mathbb{1}_\Lambda(H_0)u, \overline{f}(H_1)\mathbb{1}_\Lambda(H_1)v) - (f(H_0)\mathbb{1}_\Lambda(H_0)u, \mathbb{1}_\Lambda(H_1)v),$$

for $u \in \mathrm{Dom}\, f(H_0)$, $v \in \mathrm{Dom}\, f(H_1)$. Of course, if f is bounded, we can define $D_\Lambda(f)$ directly and then

$$d_{\Lambda,f}[u,v] = (D_\Lambda(f)u, v) \tag{2.7}$$

for all u and v as above. We use the standard convention that if the norms in the right hand side of an upper bound are all finite, then the bound includes the statement that the norms in the left hand side are also finite. The following theorem is a combination of Theorems 7.5 and 7.6 from [7].

Theorem 2.1. *Let H_0, H_1, G_0, G_1, Λ, $d_{\Lambda,f}$ be as above.*

(i) *For any $f \in \mathrm{BMO}(\mathbb{R})$, the sesquilinear form $d_{\Lambda,f}[u,v]$ satisfies the bound*

$$|d_{\Lambda,f}[u,v]| \leq C\|f\|_{\mathrm{BMO}(\mathbb{R})}\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}(H_0)}\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}(H_1)}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}},$$

for any $u \in \mathrm{Dom}\, f(H_0)$, $v \in \mathrm{Dom}\, f(H_1)$, where the constant C depends only on the choice of the norm in $\mathrm{BMO}(\mathbb{R})$. *Thus, the form $d_{\Lambda,f}$ corresponds to a bounded linear operator $D_\Lambda(f)$ in $\mathcal{H}$ (in the sense of* (2.7)*), and this operator satisfies*

$$\|D_\Lambda(f)\| \leq C\|f\|_{\mathrm{BMO}(\mathbb{R})}\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}(H_0)}\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}(H_1)}. \tag{2.8}$$

(ii) *Assume that* $G_0\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}(H_0)$, $G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}(H_1)$ *and at least one of the inclusions*

$$G_0\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_\infty(H_0), \quad G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_\infty(H_1)$$

holds. Then for any $f \in \mathrm{VMO}(\mathbb{R})$ *the operator* $D_\Lambda(f)$ *is compact.*

(iii) *Let* p, q, r *be finite positive indices such that* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. *Then for any* $f \in B^{1/p}_{p,p}(\mathbb{R}) \cap \mathrm{BMO}(\mathbb{R})$, *one has*

$$\|D_\Lambda(f)\|_p \le C(p)\|f\|_{B^{1/p}_{p,p}(\mathbb{R})}\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}_q(H_0)}\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)},$$

where the constant $C(p)$ *depends only on the choice of the function* w *in* (2.2).

3. Off-diagonal terms

Let H_0, H_1 be self-adjoint operators in $\mathcal{H}$, with

$$H_1 - H_0 = G_1^*G_0 = G_0^*G_1,$$

where G_0 and G_1 are bounded operators in $\mathcal{H}$.

Let $\Lambda = (a-b, a+b)$ be a bounded open interval, and let f be a function supported in Λ. In this section we estimate the norms of the off-diagonal terms in (1.6), namely,

$$\mathbb{1}_{\Lambda^c}(H_1)f(H_0) \quad \text{and} \quad f(H_1)\mathbb{1}_{\Lambda^c}(H_0). \tag{3.1}$$

As in the previous section, since f need not be bounded, we have to take care about defining the operators (3.1). We define $\mathbb{1}_{\Lambda^c}(H_1)f(H_0)$ initially on $\mathrm{Dom}\, f(H_0)$. Further, instead of $f(H_1)\mathbb{1}_{\Lambda^c}(H_0)$ we will consider initially its formal adjoint $\mathbb{1}_{\Lambda^c}(H_0)f(H_1)$, defined on $\mathrm{Dom}\, f(H_1)$.

The following preliminary lemma establishes a series representation for these two operators. This representation plays the same role here as the double operator integrals in the proof of Theorem 2.1 (see [7]): it allows us to estimate the operator norms. Then we will refine this representation and estimate the Schatten norms in Lemma 3.2.

In what follows we denote $R_0(z) = (H_0 - z)^{-1}$, $R_1(z) = (H_1 - z)^{-1}$.

Lemma 3.1. *Let* H_0, H_1, G_0, G_1, Λ *be as described above, and let* $f \in L^2(\mathbb{R})$, $\mathrm{supp}\, f \subset \Lambda$. *Assume that*

$$G_0\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}(H_0) \quad \text{and} \quad G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}(H_1).$$

Then the operator $\mathbb{1}_{\Lambda^c}(H_1)f(H_0)$, *defined initially on* $\mathrm{Dom}\, f(H_0)$, *and the operator* $\mathbb{1}_{\Lambda^c}(H_0)f(H_1)$, *defined initially on* $\mathrm{Dom}\, f(H_1)$, *extend to bounded operators on* $\mathcal{H}$.

Moreover, we have the series representations

$$\mathbb{1}_{\Lambda^c}(H_0)f(H_1) = -\sum_{m=0}^{\infty}(H_0-a)^{-m-1}\mathbb{1}_{\Lambda^c}(H_0)G_0^*G_1(H_1-a)^m f(H_1)\,, \tag{3.2}$$

$$\mathbb{1}_{\Lambda^c}(H_1)f(H_0) = \sum_{m=0}^{\infty}(H_1-a)^{-m-1}\mathbb{1}_{\Lambda^c}(H_1)G_1^*G_0(H_0-a)^m f(H_0)\,, \tag{3.3}$$

where both series converge absolutely in the operator norm. Furthermore, with $\delta = \operatorname{dist}(\operatorname{supp} f, \Lambda^c)$ *and* $z = a + ib$*, we have the estimates*

$$\|\mathbb{1}_{\Lambda^c}(H_0)f(H_1)\| \le \sqrt{2}(b/\delta)\|f\|_{L^2}\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}(H_1)}\|G_0R_0(z)\|, \tag{3.4}$$

$$\|\mathbb{1}_{\Lambda^c}(H_1)f(H_0)\| \le \sqrt{2}(b/\delta)\|f\|_{L^2}\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}(H_0)}\|G_1R_1(z)\|. \tag{3.5}$$

If, in addition,

$$G_0\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_\infty(H_0) \quad \text{and} \quad G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_\infty(H_1),$$

then

$$\mathbb{1}_{\Lambda^c}(H_0)f(H_1) \in \mathbf{S}_\infty \quad \text{and} \quad \mathbb{1}_{\Lambda^c}(H_1)f(H_0) \in \mathbf{S}_\infty.$$

We note that although the stand-alone operator $(H_0 - a)^{-m-1}$ does not necessarily make sense, the product $(H_0 - a)^{-m-1}\mathbb{1}_{\Lambda^c}(H_0)$ in (3.2) is well defined and bounded, because $a \in \Lambda$. The same comment applies to the operator $(H_1 - a)^{-m-1}\mathbb{1}_{\Lambda^c}(H_1)$ in (3.3).

Proof. For simplicity of notation, we assume $a = 0$, so $\operatorname{supp} f \subset [-b_0, b_0]$ with $b_0 = b - \delta$. First observe that formally, we have

$$\sum_{m=0}^{\infty} H_0^{-m-1}G_0^*G_1H_1^m = \sum_{m=0}^{\infty} H_0^{-m-1}(H_1 - H_0)H_1^m$$
$$= \sum_{m=0}^{\infty}(H_0^{-m-1}H_1^{m+1} - H_0^{-m}H_1^m) = -I.$$

After multiplication by $\mathbb{1}_{\Lambda^c}(H_0)$ on the left and by $f(H_1)$ on the right, we obtain (3.2). Now let us prove the norm convergence of the series in (3.2). For each term, we have the estimate

$$\begin{aligned}\|\mathbb{1}_{\Lambda^c}(H_0)H_0^{-m-1}G_0^*G_1H_1^m f(H_1)\| &\le \|\mathbb{1}_{\Lambda^c}(H_0)H_0^{-m-1}G_0^*\|\|G_1H_1^m f(H_1)\| \\ &\le b^{-m}\|\mathbb{1}_{\Lambda^c}(H_0)H_0^{-1}G_0^*\|\; b_0^m\|G_1 f(H_1)\| \\ &\le (b_0/b)^m\|f\|_{L^2}\|G_0H_0^{-1}\mathbb{1}_{\Lambda^c}(H_0)\|\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}(H_1)}\,.\end{aligned} \tag{3.6}$$

Since $b_0 < b$, we have the norm convergence of the series in (3.2), and

$$\sum_{m=0}^{\infty} b_0^m b^{-m} = 1/(1 - b_0/b) = b/\delta$$

gives the factor b/δ in (3.4). Finally,

$$\|G_0H_0^{-1}\mathbb{1}_{\Lambda^c}(H_0)\| \leq \|G_0R_0(ib)\|\|H_0^{-1}\mathbb{1}_{\Lambda^c}(H_0)(H_0-ib)\| \leq \sqrt{2}\|G_0R_0(ib)\|,$$

since

$$\sup_{|\lambda|>b}|(\lambda-ib)/\lambda| \leq \sqrt{2}. \tag{3.7}$$

This gives the estimate (3.4).

The identity (3.3) and the estimate (3.5) are considered similarly. Finally, the compactness statement follows from the fact that by (2.6), each term in the norm convergent series (3.2), (3.3) is compact. □

Now we come to the Schatten class estimate. It is not difficult to estimate the Schatten norm of the off-diagonal terms (3.1) by the expressions similar to the right sides of (3.4), (3.5) but with Schatten norms instead of the operator norms. However, in application to the Schrödinger operator, this is not sufficient, as the operators $G_1R_1(z)$, $G_0R_0(z)$ will not necessarily be in the required Schatten classes. The standard way to deal with this problem is to consider powers of the resolvent, i.e., to consider $G_1R_1(z)^m$, $G_0R_0(z)^m$ for sufficiently high m; these operators will be in the required Schatten class. This is what we do below. The price to pay are the additional terms in the right sides of (3.8) and (3.9).

Lemma 3.2. *Assume the hypothesis of Lemma 3.1, and let p, q, r be positive finite exponents satisfying $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. Then for $z=a+ib$ and any integer $k\geq 0$,*

$$\begin{aligned}\|\mathbb{1}_{\Lambda^c}(H_0)f(H_1)\|_p \leq C(b,\delta,p,k)\big(&\|f\|_{L^2}\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)}\|G_0R_0(z)^{k+1}\|_q \\ &+\left\|\left(R_1(z)^k-R_0(z)^k\right)f(H_1)\right\|_p\big),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\|\mathbb{1}_{\Lambda^c}(H_1)f(H_0)\|_p \leq C(b,\delta,p,k)\big(&\|f\|_{L^2}\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}_q(H_0)}\|G_1R_1(z)^{k+1}\|_r \\ &+\left\|\left(R_1(z)^k-R_0(z)^k\right)f(H_0)\right\|_p\big).\end{aligned} \tag{3.9}$$

Proof. For simplicity of notation, we assume $a=0$ and let $\operatorname{supp} f\subset[-b_0,b_0]$, $b_0=b-\delta$. We will prove the first bound (3.8); the second bound (3.9) is proved in the same way.

Step 1. We prove the lemma for $k=0$.

We need to estimate the $\mathbf{S}_p$ norm of each term in the series in (3.2). Similarly to (3.6), we have

$$\begin{aligned}\|\mathbb{1}_{\Lambda^c}(H_0)H_0^{-m-1}G_0^*G_1H_1^mf(H_1)\|_p &\leq \|G_0H_0^{-m-1}\mathbb{1}_{\Lambda^c}(H_0)\|_q\|G_1H_1^mf(H_1)\|_r \\ &\leq b^{-m}\|G_0H_0^{-1}\mathbb{1}_{\Lambda^c}(H_0)\|_q b_0^m\|G_1f(H_1)\|_r \\ \leq (b_0/b)^m\|f\|_{L^2}\|(H_0-ib)H_0^{-1}&\mathbb{1}_{\Lambda^c}(H_0)\|\|G_0R_0(ib)\|_q\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)} \\ &\leq \sqrt{2}(b_0/b)^m\|f\|_{L^2}\|G_0R_0(ib)\|_q\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)},\end{aligned}$$

where the last estimate uses (3.7). For $p \geq 1$, this yields

$$\|\mathbb{1}_{\Lambda^c}(H_0)f(H_1)\|_p \leq \sum_{m=0}^{\infty} \sqrt{2}(b_0/b)^m \|f\|_{L^2} \|G_0 R_0(ib)\|_q \|G_1 \mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)}$$
$$= \sqrt{2}(b/\delta)\|f\|_{L^2}\|G_0R_0(ib)\|_q\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)}.$$

For $0 < p < 1$ we use the modified triangle inequality (2.3) in $\mathbf{S}_p$, which yields the same estimate with a different constant. Thus we get the required estimate for $k = 0$.

Step 2. We now consider $k > 0$. Let $g(\lambda) = (\lambda - z)^k f(\lambda)$, so that

$$f(H_1) = R_0(z)^k g(H_1) + \left(R_1(z)^k - R_0(z)^k\right) g(H_1)$$

and therefore

$$\mathbb{1}_{\Lambda^c}(H_0)f(H_1) = \mathbb{1}_{\Lambda^c}(H_0)R_0(z)^k g(H_1) + \mathbb{1}_{\Lambda^c}(H_0)\left(R_1(z)^k - R_0(z)^k\right) g(H_1)\,. \tag{3.10}$$

Let us discuss the two terms on the right side of (3.10) separately.

The first term can be estimated by the same technique as in Step 1. This yields

$$\left\|\mathbb{1}_{\Lambda^c}(H_0)R_0(z)^k g(H_1)\right\|_p \leq C(b,\delta,p)\,\|G_1 g(H_1)\|_r \left\|G_0 H_0^{-1} R_0(z)^k \mathbb{1}_{\Lambda^c}(H_0)\right\|_q$$
$$\leq C(b,\delta,p,k)\|f\|_{L^2}\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_r(H_1)} \left\|G_0 R_0(z)^{k+1}\right\|_q.$$

The second term in (3.10) is simply estimated by

$$\left\|\mathbb{1}_{\Lambda^c}(H_0)\left(R_1(z)^k - R_0(z)^k\right) g(H_1)\right\|_p \leq 2^{k/2} b^k \left\|\left(R_1(z)^k - R_0(z)^k\right) f(H_1)\right\|_p.$$

This completes the proof of the lemma. □

4. $\mathbf{S}_p$-VALUED SMOOTHNESS FOR THE SCHRÖDINGER OPERATOR

In this section H_0, H_1 are as in (1.1). We set $\langle x\rangle = \sqrt{1+|x|^2}$ and assume that $V(x)$ is real-valued and satisfies the condition

$$|V(x)| \leq C\langle x\rangle^{-\rho}, \quad \rho > 1. \tag{4.1}$$

As in Section 3, we denote the resolvents by $R_0(z) = (H_0 - z)^{-1}$, $R_1(z) = (H_1 - z)^{-1}$.

4.1. **The LAP and its consequences.** First we recall the limiting absorption principle (LAP) for the Schrödinger operator and translate it into statements about $\mathbf{S}_p$-valued smoothness.

Lemma 4.1. *Let H_0, H_1 be as above, with some $\rho > 1$. Then for any $\lambda > 0$, the limits*

$$\langle x\rangle^{-\rho/2} R_0(\lambda \pm i0)\langle x\rangle^{-\rho/2}, \quad \langle x\rangle^{-\rho/2} R_1(\lambda \pm i0)\langle x\rangle^{-\rho/2} \tag{4.2}$$

exist in the operator norm and are continuous (in the operator norm) in $\lambda > 0$. *Further, for any* $p \geq 1$, $p > \frac{d-1}{\rho-1}$, *we have the inclusions*

$$\mathrm{Im}\,(\langle x\rangle^{-\rho/2} R_0(\lambda + i0)\langle x\rangle^{-\rho/2}) \in \mathbf{S}_p, \tag{4.3}$$

$$\mathrm{Im}\,(\langle x\rangle^{-\rho/2} R_1(\lambda + i0)\langle x\rangle^{-\rho/2}) \in \mathbf{S}_p, \tag{4.4}$$

and these operators are continuous in $\lambda > 0$ *in* $\mathbf{S}_p$. *Finally, for the same range of* p *we have the inclusions*

$$\langle x\rangle^{-\rho/2} \mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_{2p}(H_0), \qquad \langle x\rangle^{-\rho/2} \mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_{2p}(H_1) \tag{4.5}$$

for any bounded interval $\Lambda \subset \mathbb{R}$ *with* $\mathrm{clos}(\Lambda) \subset (0, \infty)$.

Proof. The existence and continuity of the limits (4.2) is the standard LAP, see e.g. [22, Proposition 1.7.1, Theorem 6.2.1]. The inclusion (4.3) and the corresponding continuity in $\lambda > 0$ is also well-known; see e.g. [22, Lemma 8.1.2].

In order to deal with the operator in (4.4), we need a version of the resolvent identity. For $\mathrm{Im}\, z > 0$, we have

$$R_1(z) = (I + R_0(z)V)^{-1} R_0(z), \quad (I + R_0(z)V)^{-1} = I - R_1(z)V.$$

Taking the imaginary part in the first identity here and subsequently using the second identity, we obtain

$$\begin{aligned} \mathrm{Im}\, R_1(z) &= (I + R_0(z)V)^{-1}(\mathrm{Im}\, R_0(z))(I + VR_0(z)^*)^{-1} \\ &= (I - R_1(z)V)(\mathrm{Im}\, R_0(z))(I - VR_1(z)^*). \end{aligned} \tag{4.6}$$

Let us denote for brevity

$$W(x) = \langle x\rangle^{-\rho/2}, \quad V_1(x) = V(x)\langle x\rangle^{\rho/2}.$$

Multiplying (4.6) by W both on the right and on the left, we obtain

$$\begin{aligned} \mathrm{Im}\,(WR_1(z)W) &= W(I - R_1(z)V)(\mathrm{Im}\, R_0(z))(I - VR_1(z)^*)W \\ &= (I - WR_1(z)V_1)\mathrm{Im}\,(WR_0(z)W)(I - V_1 R_1(z)^* W). \end{aligned} \tag{4.7}$$

Now observe that $|V_1(x)| \leq C\langle x\rangle^{-\rho/2}$, and so, by the LAP (4.2), we can pass to the limit in the operator norm on both sides of (4.7) as $z \to \lambda + i0$, $\lambda > 0$. By (4.2) and (4.3), this yields the inclusion (4.4) and the continuity in $\lambda > 0$.

Let us prove the first inclusion in (4.5). By the LAP, for any $\varphi \in L^2(\mathbb{R})$, $\mathrm{supp}\,\varphi \subset \Lambda$, we have

$$W\varphi(H_0)(W\varphi(H_0))^* = W|\varphi(H_0)|^2 W = \frac{1}{\pi}\int_\Lambda |\varphi(\lambda)|^2 \mathrm{Im}\,(WR_0(\lambda + i0)W)d\lambda,$$

and therefore, by (4.3),

$$\|W\varphi(H_0)\|_{2p}^2 = \|W|\varphi(H_0)|^2 W\|_p \leq \frac{1}{\pi}\sup_{\lambda\in\Lambda}\|\mathrm{Im}\, WR_0(\lambda + i0)W\|_p \int_\Lambda |\varphi(\lambda)|^2 d\lambda.$$

This gives the inclusion $W\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_{2p}(H_0)$. The second inclusion in (4.5) follows from (4.4) in the same way. $\square$

4.2. **Estimates for $g(x)h(-i\nabla)$ and their consequences.** Let us we recall two estimates for operators of the form

$$g(x)h(-i\nabla) \quad \text{in } L^2(\mathbb{R}^d), \tag{4.8}$$

where g, h are complex-valued functions on $\mathbb{R}^d$ of the class to be specified below. Notation (4.8) is a common shorthand for operators defined by

$$\varphi \mapsto g(x)(\widecheck{h\widehat{\varphi}})(x), \quad x \in \mathbb{R}^d, \quad \varphi \in L^2(\mathbb{R}^d),$$

where $\varphi \mapsto \widehat{\varphi}$ is the standard (unitary) Fourier transform and $\varphi \mapsto \check{\varphi}$ is the inverse Fourier transform. See e.g. [18, Chapter 4] for the details. For $q > 0$ and a complex-valued function g on $\mathbb{R}^d$, we will use the notation

$$\|g\|_{\ell^q(L^2)}^q := \sum_{k\in\mathbb{Z}^d} \left(\int_{(0,1)^d+k} |g(x)|^2 dx \right)^{q/2};$$

the space $\ell^q(L^2)$ is the set of functions g with $\|g\|_{\ell^q(L^2)} < \infty$.

Proposition 4.2. (i) *Let $2 \le q < \infty$ and $g, h \in L^q(\mathbb{R}^d)$. Then $g(x)h(-i\nabla) \in \mathbf{S}_q$ and*

$$\|g(x)h(-i\nabla)\|_q \le C_{d,q}\|g\|_{L^q}\|h\|_{L^q}.$$

(ii) *Let $0 < q \le 2$ and $g, h \in \ell^q(L^2)$. Then $g(x)h(-i\nabla) \in \mathbf{S}_q$ and*

$$\|g(x)h(-i\nabla)\|_q \le C_{d,q}\|g\|_{\ell^q(L^2)}\|h\|_{\ell^q(L^2)}.$$

Part (i) is the Kato-Seiler-Simon inequality, see [17] or [18, Thm. 4.1]; part (ii) is the Birman-Solomyak inequality, see [2, Thm. 11.1] (or [18, Thm. 4.5] for $1 \le q \le 2$). Part (ii) is used in the next lemma, and part (i) is used in the following Section.

Lemma 4.3. *Let $\sigma > 0$ and $d/\sigma < q \le 2$. Then $\langle x\rangle^{-\sigma}\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_q(H_0)$ for any bounded interval $\Lambda \subset \mathbb{R}$ with $\mathrm{clos}(\Lambda) \subset (0,\infty)$.*

Proof. By Proposition 4.2(ii), we have

$$\|\langle x\rangle^{-\sigma}\mathbb{1}_\Lambda(H_0)\varphi(H_0)\|_q \le C\|\langle x\rangle^{-\sigma}\|_{\ell^q(L^2)}\|\varphi(|\xi|^2)\|_{\ell^q(L^2)}.$$

As Λ is bounded, the support of the function $\varphi(|\xi|^2)$ in $\mathbb{R}^d$ is also bounded. It follows that the sum in the expression for the norm $\|\varphi(|\xi|^2)\|_{\ell^q(L^2)}$ contains only finitely many terms. From here it easily follows that

$$\|\varphi(|\xi|^2)\|_{\ell^q(L^2)} \le C_\Lambda\|\varphi\|_{L^2}, \qquad \mathrm{supp}\,\varphi \subset \mathrm{clos}\,\Lambda,$$

which completes the proof. ☐

5. Global $\mathbf{S}_p$ conditions

Here H_0, H_1, V are as in the previous section.

Lemma 5.1. *Let $\sigma > 0$, $q > 0$, $m \in \mathbb{N}$ be such that*

$$\sigma q > d \qquad \text{and} \qquad 2mq > d\,.$$

Then for $\operatorname{Im} z \neq 0$, we have the inclusion $\langle x\rangle^{-\sigma} R_0(z)^m \in \mathbf{S}_q$. Further, if $f \in \mathrm{BMO}(\mathbb{R})$ has compact support in $(0,\infty)$, then also $\langle x\rangle^{-\sigma} R_0(z)^m f(H_0) \in \mathbf{S}_q$.

Proof. For $q \geq 2$ we use Proposition 4.2(i):

$$\big\|\langle x\rangle^{-\sigma} R_0(z)^m\big\|_q^q \leq C_{q,d} \|\langle x\rangle^{-\sigma}\|_{L^q}^q \, \|(|\xi|^2 - z)^{-m}\|_{L^q}^q\,.$$

This proves the first assertion since $\|\langle x\rangle^{-\sigma}\|_{L^q} < \infty$ if $\sigma q > d$ and $\|(|\xi|^2 - z)^{-m}\|_{L^q} < \infty$ if $2mq > d$.

For $0 < q < 2$ we use Proposition 4.2(ii):

$$\|\langle x\rangle^{-\sigma} R_0(z)^{-m}\|_q^q \leq C_{d,q} \|\langle x\rangle^{-\sigma}\|_{\ell^q(L^2)}^q \, \|(|\xi|^2 - z)^{-m}\|_{\ell^q(L^2)}^q\,.$$

Again, we have $\|\langle x\rangle^{-\sigma}\|_{\ell^q(L^2)} < \infty$ if $\sigma q > d$ and $\|(|\xi|^2 - z)^{-1}\|_{\ell^q(L^2)} < \infty$ if $2mq > d$.

The assertion with an additional term in BMO follows in the same way since the L^q or $\ell^q(L^2)$ norm of $(|\xi|^2 - z)^{-1} f(|\xi|^2)$ is still finite if $2mq > d$. □

We also need an analogue of Lemma 5.1 with R_1^m instead of R_0^m. In order to prove it, we need to consider the difference $R_1^m - R_0^m$. The following lemma is essentially contained in [21]. We include its proof for the sake of completeness.

Lemma 5.2. *Let V satisfy (4.1) with some $\rho > 0$, let $r > 0$ and let $m \geq 0$ be an integer such that*

$$\rho r > d \qquad \text{and} \qquad 2(m+1)r > d\,.$$

Then for $\operatorname{Im} z \neq 0$ we have the inclusion $R_1(z)^m - R_0(z)^m \in \mathbf{S}_r$, and, if $f \in \mathrm{BMO}(\mathbb{R})$ has compact support, then also $f(H_0)(R_1(z)^m - R_0(z)^m) \in \mathbf{S}_r$.

Proof. Throughout the proof, we suppress the dependence on z, writing $R_0 = R_0(z)$ and $R_1 = R_1(z)$. We use induction on m. For $m = 0$ the statement is trivial. Now let $m \geq 1$ and assume the claim has already been proved for all smaller values of m. We have

$$\begin{aligned} R_1^m - R_0^m &= \sum_{l=1}^m R_1^{l-1}(R_1 - R_0) R_0^{m-l} = -\sum_{l=1}^m R_1^l V R_0^{m-l+1} \\ &= -\Big(\sum_{l=1}^m R_0^l V R_0^{m-l+1} + \sum_{l=1}^m (R_1^l - R_0^l) V R_0^{m-l+1}\Big). \end{aligned}$$

Separating the $l = m$ term in the second sum on the right, combining it with the left hand side and inverting $I + VR_0$ (the inverse exists and is bounded since $\operatorname{Im} z \neq 0$) we obtain

$$R_1^m - R_0^m = -\Big(\sum_{l=1}^{m} R_0^l V R_0^{m-l+1} + \sum_{l=1}^{m-1} (R_1^l - R_0^l) V R_0^{m-l+1}\Big)(I + VR_0)^{-1}. \quad (5.1)$$

Let us consider the first sum in the right hand side here. Let us check the inclusions

$$R_0^l V R_0^{m-l+1} \in \mathbf{S}_r \quad (5.2)$$

for each $1 \leq l \leq m$. We write

$$R_0^l V R_0^{m-l+1} = \big(R_0^l |V|^\alpha \operatorname{sign}(V)\big)\big(|V|^\beta R_0^{m-l+1}\big)$$

with $\alpha = \frac{l}{m+1}$, $\beta = \frac{m-l+1}{m+1}$. Setting $r_1 = r(m+1)/l$ and $r_2 = r(m+1)/(m-l+1)$, and using Lemma 5.1, we obtain

$$R_0^l |V|^\alpha \in \mathbf{S}_{r_1}, \quad |V|^\beta R_0^{m-l+1} \in \mathbf{S}_{r_2}.$$

Now (5.2) follows by application of the Hölder inequality in trace ideals (2.4).

Next, we consider the second sum in (5.1). Let us show the inclusion

$$(R_1^l - R_0^l) V R_0^{m-l+1} \in \mathbf{S}_{r(m+1)/(m+2)} \subset \mathbf{S}_r \quad (5.3)$$

for each $1 \leq l \leq m-1$. Let $r_1 = r(m+1)/(l+1)$ and $r_2 = r(m+1)/(m-l+1)$. Then $r_1 \geq r$ and therefore $\rho r_1 > d$. Moreover,

$$2(l+1)r_1 = 2(m+1)r > d\,.$$

Therefore, by induction hypothesis, $R_1^l - R_0^l \in \mathbf{S}_{r_1}$. On the other hand, $r_2 \geq r$ and therefore $\rho r_2 > d$. Moreover,

$$2(m-l+1)r_2 = 2(m+1)r > d\,.$$

Therefore, by Lemma 5.1, $VR_0^{m-l+1} \in \mathbf{S}_{r_2}$. By Hölder's inequality in trace ideals, since $r_1^{-1} + r_2^{-1} = ((m+2)/(m+1))r^{-1}$, we obtain the inclusion (5.3). Thus, the right hand side in (5.1) is in $\mathbf{S}_r$; we have completed the induction argument and thereby proved the first claim of the lemma.

The second claim is proven in the same way: one checks without difficulty that (5.2), (5.3) hold true (for the same reasons as above) with an extra $f(H_0)$ term on the left. □

Lemma 5.3. *Let $\sigma > 0$, $q > 0$, $m \in \mathbb{N}$ be such that*

$$\rho q > d, \qquad \sigma q > d \qquad \text{and} \qquad 2mq > d\,.$$

Then for $\operatorname{Im} z \neq 0$, we have the inclusion $\langle x\rangle^{-\sigma} R_1(z)^m \in \mathbf{S}_q$.

Proof. We write

$$\langle x\rangle^{-\sigma}R_1(z)^m = \langle x\rangle^{-\sigma}R_0(z)^m + \langle x\rangle^{-\sigma}\left(R_1(z)^m - R_1(z)^m\right).$$

According to Lemma 5.1, the first term is in $\mathbf{S}_q$. The second term is in $\mathbf{S}_q$ by Lemma 5.2 (with $r = q$). □

6. Putting it all together

Proof of Theorem 1.1. Throughout the proof, we set

$$V = G_1^* G_0, \quad G_0 = |V|^{1/2}, \quad G_1 = \operatorname{sign}(V)|V|^{1/2}, \tag{6.1}$$

and let $\Lambda \subset \mathbb{R}$ be a bounded open interval such that $\operatorname{supp} f \subset \Lambda$ and the closure of Λ is contained in $(0,\infty)$. We consider the three terms in the right hand side of the decomposition (1.6).

First, consider the diagonal term

$$\mathbb{1}_\Lambda(H_1)D(f)\mathbb{1}_\Lambda(H_0). \tag{6.2}$$

By Lemma 4.1, we have

$$G_0\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_\infty(H_0) \quad \text{and} \quad G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_\infty(H_1).$$

Now we can use Theorem 2.1, which ensures that for $f \in \mathrm{BMO}(\mathbb{R})$ the product (6.2) is bounded, and for $f \in \mathrm{VMO}(\mathbb{R})$ it is compact.

Next, the off-diagonal terms

$$\mathbb{1}_{\Lambda^c}(H_1)f(H_0), \quad f(H_1)\mathbb{1}_{\Lambda^c}(H_0)$$

are compact by Lemma 3.1. □

Proof of Theorem 1.2. Again, we decompose $f(H_1) - f(H_0)$ as in (1.6) and treat the three terms separately. Instead of following the cases (i) and (ii) as in the statement of the theorem, it will be convenient to split the range of variables as follows: $p \geq 1$ and $0 < p < 1$.

Case $p \geq 1$. Throughout the consideration of this case we use the factorisation (6.1). Observe that for $p \geq 1$ both in case (i) and in case (ii) we have

$$p > \frac{d}{\rho} \quad \text{and} \quad p > \frac{d-1}{\rho - 1}.$$

The diagonal term. We use Theorem 2.1(iii) and take $q = r = 2p$. Both terms $\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}_{2p}(H_0)}$ and $\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_{2p}(H_1)}$ are finite as shown in Lemma 4.1.

The term $\mathbb{1}_{\Lambda^c}(H_1)f(H_0)$. Let $k \geq 0$ be an integer sufficiently large such that $2(k+1)p > d$. We use the bound (3.9) from Lemma 3.2. As already mentioned, the norm $\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}_{2p}(H_0)}$ is finite. Moreover, according to Lemma 5.3, the assumptions $\rho p > d$ and $4(k+1)p > d$ imply that $G_1R_1(z)^{k+1} \in \mathbf{S}_{2p}$ for $\operatorname{Im} z \neq 0$. If $k = 0$, this already shows that $\mathbb{1}_{\Lambda^c}(H_1)f(H_0) \in \mathbf{S}_p$.

If $k \geq 1$, we still need to show that $(R_0(z)^k - R_1(z)^k)f(H_0) \in \mathbf{S}_p$. This follows from Lemma 5.2 (by taking adjoints).

The term $f(H_1)\mathbb{1}_{\Lambda^c}(H_0)$. The argument in this case is similar to that for the second term and we will be brief. We choose k as before and this time, we use bound (3.8) from Lemma 3.2. We already know that $G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_{2p}(H_1)$ and we infer that $G_0 R_0(z)^{k+1} \in \mathbf{S}_{2p}$ from Lemma 5.1. This concludes the proof for $k = 0$.

For $k \geq 1$, we still need to show that $f(H_1)(R_1(z)^k - R_0(z)^k) \in \mathbf{S}_p$. We write

$$f(H_1)(R_1(z)^k - R_0(z)^k) = f(H_0)\left(R_1(z)^k - R_0(z)^k\right) + D(f)\left(R_1(z)^k - R_0(z)^k\right).$$

Since f is compactly supported and $f \in B^{1/p}_{p,p}$, we have $f \in \mathrm{BMO}$ and therefore the operator $D(f)$ is bounded by Theorem 1.1. Thus, it suffices to prove that

$$f(H_0)\left(R_1(z)^k - R_0(z)^k\right),\ R_1(z)^k - R_0(z)^k \in \mathbf{S}_p\,.$$

This is again a consequence of Lemma 5.2.

Case $0 < p < 1$**.** Here we are in the setting of part (ii) where $\rho > d$. Again, we treat separately the three terms in (1.6). This time we split the perturbation $V = G_1^* G_0$ with

$$G_0 = (\operatorname{sgn} V)|V|^\theta \qquad \text{and} \qquad G_1 = |V|^{1-\theta}\,.$$

Here $0 < \theta < 1$ is chosen such that, with $q = 2p/(2-p)$, we have $\theta\rho q > d$ and $(1-\theta)\rho > d/2$. (Such choice of θ is possible since $p > d/\rho$.)

The diagonal term. We use Theorem 2.1(iii) with $q = 2p/(2-p)$ and $r = 2$. The term $\|G_0\mathbb{1}_\Lambda(H_0)\|_{\mathrm{Smooth}_q(H_0)}$ is finite by Lemma 4.3 since $\theta\rho q > d$. Let us check that the term $\|G_1\mathbb{1}_\Lambda(H_1)\|_{\mathrm{Smooth}_2(H_1)}$ is finite.

Let $\widetilde{\rho} = \min\{\rho, 2(1-\theta)\rho\}$. Then V satisfies (4.1) with $\widetilde{\rho}$ instead of ρ. Moreover, $\widetilde{\rho} > 1$ (since $\rho > 1$ and $2(1-\theta)\rho > d \geq 1$) and $1 > (d-1)/(\widetilde{\rho}-1)$ (since $\rho > d$ and $2(1-\theta)\rho > d$). Therefore, we can apply Lemma 4.1 with $p = 1$ and with $\widetilde{\rho}$ instead of ρ. This gives $\langle x\rangle^{-\widetilde{\rho}/2}\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_2(H_1)$. On the other hand, $|V|^{1-\theta}\langle x\rangle^{\widetilde{\rho}/2}$ is bounded and therefore $G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_2(H_1)$.

The term $\mathbb{1}_{\Lambda^c}(H_1)f(H_0)$. Let $k \geq 0$ be an integer sufficiently large so that $2(k+1)p > d$. We use bound (3.9) with the exponents $q = 2p/(2-p)$, $r = 2$. We already know that $G_0\mathbb{1}_\Lambda(H_0) \in \mathrm{Smooth}_q(H_0)$. Further, according to Lemma 5.3, the assumptions $(1-\theta)\rho > d/2$ and $4(k+1) > d$ imply that $G_1 R_1(z)^{k+1} \in \mathbf{S}_2$ for $\operatorname{Im} z \neq 0$. If $k = 0$, this already shows that $\mathbb{1}_{\Lambda^c}(H_1)f(H_0) \in \mathbf{S}_p$.

If $k \geq 1$, we argue as in the case $p \geq 1$ that $(R_0(z)^k - R_1(z)^k)f(H_0) \in \mathbf{S}_p$.

The term $f(H_1)\mathbb{1}_{\Lambda^c}(H_0)$. Again, the argument is similar and we will be brief. We choose k as before and this time, we use bound (3.8). We already know that $G_1\mathbb{1}_\Lambda(H_1) \in \mathrm{Smooth}_2(H_1)$, and we infer that $G_0 R_0(z)^{k+1} \in \mathbf{S}_q$ from Lemma 5.1 since $\theta\rho q > d$ and $2(k+1)q > d$. If $k = 0$, this already shows that $f(H_1)\mathbb{1}_{\Lambda^c}(H_0) \in \mathbf{S}_p$.

If $k \geq 1$, we argue as in the case $p \geq 1$ that $f(H_1)(R_1(z)^k - R_0(z)^k) \in \mathbf{S}_p$. This concludes the proof of the theorem. □

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