Landau's switching problem: An elementary decision theoretic resolution of the two-envelope paradox

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This is my first draft on the two-envelope problem to receive comments from friends and colleagues and to learn more about the relevant literature. My resolution comes from formalizing the two-envelope problem in a basic decision theoretic framework. Thus, I doubt that this short note involves any original contribution to the discussion on the two-envelope problem. Nevertheless, I believe that it is useful to organize my thoughts and to eventually prepare a document that can be used to teach this stimulating problem and its simple resolution at the undergraduate and postgraduate levels. Any comments would be welcome.

Abstract

In this short note, I provide an elementary decision theoretic resolution of Landau's switching problem, which is also known as the two-envelope paradox. I show that there are no general arguments for or against switching, which confirms the common intuition. This result does not depend on whether we assume the existence of probabilities or not. Nor does it depend on the variations of the problem in which the decision maker opens the first envelope or not. I also give an explanation for why (i) the popular argument for switching is actually right, though it is only part of a larger decision problem, and (ii) the fragmented literature on this problem does not necessarily mean that some opposing approaches to the problem are wrong.

1. Introduction to Landau's switching problem

The two-envelope problem or "paradox" has gained interest in various fields, including economics, statistics, philosophy, decision theory, game theory, logic, probability theory, and optimization. The problem is also known as the necktie paradox, exchange paradox, Ali Baba problem, and the box problem.

Two-envelope problem: You are offered two envelopes, one of which contains twice as much money as the other. You can choose one of the envelopes and then decide keeping the envelope or switching to the other envelope. There is variation of this problem in which after choosing one envelope you can open the envelope and see the amount inside it.

Switching argument: This problem is often called a paradox because different ways of modeling the problem seems to lead to contradictory conclusions. Perhaps the most counterintuitive argument is that by switching you can be better off. Suppose that the envelope you choose contains x units of money. Then, the other envelope contains either $2x$ or $\frac{x}{2}$, so switching gives you the following expected value:

$$\frac{1}{2}(2x + \frac{x}{2}) = \frac{5x}{4}. \qquad (1)$$

Because $\frac{5x}{4} > x$, a risk-neutral decision maker would be better off by switching. This is counterintuitive because there does not seem to be any reason to prefer one envelope over the other. Yet, this argument seems to show that no matter what envelope one chooses switching improves his or her expected value. What is worse, it appears that you can repeatedly apply the same argument to reach a conclusion that you should never stop switching!

It is fascinating that this simple-looking problem has attracted interest from not only economists but also physicists, mathematicians, and philosophers. The presentation of the problem is not formal, and I believe that the ambiguity in its presentation may have attracted researchers from very different fields. By formalizing the problem, I do not mean to hinder inter- and cross-disciplinary attempts at understanding this problem. But I use a basic decision theoretic set up in order to have a common understanding of the actions, outcomes, and the states of the world in this problem.

To summarize, I find that, whether we assume the existence of probabilities or not, switching is *not* strictly better than staying and vice versa. This result holds irrespective of whether the DM opens the first envelope or not. If we do not make any assumption about the probability distribution over the states of the world, then no action dominates the other action. If we assume that all states are equally likely, then switching and not switching (i.e., staying) give exactly the same expected values, which confirms the common intuition.

I call this problem Landau's switching problem because it appears first in a slightly different form in Kraitchik (1930, p. 253) who mentions that Belgian mathematician Alfred Errera told him the problem and that the problem was created by German mathematician Edmund Landau in 1912 in a maths course. Although Kraitchik (1930) is widely cited in the literature, it appears that Landau's contribution has been overlooked. Because all variations of this problem involve some sort of a switching argument, I dub the problem Landau's switching problem.

There is an extensive yet fragmented literature on the two-envelope problem with different conclusions. The problem was popularized in 1980s by Gardner (1982) and Nalebuff (1989), which led to a growing number of studies, including Christensen and Utts (1992), Linzer (1994), Brams and Kilgour (1995), Bruss (1996), McDonnell and Abbott (2009), Syverson (2010), in which the DM opens the first envelope, and Clark and Shackel (2000), Blachman and Kilgour (2001), Meacham and Weisberg (2003), Samet et al. (2004), Dietrich and List (2005), and Douven (2007), in which the DM does not open the first envelope. For a review of the literature, see, e.g., Nalebuff (1989), Albers et al. (2005), and Vasudevan (2019). My resolution is most similar to Gilboa's (2009, p. 164) resolution of Newcomb's paradox.

2. A resolution of the paradox

Two-envelope problem is a decision problem but its original presentation is not formal. Although its ambiguous presentation perhaps has given rise to the rich and fruitful literature, to reach a common understanding I put it into a standard decision theory framework with the precise set of acts, outcomes, and the states of the world.

Let's fix some notation. Let e_1 and e_2 denote two envelopes. Let $(i, j)\colon e_k$ denote the state of the world in which envelope e_1 contains i units of money, envelope e_2 contains j units, and the status quo (i.e., the initial envelope) is envelope e_k where $k \in \{1, 2\}$. Assume that the amount in one of the envelopes is a. Then, envelopes e_1 and e_2 must contain one of the following

	$(a,2a)\colon e_1$	$\left(a,\frac{a}{2}\right)\colon e_1$	$(a,2a)\colon e_2$	$\left(a,\frac{a}{2}\right)\colon e_2$	$(2a,a)\colon e_1$	$\left(\frac{a}{2},a\right)\colon e_1$	$(2a,a)\colon e_2$	$\left(\frac{a}{2},a\right)\colon e_2$
Stay	a	a	$2a$	$\frac{1}{2}a$	$2a$	$\frac{1}{2}a$	a	a
Switch	$2a$	$\frac{1}{2}a$	a	a	a	a	$2a$	$\frac{1}{2}a$

Table 1. Decision matrix of the two-envelope problem. Columns represent the states of the world and rows the actions of the DM, Stay (at the status quo) and Switch. Notation $(i,j)\colon e_k$ denotes the state of the world in which envelope e_1 contains i units, envelope e_2 contains j units, and the status quo is envelope e_k where $k \in \{1,2\}$.

combinations: $(a,2a)$, $(a,\frac{a}{2})$, $(2a,a)$, and $(\frac{a}{2},a)$. In addition, there are two possible status quos e_1 and e_2 from which a DM makes her Stay or Switch decision. As a result, there are $4 \times 2 = 8$ possible states of the world.[1] The decision maker (DM) has two actions: 'Stay' at the status quo or 'Switch' from the status quo. (Note that actions Stay and Switch imply the existence of some status quo to stay at or to switch from and that at this point we make no assumptions on how these status quos are chosen.)

To see that the states of the world in Table 1 are well-defined, consider the following hypothetical states of the world and confirm that they are consistent with the two-envelope problem. Suppose that the amount in one of the envelopes is 10. Then, all of the following combinations are possible: (10, 20), (10, 5), (20, 10), and (5, 10). Besides, there are two possible status quos. Thus, there are 8 states of the world as illustrated below:

	$(10,20)\colon e_1$	$(10,5)\colon e_1$	$(10,20)\colon e_2$	$(10,5)\colon e_2$	$(20,10)\colon e_1$	$(5,10)\colon e_1$	$(20,10)\colon e_2$	$(5,10)\colon e_2$
Stay	10	10	20	5	20	5	10	10
Switch	20	5	10	10	10	10	20	5

[1] It is possible to make the states of the world smaller or larger; I prefer the current form because it does not require any probabilistic assumptions and hence expected value calculation. Note that the original problem, too, does not make any assumptions about probabilities.

Table 1 describes the decision problem we have, but note that so far we have not assumed anything about the probability of those states happening. Nor does the description of the two-envelope problem assume any specific probability. I believe that Table 1 is a reasonable formalization of the two-envelope problem. Without assuming any probabilities, it can already be seen that there is no dominance relationship between Stay and Switch. On the other hand, *if* we assume that all states are equally likely, then the expected values can be calculated as follows.

$$EV(\text{Stay}) = \frac{1}{8}\left(4 \times a + 2 \times 2a + 2 \times \frac{1}{2}a\right) = \frac{9a}{8},$$

$$EV(\text{Switch}) = \frac{1}{8}\left(4 \times a + 2 \times 2a + 2 \times \frac{1}{2}a\right) = \frac{9a}{8}.$$

In conclusion, with or without probabilities, Switch is *not* strictly better than Stay and vice versa, irrespective of whether the DM opens the status quo envelope or not. If we do not make any assumption about probabilities, then no action dominates the other. If we assume that all states are equally likely, then Stay and Switch give exactly the same expected values. Note that the conclusion is *not* that every decision maker will be indifferent between the two envelopes. Different decision makers may have different utility functions (they need not be risk-neutral) and different beliefs about the states of the world which may depend on the status quo, and hence affect their choices. Thus, they may prefer Switch over Stay, or vice versa (see, e.g., Brams and Kilgour, 1995). What we find is that (i) there is no dominance relationship between Switch and Stay, and (ii) if the states of the world are equally likely, then Switch and Stay lead to the same expected value, which is consistent with our common sense.

3. What is wrong with the argument in (1) in favor of switching?

The essence of the paradox is also to show what is wrong with the switching argument. I will next show that the argument and its calculations are actually correct, but they form only a part of a larger decision problem.

As before e_1 and e_2 denote two envelopes. Table 2 illustrates the same decision problem as in Table 1, but with fewer number of states of the world. For example, the first and the second

	S_1	S_2	S_3	S_4
Stay	x	$\frac{5x}{4}$	x	$\frac{5x}{4}$
Switch	$\frac{5x}{4}$	x	$\frac{5x}{4}$	x

Table 2. Two-envelope problem decision matrix with four states of the world.

states in Table 1 are merged into one state, denoted as S_1, in Table 2, and the third and the fourth states in Table 1 are merged into another state, denoted as S_2, in Table 2, and so on.

- S_1: State $(a, 2a)$: e_1 or state $\left(a, \frac{1}{2}a\right)$: e_1—i.e, e_1 contains a, e_2 contains either $2a$ or $\frac{1}{2}a$, and the status quo is e_1.
- S_2: State $(a, 2a)$: e_2 or state $\left(a, \frac{a}{2}\right)$: e_2.
- S_3: State $(2a, a)$: e_1 or state $\left(\frac{a}{2}, a\right)$: e_1.
- S_4: State $(2a, a)$: e_2 or state $\left(\frac{a}{2}, a\right)$: e_2.

In the original switching argument, it is assumed that the status quo envelope contains x, as illustrated in Table 2 at the intersection of Stay and S_1. Indeed, in this state of the world—i.e., e_1 contains a, e_2 contains either $2x$ or $\frac{1}{2}x$, and the status quo is e_1—switching gives a greater expected value $\frac{5x}{4}$ which is illustrated at the intersection of Switch and S_1.

True, the argument in favor of switching shows that Switch gives you a greater expected value, but only in a specific state of the world. With this argument, you cannot reach the conclusion that switching would always give you a greater expected value because you are in one of the four states of the world, and *you do not actually know in which one you are*, irrespective of you open the envelope or not. Note that when you are in a state of the world which you do not know and assume that Stay gives you "x" under that state (e.g., state S_1 in Table 2), all other outcomes in the decision problem are automatically defined from this very assumption.

Going back to Table 2, if the state of the world is S_2, then Stay (at envelope e_2) gives a greater expected value $\frac{5x}{4}$ than Switch, because we assumed that e_1 contains x, which can be obtained by Switch in state S_2. Calculations of the outcomes under S_3 and S_4 are analogous to those under S_1 and S_2, respectively. If we assume that all states are equally likely, then the expected value of Stay equals the expected value of Switch:

$$EV(\text{Stay}) = \frac{1}{4}\left(x + \frac{5x}{4} + x + \frac{5x}{4}\right) = \frac{9x}{8},$$

$$EV(\text{Switch}) = \frac{1}{4}\left(\frac{5x}{4} + x + \frac{5x}{4} + x\right) = \frac{9x}{8}.$$

As a result, there is no gain on average from switching irrespective of whether the DM opens the envelope or not. This also gives an explanation for why we may see completely opposing solutions to the same problem, which may all be correct as part of a larger decision problem.

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