Six Dimensional Supergravity, Spinorial Geometry and (1,0)-Superconformal Theories

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Abstract

In this thesis we explore (1,0) supersymmetric theories in six dimensions. The first part of the thesis focuses on the investigation of supersymmetric solutions of (1,0) six dimensional supergravity theory coupled to any number of tensor, vector and scalar multiplets. The methodology used to solve the Killing spinor equations will be based on the spinorial geometry technique. Therefore, we begin by giving details of the spinorial geometry approach in the first chapter. In the chapter that follows six dimensional supergravity coupled to tensor, vector and scalar multiplets is described. Once we have given details of the theory under consideration the solutions to the Killing spinor equations are discussed in some detail. In particular, we find that there are backgrounds preserving 1, 2, 3, 4 and 8 supersymmetries broadly falling into two cases; those with Killing spinors that have compact isotropy groups and those with non-compact isotropy groups. We then discuss the integrability conditions of the Killing spinor equations.

In the fourth chapter we analyse the supersymmetric near horizon geometries of (1,0) six dimensional supergravity coupled to arbitrary number of tensor and scalar multiplets. In order to do this we make use of Gaussian null coordinates as well as the solutions of the Killing spinor equations. We find that there are two classes of near horizon geometries. One class is isometric to $\mathbb{R}^{1,1} \times \mathcal{S}$, where $\mathcal{S}$ is a suitable 4-manifold, and the other class is isometric to $AdS_3 \times \Sigma^3$, where Σ^3 is a homology 3-sphere.

In the final chapter we investigate a more recent development, namely (1,0) superconformal theories in six dimensions. In particular we find the BPS solutions of (1,0) superconformal theory in all cases. In addition, we analyse the half-supersymmetric solutions to some specific models in detail and give examples of string and 3-brane solutions.

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Dedicated to my family.

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Introduction

There are four known fundamental forces of nature; gravity, electromagnetism, the weak nuclear force and the strong nuclear force. At low energies these forces describe the different types of interactions experienced by particles that exists in the universe. However, at high energies it is believed that our knowledge is incomplete. Additionally, there are objects within the universe for which the physics is not well understood, a good example being black holes. The unification of these fundamental forces has therefore been one of the driving forces behind research in theoretical physics.

In fact this started some time ago with the work of James Clerk Maxwell who unified electricity and magnetism to the single theory of electromagnetism. This was a significant achievement, bringing together what was previously thought of as two different phenomenon into one. The theory of electromagnetism was later combined with that of the weak nuclear force to obtain the mathematical framework of the electroweak force. This was primarily due to the works of Sheldon Glashow, Abdus Salam and Steven Weinberg. The Standard Model of particle physics then incorporated the strong, weak and electromagnetic forces into a single successful theory describing three of the fundamental forces.

The fourth known force, gravity, is described by Einstein's General Theory of Relativity. It is the "dominant" force experienced by objects over large distances. A lot of effort and time has been put into formulating a single theory capable of successfully describing all the fundamental forces of nature, including gravity.

String/M-theory has been proposed as one of the most promising candidates for the unification of all four fundamental forces of nature. However, it has not been easy to reconcile string/M-theory with the four dimensional spacetime which we are familiar with. These theories are formulated in ten/eleven dimensions and in order to arrive at a four dimensional spacetime six/seven of the dimensions have to be compactified so that they are invisible to the naked eye but are present at every point in spacetime. In addition, in four dimensions we think of the constituents of matter as some fundamental particles, but in these extra-dimensional theories the fundamental objects are generalised to p-branes, which are p-spatial dimensional objects traversing ten/eleven spacetime dimensions. For example, a point particle corresponds to a 0-brane, a 1-brane describes a string, a 2-brane describes a membrane and so on. Although the first features of string theory were introduced in the late 1960's, it was not initially aimed for the unification of the forces but to develop a better understanding of the strong nuclear force. However, after the development of an alternative theory, quantum chromodynamics (QCD), to explain the strong nuclear force very little attention was paid to string theory until the 1980's.

In the 1970's there was another new development which would become an integral part of modern day string/M-theory, supersymmetry. This was the idea of a new symmetry relating the two types particles that exist in the universe, namely bosons with integer spin and fermions with half-integer spins. In the 1980's string theory saw a rapid development with the inclusion of supersymmetry. This led to the discovery of five different but consistent types of superstring theories. These are referred to as the type I, type IIA, type IIB, heterotic $E_8 \times E_8$ and the heterotic $SO(32)$ superstring theories, see for example [1].

However, the aim of unification is to have a single theory describing all the fundamental forces; so which one of the superstring theories, if any, is this desired theory? It turns out that all these theories are related by a web of dualities, and moreover, they point to a single theory in eleven dimensions called M-theory [2, 3]. One key thing known about this eleven dimensional M-theory is that its low energy approximation is eleven dimensional supergravity [4]. Further investigations of eleven dimensional supergravity led to the discovery of the electric 2-brane [5] called the M2 brane and its magnetic dual 5-brane called the M5 brane [6]. These are believed to be two important ingredients of M-theory and a lot of research has focused on developing a better understanding of these objects.

Supergravity theories were themselves introduced in the 1970's after incorporating supersymmetry into the framework of General Relativity. In addition to eleven dimensional supergravity, each of the superstring theories have their own low energy limits corresponding to a supergravity theory. These have played very important roles in our understanding of string/M-theory. Since we will primarily focus on supergravities let us briefly mention what some these theories are:

- **Eleven Dimensional Supergravity**: This is the low energy limit of M-theory and consists of a Majorana spinor gravitino ψ_M, the metric g_{MN} and a 3-form gauge potential H_{MNP}, where M, N, P are the spacetime indices running over the spacetime dimensions.

- **Type I Supergravity**: This is a $\mathcal{N} = (1, 0)$ theory in ten dimensions and so corresponds to a theory with 16 supersymmetries. We consider the low energy limit of the type I string theory coupled to super Yang-Mills and the two heterotic string theories. The field contents are a metric g_{MN}, a real scalar ϕ, a 2-form gauge potential B_{MN} and a 1-form gauge potential A_M, along with the following fermions: a Majorana-Weyl gravitino ψ_M, a Majorana-Weyl gaugino χ and a Majorana-Weyl dilatino λ.

- **Type IIA Supergravity**: This is a $\mathcal{N} = (1, 1)$ theory, i.e. a non-chiral theory with 32 supersymmetries. It contains two Majorana-Weyl spinors of opposite chirality and two Majorana-Weyl dilatinos of opposite chirality. On the bosonic side it consists of the metric g_{MN}, a real scalar ϕ, a 1-form, a 2-form and a 3-form gauge potential. Type IIA supergravity can be obtained from eleven dimensional supergravity by dimensional reduction.

- **Type IIB Supergravity**: This is a $\mathcal{N} = (2, 0)$ theory, i.e. a chiral theory with 32 supersymmetries. It contains two Majorana-Weyl spinors of the same chirality and two Majorana-Weyl dilatinos of the same chirality. The bosonic part comes in two groups; the NS-NS section consists of the metric g_{MN}, a 2-form gauge potential B_{MN} and a real scalar ϕ, the R-R sector contains a 0-form, 2-form and 4-form gauge potential with self-dual 5-form field strength.

These particular supergravity theories are important because of their direct link to string/M-theory, but supergravity theories can be formulated in other dimensions or can be obtained by dimensional reduction of the ones we have mentioned above, see for example [1, 7, 8, 9].

Eleven dimensional supergravity formulated by Cremmer, Julia and Scherk [4] is the unique supergravity theory in eleven dimensions. The reason for eleven being the largest dimension in which a consistent supergravity is formulated is due to supersymmetry. In four dimensional supergravity theories the supersymmetry parameter is a Majorana spinor and so has four real components, i.e. there are four real supercharges. It is believed that there are no consistent supergravity theories containing particles with spin greater than two. This means $\mathcal{N} = 8$ theories are the most supersymmetric theories that can be constructed in four dimensions that contain particles with spin ≤ 2, this corresponds to 32 supersymmetries in total. Majorana spinors have $2^{[D/2]}$ real components where D is the spacetime dimension. In eleven dimensions this corresponds to a total of 32 supersymmetries. If $D \geq 12$ then the total number of supersymmetries for an $\mathcal{N} = 1$ theory is more than 32, so upon dimensional reduction to four dimensions this will lead to the appearance of particles with spin greater than two. Therefore, eleven dimensions is the maximum dimension in which this does not happen.

One of the aims of supergravity theories was to tackle the ultraviolet divergences in gravity, see for example [10]. In addition, there was the prospect that they could lead to a quantum theory of gravity. After the initial flurry of activity on supergravity in the late 1970's and early 1980's interest on them was superseded by the developments in string theory. However, supergravity continues to play a crucial role in our understanding of string/M-theory by providing an effective low energy description of these theories.

One of the main applications of supergravities in string/M-theory has been via the investigation of supersymmetric supergravity solutions. These are solutions of supergravity theories which in addition to solving the field equations also solve a set of first order but non-linear equations called the Killing spinor equations. The Killing spinor equations arise from the supersymmetry transformations that leave the theory invariant. We will discuss the basics of supersymmetric solutions in more detail in the first chapter. These classes of solutions have now been prominently examined for a number of years and have given us a better insight into string/M-theory.

In particular, they have been useful in understanding string/M-theory compactifications, branes, string/M-theory dualities, the AdS/CFT correspondence, and in the investigation of black holes, see for example [11] for a review.

Outline

The focus of this thesis will be on the $\mathcal{N} = (1, 0)$ supergravity in six dimensions. These have eight real supercharges, and so backgrounds preserve a maximum of eight supersymmetries. In particular, we will solve the Killing spinor equations of this theory with the most general couplings possible. This is the $\mathcal{N} = (1, 0)$ supergravity coupled to an arbitrary number of tensor, vector and scalar multiplets. In addition to solving the Killing spinor equations (KSEs), we will make use of the solutions to investigate the near horizon geometries of black holes in (1,0) supergravity. Furthermore, we will make use of the same techniques to investigate the KSEs of the (1,0) superconformal theories in six dimensions. This will allow us to obtain the BPS conditions in all cases.

This thesis is primarily based on the three papers published in [12, 13, 14], written in collaboration with my supervisor:

- M. Akyol and G. Papadopoulos, "Spinorial geometry and Killing spinor equations of 6-D Supergravity," Class. Quant. Grav. **28** (2011) 105001, [arXiv: 1010.2632 [hep-th]].
- M. Akyol and G. Papadopoulos, "Topology and geometry of 6-dimensional (1,0) supergravity black hole horizons," Class. Quantum Grav. **29** (2012) 055002, [arXiv: 1109.4254 [hep-th]].
- M. Akyol and G. Papadopoulos, "(1,0) superconformal theories in six dimensions and Killing spinor equations," JHEP **1207** (2012) 070 , [arXiv: 1204.2167 [hep-th]].

The outline of the thesis will take the following format:

- **Chapter One**: In the first chapter we mainly focus on giving technical details that will be required in the rest of the thesis. We will begin by reviewing the classification programme for supersymmetric supergravity solutions. This will involve briefly discussing the different approaches to the classification of supersymmetric solutions. We will in addition talk about black holes and the uniqueness problems. A detailed exposition to the spinorial geometry method of solving the KSEs will also be given. We will conclude the chapter with an example of how the spinorial geometry method is used in ten dimensions.

- **Chapter Two**: The second chapter is mainly based on [12]. In this chapter we will introduce six dimensional (1,0) supergravity coupled to any number of tensor, vector and scalar multiplets, giving details of the KSEs and how we go about solving them. We then solve the KSEs and give details of the conditions on the fluxes and discuss the geometry of the backgrounds. In particular, we find that there are backgrounds preserving 1, 2, 3, 4 and 8 supersymmetries. The conditions imposed on the fields in each case are given.

- **Chapter Three**: In this chapter we give a detailed derivation of the integrability conditions of the Killing spinor equations.

- **Chapter Four**: The fourth chapter will follow the paper in [13]. We begin this chapter by giving some general details on near horizon geometries and the use of Gaussian null coordinates. This will be followed with the analysis of all possible near horizon geometries admitted by (1,0) supergravity coupled to any number of tensor and scalar multiplets. To do this we make use of the results coming from the KSE in the second chapter as well as the field equations that arise from the integrability conditions.

- **Chapter Five**: The fifth chapter is based on the work done in [14]. This investigates (1,0) superconformal theories in six dimensions and KSEs, which have been the focus of more recent research. We will begin by discussing the construction of the (1,0) superconformal models. Then the KSEs are solved in all cases to obtain the BPS conditions. Following this we investigate the half supersymmetric solutions of a number of models in more detail aiming to give an M-theoretic interpretation to these solutions.

- **Chapter Six**: In the final chapter we give our conclusions. We will give a summary and discuss the main conclusions of the thesis. Furthermore we outline some open problems for (1,0) supersymmetric theories in six dimensions and discuss the possibilities of further work.

- **Appendices A and B**: In appendix A we give details of numerous identities used in the derivation of the integrability conditions of (1,0) supergravity. In appendix B we derive the field equations of (1,0) superconformal theory from the KSEs and the Bianchi identities.

Chapter 1

Preliminaries

1.1 Introduction

The aim of this chapter is to give details of the methods and techniques that we will use extensively in the chapters that are to follow. It will also provide a good opportunity to present the conventions and notations we follow.

We will begin with a general discussion of supersymmetric supergravity solutions explaining some of the key terminology and reviewing some of the literature. As we will be discussing the near horizon geometries of black holes later, it is also appropriate to briefly give an introduction and discuss some aspects of black holes in higher dimensions and searches for black hole uniqueness theorems.

The main focus of this chapter, however, is to introduce the spinorial geometry method [15]. This method has proved to be an effective tool in the investigation of supersymmetric supergravity solutions, and it will be the approach we take in order to solve the Killing spinor equations of six dimensional supergravity.

To introduce this we will mainly follow the discussion of the spinorial geometry method used in the investigation of ten and eleven dimensional supergravities in [16, 15] but our results will be generalised to arbitrary Lorentzian signatures, a similar analysis in the case of Euclidean signatures can be found in [17]. Further mathematical expositions can be found in [18, 19, 20]. As an explicit example we will discuss the ten dimensional case in more detail. The results of this example will be needed in the next chapter when we discuss the KSEs of (1,0) six dimensional supergravity.

1.2 Supersymmetric Supergravity Solutions

We begin this section with a general discussion of supersymmetric supergravity solutions. Supergravity theories have been formulated and studied in diverse dimensions since the 1970s, for some introductory texts on supersymmetry and supergravity see for example [7, 21, 22, 23]. As we mentioned in the introduction they were initially investigated in their own rights but with progress in the development of string theory and later M-theory supergravity naturally arose as the low-energy limit to these theories, and since, they have been studied to build a better understanding of these theories.

1.2.1 Killing Spinor Equations

One aspect of the study in supergravity theories has focused on the investigation of supersymmetric solutions. The reason they are called supergravity theories is because they incorporate supersymmetry and this in turn means they are invariant under a set of supersymmetry transformations. In other words, the fermions in the theory transform into some bosonic configuration and similarly the bosons transform into some fermionic configuration. Schematically this can be expressed in the following abstract way

$$\begin{aligned} \delta F &= B\epsilon \ , \\ \delta B &= \bar{\epsilon} F \ , \end{aligned} \tag{1.1}$$

where B and F denote the bosons and fermions, respectively, in the theory and ϵ is the local supersymmetry parameter.

When searching for supersymmetric solutions one looks for non-trivial ϵ such that the supersymmetry transformations vanish. As these are classical bosonic solutions the fermions in the theory vanish, which means the supersymmetry transformation of the bosons, the second equation in (1.1), are automatically zero and so we need not worry further about these.

However, we now need to consider the supersymmetry variations of the fermions in more detail. To find supersymmetric solutions we impose that

$$\delta F = B\epsilon = 0 \ , \tag{1.2}$$

these equations are called *Killing spinor equations* and each spinor ϵ that satisfies these equations are called *Killing spinors*.

All supergravity theories have a supergravity multiplet which contains the graviton and at least one gravitino, in addition it may contain other fields. Depending on the theory there could be possible couplings to vector, tensor, and scalar multiplets. There will be a corresponding Killing spinor equation (KSE) for each fermion that appears in the theory. For supergravity theories the most important one is the gravitino KSE, which is a first order differential equation and it is the KSE corresponding to the supersymmetry transformation of the gravitinos, taking the general form,

$$\mathcal{D}\epsilon = 0 \ , \tag{1.3}$$

where $\mathcal{D}$ is the supercovariant derivative, which in addition to the standard Levi-Civita connection contains terms that depend on the matter content of the theory. The KSEs for the other fermions in the theory are algebraic expressions, which we collectively denote as

$$\mathcal{A}\epsilon = 0 \ . \tag{1.4}$$

Solving these gives us constraints on the fields as well as information about the background geometry that the theory lives on.

In addition to the constraints arising from the KSEs we have to consider the field equations of the theory. The aim is then to impose the constraints from the KSEs and analyse further the field equations. Usually a number of the field equations are implied from the KSEs, however, those that are not implied give further restrictions and these are all used altogether to determine the type of supersymmetric solutions the theory admits.

1.2.2 Six Dimensional Supergravity in Context

We will now give a brief summary of some key results that have been obtained in the context of supersymmetric solutions and outline how the investigation in six dimensions fits into this picture.

This began with the work of Tod, who in 1983 was able to classify the supersymmetric solutions of $\mathcal{N} = 2$ D=4 supergravity [24], this was followed by further work on D=4 by Tod in [25]. Since then there has been gradual progress towards the classification of supersymmetric supergravity solutions and several methods have been developed for this purpose.

One such method is the use of G-structures and Killing spinor bilinears see for example [26, 27, 28, 29]. This was initially used in the classification of supersymmetric solutions for minimal supergravity in five dimensions [28] and for eleven dimensional supergravity [29], but it has in general been effective in finding solutions preserving low numbers of supersymmetries. It is based on the assumption that there is at least one Killing spinor, which is used to construct differential forms using spinor bilinears. These are then used to investigate the geometries admitted by the supergravity theory in question. This method has been used in the classification of supergravity solutions in a diverse range of dimensions [30, 28, 31, 32, 33, 34, 35, 36, 37, 29, 38], and includes both gauged and ungauged supersymmetric supergravity solutions. A review of the use of this method in the classification of supergravity solutions can be found in [39].

The maximally supersymmetric solutions of ten and eleven dimensional supergravities have also been classified [40], see also [41, 42, 43], which in particular makes use of the integrability conditions of the Killing spinor equations.

However, the classification of supersymmetric solutions preserving all possible fractions of supersymmetry remains a difficult task. An approach to tackling this problem was proposed in [15], called the spinorial geometry method. We will discuss this technique in more detail below and will be making use of this method throughout this thesis in order to solve the KSEs. In brief, this technique makes use of the realisation of spinors in terms of differential forms and, in particular, an oscillator basis can be used in the space of spinors. Furthermore, the gauge group which leaves the KSEs invariant is used to simplify the form the Killing spinors take which in turn leads to considerable simplification of calculations.

This method has been used to investigate the supersymmetric solutions of ten and eleven dimensional supergravities. One of the most significant achievements of this was in the classification of all supersymmetric type I and heterotic supergravities backgrounds [44, 45, 46]. It has also been used to investigate supersymmetric solutions of type IIB supergravity admitting one Killing spinor [16, 47]. The analysis in [47] also looks into the supersymmetric solutions with extended supersymmetry. In addition, the spinorial geometry technique has been used to investigate near maximally supersymmetric backgrounds, see for example [48, 49, 50, 51, 52].

The progress of the last twenty years or so shows that there has been much development in the classification of supersymmetric solutions in the lower and higher dimensional supergravity theories. Although the initial works involved the classification of ungauged theories or minimal gauging many further works incorporated couplings to vector, tensor and scalar multiplets. In the first part of the thesis we aim to add to this list of classifications by solving the KSEs of the most general (1,0) supergravity in six dimensions.

We will focus on (1,0) supergravity in six dimensions coupled to arbitrary numbers of tensor, vector and scalar multiplets. This theory was constructed successively in [53, 54, 55, 56, 57]. We will give details of the theory in chapter two. The KSEs of six dimensional (1,0) supergravity have previously been solved in various special cases. In particular, the KSEs of minimal (1,0) supergravity have been solved in [34], and the maximally supersymmetric backgrounds have been classified in [34, 58]. The method followed in [34] is that of G-structures and spinor bilinears, whereas [58] makes use of the integrability conditions and uses a Lie algebra approach. The KSEs of the (1,0) theory coupled to a tensor multiplet and some vector multiplets have been solved in [59] for backgrounds preserving one supersymmetry. The KSEs of (1,0) supergravity coupled to a tensor, some vector and hypermultiplets have been solved for backgrounds preserving one supersymmetry in [60], see also [61].

We extend these works by solving the KSEs of (1,0) supergravity coupled to any number of tensor, vector and scalar multiplets for backgrounds preserving any number of supersymmetries.

1.3 Black Holes and Uniqueness Theorems

One application of supersymmetric supergravity solutions has been in the investigation of supersymmetric black hole horizons. We will be pursuing this in a later chapter for the case of six dimensional supergravity, but first let us give a general introduction to this topic.

Ever since the conception of General Relativity the prediction of black holes and their study has been one of the most interesting and researched areas of gravity. The main focus has been on the black hole solutions in four dimensions and they have been extensively studied. In particular, the first static solution to the vacuum Einstein equations gave rise to the discovery of the Schwarzschild solution which was soon generalised to the Reisner-Nordstorm solution via the coupling to electromagnetic fields. These were discovered very soon after the introduction of General Relativity, see for example [62, 63] for more details. The Kerr solution which generalises the Schwarzschild solution by allowing the black hole to have some angular momentum was discovered in 1963 and further generalised to the Kerr-Newman solution with the coupling to electromagnetism.

In the 1960's and 70's work was done to prove the uniqueness of these solutions, which was established in [64, 65, 66, 67, 68, 69], see also [70]. However, these results do not extend to five and higher dimensions. The four dimensional black hole solutions have spherical near horizon topologies. The five dimensional black holes, for example, in addition to the spherical horizon topologies [71, 72] also admit near horizon topologies of $S^1 \times S^2$, the black rings [73, 74]. It is also expected that in dimensions higher than five there may exist black holes with exotic horizon topologies [75, 76, 77, 78, 79, 80], for a detailed review see [81].

We are more interested in black holes in the context of supergravities. In particular, we are interested in supersymmetric black holes, i.e. black hole solutions that preserve some degree of supersymmetry. Finding explicit black hole solutions in higher dimensions is a significant challenge. It is easier to identify all near horizon geometries and then to find ways to see if these correspond to particular black hole solutions, this approach was for example taken in the identification of supersymmetric black holes in five dimensions [82, 83, 84, 85, 86]. Although finding black hole solutions in higher dimensions remains a difficult challenge there has been some progress towards the classification of all near horizon black hole geometries. Following the approach in [82] investigation of near horizon geometries in lower dimensions [87, 88, 34] and more recently in higher, ten and eleven, dimensions [89, 90, 91, 92, 93, 94] have been carried out.

Another tool which has been used to investigate black holes in the context of string theory has been the attractor mechanism, which was discovered in [95, 96]. We will not use this and so do not require any further details on this, the interested reader is referred to [95, 96, 1, 7].

In this context, we investigate the near horizon geometries of six dimensional supergravity coupled to tensor and scalar multiplets in chapter 4.

1.4 Spinorial Geometry

For most supersymmetric theories the Killing spinor equations can be very difficult to solve. It is even more challenging to get a complete classification of solutions. This involves analysing all possible backgrounds that may arise by the admission of different numbers of Killing spinors, where the number of Killing spinors corresponds to the number of supersymmetries preserved. One way of tackling this challenge has been via the spinorial geometry method. We will now discuss this technique in some detail.

The spinorial geometry method is a procedure that makes use of the fact that spinors can be written in terms of differential forms, and in particular one can use an oscillator basis, in the space of spinors, to carry out calculations in a straightforward way when searching for supersymmetric solutions. The calculations that need to be made are further simplified by making use of the gauge group that leaves the Killing spinor equations invariant to choose an expression for the Killing spinors. Next, we will give an outline of this method.

Let us begin with a theory covariant under the Spin group $Spin(2n-1,1)$; note that this is even dimensional and of Lorentzian signature. However, the method can be easily generalised to odd dimensions, which will be discussed later. We will also be working with a mostly plus metric, which is the convention we adopt throughout. Now, we consider the real vector space $V = \mathbb{R}^{2n-1,1}$, which naturally comes with a Lorentzian inner product. On this vector space we introduce an orthonormal basis $e_0, e_1, \ldots, e_{2n-1}$, where e_0 denotes the time direction. The Lorentzian inner product, $\langle -, - \rangle$, is then defined by

$$\langle w^a e_a, z^b e_b \rangle = \sum_{a,b=0}^{2n-1} w^a z^a \eta_{ab} \ , \tag{1.5}$$

where $\eta_{ab} = diag(-1, +1, \ldots, +1)$.

To construct the Dirac representation of $Spin(2n-1,1)$ we take the complexified space $U = \mathbb{C}\langle e_1, \ldots, e_n \rangle$, we then denote the space of Dirac spinors by $\Delta_c = \Lambda^*(U)$, where $\Lambda^*(U)$ denotes the space of differential forms. A basis for the Dirac spinor is given by

$$\begin{aligned} 1 \ &, \\ e_1, \ e_2, \ldots, \ e_n \ &, \\ e_{12}, \ e_{13}, \cdots \ , \ e_{(n-1)(n)} \ &, \\ \vdots \ & \end{aligned}$$

$$e_{123\ldots n} \ , \tag{1.6}$$

which corresponds to a total of 2^n elements. We have used the short hand notation

$$e_{ijk} = e_i \wedge e_j \wedge e_k \ , \tag{1.7}$$

where $\wedge$ is the wedge product. More explicitly, this means that a general Dirac spinor $\chi \in \Delta_c$ can be written as

$$\chi = a1 + b^i e_i + c^{ij} e_{ij} + \ldots + d^{1\ldots n} e_{1\ldots n} \ , \tag{1.8}$$

where the coefficients $a, b, c, d \in \mathbb{C}$. Therefore it is easy to see that a Dirac spinor of $Spin(2n{-}1, 1)$ has 2^n complex components. The number of independent components of a spinor can be reduced by imposing Weyl and Majorana conditions leading to Weyl and Majorana spinors, and we will discuss these in a bit more detail later. However we first point out that the space of Dirac spinors naturally decomposes into two chiral subspaces, i.e. the Weyl representations,

$$\Delta_c = \Delta_c^+ + \Delta_c^- \ , \tag{1.9}$$

where Δ_c^+ denotes the positive chiral space, generated by even degree differential forms, and Δ_c^- denotes the negative chiral space, generated by odd degree differential forms.

The Clifford algebra gamma matrices can also be constructed using these basis elements and are defined in the following way,

$$\begin{aligned} \Gamma_0 &= -e_n \wedge \ + \ e_n \lrcorner \ , \qquad \Gamma_n = e_n \wedge \ + \ e_n \lrcorner \ , \\ \Gamma_i &= e_i \wedge \ + \ e_i \lrcorner \ , \qquad \Gamma_{i+n} = i e_i \wedge \ - \ i e_i \lrcorner \ , \\ i &= 1, \ldots n-1 \ , \end{aligned} \tag{1.10}$$

where $\lrcorner$ denotes the adjoint operation to $\wedge$. In particular it satisfies the property

$$e_i \lrcorner (e_j \wedge e_k) \ = \ \delta_{ij} e_k - \delta_{ik} e_j \ . \tag{1.11}$$

It is straightforward to verify that the gamma matrices constructed here satisfy the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = \Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB} \ , \tag{1.12}$$

where $A, B = 0, 1, \ldots, 2n-1$ and $\eta_{AB} = diag(-1, +1, \ldots, +1)$. An extensive discussion of spinors and Clifford algebras in general can be found in [97].

The Lorentzian inner product defined in (1.5) can be extended to a Hermitian inner product on the space of Dirac spinors, defined by

$$\langle w^a e_a, z^b e_b \rangle = \sum_{a=1}^{n} (w^a)^* z^a \ , \tag{1.13}$$

where $(w^a)^*$ is the complex conjugate of w^a. In addition, note that Γ_0 is anti-Hermitian and Γ_i, for $i = 1, ..., 2n-1$, are Hermitian. In regards to the inner product in (1.13) this means

$$\begin{aligned} \langle \Gamma_0 \eta, \theta \rangle &= -\langle \eta, \Gamma_0 \theta \rangle \ , \\ \langle \Gamma_i \eta, \theta \rangle &= \langle \eta, \Gamma_i \theta \rangle \ , \end{aligned} \tag{1.14}$$

where η and θ are two spinors.

We now give some further definitions that will be needed later. Firstly, we define the Dirac inner product as [15]

$$D(\eta, \theta) = \langle \Gamma_0 \eta, \theta \rangle \ , \tag{1.15}$$

which, unlike the Hermitian inner product in (1.13), is $Spin(2n-1,1)$ invariant. This means

$$D(\Gamma_{AB}\eta, \theta) + D(\eta, \Gamma_{AB}\theta) = 0 \ , \tag{1.16}$$

where $\Gamma_{AB} \in Spin(2n-1,1)$ and $\Gamma_{AB} = \Gamma_{[A}\Gamma_{B]} = \frac{1}{2}(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A)$. In general we use this notation to denote antisymmetrisation in the lower indices

$$\Gamma_{A_1 \dots A_n} = \Gamma_{[A_1}\Gamma_{A_2} \dots \Gamma_{A_n]} \ . \tag{1.17}$$

In addition to the Dirac inner product we can define two Majorana inner products, which is the case since we are considering an even dimensional space. These correspond to the two Majorana conjugates which can be defined in even dimensions, see for example [8]. The first one is defined as

$$M_1(\eta, \theta) = \langle A(\eta)^*, \theta \rangle \ , \tag{1.18}$$

and the second one is

$$M_2(\eta, \theta) = \langle B(\eta)^*, \theta \rangle \ , \tag{1.19}$$

where the two maps A and B are defined as

$$\begin{aligned} A &= \Gamma_{12\dots n} \ , \\ B &= \Gamma_{0(n+1)\dots,(2n-1)} \ , \end{aligned} \tag{1.20}$$

and are related to the two charge conjugation matrices that one finds in even dimensions. Both $M_{1,2}$ are $Spin(2n-1,1)$ invariant, i.e.

$$M_{1,2}(\Gamma_{AB}\eta,\theta)+M_{1,2}(\eta,\Gamma_{AB}\theta)=0\ . \tag{1.21}$$

When considering higher dimensional theories one aims to work with the simplest representation of fermions. This depends on the dimension in which the theory is formulated in, but the most general spinor that one can consider is the Dirac spinor, which has $2^{[D/2]}$ complex components, where D is the dimension we are working in. In even dimensions one can impose a chirality condition that halves the number of degrees of freedom. One can also generally impose a reality condition called the Majorana condition which also halves the number of degrees of freedom. In some dimensions one can impose both of these condition to obtain Majorana-Weyl spinors which have one quarter of the degrees of freedom of the original Dirac spinor. In certain special cases when the Majorana condition cannot be imposed an alternative "symplectic Majorana" condition can be applied, as a result one obtains symplectic Majorana spinors. This notably happens in six dimensions and we will discuss this in more detail further on. We now give details on how these conditions are imposed in the formulation we have been describing.

The standard way to impose the Majorana condition is to equate the Majorana conjugate of a spinor with the Dirac conjugate. To do this in the spinorial language that we have been describing thus far we first introduce the following two anti-linear maps [16]

$$L_{+} \;=\; e^{i\psi_{+}}\Gamma_0 A\,*\ , \tag{1.22}$$
$$L_{-} \;=\; e^{i\psi_{-}}\Gamma_0 B\,*\ , \tag{1.23}$$

where A and B are the maps defined in (1.20), $\psi_{\pm}$ are arbitrary phases chosen such that the final expression is simplified and when acting on a spinor the $*$ takes the complex conjugate of the spinor, see the next subsection. The Majorana condition is defined as

$$L_{\pm}(\eta)=\eta\ , \tag{1.24}$$

for a spinor η. In the next subsection we shall explicitly show how these maps act on spinors in order to get the general expressions for the charge conjugation matrices. In addition, we will discuss some further details of the spinorial geometry method.

1.4.1 The Majorana Condition

Let us now demonstrate what the condition in (1.24) implies. The anti-linear map acts in the following way

$$\begin{aligned} \eta &= L_+(\eta) \ , \\ &= e^{i\psi_+}\Gamma_0 A(\eta)^* \ , \\ &= e^{i\psi_+}\Gamma_0\Gamma_{12\dots n}\eta^* \ , \end{aligned} \tag{1.25}$$

rearranging this we find

$$\eta^* = (-1)^{\frac{1}{2}n(n+1)}\Gamma_{012\dots n}\eta \ , \tag{1.26}$$

where we have chosen the phase so that $e^{i\psi_+} = -1$. This is the more familiar Majorana condition that we know

$$\eta^* = C_+\eta \ , \tag{1.27}$$

where we identify

$$C_+ = (-1)^{\frac{1}{2}n(n+1)}\Gamma_{012\dots n} \tag{1.28}$$

with the charge conjugation matrix.

An analogues calculation for the L_- map gives

$$\eta^* = (-1)^{\frac{1}{2}(n-1)(n-2)}\Gamma_{(n+1)\dots(2n-1)}\eta \ , \tag{1.29}$$

therefore, the alternative charge conjugation matrix is

$$C_- = (-1)^{\frac{1}{2}(n-1)(n-2)}\Gamma_{(n+1)\dots(2n-1)} \ . \tag{1.30}$$

In even dimensions either condition can be used to impose the Majorana condition.

1.4.2 Oscillator Basis

An alternative formulation of the gamma matrices given in (1.10) is in terms of an oscillator basis, related to those in (1.10) by [15, 16]

$$\begin{aligned}
\Gamma_- &= \frac{1}{\sqrt{2}}(\Gamma_n - \Gamma_0) = \sqrt{2}e_n \wedge \ , \\
\Gamma_+ &= \frac{1}{\sqrt{2}}(\Gamma_n + \Gamma_0) = \sqrt{2}e_n \lrcorner \ , \\
\Gamma_\alpha &= \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+n}) = \sqrt{2}e_\alpha \wedge \ , \\
\Gamma_{\bar{\alpha}} &= \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+n}) = \sqrt{2}e_\alpha \lrcorner \ ,
\end{aligned} \tag{1.31}$$

where $\alpha = 1, \ldots, n-1$. The Clifford algebra now becomes

$$\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2g_{AB} \ , \tag{1.32}$$

where g_{AB} is non-zero in the case when $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ and $g_{+-} = 1$. Note that this means a lowered (raised) + index becomes − raised (lowered) index, and similarly a lowered (raised) α becomes $\bar{\alpha}$ when raised (lowered).

This basis is often referred to as the creation/annihilation basis because of the action of the gamma matrices on the space of spinors. Acting with Γ_- or Γ_α increases the degree of the forms by one whereas acting with Γ_+ or $\Gamma_{\bar{\alpha}}$ reduces the degree of the forms by one. Therefore, starting with a vacuum state 1, a generic spinor can be written as [15, 16]

$$\chi = \sum_{k=0}^{n-1} \frac{1}{k!}\chi^{a_1\ldots a_k}\Gamma_{a_1\ldots a_k}1 \ , \qquad a = -, \alpha \ . \tag{1.33}$$

1.4.3 Γ_{2n+1}

In even dimensions we can construct the chirality operator in the usual way

$$\Gamma_{2n+1} = c\Gamma_0\Gamma_1 \ldots \Gamma_{2n-1} \ , \tag{1.34}$$

where $c \in \mathbb{C}$ and is fixed by requiring $(\Gamma_{2n+1})^2 = 1$. For the group $Spin(2n-1,1)$ this is

$$\Gamma_{2n+1} = (-1)^{\frac{1}{2}(n-1)}\Gamma_0\Gamma_1 \ldots \Gamma_{2n-1} \ . \tag{1.35}$$

In addition to squaring to one, this operator, by construction, also anti-commutes with all the elements of the Clifford algebra Γ_A

$$\{\Gamma_{2n+1}, \Gamma_A\} = 0 \ . \tag{1.36}$$

This means the formalism that we have described can naturally be extended to odd dimensions by taking Γ_{2n+1} as the additional gamma matrix of the Clifford algebra. However, a consequence of extending the Clifford algebra to odd dimensions is that there no longer exists a chirality operator and hence there are no Weyl spinors in odd dimensions.

1.4.4 Spacetime Form Bilinears

The spacetime form bilinears can be calculated using [15, 16]

$$\alpha(\eta,\theta) = \frac{1}{k!} M_2(\eta, \Gamma_{A_1 \dots A_k}\theta) e^{A_1} \wedge \dots \wedge e^{A_k} \ , \qquad k = 0, 1, \dots, 2n-1 \ , \tag{1.37}$$

where α corresponds to a k-form. Using this expression one can calculate all the spacetime form bilinears associated with the spinors η and θ. The Hodge duality means the bilinear forms for $k \geq n$ are related to those for $k \leq n$ and, therefore, it is enough to determine the bilinear forms upto $k = n$.

This completes our analysis of writing spinors in terms of forms, and it will a play crucial role in the investigation that is to follow in later chapters.

1.5 Spinorial Geometry Method in 10D

In this section the spinorial geometry method introduced in the previous part will be adapted to the ten dimensional case. This will allow us to provide an explicit example of how this method works, but more importantly, the discussion of this example will demonstrate one of the crucial ingredients needed in the next chapter. A more thorough discussion of what follows can be found in [16, 44], where supersymmetric supergravity solutions in ten dimensions were investigated.

We begin by setting $n = 5$ and investigating the spinors of $Spin(9,1)$. First we consider the real vector space $V = \mathbb{R}^{9,1}$ and choose an orthonormal basis on this space given by $e_0, e_1, \dots, e_9$. In order to formulate the Dirac spinors of $Spin(9,1)$ we take the complexified subspace $U = \mathbb{C}\langle e_1, \dots, e_5 \rangle$. We can now write a general Dirac spinor $\chi \in \Delta_c = \Lambda^*(U)$ as

$$\chi = a1 + b^i e_i + c^{ij} e_{ij} + d^{ijk} e_{ijk} + f^{ijkl} e_{ijkl} + g e_{12345} \ , \tag{1.38}$$

where $i, j = 1, \dots, 5$ and the coefficients before each basis $a, b, c, d, f, g \in \mathbb{C}$.

The gamma matrices of the Clifford algebra, $\text{Clif}(\mathbb{R}^{9,1})$, are obtained by setting

$n = 5$ in (1.10) and are given by

$$\begin{aligned}\Gamma_0 &= -e_5 \wedge \ + \ e_5 \lrcorner \ , \qquad \Gamma_5 = e_5 \wedge \ + \ e_5 \lrcorner \ , \\ \Gamma_i &= e_i \wedge \ + \ e_i \lrcorner \ , \qquad \Gamma_{i+n} = i e_i \wedge \ - \ i e_i \lrcorner \ , \\ i &= 1, \ldots, 4 \ . \end{aligned} \tag{1.39}$$

The Dirac inner product and the two Majorana inner products are as in (1.15), (1.18) and (1.19) respectively. The map A in (1.18) and B in (1.19) become

$$\begin{aligned} A &= \Gamma_{12345} \ , \\ B &= \Gamma_{06789} \ . \end{aligned} \tag{1.40}$$

The corresponding Majorana condition is given by

$$\eta^* = C_\pm \eta \ , \tag{1.41}$$

where

$$\begin{aligned} C_+ &= -\Gamma_{012345} \ , \\ C_- &= \Gamma_{6789} \ . \end{aligned} \tag{1.42}$$

We are free to choose the Majorana inner product, and thus the map A or B, we want to work with. Following the convention in [16, 44] we will work with the second Majorana inner product involving the map B. The corresponding Majorana condition is

$$\eta^* = \Gamma_{6789} \eta \ . \tag{1.43}$$

The gamma matrices of the oscillator basis are

$$\begin{aligned} \Gamma_- &= \sqrt{2} e_5 \wedge \ , \qquad \Gamma_+ = \sqrt{2} e_5 \lrcorner \ , \\ \Gamma_\alpha &= \sqrt{2} e_\alpha \wedge \ , \qquad \Gamma_{\bar{\alpha}} = \sqrt{2} e_\alpha \lrcorner \ , \end{aligned} \tag{1.44}$$

where $\alpha = 1, 2, 3, 4$. Observe how much simpler the gamma operations have become in this basis, the calculations also become correspondingly more straightforward.

For completeness we give the chirality operator in ten dimensions

$$\Gamma_{11} = \Gamma_0 \Gamma_1 \ldots \Gamma_9 \ . \tag{1.45}$$

1.5.1 The Real Spinors

Where possible one tends to work with real spinors, and this is obtained by enforcing the Majorana condition. In this section we outline how the Majorana condition is used to determine real spinors in terms of forms. The Majorana condition is given in (1.43). This condition illustrates the action of the charge conjugation matrix. For example, given the basis 1 the charge conjugation matrix changes it to e_{1234} since

$$\begin{aligned}\Gamma_6\Gamma_7\Gamma_8\Gamma_9 1 = \Gamma_6\Gamma_7\Gamma_8(ie_4\wedge -ie_4\lrcorner)1 &= i\Gamma_6\Gamma_7\Gamma_8 e_4\\ &= -\Gamma_6\Gamma_7 e_{34}\\ &= -i\Gamma_6 e_{234} = e_{1234}\ .\end{aligned} \quad (1.46)$$

Similarly the basis e_{1234} changes to 1. Therefore, lets consider the following spinor

$$\eta = a1 + be_{1234}\ , \quad (1.47)$$

where $a, b \in \mathbb{C}$, this spinor is also a Weyl spinor since it is composed of even degree forms, $\eta \in \Delta^{even}$ and can be checked using the chirality operator in (1.45). We now apply the Majorana condition to reduce this to its real parts. The left hand side of (1.43) uses the standard complex conjugate and so

$$\eta^* = a^*1 + b^*e_{1234}\ , \quad (1.48)$$

whereas the right hand side of the Majorana condition gives

$$\Gamma_{6789}\eta = ae_{1234} + b1\ . \quad (1.49)$$

Comparing these two expressions we find

$$a = b^*\ , \quad (1.50)$$

and so

$$\eta = a1 + a^*e_{1234}\ . \quad (1.51)$$

This means the two real spinors are

$$\eta_1 = 1 + e_{1234}\ , \qquad \eta_2 = i(1 - e_{1234})\ . \quad (1.52)$$

These are two of the basis spinors for the space of Majorana-Weyl spinors in ten dimensions. In a similar fashion to the procedure outlined above the remaining basis spinors can be found, see for example [44, 16].

1.5.2 Spinor Forms and KSEs

Once an explicit basis for the spinors has been established gauge transformation that leave the KSEs invariant can be used to obtain simple expressions for Killing spinors. These Killing spinors, written in terms of forms, are substituted directly into the KSEs. Since the gamma matrices are also expressed in terms of form operators one is able to directly solve the KSEs to obtain purely algebraic relations on the field content of the theory, which can then be translated to conditions on the geometry. This procedure can be repeated for more Killing spinors and the aim is to obtain a full classification of the solutions to the KSEs. In the case of heterotic supergravity [44, 45, 46] and six dimensional (1,0) supergravity a classification of the solutions to the KSEs was achieved according to the isotropy group of the Killing spinors. However, this is not always possible, for example this has not been done for the IIB supergravity.

1.5.3 Ten Dimensional Super Yang-Mills

In this section we give a simple example to demonstrate how one uses the machinery introduced above to solve the KSE of ten dimensional super Yang-Mills to find its supersymmetric solutions. The KSE for this theory is

$$F_{AB}\Gamma^{AB}\epsilon = 0 \ , \tag{1.53}$$

where ϵ is a Majorana-Weyl spinor of $Spin(9,1)$ and F_{AB} is the gauge field strength. This group has one type of orbit with stability subgroup $Spin(7) \ltimes \mathbb{R}^8$, see [98, 99]. A representative spinor can be chosen as [44, 16]

$$\epsilon = 1 + e_{1234} \ . \tag{1.54}$$

Substituting this spinor into the above KSE and expanding the repeated indices we find,

$$\begin{aligned}\big(2F_{-+}\Gamma^{-+} + 2F_{-\alpha}\Gamma^{-\alpha} + 2F_{+\alpha}\Gamma^{+\alpha} + 2F_{-\bar\alpha}\Gamma^{-\bar\alpha} + 2F_{+\bar\alpha}\Gamma^{+\bar\alpha} + \\ F_{\alpha\beta}\Gamma^{\alpha\beta} + 2F_{\alpha\bar\beta}\Gamma^{\alpha\bar\beta} + F_{\bar\alpha\bar\beta}\Gamma^{\bar\alpha\bar\beta}\big)(1+e_{1234}) \ &= \ 0 \ .\end{aligned} \tag{1.55}$$

The gamma matrices are given in (1.39), these are used to find

$$\begin{aligned}
\Gamma^{-+}(1+e_{1234}) = (1+e_{1234}) \ , \quad \Gamma^{-\alpha}(1+e_{1234}) = 0 \ , \quad \Gamma^{-\bar\alpha}(1+e_{1234}) = 0 \ , \\
\Gamma^{+1}(1+e_{1234}) = -2e_{2345} \ , \quad \Gamma^{+2}(1+e_{1234}) = 2e_{1345} \ , \\
\Gamma^{+3}(1+e_{1234}) = -2e_{1245} \ , \quad \Gamma^{+4}(1+e_{1234}) = 2e_{1235} \ , \\
\Gamma^{+\bar1}(1+e_{1234}) = 2e_{51} \ , \quad \Gamma^{+\bar2}(1+e_{1234}) = 2e_{52} \ , \\
\Gamma^{+\bar3}(1+e_{1234}) = 2e_{53} \ , \quad \Gamma^{+\bar4}(1+e_{1234}) = 2e_{54} \ , \\
\Gamma^{\bar\alpha\bar\beta}(1+e_{1234}) = 2e_{\alpha\beta} \ , \quad \Gamma^{\alpha\bar\beta}(1+e_{1234}) = \delta^{\alpha\bar\beta}(1-e_{1234}) \ ,
\end{aligned}$$

$$\Gamma^{\alpha\beta}(1+e_{1234}) = -\epsilon^{\alpha\beta\gamma\delta}e_{\gamma\delta}\ , \quad (1.56)$$

where $\epsilon^{1234} = 1$. Substituting these in and equating the coefficients before each basis to zero we find

$$F_{-+} = \delta^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}} = F_{+\alpha} = F_{+\bar{\alpha}} = 0\ , \quad F_{\bar{\alpha}\bar{\beta}} = \frac{1}{2}\epsilon_{\bar{\alpha}\bar{\beta}}{}^{\gamma\delta}F_{\gamma\delta}\ . \quad (1.57)$$

In addition, the $F_{-\alpha}$ and $F_{-\bar{\alpha}}$ components are not constrained apart from being hermitian conjugates of each other. Therefore, we can use this to write the gauge field strength as

$$F = F_{-i}e^{-}\wedge e^{i} + F_{ij}e^{i}\wedge e^{j}\ , \quad (1.58)$$

where the F_{ij} components are constrained according to (1.57), i.e. it is contained in $Spin(7)$ and this group is fixed as we have chosen the positive chiral spinor in (1.54), i are the transverse directions to the light-cone. This represents the form that solutions which preserve one supersymmetry must take. This is a simple example which we have given to demonstrate how the spinorial geometry method can be used, the equations that we come across will be a lot more involved.

1.6 Summary

In this first chapter we have discussed the basic aspects of supersymmetric supergravity solutions and what we mean by terms like Killing spinors and Killing spinor equations. In addition, an overview of some of the relevant literature was given in which the six dimensional theory was put into context.

However, the main part of the chapter was to introduce the spinorial geometry approach to solving the KSEs. In particular, we discussed how spinors can be written in terms of forms and also detailed some technical tools used in the manipulations of spinors in the language of spinorial geometry. Furthermore, we demonstrated how this method works in ten dimensions giving the specific example of finding a supersymmetric solution of ten dimensional super Yang-Mills theory.

Throughout the remainder of the thesis this technique will be used in the analysis of different aspects of (1,0)-supersymmetric theories in six dimensions. In the next chapter we will describe (1,0) supergravity in six dimensions and use this method to solve the KSEs for this theory.

Chapter 2

Six Dimensional Supergravity and Killing Spinor Equations

2.1 Introduction

In this chapter we will solve the Killing spinor equations of six dimensional (1,0) supergravity coupled to any number of tensor, vector and scalar multiplets in all cases. As we mentioned in the previous chapter the supersymmetric solutions of (1,0) six dimensional supergravity have been considered before but these were restricted in their couplings to matter fields and solutions considered preserved only one supersymmetry or maximal supersymmetry, see [34, 58, 59, 60]. We will solve the KSEs of the most general theory and for all possible fractions of supersymmetry preserved. In each case we will give the constraints imposed on the matter content of the theory and discuss the restrictions on the geometry of spacetime. To do this we will make use of the spinorial geometry method and we will use the analogy that exists between the KSEs of heterotic supergravity and those of six dimensional (1,0) supergravity. This was one of the reasons why we used the example in section 1.5.

We shall begin the chapter by introducing the (1,0) six dimensional supergravity, discussing the field content and the KSEs that we focus on. Then we will discuss spinors and the techniques we use to solve the KSEs. In particular, we will outline the relation between heterotic supergravity and the theory under consideration, and furthermore, discuss how this is used to rewrite the KSEs of the theory.

As a next step, we investigate the isotropy groups of the different number of Killing spinors admitted, and in each case we determine the corresponding representative spinors. This analysis will also allow us to look for descendant solutions; these are solutions which have less Killing spinors than parallel spinors, i.e. some of the parallel spinors may not necessarily be solutions to the other KSEs. This analysis will demonstrate that the solutions of the KSEs can be classified uniquely according to the isotropy group of the Killing spinors in $Spin(5,1) \cdot Sp(1)$, which corresponds to the holonomy of the supercovariant connection of a generic background. This is true except in one case, which becomes clear when the existence of descendants is investigated. The dot in this group and in the groups which we will use later denote a mod $\mathbb{Z}_2$, i.e. for two groups, G_1 and G_2 we have $G_1 \cdot G_2 = \frac{G_1 \times G_2}{\mathbb{Z}_2}$. The isotropy group of the Killing spinors are found to be

$$\begin{aligned} & Sp(1)\cdot Sp(1) \ltimes \mathbb{H}\ (1)\ ,\quad U(1)\cdot Sp(1)\ltimes \mathbb{H}\ (2)\ ,\quad Sp(1)\ltimes \mathbb{H}\ (3,4)\ ; \\ & Sp(1)\ (2)\ ,\quad U(1)\ (4)\ ,\quad \{1\}\ (8)\ , \end{aligned} \tag{2.1}$$

where in parenthesis we indicate the number of Killing spinors that are left invariant. The representative spinors which are left invariant under these subgroups will be discussed later.

In the sections that follow this analysis we consider each of the isotropy groups and the corresponding Killing spinors separately. In each case we solve the KSEs and discuss the implications of the solutions on the fields and the geometry, giving a detailed description of the backgrounds that arise.

2.2 (1,0) Supergravity

In this section we give details of the theory.

2.2.1 Fields and KSEs

In six dimensions there are four types of (1,0)-supersymmetry multiplets; the gravitational, tensor, vector and scalar[1] multiplets. The theory that we shall consider is (1,0) supergravity coupled to n_T tensor, n_V vector and n_H scalar multiplets. We are interested in the bosonic fields of the theory and so we shall focus on the bosonic components of each multiplet, however, the construction of this theory including a description of the fermions can be found in [55, 57]. We will mainly follow the construction in [57]. The gravitational multiplet apart from the graviton has a 2-form gauge potential; each tensor multiplet contains a 2-form gauge potential and a real scalar; the vector multiplet has a vector and each scalar multiplet contains four real scalars. The bosonic fields of the scalar multiplet take values in a Quaternionic Kähler manifold which has real dimension $4n_H$.

The spinors in six dimensions can be described by symplectic Majorana spinors. Therefore, before proceeding to describe the KSEs, we give the symplectic Majorana condition satisfied by the fermions that appear in (1,0) supergravity. This condition utilises the invariant $Sp(1)$ and $Sp(n_H)$ forms to impose a reality condition. Let us suppose that the Dirac or Weyl spinors λ and χ transform under the fundamental representations of $Sp(1)$ and $Sp(n_H)$ respectively. The symplectic Majorana

[1] Note that we will also refer to scalar multiplets as hypermultiplets, these terms will be used interchangeably throughout.

condition is given by

$$\lambda^{\underline{A}} = \epsilon^{\underline{AB}} C \bar{\lambda}^T_{\underline{B}} \ , \quad \chi^{\underline{a}} = \epsilon^{\underline{ab}} C \bar{\chi}^T_{\underline{b}} \ , \tag{2.2}$$

where C is the charge conjugation matrix and $\epsilon^{\underline{AB}}$ and $\epsilon^{\underline{ab}}$ are the symplectic invariant forms of $Sp(1)$ and $Sp(n_H)$, respectively, and $\underline{A}, \underline{B} = 1, 2$ and $\underline{a}, \underline{b} = 1, \dots, 2n_H$.

The supersymmetry transformations of the fermions evaluated at the bosonic fields are

$$\begin{aligned}
\delta\Psi^{\underline{A}}_\mu &= \nabla_\mu \epsilon^{\underline{A}} - \frac{1}{8} H_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon^{\underline{A}} + \mathcal{C}_\mu{}^{\underline{A}}{}_{\underline{B}}\,\epsilon^{\underline{B}} \ , \\
\delta\chi^{\underline{MA}} &= \frac{i}{2} T^{\underline{M}}_\mu \gamma^\mu \epsilon^{\underline{A}} - \frac{i}{24} H^{\underline{M}}_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon^{\underline{A}} \ , \\
\delta\psi^{\underline{a}} &= i\gamma^\mu \epsilon_{\underline{A}} V^{\underline{aA}}_\mu \ , \\
\delta\lambda^{a'\underline{A}} &= -\frac{1}{2\sqrt{2}} F^{a'}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{\underline{A}} - \frac{1}{\sqrt{2}} (\mu^{a'})^{\underline{A}}{}_{\underline{B}}\epsilon^{\underline{B}} \ ,
\end{aligned} \tag{2.3}$$

where Ψ is the gravitino, χ are the tensorini, ψ are the hyperini and λ are the gaugini, ϵ is the supersymmetry parameter and the index $a' = 1, \dots, n_V$. The remaining coefficients that appear in the supersymmetry transformations depend on the fundamental fields of the theory. This in turn means their explicit expressions depend on the formulation of the theory. The structure of the supersymmetry transformations that we have stated above includes all known formulations. As a result, most of the analysis on the solutions of the KSEs that follows is independent on the precise expression of the supersymmetry transformations in terms of the fields. Because of this, the conditions that arise from the KSEs will be given in generality. We will also state explicitly where the expressions of the KSEs in terms of the fields is used. In what follows, we shall always assume that ∇ is the spin connection of the spacetime and $\mathcal{C}$ is a $Sp(1)$ connection.

In order to give an example of how the supersymmetry transformations, (2.3), depend on the fundamental fields of the theory, we use the formulation proposed in [57]. However, we use a different normalization[2] for some of the fields from that in [57]. The formulation in [57] organises the fields in the following way. The theory has $n_T + 1$ 2-form gauge potentials denoted as $B^{\underline{r}}$ where $\underline{r} = 0, 1, \dots, n_T$. One of these 2-form potentials comes from the gravitational multiplet and the remaining n_T are associated to the tensor multiplets. We denote the corresponding 3-form field strengths with $G^{\underline{r}}$. The precise relation between $B^{\underline{r}}$ and $G^{\underline{r}}$ as well as the duality conditions on $G^{\underline{r}}$ will be given later. The n_T scalars of the tensor multiplets parametrise the coset space $SO(1, n_T)/SO(n_T)$. A convenient way to describe this coset space was introduced in [100], where the scalars are described by an $SO(1, n_T)$

[2] Our normalization is similar to that of heterotic supergravity.

matrix

$$S = \begin{pmatrix} v_{\underline{r}} \\ x_{\underline{r}}^{\underline{M}} \end{pmatrix} \ , \quad \underline{M} = 1, \dots n_T \ . \tag{2.4}$$

Since the matrix $S \in SO(1, n_T)$, one has $S^T \eta S = \eta$ where η is the Lorentzian metric in $(1, n_T)$-dimensions. In particular, the components of the matrix S satisfy

$$v_{\underline{r}} v^{\underline{r}} = 1 \ , \quad v_{\underline{r}} v_{\underline{s}} - \sum_{\underline{M}} x_{\underline{r}}^{\underline{M}} x_{\underline{s}}^{\underline{M}} = \eta_{\underline{rs}} \ , \quad v^{\underline{r}} x_{\underline{r}}^{\underline{M}} = 0 \ . \tag{2.5}$$

The canonical $SO(n_T)$ connection of the coset is $\sum_{\underline{r}} x_{\underline{r}}^{\underline{M}} dx_{\underline{r}}^{\underline{N}}$.

On the other hand, the scalars of the hypermultiplet parametrise a Quaternionic Kähler manifold which has holonomy $Sp(n_H) \cdot Sp(1)$. This manifold admits a frame E such that the metric can be written as

$$g_{\underline{IJ}} = E_{\underline{I}}^{\underline{aA}} E_{\underline{J}}^{\underline{bB}} \epsilon_{\underline{ab}} \epsilon_{\underline{AB}} \ , \tag{2.6}$$

where $\underline{I} = 1, \dots, 4n_H$ and, $\epsilon_{\underline{ab}}$ and $\epsilon_{\underline{AB}}$ are the invariant $Sp(n_H)$ and $Sp(1)$ 2-forms, respectively. The associated spin connection has holonomy $Sp(n_H) \cdot Sp(1)$ and so decomposes as $(\mathcal{A}_{\underline{I}}{}^{\underline{a}}{}_{\underline{b}}, \mathcal{A}_{\underline{I}}{}^{\underline{A}}{}_{\underline{B}})$. For general details on geometry and holonomy see for example [101, 102, 103], we will tend not to give specific details but discuss only what is required to continue without affecting the discussion.

In [57] to include vector multiplets with (non-abelian) gauge potential $A_\mu^{a'}$, one assumes that the quaternionic Kähler manifold of the hypermultiplet is $Sp(1, n_H)/Sp(1) \times Sp(n_H)$ and gauges the maximal compact isometry subgroup $Sp(1) \times Sp(n_H)$. So the gauge group of the theory is $H = Sp(1) \times Sp(n_H) \times K$, where K is a product of semi-simple groups which does not act on the scalars. Let $\xi_{a'_1}$ and $\xi_{a'_2}$ be the vector fields generated on $Sp(1, n_H)/Sp(1) \times Sp(n_H)$ by the action of $Sp(1)$ and $Sp(n_H)$, respectively. Under these assumptions, one has the following

$$\begin{array}{l} H_{\mu\nu\rho} = v_{\underline{r}} G^{\underline{r}}_{\mu\nu\rho} \ , \quad H^{\underline{M}}_{\mu\nu\rho} = x^{\underline{M}}_{\underline{r}} G^{\underline{r}}_{\mu\nu\rho} \ , \quad \mathcal{C}_\mu{}^{\underline{A}}{}_{\underline{B}} = D_\mu \phi^{\underline{I}} \mathcal{A}_{\underline{I}}{}^{\underline{A}}{}_{\underline{B}} \ , \\ T^{\underline{M}}_\mu = x^{\underline{M}}_{\underline{r}} \partial_\mu v^{\underline{r}} \ , \quad V^{\underline{aA}}_\mu = E^{\underline{aA}}_{\underline{I}} D_\mu \phi^{\underline{I}} \ , \quad F^{a'}_{\mu\nu} = \partial_\mu A^{a'}_\nu - \partial_\nu A^{a'}_\mu + f^{a'}{}_{b'c'} A^{b'}_\mu A^{c'}_\nu \ , \\ (\mu^{a'_1})^{\underline{A}}{}_{\underline{B}} = \dfrac{1}{v_{\underline{r}} c^{\underline{r}1}} \mathcal{A}_{\underline{I}}{}^{\underline{A}}{}_{\underline{B}} \xi^{\underline{I}a'_1} \ , \quad (\mu^{a'_2})^{\underline{A}}{}_{\underline{B}} = \dfrac{1}{v_{\underline{r}} c^{\underline{r}2}} \mathcal{A}_{\underline{I}}{}^{\underline{A}}{}_{\underline{B}} \xi^{\underline{I}a'_2} \ , \quad (\mu^{a'_3})^{\underline{A}}{}_{\underline{B}} = 0 \ , \end{array} \tag{2.7}$$

where the gauge index a'_3 ranges over the gauge subgroup K, $\phi^{\underline{I}}$ are the scalars of the hypermultiplet,

$$\begin{aligned} \nabla_\mu \epsilon^{\underline{A}} &= \partial_\mu \epsilon^{\underline{A}} + \frac{1}{4} \Omega_{\mu, mn} \gamma^{mn} \epsilon^{\underline{A}} \ , \\ D_\mu \phi^{\underline{I}} &= \partial_\mu \phi^{\underline{I}} - A^{a'}_\mu \xi^{\underline{I}}_{a'} \ , \end{aligned} \tag{2.8}$$

respectively, and Ω is the frame connection of spacetime. It is also understood that

$\xi_{a'_3} = 0$ as K does not act on the scalars of the hypermultiplet. Furthermore, $F^{a'}$ are the field strengths of the gauge potentials $A^{a'}$ and f are the structure constants of the gauge group H.

We now define the field strengths $G^{\underline{r}}$. These are given by [57]

$$G^{\underline{r}}_{\mu\nu\rho} = 3\partial_{[\mu}B^{\underline{r}}_{\nu\rho]} + c^{\underline{r}1}CS(A^{Sp(1)})_{\mu\nu\rho} + c^{\underline{r}2}CS(A^{Sp(n_H)})_{\mu\nu\rho} + c^{\underline{r}K}CS(A^K)_{\mu\nu\rho} \ , \quad (2.9)$$

where $c^{\underline{r}}$'s are constants, one for each copy of the gauge group, and $CS(A)$'s are the Chern-Simons 3-forms. Note also that the constants $c^{\underline{r}1}$ and $c^{\underline{r}2}$ appear in the definition of μ's in (2.7).

The duality condition on G is given by

$$\zeta_{\underline{rs}}G^{\underline{s}}_{\mu_1\mu_2\mu_3} = \frac{1}{3!}\epsilon_{\mu_1\mu_2\mu_3}{}^{\nu_1\nu_2\nu_3}G_{\underline{r}\nu_1\nu_2\nu_3} \ , \quad (2.10)$$

where

$$\zeta_{\underline{rs}} = v_{\underline{r}}v_{\underline{s}} + \sum_{\underline{M}} x^{\underline{M}}_{\underline{r}}x^{\underline{M}}_{\underline{s}} \ . \quad (2.11)$$

Note that the duality conditions for H and $H^{\underline{M}}$ are opposite. In our conventions, H is anti-self-dual while $H^{\underline{M}}$ is self-dual. More explicitly, in terms of H and $H^{\underline{M}}$ we have

$$\begin{aligned} H_{\mu_1\mu_2\mu_3} &= -\frac{1}{3!}\epsilon^{\nu_1\nu_2\nu_3}{}_{\mu_1\mu_2\mu_3}H_{\nu_1\nu_2\nu_3} \ , \\ H^{\underline{M}}_{\mu_1\mu_2\mu_3} &= \frac{1}{3!}\epsilon^{\nu_1\nu_2\nu_3}{}_{\mu_1\mu_2\mu_3}H^{\underline{M}}_{\nu_1\nu_2\nu_3} \ . \end{aligned} \quad (2.12)$$

The definition of the fields in (2.7) can be used to interpret the components appearing in the supersymmetry transformations in (2.3) in terms of physical fields. We will be using these expressions when we want to discuss the solutions of the KSEs in terms of the physical fields

2.2.2 Spinors

To make effective use of the spinorial geometry method in solving the Killing spinor equations we need to express the spinors in terms of forms. In the case of the above theory in six dimensions we need to find a way to impose the symplectic Majorana condition on the spinors. This is where the relation between six dimensional supergravity and heterotic supergravity plays an important role. Firstly, we identify the symplectic Majorana-Weyl $Spin(5,1)$ spinors with the $SU(2)$-invariant Majorana-Weyl $Spin(9,1)$ spinors. Under this identification the symplectic Majorana condition of $Spin(5,1)$ spinors is replaced by the Majorana condition on the $Spin(9,1)$ spinors, which we described in section 1.5. We will now demonstrate this more explicitly. Recall that the Dirac spinors of $Spin(9,1)$ are identified with $\Lambda^*(\mathbb{C}^5)$, and the positive and negative chirality spinors are the even and odd degree forms, respectively. The gamma matrices of $\text{Clif}(\mathbb{R}^{9,1})$ are as in (1.39), which we give here again for ease of reference

$$\begin{aligned} \Gamma_0 &= -e_5 \wedge + e_5 \lrcorner \ , \quad \Gamma_5 = e_5 \wedge + e_5 \lrcorner \ , \\ \Gamma_i &= e_i \wedge + e_i \lrcorner \ , \quad \Gamma_{i+5} = i(e_i \wedge - e_i \lrcorner) \ , \quad i = 1,2,3,4 \ , \end{aligned} \tag{2.13}$$

where we recall that e_i for $i = 1, \ldots, 5$, is a Hermitian basis in $\mathbb{C}^5$. Now, we identify the gamma matrices of $\text{Clif}(\mathbb{R}^{5,1})$ in the following way

$$\gamma_\mu = \Gamma_\mu \ , \quad \mu = 0,1,2 \ ; \qquad \gamma_\mu = \Gamma_{\mu+2} \ , \quad \mu = 3,4,5 \ . \tag{2.14}$$

Therefore, the positive chirality Weyl spinors of $Spin(5,1) \cong SL(2,\mathbb{H})$ are $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5 \rangle) \cong \mathbb{H}^2$. Furthermore, we identify the symplectic Majorana-Weyl condition of $Spin(5,1)$ with the Majorana-Weyl condition of $Spin(9,1)$ spinors, i.e.

$$\epsilon^* = \Gamma_{67}\Gamma_{89}\epsilon \ , \tag{2.15}$$

where $\epsilon \in \Lambda^{\text{ev}}\mathbb{C}\langle e_1, e_2, e_5 \rangle \otimes \Lambda^*\mathbb{C}\langle e_{34} \rangle$. In particular, a basis for the symplectic Majorana-Weyl spinors is

$$\begin{aligned} &1 + e_{1234} \ , \quad i(1 - e_{1234}) \ , \quad e_{12} - e_{34} \ , \quad i(e_{12} + e_{34}) \ , \\ &e_{15} + e_{2534} \ , \quad i(e_{15} - e_{2534}) \ , \quad e_{25} - e_{1534} \ , \quad i(e_{25} + e_{1534}) \ . \end{aligned} \tag{2.16}$$

Observe that the above basis selects the diagonal of two copies of the Weyl representation of $Spin(5,1)$, where the first copy is in $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5 \rangle)$ while the second copy includes the auxiliary direction e_{34}. The $SU(2)$ acting on the auxiliary directions e_3 and e_4 leaves the basis invariant. These basis spinors are derived in the same way that the Majorana-Weyl spinors in ten dimensions are obtained, which we demonstrated when discussing real spinors in section 1.5, but the spinors in (2.16) correspond only to a subset of these spinors, specifically the $SU(2)$-invariant $Spin(9,1)$ spinors, see [44].

2.2.3 KSEs Revisited

We now rewrite the KSEs of six dimensional supergravity in terms of the ten dimensional notation we have introduced above. To do this we first define $\rho^{r'}$, $r' = 1,2,3$, such that

$$\rho^1 = \frac{1}{2}(\Gamma_{38} + \Gamma_{49}) \ , \quad \rho^2 = \frac{1}{2}(\Gamma_{89} - \Gamma_{34}) \ , \quad \rho^3 = \frac{1}{2}(\Gamma_{39} - \Gamma_{48}) \ , \tag{2.17}$$

which satisfy

$$\left[\rho^{r'}, \rho^{s'}\right] = 2\epsilon^{r's't'}\rho^{t'} \ , \tag{2.18}$$

where $\epsilon^{123} = 1$. Note that these are the generators of the Lie algebra $Sp(1)$ as it acts on the basis (2.16). Using this we can rewrite the KSEs as

$$\begin{aligned}
\mathcal{D}\epsilon \equiv \big(\nabla_\mu - \frac{1}{8}H_{\mu\nu\rho}\gamma^{\nu\rho} + \mathcal{C}_\mu^{r'}\rho_{r'}\big)\epsilon &= 0, \\
\left(\frac{i}{2}T_\mu^{\underline{M}}\gamma^\mu - \frac{i}{24}H_{\mu\nu\rho}^{\underline{M}}\gamma^{\mu\nu\rho}\right)\epsilon &= 0, \\
i\gamma^\mu\epsilon_{\underline{A}}V_\mu^{a\underline{A}} &= 0 \ , \\
\left(\frac{1}{4}F_{\mu\nu}^{a'}\gamma^{\mu\nu} + \frac{1}{2}\mu_{r'}^{a'}\rho^{r'}\right)\epsilon &= 0 \ ,
\end{aligned} \tag{2.19}$$

where we have made use of the fact that

$$\mathcal{C}_\mu{}^{\underline{A}}{}_{\underline{B}}\,\epsilon^{\underline{B}} = \mathcal{C}_\mu^{r'}(\rho_{r'})^{\underline{A}}{}_{\underline{B}}\,\epsilon^{\underline{B}} \ , \qquad (\mu^{a'})^{\underline{A}}{}_{\underline{B}}\epsilon^{\underline{B}} = (\mu^{a'})^{r'}(\rho_{r'})^{\underline{A}}{}_{\underline{B}}\epsilon^{\underline{B}} \ . \tag{2.20}$$

In the hyperini KSE, it should be understood that

$$\epsilon_1 = -\epsilon^2 \ , \quad \epsilon_2 = \Gamma_{34}\epsilon^1 \ , \tag{2.21}$$

where ϵ^1 and ϵ^2 are the components of ϵ in the two copies of the Weyl representation used to construct the symplectic-Majorana representation.

2.3 Parallel and Killing Spinors

2.3.1 Parallel Spinors

The gravitino KSE is

$$\mathcal{D}_\mu\epsilon = 0 \ , \tag{2.22}$$

and we say that any spinor ϵ that solves this equation is a parallel spinor with respect to the supercovariant derivative $\mathcal{D}$. The (reduced) holonomy[3] of six dimensional supergravity supercovariant connection $\mathcal{D}$, (2.19), is contained in $Spin(5,1)\cdot Sp(1)$. This is also the same as the gauge group of the theory. Therefore, there are two general possibilities for the isotropy group of parallel spinors. Either the parallel spinors have a trivial isotropy group in $Spin(5,1)\cdot Sp(1)$ or the parallel spinors have a non-trivial isotropy group in $Spin(5,1)\cdot Sp(1)$. To investigate these two case we

[3]We assume that the backgrounds are simply connected or equivalently we consider the universal cover.

first consider the integrability condition of the gravitino Killing spinor equation,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\,\epsilon = 0 \ , \tag{2.23}$$

which in turn gives

$$\frac{1}{4}\hat{R}_{\mu\nu,\rho\sigma}\gamma^{\rho\sigma}\epsilon + \mathcal{F}^{r'}_{\mu\nu}\rho_{r'}\epsilon = 0 \ , \tag{2.24}$$

where

$$\mathcal{F}^{r'}_{\mu\nu} = \partial_\mu \mathcal{C}^{r'}_\nu - \partial_\nu \mathcal{C}^{r'}_\mu + 2\epsilon^{r'}{}_{s't'}\mathcal{C}^{s'}_\mu \mathcal{C}^{t'}_\nu \ , \tag{2.25}$$

and $\hat{R}$ is the curvature of the connection, $\hat{\nabla}$, with skew-symmetric torsion H defined as

$$\hat{\nabla}_\mu Y^\nu = \nabla_\mu Y^\nu + \frac{1}{2}H^\nu{}_{\mu\lambda}Y^\lambda \ . \tag{2.26}$$

Trivial Isotropy Group

Now, if the isotropy group of the parallel spinors is the trivial group $\{1\}$, the integrability condition in (2.24) means that

$$\hat{R} = 0 \ , \qquad \mathcal{F} = 0 \ . \tag{2.27}$$

The spacetime is parallelisable with respect to a connection with skew-symmetric torsion and admits eight parallel spinors. Moreover, the torsion is anti-self-dual. All such spacetimes are group manifolds with anti-self-dual structure constants. We will discuss these backgrounds in more detail in section 2.9.

Non-Trivial Isotropy Group

We will now consider the case where the parallel spinors have a non-trivial isotropy group in $Spin(5,1)\cdot Sp(1)$. There are two ways to tackle this problem which are based on the approaches in [98] and [99]. The different possible spinor orbits of $Spin(5,1)$ have been discussed in [98]. In this section we will consider the possible isotropy groups as well as the representative spinors in $Spin(5,1)\cdot Sp(1)$. First, we note that $Spin(5,1) \cong SL(2,\mathbb{H})$ and that the action of $Spin(5,1)\cdot Sp(1)$ on the symplectic Majorana-Weyl spinors can be described in terms of quaternions. In particular, the symplectic Majorana-Weyl spinors can be identified with $\mathbb{H}^2$ where $Spin(5,1) \cong SL(2,\mathbb{H})$ acts from the left with quaternionic matrix multiplication while $Sp(1)$ acts on the right with the conjugate quaternionic multiplication. Let us give some further details before discussing the isotropy groups. An element

$S \in SL(2, \mathbb{H})$ can be represented by

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad a, b, c, d \in \mathbb{H} . \tag{2.28}$$

Each element of the quaternions can be written as $a = u_1 + u_2 i + u_3 j + u_4 k$ where $u \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = -1 , \quad ij = -ji = k , \quad jk = -kj = i , \quad ki = -ik = j . \tag{2.29}$$

An element of $Sp(1)$ say $f \in Sp(1)$ satisfies $f\bar{f} = 1$ where $\bar{f}$ is the quaternionic conjugate of f, i.e. if $f = v_1 + v_2 i + v_3 j + v_4 k$ then $\bar{f} = v_1 - v_2 i - v_3 j - v_4 k$ and the v's take values in $\mathbb{R}$. Furthermore, the space of symplectic Majorana-Weyl spinors spans the space $\mathbb{H} \oplus \mathbb{H}$, the first $\mathbb{H}$ is spanned by the basis spinors in the first line of (2.16) and the second $\mathbb{H}$ is spanned by basis spinors in the second line of (2.16). The action of the group $Spin(5,1) \cdot Sp(1)$ on this space is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \bar{f} , \tag{2.30}$$

where X takes values in the first copy of $\mathbb{H}$ and Y takes values in the second copy of $\mathbb{H}$. Using this we find that there is a single non-trivial orbit of $Spin(5,1) \cdot Sp(1)$ on the symplectic Majorana-Weyl spinors with isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$. A representative symplectic Majorana-Weyl spinor can be chosen as $1 + e_{1234}$.

The alternative way of deriving this isotropy group is by making use of the method in [99] where one needs to solve the infinitesimal equation

$$M_{\mu\nu}\gamma^{\mu\nu}(1 + e_{1234}) + \Lambda_{r'}\rho^{r'}(1 + e_{1234}) = 0 . \tag{2.31}$$

In this equation $M_{\mu\nu}$ parametrise the infinitesimal $Spin(5,1)$ transformations and $\Lambda_{r'}$ parametrise the infinitesimal $Sp(1)$ transformations. Solving this equation we find the following constraints

$$\begin{aligned} M_{-+} = M_{+\alpha} = M_{+\bar{\alpha}} = 0 , \quad M_{\alpha}{}^{\alpha} + i\Lambda_3 = 0 , \\ M_{\bar{1}\bar{2}} - M_{12} + i\Lambda_2 = 0 , \quad M_{\bar{1}\bar{2}} + M_{12} + \Lambda_1 = 0 , \end{aligned} \tag{2.32}$$

and the components $M_{-\alpha}$ and $M_{-\bar{\alpha}}$ are not constrained apart from being Hermitian conjugates of each other. From these constraints one can infer that the spinor $1 + e_{1234}$ is left invariant by the Lie algebra $\mathfrak{su}(2) \times \mathfrak{su}(2) \oplus \mathbb{H}$. Furthermore, this highlights the possible isotropy groups of the spinor under consideration; using this result, with verification from the previous method, we conclude that the isotropy group of one spinor is $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$.

To continue we have to determine the action of $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ on $\mathbb{H}^2$. Decomposing $\mathbb{H}^2 = \mathbb{R} \oplus \mathrm{Im}\mathbb{H} \oplus \mathbb{H}$, where $\mathbb{R}$ is chosen to be along the first invariant spinor, we find that the action of the isotropy group is

$$\mathrm{Im}\mathbb{H} \oplus \mathbb{H} \rightarrow a\mathrm{Im}\mathbb{H}\bar{a} \oplus b\mathbb{H}\bar{a} \ , \tag{2.33}$$

where $(a, b) \in Sp(1) \cdot Sp(1)$ and $\bar{a}$ is the quaternionic conjugate of $a \in Sp(1)$. Now, there are two possibilities; either the second invariant spinor lies in $\mathrm{Im}\mathbb{H}$ or in $\mathbb{H}$. It cannot lie in both because if it does then there is a $\mathbb{H}$ transformation in $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ such that the component in $\mathrm{Im}\mathbb{H}$ can be set to zero. Now, if the second spinor is in $\mathrm{Im}\mathbb{H}$ then it can be arranged so that a representative is given by $i(1 - e_{1234})$. This is because $Sp(1) \cdot Sp(1) \subset Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ acts on $\mathrm{Im}\mathbb{H}$ with the three dimensional representation and so can rotate any spinor to a particular direction. The isotropy group of the two spinors reduces to $U(1) \cdot Sp(1) \ltimes \mathbb{H}$. On the other hand, if the non-trivial component lies in $\mathbb{H}$ then a representative spinor can be chosen as $e_{15} + e_{2345}$ as $Sp(1) \cdot Sp(1) \subset Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ acts on $\mathbb{H}$ with left and right quaternionic multiplication allowing any spinor to be rotated to a particular direction. The isotropy group in this case is $Sp(1)$.

There are two possible routes to take from here, depending on which of the two isotropy groups we decompose the space $\mathbb{H}^2$ with respect to. In both cases we will find that if there are three invariant spinors, then there always exist an additional invariant spinor. Let us first consider the case when the two invariant spinors have isotropy group $U(1) \cdot Sp(1) \ltimes \mathbb{H}$. Then the space of spinors decomposes under the action of this group as $\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{H}$ where the first $\mathbb{R}^2$ is spanned by the two invariant spinors we have already mentioned. Therefore, the third invariant spinor lies in $\mathbb{R}^2 \oplus \mathbb{H}$. Again there are two options; either the third invariant spinor is in $\mathbb{R}^2$ or in $\mathbb{H}$. It cannot be in both because if it does then there is a $\mathbb{H}$ transformation in $U(1) \cdot Sp(1) \ltimes \mathbb{H}$ such that the component in $\mathbb{R}^2$ can be set to zero. Now, if the third spinor is in $\mathbb{R}^2$ then we can always use the $U(1) \subset U(1) \cdot Sp(1) \ltimes \mathbb{H}$ to arrange so that a representative for this is given by $e_{12} - e_{34}$, but this means that we get an additional singlet in the decomposition of spinors for free. Therefore, we find in this case that there are four invariant spinors with isotropy group $Sp(1) \ltimes \mathbb{H}$ and with representative spinors given by the first line in (2.16).

On the other hand, if we choose the first two invariant spinors to have isotropy group $Sp(1)$ the space of spinors decomposes as $\mathbb{R} \oplus \mathrm{Im}\mathbb{H} \oplus \mathbb{R} \oplus \mathrm{Im}\mathbb{H}$ under the action of this group. The subspace $\mathbb{R} \oplus \mathbb{R}$ is spanned by the first two invariant spinors. In this case, the third invariant spinor lies in $\mathrm{Im}\mathbb{H} \oplus \mathrm{Im}\mathbb{H}$, however, we find that this also leads to two additional singlets, and therefore, four linearly independent invariant spinors in total, with isotropy group $U(1)$. The representative spinors for the additional singlets can be chosen as $i(1 - e_{1234})$ and $i(e_{15} - e_{2345})$.

The isotropy group of more than four linearly independent spinors is $\{1\}$. The results of this section have been summarised in table 2.1.

N	Isotropy Groups	Spinors
1	$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234}$
2	$(U(1) \cdot Sp(1)) \ltimes \mathbb{H}$	$1 + e_{1234}\ ,\ i(1 - e_{1234})$
4	$Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234}\ ,\ i(1 - e_{1234})\ ,\ e_{12} - e_{34}\ ,\ i(e_{12} + e_{34})$
2	$Sp(1)$	$1 + e_{1234}\ ,\ e_{15} + e_{2345}$
4	$U(1)$	$1 + e_{1234}\ ,\ i(1 - e_{1234})\ ,\ e_{15} + e_{2345}\ ,\ i(e_{15} - e_{2345})$

Table 2.1: The first column gives the number of invariant spinors, the second column the associated isotropy groups and the third representatives of the invariant spinors. Observe that if 3 spinors are invariant, then there is a fourth one. Moreover the isotropy group of more than 4 spinors is the identity.

2.3.2 Descendants

From the analysis we have carried out above one can see that the gravitino KSE has solutions that leave 1, 2, 4 and 8 spinors invariant, see table 2.1. These are the parallel spinors and in each case the holonomy of the supercovariant derivative reduces to one that is contained in the isotropy group of the invariant spinors. Supersymmetric backgrounds where all the parallel spinors, given in table 2.1, also solve the remaining Killing spinor equations are referred to as Killing, and this set of solutions form a particularly distinguished class of solutions. This is because it is not necessarily the case that solutions of the gravitino KSE are also solutions of the other three KSEs. Typically, only some or a particular linear combination of the parallel spinors are Killing. Backgrounds where there are less Killing than parallel spinors will be called descendant backgrounds. Normally the identification of such backgrounds requires an extensive analysis as was the case in the investigation of the descendant solutions of heterotic supergravity in [45]. The analysis of the descendant solutions of six dimensional supergravity is less involved. Although we will aim to give a thorough and self contained analysis of the descendant solutions for six dimensional supergravity we will avoid specific details, see [45] for full specific details of the investigation of descendants.

We will see that there are many descendants but in most cases the Killing spinors of the descendants are given in terms of the parallel spinors of table 2.1. Such descendant backgrounds are special cases of solutions for which all parallel spinors are Killing. Our aim in these sections will be to see if there are background which have Killing spinors that differ from those given in table 2.1. If they exist, such backgrounds will be called independent descendant solutions or simply "independent".

We first note that in all cases if a solution has just one Killing spinor, irrespective of the number of parallel spinors, it is always possible to rotate it so that it is identified with $1 + e_{1234}$. Therefore, such descendant backgrounds are included in those for which $1 + e_{1234}$ is both a parallel and Killing spinor and so they are not independent. Using this, the cases we have to examine are those with two or more Killing and with four or more parallel spinors.

2.3.3 Descendants of four parallel spinors

There are two cases to consider depending on whether the isotropy group of the four parallel spinors is $Sp(1) \ltimes \mathbb{H}$ or $U(1)$. We will consider them in turn.

$Sp(1) \ltimes \mathbb{H}$

In order to identify descendant solutions we need to determine the sigma group [45]. The sigma group is the group which can act non-trivially on the space of parallel spinors and preserves the subspace spanned by them. For the purpose of the spinorial geometry method it plays the same role as the gauge group when investigating descendant solutions, see [45] for a detailed discussion.

If the isotropy group of the four parallel spinors is $Sp(1) \ltimes \mathbb{H}$, then the sigma group [45] is $Spin(1,1) \times Sp(1) \cdot Sp(1)$. This is the group which leaves these four parallel spinors invariant and will play the same role as the gauge group for the purpose of the spinorial geometry method, see [45]. The $Spin(1,1)$ is generated by γ^{+-}, one of the $Sp(1)$ are generated by the generators in (2.17) and the other $Sp(1)$ group is generated in a similar way to (2.17) but with the gamma matrices in the directions $1, 2, 6, 7$.

First we consider the case where there are four parallel spinors but only two Killing spinors. The subgroup $Sp(1) \cdot Sp(1) = SO(4)$ acts with the vector representation on the four parallel spinors. In such a case, it is always possible to arrange so that the first two Killing spinors are

$$1 + e_{1234} \ , \quad i(1 - e_{1234}) \ . \tag{2.34}$$

Therefore such solutions are special cases of backgrounds with two supersymmetries associated with two parallel spinors with isotropy group $U(1) \cdot Sp(1)) \ltimes \mathbb{H}$, and so are not independent.

Next we consider the possibility of a solution with three Killing spinors. Once again the subgroup $Sp(1) \cdot Sp(1)$ of the sigma group acts with the vector representation allowing the three Killing spinors to be chosen as

$$1 + e_{1234} \ , \quad i(1 - e_{1234}) \ , \quad e_{12} - e_{34} \ . \tag{2.35}$$

We will see that if the gravitino, tensorini and gaugini KSEs admit (2.35) as a solution, then they also admit $i(e_{12} - e_{34})$ as a solution. So all the parallel spinors of this case solve three out of the four KSEs. It then remains to investigate the hyperini KSE. We shall see that the conditions that arise from the hyperini KSE evaluated on (2.35) are in fact different from those that one finds when the same KSE is evaluated on all four $Sp(1) \ltimes \mathbb{H}$-invariant spinors. As a result, the KSEs allow for backgrounds that preserve three supersymmetries. However, the existence of such backgrounds depends also on the field equations.

$U(1)$

We now move on to investigate the case for which the four parallel spinors have isotropy group $U(1)$. The sigma group [45] in this case is $Spin(3,1) \times U(1)$. One way to see this is to treat the directions $2, 3$ and 4 in the $U(1)$-invariant spinors in table 2.1 as auxiliary and suppress them. Then the spinors can be identified with the Majorana spinors of $Spin(3,1)$. The $U(1)$ subgroup of the sigma group is generated by the spin transformations along the auxiliary directions. The analysis of the orbits of the sigma group is then identical to that of the gauge group of four dimensional supergravity in [104]. In this case there are two different cases where descendants with two supersymmetries arise. However, one can arrange such that the Killing spinors of the two cases are identical to the parallel spinors of table 2.1 with isotropy groups $U(1) \cdot Sp(1) \ltimes \mathbb{H}$ and $Sp(1)$, respectively. Therefore, both cases are special cases of other backgrounds with less parallel spinors and so are not independent.

The other case we consider are backgrounds with three Killing spinors. The existence of such backgrounds depends on the details of the KSEs. Without going into detail, see [45, 104, 48] for details, the sigma group can be used to choose the three Killing spinors as

$$1 + e_{1234} \ , \ i(1 - e_{1234}) \ , \ e_{15} + e_{2345} \ . \tag{2.36}$$

Once again it is easy to check that if (2.36) solves the gravitino, tensorini and gaugini KSEs, then $i(e_{15} - e_{2345})$ is also a solution. Therefore, all four $U(1)$-invariant spinors once more solve three out of the four KSEs. It now remains to examine the hyperini KSE. However, unlike the previous case, the hyperini KSE evaluated on (2.36) gives the same conditions as those obtained for all four $U(1)$-invariant spinors. Thus, in this case there are no descendants that preserve three supersymmetries.

2.3.4 Descendants of Eight Parallel Spinors

Finally we examine the descendants of backgrounds with eight parallel spinors. To do this it is most convenient to consider the KSEs in the following order

$$\text{gravitino} \rightarrow \text{gaugini} \rightarrow \text{tensorini} \rightarrow \text{hyperini} \ . \tag{2.37}$$

We have already stated that the gravitino KSE admits eight parallel spinors. Thus, it remains to investigate the other three KSEs.

Gaugini

The solutions of the gaugini KSE are spinors which are invariant under some subgroup of $Spin(5,1) \cdot Sp(1)$. This is because the gauge field and moment maps, $\mu_{r'}$, can be viewed as maps from $\mathfrak{spin}(5,1) \oplus \mathfrak{sp}(1)$ to the Lie algebra of the gauge group, where $\mathfrak{spin}(5,1) = \Lambda^2(\mathbb{R}^{5,1})$. But all such spinors and their isotropy groups have been tabulated in table 2.1. Therefore, the gaugini KSE can preserve 1, 2 (2), 4 (2) and 8 out of the total of eight parallel spinors, where the number in the parenthesis states the multiplicity of each case.

Having established that the gaugino KSE has solutions given by the spinors in table 2.1, we are left to investigate the remaining two KSEs. If the gaugini KSE has up to four solutions, the investigation of the descendants for the tensorini and hyperini KSE follow the same argument as that presented in section 2.3.3. In particular, there is one descendant with three supersymmetries which arises in the case of four Killing spinors with isotropy group $Sp(1) \ltimes \mathbb{H}$. The three Killing spinors are given in (2.35), but this case can be thought of as a special case of backgrounds with four parallel spinors and Killing spinors as in (2.35). Since we have dealt with all descendants of the gaugini KSE from now on we shall take that the gaugini KSE preserves all eight parallel spinors.

Tensorini

Assuming that the gravitino and gaugini KSEs admit eight Killing spinors we continue now by discussing the solutions of the tensorini KSE. First we note that the tensorini KSE commutes with all three of the ρ operations given in (2.17). This means that it preserves either four or eight supersymmetries. In addition, when it preserves four supersymmetries the Killing spinors can be given in terms of the $Sp(1) \ltimes \mathbb{H}$-invariant spinors of table 2.1. We can then use this to solve the hyperini KSE to find backgrounds that preserve 1, 2, 3 and 4 supersymmetries. All of these are special cases of the solutions that we have already investigated above. In particular, if the solutions preserve one supersymmetry, then it is a special case of backgrounds with one parallel spinor which is also Killing. If the background preserves two supersymmetries then they are special cases of solutions with two parallel spinors which are also Killing and have isotropy group $U(1) \cdot Sp(1) \ltimes \mathbb{H}$. For three supersymmetries the backgrounds are special cases of those with $Sp(1) \ltimes \mathbb{H}$-invariant parallel spinors and the three Killing spinors are given in (2.35). The case of four supersymmetries is included in that for which the four $Sp(1) \ltimes \mathbb{H}$-invariant parallel spinors are also Killing. This concludes the analysis of the descendants of the tensorini KSE and from now we shall assume that the tensorini KSE admits eight Killing spinors.

Hyperini

Let us now assume that the gravitino, gaugini and tensorini KSEs admit eight Killing spinors. We finally need to investigate solutions of the hyperini KSE. In order to do this we need to identify the orbits of the sigma group, which in this case is $Spin(5,1) \cdot Sp(1)$, on the space of spinors. The descendants preserving one supersymmetry have already been considered. In this case the Killing spinor can be identified with $1 + e_{1234}$. To investigate the case with two supersymmetries, we first recall that the sigma group $Spin(5,1) \cdot Sp(1)$ has one orbit in the space of symplectic-Majorana spinors with isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$, and the representative can be chosen as $1 + e_{1234}$. The action of the isotropy group on the space of spinors is given in (2.33). This isotropy group has two non-trivial orbits on the space of spinors and the representatives can be chosen as either $i(1 - e_{1234})$ or $e_{15} + e_{2345}$, as was discussed before. From this we can see that solutions with Killing spinors $1 + e_{1234}$ and $i(1 - e_{1234})$ or $1 + e_{1234}$ and $e_{15} + e_{2345}$ are not independent descendants. Thus, there are no independent descendants with two supersymmetries.

We now consider the case with three supersymmetries. There are two cases to investigate. Firstly, we consider the case where the first two spinors have isotropy group $U(1) \cdot Sp(1) \ltimes \mathbb{H}$. This group has two different orbits on the rest of the spinors where the representatives can be chosen as $e_{12} - e_{34}$ and $e_{15} + e_{2345}$, respectively. However, these two cases are not new as the Killing spinors are identical to those found in (2.35) and (2.36), respectively. In addition, one can show that if the hyperini KSE admits (2.36) as Killing spinors, then it preserves four supersymmetries with Killing spinors the $U(1)$-invariant spinors of table 2.1.

The second case is when the isotropy of the first two Killing spinors is $Sp(1)$. It can be seen from (2.33) that $Sp(1)$ acts with two copies of the 3-dimensional representation on the remaining six spinors. As a result it can be arranged such that the third spinor can be chosen in such a way that the three Killing spinors are

$$1 + e_{1234} \;, \quad e_{15} + e_{2345} \;, \quad c_1 i(1 - e_{1234}) + i c_2(e_{15} - e_{2345}) + c_3(e_{25} - e_{1345}) \;, \tag{2.38}$$

where c's are constants. If $c_1 = 0$, then the third spinor can be simplified further by choosing $c_3 = 0$. As we will see, this does not give rise to a new descendant. The hyperini KSE evaluated on the above spinors implies that either it preserves four supersymmetries with Killing spinors as the $U(1)$-invariant spinors of table 2.1 or it preserves all eight supersymmetries. This depends on the coefficients c, which we discuss further in section 2.9.

It remains to investigate descendants with four supersymmetries. First suppose that the first three Killing spinors are chosen as in (2.35), which have isotropy group $Sp(1) \ltimes \mathbb{H}$. This has two orbits on the remaining spinors. In one case the representatives can be chosen such that the four Killing spinors are given by the four

$Sp(1) \ltimes \mathbb{H}$-invariant spinors of table 2.1 and in the other case they can be chosen as

$$1 + e_{1234} \ , \quad i(1 - e_{1234}) \ , \quad e_{12} - e_{34} \ , \quad e_{15} + e_{2345} \ . \tag{2.39}$$

This can potentially be a new descendant. However, it turns out that if the hyperini KSE preserves the above four spinors, then it preserves all eight supersymmetries.

Next suppose that the first three Killing spinors are given in (2.36), these have isotropy group $U(1)$. Then, the fourth spinor can be chosen as

$$c_1(e_{12} - e_{34}) + c_2 i(e_{15} - e_{2345}) + c_3(e_{25} - e_{1345}) + c_4 i(e_{25} + e_{1345}) \ . \tag{2.40}$$

It turns out that depending on the choice of the coefficients c the hyperini KSE preserves either four supersymmetries with Killing spinors given by the $U(1)$-invariant spinors of table 2.1 or all eight supersymmetries. So again there are no new descendants. A similar conclusion also holds for the case when the third Killing spinor is chosen as in (2.38).

To conclude, if the isotropy group of parallel spinors is $\{1\}$ then there are descendant backgrounds which preserve 1, 2, 3 and 4 supersymmetries. However, these are not independent. All of them appear as special cases of backgrounds that admit less parallel spinors. The results for all possible descendants have been tabulated in table 2.2.

$\mathrm{hol}(\mathcal{D})$	N
$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	1
$U(1) \cdot Sp(1) \ltimes \mathbb{H}$	$*$, 2
$Sp(1) \ltimes \mathbb{H}$	$*$, $*$, 3, 4
$Sp(1)$	$*$, 2
$U(1)$	$*$, $*$, $-$, 4
$\{1\}$	$*$,$*$,$*$, $*$,$-$, $-$, $-$, 8

Table 2.2: In the columns are the holonomy groups that arise from the solution of the gravitino KSE and the number N of supersymmetries, respectively. $*$ entries denote the cases that occur but are special cases of others with the same number of supersymmetries but with less parallel spinors. The $-$ entries denote cases which do not occur. The Killing spinors for $N = 1, 2, 4$ are the same as those given in table 2.1 while for $N = 3$ in (2.35).

2.4 N=1 Backgrounds

In the next couple of sections we will consider the possible backgrounds that can arise in some detail. In particular, we will solve all four of the KSEs equations and discuss what the constraints coming from these mean for the matter content of the theory and the geometry of spacetime. In each case we will start by outlining the constraints imposed by each of the KSEs and then move onto discuss the spacetime geometry. We begin here with backgrounds preserving one supersymmetry.

2.4.1 Gravitino KSE

As the gauge group of the theory is the same as the holonomy of the supercovariant connection of generic backgrounds, the Killing spinor of $N = 1$ backgrounds can be chosen as $\epsilon = 1 + e_{1234}$, more details can be found in [44, 45]. The gravitino KSE requires that this spinor is parallel. As a result the holonomy of $\mathcal{D}$ reduces to a subgroup of the isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ of the parallel spinor, i.e.

$$\mathrm{hol}(\mathcal{D}) \subseteq Sp(1) \cdot Sp(1) \ltimes \mathbb{H} \ . \tag{2.41}$$

This is the full content of the gravitino KSE. The restrictions that this imposes on the geometry will be examined later.

2.4.2 Gaugini KSE

Recall that the gaugini KSE is

$$\left(\frac{1}{4}F^{a'}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}\mu^{a'}_{r'}\rho^{r'}\right)\epsilon \ = \ 0 \ . \tag{2.42}$$

To solve this for the $N = 1$ case we substitute in the spinor $\epsilon = 1 + e_{1234}$ and sum over the repeated indices. After that we determine the action of the gamma matrices and the ρ operators on the spinor. This calculation is most easily done using the oscillator basis for the gamma matrices described in section 1.4.2. We are then left with an algebraic equation. In this algebraic equation we set the coefficients before each spinor basis to zero in order to find the constraints on the components of $F_{\mu\nu}$ and $\mu_{r'}$. After applying this technique we find the conditions arising from the gaugini KSE are

$$F^{a'}_{+i} = F^{a'}_{+-} = 0 \ , \quad F^{a'}_{\alpha}{}^{\alpha} + i\mu^1 = 0 \ , \quad 2F^{a'}_{12} + \mu^2 - i\mu^3 = 0 \ . \tag{2.43}$$

Note that the gauge field strength vanishes along one of the light-cone directions, and $F^{a'}_{-i}$ are not constrained.

2.4.3 Tensorini KSE

The tensorini KSE is given by

$$\left(\frac{i}{2}T^{M}_{\mu}\gamma^{\mu} - \frac{i}{24}H^{M}_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right)\epsilon \ = \ 0 \ , \tag{2.44}$$

substituting the spinor $\epsilon = 1 + e_{1234}$ and repeating the procedure that was done for the gaugini KSE above one obtains the following constraints

$$T^{\underline{M}}_{+} = 0 \ , \quad H^{\underline{M}}_{+\alpha}{}^{\alpha} = H^{\underline{M}}_{+\alpha\beta} = 0 \ ,$$
$$T^{\underline{M}}_{\bar{\alpha}} - \frac{1}{2}H^{\underline{M}}_{-+\bar{\alpha}} - \frac{1}{2}H^{\underline{M}}_{\bar{\alpha}\beta}{}^{\beta} = 0 \ . \tag{2.45}$$

As we have already mentioned, the tensorini KSE commutes with the Clifford algebra operations $\rho^{r'}$ in (2.17). As a result, if the tensorini KSE admits a solution ϵ, then $\rho^{r'}\epsilon$ also solve the tensorini KSE. This means the four spinors

$$1 + e_{1234} \ , \quad \rho^{r'}(1 + e_{1234}) \ , \quad r' = 1, 2, 3, \tag{2.46}$$

are all solutions to the tensorini KSE with the same constraints as in (2.45). Moreover, the 3-form field strengths $H^{\underline{M}}$ are further restricted by the self-duality condition given in (2.12), which we discuss later.

2.4.4 Hyperini KSE

The hyperini KSE is given by

$$i\gamma^{\mu}\epsilon_{\underline{A}}V^{\underline{aA}}_{\mu} \;=\; 0 \ . \tag{2.47}$$

To understand the hyperini KSE, one has to identify the $\epsilon_{\underline{A}}$ components of the Killing spinor in the context of spinorial geometry. In our notation $\epsilon^1 = 1$ and $\epsilon^2 = e_{1234}$ and since $\epsilon_1 = -\epsilon^2$ and $\epsilon_2 = \Gamma_{34}\epsilon^1$, one has $\epsilon_1 = -e_{1234}$ and $\epsilon_2 = e_{34}$. Substituting these into the KSE, one finds the conditions

$$V^{\underline{aA}}_{+} = 0 \ , \quad -V^{\underline{a1}}_{1} + V^{\underline{a2}}_{\bar{2}} = 0 \ , \quad V^{\underline{a1}}_{2} + V^{\underline{a2}}_{\bar{1}} = 0 \ . \tag{2.48}$$

Expressing the coefficients of the KSEs in terms of the fundamental fields as in (2.7), it is clear from the first condition in (2.48) that

$$D_{+}\phi^{\underline{I}} = 0 \ . \tag{2.49}$$

2.4.5 Geometry

Form Spinor Bilinears

In order to investigate the geometry of spacetime further, one has to compute the form spinor bilinears. The form spinor bilinears of two spinors in six dimensions are given by

$$\tau = \frac{1}{k!}B(\epsilon_1, \gamma_{\mu_1\ldots\mu_k}\epsilon_2)\ e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \ , \tag{2.50}$$

where τ is a k-form and B is the Majorana inner product as for the heterotic supergravity [44] in ten dimensions. We have discussed this in section 1.4, where we denoted this inner product with M_2 as in equation (1.37). Assuming that ϵ_1 and ϵ_2 satisfy the gravitino KSE, it is easy to see that

$$\hat{\nabla}_\nu \tau = 0 \ . \tag{2.51}$$

The form τ is covariantly constant with respect to $\hat{\nabla}$ and the $Sp(1)$ connection $\mathcal{C}^{r'}$ does not contribute in the parallel transport equation.

On the other hand, one may also consider the $\mathfrak{sp}(1)$-valued form bilinears

$$\tau^{r'} = \frac{1}{k!} B(\epsilon_1, \gamma_{\mu_1 \dots \mu_k} \rho^{r'} \epsilon_2) \ e^{\mu_1} \wedge \dots \wedge e^{\mu_k} \ . \tag{2.52}$$

Assuming again that ϵ_1 and ϵ_2 satisfy the gravitino KSE, one finds that

$$\hat{\nabla}_\nu \tau^{r'} + 2 \, \mathcal{C}_\nu^{s'} \epsilon^{r'}{}_{s't'} \tau^{t'} = 0 \ . \tag{2.53}$$

Observe that the $\mathfrak{sp}(1)$-valued form bilinears are twisted with respect to the $Sp(1)$ connection $\mathcal{C}^{r'}$. So $\nabla_\nu \tau^{r'}$ are not forms but rather vector bundle valued forms. However, for simplicity in what follows, we shall refer to both τ and $\tau^{r'}$ as forms.

Example

As an example let us calculate the bilinear 1-form associated to $\epsilon = 1 + e_{1234}$, which will be needed in the investigation of backgrounds that are to follow. Using (2.50) we have

$$\tau = B(\epsilon, \gamma_\mu \epsilon) \ e^\mu \ , \tag{2.54}$$

expanding this we find

$$\begin{aligned} \tau \ = \ & B(\epsilon, \gamma_- \epsilon) \ e^- + B(\epsilon, \gamma_+ \epsilon) \ e^+ + B(\epsilon, \gamma_1 \epsilon) \ e^1 \\ & + B(\epsilon, \gamma_2 \epsilon) \ e^2 + B(\epsilon, \gamma_{\bar{1}} \epsilon) \ e^{\bar{1}} + B(\epsilon, \gamma_{\bar{2}} \epsilon) \ e^{\bar{2}} \ . \end{aligned} \tag{2.55}$$

Recall that the inner product is defined as

$$B(\epsilon_1, \epsilon_2) = \langle \Gamma_{06789} (\epsilon_1)^*, \epsilon_2 \rangle \ , \tag{2.56}$$

and

$$\Gamma_{06789} (1 + e_{1234}) = -(e_5 + e_{12345}) \ . \tag{2.57}$$

Using this and the action of γ_μ on ϵ we find only the first term in (2.55) to be non-vanishing and it gives

$$
\begin{aligned}
\tau &= \langle -(e_5 + e_{12345}), \sqrt{2}(e_5 + e_{12345}) \rangle \; e^- \\
&= -2\sqrt{2} e^- \; .
\end{aligned} \tag{2.58}
$$

Therefore, we find that there is a 1-form which is given by

$$
e^- \; . \tag{2.59}
$$

The same method can be used to calculate the remaining form bilinears in addition to the twisted bilinear forms. One more thing we mention before discussing the $N = 1$ backgrounds is the relation between the frames $e^0, e^1, \ldots, e^5$ and $e^-, e^+, e^\alpha, e^{\bar{\alpha}}$,

$$
\begin{aligned}
e^- &= \frac{1}{\sqrt{2}}(-e^0 + e^3) \; , \quad e^+ = \frac{1}{\sqrt{2}}(e^0 + e^3) \; , \\
e^1 &= \frac{1}{\sqrt{2}}(e^1 + ie^4) \; , \quad e^2 = \frac{1}{\sqrt{2}}(e^2 + ie^5) \; , \\
e^{\bar{1}} &= \frac{1}{\sqrt{2}}(e^1 - ie^4) \; , \quad e^{\bar{2}} = \frac{1}{\sqrt{2}}(e^2 - ie^5) \; ,
\end{aligned} \tag{2.60}
$$

which we often use. Note also that we will often use a tilde on top of the real directions, i.e. those on the rhs, to distinguish them from the complex ones, this will be clearer when we discuss it later.

Spacetime Geometry of N=1 Backgrounds

In this case we have to find the bilinears associated with the spinor $1+e_{1234}$. Putting this into (2.50) and (2.52) and following the example given above we find the algebraic independent bilinears of backgrounds preserving one supersymmetry to be given by

$$
e^- \; , \quad e^- \wedge \omega_I \; , \quad e^- \wedge \omega_J \; , \quad e^- \wedge \omega_K \; , \tag{2.61}
$$

where e^- is a null one-form and

$$
\omega_I = -i\delta_{\alpha\bar{\beta}} e^\alpha \wedge e^{\bar{\beta}} \; , \quad \omega_J = -e^1 \wedge e^2 - e^{\bar{1}} \wedge e^{\bar{2}} \; , \quad \omega_K = i(e^1 \wedge e^2 - e^{\bar{1}} \wedge e^{\bar{2}}) \; . \tag{2.62}
$$

The three 2-forms ω_I, ω_J and ω_K are Hermitian forms in the directions transverse to the light-cone. In what follows, we also set $\omega^1 = \omega_I$, $\omega^2 = \omega_J$ and $\omega^3 = \omega_K$.

The conditions that the gravitino KSE imposes on the spacetime geometry can be rewritten as

$$
\hat{\nabla}_\mu e^- = 0 \; , \quad \hat{\nabla}_\mu (e^- \wedge \omega^{r'}) + 2\, \mathcal{C}_\mu^{s'} \epsilon^{r'}{}_{s't'} (e^- \wedge \omega^{t'}) = 0 \; . \tag{2.63}
$$

Therefore, we can see the 1-form is parallel with respect to $\hat{\nabla}$, whereas the 3-forms are twisted with respect to the $Sp(1)$ connection. The integrability conditions to these parallel transport equations are

$$\hat{R}_{\mu_1\mu_2,+\nu} = 0 \ , \qquad - \hat{R}_{\mu_1\mu_2,}{}^k{}_i \omega^{r'}{}_{kj} + (j,i) + 2\mathcal{F}^{s'}_{\mu_1\mu_2}\epsilon^{r'}{}_{s't'}\omega^{t'}_{ij} = 0 \ . \tag{2.64}$$

In addition to this, the torsion H has to be anti-self-dual in six dimensions as we pointed out in (2.12). The conditions that come from the anti-self-duality can be written as

$$H_{+\alpha\beta} = H_{+\alpha}{}^{\alpha} = 0 \ , \quad H_{-+\bar{\alpha}} + H_{\bar{\alpha}\beta}{}^{\beta} = 0 \ , \quad H_{-1\bar{1}} - H_{-2\bar{2}} = 0 \ , \quad H_{-1\bar{2}} = 0 \ , \tag{2.65}$$

where $\epsilon_{-+1\bar{1}2\bar{2}} = \epsilon_{013245} = -1$. Notice also that from the four dimensional perspective H_{+ij} is an anti-self-dual while H_{-ij} is a self-dual 2-form, respectively. Using (2.12) we can demonstrate this explicitly for H_{+ij} where we have

$$H_{+ij} = \frac{1}{2}\epsilon_{+ij-kl}H^{-kl} \ , \tag{2.66}$$

which in turn gives

$$H_{+ij} = -\frac{1}{2}\epsilon_{ij}{}^{kl}H_{+kl} \ , \tag{2.67}$$

where we have used $\epsilon_{-+ijkl} = \epsilon_{ijkl}$, similarly

$$H_{-ij} = \frac{1}{2}\epsilon_{ij}{}^{kl}H_{-kl} \ . \tag{2.68}$$

To specify the spacetime geometry, we have to solve (2.63) subject to (2.65). For this we adapt a frame basis on the spacetime such that one of the light-cone frames is the parallel 1-form e^-, i.e. the metric is written as

$$ds^2 = 2e^-e^+ + \delta_{ij}e^ie^j \ . \tag{2.69}$$

The first condition in (2.63) then implies that the dual vector field X to e^- is *Killing* and

$$de^- = i_X H \ . \tag{2.70}$$

Using these we can write the torsion 3-form as

$$H = e^+ \wedge de^- + \frac{1}{2}H_{-ij}e^- \wedge e^i \wedge e^j + \tilde{H} \ , \quad \tilde{H} = \frac{1}{3!}\tilde{H}_{ijk}e^i \wedge e^j \wedge e^k \ . \tag{2.71}$$

Moreover, the anti-self-duality of H can be used to relate the $\tilde{H}$ component to de^-.

In particular, we find that

$$\tilde{H} = -\frac{1}{3!}(de^-)_{-\ell}\, \epsilon^{\ell}{}_{ijk}\, e^i \wedge e^j \wedge e^k \ . \tag{2.72}$$

This solves the first condition in (2.63). To solve the remaining three conditions, we consider the parallel transport equation in (2.63) first along the light-cone directions. Since H_{+ij} is anti-self-dual, one can show that

$$\mathcal{D}_+\omega^{r'} = \nabla_+\omega^{r'} + 2\,\mathcal{C}_+^{s'}\epsilon^{r'}{}_{s't'}\omega^{t'} = 0 \ . \tag{2.73}$$

This condition can be used to express $\mathcal{C}_+$ in terms of the geometry of spacetime. Next we consider the $-$ light-cone direction to find

$$\mathcal{D}_-\omega^{r'}_{ij} = \nabla_-\omega^{r'}_{ij} - H_-{}^k{}_{[i}\omega^{r'}_{j]k} + 2\,\mathcal{C}_-^{s'}\epsilon^{r'}{}_{s't'}\omega^{t'}_{ij} = 0 \ . \tag{2.74}$$

Since H_{-ij} is self-dual, this implies that it can be written as

$$H_{-ij} = w_{r'}\omega^{r'}_{ij} \ , \tag{2.75}$$

for some functions $w_{r'}$, and $\omega^{r'}$ act as a constant basis of self-dual 2-forms in $\mathbb{R}^4$. Substituting this expression into (2.74) we find

$$\nabla_-\omega^{r'}_{ij} + w^{s'}\epsilon^{r'}{}_{s't'}\omega^{t'}_{ij} + 2\,\mathcal{C}_-^{s'}\epsilon^{r'}{}_{s't'}\omega^{t'}_{ij} = 0 \ , \tag{2.76}$$

where we have made use of the fact that

$$\omega^{r'}_{ij} = \delta_{ik}(I^{r'})^k{}_j \ , \tag{2.77}$$

where $I^{r'}$ are three almost complex structures associated to the three Hermitian forms which satisfy the algebra of the imaginary quaternions

$$(I^{r'})^i{}_k(I^{s'})^k{}_j = -\delta^{r's'}\delta^i{}_j + \epsilon^{r's'}{}_{t'}(I^{t'})^i{}_j \ . \tag{2.78}$$

The condition in (2.76) can be interpreted as a condition which relates $\mathcal{C}_-^{s'}$ to the H_{-ij} components of the torsion. As a result, it can be solved to express H_{-ij} in terms of other fields and the geometry of spacetime. In particular, after solving for $w^{t'}$ we find

$$w^{t'} = -\frac{1}{8}\big(\omega^{r'}_{kl}\nabla_-\omega^{s'kl}\epsilon_{r's'}{}^{t'} + 2\mathcal{C}_-^{t'}\big) \ , \tag{2.79}$$

which in turn means

$$H_{-ij} = -\frac{1}{8}\big(\omega^{r'}_{kl}\nabla_-\omega^{s'kl}\epsilon_{r's'}{}^{t'} + 2\mathcal{C}_-^{t'}\big)\omega_{t'ij} \ . \tag{2.80}$$

To determine the conditions imposed on the geometry from the gravitino KSE in the directions transverse to the light-cone, we first observe that a generic metric connection in four dimensions has holonomy contained in $Sp(1)\cdot Sp(1)$. Therefore, the only condition required is the identification of the $Sp(1)$ part of the metric spacetime connection with the $Sp(1)$ part of the induced connection from the Quaternionic Kähler manifold of the hyper-multiplets. This also follows from the integrability conditions (2.64).

To summarise, we have found that the spacetime admits a null Killing vector field X whose rotation in the directions transverse to the light-cone is anti-self-dual. The geometry is restricted by (2.73). Furthermore, (2.76) relates the self-dual H_{-ij} component of the torsion to a component of the induced $Sp(1)$ connection from the Quaternionic Kähler manifold of the hypermultiplets as in (2.80). The metric and torsion of the spacetime can be written as

$$\begin{aligned} ds^2 &= 2e^-e^+ + \delta_{ij}e^ie^j \ , \\ H &= e^+\wedge de^- - \big(\frac{1}{16}\omega^{r'}_{kl}\nabla_-\omega^{s'kl}\epsilon_{r's'}{}^{t'} + \mathcal{C}^{t'}_-\big)\,\omega_{t'ij}\ e^-\wedge e^i\wedge e^j \\ &\qquad -\frac{1}{3!}(de^-)_{-\ell}\,\epsilon^{\ell}{}_{ijk}\ e^i\wedge e^j\wedge e^k \ . \end{aligned} \tag{2.81}$$

The remaining conditions that come from the KSEs are restrictions on the matter content of the theory. We begin with the gaugino KSE. To analyse the conditions further, one can choose the gauge

$$A_+ = 0 \ . \tag{2.82}$$

Therefore, using the first two conditions in (2.43), $F^{a'}_{+\mu}=0$, we find that the components of the gauge connections do not depend on the coordinate adapted to the Killing vector field $X=\partial_u$. The components $F^{a'}_{-i}$ are not restricted by the KSE. The components of the field strength in the directions transverse to the light-cone, $F^{r'}_{ij}$, can be decomposed into the self-dual (sd) and anti-self-dual (asd) parts using

$$\begin{aligned} (F^{sd})^{a'}_{ij} &= \frac{1}{2}\left(F^{a'}_{ij}+\frac{1}{2}\epsilon^{kl}{}_{ij}F^{a'}_{kl}\right) \ , \\ (F^{asd})^{a'}_{ij} &= \frac{1}{2}\left(F^{a'}_{ij}-\frac{1}{2}\epsilon^{kl}{}_{ij}F^{a'}_{kl}\right) \ . \end{aligned} \tag{2.83}$$

From the restrictions coming from the KSE we find that the self-dual part of $F^{a'}_{ij}$ is given in terms of the moment maps, $\mu_{r'}$, while the anti-self-dual part is not restricted. So we can write

$$F^{a'} = F^{a'}_{-i}e^-\wedge e^i + \frac{1}{2}\mu_{r'}\omega^{r'} + (F^{\rm asd})^{a'} \ . \tag{2.84}$$

Now let us consider the tensorini KSE. In the gauge (2.82), we can see from the first condition in (2.45) that the tensorini scalars are invariant under the isometries of the spacetime, i.e. they do not depend on the coordinate u. The 3-form field strengths, $H^{\underline{M}}$, are self-dual in six dimensions (2.12). This implies that

$$H^{\underline{M}}_{-\alpha\beta} = H^{\underline{M}}_{-\alpha}{}^{\alpha} = 0 \ , \quad H^{\underline{M}}_{-+\bar{\alpha}} - H^{\underline{M}}_{\bar{\alpha}\beta}{}^{\beta} = 0 \ , \quad H^{\underline{M}}_{+1\bar{1}} - H^{\underline{M}}_{+2\bar{2}} = 0 \ , \quad H^{\underline{M}}_{+1\bar{2}} = 0 \ . \tag{2.85}$$

Combining these conditions with those coming from the tensorini KSE, (2.45), we find that

$$H^{\underline{M}}_{+ij} = 0 \ . \tag{2.86}$$

We also note that $H^{\underline{M}}_{-ij}$ is anti-self-dual in the directions transverse to the light-cone and the remaining components are determined in terms of $T^{\underline{M}}$. Putting these together we therefore have

$$H^{\underline{M}} = \frac{1}{2} H^{\underline{M}}_{-ij}\, e^- \wedge e^i \wedge e^j + T^{\underline{M}}_i e^- \wedge e^+ \wedge e^i - \frac{1}{3!} T^{\underline{M}}_{\ell}\, \epsilon^{\ell}{}_{ijk}\; e^i \wedge e^j \wedge e^k \ , \tag{2.87}$$

where we have used the self-duality of $H^{\underline{M}}$ to relate the $H^{\underline{M}}_{ijk}$ component to the $H^{\underline{M}}_{-+i}$ component.

We can use the definitions of the fundamental fields in (2.7) to obtain some further simplifications. In particular, (2.49) implies that $\mathcal{C}^{r'}_{+} = 0$ and so (2.73) leads to the geometric conditions

$$\nabla_+ \omega^{r'} = 0 \ , \quad r' = 1, 2, 3 \ . \tag{2.88}$$

In addition, $T^{\underline{M}}_i = x^{\underline{M}}_{\underline{r}} \partial_i v^{\underline{r}}$. Substituting this in (2.87) we see that most of the components of $H^{\underline{M}}$ are determined in terms of the scalars. Furthermore, the conditions of the hyperini KSE in the gauge (2.82) imply that the scalars of the multiplet are invariant under the action of isometries generated by X, i.e.

$$D_+ \phi^{\underline{I}} = \partial_u \phi^{\underline{I}} = 0 \ . \tag{2.89}$$

The remaining restrictions coming from the hyperini KSE give a holomorphicity-like condition for the imbedding scalars.

2.5 N=2 Non-Compact Backgrounds

There are two cases with $N = 2$ supersymmetry, each distinguished by the isotropy group of the Killing spinors. The title of this section and the ones that follow refer to the compact and non-compact nature of the isotropy groups. If the isotropy group is non-compact $U(1) \cdot SU(2) \ltimes \mathbb{H}$, the two Killing spinors are

$$\epsilon_1 = 1 + e_{1234} \ , \quad \epsilon_2 = i(1 - e_{1234}) = \rho^1 \epsilon_1 \ . \tag{2.90}$$

Therefore, the additional conditions on the fields which arise from the second Killing spinor can be expressed as the requirement that the KSEs commute with the Clifford algebra operation ρ^1.

2.5.1 Gravitino KSE

The gravitino KSE commutes with ρ^1, if and only if,

$$\mathcal{C}^2 = \mathcal{C}^3 = 0 \ . \tag{2.91}$$

Equivalently, the gravitino KSE implies that the holonomy of the supercovariant connection is included in $U(1) \cdot Sp(1) \ltimes \mathbb{H}$, $\mathrm{hol}(\mathcal{D}) \subseteq U(1) \cdot Sp(1) \ltimes \mathbb{H}$. The restrictions that this imposes on the geometry will be investigated later.

2.5.2 Gaugini KSE

The gaugini KSE commutes with ρ^1, iff

$$\mu_2 = \mu_3 = 0 \ . \tag{2.92}$$

These restrictions are in addition to the conditions given in (2.43). When combined these become

$$F^{a'}_{+i} = F^{a'}_{+-} = 0 \ , \quad F^{a'\alpha}_{\alpha} + i\mu^1 = 0 \ , \quad F^{a'}_{12} = 0 \ . \tag{2.93}$$

Once again the $F^{a'}_{-i}$ components are not restricted.

2.5.3 Tensorini KSE

A direct substitution of the second Killing spinor, $\epsilon_2 = i(1 - e_{1234})$, into the tensorini KSE reveals that there are in fact no additional conditions on top of the ones given in (2.45). This agrees with what we have already mentioned; that the tensorini KSE commutes with all the ρ operators. Hence, if $\epsilon_1 = 1 + e_{1234}$ is a solution then so is $\epsilon_2 = \rho^1 \epsilon_1$.

2.5.4 Hyperini KSE

Using $\epsilon_2 = i(1 - e_{1234})$ in the hyperini KSE leads to the following restrictions,

$$V^{\underline{a}A}_{+} = 0 \ , \quad V^{\underline{a}1}_{1} + V^{\underline{a}2}_{\bar{2}} = 0 \ , \quad V^{\underline{a}1}_{2} - V^{\underline{a}2}_{\bar{1}} = 0 \ . \tag{2.94}$$

Combining these conditions with those in (2.48) obtained for the first Killing spinor gives

$$V_+^{aA} = 0 \ , \quad V_\alpha^{a1} = 0 \ , \quad V_{\bar{\alpha}}^{a2} = 0 \ . \tag{2.95}$$

2.5.5 Geometry

The form spinor bilinears are given in (2.63). The only difference now is that the full content of the gravitino KSE can be expressed as

$$\begin{aligned} &\hat{\nabla} e^- = 0 \ , \quad \hat{\nabla}(e^- \wedge \omega) = 0 \ , \\ &\hat{\nabla}(e^- \wedge \omega^2) - 2\,\mathcal{C} e^- \wedge \omega^3 = 0 \ , \hat{\nabla}(e^- \wedge \omega^3) + 2\,\mathcal{C} e^- \wedge \omega^2 = 0 \ , \end{aligned} \tag{2.96}$$

where we have imposed the additional conditions coming from the gravitino KSE, $\mathcal{C}^2 = \mathcal{C}^3 = 0$. We have also set $\omega = \omega^1$ and $\mathcal{C} = \mathcal{C}^1$, and so we find that the form $e^- \wedge \omega$ is covariantly constant with respect to the connection with skew-symmetric torsion only.

The discussion of the geometry here follows along the same lines as in section 2.4.5 for $N = 1$ backgrounds. In particular, it is clear from the first condition in (2.96) that the spacetime admits a null Killing vector field X, which is the dual of the 1-form e^-, and that (2.70) is valid. The metric and torsion 3-form can again be written as in (2.69) and (2.71), respectively.

Next we consider the other three parallel transport equations in (2.96). As in the previous $N = 1$ case, the parallel transport equations along the + light-cone direction leads to (2.73) but with $\mathcal{C}^2 = \mathcal{C}^3 = 0$. These become

$$\nabla_+ \omega^1_{ij} = 0 \ , \quad \nabla_+ \omega^2_{ij} - 2\,\mathcal{C}_+ \omega^3_{ij} = 0 \ , \quad \nabla_+ \omega^3_{ij} + 2\,\mathcal{C}_+ \omega^2_{ij} = 0 \ . \tag{2.97}$$

The first condition is a restriction on the geometry. The second can be solved for $\mathcal{C}_+$ to give

$$\mathcal{C}_+ = \frac{1}{8} (\omega^3)^{ij} \nabla_+ \omega^2_{ij} \ . \tag{2.98}$$

The third equation in (2.97) is automatically satisfied. The $-$ component of the second equation in (2.96) gives

$$\hat{\nabla}_- \omega_{ij} = \nabla_- \omega_{ij} - H_-{}^k{}_{[i} \omega_{j]k} = 0 \ , \tag{2.99}$$

which we can use along with the fact that H_{-ij} is self-dual (2.75) to obtain

$$H_{-ij} = -\nabla_- \omega_{ik} \, I^k{}_j \ . \tag{2.100}$$

The two remaining conditions along the $-$ light-cone direction can be used to express

$\mathcal{C}_-$ in terms of the geometry and give some additional restrictions on the geometry of spacetime. To do this we start by writing these equations in component form to obtain

$$\begin{aligned}&\nabla_-\omega^2_{ij} - H_-{}^k{}_{[i}\omega^2_{j]k} - 2\,\mathcal{C}_-\omega^3_{ij} = 0\ ,\\&\nabla_-\omega^3_{ij} - H_-{}^k{}_{[i}\omega^3_{j]k} + 2\,\mathcal{C}_-\omega^2_{ij} = 0\ ,\end{aligned}\tag{2.101}$$

working with these and using the expression in (2.100) we find

$$\begin{aligned}&\mathcal{C}_- = \frac{1}{8}\nabla_-\omega^2_{ij}\omega^{3ij}\ ,\\&\nabla_-\omega^2_{ij} - \nabla_-\omega^1_{k[i}(I^3)^k{}_{j]} - \frac{1}{4}\nabla_-\omega^2_{k\ell}\omega^{3k\ell}\omega^3_{ij} = 0\ ,\\&\nabla_-\omega^3_{ij} + \nabla_-\omega^1_{k[i}(I^2)^k{}_{j]} + \frac{1}{4}\nabla_-\omega^2_{k\ell}\omega^{3k\ell}\omega^2_{ij} = 0\ .\end{aligned}\tag{2.102}$$

The second condition in (2.96) along the transverse to the light-cone directions gives

$$\tilde{H} = -i_I\tilde{d}\omega\ ,\tag{2.103}$$

where $\tilde{d}$ is the exterior derivative projected in the directions transverse to the light-cone. This together with the anti-self-duality condition for H turn (2.72) into a condition on the geometry of spacetime

$$(de^-)_{-\ell}\,\epsilon^\ell{}_{ijk} = (i_I\tilde{d}\omega)_{ijk}\ .\tag{2.104}$$

The remaining two parallel transport equations are automatically satisfied provided that the $U(1)$ part of the curvature tensor of the spacetime connection with torsion is identified with the curvature of $U(1)$ connection $\mathcal{C}$. To see this note that the integrability conditions of the gravitino KSE can be written as

$$\begin{aligned}&\hat{R}_{\mu_1\mu_2,+\nu} = 0\ ,\quad \hat{R}_{\mu_1\mu_2,ki}\,I^k{}_j - \hat{R}_{\mu_1\mu_2,kj}\,I^k{}_i = 0\ ,\\&-\hat{R}_{\mu_1\mu_2,ki}\,(I^2)^k{}_j + \hat{R}_{\mu_1\mu_2,kj}(I^2)^k{}_i - 2\mathcal{F}_{\mu_1\mu_2}\omega^3_{ij} = 0\ .\end{aligned}\tag{2.105}$$

The second condition implies that the holonomy of the $\hat{\nabla}$ connection in the directions transverse to the light-cone is contained in $U(2) = U(1)\cdot Sp(1)$. The last condition identifies the $U(1)$ part of the curvature with the curvature of $\mathcal{C}$.

In summary, the gravitino KSE implies that the metric and torsion can be written as

$$\begin{aligned}ds^2 &= 2e^-e^+ + \delta_{ij}e^ie^j\ ,\\H &= e^+\wedge de^- - \nabla_-\omega_{ik}\,I^k{}_j\ e^-\wedge e^i\wedge e^j - \frac{1}{3!}(de^-)_{-\ell}\,\epsilon^\ell{}_{ijk}\ e^i\wedge e^j\wedge e^k\end{aligned}\tag{2.106}$$

As with the $N=1$ case, the spacetime admits a null Killing vector field X which also determines components of H and the geometric condition (2.73) is satisfied.

Furthermore, one has to impose the geometric conditions (2.102), (2.104) and the restrictions implied by (2.105).

We now move onto the restrictions imposed on the matter content by the other KSEs. As we have mentioned, the tensorini KSE does not impose any new conditions on the matter field. As a result, the restrictions are summarised in (2.45) and the fields are expressed as in (2.87).

The gaugino KSE gives the condition in (2.93). So in the gauge $A_+ = 0$, one has

$$F^{a'} = F^{a'}_{-i}\, e^- \wedge e^i + \frac{1}{2}\mu\, \omega + (F^{\rm asd})^{a'} \ , \quad \mu^2 = \mu^3 = 0 \ , \tag{2.107}$$

where $\mu = \mu^1$.

The hypernini KSE imposes a restriction on the + light-cone direction. The other conditions are Cauchy-Riemann type of equations on the scalars.

As in the $N = 1$ case, by expressing the KSEs in terms of the fundamental fields (2.7), we can improve somewhat on the solutions to the KSEs. In particular, the hyperini KSE condition $D_+\phi = 0$, (2.49), implies that $\mathcal{C}_+ = 0$. Then (2.97) gives rise to the geometric conditions

$$\nabla_+\omega^1_{ij} = \nabla_+\omega^2_{ij} = \nabla_+\omega^3_{ij} = 0 \ . \tag{2.108}$$

Writing $X = \partial_u$ and taking the gauge $A_+ = 0$, we can again conclude that ϕ are independent from u, (2.89).

2.6 N=2 Compact Backgrounds

The other case where two supersymmetries are preserved is when the two Killing spinors have isotropy group $Sp(1)$. The two Killing spinors in this can be chosen as, table 2.1,

$$\epsilon_1 = 1 + e_{1234} \ , \quad \epsilon_2 = e_{15} + e_{2345} \ . \tag{2.109}$$

We now give the conditions arising from each of the KSEs.

2.6.1 Gravitino KSE

The full content of the gravitino KSE can be summarised as

$$\text{hol}(\mathcal{D}) \subseteq Sp(1) \ . \tag{2.110}$$

The implications of this condition on the spacetime geometry will be investigated later.

2.6.2 Gaugini KSE

Evaluating the gaugini KSE on $\epsilon_2 = e_{15} + e_{2345}$, we find

$$-2F^{a'}_{1\bar{2}} + \mu^2 + i\mu^3 = 0 \ , \quad - F^{a'}_{1\bar{1}} + F^{a'}_{2\bar{2}} + i(\mu^{a'})^1 = 0 \ , \quad F^{a'}_{-i} = 0 \ . \tag{2.111}$$

Combining the above conditions with those coming from ϵ_1 given in (2.43), we get

$$\begin{aligned} F^{a'}_{+-} = F^{a'}_{+i} = F^{a'}_{-i} = 0 \ , \quad F^{a'}_{1\bar{1}} = 0 \ , \quad F^{a'}_{2\bar{2}} + i(\mu^{a'})^1 = 0 \ , \\ F^{a'}_{12} - F^{a'}_{1\bar{2}}(\mu^{a'})^2 = 0 \ , \quad F^{a'}_{12} + F^{a'}_{1\bar{2}} + (\mu^{a'})^2 = 0 \ , \end{aligned} \tag{2.112}$$

To write this in a more compact notation we change to real coordinates to obtain

$$F^{a'}_{ab} = 0 \ , \quad F^{a'}_{ai} = 0 \ , \quad F^{a'}_{ij} = -\epsilon_{ijk}\mu^{a'k} \ , \quad a = -, +, \tilde{1} \ , \tag{2.113}$$

where $i = 4, \tilde{2}, 5$ and $\epsilon_{\tilde{2}45} = -1$. Each of the indices a and i label 3 real directions, note that we have also used $\tilde{1}$ and $\tilde{2}$ to distinguish the real directions from the complex directions 1 and 2 which naturally appear in the various conditions coming from the KSEs. In addition, the $r' = 1, 2, 3$ index of μ has been replaced with $k = 4, 2, 5$ after an appropriate adjustment of the ranges and identification of the components of μ. In particular, $\mu^1 = \mu^4$, $\mu^2 = \mu^{\tilde{2}}$, and $\mu^3 = \mu^5$. We can express these as

$$F^{a'} = -\frac{1}{2}\epsilon_{ijk}\mu^{a'k} e^i \wedge e^j \ . \tag{2.114}$$

2.6.3 Tensorini KSE

A direct substitution of the second Killing spinor, $\epsilon_2 = e_{15} + e_{2345}$, into the tensorini KSE gives

$$\begin{aligned} T^{\underline{M}}_{-} = 0 \ , \quad H^{\underline{M}}_{-1\bar{1}} - H^{\underline{M}}_{-2\bar{2}} = 0 \ , \quad H^{\underline{M}}_{-1\bar{2}} = 0 \ , \\ T^{\underline{M}}_{\bar{\alpha}} + \frac{1}{2}H_{-+\bar{\alpha}} + \frac{1}{2}H_{\bar{\alpha}\beta}{}^{\beta} = 0 \ . \end{aligned} \tag{2.115}$$

Combining these conditions with those we derived for $\epsilon_1 = 1 + e_{1234}$ in (2.45) and using the self-duality of $H^{\underline{M}}$ given in (2.85), we find

$$T^{\underline{M}}_{\mu} = 0 \ , \quad H^{\underline{M}}_{\mu\nu\rho} = 0 \ . \tag{2.116}$$

This means the tensorini KSE vanishes identically. As a result all eight supersymmetries are preserved. In turn using the expression of $T^{\underline{M}}$ and $H^{\underline{M}}$ in terms of the physical fields (2.7), we find the scalars to be constant and the 3-form field strengths of the tensor multiplet to vanish.

2.6.4 Hyperini KSE

Evaluating the hyperini KSE on $\epsilon_2 = e_{15} + e_{2345}$, we find the conditions

$$V_{-}^{\underline{aA}} = 0 \ , \quad -V_{2}^{\underline{a1}} + V_{1}^{\underline{a2}} = 0 \ , \quad V_{\bar{1}}^{\underline{a1}} + V_{\bar{2}}^{\underline{a2}} = 0 \ . \tag{2.117}$$

Combining these conditions with those coming from the first Killing spinor given in (2.48), we get

$$V_{a}^{\underline{aA}} = 0 \ , \quad a = -, +, \tilde{1} \ , \tag{2.118}$$

where again we have converted back to real coordinates to derive these conditions. The remaining conditions can be derived by substituting (2.118) in either (2.48) or (2.117).

Expressing the KSE in terms of the physical fields as in (2.7), one finds that (2.118) implies

$$D_a \phi^{\underline{I}} = 0 \ , \quad a = -, +, \tilde{1} \ . \tag{2.119}$$

Therefore, the hypermultiplet scalars do not depend on three spacetime directions.

2.6.5 Geometry

Firstly, we have to determine the algebraic independent form bilinears. To do this we use the spinors $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = e_{15} + e_{2345}$ in (2.50) and (2.52). Doing this we find that the independent form bilinears are given by

$$e^a \ , \quad a = -, +, \tilde{1} \ ; \quad e^i \ , \quad i = 4, \tilde{2}, 5 \ , \tag{2.120}$$

where e^a and e^i are 1-forms. The e^i are twisted with respect to the $Sp(1)$ connection. This means the conditions implied by the gravitino Killing spinor equation can be rewritten as

$$\begin{aligned} &\hat{\nabla}_\mu e^a = 0 \ , \\ &\hat{\nabla}_\mu e^i + 2\epsilon^i{}_{jk} \mathcal{C}^j_\mu e^k = 0 \ , \end{aligned} \tag{2.121}$$

where as in the gaugini KSE case the indices r', s' and t' have been replaced with i, j and k, the ranges have been adjusted, and the components of $\mathcal{C}$ have been appropriately identified. It is clear that the spacetime admits a $3+3$ "split". In particular, the tangent space, TM, of spacetime decomposes as

$$TM = I \oplus \xi \ , \tag{2.122}$$

where I is a topologically trivial vector bundle spanned by the vector fields associated to the three 1-forms e^a.

The 1-forms e^a and e^i can be used as a spacetime frame and so we can choose to write the metric as

$$ds^2 = \eta_{ab}e^a e^b + \delta_{ij}e^i e^j \ . \tag{2.123}$$

We now focus on the first equation in (2.121). These imply that the associated vector fields to e^a are Killing. In addition, using the anti-self-duality of H, all of the components of H can be determined in terms of e^a and its first derivatives. In particular, we have

$$de^a = \eta^{ab} i_b H \ , \tag{2.124}$$

where $\eta^{ab} = g(e^a, e^b)$, and so this gives

$$H_{a_1 a_2 a_3} = \eta_{a_1 b} de^b_{a_2 a_3} \ , \quad H_{a_1 a_2 i} = \eta_{a_1 b} de^b_{a_2 i} \ , \quad H_{aij} = \eta_{ab} de^b_{ij} \ . \tag{2.125}$$

Therefore, using the anti-self-duality condition of H we get

$$H_{a_1 a_2 a_3}\epsilon^{a_1 a_2 a_3} = H_{ijk}\epsilon^{ijk} \ , \quad \epsilon_b{}^{a_1 a_2} H_{a_1 a_2 i} = -\epsilon_i{}^{jk} H_{bjk} \ , \tag{2.126}$$

where $\epsilon_{013} = \epsilon_{245} = 1$. This in turn means H can be rewritten as

$$H = K - \star K \ , \quad K = \frac{1}{3!} H_{a_1 a_2 a_3} e^{a_1} \wedge e^{a_2} \wedge e^{a_3} + \frac{1}{2} H_{i a_1 a_2} e^i \wedge e^{a_1} \wedge e^{a_3} \ , \tag{2.127}$$

subject to the geometric condition

$$(de_{a_1})_{a_2 i_1} \epsilon^{a_1 a_2}{}_{a_3} = -\epsilon_{i_1}{}^{i_2 i_3} (de_{a_3})_{i_2 i_3} \ . \tag{2.128}$$

We now return to the second equation in (2.121), decomposing this equation along the two types of spacetime directions, a and i, we find it is equivalent to

$$\begin{aligned} \nabla_b e^i_j - \frac{1}{2} H^i{}_{bj} + 2\epsilon^i{}_{kj} \mathcal{C}^k_b = 0 \ , \\ \nabla_j e^i_k - \frac{1}{2} H^i{}_{jk} + 2\epsilon^i{}_{sk} \mathcal{C}^s_j = 0 \ . \end{aligned} \tag{2.129}$$

The first condition again expresses a component of H in terms of the geometry and $\mathcal{C}$. Substituting the expression we have for $H^i{}_{bj}$ in (2.125), we find

$$\nabla_a e^i_j + 2\epsilon^i{}_{jk} \mathcal{C}^j_a e^k = -\frac{1}{2} \eta_{ab} \, de^b_{kj} \, \delta^{ki} \ . \tag{2.130}$$

The last condition in (2.129) identifies the spin connection $\hat{\Omega}$ of the spacetime in the directions transverse to the Killing with the induced $Sp(1)$ connection of the scalars. Another way to see this is by looking at the integrability conditions of the gravitino KSE. In particular, we get

$$\hat{R}_{\mu_1\mu_2,a\nu} = 0 \ , \quad \hat{R}_{\mu\nu,j_1j_2} = -2\mathcal{F}^k_{\mu\nu}\epsilon_{kj_1j_2} \ , \tag{2.131}$$

which is obtained by taking the integrability conditions of (2.121) on e^a and e^i, respectively.

As before, we can express the KSEs in terms of the physical fields in (2.7) and use the restrictions coming from the gaugini and hyperini KSEs to further simplify things, particularly we find that

$$\hat{R}_{a\mu,\nu_1\nu_2} = 0 \ . \tag{2.132}$$

Also using (2.118) we find

$$\mathcal{C}^i_a = 0 \ , \tag{2.133}$$

and this therefore means (2.130) turns into a condition on the geometry. It is clear that the only non-trivial components of the curvature with torsion are those along the transverse to the Killing vector directions and these are specified in terms of the curvature of $\mathcal{C}$.

Finally let us summarise; the spacetime admits three Killing vector fields and the torsion H is completely determined in terms of these and their first derivatives. In particular, we have

$$\begin{aligned} &ds^2 = \eta_{ab}e^a e^b + \delta_{ij}e^i e^j \ , \\ &H = K - \star K \ , \quad K = \frac{1}{3!}H_{a_1a_2a_3}e^{a_1}\wedge e^{a_2}\wedge e^{a_3} + \frac{1}{2}H_{ia_1a_2}e^i \wedge e^{a_1}\wedge e^{a_2} \end{aligned} \tag{2.134}$$

In addition, the spacetime geometry is restricted by (2.128), (2.130) and the last condition in (2.129) or equivalently (2.131).

Examples

The spacetime geometry can be further analysed under some additional conditions. We will not go into specific details but briefly mention how this can be achieved, for details see [44, 105, 12]. As we mentioned, the spacetime admits three Killing vector fields e_a, $a = +, -, \tilde{1}$, the commutator of these vector fields does not necessarily close under the Lie bracket. However, if one imposes the requirement that the algebra of the vector fields closes under the Lie bracket then it can be shown, in analogy with the results of [44], that the spacetime can be described in terms of principle bundles, where one also needs to make use of the classification of Lorentzian Lie algebras [106, 107, 108]. In addition, the closure of the algebra requires

$$H_{abi} = 0 \ , \tag{2.135}$$

and in turn the anti-self-duality of H requires

$$H_{aij} = 0 \ . \tag{2.136}$$

These then have further implications, see [12, 105]. In particular, the spacetime can be locally identified as $G \times \Sigma$, where G is either $\mathbb{R}^{2,1}$ or $SL(2,\mathbb{R})$ and Σ is 3-dimensional Riemannian manifold [12].

2.7 N=4 Non-Compact Backgrounds

There are two cases where the background preserves four supersymmetries. The first case we consider is when the Killing spinors have isotropy group $Sp(1) \ltimes \mathbb{H}$ with the invariant spinors given as in table 2.1. These spinors can be written as

$$1 + e_{1234} \ , \quad \rho^1(1 + e_{1234}) \ , \quad \rho^2(1 + e_{1234}) \ , \quad \rho^3(1 + e_{1234}) \ . \tag{2.137}$$

For these to be solutions to the KSEs, we require the KSEs to commute with the Clifford algebra operations $\rho^{r'}$. We shall use this together with the conditions imposed on backgrounds preserving one supersymmetry to derive all the conditions implied by the KSEs in this case.

2.7.1 Gravitino KSE

The gravitino KSE commutes with the $\rho^{r'}$ operations iff

$$\mathcal{C} = 0 \ . \tag{2.138}$$

As a result the curvature of $\mathcal{C}$ must vanish, $\mathcal{F} = 0$. The full content of the gravitino KSE can be expressed as $\mathrm{hol}(\hat{\nabla}) \subseteq Sp(1) \ltimes \mathbb{H}$. The restrictions that this condition imposes on the spacetime geometry will be examined later.

2.7.2 Gaugini KSE

The gaugini KSE commutes with $\rho^{r'}$, iff

$$\mu_1 = \mu_2 = \mu_3 = 0 \ . \tag{2.139}$$

These are of course in addition to the conditions in (2.43). The same conditions can derived by explicitly substituting in the four Killing spinors into the gaugini KSE.

This in turn means we have

$$F^{a'} = F^{a'}_{-i} e^- \wedge e^i + (F^{\rm asd})^{a'} \ , \tag{2.140}$$

where we have just imposed the conditions in (2.139) on the expression in (2.84).

2.7.3 Tensorini KSE

As mentioned previously, the tensorini KSE commutes with the Clifford algebra operations $\rho^{r'}$. Thus, there are no additional conditions on top of those given in (2.45).

2.7.4 Hyperini KSE

Substituting the Killing spinors in (2.137) into the hyperini KSE we find, in addition to the conditions in (2.95), that

$$V^{\underline{a1}}_{\bar{\alpha}} = 0 \ , \quad V^{\underline{a2}}_{\alpha} = 0 \ . \tag{2.141}$$

This means the only non-vanishing component is

$$V^{\underline{aA}}_{-} \ . \tag{2.142}$$

Imposing the conditions of the hyperini KSE on the physical fields using (2.7), we find that the only non-vanishing derivative on the scalars is

$$D_- \phi^{\underline{I}} \ . \tag{2.143}$$

This means that the scalars depend only on one light-cone direction.

2.7.5 Geometry

The spacetime form bilinears associated to the spinors in this case are the same as those of the $N = 2$ non-compact case. However, the important difference here is that $\mathcal{C} = 0$ and so the conditions imposed by the gravitino KSE can be rewritten as

$$\hat{\nabla} e^- = 0 \ , \quad \hat{\nabla}(e^- \wedge \omega^{r'}) = 0 \ , \tag{2.144}$$

i.e. there are no twists with respect to the $Sp(1)$ connection since this vanishes. The analysis of the solution to these conditions is similar to that of the non-compact $N = 2$. Following the $N = 1$ and $N = 2$ non-compact cases but in addition imposing the condition $\mathcal{C} = 0$ means we can write

$$ds^2 \ = \ 2e^- e^+ + \delta_{ij} e^i e^j \ ,$$

$$H = e^+ \wedge de^- - \frac{1}{16}\omega^{r'}_{kl}\nabla_-\omega^{s'kl}\epsilon_{r's'}{}^{t'}\,\omega_{t'ij}\, e^- \wedge e^i \wedge e^j - \frac{1}{3!}(de^-)_{-\ell}\,\epsilon^\ell{}_{ijk}\, e^i \wedge e^j \wedge e^k \ . \qquad (2.145)$$

We have used the anti-self-duality of H to relate the $\tilde{H}$ component to de^- as in (2.72).

We now discuss the geometric conditions imposed on the spacetime. We have already dealt with the first condition in (2.144). To solve the last three conditions in (2.144), we first consider the + light-cone direction which gives

$$\hat{\nabla}_+\omega^{r'} = \nabla_+\omega^{r'} = 0 \ . \qquad (2.146)$$

This is a condition on the geometry. From the − component we find

$$\nabla_-\omega^{r'}_{ij} - H_-{}^k{}_{[i}\omega^{r'}_{j]k} = 0 \ . \qquad (2.147)$$

This together with the self-duality of H_{-ij} can be used to express H_{-ij} in terms of the geometry as in (2.145), and was discussed in detail for the $N = 1$ backgrounds. There are no conditions on the geometry along this light-cone direction.

Next, considering the conditions along the transverse to light-cone directions we find

$$\tilde{H} = -i_{I_{r'}}\tilde{d}\omega^{r'} \ , \quad (\text{no r}' \text{ summation}) \ . \qquad (2.148)$$

These appear as three independent conditions but actually they are not. One of them implies the other two. In turn, this condition together with (2.72) imply

$$de^-_{-j}\,\epsilon^j{}_{i_1 i_2 i_3} = (i_{I_{r'}}\tilde{d}\omega^{r'})_{i_1 i_2 i_3} \ , \quad (\text{no r}' \text{ summation}) \ . \qquad (2.149)$$

This is another condition on the geometry. The restrictions on the fields imposed by the other three KSEs have already been explained.

2.7.6 N=3 Descendant

Unlike in all the other cases, the $N = 4$ backgrounds with $Sp(1) \ltimes \mathbb{H}$-invariant parallel spinors exhibit an independent descendant with three supersymmetries. This was discussed earlier when descendants of four parallel spinors was analysed in section 2.3.3. We noted then that the conditions on the fields arising from the gravitino, gaugini and tensorini KSEs remain the same as those for backgrounds with four Killing spinors (2.137). However, when the hyperini KSE is considered different conditions appear for backgrounds admitting three and four Killing spinors.

The three Killing spinors were given in (2.35). Substituting these spinors into the hyperini KSE gives us the constraints

$$V_{+}^{aA}=0\ ,\quad V_{\alpha}^{a1}=V_{\bar{\alpha}}^{a2}=0\ ,\quad V_{\bar{1}}^{a1}-V_{2}^{a2}=0\ ,\quad V_{\bar{2}}^{a1}+V_{1}^{a2}=0\ .(2.150)$$

These are indeed different from the conditions that we found for the four $Sp(1)\ltimes\mathbb{H}$-invariant Killing spinors given in (2.95) and (2.141). We can express these conditions in terms of the physical fields using (2.7). The first condition, for example, can be written as in (2.89). However, note that the analysis of the geometry of spacetime given in the previous section does not change. The difference here is in the conditions the scalars of the hypermultiplets satisfy compared to backgrounds that preserve four supersymmetries.

2.8 N=4 Compact Backgrounds

The other case of $N=4$ backgrounds is when the four Killing spinors are chosen as the $U(1)$-invariant spinors of table 2.1. These can be written in the following way using the ρ^1 operator

$$1+e_{1234}\ ,\quad e_{15}+e_{2345}\ ,\quad \rho^1(1+e_{1234})\ ,\quad \rho^1(e_{15}+e_{2345})\ . \qquad (2.151)$$

The conditions imposed on the fields by the KSEs evaluated on these spinors can be calculated from the conditions we found for the $Sp(1)$-invariant Killing spinors and by the additional requirement that the KSEs commute with the ρ^1 Clifford algebra operator.

2.8.1 Gravitino KSE

For the gravitino KSE to commute with the Clifford algebra operation ρ^1 we require

$$\mathcal{C}^2=\mathcal{C}^3=0\ . \qquad (2.152)$$

The full content of the gravitino KSE can be expressed as the requirement that hol($\mathcal{D}\subseteq U(1)$. We will examine the implications on the geometry later.

2.8.2 Gaugini KSE

The gaugini KSE commutes with ρ^1 iff $\mu^2=\mu^3=0$. Combining this with the conditions coming from the $N=2$ case in (2.113), we find

$$F^{a'}_{2\bar{2}}+i\mu^{a'}=0\ , \qquad (2.153)$$

where after suppressing the gauge index we set $\mu=\mu^1$.

2.8.3 Tensorini KSE

The tensorini KSE commutes with all the Clifford algebra $\rho^{r'}$ operators. Since both $1+e_{1234}$ and $e_{15}+e_{2345}$ are Killing spinors, we conclude that all eight supersymmetries are preserved. Therefore, $T^{\underline{M}} = H^{\underline{M}} = 0$ as in (2.116) for the $N=2$ compact case. In turn, this means the tensor multiplet scalars are constant and the 3-form field strengths, $H^{\underline{M}}$, vanish.

2.8.4 Hyperini KSE

To find the conditions arising from the hyperini KSE we have to evaluate it on the four spinors given in (2.151), which is equivalent to simultaneously imposing (2.117) and (2.95). This gives

$$V_a^{\underline{aA}} = 0 \ , \quad a = -, +, 1, \bar{1} \ , \tag{2.154}$$

and

$$V_2^{\underline{a1}} = V_{\bar{2}}^{\underline{a2}} = 0 \ . \tag{2.155}$$

In other words, the only non-vanishing components are $V_{\bar{2}}^{\underline{a1}}$ and $V_2^{\underline{a2}}$.

Using the physical fields in (2.7) these conditions can be expressed as

$$D_a \phi^{\underline{I}} = 0 \ , \quad a = -, +, 1, \bar{1} \ , \tag{2.156}$$

and

$$D_2 \phi^{\underline{I}} E_{\underline{I}}^{\underline{a1}} = D_{\bar{2}} \phi^{\underline{I}} E_{\underline{I}}^{\underline{a2}} = 0 \ , \tag{2.157}$$

respectively. So we find the scalar fields not to depend on four spacetime directions. The last two conditions are Cauchy-Riemann type of equations along the remaining two directions.

2.8.5 Geometry

Using the four Killing spinors that we have in this case, we find that a basis for the algebraically independent spacetime form bilinears is spanned by the 1-forms

$$e^a \ , \quad a = -, +, 1, \bar{1} \ , \quad e^i \ , \quad i = 2, \bar{2} \ . \tag{2.158}$$

The gravitino KSE can then be rewritten as

$$\hat{\nabla} e^a = 0 \ , \quad \hat{\nabla} e^i - 2\,\mathcal{C}\, \epsilon^i{}_j e^j = 0 \ , \tag{2.159}$$

where we have set $\mathcal{C} = \mathcal{C}^1$, and note e^i are the twisted bilinears.

As in the cases we have already investigated, the first equation implies that the vector fields X_a associated to the 1-forms e^a are Killing and

$$i_a H = \eta_{ab} de^b \ . \tag{2.160}$$

In this case we find that the spacetime admits a 4 + 2 split. In particular, this means the tangent space $TM = I \oplus \xi$, where now I is a rank 4 trivial vector bundle spanned by the four Killing vectors X_a.

The second equation in (2.159) can decomposed in terms of equations along the two types of spacetime directions $\mu = a, i$ to give

$$\begin{aligned} &(\nabla_a e^i)_j - \frac{1}{2} H^i{}_{aj} - 2\,\mathcal{C}_a\, \epsilon^i{}_j = 0\ , \\ &(\nabla_j e^i)_k - 2\mathcal{C}_j \epsilon^i{}_k = 0\ . \end{aligned} \tag{2.161}$$

In turn, the first condition in (2.161) gives

$$(\nabla_a e^i)_j - 2\,\mathcal{C}_a\, \epsilon^i{}_j = -\frac{1}{2}\eta_{ab}(de^b)_{kj}\delta^{ki}\ , \tag{2.162}$$

since we can use (2.160) to write $H^i{}_{aj} = -\eta_{ab}(de^b)_{kj}\delta^{ki}$. We also know that H is anti-self-dual and this implies

$$H_{aij} = \frac{1}{3!}\epsilon_{ij}\, \epsilon_a{}^{b_1 b_2 b_3} H_{b_1 b_2 b_3}\ , \quad H_{a_1 a_2 i} = \frac{1}{2}\epsilon_{a_1 a_2}{}^{b_1 b_2}\epsilon_i{}^j H_{b_1 b_2 j}\ , \tag{2.163}$$

where $\epsilon_{2\bar{2}} = i$ and $\epsilon_{-+1\bar{1}} = i$. All the components of H are determined in terms of e^a and its first derivative, and this leads to more restrictions on the spacetime geometry. Using (2.160) and (2.163) these can be expressed as

$$de^a_{ij} = \frac{1}{3!}\epsilon_{ij}\, \epsilon^{ab_1 b_2}{}_{b_3} de^{b_3}_{b_1 b_2}\ , \quad de^{a_1}_{a_2 i} = \frac{1}{2}\epsilon_{a_2 b_2}{}^{a_1 b_1}\epsilon_i{}^j de^{b_2}_{b_1 j}\ . \tag{2.164}$$

One thing to note here is that the rhs of the first equation depends on the structure constants of the algebra of the four Killing vector fields.

The last condition in (2.161) identifies the spacetime connection along the directions transverse to the Killing with a $U(1)$ component of the induced $Sp(1)$ quaternionic Kähler connection. This can also be seen from the integrability conditions of (2.159). In particular, we find that

$$\hat{R}_{\mu_1\mu_2, a\nu} = 0\ , \quad \hat{R}_{\mu\nu, j_1 j_2} = -2\mathcal{F}_{\mu\nu}\, \epsilon_{j_1 j_2}\ . \tag{2.165}$$

The derivation of these conditions is similar to that of the $Sp(1)$ holonomy case investigated for the $N = 2$ compact backgrounds.

If we use (2.7) to express the above conditions in terms of the physical fields we find that there are some additional simplifications. In particular, using the hypernini and gaugini KSEs, we find that apart from (2.165)

$$\hat{R}_{a\mu,\nu_1\nu_2} = 0 \ . \qquad (2.166)$$

Similarly using (2.156) we find $\mathcal{C}_a = 0$ and so (2.162) becomes a condition on the geometry of spacetime.

Examples

Once again examples can be constructed following a similar argument to that given in the $N = 2$ compact case. In this case we have four Killing vector fields and imposing the closure of the algebra requires

$$H_{abi} = 0 \ . \qquad (2.167)$$

Furthermore, the Lie algebra of the Killing vector fields is isomorphic [106, 107, 108] to one of the following

$$\mathbb{R}^{3,1} \ , \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1) \ , \quad \mathbb{R} \oplus \mathfrak{su}(2) \ , \quad \mathfrak{cw}_4 \ . \qquad (2.168)$$

We have not discussed these examples in any detail, but have mentioned them to indicate some of the possibilities available for further investigation, see [12].

2.9 Trivial Isotropy Group

Backgrounds with parallel spinors which have a trivial isotropy group admit eight parallel spinors. These are maximally supersymmetric backgrounds. The spacetime is a Lorentzian Lie group with anti-self-dual structure constants. They have been classified in a similar context in [58]. In particular, the spacetime is locally isometric to

$$\mathbb{R}^{5,1} \ , \quad AdS_3 \times S^3 \ , \quad CW_6 \ , \qquad (2.169)$$

where the radii of AdS_3 and S^3 are equal, and the structure constants of CW_6 are given by a constant self-dual 2-form on $\mathbb{R}^4$. Moreover,

$$\mathcal{F}(\mathcal{C}) = 0 \ , \qquad (2.170)$$

which we infer from the integrability condition (2.24). This concludes the conditions which arise from the gravitino KSE.

The gaugino KSE implies the gauge field strength vanishes and $\mu^{r'} = 0$. The tensorini KSE implies that the 3-form field strengths vanish and the tensor multiplet scalars are constants. Similarly, the hyperini KSE implies that the scalars of the hypermultiplet are constant. In turn using (2.7), the latter gives $\mathcal{C} = 0$.

2.9.1 Descendants

The case of trivial isotropy group does give rise to descendants, which we discussed in some detail in section 2.3.4. In particular, the KSEs allow for backgrounds with $1, 2, 3$ and 4 supersymmetries, but we argued that none of these were independent from the backgrounds we have discussed and examined thus far. In this section we will give a proof of this to establish the results of section 2.3.4. We will give the proof for one of the cases and the rest follow in a similar way. Let us consider the descendants with three supersymmetries for which the Killing spinors are those given in (2.38). To show that there are no independent descendants we have to solve the hyperini KSE for these three spinors and establish the fact that the conditions are the same as the constraints arising from one of the backgrounds we have already discussed. The first two Killing spinors give

$$\begin{aligned} &V_{+}^{aA} = 0 \ , \quad - V_{1}^{a1} + V_{\bar{2}}^{a2} = 0 \ , \quad V_{2}^{a1} + V_{\bar{1}}^{a2} = 0 \ , \\ &V_{-}^{aA} = 0 \ , \quad - V_{2}^{a1} + V_{1}^{a2} = 0 \ , \quad V_{\bar{1}}^{a1} + V_{\bar{2}}^{a2} = 0 \ , \end{aligned} \tag{2.171}$$

which follows from (2.48) and (2.117). Substituting the third Killing spinor in (2.38) into the hyperini KSE we find

$$\begin{aligned} &c_1 V_{1}^{a1} + c_1 V_{\bar{2}}^{a2} = 0 \ , \quad - c_1 V_{2}^{a1} + c_1 V_{\bar{1}}^{a2} = 0 \ , \\ &i c_2 V_{\bar{1}}^{a1} - c_3 V_{\bar{2}}^{a1} - i c_2 V_{\bar{2}}^{a2} + c_3 V_{\bar{1}}^{a2} = 0 \ , \\ &i c_2 V_{2}^{a1} + c_3 V_{1}^{a1} + i c_2 V_{1}^{a2} + c_3 V_{2}^{a2} = 0 \ . \end{aligned} \tag{2.172}$$

Now, there are two cases to consider; if $c_1 \neq 0$, then the V's vanish and so the hyperini KSE preserves all eight supersymmetries. The other case is when $c_1 = 0$, this means that we can also always set $c_3 = 0$ which has been argued in section 2.3.4. Setting $c_3 = 0$ in the last two conditions in (2.172), we find that

$$V_{\bar{1}}^{a1} - V_{\bar{2}}^{a2} = 0 \ , \quad V_{2}^{a1} + V_{1}^{a2} = 0 \ . \tag{2.173}$$

Then comparing this with (2.171), we again find that all V's vanish. Therefore, once again the hyperini KSE preserves all supersymmetry and so there is no new descendant.

2.10 Summary

In this chapter we solved the Killing spinor equations of six dimensional supergravity with eight real supercharges coupled to any number of vector, tensor and scalar multiplets in all cases. To do so we made use of the spinorial geometry method that we discussed in chapter 1 as well as the similarity of the KSEs of six dimensional supergravity with those of heterotic supergravity.

We began the chapter by briefly discussing the (1,0) supergravity and its coupling to arbitrary numbers of vector, tensor and scalar multiplets. In particular, we gave the gravitino, gaugini, tensorini and hyperini KSEs arising from the supersymmetry variations of the fermions in the theory. To continue, the symplectic Majorana-Weyl spinors of $Spin(5,1)$ were identified with the $SU(2)$-invariant Majorana-Weyl spinors of $Spin(9,1)$. The gamma matrices of Clif($\mathbb{R}^{5,1}$) were identified from a subset of Clif($\mathbb{R}^{9,1}$) and the remaining four gamma matrices were used to define $\rho^{r'}$ which generate an $SU(2)$ algebra and play an important role in the analysis of the KSEs.

We found, apart from one case, that the solutions can be uniquely characterised by the isotropy group of the Killing spinors in $Spin(5,1) \cdot Sp(1)$, these are given in table 2.1. The one case where an independent descendant arises is with three Killing spinors and it is when the isotropy group of the four parallel spinors is $Sp(1) \ltimes \mathbb{H}$. The difference is due to the conditions that come from the hyperini KSE.

The geometry of the solutions fall into two groups; those where the isotropy group of the Killing spinors is compact and those where it is non-compact. In the non-compact case the spacetime always admits a parallel null 1-form with respect to the connection with skew-symmetric torsion given by the 3-form of the gravitational multiplet. In this case there are backgrounds which preserve $1, 2, 3$ and 4 supersymmetries. Each of these were considered in turn and the conditions imposed on the geometry of spacetime were discussed. In addition, the constraints imposed on the fields by the KSE were given in all cases.

On the other hand, when the isotropy group of the Killing spinors is compact we found the solutions to preserve $2, 4$ and 8 supersymmetries. In the case of two supersymmetries the spacetime admits a $3+3$ split where the first three directions are spanned by the three parallel vector fields with respect to the connection with the skew-symmetric torsion given by the 3-form of the gravitational multiplet. There is a natural frame on the spacetime given by six 1-form spinor bilinears. Similarly, when four supersymmetries are preserved the spacetime allows a $4+2$ split where the four directions are spanned by the four parallel vectors with respect to the connection with skew-symmetric torsion. Once again there is a natural frame for the spacetime. Backgrounds that preserve eight supersymmetries admit spacetimes that are locally isometric to $\mathbb{R}^{5,1}$, $AdS_3 \times S^3$ and CW_6.

This concludes our analysis of the KSEs of six dimensional supergravity coupled to arbitrary numbers of vector, tensor and scalar multiplets. In one of the chapters that follows we make use of the results of this chapter to investigate near horizon geometries arising in six dimensional supergravity. In a later chapter we will use the techniques and results discussed here to investigate the BPS conditions of

(1,0) superconformal models in six dimensions. Next, we focus on the integrability conditions of the KSEs.

Chapter 3

Integrability Conditions

3.1 Introduction

Integrability conditions have played an important role in finding supersymmetric solutions; they were, for example, used in determining the maximally supersymmetric solutions of supergravities in ten and eleven dimensions [40]. As we have mentioned, supersymmetric supergravity solutions are obtained after solving the KSEs as well as imposing the field equations of the theory. The integrability conditions can be used to determine which of the field equations are implied by the solutions of the KSEs. Once this is done the supergravity solutions are obtained by imposing the components of the field equations that are not implied by the KSEs. In addition, deriving the integrability conditions provides an important consistency check for the theory in question and are also needed for the consistency of the KSEs.

In this chapter we give a detailed derivation of the integrability conditions arising from the four Killing spinor equations discussed in the previous chapter. This will be a technical chapter and involve a lot of detailed calculations. Where possible we emphasise on the most important parts. In what follows we first give a general outline of the approach we take in finding the integrability conditions. Then we consider the integrability condition of each KSE in turn and derive the field equations from them. The field equations derived in this chapter will be required for the analysis of near horizon geometries that follow in chapter 4.

3.2 The Integrability Conditions

The integrability conditions are obtained by taking the commutator of the gravitino KSE with itself and the other KSE equations. This means we need to evaluate the following

$$\begin{aligned}\Big[\mathcal{D}_\mu, \mathcal{D}_\nu\Big]\epsilon &= 0 \,, \\ \Big[\mathcal{D}_\mu, T^{M}_{\nu}\gamma^\nu - \frac{1}{12}H^{M}_{\nu\rho\sigma}\gamma^{\nu\rho\sigma}\Big]\epsilon &= 0 \,,\end{aligned}$$

$$\Big[\mathcal{D}_\mu, \gamma^\nu V_\nu^{\underline{aA}}\Big]\epsilon_{\underline{A}} = 0 \,,$$
$$\Big[\mathcal{D}_\mu, \frac{1}{2}F^{a'}_{\nu\rho}\gamma^{\nu\rho} + \mu^{a'}_{r'}\Big]\epsilon = 0 \,, \tag{3.1}$$

The KSEs can also be used to derive various algebraic identities. These identities can in turn be used to rewrite some of the terms appearing in the calculation of the integrability conditions.

We have primarily been following the construction presented in [57] for the coupling of (1,0) six dimensional supergravity to n_V vector, n_T tensor and n_H hypermultiplets. The bosonic sector of the Lagrangian for this theory [57], after imposing our normalisation conventions, is given by

$$\begin{aligned} e^{-1}\mathcal{L} = & -\frac{1}{4}R + \frac{1}{48}\varsigma_{\underline{rs}}G^{\underline{r}}_{\mu\nu\rho}G^{\underline{s}\;\mu\nu\rho} - \frac{1}{4}\partial_\mu v^{\underline{r}}\partial^\mu v_{\underline{r}} + \frac{1}{8}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\nu}F^{a'\mu\nu} \\ & -\frac{1}{64e}\epsilon^{\mu\nu\rho\sigma\delta\tau}B^{\underline{r}}_{\mu\nu}c_{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\tau} + \frac{1}{2}g_{\underline{IJ}}D_\mu\phi^{\underline{I}}D^\mu\phi^{\underline{J}} \\ & -\frac{1}{2v_{\underline{r}}c^{\underline{r}}}\mathcal{A}^{\underline{A}}_{\underline{I}\;\;B}\mathcal{A}^{\underline{B}}_{\underline{J}\;\;\underline{A}}\xi^{a'\underline{I}}\xi^{a'\underline{J}} \,. \end{aligned} \tag{3.2}$$

Note that we have ignored the subtlety arising from the (anti-)self-duality of the 3-form gauge field strengths when writing a term for these in the Lagrangian. One has to of course keep in mind that the (anti-)self-duality of the 3-form gauge field strengths has to be imposed after the equations of motions are derived.

When the integrability conditions are used to derive the field equations it is not always easy to know which terms appear in which of the field equations. Therefore, as a point reference we will calculate the field equations obtained from varying the Lagrangian with respect to the different fields appearing in the theory. This will help us to group terms that belong together. Firstly, varying the Lagrangian with respect to $g^{\mu\nu}$ gives rise to the Einstein equation, which is given by

$$\begin{aligned} E_{\mu\nu} = & -\frac{1}{2}R_{\mu\nu} + \frac{1}{8}\varsigma_{\underline{rs}}G^{\underline{r}\;\;\alpha\beta}_{\mu}G^{\underline{s}}_{\nu\alpha\beta} - \frac{1}{2}\partial_\mu v^{\underline{r}}\partial_\nu v_{\underline{r}} \\ & +\frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'\lambda}_{\mu}F^{a'}_{\nu\lambda} + g_{\underline{IJ}}D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}} - \frac{1}{16}v_{\underline{r}}c^{\underline{r}}F^{a'\alpha\beta}F^{a'}_{\alpha\beta}g_{\mu\nu} \\ & -\frac{1}{4}v_{\underline{r}}c^{\underline{r}}\mathcal{A}^{\underline{A}}_{\underline{I}\;\;\underline{B}}\mathcal{A}^{\underline{B}}_{\underline{J}\;\;\underline{A}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu} = 0 \,. \end{aligned} \tag{3.3}$$

Varying the Lagrangian with respect to $v_{\underline{r}}$ we find

$$\begin{aligned} (Ev)^{\underline{r}} = & \nabla^\mu\partial_\mu v^{\underline{r}} + \frac{1}{6}v_{\underline{s}}G^{\underline{s}\mu\nu\rho}G^{\underline{r}}_{\mu\nu\rho} + \frac{1}{4}c^{\underline{r}}F^{a'}_{\mu\nu}F^{a'\mu\nu} \\ & +\frac{c^{\underline{r}}}{(v_{\underline{s}}c^{\underline{s}})^2}\mathcal{A}^{\underline{A}}_{\underline{I}\;\;\underline{B}}\mathcal{A}^{\underline{B}}_{\underline{J}\;\;\underline{A}}\xi^{\underline{I}a'}\xi^{\underline{J}a'} = 0. \end{aligned} \tag{3.4}$$

This corresponds to the equation of motion for the scalars of the tensor multiplet. To find the equation of motion of the hypermultiplet scalars we vary the Lagrangian with respect to $\phi^{\underline{I}}$ to find

$$(E\phi)_{\underline{K}} = \frac{1}{2}\partial_{\underline{K}} g_{\underline{IJ}} D_\mu \phi^{\underline{I}} D^\mu \phi^{\underline{J}} - \nabla_\mu \left(g_{\underline{IK}} D^\mu \phi^{\underline{I}}\right) - g_{\underline{IJ}} D_\mu \phi^{\underline{I}} A^\mu_{a'} \partial_{\underline{K}} \xi^{\alpha \underline{J}} + \frac{2}{v_{\underline{r}} c^{\underline{r}}} \mathcal{F}^{r'}_{\underline{KJ}} \mathcal{A}_{r' \underline{I}} \xi^{a' \underline{I}} \xi^{a' \underline{J}} = 0 \ , \tag{3.5}$$

where $\partial_{\underline{I}} = \partial / \partial \phi^{\underline{I}}$. In the case of the gauge field A_μ to find the covariant equation of motion we need to evaluate

$$\frac{\delta \mathcal{L}}{\delta A_\mu} - f^\mu = 0 \ , \tag{3.6}$$

where f^μ is a contribution that needs to be included due to the supersymmetry anomaly, i.e. the anomaly that arise from the variation of the Lagrangian with respect to the supersymmetry transformations of the theory, further details of this can be found in [56, 109], and this contribution is given by

$$f^\mu = -\frac{1}{24} c_{\underline{r}} c^{\underline{r}} \epsilon^{\mu\nu\rho\sigma\delta\tau} F^{a'}_{\nu\rho} (CS)_{\sigma\delta\tau} + \frac{1}{32} c_{\underline{r}} c^{\underline{r}} \epsilon^{\mu\nu\rho\sigma\delta\tau} A^{a'}_\nu F^{b'}_{\rho\sigma} F^{b'}_{\delta\tau} \ . \tag{3.7}$$

Taking this into account we find the vector gauge field equation to be

$$(EF)^{a'\mu} = \nabla_\lambda \left(c_{\underline{r}} v^{\underline{r}} F^{a'\lambda\mu}\right) + \frac{1}{2} \varsigma_{\underline{rs}} c^{\underline{r}} G^{\underline{s}\alpha\beta\mu} F^{a'}_{\alpha\beta} - v_{\underline{r}} c^{\underline{r}} f^{a'b'c'} F^{b'\lambda\mu} A^{c'}_\lambda + 2 g_{\underline{IJ}} D^\mu \phi^{\underline{I}} \xi^{a' \underline{J}} = 0 \ . \tag{3.8}$$

Finally, we note that the second order equation of motion for the 2-form gauge potentials $B^r_{\mu\nu}$ is given by

$$(EG)^{\mu\nu}_{\underline{r}} = \nabla_\lambda \left(\varsigma_{\underline{rs}} G^{\underline{s}\lambda\mu\nu}\right) + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma\delta\tau} c_{\underline{r}} F^{a'}_{\rho\sigma} F^{a'}_{\delta\tau} = 0 \ . \tag{3.9}$$

We shall now consider each of the integrability conditions in turn and derive the field equations from the KSEs.

3.3 Integrability of the Gravitino KSE

We begin with the first condition in (3.1) which has already been discussed in the previous chapter and is given in equation (2.24). We now contract this equation with γ^ν

$$\gamma^\nu \left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] \epsilon = \gamma^\nu \Big(\frac{1}{4} \hat{R}_{\mu\nu,\rho\sigma} \gamma^{\rho\sigma} \epsilon + \mathcal{F}^{r'}_{\mu\nu} \rho_{r'} \epsilon\Big) = 0 \ . \tag{3.10}$$

Writing these in terms of the fundamental fields of (2.7) we find

$$\gamma^\nu \left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] \epsilon = \frac{1}{8} \nabla_\lambda H^\lambda{}_{\rho\sigma} g_{\mu\nu} \gamma^{\nu\rho\sigma} \epsilon + \Big(-\frac{1}{2} R_{\mu\nu} + \frac{1}{8} H_{\mu\varepsilon\rho} H^{\varepsilon\rho}{}_\nu \Big) \gamma^\nu \epsilon$$

$$-\frac{1}{4}\nabla^\sigma H_{\sigma\mu\nu}\gamma^\nu\epsilon - F^{a'}_{\mu\nu}\xi^{\underline{I}}_{a'}\mathcal{A}^{r'}_{\underline{I}}\gamma^\nu\rho_{r'}\epsilon$$
$$+D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\mathcal{F}^{r'}_{\underline{IJ}}\gamma^\nu\rho_{r'}\epsilon = 0 \ , \tag{3.11}$$

where

$$\mathcal{F}^{r'}_{\underline{IJ}} = \partial_{\underline{I}}\mathcal{A}^{r'}_{\underline{J}} - \partial_{\underline{J}}\mathcal{A}^{r'}_{\underline{I}} + 2\epsilon^{r'}{}_{s't'}\mathcal{A}^{s'}_{\underline{I}}\mathcal{A}^{t'}_{\underline{J}} \ , \tag{3.12}$$

and the other terms have been defined in (2.7). Note that we have already started to group terms according to their symmetry properties as well as the rank of gamma matrices they appear with. To rewrite some of these terms we make use of the algebraic identities that one can obtain from the KSEs. In particular, multiplying the gaugini KSE with $v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\nu}\gamma^{\alpha\mu\nu}$ we obtain one expression, multiplying it with $F^{a'}_{\mu\nu}\gamma^\nu$ we get another and multiplying it with $(\frac{1}{4}F^{a'}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}\mu^{a'}_{r'}\rho^{r'})$ gives another, expressions for these along with various other identities can be found in appendix A. Using a combination of these expressions we rewrite the fifth term appearing in (3.11) as

$$\begin{aligned}
-F^{a'}_{\mu\nu}\xi^{\underline{I}a'}\mathcal{A}^{r'}_{\underline{I}}\rho_{r'}\gamma^\nu\epsilon \quad = \quad & -\frac{1}{32}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^\nu\epsilon + \frac{1}{64}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'}_{\delta\gamma}\epsilon^{\alpha\beta\delta\gamma}{}_{\rho\sigma}g_{\mu\nu}\gamma^{\nu\rho\sigma} \\
& +\frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu}\gamma^\nu\epsilon - \frac{1}{16}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu}\gamma^\nu\epsilon \\
& +\frac{1}{2v_{\underline{r}}c^{\underline{r}}}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu}\gamma^\nu\epsilon \ ,
\end{aligned} \tag{3.13}$$

and once this is substituted into (3.11) we find

$$\begin{aligned}
&\Big(-\frac{1}{2}R_{\mu\nu} + \frac{1}{8}H_{\mu\varepsilon\rho}H^{\varepsilon\rho}{}_\nu + \frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu} \\
&-\frac{1}{16}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu} + \frac{1}{2v_{\underline{r}}c^{\underline{r}}}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu}\Big)\gamma^\nu\epsilon \\
&+\frac{1}{8}\Big(\nabla_\lambda H^\lambda{}_{\rho\sigma} + \frac{1}{8}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'}_{\delta\gamma}\epsilon^{\alpha\beta\delta\gamma}{}_{\rho\sigma}\Big)g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon \\
&-\frac{1}{4}\Big(\nabla^\sigma H_{\sigma\mu\nu} + \frac{1}{8}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\Big)\gamma^\nu\epsilon \\
&+D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\mathcal{F}^{r'}_{\underline{IJ}}\gamma^\nu\rho_{r'}\epsilon \quad = \quad 0 \ .
\end{aligned} \tag{3.14}$$

To continue we make use of an identity that comes from multiplying the tensorini KSE with $H^{\underline{M}}_{\mu\nu\rho}\gamma^{\nu\rho}$, which is given by

$$\begin{aligned}
&-\frac{1}{2}\partial_\mu v_{\underline{r}}\partial_\nu v^{\underline{r}}\gamma^\nu\epsilon - \frac{1}{4}v^{\underline{r}}\nabla_\lambda\left(x^{\underline{M}}_{\underline{r}}H^{\underline{M}\lambda}{}_{\mu\nu}\right)\gamma^\nu\epsilon + \frac{1}{8}v^{\underline{r}}\nabla_\lambda\left(x^{\underline{M}}_{\underline{r}}H^{\underline{M}\lambda}{}_{\rho\sigma}\right)g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon \\
&+\frac{1}{8}H^{\underline{M}}_{\mu\lambda\sigma}H^{\underline{M}\lambda\sigma}{}_\nu\gamma^\nu\epsilon + \frac{1}{16}H^{\underline{M}}_{\mu\lambda\nu}H^{\underline{M}\lambda}{}_{\rho\sigma}\gamma^{\nu\rho\sigma}\epsilon = 0 \ ,
\end{aligned} \tag{3.15}$$

in writing this we have also used

$$x^{\underline{M}}_{\underline{r}}\partial_\lambda v^{\underline{r}}H^{\underline{M}\lambda}{}_{\mu\nu} = -v^{\underline{r}}\nabla_\lambda\left(x^{\underline{M}}_{\underline{r}}H^{\underline{M}\lambda}{}_{\mu\nu}\right) \ . \tag{3.16}$$

Note also that the last term in (3.15) vanishes because the duality of the 3-form implies

$$H^{\underline{M}}_{\mu\lambda[\nu}H^{\underline{M}\lambda}{}_{\rho\sigma]} = 0 \ . \tag{3.17}$$

Using these and

$$\nabla_\lambda H^\lambda{}_{\mu\nu} = v^{\underline{r}}\nabla_\lambda(\varsigma_{\underline{rs}}G^{\underline{s}\lambda}{}_{\mu\nu}) - v^{\underline{r}}\nabla_\lambda\left(x^{\underline{M}}_{\underline{r}}H^{\underline{M}\lambda}{}_{\mu\nu}\right) \ , \tag{3.18}$$

the integrability condition now becomes

$$\begin{aligned}\Big(-\frac{1}{2}R_{\mu\nu} - \frac{1}{2}\partial_\mu v_{\underline{r}}\partial_\nu v^{\underline{r}} + \frac{1}{8}\varsigma_{\underline{rs}}G^{\underline{r}}_{\mu\alpha\beta}G^{\underline{s}\,\alpha\beta}_{\nu} + \frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu} \\ -\frac{1}{16}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu} + \frac{1}{2v_{\underline{r}}c^{\underline{r}}}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu}\Big)\gamma^\nu\epsilon \\ +\frac{1}{8}v^{\underline{r}}\Big(\nabla_\lambda\left(\varsigma_{\underline{rs}}G^{\underline{s}\lambda}{}_{\mu\nu}\right) + \frac{1}{8}c_{\underline{r}}F^{a'}_{\alpha\beta}F^{a'}_{\delta\gamma}\epsilon^{\alpha\beta\delta\gamma}{}_{\rho\sigma}\Big)g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon \\ -\frac{1}{4}v^{\underline{r}}\Big(\nabla_\lambda\left(\varsigma_{\underline{rs}}G^{\underline{s}\lambda}{}_{\mu\nu}\right) + \frac{1}{8}c_{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\Big)\gamma^\nu\epsilon \\ D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\mathcal{F}^{r'}_{\underline{IJ}}\gamma^\nu\rho_{r'}\epsilon &= 0.\end{aligned} \tag{3.19}$$

For the final step we need to find a way to rewrite the last term. To do so we make use of a number of relations that the vielbeins $E^{\underline{aA}}_{\underline{I}}$ on the Quaternionic Kähler manifold satisfy in order for them to be covariantly constant, these are [55, 57]

$$\begin{aligned} E^{\underline{I}}_{\underline{aA}}E^{\underline{J}}_{\underline{bB}}g_{\underline{IJ}} &= \epsilon_{\underline{AB}}\epsilon_{\underline{ab}} \ , \\ E^{\underline{I}}_{\underline{aA}}E^{\underline{JbA}} + E^{\underline{J}}_{\underline{aA}}E^{\underline{IbA}} &= \frac{1}{n_H}g^{\underline{IJ}}\delta^{\underline{a}}_{\underline{b}} \ , \\ E^{\underline{I}}_{\underline{aA}}E^{\underline{JaB}} + E^{\underline{J}}_{\underline{aA}}E^{\underline{IaB}} &= g^{\underline{IJ}}\delta^{\underline{B}}_{\underline{A}} \ . \end{aligned} \tag{3.20}$$

We also note that $\mathcal{F}^{\underline{AB}}_{\underline{IJ}}$ can be written in terms of $E^{\underline{aA}}_{\underline{I}}$ as [55, 57]

$$\mathcal{F}_{\underline{IJAB}} = E_{\underline{IaA}}E^{\underline{a}}_{\underline{JB}} + E_{\underline{IaB}}E^{\underline{a}}_{\underline{JA}} \ . \tag{3.21}$$

Using these expressions we find

$$\mathcal{F}_{\underline{IJAB}} = 2E_{\underline{IaA}}E^{\underline{a}}_{\underline{JB}} - g_{\underline{IJ}}\epsilon_{\underline{AB}} \ , \tag{3.22}$$

and this in turn means

$$D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\mathcal{F}^{r'}_{\underline{IJ}}\gamma^\nu\rho_{r'}\epsilon = g_{\underline{IJ}}D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\gamma^\nu\epsilon + 2E_{\underline{IaA}}E^{\underline{a}}_{\underline{JB}}D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\gamma^\nu\epsilon^{\underline{B}} \ , \tag{3.23}$$

but the last term here vanishes due to the hyperini KSE and so we find

$$D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\mathcal{F}^{r'}_{\underline{IJ}}\gamma^\nu\rho_{r'}\epsilon = g_{\underline{IJ}}D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\gamma^\nu\epsilon \ . \tag{3.24}$$

Therefore, the integrability condition in (3.19) becomes

$$\Big(- \frac{1}{2}R_{\mu\nu} - \frac{1}{2}\partial_\mu v_{\underline{r}} \partial_\nu v^{\underline{r}} + \frac{1}{8}\varsigma_{\underline{rs}} G^{\underline{r}}_{\mu\alpha\beta} G^{\underline{s}}_{\nu}{}^{\alpha\beta} + \frac{1}{2} v_{\underline{r}} c^{\underline{r}} F^{a'}_{\mu\lambda} F^{a'}_{\nu}{}^{\lambda} + g_{\underline{IJ}} D_\mu \phi^{\underline{I}} D_\nu \phi^{\underline{J}}$$
$$- \frac{1}{16} v_{\underline{r}} c^{\underline{r}} F^{a'}_{\alpha\beta} F^{a'\alpha\beta} g_{\mu\nu} + \frac{1}{2 v_{\underline{r}} c^{\underline{r}}} \mathcal{A}_{\underline{I}r'} \mathcal{A}^{r'}_{\underline{J}} \xi^{\underline{I}a'} \xi^{\underline{J}a'} g_{\mu\nu} \Big) \gamma^\nu \epsilon$$
$$+ \frac{1}{8} v^{\underline{r}} \Big(\nabla_\lambda \left(\varsigma_{\underline{rs}} G^{\underline{s}\lambda}{}_{\rho\sigma} \right) + \frac{1}{8} c_{\underline{r}} F^{a'}_{\alpha\beta} F^{a'}_{\delta\gamma} \epsilon^{\alpha\beta\delta\gamma}{}_{\rho\sigma} \Big) g_{\mu\nu} \gamma^{\nu\rho\sigma} \epsilon$$
$$- \frac{1}{4} v^{\underline{r}} \Big(\nabla_\lambda \left(\varsigma_{\underline{rs}} G^{\underline{s}\lambda}{}_{\mu\nu} \right) + \frac{1}{8} c_{\underline{r}} F^{a'}_{\rho\sigma} F^{a'}_{\delta\gamma} \epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu} \Big) \gamma^\nu \epsilon = 0 \tag{3.25}$$

In the short hand notation introduced in the previous section we can write this as

$$\gamma^\nu \left[\mathcal{D}_\mu, \mathcal{D}_\nu \right] \epsilon = E_{\mu\nu} \gamma^\nu \epsilon + \frac{v^{\underline{r}}}{8} (EG)_{\underline{r}\rho\sigma} g_{\mu\nu} \gamma^{\nu\rho\sigma} \epsilon - \frac{v^{\underline{r}}}{4} (EG)_{\underline{r}\mu\nu} \gamma^\nu \epsilon = 0 \ . \tag{3.26}$$

3.4 Integrability of the Tensorini KSE

Next, we derive the scalar field equation of the tensor multiplets using the integrability condition. For this we consider

$$\gamma^\mu \Big[\mathcal{D}_\mu, T^{\underline{M}}_{\nu} \gamma^\nu - \frac{1}{12} H^{\underline{M}}_{\nu\rho\sigma} \gamma^{\nu\rho\sigma} \Big] \epsilon \ = \ 0 \ . \tag{3.27}$$

When evaluated using the fundamental fields in (2.7) this becomes

$$\Big(\partial^\mu x^{\underline{M}}_{\underline{r}} \partial_\mu v^{\underline{r}} + x^{\underline{M}}_{\underline{r}} \nabla^\mu \partial_\mu v^{\underline{r}} \Big) \epsilon$$
$$+ \Big(\partial_\mu x^{\underline{M}}_{\underline{r}} \partial_\nu v^{\underline{r}} + x^{\underline{M}}_{\underline{r}} \nabla_\mu \partial_\nu v^{\underline{r}} - \frac{1}{2} H_{\mu\nu}{}^{\lambda} x^{\underline{M}}_{\underline{r}} \partial_\lambda v^{\underline{r}}$$
$$- \frac{1}{2} \nabla^\lambda H^{\underline{M}}_{\lambda\mu\nu} - \frac{1}{2} H_\mu{}^{\lambda\sigma} H^{\underline{M}}_{\lambda\sigma\nu} \Big) \gamma^{\mu\nu} \epsilon \ = \ 0 \ . \tag{3.28}$$

To continue, we need to make use of some identities coming from the KSEs. Firstly, note that multiplying the tensorini KSE with $H_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}$ allows us to write the last term as

$$- \frac{1}{2} H_{\mu\alpha\beta} H^{\underline{M}\alpha\beta}_{\nu} \gamma^{\mu\nu} \epsilon \ = \ x^{\underline{M}}_{\underline{r}} \nabla_\lambda \left(v^{\underline{r}} H^{\lambda}{}_{\mu\nu} \right) \gamma^{\mu\nu} + \frac{1}{6} v_s x^{\underline{M}}_{\underline{r}} G^{\underline{s}\mu\nu\rho} G^{\underline{r}}_{\mu\nu\rho} \ , \tag{3.29}$$

and multiplying the tensorini KSE with $x^{\underline{M}}_{\underline{s}} \partial_\mu x^{\underline{sN}} \gamma^\mu$ means

$$- \partial_\mu x^{\underline{M}}_{\underline{r}} \partial_\nu v^{\underline{r}} \gamma^{\mu\nu} \epsilon - \partial_\mu x^{\underline{M}}_{\underline{r}} \partial^\mu v^{\underline{r}} \epsilon - \frac{1}{2} x^{\underline{N}}_{\underline{s}} x^{\underline{N}}_{\underline{r}} \partial_\lambda x^{\underline{sM}} G^{\underline{r}\lambda}{}_{\mu\nu} \gamma^{\mu\nu} \epsilon = 0 \ . \tag{3.30}$$

In addition multiplying the gaugini KSE with $x^{\underline{M}}_{\underline{r}} c^{\underline{r}} (\frac{1}{2} F^{a'}_{\mu\nu} \gamma^{\mu\nu} - \mu^{a'}_{r'} \rho^{r'})$ gives

$$\frac{1}{16} x^{\underline{M}}_{\underline{r}} c^{\underline{r}} F^{a'}_{\rho\sigma} F^{a'}_{\delta\gamma} \epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu} \gamma^{\mu\nu} \epsilon + \frac{1}{4} x^{\underline{M}}_{\underline{r}} c^{\underline{r}} F^{a'}_{\mu\nu} F^{a'\mu\nu} \epsilon$$
$$- \frac{2 x^{\underline{M}}_{\underline{r}} c^{\underline{r}}}{(v_{\underline{s}} c^{\underline{s}})^2} \mathcal{A}_{\underline{I}r'} \mathcal{A}^{r'}_{\underline{J}} \xi^{\underline{I}a'} \xi^{\underline{J}a'} \epsilon \ = \ 0 \ . \tag{3.31}$$

Using these three identities allows us to write the integrability condition as

$$x^{\underline{M}}_{\underline{r}}\Big(\nabla^{\mu}\partial_{\mu}v^{\underline{r}}+\frac{1}{4}c^{\underline{r}}F^{a'}_{\mu\nu}F^{a'\mu\nu}+\frac{1}{6}v_{\underline{s}}G^{\underline{s}\mu\nu\rho}G^{\underline{r}}_{\mu\nu\rho}-\frac{2c^{\underline{r}}}{(v_{\underline{s}}c^{\underline{s}})^2}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}\Big)\epsilon$$
$$+\frac{1}{2}x^{\underline{M}}_{\underline{r}}\Big(\nabla_{\lambda}\left(\varsigma^{\underline{rs}}G^{\lambda}_{\underline{s}\,\mu\nu}\right)+\frac{1}{8}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\Big)\gamma^{\mu\nu}\epsilon=0\ , \tag{3.32}$$

where we have also used

$$\frac{1}{2}x^{\underline{M}}_{\underline{r}}\nabla_{\lambda}\left(\varsigma^{\underline{rs}}G^{\lambda}_{\underline{s}\,\mu\nu}\right)=-\frac{1}{2}\nabla^{\lambda}H^{\underline{M}}_{\lambda\mu\nu}+\frac{1}{2}x^{\underline{M}}_{\underline{r}}\nabla_{\lambda}\left(v^{\underline{r}}H^{\lambda}{}_{\mu\nu}\right)-\frac{1}{2}x^{\underline{N}}_{\underline{s}}x^{\underline{N}}_{\underline{r}}\partial_{\lambda}x^{\underline{sM}}G^{\underline{r}\lambda}{}_{\mu\nu}\ . \tag{3.33}$$

Using the compact notation, the integrability condition in (3.32) can be written as

$$\gamma^{\mu}\Big[\mathcal{D}_{\mu},T^{\underline{M}}_{\nu}\gamma^{\nu}-\frac{1}{12}H^{\underline{M}}_{\nu\rho\sigma}\gamma^{\nu\rho\sigma}\Big]\epsilon=x^{\underline{M}}_{\underline{r}}(Ev)^{\underline{r}}\epsilon+\frac{1}{2}x^{\underline{M}}_{\underline{r}}(EG)^{\underline{r}}_{\mu\nu}\gamma^{\mu\nu}\epsilon=0\ . \tag{3.34}$$

3.5 Integrability of the Hyperini KSE

To derive the integrability condition of the hyperini KSE we consider

$$\gamma^{\mu}\Big[\mathcal{D}_{\mu},\gamma^{\nu}V^{\underline{aA}}_{\nu}\Big]\epsilon_{\underline{A}}\ =\ 0\ , \tag{3.35}$$

which, when evaluated using the physical fields becomes

$$\Big(D^{\mu}\phi^{\underline{I}}D_{\mu}\phi^{\underline{J}}\partial_{\underline{J}}E^{\underline{aA}}_{\underline{I}}+A^{a'}_{\mu}\xi^{\underline{J}}_{a'}\partial_{\underline{J}}E^{\underline{aA}}_{\underline{I}}D^{\mu}\phi^{\underline{I}}+E^{\underline{aA}}_{\underline{I}}\nabla_{\mu}D^{\mu}\phi^{\underline{I}}\Big)\epsilon_{\underline{A}}$$
$$+\Big(D_{\mu}\phi^{\underline{I}}D_{\nu}\phi^{\underline{J}}\partial_{\underline{I}}E^{\underline{aA}}_{\underline{J}}+A^{a'}_{\mu}\xi^{\underline{J}}_{a'}\partial_{\underline{J}}E^{\underline{aA}}_{\underline{I}}D_{\nu}\phi^{\underline{I}}$$
$$+E^{\underline{aA}}_{\underline{I}}\nabla_{\mu}D_{\nu}\phi^{\underline{I}}-\frac{1}{2}H_{\mu\nu}{}^{\lambda}E^{\underline{aA}}_{\underline{I}}D_{\lambda}\phi^{\underline{I}}\Big)\gamma^{\mu\nu}\epsilon_{\underline{A}}\ =\ 0\ . \tag{3.36}$$

Note that by expanding and using the antisymmetry in the spacetime indices we can write the fifth and sixth terms that appear in the above expression as

$$\Big(A^{a'}_{\mu}\xi^{\underline{J}}_{a'}\partial_{\underline{J}}E^{\underline{aA}}_{\underline{I}}D_{\nu}\phi^{\underline{I}}+E^{\underline{aA}}_{\underline{I}}\nabla_{\mu}D_{\nu}\phi^{\underline{I}}\Big)\gamma^{\mu\nu}\epsilon_{\underline{A}}=-\frac{1}{2}E^{\underline{aA}}_{\underline{I}}F^{a'}_{\mu\nu}\xi^{\underline{I}}_{a'}\gamma^{\mu\nu}\epsilon_{\underline{A}}\ . \tag{3.37}$$

Multiplying the hyperini KSE with $H_{\mu\nu\rho}\gamma^{\mu\nu\rho}$ we find

$$E^{\underline{aA}}_{\underline{I}}H_{\mu\nu}{}^{\lambda}D_{\lambda}\phi^{\underline{I}}\gamma^{\mu\nu}\epsilon_{\underline{A}}=0\ , \tag{3.38}$$

which means the last term in (3.36) vanishes. We also have

$$\xi^{\underline{I}}_{a'}\partial_{\underline{I}}E^{\underline{aA}}_{\underline{J}}+\partial_{\underline{J}}\xi^{\underline{I}}_{a'}E^{\underline{aA}}_{\underline{I}}=0\ . \tag{3.39}$$

Using the above three expressions the integrability condition in (3.36) can be written as

$$\Big(D^\mu\phi^{\underline{I}}D_\mu\phi^{\underline{J}}\partial_{\underline{I}}E^{aA}_{\underline{J}} - A^{a'}_\mu\partial_{\underline{I}}\xi^{\underline{J}}_{a'}E^{aA}_{\underline{J}}D^\mu\phi^{\underline{I}} + E^{aA}_{\underline{I}}\nabla_\mu D^\mu\phi^{\underline{I}}\Big)\epsilon_{\underline{A}} + \Big(D_\mu\phi^{\underline{I}}D_\nu\phi^{\underline{J}}\partial_{\underline{I}}E^{aA}_{\underline{J}} - \frac{1}{2}E^{aA}_{\underline{I}}F^{a'}_{\mu\nu}\xi^{\underline{I}}_{a'}\Big)\gamma^{\mu\nu}\epsilon_{\underline{A}} = 0. \quad (3.40)$$

The fact that the vielbeins $E^{\underline{I}}_{\underline{aA}}$ are covariantly constant means

$$\nabla_{\underline{I}}E^{aA}_{\underline{J}} = \partial_{\underline{I}}E^{aA}_{\underline{J}} - \Gamma^{\underline{K}}_{\underline{IJ}}E^{aA}_{\underline{K}} + \mathcal{A}^{a}_{\underline{I}b}E^{bA}_{\underline{J}} + \mathcal{A}^{A}_{\underline{I}\,\underline{B}}E^{aB}_{\underline{J}} = 0\ , \quad (3.41)$$

where

$$\Gamma^{\underline{K}}_{\underline{IJ}} = \frac{1}{2}g^{\underline{KL}}(\partial_{\underline{I}}g_{\underline{JL}} + \partial_{\underline{J}}g_{\underline{IL}} - \partial_{\underline{L}}g_{\underline{IJ}})\ , \quad (3.42)$$

is the Christoffel connection of the Quaternionic Kähler manifold parametrised by the hypermultiplets. We can use this to rewrite (3.40) as

$$E^{aA}_{\underline{K}}\Big(\Gamma^{\underline{K}}_{\underline{IJ}}D^\mu\phi^{\underline{I}}D_\mu\phi^{\underline{J}} - D^\mu\phi^{\underline{I}}A^{a'}_\mu\partial_{\underline{I}}\xi^{\underline{K}}_{a'} + \nabla_\mu D^\mu\phi^K\Big)\epsilon_{\underline{A}} - \frac{1}{2}E^{aA}_{\underline{I}}F^{a'}_{\mu\nu}\xi^{\underline{I}}_{a'}\gamma^{\mu\nu}\epsilon_{\underline{A}} = 0\ . \quad (3.43)$$

The gaugini KSE together with the identities in (3.20) and (3.21) can be used to show

$$\frac{1}{2}E^{aA}_{\underline{I}}F^{a'}_{\mu\nu}\xi^{\underline{I}}_{a'}\gamma^{\mu\nu}\epsilon_{\underline{A}} = \frac{2}{v_{\underline{r}}c^{\underline{r}}}E^{aA}_{\underline{K}}g^{\underline{KL}}\mathcal{F}^{r'}_{\underline{LJ}}\mathcal{A}_{r'\underline{I}}\xi^{a'\underline{I}}\xi^{a'\underline{J}}\epsilon_{\underline{A}}\ , \quad (3.44)$$

this means the integrability condition becomes

$$\begin{aligned}\gamma^\mu\Big[\mathcal{D}_\mu, \gamma^\nu V^{aA}_\nu\Big]\epsilon_{\underline{A}} &= E^{aA}_{\underline{K}}\Big(\Gamma^{\underline{K}}_{\underline{IJ}}D^\mu\phi^{\underline{I}}D_\mu\phi^{\underline{J}} - D^\mu\phi^{\underline{I}}A^{a'}_\mu\partial_{\underline{I}}\xi^{\underline{K}}_{a'} \\ &\quad + \nabla_\mu D^\mu\phi^K - \frac{2}{v_{\underline{r}}c^{\underline{r}}}E^{aA}_{\underline{K}}g^{\underline{KL}}\mathcal{F}^{r'}_{\underline{LJ}}\mathcal{A}_{r'\underline{I}}\xi^{a'\underline{I}}\xi^{a'\underline{J}}\Big)\epsilon_{\underline{A}} \\ &= E^{aA}_{\underline{K}}(E\phi)^{\underline{K}}\epsilon_{\underline{A}} = 0\ . \end{aligned} \quad (3.45)$$

Note that this agrees with the expression for the field equation in (3.5). To see this expand the second term in (3.5) and rearrange so that (3.42) is used to write the equation in (3.5) as in the lhs of (3.45).

3.6 Integrability of the Gaugini KSE

Lastly, we consider the integrability condition of the gaugini KSE. This is given by

$$\gamma^\mu\Big[\mathcal{D}_\mu, \frac{1}{2}F^{a'}_{\nu\rho}\gamma^{\nu\rho} + \mu^{a'}_{r'}\Big]\epsilon = 0\ , \quad (3.46)$$

evaluating this and using the physical fields we find

$$
\begin{aligned}
&\frac{1}{2}\hat{\nabla}_\mu F^{a'}_{\nu\rho}\gamma^{\mu\nu\rho}\epsilon + \nabla_\lambda F^{a'\lambda}{}_\mu\gamma^\mu\epsilon + \frac{1}{2}H_\mu{}^{\alpha\beta}F^{a'}_{\alpha\beta}\gamma^\mu\epsilon \\
&+\frac{2\left(\partial_\mu v_{\underline{r}}\right)c^{\underline{r}}}{\left(v_{\underline{s}}c^{\underline{s}}\right)^2}\mathcal{A}^{r'}_{\underline{I}}\xi^{\underline{I}}_{a'}\rho_{r'}\gamma^\mu\epsilon + \frac{2}{v_{\underline{s}}c^{\underline{s}}}\mathcal{F}^{r'}_{\underline{IJ}}D_\mu\phi^{\underline{I}}\xi^{a'\underline{J}}\rho_{r'}\gamma^\mu\epsilon \\
&+\frac{2}{v_{\underline{s}}c^{\underline{s}}}\mathcal{A}^{r'}_{\underline{I}}A^{b'}_\mu f^{a'b'c'}\xi^{\underline{I}}_{c'}\rho_{r'}\gamma^\mu\epsilon \quad = \quad 0.
\end{aligned}
\tag{3.47}
$$

To continue, we make use of two identities derived from the gaugini KSE. These identities are obtained by multiplying the gaugini KSE with $\frac{\partial_\mu v_{\underline{s}}c^{\underline{s}}}{v_{\underline{r}}c^{\underline{r}}}\gamma^\mu$ and with $f^{a'b'c'}A^{b'}_\mu\gamma^\mu$, see appendix A for the explicit expressions. This then allows us to write the fourth and sixth terms in (3.47) as

$$
\begin{aligned}
&\frac{2\left(\partial_\mu v_{\underline{r}}\right)c^{\underline{r}}}{\left(v_{\underline{s}}c^{\underline{s}}\right)^2}\mathcal{A}^{r'}_{\underline{I}}\xi^{a'\underline{I}}\rho_{r'}\gamma^\mu\epsilon + \frac{2}{v_{\underline{s}}c^{\underline{s}}}\mathcal{A}^{r'}_{\underline{I}}A^{b'}_\mu f^{a'b'c'}\xi^{\underline{I}}_{c'}\rho_{r'}\gamma^\mu\epsilon = \\
&\qquad\frac{1}{2}\left(\frac{\left(\partial_\mu v_{\underline{r}}\right)c^{\underline{r}}}{\left(v_{\underline{s}}c^{\underline{s}}\right)}F^{a'}_{\nu\rho} + A^{b'}_\mu f^{a'b'c'}F^{c'}_{\nu\rho}\right)\gamma^{\mu\nu\rho}\epsilon \\
&\qquad+\left(\frac{\left(\partial^\lambda v_{\underline{r}}\right)c^{\underline{r}}}{\left(v_{\underline{s}}c^{\underline{s}}\right)}F^{a'}_{\lambda\mu} + A^{\lambda b'}f^{a'b'c'}F^{c'}_{\lambda\mu}\right)\gamma^\mu\epsilon\ .
\end{aligned}
\tag{3.48}
$$

Furthermore, multiplying the tensorini KSE with $x^{\underline{M}}_{\underline{s}}c^{\underline{s}}F^{a'}_{\alpha\beta}\gamma^{\alpha\beta}$ and simplifying we find

$$
\begin{aligned}
&\left(\frac{\left(\partial^\lambda v_{\underline{s}}\right)c^{\underline{s}}}{\left(v_{\underline{r}}c^{\underline{r}}\right)}F^{a'}_{\lambda\mu} - \frac{1}{2}H_\mu{}^{\alpha\beta}F^{a'}_{\alpha\beta} + \frac{1}{2v_{\underline{t}}c^{\underline{t}}}\varsigma_{\underline{rs}}c^{\underline{r}}G^{\underline{s}\alpha\beta}{}_\mu F^{a'}_{\alpha\beta}\right)\gamma^\mu\epsilon \\
&\quad+\left(-\frac{\left(\partial_\mu v_{\underline{s}}\right)c^{\underline{s}}}{2\left(v_{\underline{r}}c^{\underline{r}}\right)}F^{a'}_{\nu\rho} + \frac{1}{4v_{\underline{t}}c^{\underline{t}}}\varsigma_{\underline{rs}}c^{\underline{r}}G^{\underline{s}\lambda}{}_{\nu\rho}F^{a'}_{\lambda\mu}\right)\gamma^{\mu\nu\rho}\epsilon \quad = \quad 0\ .
\end{aligned}
\tag{3.49}
$$

Now, we can use this along with (3.48) to write the integrability condition in (3.47) as

$$
\begin{aligned}
&\frac{1}{2}\left(\hat{\nabla}_\mu F^{a'}_{\nu\rho} + A^{b'}_\mu f^{a'b'c'}F^{c'}_{\nu\rho}\right)\gamma^{\mu\nu\rho}\epsilon \\
&+\left(\nabla_\lambda F^{a'\lambda}{}_\mu + \frac{2\left(\partial^\lambda v_{\underline{s}}\right)c^{\underline{s}}}{\left(v_{\underline{r}}c^{\underline{r}}\right)}F^{a'}_{\lambda\mu} + A^{\lambda b'}f^{a'b'c'}F^{c'}_{\lambda\mu} + \frac{1}{2v_{\underline{t}}c^{\underline{t}}}\varsigma_{\underline{rs}}c^{\underline{r}}G^{\underline{s}\alpha\beta}{}_\mu F^{a'}_{\alpha\beta}\right)\gamma^\mu\epsilon \\
&+\frac{1}{4v_{\underline{t}}c^{\underline{t}}}\varsigma_{\underline{rs}}c^{\underline{r}}G^{\underline{s}\lambda}{}_{\nu\rho}F^{a'}_{\lambda\mu}\gamma^{\mu\nu\rho}\epsilon + \frac{2}{v_{\underline{r}}c^{\underline{r}}}\mathcal{F}^{r'}_{\underline{IJ}}D_\mu\phi^{\underline{J}}\xi^{a'\underline{I}}\rho_{r'}\gamma^\mu\epsilon = 0\ .
\end{aligned}
\tag{3.50}
$$

In a similar way to deriving the expression in (3.24) we find that the last term can be written as

$$
\frac{2}{v_{\underline{r}}c^{\underline{r}}}\mathcal{F}^{r'}_{\underline{IJ}}D_\mu\phi^{\underline{J}}\xi^{\underline{I}}_{a'}\rho_{r'}\gamma^\mu\epsilon = \frac{2}{v_{\underline{r}}c^{\underline{r}}}g_{\underline{IJ}}D_\mu\phi^{\underline{I}}\xi^{\underline{J}}_{a'}\gamma^\mu\epsilon\ .
\tag{3.51}
$$

In addition, taking the commutator between the tensorini and gaugini KSEs and multiplying the result with $x_{\underline{s}}^{\underline{M}} c^{\underline{s}}$ we find

$$\left(c_{\underline{r}} \partial_\lambda v^{\underline{r}} F^{a'\lambda}{}_\mu \gamma^\mu + \frac{1}{4} \varsigma_{\underline{rs}} c^{\underline{r}} G^{\underline{s}\lambda}{}_{\nu\rho} F^{a'}_{\lambda\mu} \gamma^{\mu\nu\rho}\right) \epsilon = 0 \ , \tag{3.52}$$

where we have also used

$$H_{\lambda\mu\nu} F^{a'\lambda}{}_\rho \Gamma^{\mu\nu\rho} \epsilon = 0 \ , \tag{3.53}$$

coming as a consequence of the duality relations. Using (3.51) and (3.52) means (3.50) becomes

$$\begin{aligned} \gamma^\mu \Big[\mathcal{D}_\mu, \frac{1}{2} F^{a'}_{\nu\rho} \gamma^{\nu\rho} + \mu^{a'}_{r'}\Big] \epsilon \ &= \ \Big(\nabla_\lambda \left(v_{\underline{r}} c^{\underline{r}} F^{a'\lambda}_\mu\right) - v_{\underline{r}} c^{\underline{r}} A^{\lambda c'} f^{a'b'c'} F^{b'}_{\lambda\mu} \\ &\quad + \frac{1}{2} \varsigma_{\underline{rs}} c^{\underline{r}} G^{\underline{s}\alpha\beta}{}_\mu F^{a'}_{\alpha\beta} + 2 g_{\underline{IJ}} D_\mu \phi^{\underline{I}} \xi^{a'\underline{J}}\Big) \gamma^\mu \epsilon \\ &\quad + \frac{1}{2} v_{\underline{r}} c^{\underline{r}} \left(\partial_\mu F^{a'}_{\nu\rho} + f^{a'b'c'} A^{b'}_\mu F^{c'}_{\nu\rho}\right) \gamma^{\mu\nu\rho} \epsilon \\ &= \ (EF)^{a'}_\mu \gamma^\mu \epsilon + \frac{1}{2} v_{\underline{r}} c^{\underline{r}} (BF)^{a'}_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon = 0 \ . \end{aligned} \tag{3.54}$$

where $(BF)^{a'}_{\mu\nu\rho}$ is the Bianchi identity of $F^{a'}$. This concludes our analysis of the integrability conditions.

3.7 Summary

In this chapter the integrability conditions of the KSEs were derived, these provide an important consistency check for the theory. In order to obtain the integrability conditions a detailed analysis of the KSEs was carried out to determine various identities. The integrability conditions can be summarised as

$$\begin{aligned} \gamma^\nu \left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] \epsilon \ &= \ E_{\mu\nu} \gamma^\nu \epsilon + \frac{v^{\underline{r}}}{8} (EG)_{\rho\sigma} g_{\mu\nu} \gamma^{\nu\rho\sigma} \epsilon - \frac{v^{\underline{r}}}{4} (EG)_{\mu\nu} \gamma^\nu \epsilon = 0 \ , \\ \gamma^\mu \Big[\mathcal{D}_\mu, T^{\underline{M}}_\nu \gamma^\nu - \frac{1}{12} H^{\underline{M}}_{\nu\rho\sigma} \gamma^{\nu\rho\sigma}\Big] \epsilon \ &= \ x^{\underline{M}}_{\underline{r}} (Ev)^{\underline{r}} \epsilon + \frac{1}{2} x^{\underline{M}}_{\underline{r}} (EG)^{\underline{r}}_{\mu\nu} \gamma^{\mu\nu} \epsilon = 0 \ , \\ \gamma^\mu \Big[\mathcal{D}_\mu, \gamma^\nu V^{\underline{aA}}_\nu\Big] \epsilon_{\underline{A}} \ &= \ E^{\underline{aA}}_{\underline{K}} (E\phi)^{\underline{K}} \epsilon_{\underline{A}} = 0 \ , \\ \gamma^\mu \Big[\mathcal{D}_\mu, \frac{1}{2} F^{a'}_{\nu\rho} \gamma^{\nu\rho} + \mu^{a'}_{r'}\Big] \epsilon \ &= \ (EF)^{a'}_\mu \gamma^\mu \epsilon + \frac{1}{2} v_{\underline{r}} c^{\underline{r}} (BF)^{a'}_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon = 0 \ , \end{aligned} \tag{3.55}$$

where $E_{\mu\nu}$, $(Ev)^{\underline{r}}$, $(E\phi)^{\underline{I}}$, $(EF)^{a'}_\mu$ and $(EG)^{\underline{r}}_{\mu\nu}$ have been defined in (3.3), (3.4), (3.5), (3.8) and (3.9) respectively. These field equations will play an important role in the next chapter when we analyse the near horizon geometries of (1,0) supergravity black holes.

Chapter 4

Near Horizon Geometries of Six Dimensional (1,0) Supergravity

4.1 Introduction

In this chapter we focus on the investigation of the topology and geometry of six dimensional (1,0) supergravity black hole horizons. It is primarily based on the paper [13] and forms a natural extension of the work described in the previous chapters. In particular, we make use of the solutions to the KSEs to analyse near horizon geometries (NHG) in six dimensions. We focus on (1,0) supergravity coupled to any number of tensor and scalar multiplets. Therefore, we consider horizons for which the vector multiplets can be consistently set to zero. The reason for setting the vector multiplets to zero is due to the fact that in the presence of active vectors the field equations of the theory cannot be put into a form that allows the application of the maximum principle.

We will begin by discussing the method that is used. This will involve a discussion of Gaussian null coordinates [110] and how these are used to characterise the near horizon geometries [82, 89] of black holes. Using regularity arguments we will also determine the general form that the other fields of the theory have to take. Using this data we will solve the Killing spinor equations along with the field equations of the theory to determine the near horizon geometries. In particular, we will show that there are two classes of near horizon geometries; the first class of geometries that we discuss are locally $AdS_3 \times \Sigma^3$, where we find Σ^3 to be diffeomorphic[1] to S^3. Furthermore, we show that this class of geometries preserve 2, 4 or 8 supersymmetries and the amount of supersymmetry preserved depends on the properties of the geometry of Σ^3. The hypermultiplet scalars are also accordingly constrained.

The second class of near horizon geometries that we will discuss are $\mathbb{R}^{1,1} \times \mathcal{S}$, where the horizon section $\mathcal{S}$ is a 4-manifold whose geometry depends on the hypermultiplet scalars. We will show the tensor multiplet scalars are constant and

[1]Throughout we use S^n to denote the n-sphere with the standard "round" metric

all 3-form field strengths, including that of the gravitational multiplet, vanishes. This class of geometries preserve 1, 2 and 4 supersymmetries. These are the only two classes of near horizon geometries we find.

As a general note, when we discuss aspects of NHG the specific details of differential geometry are not given. However, we aim to be self-contained and try to give details that are required in order to continue the discussion. More details can be found for example in [62, 63, 101].

4.2 Gaussian Null Coordinates and NHG

4.2.1 Gaussian Null Coordinates

The investigation of NHGs can be carried out by adapting to Gaussian null coordinates. These are similar to Gaussian normal coordinates [62, 63] but adapted to the case of null hypersurfaces. Therefore let us start with an introduction to Gaussian normal coordinates, we mainly follow the discussion in [62, 63]. We begin with the spacetime $(\mathcal{M}, g_{\mu\nu})$, where $\mathcal{M}$ is an n-dimensional manifold, the hypersurface Σ is defined to be an $(n-1)$-dimensional submanifold. Now, consider the tangent space at a point $p \in \Sigma$ and denote this with $T\Sigma_p$, this space can also be considered as a subspace of the tangent space of $T\mathcal{M}_p$ of $\mathcal{M}$. This means that for all such vector spaces one can find a n_μ in $T\mathcal{M}_p$ which is orthogonal to all the elements of $T\Sigma_p$. To define Gaussian normal coordinates one constructs geodesics which pass through $p \in \Sigma$ with n_μ tangent to these. In particular, to parametrise a small region of space in $\mathcal{M}$ one can choose some coordinates $y^i = \{y^1, \ldots, y^{n-1}\}$ on Σ and then each point, say q, in the neighbourhood of this part of Σ lies on the unique geodesic constructed with coordinates given by $y^\mu = \{t, y^1, \ldots, y^{n-1}\}$, where t is the affine parameter of the geodesic that has been defined at each point $p \in \Sigma$ with coordinate y^i. This coordinate system is only well defined in the small region around Σ [62, 63].

Since the geodesics that have been defined at each point are unique they may eventually cross over, but until that happens Gaussian normal coordinates hold [62]. This is true because of the fact that the geodesics are orthogonal to all hypersurfaces, parametrised by Σ_t with $t = constant$, see [62, 63] for details. To demonstrate this, we first note that on the hypersurface $\Sigma_{t=0}$ we have $n_\mu n^\mu = \pm 1$ and $n_\mu Y^\mu = 0$ by construction, where Y^μ denote a set of basis vector fields. One now needs to show the directional derivative of $n_\mu Y^\mu_{(i)}$, where $Y^\mu_{(i)}$ are the basis vectors of the tangent space at Σ_t, vanishes; this will mean the inner product $n_\mu Y^\mu_{(i)}$ is preserved and so $n_\mu Y^\mu_{(i)} = 0$ for the hypersurface Σ_t. To start with

$$n^\mu \nabla_\mu (n_\nu Y^\nu_{(i)}) \;=\; n^\mu \nabla_\mu n_\nu Y^\nu_{(i)} + n^\mu n_\nu \nabla_\mu Y^\nu_{(i)} \; , \tag{4.1}$$

where we have used the Leibniz rule. Then using the fact $n^\mu \nabla_\mu n_\nu = 0$ and

$n^\mu \nabla_\mu Y^\nu_{(i)} = Y^\mu_{(i)} \nabla_\mu n^\nu$ gives [63]

$$
\begin{aligned}
n^\mu \nabla_\mu (n_\nu Y^\nu_{(i)}) &= n_\nu Y^\mu_{(i)} \nabla_\mu n^\nu \ , \\
&= \frac{1}{2} Y^\mu_{(i)} \nabla_\mu (n_\nu n^\nu) \ ,
\end{aligned}
\tag{4.2}
$$

and this vanishes since $n_\nu n^\nu$ is a constant. Note that due to the construction of the coordinate system $g_{tt} = g(\partial_t, \partial_t) = \pm 1$ and $g_{ti} = 0$ and so the metric can be written as

$$
ds^2 = \pm dt^2 + \gamma_{ij} dy^i dy^j \ , \tag{4.3}
$$

where the $\pm$ in the first term depends on whether we have a timelike or a spacelike normal vector and $\gamma_{ij}(t, y) = g(\partial_i, \partial_j)$.

We now move onto discuss the construction of Gaussian null coordinates [110]. In particular, it can be adapted to the case where there exist a time-like Killing vector field k_μ which becomes null on the horizon of a stationary black hole [82]. To do so we begin by taking an $(n-1)$-dimensional hypersurface, Σ, where the normal vector to it, n_μ, is a null vector $n_\mu n^\mu = 0$. Also note that in the case of null hypersurfaces one finds the integral curves of the null vector field to correspond to null geodesics, called the generators of the hypersurface, and the union of these correspond to the null hypersurface itself [62, 63]. Next we consider an $(n-2)$-dimensional hypersurface, denoted with Ξ, that is embedded in Σ. Following the same argument as in the discussion of Gaussian normal coordinates we consider the tangent space $T\Xi_p$ that is a subset of $T\Sigma_p$ which in turn is contained in $T\mathcal{M}_p$. For all elements of $T\Xi_p$ one can find a normal vector field k_μ whose integral curves are the null geodesics which generate Ξ. On a particular chart of Ξ one can then choose the coordinates $y^i = \{y^1, \ldots, y^{n-2}\}$ so that in a small neighbourhood of this region the geodesics with tangent k_μ and affine parameter u allow the coordinates of this region of Σ, denoted as $\tilde{\Sigma}$, to be given by $\{u, y^1, \ldots, y^{n-2}\}$ [110, 82].

Now, at each point $p \in \tilde{\Sigma}$ there is a null vector field n_μ such that $n_\mu k^\mu = 1$ and $n_\mu Y^\mu = 0$ for all Y_μ that is tangent to $\tilde{\Sigma}$, and that satisfies $Y^\mu \nabla_\mu u = 0$. For all points (u, y^i) in $\tilde{\Sigma}$ we can find geodesics with tangent n_μ and with affine parameter r so that in this small region of $\mathcal{M}$ the coordinates of the space can be written as $y^\mu = \{r, u, y^1, \ldots, y^{n-2}\}$, where the values of (u, y^i) are kept constant along these geodesics. These are referred to as Gaussian null coordinates, see [110, 82] for more details.

On $\tilde{\Sigma}$, by definition we have $k^\mu = (\partial/\partial u)^\mu$ and the other coordinates have been defined so that $n^\mu = (\partial/\partial r)^\mu$ and $g_{rr} = g(\partial_r, \partial_r) = 0$, in addition the components $g_{ru}, g_{r3}, \ldots, g_{rn}$ are all independent of r. The condition $n_\mu k^\mu = 1$ gives $g_{ru} = 1$ and $n_\mu Y^\mu = 0$ means $g_{rI} = 0$ where $I = 3, \ldots, n$. On the hypersurface $\tilde{\Sigma}$, which could be thought of as the surface where $r = 0$, one has $g_{uu} = 0$ and $g_{uI} = 0$. This means there are functions f and h_I on $\mathcal{M}$ such that when evaluated on $\tilde{\Sigma}$ we get $f|_{\tilde{\Sigma}} = (\partial g_{uu}/\partial r)|_{r=0}$ and $h_I|_{\tilde{\Sigma}} = \partial g_{uI}/\partial r|_{r=0}$ [110]. Putting these together means the metric can be written as

$$ds^2 = rfdu^2 + 2drdu + 2rh_I dudy^I + \gamma_{IJ}dy^I dy^J \ , \tag{4.4}$$

note that the vector field $k_\mu = \partial_u$ is the Killing vector field and the components f, h_I and γ_{IJ} depend on r and y^I [110, 82].

4.2.2 Near Horizon Limit

By construction we have chosen so that the horizon is located at $r = 0$. The metric γ_{IJ} is the metric on $\mathcal{M}^{n-2}$ which is defined by $u = constant$ and $r = 0$. The requirement of regularity at the horizon means the metric needs to be well behaved at $r = 0$. We assume the metric is analytic in r and this means we can expand the different components of the metric [89] as follows

$$\begin{aligned} f(r,y) &= \sum_{n=0}^{\infty} \frac{r^n}{n!}\partial_r^n f|_{r=0} \ , \\ h_I(r,y) &= \sum_{n=0}^{\infty} \frac{r^n}{n!}\partial_r^n h_I|_{r=0} \ , \\ \gamma_{IJ}(r,y) &= \sum_{n=0}^{\infty} \frac{r^n}{n!}\partial_r^n \gamma_{IJ}|_{r=0} \ . \end{aligned} \tag{4.5}$$

To determine the near horizon geometry we first perform the following coordinate transformation

$$r \to \epsilon r \ , \quad u \to \epsilon^{-1} u \ , \quad y^I \to y^I \ , \tag{4.6}$$

and then take the limit $\epsilon \to 0$ [82]. This means the metric in (4.4) becomes

$$ds^2 = r^2 F(y)du^2 + 2drdu + 2rh_I(y)dudy^I + \gamma_{IJ}(y)dy^I dy^J \ , \tag{4.7}$$

where $F(y) = \partial_r f|_{r=0}$ and all the components are independent of r. In addition, regularity requires $f(0,y) = 0$. To obtain this expression for the metric we have made use of the field expansions in (4.5). This is the near horizon limit for extreme black holes.

The six dimensional supergravity which we are considering has other fields like the 3-form gauge field strengths. We can also define a near horizon limit for these fields, which will be discussed in the next section.

4.3 Supersymmetric Horizons

4.3.1 Fields and KSEs of Six dimensional Supergravity

In this section we briefly recap the fields and the KSEs of six dimensional supergravity but in the absence of vector multiplets, i.e. coupled only to tensor and scalar multiplets. In the absence of vector multiplets the KSEs of the theory become

$$\begin{aligned} \mathcal{D}_\mu \epsilon \equiv \big(\nabla_\mu - \frac{1}{8} H_{\mu\nu\rho}\gamma^{\nu\rho} + \mathcal{C}_\mu^{r'}\rho_{r'}\big)\epsilon &= 0 \ , \\ \left(\frac{i}{2} T_\mu^{\underline{M}}\gamma^\mu - \frac{i}{24} H_{\mu\nu\rho}^{\underline{M}}\gamma^{\mu\nu\rho}\right)\epsilon &= 0 \ , \\ i\gamma^\mu \epsilon_{\underline{A}} V_\mu^{\underline{aA}} &= 0 \ , \end{aligned} \tag{4.8}$$

where the first equation is the gravitino KSE, the second is the tensorini and the third is the hyperini KSE. In addition, the physical fields defined in (2.7) reduce to

$$\begin{aligned} H_{\mu\nu\rho} &= v_{\underline{r}} G^{\underline{r}}_{\mu\nu\rho}, \quad H^{\underline{M}}_{\mu\nu\rho} = x^{\underline{M}}_{\underline{r}} G^{\underline{r}}_{\mu\nu\rho} \ , \quad \mathcal{C}_\mu{}^{\underline{A}}{}_{\underline{B}} = \partial_\mu \phi^{\underline{I}} \mathcal{A}_{\underline{I}}{}^{\underline{A}}{}_{\underline{B}} \ , \\ T^{\underline{M}}_\mu &= x^{\underline{M}}_{\underline{r}} \partial_\mu v^{\underline{r}} \ , \quad V^{\underline{aA}}_\mu = E^{\underline{aA}}_{\underline{I}} \partial_\mu \phi^{\underline{I}} \ . \end{aligned} \tag{4.9}$$

The $n_T + 1$ 3-form field strengths of the supergravity theory still satisfy the duality relation given in (2.10) but are now defined as

$$G^{\underline{r}}_{\mu\nu\rho} = 3\partial_{[\mu} B^{\underline{r}}_{\nu\rho]} \ , \quad r = 0, \ldots, n_T \ , \tag{4.10}$$

since the Chern-Simons terms vanish after setting the vector multiplets to zero. The rest of the theory is defined as in chapter 2.

4.3.2 Near Horizon Geometry

The form the spacetime metric has to take near the horizon is given in (4.7). We will now derive the form the remaining fields of the theory take in this limit. The null coordinates $\{r, u, y^1, \ldots, y^{n-2}\}$ defined in the previous section allow us to write the 2-form gauge potentials, $B^{\underline{r}}_{\mu\nu}$ as

$$B^{\underline{r}} = b^{\underline{r}} du \wedge dr + b^{\underline{r}}_I du \wedge dy^I + c^{\underline{r}}_I dr \wedge dy^I + b^{\underline{r}}_{IJ} dy^I \wedge dy^I \ , \tag{4.11}$$

where all the coefficients are functions of r and y. Then assuming analyticity of these components in the r coordinate allows us to write these components in a similar way to the components of the metric in (4.5). Using this and the coordinate transformation in (4.6) we find

$$
\begin{aligned}
B^{\underline{r}} \;=\;& \sum_{n=0}^{\infty} \epsilon^{n} \frac{r^n}{n!} \partial_r^n b^{\underline{r}} du \wedge dr + \sum_{n=0}^{\infty} \epsilon^{n-1} \frac{r^n}{n!} \partial_r^n b^{\underline{r}}_I du \wedge dy^I \\
&+ \sum_{n=0}^{\infty} \epsilon^{n+1} \frac{r^n}{n!} \partial_r^n c^{\underline{r}}_I dr \wedge dy^I + \sum_{n=0}^{\infty} \epsilon^{n} \frac{r^n}{n!} \partial_r^n b^{\underline{r}}_{IJ} dy^I \wedge dy^I \ . \qquad (4.12)
\end{aligned}
$$

Next taking the near horizon limit $\epsilon \to 0$ we find

$$
B^{\underline{r}} = b^{\underline{r}}(y) du \wedge dr + r\partial_r b^{\underline{r}}_I(y) du \wedge dy^I + b^{\underline{r}}_{IJ}(y) dy^I \wedge dy^J \ . \qquad (4.13)
$$

Note that we have also used the regularity condition $b^{\underline{r}}_I(0, y) = 0$. This in turn can be written more conveniently [89] as

$$
B^{\underline{r}} = r du \wedge N^{\underline{r}} + S^{\underline{r}} du \wedge \big(dr + r h_I dy^I\big) + W^{\underline{r}} \ , \qquad (4.14)
$$

where $S^{\underline{r}}$ are scalars, h, $N^{\underline{r}}$ are 1-forms and $W^{\underline{r}}$ are 2-forms on the horizon section and depend only on y. Moreover, in the near horizon limit the scalars of the hypermultiplet and those of the tensor multiplet only depend on y. The black hole horizon section $\mathcal{S}$ has been defined as the surface given by $r = u = 0$. In addition, it is assumed to be compact, connected and without boundary. Since $G^{\underline{r}} = dB^{\underline{r}}$, we take the exterior derivative of (4.14) and collect the other near horizon data to find

$$
\begin{aligned}
ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ , \\
G^{\underline{r}} &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge \big(dS^{\underline{r}} - N^{\underline{r}} - S^{\underline{r}} h\big) \\
&\quad + r\mathbf{e}^+ \wedge \big(h \wedge N^{\underline{r}} - dN^{\underline{r}} - S^{\underline{r}} dh\big) + dW^{\underline{r}} \ , \\
\phi^{\underline{I}} &= \phi^{\underline{I}}(y) \ , \qquad \varphi = \varphi(y) \ , \qquad (4.15)
\end{aligned}
$$

where

$$
\mathbf{e}^+ = du \ , \quad \mathbf{e}^- = dr + rh + r^2 F(y) du \ , \quad \mathbf{e}^i \;=\; e^i{}_I dy^I \ , \qquad (4.16)
$$

$\phi^{\underline{I}}$ are the scalars of the hypermultiplet and φ are the scalars of the tensor multiplet, and $\mathbf{e}^i$ is a frame on $\mathcal{S}$ that depends only on y.

To find the supersymmetric horizons of six dimensional (1,0) supergravity, one needs to solve the field and KSEs of the theory for the data given in (4.15). We will first consider the solutions of KSEs.

4.3.3 Solution of KSEs

To continue, we substitute (4.15) into the KSEs (4.8) and assume that backgrounds preserve at least one supersymmetry. Moreover, we identify the stationary Killing vector field ∂_u of the near horizon geometry with the Killing vector constructed as a Killing spinor bilinear. We now discuss this in some detail.

Light-cone Integrability of KSEs

As we will identify the Killing vector field of the black hole ∂_u with the Killing vector constructed from the Killing spinor bilinears, and since this is null it means $F(y) = 0$ in (4.15). As a result, we find the non-vanishing components of the frame connection associated to the Levi-Civita connection of the spacetime to be given by

$$\begin{aligned} \Omega_{+,-i} &= -\frac{1}{2}h_i \ , \quad \Omega_{+,ij} = -\frac{1}{2}r(dh)_{ij} \ , \quad \Omega_{-,+i} = -\frac{1}{2}h_i \ , \\ \Omega_{i,+-} &= \frac{1}{2}h_i \ , \quad \Omega_{i,+j} = -\frac{1}{2}r(dh)_{ij} \ , \quad \Omega_{i,jk} = \tilde{\Omega}_{i,jk} \ , \end{aligned} \tag{4.17}$$

where $\tilde{\Omega}_{i,jk}$ is the connection associated to the horizon section. For later use, we note the anti-self duality of H implies

$$\begin{aligned} H_{+-i} &= \frac{1}{3!}\epsilon_{+-ijkl}H^{jkl} \ , \\ &= \frac{1}{3!}\epsilon_{+-ijkl}v_{\underline{r}}dW^{\underline{r}jkl} \ . \end{aligned} \tag{4.18}$$

We now use these, as well as the components of $H_{\mu\nu\rho}$, to integrate the gravitino KSE along the two light-cone directions. For this we first decompose the Killing spinor ϵ as

$$\epsilon = \epsilon_+ + \epsilon_- \ , \qquad \gamma_\pm \epsilon_\pm = 0 \ . \tag{4.19}$$

The $-$ component of the gravitino KSE is

$$\partial_- \epsilon + \frac{1}{4}\Omega_{-,\nu\rho}\gamma^{\nu\rho}\epsilon - \frac{1}{8}H_{-\nu\rho}\gamma^{\nu\rho}\epsilon + \mathcal{C}_-^{r'}\rho_{r'}\epsilon = 0 \ , \tag{4.20}$$

and this becomes

$$\partial_- \epsilon - \frac{1}{4}(v_{\underline{r}}dS^{\underline{r}} - N - (S+1)h)_i \Gamma^i \Gamma_- \epsilon_+ = 0 \ , \tag{4.21}$$

where we have used the expression for the frame connection stated above, the expression for the fields in (4.15) and the fact $\mathcal{C}_-^{r'} = \partial_- \phi^{\underline{I}} \mathcal{A}_{\underline{I}}^{r'} = 0$ since the scalars do not depend on (u, r) in the near horizon limit. We have also set $N = v_{\underline{r}}N^{\underline{r}}$ and $S = v_{\underline{r}}S^{\underline{r}}$. Noting that $\partial_- = \partial_r$ and $\partial_+ = \partial_u$, upon integration we find

$$\begin{aligned} \epsilon_+ &= \phi_+ \ , \\ \epsilon_- &= \phi_- + \frac{1}{4}r(v_{\underline{r}}dS^{\underline{r}} - N - (S+1)h)_i \gamma^i \gamma_- \phi_+ \ , \end{aligned} \tag{4.22}$$

where $\phi_\pm$ are independent of r.

Similarly, the + component of the gravitino KSE gives

$$\begin{aligned}\partial_+\epsilon+\frac{1}{4}(v_{\underline{r}}dS^{\underline{r}}-N-(S-1)h)_i\gamma^i\gamma_+\epsilon_-\\ -\frac{1}{8}r(h\wedge N-v_{\underline{r}}dN^{\underline{r}}-(S-1)dh)_{ij}\gamma^{ij}\epsilon &= 0\ .\end{aligned} \tag{4.23}$$

Substituting in (4.22) into the + component of the gravitino KSE, we get

$$\begin{aligned}&\partial_+\Big(\phi+\frac{1}{4}r(v_{\underline{r}}dS^{\underline{r}}-N-(S+1)h)_i\gamma^i\gamma_-\phi_+\Big)\\ &+\frac{1}{4}\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S-1)h\Big)_i\gamma^i\gamma_+\Big(\phi_-+\frac{1}{4}r\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S+1)h\Big)_j\gamma^j\gamma_-\phi_+\Big)\\ &-\frac{1}{8}r\Big(h\wedge N-v_{\underline{r}}dN^{\underline{r}}-(S-1)dh\Big)_{ij}\gamma^{ij}\Big(\phi+\frac{1}{4}r\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S+1)h\Big)_k\gamma^k\gamma_-\phi_+\Big)\\ &\qquad=0\ .\end{aligned} \tag{4.24}$$

This equation is valid in every order in r. As a result the $\mathcal{O}(r^0)$ order term gives

$$\partial_+\phi+\frac{1}{4}\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S-1)h\Big)_i\gamma^i\gamma_+\phi_-=0\ , \tag{4.25}$$

which can be solved to find

$$\begin{aligned}\phi_+ &= \eta_+-\frac{1}{4}u\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S-1)h\Big)_i\gamma^i\gamma_+\eta_-\ ,\\ \phi_- &= \eta_-\ ,\end{aligned} \tag{4.26}$$

where $\eta_\pm$ is independent of r and u. Substituting these into (4.22) means the $\epsilon_\pm$ component of the Killing spinor can be written in terms of $\eta_\pm$ as

$$\begin{aligned}\epsilon_+ &= \eta_+-\frac{1}{4}u\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S-1)h\Big)_i\gamma^i\gamma_+\eta_-\ ,\\ \epsilon_- &= \eta_-+\frac{1}{4}r\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S+1)h\Big)_i\gamma^i\gamma_-\eta_+\\ &\quad+\frac{1}{8}ur\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S+1)h\Big)_i\Big(v_{\underline{r}}dS^{\underline{r}}-N-(S-1)h\Big)_j\gamma^i\gamma^j\eta_-\end{aligned} \tag{4.27}$$

The remaining conditions implied by (4.24) are algebraic which will be considerably simplified after the analysis of the next section. These are

$$\alpha_i\beta_j\gamma^i\gamma^j\eta_++\lambda_{ij}\gamma^{ij}\eta_+ = 0\ , \tag{4.28}$$

$$\lambda_{ij}\beta_k\gamma^{ij}\gamma^k\eta_+ = 0\ , \tag{4.29}$$

$$\beta_i\alpha_j\gamma^i\gamma^j\eta_--\lambda_{ij}\gamma^{ij}\eta_- = 0\ , \tag{4.30}$$

$$\alpha_i\beta_j\alpha_k\gamma^i\gamma^j\gamma^k\eta_-+\lambda_{ij}\alpha_k\gamma^{ij}\gamma^k\eta_- = 0\ , \tag{4.31}$$

$$\lambda_{ij}\beta_k\alpha_l\gamma^{ij}\gamma^k\gamma^l\eta_- = 0\ , \tag{4.32}$$

where

$$\alpha_i = (v_{\underline{r}} dS^{\underline{r}} - N - (S-1)h)_i \ , \tag{4.33}$$
$$\beta_i = (v_{\underline{r}} dS^{\underline{r}} - N - (S+1)h)_i \ , \tag{4.34}$$
$$\lambda_{ij} = (h \wedge N - v_{\underline{r}} dN^{\underline{r}} - (S-1)dh)_{ij} \ . \tag{4.35}$$

These are in fact the same constraints as those found for the heterotic horizons in [89].

Stationary and Spinor Bilinear Vector Fields

The argument presented in this section follows a similar argument made for the investigation of heterotic horizons given in [89]. As we have mentioned, additional restrictions on $\eta_\pm$ can be derived for horizons preserving one supersymmetry arising from the identification of stationary black hole Killing vector field ∂_u with that constructed as a Killing spinor bilinear. This identification implies that the components of the 1-form associated with the latter are

$$X_+ = 0 \ , \quad X_- = 1 \ , \quad X_i = 0 \ . \tag{4.36}$$

Furthermore, the $\eta_\pm$ spinors can be expanded in the basis of symplectic Majorana-Weyl spinors as

$$\begin{aligned} \eta_+ &= a_1(1+e_{1234}) + a_2 i(1-e_{1234}) + a_3(e_{12}-e_{34}) + a_4 i(e_{12}+e_{34}) \ , \\ \eta_- &= b_1(e_{15}+e_{2345}) + b_2 i(e_{15}-e_{2345}) + b_3(e_{25}-e_{1345}) + b_4 i(e_{15}+e_{2345}) \end{aligned} \tag{4.37}$$

where all components depend on the coordinates y of $\mathcal{S}$. The field data (4.15) are covariant under local $Spin(4) \cdot Sp(1)$ gauge transformations of $\mathcal{S}$. So these can be used to choose $\eta_\pm$ as

$$\begin{aligned} \eta_+ &= a(y)(1+e_{1234}) \ , \\ \eta_- &= b(y)(e_{15}+e_{2345}) \ . \end{aligned} \tag{4.38}$$

The next step to consider is the spinor bilinear associated to the Killing spinor ϵ,

$$Y_\mu e^\mu = \langle B\epsilon^*, \gamma_\mu \epsilon \rangle e^\mu \ , \tag{4.39}$$

where $B = \Gamma_{06789}$. In order to satisfy the relations in (4.36), we require the + component for the spinor bilinear to vanish. This in particular means that $Y_+|_{r=0} = 0$, and as a consequence we find

$$\eta_- = 0 \ . \tag{4.40}$$

This then means we can write the spinor in (4.27) as

$$\epsilon \;=\; \eta_+ + \frac{1}{4} r (v_{\underline{r}} dS^{\underline{r}} - N - (S+1)h)_i \gamma^i \gamma_- \eta_+ \tag{4.41}$$

Since the bilinear components on the horizon are independent of r, the next requirement we impose is for the $\mathcal{O}(r)$ term in the bilinear to vanish. This means

$$\langle B(1+e_{1234}), \gamma_\mu (v_{\underline{r}} dS^{\underline{r}} - N - (S+1)h)_i \gamma^i \gamma_- \eta_+ \rangle = 0 \;, \tag{4.42}$$

from which we obtain the condition

$$v_{\underline{r}} dS^{\underline{r}} - N - (S+1)h = 0 \;. \tag{4.43}$$

This further simplifies the Killing spinor to

$$\epsilon = \eta_+ = a(y)(1+e_{1234}) \;. \tag{4.44}$$

Finally, calculating Y_- and comparing this to X_-, we find

$$-2\sqrt{2}a^2 = 1 \;, \tag{4.45}$$

i.e. a is a constant, which without loss of generality can be set to 1. This means

$$\epsilon = 1 + e_{1234} \;. \tag{4.46}$$

This choice of Killing spinor for the horizon geometries is the same as that for general solutions of the KSEs of six dimensional supergravity preserving one supersymmetry, as discussed in chapter 2. Therefore, this will be used to simplify the analysis of near horizon geometries.

Further analysis of the Gravitino KSE

Revisiting the + component of the gravitino KSE with $\epsilon = 1 + e_{1234}$, we find that

$$\left(h \wedge N - v_{\underline{r}} dN^{\underline{r}} - (S-1)dh\right)_{ij} \gamma^{ij} \eta_+ = 0 \;. \tag{4.47}$$

As a consequence all algebraic conditions (4.28-4.32) are also satisfied.

Next we consider the i-component of the gravitino KSE. After separating the various orders in r, we have

$$\tilde{\mathcal{D}}_i \eta_+ = \left(\partial_i + \frac{1}{4} \tilde{\Omega}_{i,jk} \gamma^{jk} - \frac{1}{8} v_{\underline{r}} (dW^{\underline{r}})_{ijk} \gamma^{jk} + \mathcal{C}_i^{r'} \rho_{r'} \right) \eta_+ = 0 \;. \tag{4.48}$$

and

$$(h \wedge N - v_{\underline{r}} dN^{\underline{r}} - (S+1)dh)_{ij}\gamma^j \eta_+ = 0 \ . \tag{4.49}$$

Using $\eta_+ = 1 + e_{1234}$ in the last equation, we find that

$$h \wedge N - v_{\underline{r}} dN^{\underline{r}} - (S+1)dh = 0 \ . \tag{4.50}$$

As a result, the 3-form H simplifies to

$$H = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge h + r\mathbf{e}^+ \wedge dh + v_{\underline{r}} dW^{\underline{r}} \ . \tag{4.51}$$

This completes the analysis of the gravitino KSE, we now consider the tensorini and hyperini KSEs in turn.

Tensorini KSE

We have already solved the tensorini KSE for the spinor $\epsilon = \eta_+ = 1 + e_{1234}$ and the results are given in (2.45). Comparing the expression for $H^{\underline{M}}$ in (2.87) and (4.15), we find

$$x^{\underline{M}}_{\underline{r}} dS^{\underline{r}} - N^{\underline{M}} - S^{\underline{M}} h = -T^{\underline{M}} \ , \quad h \wedge N^{\underline{M}} - x^{\underline{M}}_{\underline{r}} dN^{\underline{r}} - S^{\underline{M}} dh = 0 \ , \tag{4.52}$$

where $N^{\underline{M}} = x^{\underline{M}}_{\underline{r}} N^{\underline{r}}$ and $S^{\underline{M}} = x^{\underline{M}}_{\underline{r}} S^{\underline{r}}$. As a result the 3-form $H^{\underline{M}}$ can be written as

$$H^{\underline{M}} \ = \ T^{\underline{M}}_i \, \mathbf{e}^- \wedge \mathbf{e}^+ \wedge \mathbf{e}^i + x^{\underline{M}}_{\underline{r}} dW^{\underline{r}} \ . \tag{4.53}$$

Hyperini KSE

Applying the results of chapter 2, in particular (2.48), and using the fact that the scalars of the hypermultiplet do not depend on the coordinates (u, r) means the hyperini KSE implies

$$-V^{\underline{a1}}_{1} + V^{\underline{a2}}_{\bar{2}} = 0 \ , \quad V^{\underline{a1}}_{2} + V^{\underline{a2}}_{\bar{1}} = 0 \ , \tag{4.54}$$

for $\epsilon = 1 + e_{1234}$.

Summary

In summary, at the end of this detailed analysis we have found that the Killing spinor can be chosen as

$$\epsilon = 1 + e_{1234} \ , \tag{4.55}$$

and the fields can be rewritten as

$$\begin{aligned}
ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{e}^i\mathbf{e}^j \ , \\
H &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge h + r\mathbf{e}^+ \wedge dh - \frac{1}{3!} h_\ell \, \epsilon^\ell{}_{ijk} \, \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \quad , \\
H^{\underline{M}} &= T_i^{\underline{M}} \, \mathbf{e}^- \wedge \mathbf{e}^+ \wedge \mathbf{e}^i - \frac{1}{3!} T_\ell^{\underline{M}} \, \epsilon^\ell{}_{ijk} \, \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \ . \\
\phi^{\underline{I}} &= \phi^{\underline{I}}(y) \ , \quad \varphi = \varphi(y) \ ,
\end{aligned} \tag{4.56}$$

where we have used the duality relations of the 3-form field strengths. In particular $h_\ell \, \epsilon^\ell{}_{ijk} = -v_{\underline{r}} dW^{\underline{r}}_{ijk}$ and similarly for $H^{\underline{M}}$. In addition the anti-self duality of H requires that

$$dh_{ij} = -\frac{1}{2}\epsilon_{ij}{}^{kl} dh_{kl} \ . \tag{4.57}$$

It is clear that H is entirely determined in terms of h while $H^{\underline{M}}$ is entirely determined in terms of the scalars φ of the tensor multiplets.

Furthermore, the gravitino KSE along the horizon section directions, (4.48), requires that

$$\tilde{\mathcal{D}}_i(1 + e_{1234}) = 0 \ , \tag{4.58}$$

where

$$\tilde{\mathcal{D}}_i = \hat{\tilde{\nabla}}_i + \mathcal{C}_i^{r'} \rho_{r'} \ , \tag{4.59}$$

and $\hat{\tilde{\nabla}}$ is the connection on $\mathcal{S}$ with skew-symmetric torsion $- \star_4 h$, where $\star_4$ denotes the hodge dual in the directions transverse to the light-cone. One can consider the geometric content of this equation by using the Hermitian 2-forms on $\mathcal{S}$, given in (2.62), which can be constructed as twisted Killing spinor bilinears, see section 2.4. In particular, the integrability condition of (4.58) can be expressed as

$$-\hat{\tilde{R}}_{mn,}{}^k{}_i \omega^{r'}{}_{kj} + (j,i) + 2\mathcal{F}^{s'}_{mn} \epsilon^{r'}{}_{s't'} \omega^{t'}_{ij} = 0 \ , \tag{4.60}$$

where

$$\mathcal{F}^{s'}_{mn} = \partial_m \phi^{\underline{I}} \partial_n \phi^{\underline{J}} \mathcal{F}^{s'}_{\underline{IJ}} \ . \tag{4.61}$$

The integrability condition identifies the $Sp(1) \subset Sp(1) \cdot Sp(1)$ component of the curvature $\hat{\tilde{R}}$ of the 4-dimensional manifold $\mathcal{S}$ with the pull back with respect to ϕ of the $Sp(1)$ component of the curvature of the Quaternionic Kähler manifold, $\mathcal{Q}$. The restriction imposed on the geometry of $\mathcal{S}$ by (4.60) depends on the scalars $\phi^{\underline{I}}$. In particular, if $\phi^{\underline{I}}$ are constant, then $\mathcal{F}_{mn} = 0$ and (4.60) implies that $\mathcal{S}$ is a hyper-Kähler with torsion manifold [111].

We will analyse these conditions further in the sections that follow. This will be done after imposing the restrictions on the fields implied by the field equations of the theory and the compactness of $\mathcal{S}$.

4.4 Horizons with $h \neq 0$

There are two classes of horizons to consider depending on whether or not h vanishes. First, we will consider the case where $h \neq 0$.

4.4.1 Holonomy Reduction

If $h \neq 0$, we shall demonstrate that, as in the heterotic case, the number of supersymmetries preserved by the near horizon geometries is always even. For this we shall use the results we have obtained from the KSEs for horizons preserving one supersymmetry and the field equations of the theory. The methodology we shall follow to prove this is to compute $\tilde{\nabla}^2 h^2$ and apply the maximum principle utilising the compactness of $\mathcal{S}$.

There are a number of different formulations of the maximum principle, see for example [112], but the one that we will need states that if

$$\tilde{\nabla}^2 a + b^i \tilde{\nabla}_i a \geq 0 \ , \tag{4.62}$$

where a and b are two smooth functions defined on $\mathcal{S}$, holds true then the compactness of $\mathcal{S}$ means the function a must be a constant. In what follows we aim to show this for the case where $a = h^2$.

The field equations of six dimensional supergravity, which were given in chapter 3, in the absence of vector multiplets are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4}\varsigma_{\underline{rs}} G^{\underline{r}}_{\mu}{}^{\alpha\beta} G^{\underline{s}}_{\nu\alpha\beta} + \partial_\mu v^{\underline{r}} \partial_\nu v_{\underline{r}} - 2 g_{\underline{IJ}} \partial_\mu \phi^{\underline{I}} \partial_\nu \phi^{\underline{J}} &= 0 \ , \\ \nabla_\lambda \big(\varsigma_{\underline{rs}} G^{\underline{s}\lambda\mu\nu} \big) &= 0 \ , \\ \nabla^\mu \partial_\mu v^{\underline{r}} + \frac{1}{6} v_{\underline{s}} G^{\underline{s}\mu\nu\rho} G^{\underline{r}}_{\mu\nu\rho} &= 0 \ , \\ D_\mu \partial^\mu \phi^{\underline{I}} &= 0 \ , \end{aligned} \tag{4.63}$$

where in the last equation it is understood that the Levi-Civita connections of both the spacetime and the hypermultiplets Quaternionic Kähler manifold metrics have been used to covariantise the expression, and the covariant derivative is denoted as D_μ.

To make use of the maximum principle we start with $h^2 = h_i h^i$ and take the Laplacian of this to find

$$\tilde{\nabla}^2 h^2 = 2\tilde{\nabla}^i h^j \tilde{\nabla}_i h_j + 2\tilde{\nabla}^i (dh)_{ij} h^j + 2\tilde{R}_{ij} h^i h^j + 2h^j \tilde{\nabla}_j \tilde{\nabla}_i h^i \ , \tag{4.64}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\mathcal{S}$ with respect to $ds^2(\mathcal{S}) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$ and $\tilde{R}$ is the associated Ricci tensor. The proof of this is given in [89]. To proceed, we use the field equations of the theory to rearrange the above expression in such a way so that one can apply the maximum principle. Using the Einstein equation and the fact [113]

$$\tilde{R}_{ij} = R_{ij} - \tilde{\nabla}_{(i}h_{j)} + \frac{1}{2}h_ih_j \ , \tag{4.65}$$

we find

$$2\tilde{R}_{ij}h^ih^j \ = \ -h^2\partial_k v_{\underline{r}}\partial^k v^{\underline{r}} + 4\partial_i\phi^{\underline{I}}\partial_j\phi^{\underline{J}}g_{\underline{IJ}}h^ih^j - h^i\tilde{\nabla}_ih^2 \ . \tag{4.66}$$

The $\mu\nu = +-$ component of the field equation $\nabla_\lambda\big(\varsigma_{\underline{rs}}G^{\underline{s}\lambda\mu\nu}\big)$ together with $H^{i+-} = -h^i$ and $H^{\underline{M}i+-} = T^{i\underline{M}}$ give

$$\partial_i v_{\underline{r}}h^i + v_{\underline{r}}\tilde{\nabla}_ih^i + \tilde{\nabla}_i\partial^i v_{\underline{r}} = 0 \ . \tag{4.67}$$

Acting on this expression with $v^{\underline{r}}$, we find

$$\tilde{\nabla}_ih^i + v^{\underline{r}}\tilde{\nabla}_i\partial^i v_{\underline{r}} = 0 \ , \tag{4.68}$$

where we have used $v_{\underline{r}}v^{\underline{r}} = 1$.

The field equation of the scalars of the tensor multiplet gives

$$v_{\underline{r}}\tilde{\nabla}_i\partial^i v^{\underline{r}} = 0 \ , \tag{4.69}$$

which when combined with (4.68) implies that

$$\tilde{\nabla}_ih^i = 0 \ . \tag{4.70}$$

In addition (4.69) and $v_{\underline{r}}v^{\underline{r}} = 1$ give

$$\partial_k v_{\underline{r}}\partial^k v^{\underline{r}} = 0 \ . \tag{4.71}$$

Thus substituting (4.66) into (4.64) and using (4.70) and (4.71), we find that

$$\tilde{\nabla}^2h^2 + h^i\tilde{\nabla}_ih^2 = 2\tilde{\nabla}^ih^j\tilde{\nabla}_ih_j + 2\tilde{\nabla}^i(dh)_{ij}h^j + 4\partial_i\phi^{\underline{I}}\partial_j\phi^{\underline{J}}g_{\underline{IJ}}h^ih^j \ . \tag{4.72}$$

This expression is close to the one required in order to apply the maximum principle. Next, we need to determine dh. For this, consider the jk-component of the 3-form field equation to find

$$\nabla^i(v_{\underline{r}}H_{ijk} + x^{\underline{M}}_{\underline{r}}H^{\underline{M}}_{ijk}) = \epsilon_{ijkl}\partial^i v_{\underline{r}}h^l + v_{\underline{r}}\epsilon_{ijkl}\nabla^ih^l = 0 \ , \tag{4.73}$$

multiplying this with $v^{\underline{r}}$ we find

$$dh = 0 \ , \tag{4.74}$$

Substituting this into (4.72), we get

$$\tilde{\nabla}^2 h^2 + h^i \tilde{\nabla}_i h^2 = 2\tilde{\nabla}^i h^j \tilde{\nabla}_i h_j + 4\partial_i \phi^{\underline{I}} \partial_j \phi^{\underline{J}} g_{\underline{IJ}} h^i h^j \ . \tag{4.75}$$

Now, applying the maximum principle using the compactness of $\mathcal{S}$, we find that h^2 is constant and so the right hand side of (4.75) is equal to zero, which in turn means

$$\begin{aligned} \tilde{\nabla}_i h_j &= 0 \ , \\ h^i \partial_i \phi^{\underline{I}} &= 0 \ . \end{aligned} \tag{4.76}$$

To establish the second equation, we have used the fact that the metric of the hypermultiplets Quaternionic Kähler manifold is positive definite. The first condition tells us that h is a parallel 1-form on $\mathcal{S}$ with respect to the Levi-Civita connection and the second one implies the scalars of the hypermultiplets are invariant under the action of h. Note also that $\hat{\tilde{\nabla}} h = 0$ as $i_h \tilde{H} = 0$.

The existence of a parallel 1-form on the horizon section $\mathcal{S}$ with respect to the Levi-Civita connection is a strong restriction. Firstly, it implies that the holonomy of $\tilde{\nabla}$ is contained in $SO(3) \subset SO(4)$,

$$\mathrm{hol}(\tilde{\nabla}) \subseteq SO(3) \ . \tag{4.77}$$

Moreover, $\mathcal{S}$ metrically (locally) splits into a product $S^1 \times \Sigma^3$, where Σ^3 is a 3-dimensional manifold. In turn, as we shall see, the near horizon geometry is locally a product $AdS_3 \times \Sigma^3$. More elegantly the near horizon geometry allows supersymmetry enhancement from one supersymmetry to two.

4.4.2 Supersymmetry Enhancement

In order to demonstrate supersymmetry enhancement for backgrounds with $h \neq 0$, let us re-investigate the KSEs for the fields given in (4.56). It is straightforward to see by substituting (4.56) into the KSEs and following the calculation that we gave in section 4.3.3 that the general form of a Killing spinor is

$$\epsilon = \eta_+ - \frac{u}{2} h_i \gamma^i \gamma_+ \eta_- + \eta_- \tag{4.78}$$

where $\eta_\pm$ depend only on the coordinates of $\mathcal{S}$. In addition, the gravitino KSE requires that

$$\hat{\tilde{\nabla}}_i \epsilon + \mathcal{C}_i^{r'} \rho_{r'} \epsilon = 0 \ , \tag{4.79}$$

the tensorini KSE implies that

$$(1 \pm \frac{1}{2}) T_i^{\underline{M}} \gamma^i \epsilon_\pm - \frac{1}{12} H_{ijk}^{\underline{M}} \gamma^{ijk} \epsilon_\pm = 0 \ , \tag{4.80}$$

and the hyperini KSE gives

$$i\gamma^i \epsilon_{\pm \underline{A}} V_i^{aA} = 0 \ . \tag{4.81}$$

Next we shall show that both

$$\epsilon_1 = 1 + e_{1234} \ , \quad \epsilon_2 = \gamma_- h_i \gamma^i (1 + e_{1234}) - uk^2 (1 + e_{1234}) \ , \tag{4.82}$$

are Killing spinors, where we have set $k^2 = h^2$ for the constant length of h. Note that the second Killing spinor is constructed by setting $\eta_+ = 0$ and $\eta_- = \gamma_- h_i \gamma^i (1 + e_{1234})$ in (4.78).

The KSEs have already been solved for ϵ_1. Next, observe that ϵ_2 solves the gravitino KSE since the Clifford algebra operation $h_i \gamma^i \gamma_-$ commutes with the supercovariant derivative in (4.79) as a consequence of the reduction of holonomy demonstrated in the previous section. Furthermore, the same Clifford operation commutes with the hyperini KSE as a result of the second equation in (4.76) and (4.81).

Now it remains to show that ϵ_2 solves the tensorini KSE as well. This is a consequence of the relation in (4.71). This is because the metric induced on $SO(n_T, 1)/SO(n_T)$ by the algebraic equation $\eta_{\underline{rs}} v^{\underline{r}} v^{\underline{s}} = 1$ is the standard hyperbolic metric and so it has Euclidean signature. Therefore, as a result,

$$\partial_i v^{\underline{r}} = 0 \ . \tag{4.83}$$

Then we conclude that the scalar fields are constant and the 3-form field strengths, $H^{\underline{M}}$ given in (4.56), of the tensorini multiplet vanish. This agrees with the classification results of chapter 2 for solutions of the KSEs of six dimensional supergravity preserving at least two supersymmetries whose Killing spinors have compact isotropy group. Some of the results of this section are tabulated in table 4.1.

4.4.3 Geometry

To investigate the geometry of spacetime, we compute the form bilinears associated with the Killing spinors in (4.82). As in the analysis of the general backgrounds given in chapter 2, we find that the spacetime admits three $\hat{\nabla}$-parallel 1-forms corresponding to the bilinears $\alpha(\epsilon_1, \epsilon_1)$, $\alpha(\epsilon_1, \epsilon_2)$ and $\alpha(\epsilon_2, \epsilon_2)$ and these are given by

$$\lambda^- = \mathbf{e}^- \ , \quad \lambda^+ = \mathbf{e}^+ - \frac{1}{2} k^2 u^2 \mathbf{e}^- - uh \ , \quad \lambda^1 = k^{-1}(h + k^2 u \mathbf{e}^-) \ . \tag{4.84}$$

$\mathrm{Iso}(\eta_+)$	$\mathrm{hol}(\tilde{\mathcal{D}})$	N	η_+
$Sp(1)\cdot Sp(1)\ltimes \mathbb{H}$	$Sp(1)$	2	$1+e_{1234}$
$U(1)\cdot Sp(1)\ltimes \mathbb{H}$	$U(1)$	4	$1+e_{1234}, i(1-e_{1234})$
$Sp(1)\ltimes \mathbb{H}^4$	$\{1\}$	8	$1+e_{1234}, i(1-e_{1234}), e_{12}-e_{34}, i(e_{12}+e_{34})$

Table 4.1: Some of the geometric data used to solving the gravitino KSE are described. In the first column, we give the isotropy groups, $\mathrm{Iso}(\eta_+)$, of $\{\eta_+\}$ spinors in $Spin(5,1)\cdot Sp(1)$. In the second column we state the holonomy of the supercovariant connection $\tilde{\mathcal{D}}$ of the horizon section $\mathcal{S}$ in each case. The holonomy of $\hat{\tilde{\nabla}}$ is identical to that of $\hat{\nabla}$. In the third column, we present the number of $\mathcal{D}$-parallel spinors and in the last column we give representatives of the $\{\eta_+\}$ spinors.

Moreover, the Lie algebra of the associated vector fields closes in $\mathfrak{sl}(2,\mathbb{R})$, this has been verified in [89]. Since h is $\tilde{\nabla}$-parallel, the spacetime is locally metrically a product $SL(2,\mathbb{R})\times\Sigma^3$, i.e.

$$\begin{aligned} ds^2 &= ds^2(SL(2,\mathbb{R}))+ds^2(\Sigma^3)\ , \\ H &= d\mathrm{vol}(SL(2,\mathbb{R}))+d\mathrm{vol}(\Sigma^3)\ , \\ \phi^{\underline{I}} &= \phi^{\underline{I}}(z)\ , \end{aligned} \tag{4.85}$$

where the scalars of the hypermultiplet depend only on the coordinates z of Σ^3.

In addition to the 1-forms given in (4.84), the spacetime admits three more twisted 1-forms bilinears, compare with the $N=2$ compact case of chapter 2. For the Killing spinors (4.82), these are given by

$$e^{r'}=k^{-1}h_j(I^{r'})^j{}_i\mathbf{e}^i\ , \tag{4.86}$$

where $I^{r'}$ are quaternionic structures on $\mathcal{S}$ associated to the Quaternionic-Hermitian 2-forms (2.62). As it has been already mentioned, these Quaternionic-Hermitian 2-forms arise from the construction of twisted spinor bilinears and so rotate to each other under patching conditions. From (4.86) one can see that the frame $e^{r'}$ is orthogonal to h and so the rotation between the frame $\mathbf{e}^i$ and $(h,e^{r'})$ is in $SO(4)$. Therefore $(k^{-1}h,e^{r'})$ is another frame on $\mathcal{S}$ with $e^{r'}$ adapted to Σ^3. Thus, we can write $ds^2(\mathcal{S})=k^{-2}h^2+ds^2(\Sigma^3)$ with $ds^2(\Sigma^3)=\delta_{r's'}e^{r'}e^{s'}$.

The metric on Σ^3 is restricted by the Einstein equation (4.63) and the integrability condition (4.60). From the Einstein equation we get

$$R^{(3)}_{r's'}-\frac{1}{2}k^2\delta_{r's'}-2\partial_{r'}\phi^{\underline{I}}\partial_{s'}\phi^{\underline{J}}g_{\underline{IJ}}=0\ , \tag{4.87}$$

where r',s' are the indices of Σ^3, and $R^{(3)}$ is the Ricci tensor of Σ^3. This is an equation which determines the metric on Σ^3 in terms of h and the hypermultiplet scalars ϕ. The integrability condition (4.60) does not give an independent condition on the metric of Σ^3.

It now remains to find the restriction imposed by supersymmetry on the scalars ϕ of the hypermultiplet. These scalars only depend on the coordinates of Σ^3, i.e. they depend only on three of the spacetime directions. A direct observation reveals that, after an appropriate identification of the frame directions of $\mathcal{S}$ with that of the Pauli matrices $\sigma_{s'}$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.88}$$

the supersymmetry conditions coming from the hyperini KSE can be rewritten as

$$\partial_{r'}\phi^{\underline{I}} = -\epsilon_{r'}{}^{s't'}\,(J_{s'})^{\underline{I}}{}_{\underline{J}}\,\partial_{t'}\phi^{\underline{J}}\ , \tag{4.89}$$

where we have used that $(J_{s'})^{\underline{Aa}}{}_{\underline{Bb}} = -i\,\sigma_{s'}{}^{\underline{A}}{}_{\underline{B}}\delta^{\underline{a}}{}_{\underline{b}}$. This is a natural condition which constrains the maps ϕ from Σ^3 into the Quaternionic Kähler manifold of the hypermultiplets. Constant maps are also solutions.

As mentioned, the geometry on Σ^3 is determined by (4.87), and depends on the solutions of (4.89). In the case of constant map solutions of (4.89), we get

$$R^{(3)}_{r's'} - \frac{1}{2}k^2\delta_{r's'} = 0\ , \tag{4.90}$$

and so Σ^3 is locally isometric to S^3 equipped with the round metric, and the near horizon geometry becomes $AdS_3 \times S^3$.

Next, let us suppose that non-trivial solutions exist for the equation in (4.89), and upon substitution solutions exist for (4.87). Therefore, one would expect the geometry on Σ^3 to depend on the choice of quaternionic Kähler manifold for the hypermultiplets and on the choice of a solution of (4.89). However, one finds that the differential structure on Σ^3 is independent of these choices. This is because the existence of non-trivial choices ϕ does not affect the fact that the Ricci tensor $R^{(3)}$ of Σ^3 is strictly positive (4.87). This in turns allows one to determine the topology of Σ^3. To see this note that in three dimensions the Ricci tensor determines the curvature of a manifold. Also, the strict positivity of the Ricci tensor implies the (reduced) holonomy of the Levi-Civita connection of Σ^3 is $SO(3)$. Then a result of Gallot and Meyer, see [114], implies that Σ^3 is a homology 3-sphere, see [13] for further details. This along with the Poincaré conjecture [115] allows one to conclude that the universal cover of Σ^3 is diffeomorphic to the 3-sphere, see [13]. This result means that in the simply connected case and for non-constant solutions to (4.89), the geometry of the round sphere is deformed in such a way that the differential, and so topological, structure of S^3 is maintained.

The existence of non-trivial solutions to (4.89) is an open problem which may depend on the choice of quaternionic Kähler manifold of the hypermultiplets. However, as we will see, horizons that preserve eight supersymmetries require the hyperscalars

ϕ to be constant. This is compatible with the assertion made in the attractor mechanism, see [116, 117] for the six dimensional supergravity case, that all the scalars take constant values at the horizon. However, it is worth noting that the field equations and the KSEs do not a priori imply that the scalars are constant for near horizon geometries which preserve a small number of supersymmetries. For this some further investigation is required which may be case dependent, we will not pursue this further.

4.5 N=4 and N=8 Horizons

4.5.1 N=4 Horizons

We have shown that if $h \neq 0$, then the near horizon geometries preserve 2, 4 or 8 supersymmetries. The case with two supersymmetries has already been discussed in some detail above. In addition to the two Killing spinors given in (4.82), the other two Killing spinors of horizons with four supersymmetries can be chosen as

$$\epsilon_3 = i(1 - e_{1234}) \ , \quad \epsilon_4 = -ik^2 u(1 - e_{1234}) + i h_i \gamma^{+i}(1 - e_{1234}) \ . \tag{4.91}$$

Note that $\epsilon_3 = \rho^1 \epsilon_1$ and $\epsilon_4 = \rho^1 \epsilon_2$. This requires that the KSEs commute with ρ^1. These Killing spinors give rise to an additional $\hat{\nabla}$-parallel 1-form, and in this case we find the 1-form $\hat{\nabla}$-parallel spinor bilinears to be given by

$$\begin{aligned} \lambda^- &= \mathbf{e}^- \ , \quad \lambda^+ = \mathbf{e}^+ - \frac{1}{2}k^2 u^2 \mathbf{e}^- - uh \ , \quad \lambda^1 = k^{-1}(h + k^2 u \mathbf{e}^-) \ , \\ \lambda^4 &= e^1 \ , \end{aligned} \tag{4.92}$$

where the first three bilinears are those of horizons with two supersymmetries and e^1 is given in (4.86). The vector fields associated to these are Killing and their Lie algebra is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$ [89].

The spacetime is locally metrically a product $AdS_3 \times \Sigma^3$, as for horizons preserving two supersymmetries. In addition in this case, Σ^3 is an S^1 fibration over a 2-dimensional manifold Σ^2, where the fibre direction is spanned by $\lambda^4 = e^1$. Therefore,

$$ds^2(\Sigma^3) = (e^1)^2 + ds^2(\Sigma^2) \ , \quad ds^2(\mathcal{S}) = k^{-2}h^2 + (e^1)^2 + ds^2(\Sigma^2) \ . \tag{4.93}$$

Also observe that $de^1 \neq 0$ as $e^1 \wedge de^1$ is proportional to $\tilde{H} = dvol(\Sigma^3)$, and so the fibration is twisted.

It now remains to specify the topology of Σ^2. For this we first observe that from the results of chapter 2 section 2.8, the hypermultiplet scalars depend only on the coordinates of Σ^2. To specify the topology of Σ^3, we compute the Ricci tensor $R^{(2)}$

of Σ^2 using the Einstein equation and in particular (4.87) to find

$$R^{(2)}_{r's'} - \frac{1}{2} de^1_{t'u'} (de^1)^{t'u'} \delta_{r's'} - \frac{1}{2} k^2 \delta_{r's'} - 2\partial_{r'} \phi^{\underline{I}} \partial_{s'} \phi^{\underline{J}} g_{\underline{IJ}} = 0 \ , \tag{4.94}$$

where now r', s', t', u' are indices in Σ^2, and we have used the result $R^{(3)}_{r's'} = R^{(2)}_{r's'} - \frac{1}{2} de^1_{t'u'} (de^1)^{t'u'} \delta_{r's'}$ [90]. Once again, it is clear that the Ricci tensor of Σ^2 is strictly positive and so Σ^2 is topologically a sphere irrespective of the properties of the maps ϕ.

We have mentioned that the hypermultiplet scalars ϕ depend only on the coordinates of Σ^2 as a consequence of the hyperini KSE. Thus they are maps from Σ^2 into the quaternionic Kähler manifold of the hypermultiplets. In addition, the hyperini KSE implies that

$$V^{\underline{a1}}_{\bar{2}} = 0 \ , \quad V^{\underline{a2}}_{\bar{2}} = 0 \ , \tag{4.95}$$

which is equivalent to (4.89) after additionally requiring the scalars not to depend on the fibre direction λ^4. These conditions imply that ϕ are pseudo-holomorphic maps from Σ^2 into the quaternionic Kähler manifold of the hypermultiplets. Furthermore, the analysis we have made for the existence of non-constant solutions to (4.89) applies to (4.95) as well.

4.5.2 N=8 Horizons

As in the cases with two and four supersymmetries, one can show that the spacetime is locally $AdS_3 \times \Sigma^3$. In addition, for horizons with eight supersymmetries, the hyperini KSE implies that the scalars of the hypermultiplet are constant, see chapter 2. In such a case, the Einstein equation implies that Σ^3 is locally isometric to S^3. Therefore, the only near horizon geometry preserving eight supersymmetries with $h \neq 0$ is $AdS_3 \times S^3$.

4.6 Horizons with $h = 0$

We shall now consider the second class of near horizon geometries, $\mathbb{R}^{1,1} \times \mathcal{S}$, which arise when $h = 0$.

4.6.1 Geometry of N=1 Horizons

When h vanishes we can clearly see from (4.56) that the 3-form field strength of the gravitational multiplet vanishes $H = 0$, and the near horizon geometry is a product $\mathbb{R}^{1,1} \times \mathcal{S}$. It now remains to determine the geometry of $\mathcal{S}$.

To begin with we first observe that the tensor multiplet scalars are constant and the associated 3-form field strengths, $H^{\underline{M}}$ in (4.56), vanish. The proof for this is similar to that given for the horizons with $h \neq 0$. In particular, it utilises the field equations of the tensor multiplet scalars as described in equations (4.68) and (4.69), with $h = 0$, and the argument developed around (4.83).

This then means the Einstein equation in (4.63) becomes

$$R_{\mu\nu} - 2g_{\underline{IJ}}\partial_\mu\phi^{\underline{I}}\partial_\nu\phi^{\underline{J}} = 0 \ . \tag{4.96}$$

Therefore, the Einstein equation expresses the Ricci tensor $\tilde{R}$ of $\mathcal{S}$ in terms of the hypermultiplet scalars as

$$\tilde{R}_{ij} = 2g_{\underline{IJ}}\partial_i\phi^{\underline{I}}\partial_j\phi^{\underline{J}} \ . \tag{4.97}$$

The hypermultiplet scalars are also restricted by the Killing spinor equations as in (2.48). In terms of real directions these conditions are the same as

$$\begin{array}{rcl} -V^{\underline{a1}}_{\tilde{1}} + iV^{\underline{a1}}_{4} + V^{\underline{a2}}_{\tilde{2}} + V^{\underline{a2}}_{5} & = & 0 \ , \\ V^{\underline{a2}}_{\tilde{1}} + iV^{\underline{a2}}_{4} + V^{\underline{a1}}_{\tilde{2}} - V^{\underline{a1}}_{5} & = & 0 \ , \end{array} \tag{4.98}$$

which after an appropriate identification of frame directions of $\mathcal{S}$ with the matrices $(\tau^i) = (1_{2\times 2}, -i\sigma^{r'})$, can be written as

$$(\tau^i)^{\underline{A}}{}_{\underline{B}}\, \partial_i\phi^{\underline{I}}\, E^{\underline{aB}}_{\underline{I}} = 0 \ . \tag{4.99}$$

These can equivalently be written in terms of the quaternionic structures $J_{r'}$ of $\mathcal{Q}$ as

$$(K_i)^{\underline{I}}{}_{\underline{J}}\, \partial^i\phi^{\underline{J}} = 0 \ , \tag{4.100}$$

where $(K_i) = (1_{4n_H\times 4n_H}, J_{r'})$ and $1_{4n_H\times 4n_H}$ is the identity tensor.

If the hypermultiplet scalars are constant, then the rhs of (4.97) vanishes and $\mathcal{S}$ is a hyper-Kähler manifold. So it is locally isometric to either K_3 or T^4. As we will see such horizons exhibit supersymmetry enhancement to at least $N = 4$.

The existence of horizons with strictly $N = 1$ supersymmetry depends on the existence of non-trivial solutions for (4.100) such that the rhs of the Einstein equation (4.97) does not vanish. This in turn may depend on the choice of the 4-manifold $\mathcal{S}$ and that of the quaternionic Kähler manifold $\mathcal{Q}$. We will not explore this any further here.

4.6.2 Geometry of N=2 and N=4 Horizons

The second Killing spinor of $N = 2$ horizons with $h = 0$ can be chosen as

$$\epsilon_2 = i(1 - e_{1234}) \ . \tag{4.101}$$

N	$\text{hol}(\tilde{\nabla})$	Geometry of $\mathcal{S}$
1	$Sp(1)\cdot Sp(1)$	Riemann
2	$U(2)$	Kähler
4	$Sp(1)$	hyper-Kähler

Table 4.2: Some geometric data of the horizon geometries with $h = 0$ are described. In the first column, we give the number of supersymmetries preserved. In the second column, we present the holonomy groups of the Levi-Civita connection of $\mathcal{S}$, and in the third we give the geometry of $\mathcal{S}$.

In this case, which is in agreement with the general classification results given in chapter 2, $\mathcal{S}$ is a Kähler manifold. In addition, the hypermultiplet scalars are holomorphic maps from $\mathcal{S}$ into the hypermultiplets quaternionic Kähler manifold. In particular, the hyperini KSE conditions can be written as

$$\partial_i\phi^{\underline{I}}(J_3)^{\underline{Aa}}{}_{\underline{Bb}}E^{\underline{bB}}_{\underline{I}} = I^j{}_i\partial_j\phi^{\underline{I}}E^{\underline{aA}}_{\underline{I}} \ , \tag{4.102}$$

where $(J_3)^{\underline{Aa}}{}_{\underline{Bb}} = (-i\sigma_3)^{\underline{A}}{}_{\underline{B}}\delta^{\underline{a}}{}_{\underline{b}}$ and $I^j{}_i = (i\delta^\alpha{}_\beta, -i\delta^{\bar\alpha}{}_{\bar\beta})$. Once again, the existence of horizons with strictly two supersymmetries depends on the existence of such non-trivial holomorphic maps.

The two remaining Killing spinors of $N = 4$ horizons with $h = 0$ can be chosen as

$$\epsilon_3 = e_{12} - e_{34} \ , \quad \epsilon_4 = i(e_{12} + e_{34}) \ . \tag{4.103}$$

The general classification results of chapter 2 now imply that the hypermultiplet scalars are constant. Therefore, $\mathcal{S}$ is hyper-Kähler and so locally isometric to either K_3 or T^4. In the latter case, there is supersymmetry enhancement to $N = 8$. Note also that in the general classification results we found there to be a descendant preserving three supersymmetries. However, the Killing spinors in this case have the same isotropy group as that for the four Killing spinors we have given above. This means the holonomy of the Levi-Civita connection of $\mathcal{S}$ is contained in $Sp(1)$, i.e. the geometry of $\mathcal{S}$ is hyper-Kähler. This in turn implies that it must be Ricci flat and so the hypermultiplet scalars must be constant. As a result we have supersymmetry enhancement from $N = 3$ to $N = 4$.

The Killing spinors we have considered so far are those that are annihilated by the light-cone projection operator γ_+. This means they have a non-compact isotropy group in $Spin(5,1)\cdot Sp(1)$. However, we could demand that the near horizon geometries $\mathbb{R}^{1,1}\times\mathcal{S}$ admit Killing spinors with compact isotropy groups. In this case, the only solution is $\mathbb{R}^{1,1}\times T^4$ which preserves eight supersymmetries. Some of the results of this section have been tabulated in table 4.2.

4.7 Summary

In this chapter we investigated the near horizon geometries of six dimensional (1,0) supergravity coupled to arbitrary numbers of tensor and scalar multiplets, extending the results of [34]. This was done by first adapting to Gaussian null coordinates, which we described at the beginning of the chapter. In addition, we made use of the solutions of the KSEs coming from chapter 2, and the restrictions imposed by the field equations of the theory.

In particular, we found that there were two classes of near horizon geometries depending on whether h vanishes or not. The first class of solutions that we looked at were when $h \neq 0$; in this case we found that the near horizon geometries were locally isometric to $AdS_3 \times \Sigma^3$, where Σ^3 is diffeomorphic to S^3. In addition, the tensor multiplet scalars are constants and the associated 3-form field strengths, $H^{\underline{M}}$, vanish. This class of solutions preserved 2, 4 or 8 supersymmetries, and the increase in supersymmetry further restricted the geometry of Σ^3. For horizons which preserve four supersymmetries we found that Σ^3 was a non-trivial circle fibration over a topological 2-sphere and for horizons that preserve eight supersymmetries Σ^3 is locally isometric to the 3-sphere, S^3. The scalars of the hypermultiplet can be seen as maps from Σ^3 into the quaternionic Kähler manifold, $\mathcal{Q}$, which are further constrained for horizons that admit more supersymmetry.

The second class of horizons appear when we set $h = 0$ and these take the general form $\mathbb{R}^{1,1} \times \mathcal{S}$, where $\mathcal{S}$ is a 4-manifold whose geometry depends on the hypermultiplet scalars. In this class of solutions we once again find the tensor scalars to be constants and the 3-form field strengths $H^{\underline{M}}$ to vanish. In addition, we find the 3-form field strength of the gravitational multiplet H to vanish. In the case when one supersymmetry is preserved $\mathcal{S}$ is a Riemannian manifold, when two supersymmetries are preserved it is a Kähler manifold and when four supersymmetries are preserved it is a hyper-Kähler manifold, with near horizon geometry $\mathbb{R}^{1,1} \times K_3$ or $\mathbb{R}^{1,1} \times T^4$, the latter case allows supersymmetry enhancement to N=8.

Chapter 5

(1,0) Superconformal Theories and KSEs

5.1 Introduction

The focus of this chapter will be on (1,0) superconformal theories in six dimensions and their Killing spinor equations. Before giving an outline of the aims of this chapter let us briefly mention a few general things and in particular why there has recently been interest in understanding superconformal theories in six dimensions.

One of the main reasons for this interest is due to the expectation that the dynamics of multiple M5-branes is described by such a superconformal theory in six dimensions. The evidence for this comes from the fact that the near horizon geometry of the M5-brane supergravity solution [6] is $AdS_7 \times S^4$ [118] and so this means the theory which describes the worldvolume dynamics must exhibit a $SO(6,2)$ symmetry. In addition, as $AdS_7 \times S^4$ is a maximally supersymmetric solution of eleven dimensional supergravity [4], the worldvolume theory must have 16 supersymmetries. This therefore means one needs a (2,0) theory in six dimensions. In particular, the dynamics of the multiple M5-brane system is believed to be described by gauging (2,0) tensor multiplets in six dimensions in analogy to the constructions made for the M2-brane system in [119, 120]. Some progress has been made in this direction and such (2,0) theories have been suggested in [121], and a bosonic theory in [122]. However, some of the fields in [121] are required to be independent of one of the worldvolume directions as a consequence of the closure of the supersymmetry algebra, making the theory effectively 5-dimensional. The construction of these (2,0) theories are based on Lorentzian 3-Lie algebras as was used for the M2-brane case, however, the classification of 3-algebras has led to strict restrictions for the theory see [123, 124, 125]. As was done by ABJM for M2-branes [126], one can consider M5-brane systems which preserve less than maximal supersymmetry. Following this reasoning, Samtleben et al [127, 128] proposed a six dimensional (1,0)-supersymmetric superconformal theory, see also [129]. The construction used is based on gauging (1,0) tensor multiplets and relies on the introduction of appropriate Stückelberg-type couplings. This gives rise to a number of models, some of which have a Lagrangian description provided one uses a prescription to deal with the kinetic term of the self-dual 3-form field strengths of the tensor multiplet.

In this chapter we will begin by giving an outline of the construction of (1,0) six dimensional superconformal theories described in [127, 128]. In particular, we will set the problem by giving the field content, the supersymmetry transformations and the KSEs of the models that we will investigate later. We will also describe how the methodology applied in the first two chapters will be used to solve the KSEs. After this we will derive all the conditions on the fields in order to obtain a configuration preserving a fraction of supersymmetry. This will be done for all possible fractions of supersymmetry admitted by the theory. In particular for the models in [127, 128], we find that they admit solutions preserving 1, 2, 4 and 8 supersymmetries.

In the second half of the chapter we will focus on solutions which preserve four supersymmetries, i.e. the half-supersymmetric solutions. In particular, we will discuss a few specific models and investigate the field equations and the Bianchi identities to find some explicit string and 3-brane solutions.

Note that in this chapter the notation we use to denote some of the fields is different from what we have used thus far, and similarly, different indices are used in the labelling. However, we will try to make this clear as we progress. This chapter is primarily based on the paper in [14].

5.2 (1,0) Superconformal Theory and KSEs

5.2.1 Fields and Supersymmetry Transformations

The (1,0) superconformal model of [127] has been constructed by gauging an arbitrary number of tensor multiplets and the introduction of appropriate higher form fields which are used in Stückelberg-type couplings. The field content of the theory is grouped into two multiplets; the vector and the tensor multiplet. The vector multiplets consist of $(A^r_\mu, \lambda^{ir}, Y^{ijr})$, where r labels the different vector multiplets and $i, j = 1, 2$ are the $Sp(1)$ R-symmetry indices, A^r_μ are 1-form gauge potentials, λ^{ir} are symplectic Majorana-Weyl spinors and Y^{ijr} are auxiliary fields. The field content of the tensor multiplets is $(\phi^I, \chi^{iI}, B^I_{\mu\nu})$, where I labels the different tensor multiplets, ϕ^I are scalars, χ^{iI} are symplectic Majorana-Weyl spinors, of opposite chirality from those of the vector multiplets, and $B^I_{\mu\nu}$ are the 2-form gauge potentials. Note that we have used slightly different notations in labelling the fields here compared to the previous chapters.

The field strengths associated to the 1- and 2-form gauge potentials of the vector and tensor multiplet, respectively, are constructed by introducing new coupling constants which allow a general 2- and 3-form field strength to be written. In particular, these are given by

$$
\begin{aligned}
\mathcal{F}^r_{\mu\nu} &\equiv 2\partial_{[\mu}A^r_{\nu]} - f_{st}{}^r A^s_\mu A^t_\nu + h^r_I B^I_{\mu\nu} , && (5.1)\\
\mathcal{H}^I_{\mu\nu\rho} &\equiv 3D_{[\mu}B^I_{\nu\rho]} + 6d^I_{rs}A^r_{[\mu}\partial_\nu A^s_{\rho]} - 2f_{pq}{}^s d^I_{rs}A^r_{[\mu}A^p_\nu A^q_{\rho]} + g^{Ir}C_{\mu\nu\rho r} , && (5.2)
\end{aligned}
$$

where $f_{rs}{}^t$ are the structure constants, h^r_I, g^{Ir} and $d^I_{rs} = d^I_{(rs)}$ are Stückelberg-type couplings, and $C_{\mu\nu\rho r}$ are three-form gauge potentials. The covariant derivative is defined as

$$
D_\mu\Lambda^s \equiv \partial_\mu\Lambda^s + A^r_\mu(X_r)_t{}^s\Lambda^t , \quad D_\mu\Lambda^I \equiv \partial_\mu\Lambda^I + A^r_\mu(X_r)_J{}^I\Lambda^J , \tag{5.3}
$$

where X_r are given by

$$
(X_r)_t{}^s = -f_{rt}{}^s + d^I_{rt}h^s_I , \quad (X_r)_J{}^I = 2h^s_J d^I_{rs} - g^{Is}b_{Jsr} . \tag{5.4}
$$

In addition, covariance of the field strengths under the gauge transformations of gauge potentials requires that

$$
\begin{aligned}
2(d^J_{r(u}d^I_{v)s} - d^I_{rs}d^J_{uv})h^s{}_J &= 2f_{r(u}{}^s d^I_{v)s} - b_{Jsr}d^J_{uv}g^{Is} ,\\
(d^J_{rs}b_{Iut} + d^J_{rt}b_{Isu} + 2d^K_{ru}b_{Kst}\delta^J_I)h^u_J &= f_{rs}{}^u b_{Iut} + f_{rt}{}^u b_{Isu} + g^{Ju}b_{Iur}b_{Jst} ,\\
f_{[pq}{}^u f_{r]u}{}^s - \frac{1}{3}h^s_I d^I_{u[p}f_{qr]}{}^u &= 0 ,\\
h^r_I g^{Is} &= 0 ,\\
f_{rs}{}^t h^r_I - d^J_{rs}h^t_J h^r_I &= 0 ,\\
g^{Js}h^r_K b_{Isr} - 2h^s_I h^r_K d^J_{rs} &= 0 ,\\
-f_{rt}{}^s g^{It} + d^J_{rt}h^s_J g^{It} - g^{It}g^{Js}b_{Jtr} &= 0 .
\end{aligned} \tag{5.5}
$$

In order for a consistent model to exist all of these algebraic constraints need to be satisfied. We discuss some models later on.

It remains to give the supersymmetry transformations of the fields. As we are interested in the KSEs, it suffices to state the supersymmetric variations of the fermions. These are given by

$$
\begin{aligned}
\delta\lambda^{ir} &= \frac{1}{8}\mathcal{F}^r_{\mu\nu}\gamma^{\mu\nu}\epsilon^i - \frac{1}{2}Y^{ijr}\epsilon_j + \frac{1}{4}h^r_I\phi^I\epsilon^i , && (5.6)\\
\delta\chi^{iI} &= \frac{1}{48}\mathcal{H}^I_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon^i + \frac{1}{4}D_\mu\phi^I\gamma^\mu\epsilon^i - \frac{1}{2}d^I_{rs}\gamma^\mu\lambda^{ir}\bar{\epsilon}\gamma_\mu\lambda^s . && (5.7)
\end{aligned}
$$

These, as well as the remaining supersymmetry transformations, along further details of the theory can be found in [127, 128].

5.2.2 Field Equations

The field equations of the minimal system are

$$\begin{aligned} D^\mu D_\mu \phi^I &= -\frac{1}{2} d^I_{rs} (\mathcal{F}^r_{\mu\nu} \mathcal{F}^{\mu\nu s} - 4 Y^r_{ij} Y^{ijs}) - 3 d^I_{rs} h^r_J h^s_K \phi^J \phi^K \ , & (5.8) \\ g^{Kr} b_{Irs} Y^s_{ij} \phi^I &= 0 \ , & (5.9) \\ g^{Kr} b_{Irs} \mathcal{F}^s_{\mu\nu} \phi^I &= \frac{1}{4!} \epsilon_{\mu\nu\lambda\rho\sigma\tau} g^{Kr} \mathcal{H}^{(4)\lambda\rho\sigma\tau}_r \ . & (5.10) \end{aligned}$$

Observe that generically the theory has a cubic scalar field interaction and so the potential term is not bounded from below. In addition to these field equations we have the two Bianchi identities

$$\begin{aligned} D_{[\mu} \mathcal{F}^r_{\nu\rho]} &= \frac{1}{3} h^r_I \mathcal{H}^I_{\mu\nu\rho} \ , & (5.11) \\ D_{[\mu} \mathcal{H}^I_{\nu\rho\sigma]} &= \frac{3}{2} d^I_{rs} \mathcal{F}^r_{[\mu\nu} \mathcal{F}^s_{\rho\sigma]} + \frac{1}{4} g^{Ir} \mathcal{H}^{(4)}_{\mu\nu\rho\sigma r} \ , & (5.12) \end{aligned}$$

where $\mathcal{H}^{(4)}_{\mu\nu\rho\sigma r}$ is the field strength of the 3-form.

5.2.3 KSEs

As we have previously discussed, the KSEs are the vanishing conditions for the supersymmetry variations of the fermions of the theory evaluated at the locus where all the fermions vanish. In this case, we find the KSEs to be given by

$$\begin{aligned} \frac{1}{4} \mathcal{F}^r_{\mu\nu} \gamma^{\mu\nu} \epsilon^i - Y^{ijr} \epsilon_j + \frac{1}{2} h^r_I \phi^I \epsilon^i &= 0 \ , \\ \frac{1}{12} \mathcal{H}^I_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon^i + D_\mu \phi^I \gamma^\mu \epsilon^i &= 0 \ . & (5.13) \end{aligned}$$

The first condition is the vanishing condition of the supersymmetry variation of the fermions of the vector multiplets while the second is the vanishing condition of the supersymmetry variation of the fermions of the tensor multiplets. In analogy with similar variations in six dimensional supergravity which we discussed in chapter 2, we shall refer to these two equations as the gaugini and tensorini KSEs, respectively.

Note that all the spinors that appear in the theory are symplectic Majorana-Weyl and the gauge group of the theory is $Spin(5,1) \cdot Sp(1)$. We will now proceed to writing the KSEs in the formalism that was introduced in chapter 2. Recall that we identified the symplectic Majorana-Weyl spinors with the $Sp(1)$-invariant $Spin(9,1)$ Majorana-Weyl spinors and realised them as forms. In addition, a basis for the symplectic Majorana-Weyl spinors is given in (2.16). In this formalism we can make use of the ρ^a $SU(2)$ generators, with $a = 1, 2, 3$, introduced in (2.17) to rewrite the term involving the Y in the KSEs as

$$-Y^{ijr} \epsilon_j = (Y^r)_a (\rho^a \epsilon)^i \ , \qquad (5.14)$$

where we have made use of the fact that the $Sp(1)$ indices are raised and lowered by the antisymmetric tensors ε_{ij} and ε^{ij}, with $\tau^i = \varepsilon^{ij}\tau_j$. As a result the KSEs, (5.13), can be re-expressed as

$$\frac{1}{4}\mathcal{F}^r_{\mu\nu}\gamma^{\mu\nu}\epsilon + (Y^r)_a\rho^a\epsilon + \frac{1}{2}h^r_I\phi^I\epsilon = 0 \ , \tag{5.15}$$

$$\frac{1}{12}\mathcal{H}^I_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + D_\mu\phi^I\gamma^\mu\epsilon = 0 \ . \tag{5.16}$$

In what follows, we shall solve both KSEs for backgrounds preserving any number of supersymmetries following the same prescription as that given in chapter 2 for the case of supergravity backgrounds.

5.3 Killing Spinors

The supersymmetric solutions of this theory can be classified according to the isotropy group of Killing spinors, just as in the case of the supersymmetric supergravity solutions. Since the gauge group is $Spin(5,1)\cdot Sp(1)$, the methodology for obtaining the isotropy group of Killing spinors and finding representatives for these follows the same discussion as in section 2.3 of chapter 2. The results of this analysis have been summarised in table 2.1.

Let us now briefly recap this and discuss the implications of these as solutions of the KSEs of the superconformal theory under consideration here.

5.3.1 One Killing Spinor

If the KSEs (5.15) and (5.16) admit a Killing spinor, then the representative Killing spinor can be identified as $1+e_{1234}$ which has isotropy group $Sp(1)\cdot Sp(1)\ltimes\mathbb{H}$. Also note that if the tensorini KSE (5.16) admits one Killing spinor then it admits four. This is due to the fact that it commutes with the ρ operations given in (2.17). A basis for the four Killing spinors of (5.16) is given by the $Sp(1)\ltimes\mathbb{H}$ invariant spinors in table 2.1.

5.3.2 Two Killing Spinors

In this case there are two options. Firstly, the two Killing spinors can have isotropy group $U(1)\cdot Sp(1)\ltimes\mathbb{H}$ with representative spinors given by $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = i(1-e_{1234})$. In the second case the Killing spinors have isotropy group $Sp(1)$ with representative Killing spinors given by $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = e_{15} + e_{2345}$. Note that in this case the tensorini KSE preserves all eight supersymmetries since it commutes with the ρ operations but now acting on the $Sp(1)$-invariant Killing spinors.

5.3.3 Four Killing Spinors

Once again there are two cases to consider. Firstly, we find that the gaugini KSE admits $\epsilon_1 = 1+e_{1234}$, $\epsilon_2 = i(1-e_{1234})$, $\epsilon_3 = e_{12}-e_{34}$ and $\epsilon_4 = i(e_{12}+e_{34})$ as solutions with isotropy group $Sp(1) \ltimes \mathbb{H}$, these are also the four Killing spinors solution to the tensorini KSE.

The second set of supersymmetric solutions arise when we choose the $U(1)$-invariant spinors given by $\epsilon_1 = 1 + e_{1234}$, $\epsilon_2 = i(1 - e_{1234})$, $\epsilon_3 = e_{15} + e_{2345}$ and $\epsilon_4 = i(e_{15} - e_{2345})$, and we already know that the tensorini KSE preserves all eight supersymmetries.

5.3.4 More than Four Killing Spinors

If a solution admits more than four linearly independent Killing spinors then it is maximally supersymmetric. The results of this section have been summarised in table 5.1.

Isotropy Groups	Gaugini	Tensorini
$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	1	4
$U(1) \cdot Sp(1) \ltimes \mathbb{H}$	2	4
$Sp(1) \ltimes \mathbb{H}$	4	4
$Sp(1)$	2	8
$U(1)$	4	8
$\{1\}$	8	8

Table 5.1: In the first column the isotropy groups of the Killing spinors of the gaugini KSE are given. In the second and third columns the number of Killing spinors of the gaugini and tensorini KSEs are stated, respectively. The isotropy groups of the Killing spinors of the tensorini KSE are either $Sp(1) \ltimes \mathbb{H}$ or $\{1\}$. The cases that do not appear in the table do not occur.

5.4 Solutions of the Killing Spinor Equations

In this section, we shall derive the conditions imposed on the fields by the KSEs. To do this, one can substitute into the KSEs the representative Killing spinors given in the previous section and then solve the resulting equations. This can be done in a straightforward way, and closely follows the analysis made for the KSEs of six dimensional supergravity in chapter 2.

5.4.1 N=1 Solutions

The Killing spinor is $\epsilon = 1 + e_{1234}$. Substituting this into the gaugini KSE (5.15), we find the fields must satisfy

$$\mathcal{F}^r_{-+} + h^r_I \phi^I = 0 \ , \qquad \mathcal{F}^{r\alpha}_{\alpha} + 2i(Y^r)_1 = 0 \ ,$$
$$\mathcal{F}^r_{+\alpha} = 0 = \mathcal{F}^r_{+\bar{\alpha}} \ , \qquad \mathcal{F}^r_{12} + (Y^r)_2 - i(Y^r)_3 = 0 \ , \tag{5.17}$$

where we have used the oscillator basis discussed in chapter 1 to perform this calculation in terms of light-cone and complex coordinates on Minkowski spacetime $\mathbb{R}^{5,1}$. Let us briefly recall that in the 10-dimensional description of the spinors we have adopted the spacetime directions are along 0, 5, 1, 6, 2, 7, whereas 3, 4, 8, 9 are not used and taken as auxiliary directions. The Minkowski metric on $\mathbb{R}^{5,1}$ has been chosen as

$$\begin{aligned} ds^2 &= -(dx^0)^2 + (dx^5)^2 + (dx^1)^2 + (dx^6)^2 + (dx^2)^2 + (dx^7)^2 \\ &= 2e^- e^+ + \delta_{ij} e^i e^j = 2e^- e^+ + 2\delta_{\alpha\bar{\beta}} e^\alpha e^{\bar{\beta}} \ , \end{aligned} \tag{5.18}$$

where e^α, $\alpha = 1, 2$, are the differentials of complex coordinates constructed from the pairs (dx^1, dx^6) and (dx^2, dx^7), respectively, and (e^-, e^+) are the differentials of the light-cone coordinates along the directions (dx^0, dx^5). The $\mathcal{F}^r_{-i}$ components of the 2-form field strength are not restricted by the gaugini KSE. The same also applies for the anti-self dual component $\mathcal{F}^{\mathrm{asd}}$ of $\mathcal{F}_{ij}$. Moreover, the self-dual component of $\mathcal{F}$ is completely determined in terms of the auxiliary field Y. Note the similarity to the analysis made in section 2.4 of chapter 2 for $N = 1$ backgrounds. Combining the above results we can write the 2-form field strength as

$$\begin{aligned} \mathcal{F}^r &= -h^r_I \phi^I \, e^- \wedge e^+ + \mathcal{F}^r_{-i} \, e^- \wedge e^i - [(Y^r)_2 - i(Y^r)_3] e^1 \wedge e^2 \\ &\quad -[(Y^r)_2 + i(Y^r)_3] e^{\bar{1}} \wedge e^{\bar{2}} + (Y^r)_1 \omega + \mathcal{F}^{\mathrm{asd},r} \ , \end{aligned} \tag{5.19}$$

where $\omega = -i\delta_{\alpha\bar{\beta}} e^\alpha \wedge e^{\bar{\beta}}$.

Solving the tensorini KSE (5.16) for $1 + e_{1234}$ gives

$$D_{\bar{\alpha}} \phi^I + \frac{1}{2}\mathcal{H}^I_{-+\bar{\alpha}} + \frac{1}{2}\mathcal{H}^I_{\bar{\alpha}\beta}{}^{\beta} = 0 \ , \quad \mathcal{H}^I_{+\alpha\beta} = 0 \ , \quad \mathcal{H}^I_{+\alpha}{}^{\alpha} = 0 \ , \quad D_+ \phi^I = 0 \ . \tag{5.20}$$

In addition, the 3-form field strengths are restricted to be self-dual

$$\mathcal{H}^I_{\mu\nu\rho} = \frac{1}{3!} \epsilon_{\lambda\sigma\tau\mu\nu\rho} \mathcal{H}^{I\lambda\sigma\tau} \ . \tag{5.21}$$

Decomposing this condition in the $+, -, \alpha, \bar{\alpha}$ coordinates, one finds

$$\mathcal{H}^I_{-+\alpha} + \mathcal{H}^I_{\alpha\beta}{}^{\beta} = 0 \ , \qquad \mathcal{H}^I_{-+\bar{\alpha}} - \mathcal{H}^I_{\bar{\alpha}\beta}{}^{\beta} = 0 \ ,$$
$$\mathcal{H}^I_{+1\bar{1}} - \mathcal{H}^I_{+2\bar{2}} = 0 \ , \qquad \mathcal{H}^I_{+1\bar{2}} = 0 \ . \tag{5.22}$$

Combining these conditions with those from the tensorini KSE, one finds that

$$\mathcal{H}^I_{+ij} = 0 \ . \tag{5.23}$$

$\mathcal{H}^I_{-ij}$ is anti-self-dual in the directions transverse to the light-cone and the remaining components are determined in terms of $D\phi^I$. As can be seen the analysis here is almost identical to that of section 2.4. Therefore, putting these together one can write the 3-form field strength as

$$\mathcal{H}^I = \frac{1}{2}\mathcal{H}^I_{-ij}\, e^- \wedge e^i \wedge e^j - D_i\phi^I e^- \wedge e^+ \wedge e^i + \frac{1}{3!}D_\ell\phi^I\, \epsilon^\ell{}_{ijk}\; e^i \wedge e^j \wedge e^k \ , \tag{5.24}$$

where we have used the self-duality of $\mathcal{H}^I$ to write

$$\mathcal{H}^I_{ijk} = D_l\phi^I \epsilon^l{}_{ijk} \ . \tag{5.25}$$

In contrast to the gaugini KSE, the tensorini KSE exhibits supersymmetry enhancement. In particular, if it admits one Killing spinor ϵ, it admits a further three Killing spinors given by $\rho^1\epsilon$, $\rho^2\epsilon$ and $\rho^3\epsilon$. When $\epsilon = 1 + e_{1234}$ this means all four Killing spinors are given by the $Sp(1) \ltimes \mathbb{H}$ invariant spinors of table 2.1.

5.4.2 N=2 Solutions with Non-Compact Isotropy Group

The first Killing spinor is $\epsilon_1 = 1 + e_{1234}$ as in the $N = 1$ solutions described above. The second Killing spinor is $\epsilon_2 = i(1 - e_{1234})$, which can also be written as $\epsilon_2 = \rho^1\epsilon_1$. Therefore, for the solutions to admit two supersymmetries the KSEs must commute with the ρ^1 operation. As we have mentioned, the tensorini KSE commutes with all the ρ operations and so ϵ_2 is also a Killing spinor. The 3-form field strength is given as in (5.24).

This is not always the case for the gaugini KSE. For the gaugini KSE to commute with ρ^1 we require,

$$(Y^r)_2 = (Y^r)_3 = 0 \ . \tag{5.26}$$

The 2-form field strength is given as in (5.19) after imposing (5.26). Thus it becomes

$$\mathcal{F}^r = -h^r_I\phi^I\, e^- \wedge e^+ + \mathcal{F}^r_{-i}\, e^- \wedge e^i + Y^r\omega + \mathcal{F}^{\mathrm{asd},r} \ , \tag{5.27}$$

where we have also set $Y^r = (Y^r)_1$.

5.4.3 N=2 Solutions with Compact Isotropy Group

The first Killing spinor is the same as that of $N = 1$ solutions, $\epsilon_1 = 1 + e_{1234}$. The second Killing spinor is $\epsilon_2 = e_{15} + e_{2345}$. The conditions on the fields imposed by the gaugini KSE (5.15) evaluated on ϵ_2 are

$$\mathcal{F}^r_{-+} - h^r_I\phi^I = 0 \ , \qquad \mathcal{F}^r_{1\bar{1}} - \mathcal{F}^r_{2\bar{2}} - 2i(Y^r)_1 = 0 \ ,$$
$$\mathcal{F}^r_{1\bar{2}} - (Y^r)_2 - i(Y^r)_3 = 0 \ , \qquad \mathcal{F}^r_{-\alpha} = 0 \ , \qquad \mathcal{F}^r_{-\bar{\alpha}} = 0 \ . \tag{5.28}$$

Combining these conditions with those we have found for the first Killing spinor, ϵ_1, in (5.17), we get

$$\mathcal{F}^r_{-+} = 0 \ , \quad \mathcal{F}^r_{-i} = 0 \ , \quad \mathcal{F}^r_{+i} = 0 \ , \quad \mathcal{F}^r_{1\bar{1}} = 0 \ , \quad h^r_I\phi^I = 0 \ ,$$
$$\mathcal{F}^r_{2\bar{2}} + 2i(Y^r)_1 = 0 \ , \quad \mathcal{F}^r_{12} + \mathcal{F}^r_{1\bar{2}} - 2i(Y^r)_3 = 0 \ , \quad \mathcal{F}^r_{12} - \mathcal{F}^r_{1\bar{2}} + 2(Y^r)_2 = 0 \ . \tag{5.29}$$

It is convenient to rewrite the these conditions in terms of real coordinates. In particular, we find

$$\mathcal{F}^r_{-\nu} = 0 \ , \quad \mathcal{F}^r_{+\nu} = 0 \ , \quad \mathcal{F}^r_{\tilde{1}\nu} = 0 \ , \quad h^r_I\phi^I = 0 \ ,$$
$$\mathcal{F}^r_{\tilde{6}\tilde{7}} = 2(Y^r)_2 \ , \quad \mathcal{F}^r_{\tilde{2}\tilde{6}} = 2(Y^r)_3 \quad \mathcal{F}^r_{\tilde{2}\tilde{7}} = -2(Y^r)_1 \ , \tag{5.30}$$

where again we have used a tilde on the real directions to distinguish them from the complex ones we have used so far in the analysis of the KSEs in this chapter. The above conditions on the fields can be expressed more compactly by introducing a 3+3 split on the spacetime. In particular, we can introduce the coordinates x^a, $a = -, +, \tilde{1}$ and y^i, $i = \tilde{2}, \tilde{6}, \tilde{7}$. Then using these, (5.30) can be written as

$$\mathcal{F}^r_{ab} = 0 \ , \quad \mathcal{F}^r_{ai} = 0 \ , \quad h^r_I\phi^I = 0 \ , \quad \mathcal{F}^r_{ij} = -2\varepsilon_{ijk}(Y^r)^k \ , \tag{5.31}$$

where $\varepsilon_{\tilde{2}\tilde{6}\tilde{7}} = -1$ and we have set $(Y^r)^1 = (Y^r)^{\tilde{6}}$, $(Y^r)^2 = (Y^r)^{\tilde{2}}$ and $(Y^r)^3 = (Y^r)^{\tilde{7}}$. This means we have

$$\mathcal{F}^r = -\varepsilon_{ijk}(Y^r)^k \, e^i \wedge e^j \ , \quad h^r_I\phi^I = 0 \ . \tag{5.32}$$

We now need to solve the tensorini KSE (5.16) evaluated on ϵ_2. A straightforward calculation reveals that

$$D_-\phi^I = 0 \ , \quad \mathcal{H}^I_{-1\bar{2}} = 0 \ , \quad \mathcal{H}^I_{-1\bar{1}} - \mathcal{H}^I_{-2\bar{2}} = 0 \ ,$$
$$\mathcal{H}^I_{-+\bar{\alpha}} - 2D_{\bar{\alpha}}\phi^I + \mathcal{H}^I_{\bar{\alpha}\beta}{}^{\beta} = 0 \ . \tag{5.33}$$

These conditions combined with those we have found for $N = 1$ solutions in (5.20) and the self-duality condition (5.22) means that

$$\mathcal{H}^I_{\mu\nu\rho} = 0 \ , \qquad D_\mu\phi^I = 0 \ . \tag{5.34}$$

Clearly in this case, the tensorini KSE preserves all eight supersymmetries. Moreover, the integrability of the last condition in (5.34) implies that

$$F^r_{\mu\nu} X_{rJ}{}^I \phi^J = 0 \ , \tag{5.35}$$

where $F^r_{\mu\nu} = 2\partial_{[\mu} A^r_{\nu]} + X_{st}{}^r A^s_\mu A^t_\nu$.

5.4.4 N=4 Solutions with Non-Compact Isotropy Group

The Killing spinors are the $Sp(1) \ltimes \mathbb{H}$ invariant spinors of table 2.1. As mentioned, these can also be written as

$$\epsilon_1 = 1 + e_{1234} \ , \qquad \epsilon_2 = \rho^1 \epsilon_1 \ , \quad \epsilon_3 = \rho^2 \epsilon_1 \ , \quad \epsilon_4 = \rho^3 \epsilon_1 \ . \tag{5.36}$$

Therefore, if ϵ_1 is a Killing spinor, then the rest are also Killing spinors provided that the corresponding KSEs commute with the ρ operations. This is the case for the tensorini KSE (5.16) and so the 3-form flux is given as in (5.24). Note also that $D_+\phi^I = 0$.

The gaugini KSE commutes with all the ρ operations iff all Y's vanish, i.e.

$$Y^1 = Y^2 = Y^3 = 0 \ . \tag{5.37}$$

As a result using (5.19), the KSE implies that the 2-form flux is

$$\mathcal{F}^r = -h^r_I \phi^I \, e^- \wedge e^+ + \mathcal{F}^r_{-i} \, e^- \wedge e^i + \mathcal{F}^{\text{asd},r} \ . \tag{5.38}$$

5.4.5 N=4 Solution with Compact Isotropy Group

In this case the Killing spinors are the $U(1)$ invariant spinors of table 2.1. These can be written as

$$\epsilon_1 = 1 + e_{1234} \ , \qquad \epsilon_2 = e_{15} + e_{2345} \ , \qquad \epsilon_3 = \rho^1 \epsilon_1 \ , \qquad \epsilon_4 = \rho^1 \epsilon_2 \ . \tag{5.39}$$

Thus, these spinors solve the KSEs iff ϵ_1 and ϵ_2 are Killing spinors and the KSEs commute with the ρ^1 operation.

We have already shown that if ϵ_1 and ϵ_2 solve the tensorini KSE, then all eight supersymmetries are preserved. In particular, both the 3-form flux and $D\phi$ vanish,

$$\mathcal{H} = D\phi = 0 \ . \tag{5.40}$$

On the other hand, for the gaugini KSE to commute with ρ^1 we require $(Y^r)_2 = (Y^r)_3 = 0$. Substituting this into (5.32), we find that

$$\mathcal{F}^r = -2iY^r e^2 \wedge e^{\bar{2}} \ , \qquad h^r_I \phi^I = 0 \ , \tag{5.41}$$

where we have set $Y^r = (Y^r)_1$.

5.4.6 Maximally Supersymmetric Solutions

As we have mentioned all backgrounds which preserve more than four supersymmetries are maximally supersymmetric. It is straightforward to see that the conditions on the fluxes for maximally supersymmetric backgrounds are

$$D_\mu \phi^I = 0 \ , \quad h^r_I \phi^I = 0 \ , \quad \mathcal{F}^r_{\mu\nu} = 0 \ , \quad \mathcal{H}^I_{\mu\nu\rho} = 0 \ , \quad Y^{ijr} = 0 \ . \tag{5.42}$$

Thus, all the scalars ϕ^I are covariantly constant. In addition, those projected by h are required to vanish. Similarly the 2-form and 3-form field strengths vanish as well. The same applies for the auxiliary fields Y.

5.5 Half-Supersymmetric Solutions without Stückelberg Couplings

In the three sections that follow we discuss the half-supersymmetric solutions of three different models. To do this one needs to solve the field equations and the Bianchi identities in addition to imposing the constraints coming from the KSEs. One also needs to ensure that all the key algebraic conditions required for the consistency of the theory outlined in (5.5) are satisfied.

5.5.1 Summary of the Conditions

Before we proceed with the solution of the field equations and Bianchi identities for half supersymmetric backgrounds, we first summarise the restrictions on the fields imposed by the KSEs when four supersymmetries are preserved. In particular, we have found that if the isotropy group of the Killing spinors is non-compact, then

$$\begin{aligned} \mathcal{F}^r &= -h^r_I \phi^I e^- \wedge e^+ + \mathcal{F}_{-i} e^- \wedge e^i + \mathcal{F}^{\mathrm{asd},r} \ , \quad D_+ \phi^I = 0 \ , \\ \mathcal{H}^I &= \frac{1}{2} \mathcal{H}^I_{-ij} e^- \wedge e^i \wedge e^j - D_i \phi^I e^- \wedge e^+ \wedge e^i + \frac{1}{3!} D_\ell \epsilon^\ell{}_{ijk}\, e^i \wedge e^j \wedge e^k \ , \end{aligned} \tag{5.43}$$

where all the auxiliary fields Y vanish, $Y = 0$, and $\mathcal{H}_{-ij}$ and $\mathcal{F}^{\mathrm{asd},r}$ are anti-self-dual in the indices transverse to the light-cone. On the other hand, if the Killing spinors have compact isotropy group then

$$\begin{aligned} \mathcal{H}^I &= 0 \ , \quad D_\mu \phi^I = 0 \ , \quad h^r_I \phi^I = 0 \ , \\ \mathcal{F}^r &= -2i Y^r e^2 \wedge e^{\bar{2}} \ , \end{aligned} \tag{5.44}$$

where $(Y^2)^r = (Y^3)^r = 0$ and $Y^r = (Y^1)^r$. In what follows we shall take the field equation (5.10) as the definition of $\mathcal{H}^{(4)}$.

5.5.2 The Model

To start with let us consider the model with $g^{Ir} = h_I^r = 0$. The algebraic conditions in (5.5) are all satisfied provided that f are the structure constants of a Lie algebra $\mathfrak{g}$ and d^I_{rs} and b_{Irs} are invariant symmetric tensors under the action of the adjoint representation of $\mathfrak{g}$. For example, one could set $d^I_{rs} = d^I g_{rs}$ and $b_{Irs} = b_I g_{rs}$, where g_{rs} is a bi-invariant metric on $\mathfrak{g}$. Furthermore, in this case the 2-form field strength $\mathcal{F}$ is the standard curvature of a gauge connection. To identify the half-supersymmetric solutions in this case, one has to solve the field equations in addition to the KSEs. There are two cases to consider depending on whether the isotropy group of the Killing spinors is compact or not.

5.5.3 Non-Compact Isotropy Group

The condition $\mathcal{F}_{+\mu} = 0$ can be solved by fixing the gauge $A_+ = 0$ which then implies $\partial_+ A^r_\mu = 0$, i.e. A_-, A_i do not depend on the light-cone coordinate x^+. Similarly the condition $D_+\phi^I = 0$ coming from the KSE equations means the scalars ϕ^I do not depend on x^+, since the condition $D_+\phi^I = 0$ becomes $\partial_+\phi^I = 0$. Then the field equation (5.8) reduces to

$$\partial_i \partial^i \phi^I = -\frac{1}{2} d^I_{rs} \mathcal{F}^r_{ij} \mathcal{F}^{ijs} \ , \tag{5.45}$$

where $\mathcal{F}^r_{ij}$ are anti-self-dual instantons along the directions transverse to the light-cone. Observe that if $g^{Ir} = h_I^r = 0$ we find the generators $(X^r)^I{}_J = 0$, then the scalar fields are neutral (invariant) under the action of the gauge group and so they are not gauged, which means $D_\mu \phi^I = \partial_\mu \phi^I$. If $d^I_{rs} = d^I g_{rs}$ and $b_{Irs} = b_I g_{rs}$, where g_{rs} is a bi-invariant metric on $\mathfrak{g}$, then the right hand side of (5.45) can be identified with the Pontryagin density of instantons [130]. In such a case, this equation can be solved and similar equations have been solved in the context of heterotic supergravity in [130]. A more detailed anaylsis will be given in section 5.7 which can be adapted to this case, and so we do not discuss this any further here. The other two field equations (5.9) and (5.10) are automatically satisfied.

It remains to solve the Bianchi identity (5.12) for $\mathcal{H}$ subject to the restrictions imposed by the KSEs. The only independent component is

$$\partial_- \partial_\ell \phi^I \epsilon^\ell{}_{ijk} - 3\partial_{[i} \mathcal{H}^I_{jk]-} = 6 d^I_{rs} \mathcal{F}^r_{-[i} \mathcal{F}^s_{jk]} \ , \tag{5.46}$$

which arises from the $\mu\nu\rho\sigma = -ijk$ component. This completes the analysis of the Bianchi identities.

String Solutions

Take a string to span the two light-cone directions. Such classes of solutions exhibit Poincaré invariance along the directions of the string and can be found by setting $\mathcal{F}_{\pm\mu} = \mathcal{H}_{\pm\mu\nu} = 0$ and requiring all other fields to be independent from the $x^{\pm}$ coordinates of the light-cone. This means (5.46) is satisfied and the only non-trivial equation that remains to be solved is (5.45). This equation has solutions provided that $d^I_{rs} = d^I g_{rs}$ and $b_{Irs} = b_I g_{rs}$ with gauge groups that include $SU(N)$ and $Sp(N)$ and for any instanton number. See section 5.7 for a detailed analysis of an example based on a similar context.

A more general solution can be found by taking $\mathcal{F}_{\pm\mu} = 0$ and $\mathcal{H}_{-ij} \neq 0$, and the fields $\mathcal{F}$ and ϕ to be independent of $x^{\pm}$. In such a case, the Lorentz invariance on the string is broken. We know the self-duality of H^I implies the component $\mathcal{H}^I_{jk-}$ to be anti-self-dual along the transverse to light-cone directions and the Bianchi identity (5.46) requires it to be closed. Therefore it can identified with an abelian anti-self-dual field strength on $\mathbb{R}^4$ which are determined in terms of harmonic functions. There are no smooth solutions unless $\mathcal{H}^I_{jk-}$ is taken to be constant. Such a solution has the interpretation of a string with a wave propagating on it.

5.5.4 Compact Isotropy Group

In this case, the tensorini KSE (5.44) imply that the scalars ϕ^I are constant and $\mathcal{H} = 0$. Moreover, the auxiliary fields $Y^2 = Y^3 = 0$ and the only non-vanishing component of $\mathcal{F}$ is supported on a 2-dimensional subspace of the 4-dimensional space transverse to the light-cone directions. Furthermore, the KSEs imply both the field equations and Bianchi identities. Therefore, the only non-trivial field is $\mathcal{F}_{2\bar{2}}$ and it is related to Y^1 as in (5.41). This solution exhibits a $\mathbb{R}^{3,1}$ Poincaré invariance and so it has the interpretation of a 3-brane.

5.6 Half-Supersymmetric Solutions of the Adjoint Model

5.6.1 The Model

Another way to satisfy the conditions in (5.5) is by taking the number of tensor multiplets to be the same as the number of vector multiplets and setting

$$h^s_r = 0 \ , \quad d^s{}_{rt} = d_{prt} g^{ps} \ , \quad b_{prt} = f_{prt} \ , \tag{5.47}$$

where now g is a bi-invariant metric, and d is a totally symmetric bi-invariant tensor on the Lie algebra $\mathfrak{g}$ with structure constants f. This model does not admit a Lagrangian description.

5.6.2 Non-Compact Isotropy Group

The restrictions coming from the KSEs for the general model have been summarised in (5.43), so it remains to investigate the field equations and Bianchi identities of the model. Since $h = 0$ the 2-form field strength $\mathcal{F}$ reduces to the standard curvature of the gauge connection, as was the case in the previous model. As $\mathcal{F}_{+\mu} = 0$, one can again choose a gauge $A_+ = 0$ to find that all components of the gauge connection A do not depend on the light-cone coordinate x^+.

In this case we find that the field equation for the scalars is given by

$$D_i D^i \phi^r = -\frac{1}{2} d^r{}_{st} \mathcal{F}^s_{ij} \mathcal{F}^{t,ij} \ , \tag{5.48}$$

where we have used $\partial_+ \phi^r = 0$ which arises as a condition of the tensorini KSE. Unlike in the previous model, it is not apparent that anti-self-duality of $\mathcal{F}^s_{ij}$ implies that the above equation has solutions. The analysis requires details of the Lie algebra $\mathfrak{g}$ and so will not considered any further here. The second field equation (5.9) is automatically satisfied as $Y = 0$. As mentioned, one can view the last field equation (5.10) as the definition of $\mathcal{H}^{(4)}$. Upon substitution of this into the Bianchi identity (5.12) and using the solution of the KSE in (5.43), we find that the remaining independent equations are

$$\begin{aligned} D_- D_\ell \phi^r \epsilon^\ell{}_{ijk} - 3 D_{[i} \mathcal{H}^r_{jk]-} &= 6 d^r{}_{st} \mathcal{F}^s_{-[i} \mathcal{F}^t_{jk]} + \epsilon_{ijk}{}^m f^r{}_{st} \mathcal{F}^s_{-m} \phi^t \\ D_+ \mathcal{H}^r_{-ij} &= 0 \ , \end{aligned} \tag{5.49}$$

where $\epsilon_{-+ijkl} = \epsilon_{ijkl}$ and $\epsilon_{-+1\bar{1}2\bar{2}} = -1$. This concludes the analysis of the Bianchi identities of this model.

String Solutions

Solutions to (5.49) can be found by setting $\mathcal{F}_{-\mu} = \mathcal{F}_{+\mu} = 0$, choosing the gauge $A_\pm = 0$, identifying H^r_{jk-} with the curvature of an anti-self-dual connection, and taking all fields to be independent from the light-cone directions.

To find a solution of the theory, it remains to solve the field equations for the scalars. As mentioned, this depends on the choice of gauge group. However, this can be circumvented in the special case where we choose the coupling $d = 0$. This then means the field equation for the scalars becomes

$$D_i D^i \phi^r = 0 \ . \tag{5.50}$$

A class of solutions of (5.50) is given by the Green functions of the Laplace operator in an instanton background. These have been calculated for the adjoint representation in [131, 132], see also [133]. However in such a case, the scalar equation has delta function sources.

Alternatively, one can take the scalars in (5.50) to be neutral under the gauge group. This, for example, happens if

$$(X_r)_p{}^t = -g^{ts} b_{psr} \ , \tag{5.51}$$

vanishes on the active scalar fields of the solution. The covariant Laplace equation above then becomes a standard Laplace equation and ϕ^r can be expressed in terms of harmonic functions. For the structure constants (5.51) on the active scalars to vanish, one may take $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}'$, where $\mathfrak{t}$ is an abelian algebra which commutes with the subalgebra $\mathfrak{g}'$, and the ϕ's and $\mathcal{H}$'s are restricted to take values in $\mathfrak{t}$. Such solutions are singular unless ϕ is chosen to be constant.

Next, if $H^r_{jk-} = 0$, the solutions above exhibit a $\mathbb{R}^{1,1}$ Poincaré symmetry and so have an interpretation as strings. On the other hand if $H^r_{jk-} \neq 0$, the Poincaré symmetry is broken and the solutions have the interpretation as waves propagating on strings.

5.6.3 Compact Isotropy Group

Let us now consider the compact case. The scalar field equation in (5.8) and the Bianchi identity (5.11) are satisfied either as a consequence of the conditions on the field imposed by the KSEs summarised in (5.44) or as a consequence of the restrictions (5.47) on the coupling constants of the model.

The Bianchi identity of the 3-form field strength (5.12) implies that $\mathcal{H}^{(4)} = 0$, this is a consequence of the conditions summarised in (5.44). Then, the field equations (5.9) and (5.10) require that

$$[\mathcal{F}, \phi] = 0 \ . \tag{5.52}$$

As in the non-compact case discussed above, this condition can be solved by taking $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}'$, where $\mathfrak{t}$ commutes with the subalgebra $\mathfrak{g}'$, with the scalars ϕ taking values in $\mathfrak{t}$ while $\mathcal{F}$ takes values in $\mathfrak{g}'$. The only remaining condition is (5.41) which relates $\mathcal{F}$ to the auxiliary field Y. Such solutions exhibit a $\mathbb{R}^{3,1}$ Poincaré invariance and so have the interpretation of 3-branes.

5.7 Half-Supersymmetric Solutions of the $SO(5,5)$ Model

5.7.1 The Model

We will now investigate models that admit a Lagrangian description, but once again one has to keep in mind the subtleties arising from the kinetic term of the self-dual

3-form in the theory when using a Lagrangian formulation. For the existence of a Lagrangian there must exist a metric η_{IJ} such that [127, 128]

$$h^r_I = \eta_{IJ} g^{Jr} \ , \quad d^I_{rs} = \frac{1}{2}\eta^{IJ} b_{Jrs} \ , \quad \eta_{IJ} d^I_{r(u} d^J_{vs)} = 0 \ . \tag{5.53}$$

The reduction of the algebraic conditions (5.5) to this case has been done in [127]. In this section we shall focus on the $SO(5,5)$ model of [127] for which

$$b^I_{rs} = \gamma^I_{rs} \ , \quad f_{rs}{}^t = -4\gamma^{IJK}_{rs}\gamma_{IJp}{}^t g^p_K \ , \quad g^{Ir}\gamma_{Irs} = 0 \ , \tag{5.54}$$

where γ^I_{rs} are the gamma-matrices of $SO(5,5)$, and η is the $SO(5,5)$-invariant Minkowski metric. A key property of this model is that the cubic interaction of the scalars vanishes.

Before we proceed, let us first describe some properties of the spinor representation of $SO(5,5)$ which we will make use of later, and give an additional restriction on g^{Ir} which is required in order for the coupling constants to solve (5.5). To do this we will make use of the spinorial geometry approach described in chapter 1. In particular, a basis of the positive chirality $SO(5,5)$ spinors is

$$1, \quad e_{a_1a_2} \ , \quad e_{a_1a_2a_3a_4} \ , \tag{5.55}$$

and the gamma matrices along the light-cone directions act as

$$\gamma_a = \sqrt{2} e_a \wedge \ , \quad \gamma_{\dot{a}} = \sqrt{2} e_a \lrcorner \ , \qquad a = 1,2,3,4,5 \ . \tag{5.56}$$

Therefore gamma matrices along the time-like and space-like directions are

$$\Gamma_i = -e_i \wedge + e_i \lrcorner \ , \qquad \Gamma_{i+5} = e_i \wedge + e_i \lrcorner \ , \qquad i = 1,2,3,4,5, \tag{5.57}$$

respectively. In this realisation, the vector $SO(5,5)$ index decomposes as $I = (a, \dot{a})$, and the Clifford algebra relation is $\gamma_a\gamma_{\dot{b}} + \gamma_{\dot{b}}\gamma_a = 2\eta_{a\dot{b}}$. The Dirac inner product is defined as

$$D(\psi, \chi) := \langle \Gamma_{12345}\psi, \chi \rangle \ , \tag{5.58}$$

and acting on the space of spinors gives

$$D(e_{a_1\dots a_k}, e_{a_{k+1}\dots a_5}) = (-1)^{[\frac{k+1}{2}]+1}\epsilon_{a_1\dots a_5} \ . \tag{5.59}$$

Observe that the inner product is skew-symmetric in the interchange of pairs. We can use this to raise and lower spinor indices in the following way

$$\psi_{b_1\dots b_{5-k}} \ := \ \psi^{a_1\dots a_k} D_{a_1\dots a_k, b_1\dots b_{5-k}} = \frac{(-1)^{[\frac{k+1}{2}]+1}}{k!}\psi^{a_1\dots a_k}\epsilon_{a_1\dots a_k b_1\dots b_{5-k}}$$

$$:= \quad \eta_{b_1\dot{b}_1}\dots\eta_{b_{5-k}\dot{b}_{5-k}}\psi^{\dot{b}_1\dots\dot{b}_{5-k}} \; . \tag{5.60}$$

In this realisation, the positive chirality spinor representation decomposes as $\mathbf{1}+\mathbf{10}+\mathbf{5}$ under the subgroup $SO(5)\subset GL(5,\mathbb{R})\subset SO(5,5)$ acting on the light-cone decomposition of the vector representation of $SO(5,5)$ presented above. The additional restriction on g^{Ir} is that its only non vanishing component lies in the $\mathbf{15}$ representation, i.e. the non-vanishing component is

$$g^{\dot{a}\dot{b}} = -\frac{1}{4!}\epsilon^{\dot{b}}{}_{b_1\dots b_4}g^{\dot{a}b_1\dots b_4} \; , \quad g^{\dot{a}\dot{b}} = g^{(\dot{a}\dot{b})} \tag{5.61}$$

Clearly $\eta_{IJ}g^{Ir}g^{Js}=0$ as it is required by one of the conditions (5.5) on the couplings.

5.7.2 Non-Compact Isotropy Group

Using the fact that the cubic scalar interaction term vanishes and taking the conditions imposed on the fields by the KSEs in (5.43), we find the field equation for the scalars can be rewritten as

$$D_iD^i\phi^I = -\frac{1}{2}d^I_{rs}\mathcal{F}^r_{ij}\mathcal{F}^{s,ij} \; . \tag{5.62}$$

To expand this in $SO(5)$ representations we need to be able to calculate the rhs and in order to do this we note that

$$\begin{aligned}(\gamma_a)_{b_1\dots b_k,b_{k+1}\dots b_4} &= (-1)^{[\frac{k}{2}]}\sqrt{2}\,\epsilon_{ab_1\dots b_4} \; , \quad k=0,\dots 4 \; ,\\ (\gamma_{\dot{a}})_{b_1\dots b_k,b_{k+1}\dots b_6} &= (-1)^{[\frac{k}{2}]+1}k\sqrt{2}\delta_{\dot{a}[b_1}\epsilon_{b_2\dots b_k]b_{k+1}\dots b_6} \; , \quad k=1,\dots,5 \; .\end{aligned} \tag{5.63}$$

and also observe the gamma matrices to be symmetric in the interchange of spinor indices. In addition, a spinor is expanded in the basis (5.55) as

$$\Psi = \psi\,1 + \frac{1}{2}\psi^{ab}e_{ab} + \frac{1}{4!}\psi^{a_1\dots a_4}e_{a_1\dots a_4} \; . \tag{5.64}$$

Using these we can rewrite the field equation for the scalars in (5.62), for example taking $I=\dot{a}$ and using the fact that $d^I_{rs}=\frac{1}{2}\gamma^I_{rs}$ we get

$$\begin{aligned}D_iD^i\phi^{\dot{a}} &= -\frac{1}{4}\gamma^{\dot{a}}_{rs}\mathcal{F}^r_{ij}\mathcal{F}^{s,ij} \; ,\\ &= -\frac{1}{4.4!}(\gamma^{\dot{a}})_{1,b_1\dots b_4}\mathcal{F}_{ij}\mathcal{F}^{b_1\dots b_4,ij} - \frac{1}{4.4}(\gamma^{\dot{a}})_{b_1b_2,b_3b_4}\mathcal{F}^{b_1b_2}_{ij}\mathcal{F}^{b_3b_4,ij}\\ &\quad -\frac{1}{4.4!}(\gamma^{\dot{a}})_{b_1\dots b_4,1}\mathcal{F}^{b_1\dots b_4}_{ij}\mathcal{F}^{ij} \; ,\end{aligned} \tag{5.65}$$

then using the expressions in (5.63) and (5.60) we find this can be written as

$$D_iD^i\phi^{\dot{a}} = \frac{\sqrt{2}}{2}\mathcal{F}_{ij}\,\mathcal{F}^{\dot{a},ij} + \frac{\sqrt{2}}{16}\epsilon^{\dot{a}}{}_{b_1b_2b_3b_4}\mathcal{F}^{b_1b_2}_{ij}\,\mathcal{F}^{b_3b_4,ij} \; . \tag{5.66}$$

Applying the same procedure but this time with $I = a$ we find

$$D_i D^i \phi^a = \frac{\sqrt{2}}{2} \mathcal{F}^{ab}{}_{ij} \, \mathcal{F}_b{}^{ij} \ . \tag{5.67}$$

The scalar field equation is now equivalent to these two expressions. The second field equation (5.9) follows from the KSEs as the latter imply that all auxiliary fields Y vanish. The third field equation, (5.10), is taken as the definition of $\mathcal{H}^{(4)}$. In particular, using the above notation one has

$$g^{\dot{a}\dot{b}} \mathcal{H}^{(4)}_{\mu_1 \dots \mu_4, \dot{b}} = -\frac{\sqrt{2}}{2} \epsilon_{\mu_1 \dots \mu_4}{}^{\nu_1 \nu_2} [-g^{\dot{a}\dot{b}} \phi_{\dot{b}} \mathcal{F}_{\nu_1 \nu_2} + g^{\dot{a}}{}_{b_1} \phi_{b_2} \mathcal{F}^{b_1 b_2}{}_{\nu_1 \nu_2}] \ . \tag{5.68}$$

Next, setting $\mu_1 \cdots \mu_4 = ijkl$ and $\mu_1 \cdots \mu_4 = +ijk$, and using the expression for $\mathcal{F}$ in (5.43), we find

$$g^{\dot{a}\dot{b}} \mathcal{H}^{(4)}_{ijk\ell, \dot{b}} = g^{\dot{a}\dot{b}} \mathcal{H}^{(4)}_{+ijk, \dot{b}} = 0 \ . \tag{5.69}$$

Moreover, the rest of the components are determined in terms of ϕ and $\mathcal{F}$.

We now turn to the investigation of the Bianchi identities (5.11) and (5.12). In particular, expanding these in the notation that we have introduced, the Bianchi identity for $\mathcal{F}$ implies that

$$D_{[\mu_1} \mathcal{F}_{\mu_2 \mu_3]} = D_{[\mu_1} \mathcal{F}^{ab}_{\mu_2 \mu_3]} = 0 \ , \quad D_{[\mu_1} \mathcal{F}^{\dot{a}}_{\mu_2 \mu_3]} = \frac{1}{3} g^{\dot{a}}{}_b \mathcal{H}^b_{\mu_1 \mu_2 \mu_3} \ . \tag{5.70}$$

The first two conditions here give

$$D_+ \mathcal{F}_{-i} = D_+ \mathcal{F}_{ij} = 0 \ , \quad D_+ \mathcal{F}^{ab}_{-i} = D_+ \mathcal{F}^{ab}_{ij} = 0 \ . \tag{5.71}$$

Similarly using the expressions for the components of $\mathcal{H}$ given in (5.43), the last equation in (5.70) gives

$$D_+ \mathcal{F}^{\dot{a}}_{-i} = D_+ \mathcal{F}^{\dot{a}}_{ij} = 0 \ , \tag{5.72}$$

and

$$D_- \mathcal{F}^{\dot{a}}_{ij} + 2 D_{[i} \mathcal{F}^{\dot{a}}_{j]-} = g^{\dot{a}\dot{b}} \mathcal{H}_{-ij, \dot{b}} \ , \quad 3\, D_{[i} \mathcal{F}^{\dot{a}}_{jk]} = g^{\dot{a}\dot{b}} \, D_\ell \phi_{\dot{b}} \, \epsilon^\ell{}_{ijk} \ . \tag{5.73}$$

Next let us turn to the Bianchi identity for $\mathcal{H}$ (5.12). This decomposes as

$$\begin{aligned} D_{[\mu_1} \mathcal{H}^a_{\mu_2 \mu_3 \mu_4]} &= \frac{3}{4} \gamma^a_{rs} \mathcal{F}^r_{[\mu_1 \mu_2} \mathcal{F}^s_{\mu_3 \mu_4]} \\ D_{[\mu_1} \mathcal{H}^{\dot{a}}_{\mu_2 \mu_3 \mu_4]} &= \frac{3}{4} \gamma^{\dot{a}}_{rs} \mathcal{F}^r_{[\mu_1 \mu_2} \mathcal{F}^s_{\mu_3 \mu_4]} + \frac{1}{4 \cdot 4!} g^{\dot{a} b_1 \dots b_4} \mathcal{H}^{(4)}_{\mu_1 \dots \mu_4, b_1 \dots b_4} \end{aligned} \tag{5.74}$$

The independent conditions arising from these two equations are

$$D_+\mathcal{H}^a_{ijk} = 0 \ , \quad D_+\mathcal{H}^a_{-jk} = 0 \ , \quad D_-\mathcal{H}^a_{ijk} - 3D_{[i}\mathcal{H}^a_{jk]-} = 3\gamma^a_{rs}\mathcal{F}^r_{-[i}\mathcal{F}^s_{jk]} \tag{5.75}$$

and

$$\begin{aligned} D_+\mathcal{H}^{\dot a}_{ijk} &= 0 \ , \quad D_+\mathcal{H}^{\dot a}_{-jk} = 0 \ , \\ D_-\mathcal{H}^{\dot a}_{ijk} - 3D_{[i}\mathcal{H}^{\dot a}_{jk]-} &= -3\sqrt{2}[\mathcal{F}_{-[i}\mathcal{F}^{\dot a}_{jk]} + \mathcal{F}^{\dot a}_{-[i}\mathcal{F}_{jk]} + \frac{1}{4}\epsilon^{\dot a}{}_{b_1b_2c_1c_2}\mathcal{F}^{b_1b_2}_{-[i}\mathcal{F}^{c_1c_2}_{jk]}] \\ &\quad -\sqrt{2}\epsilon_{ijk}{}^{\ell}[g^{\dot a}{}_b\phi^b\mathcal{F}_{-\ell} + g^{\dot a}{}_{b_2}\phi_{b_1}\mathcal{F}^{b_1b_2}_{-\ell}] \ . \end{aligned} \tag{5.76}$$

This concludes the analysis of the field equations and Bianchi identities of the theory.

Regular String Solutions

This system has a string solution. Suppose that the string lies along the two light-cone directions, and take

$$\mathcal{F} = \mathcal{F}^{\dot a} = 0 \ , \quad \mathcal{F}^{ab}_{\pm\mu} = 0 \ . \tag{5.77}$$

The latter condition is required because of Lorentz invariance along the worldvolume directions of the string.

Moreover, we choose the gauge $A^r_\pm = 0$ and assume all non-vanishing fields to be independent of the light-cone coordinates $x^\pm$. Using these, the field equations and the Bianchi identities above reduce to

$$\begin{aligned} D_iD^i\phi^{\dot a} = \frac{\sqrt{2}}{16}\epsilon^{\dot a}{}_{b_1\dots b_4}\mathcal{F}^{b_1b_2}_{ij}\mathcal{F}^{b_3b_4,ij} \ , \quad D_iD^i\phi^a = 0 \ , \\ g^{\dot a}{}_bD_i\phi^b = 0 \ , \quad g^{\dot a\dot b}\mathcal{H}_{-ij,\dot b} = 0 \ , \quad D_{[i}\mathcal{H}^I_{jk]-} = 0 \ , \end{aligned} \tag{5.78}$$

which are the equation that we focus on now. To proceed, we moreover demand that

$$D_i\phi^b = 0 \ , \quad \mathcal{H}^I_{jk-} = 0 \ . \tag{5.79}$$

The latter condition is again required by Lorentz invariance along the string. Furthermore, the integrability condition of the first condition requires that

$$\mathcal{F}^{ab}_{ij}g_{bc}\phi^c = 0 \ . \tag{5.80}$$

One solution to this is to take ϕ^a constant with $g_{bc}\phi^c = 0$, but for simplicity, we will take $\phi^a = 0$. Then the only remaining non-trivial equation is

$$D_iD^i\phi^{\dot a} = \frac{\sqrt{2}}{16}\epsilon^{\dot a}{}_{b_1\dots b_4}\mathcal{F}^{b_1b_2}_{ij}\mathcal{F}^{b_3b_4,ij} \ , \tag{5.81}$$

where $\mathcal{F}^{b_1b_2}_{ij}$ is an anti-self-dual connection with gauge group $SO(5)$.

To solve (5.81) we choose $\phi^{\dot{a}}$ to lie along the 5-th direction and $\mathcal{F}^{b_1b_2}_{ij}$ to have gauge group $SO(4) \subset SO(5)$ orthogonal to $\phi^{\dot{5}}$. Now there are two cases to consider, depending on the restrictions on $\mathcal{F}$. First, if one restricts $\mathcal{F}^{b_1b_2}_{ij}$ to lie in one of the $\mathfrak{su}(2)$ subalgebras of $\mathfrak{so}(4)$, $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, then the field equation (5.81) can be rewritten as

$$\partial_i \partial^i \phi^{\dot{5}} = \pm \frac{\sqrt{2}}{8} \mathcal{F}_{ij,b_1b_2} \mathcal{F}^{b_1b_2,ij} \ , \quad b_1, b_2 = 1,2,3,4 \ , \tag{5.82}$$

where the sign depends on the choice of $\mathfrak{su}(2)$ subalgebra, we have also used $D_i \phi^{\dot{5}} = \partial_i \phi^{\dot{5}}$ and the duality relations to rewrite the rhs. Such equations have been solved before in the context of heterotic supergravity [130] and rely on the property that the Pontryagin density of an $SU(2)$ anti-self-dual connection can be written as the Laplace operator on a function [134, 135].

In particular, we can choose the minus sign and write the 2-form field strength as

$$\mathcal{F}^{b_1b_2}_{ij} = J^{b_1b_2}_{r'} \mathcal{F}^{r'}_{ij} \ , \quad r' = 1,2,3 \ , \tag{5.83}$$

where $J_{r'}$ is a basis of constant anti-self-dual 2-forms in $\mathbb{R}^4$, satisfying

$$J_{r'} J_{s'} = -\delta_{r's'} + \epsilon_{r's'}{}^{t'} J_{t'} \ . \tag{5.84}$$

Using this, the equation (5.82) can now be rewritten as

$$\partial_i \partial^i \phi^{\dot{5}} = -\frac{\sqrt{2}}{2} \delta_{r's'} \mathcal{F}^{r'}_{ij} \mathcal{F}^{s',ij} \ , \tag{5.85}$$

the term on the rhs, specifically,

$$-\frac{1}{2} \delta_{r's'} \mathcal{F}^{r'}_{ij} \mathcal{F}^{s',ij} \ , \tag{5.86}$$

is the Pontryagin density that we have mentioned a couple of times before and it can be written as the Laplace operator on a scalar [134, 135]. As mentioned, this can be solved for $SU(2)$ instantons, see for example [130] for further details in a similar context, an explicit solution will be discussed below. First, let us summarise the non-vanishing fields of the solution

$$\begin{aligned} &\mathcal{F}^{ab} = \frac{1}{2} \mathcal{F}^{ab}_{ij} e^i \wedge e^j \ , \quad \mathcal{H}^{\dot{5}} = -\partial_i \phi^{\dot{5}} e^- \wedge e^+ \wedge e^i + \frac{1}{3!} \partial_\ell \phi^{\dot{5}} \, \epsilon^\ell{}_{ijk} \, e^i \wedge e^j \wedge e^k \ , \\ &\phi^{\dot{5}} = \phi^{\dot{5}}(x) \ , \quad a, b = 1,2,3,4 \ . \end{aligned} \tag{5.87}$$

Observe that $g^{Ir} \mathcal{H}^{(4)}_r = 0$.

Now, the explicit solution we consider is the configuration with instanton number one and we use the results coming from [130]. The gauge connection A of $\mathcal{F}^{ab}$ and $\phi^{\dot{5}}$ can, in this case, be written as

$$\begin{aligned} A^{ab} &= 2(J^{r'})^{ab}(J_{r'})_{ij}\frac{x^j}{|x|^2+\rho^2}\ e^i\ , \quad \phi^{\dot{5}} = \phi_0 + 4\sqrt{2}\frac{|x|^2+2\rho^2}{(|x|^2+\rho^2)^2} + h_0\ , \\ h_0 &= \sum_\nu \frac{Q_\nu}{|x-x_\nu|^2}\ , \end{aligned} \tag{5.88}$$

where x are the coordinates in $\mathbb{R}^{5,1}$ transverse to the light-cone where the string lies, ϕ_0 is a constant, and ρ is the instanton modulus. In addition h_0 is a multi-centred harmonic function, which for simplicity will be set to zero. The solution then becomes smooth. In particular, at large $|x|$, which is the case when we are far away from the string, the scalar ϕ becomes constant since the second term in the solution vanishes. Also in this limit, the gauge connection becomes a pure gauge. In contrast as $|x|$ becomes small the modulus of the instanton $\rho \neq 0$ regulates the values of both the scalar ϕ and the gauge connection A. The solution can be generalised to any instanton number k which are also smooth [130].

For all solutions the dyonic string charge, denoted q_s, can be computed by integrating the 3-form flux $\mathcal{H}^{\dot{5}}$ on the 3-sphere at infinity. This in turn can be identified with the instanton number, k,

$$q_s = \int_{S^3\subset\mathbb{R}^4} \mathcal{H}^{\dot{5}} = k\ . \tag{5.89}$$

after an appropriate normalisation [130].

We can find a more general solution by allowing $\mathcal{F}$ to take values in both $\mathfrak{su}(2)$ subalgebras of $\mathfrak{so}(4)$. In particular this means we can write $\mathcal{F}$ as

$$\mathcal{F}^{b_1b_2}_{ij} = J^{b_1b_2}_{r'}\mathcal{F}^{r'}_{ij} + I^{b_1b_2}_{r'}\tilde{\mathcal{F}}^{r'}_{ij}\ , \tag{5.90}$$

where $I_{r'}$ form a basis for the self-dual 2-forms in $\mathbb{R}^4$ and $\tilde{\mathcal{F}}^{r'}$ denote the anti-self-dual fields associated to the second $\mathfrak{su}(2)$. In addition, $I_{r'}$ satisfy a similar relation to (5.84), and they commute with $J_{r'}$. This then means the scalar field equation becomes

$$\partial_i\partial^i\phi^{\dot{5}} = -\frac{\sqrt{2}}{2}\delta_{r's'}(\mathcal{F}^{r'}_{ij}\mathcal{F}^{s',ij} - \tilde{\mathcal{F}}^{r'}_{ij}\tilde{\mathcal{F}}^{s',ij})\ . \tag{5.91}$$

The 1-instanton solutions are then modified as follows

$$\begin{aligned} A^{ab} &= 2(J^{r'})^{ab}(J_{r'})_{ij}\frac{x^j}{|x|^2+\rho^2}\ e^i + 2(I^{r'})^{ab}(J_{r'})_{ij}\frac{x^j}{|x|^2+\sigma^2}\ e^i\ , \\ \phi^{\dot{5}} &= \phi_0 + 4\sqrt{2}\frac{|x|^2+2\rho^2}{(|x|^2+\rho^2)^2} - 4\sqrt{2}\frac{|x|^2+2\sigma^2}{(|x|^2+\sigma^2)^2} + h_0\ , \\ h_0 &= \sum_\nu \frac{Q_\nu}{|x-x_\nu|^2}\ , \end{aligned} \tag{5.92}$$

where σ is the modulus of the $\tilde{\mathcal{F}}$ instanton, but otherwise the notation is the same as for the solution above. Once again, setting $h_0 = 0$ means the solution is smooth.

However, in this case the calculation for the dyonic string charge is modified and is given by

$$q_s = \int_{S^3 \subset \mathbb{R}^4} \mathcal{H}^{\dot{5}} = k - \tilde{k} \ , \tag{5.93}$$

where k and $\tilde{k}$ are the instanton numbers of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, respectively.

5.7.3 Compact Isotropy Group

We now analyse the conditions coming from the field and Killing spinor equations for the compact case. Let us start with the scalar field equation. Since the cubic scalar interaction term in this model vanishes, the conditions coming from the KSEs given in (5.44) imply the field equation for the scalar fields in (5.8). Furthermore, the Bianchi identity for $\mathcal{H}$ (5.12) implies that

$$g^{Kr}\mathcal{H}^{(4)}_r = 0 \ , \tag{5.94}$$

and the Bianchi identity for the 2-form field strength $\mathcal{F}$ is automatically satisfied. The remaining conditions on the fields implied by the KSEs and field equations can be summarised as

$$D_\mu \phi^I = 0 \ , \quad g^{Kr} b_{Irs} Y^s \phi^I = 0 \ , \quad h^r{}_I \phi^I = 0 \ . \tag{5.95}$$

Note also that $\mathcal{H}^I = 0$. The integrability condition of the first restriction was given in (5.35) and requires

$$(F\phi)^I = 0 \ , \tag{5.96}$$

which means the holonomy group of the gauge connection leaves the scalars invariant.

Most of the analysis we have made thus far is independent of a particular model and applies in general to all solutions preserving four supersymmetries. We now investigate the conditions in (5.95) further for the particular case of model that is under consideration here. Firstly, the integrability condition above can be written as

$$\mathcal{F}^{ab}(X_{ab})_J{}^I \phi^J = 0 \ . \tag{5.97}$$

This can be solved by taking $\phi = 0$ or by taking a reduction of the holonomy group of the gauge connection to a subgroup of $SO(5)$. Considering the former case first means we have

$$\phi^I = \mathcal{H}^I = 0 \ , \quad g^{Kr}\mathcal{H}^{(4)}_r = 0 \ , \quad \mathcal{F}^r = -2iY^r \, e^2 \wedge e^{\bar{2}} \ , \tag{5.98}$$

as a solution for an arbitrary auxiliary field Y which depends only on the complex coordinates $(x^2, x^{\bar{2}})$.

Let us now suppose the holonomy of the connection A^{ab} reduces to $SO(4)$. Then the constant scalar field $\phi = (\phi^5, \phi^{\dot{5}})$ solves the integrability condition and the first condition in (5.95). The last condition in (5.95) gives $g^{\dot{a}}{}_5\phi^5 = 0$ and so we take $\phi^5 = 0$. Furthermore, the second condition in (5.95) is automatically satisfied. Therefore, the solution in this case is given by

$$\phi = (0, \phi^{\dot{5}}) \ , \quad \mathcal{F}^r = -2iY^r \, e^2 \wedge e^{\bar{2}} \ , \quad \mathrm{hol}(\mathcal{F}^{ab}) \subseteq SO(4) \ , \tag{5.99}$$

where the auxiliary fields Y satisfy the holonomy condition and depend on the complex coordinates $(x^2, x^{\bar{2}})$. This solution is invariant under the $\mathbb{R}^{3,1}$ Poincaré group and so has the naive interpretation of 3-branes.

5.8 Summary

In this chapter we have investigated (1,0) superconformal theories in six dimensions. We firstly introduced the theory and described briefly there constructions [127, 128]. One of the main results of this chapter were the solutions of the KSEs. In particular, we determined the conditions imposed on the fields in all cases. Although we focused on the models presented in [127] our results apply more generally. We found that these theories admit solutions preserving 1, 2, 4 and 8 supersymmetries. To achieve this classification we made use of the spinorial geometry method described in chapter 1 and techniques used in general to solve the KSEs of (1,0) six dimensional supergravity in chapter 2.

We then moved onto consider the half-supersymmetric solutions, i.e. those preserving four supersymmetries, in more detail. There were two cases to consider depending on the isotropy group of the Killing spinors. In each case we analysed the conditions imposed on the fields by the field equations of the theory in addition to the conditions imposed by the KSEs. In the investigation of these we also gave some explicit solutions which included string and 3-brane solutions. These can be given M-theoretic interpretations by using the M-brane intersection rules [136, 137]; which state that M2-branes end on M5-branes on a string [136] that appears as a defect of the M5-brane worldvolume and similarly two M5-branes intersect on a 3-brane [137] where the 3-brane is seen as defect on the M5-brane effective theory.

The aim of these superconformal models is to gain a better understanding of the dynamics of multiple M5-branes, which is believed to be described by a (2,0) superconformal theory in six dimensions. The (1,0) superconformal theory has highlighted some important features but it is not complete yet. For example, one can in addition to tensor and vector multiplets have hypermultiplets in (1,0) supersymmetric theories in six dimensions. Therefore, one will need to include couplings to hypermultiplets in order to have a complete (1,0) superconformal theory in six dimensions. Each of the hypermultiplets carry additional four scalars which could play important roles in allowing one to describe the dynamics of multiple M5-branes and giving in general an M-theoretic interpretation to some models.

Chapter 6

Conclusions

6.1 Overall Summary

In this thesis we have investigated different aspects of (1,0)-supersymmetric theories in six dimensions. In particular, we began by discussing (1,0) supergravity coupled to any number of tensor, vector and scalar multiplets. We solved the Killing spinor equations for this theory in all cases and further discussed the restrictions on the geometry imposed by backgrounds preserving different fractions of supersymmetry. In order to achieve this classification we made use of the spinorial geometry method, which we discussed in the first chapter. This technique relies on the ability to write spinors in terms of differential forms, and in particular, an oscillator basis in the space of spinors can be used. Moreover, the gauge group that leaves the KSEs covariant is used to choose the Killing spinors and this allows considerable simplification of calculations. When this method is used the KSEs reduce to purely algebraic relations which can be solved to obtain the constraints imposed on the fluxes of the theory.

Using the spinorial geometry method and the relation between six dimensional supergravity and heterotic supergravity we were able to solve the KSEs of the theory in all generality. In our analysis we found backgrounds preserving 1, 2, 3, 4 and 8 supersymmetries. These can uniquely be classified according to the isotropy groups of the Killing spinors, except in one case where a distinct descendant exists. The supersymmetric backgrounds fall into two types of categories; those with Killing spinors that have compact isotropy group and those with non-compact isotropy group. In the non-compact case we found the isotropy group of the Killing spinors to be $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ (1), $U(1) \cdot Sp(1) \ltimes \mathbb{H}$ (2), $Sp(1) \ltimes \mathbb{H}$ (3,4), and in the compact case we had $Sp(1)$ (2), $U(1)$ (4), and $\{1\}$ (8), where the numbers in parenthesis correspond to the number of supersymmetries preserved. Note that there are two backgrounds where the holonomy of the supercovariant derivative is contained in $Sp(1) \ltimes \mathbb{H}$, these differ as a consequence of the conditions imposed by the hyperini KSE.

In order to investigate the geometry we determined the spacetime form bilinears that arise in each case. In the non-compact case the spacetime always admitted one parallel 1-form with respect to the connection with skew-symmetric connection given by the 3-form of the gravitational multiplet. In addition, there are twisted 3-form bilinears. These were considered in turn and the implications on the geometry were determined. In the compact case the backgrounds with two supersymmetries admitted a spacetime with a 3+3 split, and backgrounds with four supersymmetries allowed a 4+2 split, where the first three, and respectively four, directions are spanned by parallel vector fields with respect to the connection with skew-symmetric torsion given by the 3-form of the gravitational multiplet. In each case there is a natural frame on the spacetime given by six 1-form spinor bilinears. Finally, we found backgrounds which preserve eight supersymmetries admit spacetimes that are locally isometric to $\mathbb{R}^{5,1}$, $AdS_3 \times S^3$ and CW_6. These results generalise those of [34, 58], see also [59, 60, 61].

Supersymmetric supergravity solutions are determined after solving the KSEs and the field equations of the theory under consideration. In the third chapter we used the Killing spinor equations to determine the field equations of the theory using the integrability conditions. This was a very technical discussion that involved the use of numerous identities coming from the KSEs. In addition to highlighting which components of the field equations are implied by the KSEs, the integrability conditions provide us with an important consistency check for the theory.

In the fourth chapter we investigated the near horizon geometries of (1,0) six dimensional supergravity black holes, making particular use of the solutions to the KSE of (1,0) six dimensional supergravity theory. Firstly, we briefly discussed Gaussian null coordinates and how this is used in the analysis of near horizon geometries of (1,0) six dimensional supergravity. We focused on near horizon geometries arising from (1,0) six dimensional supergravity coupled to arbitrary number of tensor and scalar multiplets, and left out couplings to vector multiplets due to the complications arising from the inclusion of Chern-Simons term.

Our analysis showed that there were two classes of near horizon geometries that depended on h, a 1-form on the horizon section. When $h \neq 0$ we find the near horizon geometries to be locally isometric to $AdS_3 \times \Sigma^3$, where Σ^3 is diffeomorphic to S^3. We also find the tensor scalars to be constant and the 3-form field strengths $H^{\underline{M}}$ to vanish. This class of solutions preserve 2, 4 and 8 supersymmetries. The geometry of Σ^3 is further restricted as the number of supersymmetries preserved increases.

The other class of horizons are of the form $\mathbb{R}^{1,1} \times \mathcal{S}$, where $\mathcal{S}$ is a 4-manifold whose geometry depends on the hypermultiplet scalars. In this case the tensor scalars are constant and $H^{\underline{M}}$ vanishes, but in addition we find the 3-form field strength of the gravitational multiplet H to vanish. This class of solutions preserve 1, 2 and 4 supersymmetries where $\mathcal{S}$ is a Riemannian, Kähler and hyper-Kähler manifold respectively.

The final part of the thesis was dedicated to the investigation of supersymmetric solutions to the (1,0) superconformal theories in six dimensions. As we mentioned, the interest in superconformal theories has arisen due to the understanding that the worldvolume dynamics of multiple M5-branes is described by a (2,0) superconformal theory in six dimensions. The particular focus on the (1,0) superconformal theories comes from an analogy that was made in the investigation of multiple M2-branes where reducing the number of supersymmetries allowed a more generic formulation of the theory behind the dynamics of multiple M2-branes to be obtained.

One important aspect of these theories are the BPS conditions. In this context, we were able to solve the KSEs of the (1,0) superconformal theory, making particular use of the tools that were developed throughout the thesis. We were able to solve the KSEs in all cases and gave the conditions imposed on the fields in each case. We found solutions preserving 1, 2, 4 and 8 supersymmetries. Furthermore, we looked at the half-supersymmetric solutions of some models in more detail. Once again there are two cases to consider depending on whether the isotropy group of the Killing spinors is compact or non-compact. In each case we solved the field equations and the Bianchi identities in addition to the KSEs to find string and 3-brane solutions, which can be given M-theoretic interpretations. However, these models do not include the most general couplings possible, since couplings to hypermultiplets are missing.

6.2 Future Work

There are a number of avenues for possible future research. In this thesis we have focused on the (1,0) theory in six dimensions which has eight supercharges, however, there are other theories in six dimensions with more supercharges. These can also be investigated in a similar way to what we have done here. In particular, the investigation of the supersymmetric solutions of the (2,0) six dimensional supergravity will form a natural extension to the analysis we made in chapter 2. Similarly, the near horizon analysis we have made for the six dimensional (1,0) supergravity can be extended to other six dimensional supergravity theories and to supergravity theories in diverse dimensions. Another way of extending the near horizon analysis is by including the couplings to vector multiplets that has been left out.

A substantial problem lies in the construction of black hole solutions with a prescribed near horizon geometry. Apart from the supergravity in five dimensions this problem remains open. Since there are many near horizon geometries it is expected that there are many black holes with exotic horizon topologies. The spinorial geometry method can be used to classify all such black hole solutions.

In regards to superconformal theories there are also a number of further investigations that can be made. Firstly, we have only discussed a limited number of models that satisfy the algebraic constraints that arise as a result of consistency requirements. One could therefore try to investigate the possibility of more generic models that satisfy these constraints and aim to see if these have better M-theoretic interpretations. We have also mentioned that couplings to hypermultiplets is missing, this is likely to play a key role in describing the dynamics of multiple M5-branes. Therefore, an extension to this work will be to include consistent couplings to hypermultiplets, this has been initiated in [127] but is not complete. Once this has been done one can perform a similar analysis of the KSEs to what we have done in the absence of hypermultiplets and investigate the consistent models in further detail aiming to obtain a better M-theoretic interpretation.

The supergravity theories in six dimensions play an important role as an intermediate dimension between the eleven dimensions in which M-theory is formulated in and the four dimensional spacetime which we are familiar with. In addition, the (2,0) superconformal theory in six dimension is believed to describe the dynamics of one of the key ingredients of M-theory, the M5-branes. We have focused particularly on the supersymmetric solutions of six dimensional theories. Supersymmetric solutions in general have been useful in the understanding of string/M-theory compactifications, branes, dualities and the AdS/CFT correspondence.

String/M-theory is a vast subject which has undergone many developments over the last forty years. Supergravity theories, which are low-energy approximations to the various string theories and to M-theory, have played particularly important roles in some of these developments. Moreover, the investigation of these theories will continue to help our understanding of what is one of the most promising candidates for the unification of all the fundamental forces of nature, M-theory.

Appendix A

Identities from the KSEs

In this appendix we give further details about the tools and identities we used in chapter 3 when deriving the integrability conditions. As well as using the identities that come from the KSEs we have made extensive use of relations satisfied by the gamma matrices. In particular, the gamma matrices in six dimensions satisfy the following duality relation

$$\gamma^{\mu_1\dots\mu_n}\epsilon = \frac{(-1)^{[\frac{n}{2}]+1}}{(6-n)!}\epsilon^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_{6-n}}\gamma^{\nu_1\dots\nu_{6-n}}\epsilon \ , \tag{A.1}$$

where $\epsilon_{012345} = 1$ and $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$. In addition, we have made use of various identities that arise from the multiplication of gamma matrices of different rank, for example

$$\begin{aligned}
\gamma^{\mu}\gamma^{\nu_1\dots\nu_n} &= \gamma^{\mu\nu_1\dots\nu_n} + n g^{\mu[\nu_1}\gamma^{\nu_2\dots\nu_n]} \ , \\
\gamma^{\mu_1\mu_2}\gamma_{\nu_1\nu_2} &= \gamma^{\mu_1\mu_2}{}_{\nu_1\nu_2} - 4\delta^{[\mu_1}{}_{[\nu_1}\gamma^{\mu_2]}{}_{\nu_2]} - 2\delta^{[\mu_1}{}_{[\nu_1}\delta^{\mu_2]}{}_{\nu_2]} \ , \\
\gamma^{\mu_1\mu_2}\gamma_{\nu_1\nu_2\nu_3} &= \gamma^{\mu_1\mu_2}{}_{\nu_1\nu_2\nu_3} - 6\delta^{[\mu_1}{}_{[\nu_1}\gamma^{\mu_2]}{}_{\nu_2\nu_3]} - 6\delta^{[\mu_1}{}_{[\nu_1}\delta^{\mu_2]}{}_{\nu_2}\gamma_{\nu_3]} \ , \\
\gamma^{\mu_1\mu_2\mu_3}\gamma_{\nu_1\nu_2\nu_3} &= \gamma^{\mu_1\mu_2\mu_3}{}_{\nu_1\nu_2\nu_3} + 9\delta^{[\mu_1}{}_{[\nu_1}\gamma^{\mu_2\mu_3]}{}_{\nu_2\nu_3]} \\
&\quad -18\delta^{[\mu_1}{}_{[\nu_1}\delta^{\mu_2}{}_{\nu_2}\gamma^{\mu_3]}{}_{\nu_3]} - 6\delta^{[\mu_1}{}_{[\nu_1}\delta^{\mu_2}{}_{\nu_2}\delta^{\mu_3]}{}_{\nu_3]}
\end{aligned} \tag{A.2}$$

and similar identities involving the commutator of gamma matrices of different rank, see for example [1].

Let us now give the identities that come from the KSE which we have found most useful, in what follows we use T-KSE to denote the tensorini KSE and G-KSE to denote the gaugini KSE:

1. $H^{\underline{M}}_{\mu\nu\rho}\gamma^{\nu\rho}\times$T-KSE:

$$\begin{aligned}
&4T^{\underline{M}}_{\mu}T^{\underline{M}}_{\nu}\gamma^{\nu}\epsilon + 2T^{\underline{M}}_{\lambda}H^{\underline{M}\lambda}{}_{\mu\nu}\gamma^{\nu}\epsilon - T^{\underline{M}}_{\lambda}H^{\underline{M}\lambda}{}_{\rho\sigma}g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon \\
&\quad + H^{\underline{M}}_{\mu\lambda\sigma}H^{\underline{M}\lambda\sigma}{}_{\nu}\gamma^{\nu}\epsilon + \frac{1}{2}H^{\underline{M}}_{\mu\lambda\nu}H^{\underline{M}\lambda}{}_{\rho\sigma}\gamma^{\nu\rho\sigma}\epsilon = 0 \ .
\end{aligned} \tag{A.3}$$

2. $H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\times$T-KSE:

$$6T^{\underline{M}}_{\lambda}H^{\lambda}{}_{\mu\nu}\Gamma^{\mu\nu} + H_{\mu\nu\rho}H^{\underline{M}\mu\nu\rho} + 3H_{\mu\alpha\beta}H^{\underline{M}\alpha\beta}_{\nu}\gamma^{\mu\nu}\epsilon \;\; = \;\; 0 \; . \tag{A.4}$$

3. $\frac{(\partial_\mu v_{\underline{r}})c^{\underline{r}}}{v_{\underline{s}}c^{\underline{s}}}\gamma^\mu\times$G-KSE:

$$\frac{2\left(\partial_\mu v_{\underline{r}}\right)c^{\underline{r}}}{\left(v_{\underline{s}}c^{\underline{s}}\right)^2}\mathcal{A}^{r'}_{\underline{I}}\xi^{a'\underline{I}}\rho_{r'}\gamma^\mu\epsilon = \frac{\left(\partial_\mu v_{\underline{r}}\right)c^{\underline{r}}}{2\left(v_{\underline{s}}c^{\underline{s}}\right)}F^{a'}_{\nu\rho}\gamma^{\mu\nu\rho}\epsilon + \frac{\left(\partial^\lambda v_{\underline{r}}\right)c^{\underline{r}}}{\left(v_{\underline{s}}c^{\underline{s}}\right)}F^{a'}_{\lambda\mu}\gamma^\mu\epsilon \; . \tag{A.5}$$

4. $f^{a'b'c'}A^{b'}_\mu\gamma^\mu\times$G-KSE:

$$\frac{2}{v_{\underline{s}}c^{\underline{s}}}\mathcal{A}^{r'}_{\underline{I}}A^{b'}_\mu f^{a'b'c'}\xi^{\underline{I}}_{c'}\rho_{r'}\gamma^\mu\epsilon = \frac{1}{2}f^{a'b'c'}A^{b'}_\mu F^{c'}_{\nu\rho}\gamma^{\mu\nu\rho}\epsilon + f^{a'b'c'}A^{\lambda b'}F^{c'}_{\lambda\mu}\gamma^\mu\epsilon \; . \tag{A.6}$$

5. $F^{a'}_{\mu\nu}\gamma^{\mu\nu}\times$T-KSE:

$$\begin{aligned} T^{\underline{M}}_{\mu}F^{a'}_{\nu\rho}\gamma^{\mu\nu\rho} + 2T^{\underline{M}}_{\lambda}F^{a'\lambda}_{\mu}\gamma^\mu\epsilon + F^{a'}_{\alpha\beta}H^{\underline{M}\alpha\beta}{}_{\mu}\gamma^\mu & \\ + \frac{1}{2}F^{a'}_{\lambda\mu}H^{\underline{M}\lambda}{}_{\nu\rho}\gamma^{\mu\nu\rho}\epsilon \;\; &= \;\; 0 \; . \end{aligned} \tag{A.7}$$

6. [T-KSE,G-KSE]:

$$T^{\underline{M}}_{\lambda}F^{a'\lambda}{}_{\mu}\gamma^\mu\epsilon - \frac{1}{4}H^{\underline{M}}_{\lambda\mu\nu}F^{a'\lambda}{}_{\rho}\gamma^{\mu\nu\rho}\epsilon = 0 \; . \tag{A.8}$$

7. $T^{\underline{M}}_{\nu}\gamma^{\mu\nu}\times$T-KSE:

$$\begin{aligned} T^{\underline{M}}_{\lambda}T^{\underline{M}\lambda}g_{\mu\nu}\gamma^\nu\epsilon - T^{\underline{M}}_{\mu}T^{\underline{M}}_{\nu}\gamma^\nu\epsilon - T^{\underline{M}}_{\lambda}H^{\underline{M}\lambda}{}_{\mu\nu}\gamma^\nu\epsilon & \\ + \frac{1}{4}T^{\underline{M}}_{\nu}H^{\underline{M}}_{\mu\rho\sigma}\gamma^{\nu\rho\sigma}\epsilon - \frac{1}{4}T^{\underline{M}}_{\lambda}H^{\underline{M}\lambda}{}_{\rho\sigma}g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon \;\; &= \;\; 0 \; . \end{aligned} \tag{A.9}$$

8. $\gamma^\lambda(T^{\underline{M}}_{\mu}\gamma^\mu - \frac{1}{12}H^{\underline{M}}_{\mu\nu\rho}\gamma^{\mu\nu\rho})\times$T-KSE:

$$T^{\underline{M}}_{\lambda}T^{\underline{M}\lambda}g_{\mu\nu}\gamma^\mu\epsilon - \frac{1}{2}T^{\underline{M}}_{\lambda}H^{\underline{M}\lambda}{}_{\rho\sigma}g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon - T^{\underline{M}}_{\lambda}H^{\underline{M}\lambda}{}_{\mu\nu}\gamma^\nu\epsilon = 0 \; . \tag{A.10}$$

9. $(\frac{1}{2}F^{a'}_{\mu\nu}\gamma^{\mu\nu} + \mu^{a'}_{r'}\rho^{r'})\times$G-KSE:

$$\begin{aligned} -\frac{1}{8}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\mu\nu}\epsilon - \frac{2}{v_{\underline{r}}c^{\underline{r}}}\mathcal{A}^{r'}_{\underline{I}}\xi^{\underline{I}a'}F^{a'}_{\mu\nu}\rho_{r'}\gamma^{\mu\nu}\epsilon & \\ -\frac{1}{2}F^{a'}_{\mu\nu}F^{a'\mu\nu}\epsilon - \frac{4}{\left(v_{\underline{r}}c^{\underline{r}}\right)^2}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}\epsilon \;\; &= \;\; 0 \; . \end{aligned} \tag{A.11}$$

10. $(\frac{1}{2}F^{a'}_{\nu\rho}\gamma^{\nu\rho} - \mu^{a'}_{r'}\rho^{r'})\times$G-KSE:

$$-\frac{1}{8}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\mu\nu}\epsilon - \frac{1}{2}F^{a'}_{\mu\nu}F^{a'\mu\nu}\epsilon + \frac{4}{\left(v_{\underline{r}}c^{\underline{r}}\right)^2}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}\epsilon = 0 \; . \tag{A.12}$$

11. $F^{a'}_{\mu\nu}\gamma^{\nu}\times$G-KSE:

$$-F^{a'}_{\mu\nu}\xi^{\underline{I}a'}\mathcal{A}^{r'}_{\underline{I}}\rho_{r'}\gamma^{\nu}\epsilon = -\frac{v_{\underline{r}}c^{\underline{r}}}{4}F^{a'}_{\mu\nu}F^{a'}_{\rho\sigma}\gamma^{\nu\rho\sigma}\epsilon + \frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu}\gamma^{\nu}\epsilon \ . \qquad (A.13)$$

12. Equations (A.12)+(A.11):

$$\frac{1}{4}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\mu\nu}\epsilon + F^{a'}_{\mu\nu}F^{a'\mu\nu}\epsilon + \frac{2}{v_{\underline{r}}c^{\underline{r}}}\mathcal{A}^{r'}_{\underline{I}}\xi^{\underline{I}a'}F^{a'}_{\mu\nu}\rho_{r'}\gamma^{\mu\nu}\epsilon = 0 \ . \qquad (A.14)$$

13. Equations (A.12)-(A.11):

$$\mathcal{A}^{r'}_{\underline{I}}\xi^{\underline{I}a'}F^{a'}_{\mu\nu}\rho_{r'}\gamma^{\mu\nu}\epsilon + \frac{4}{(v_{\underline{r}}c^{\underline{r}})}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}\epsilon \ = \ 0 \ . \qquad (A.15)$$

14. $v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\nu}\gamma^{\alpha\mu\nu}\times$G-KSE:

$$\begin{aligned}&\frac{1}{4}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\nu}\epsilon - \frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon - v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu}\gamma^{\nu}\epsilon \\ &+\frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu}\gamma^{\nu}\epsilon + \mathcal{A}^{r'}_{\underline{I}}\xi^{\underline{I}a'}F^{a'}_{\rho\sigma}g_{\mu\nu}\rho_{r'}\gamma^{\nu\rho\sigma}\epsilon \ = \ 0\end{aligned} \qquad (A.16)$$

15. Equation (A.13)+(A.16)-$\gamma^{\mu}\times$(A.15):

$$\begin{aligned}-F^{a'}_{\mu\nu}\xi^{\underline{I}a'}\mathcal{A}^{r'}_{\underline{I}}\rho_{r'}\gamma^{\nu}\epsilon \ = \ &-\frac{1}{16}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\nu}\epsilon + \frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu}\gamma^{\nu}\epsilon \\ &-\frac{1}{8}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu}\gamma^{\nu}\epsilon \\ &+\frac{1}{v_{\underline{r}}c^{\underline{r}}}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu}\gamma^{\nu}\epsilon \ .\end{aligned} \qquad (A.17)$$

16. $v_{\underline{r}}c^{\underline{r}}\gamma\mu\times$(A.14) - $\gamma^{\mu}\times$(A.15):

$$\begin{aligned}&\frac{1}{8}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'}_{\delta\gamma}\epsilon^{\alpha\beta\delta\gamma}{}_{\rho\sigma}g_{\mu\nu}\gamma^{\nu\rho\sigma}\epsilon + \frac{1}{4}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\nu}\epsilon \\ &-\frac{4}{v_{\underline{r}}c^{\underline{r}}}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu}\gamma^{\nu}\epsilon + \frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu}\gamma^{\nu}\epsilon \ = \ 0 \ .\end{aligned} \qquad (A.18)$$

17. $\frac{1}{8}\times$(A.18) + (A.17):

$$\begin{aligned}-F^{a'}_{\mu\nu}\xi^{\underline{I}a'}\mathcal{A}^{r'}_{\underline{I}}\rho_{r'}\gamma^{\nu}\epsilon \ = \ &-\frac{1}{32}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\rho\sigma}F^{a'}_{\delta\gamma}\epsilon^{\rho\sigma\delta\gamma}{}_{\mu\nu}\gamma^{\nu}\epsilon + \frac{1}{64}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'}_{\delta\gamma}\epsilon^{\alpha\beta\delta\gamma}{}_{\rho\sigma}g_{\mu\nu}\gamma^{\nu\rho\sigma} \\ &+\frac{1}{2}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\mu\lambda}F^{a'\lambda}_{\nu}\gamma^{\nu}\epsilon - \frac{1}{16}v_{\underline{r}}c^{\underline{r}}F^{a'}_{\alpha\beta}F^{a'\alpha\beta}g_{\mu\nu}\gamma^{\nu}\epsilon \\ &+\frac{1}{2v_{\underline{r}}c^{\underline{r}}}\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}}\xi^{\underline{I}a'}\xi^{\underline{J}a'}g_{\mu\nu}\gamma^{\nu}\epsilon \ ,\end{aligned} \qquad (A.19)$$

In addition we have used

$$\mathcal{A}^{\underline{A}}_{\underline{I}\ \underline{B}}\mathcal{A}^{\underline{B}}_{\underline{J}\ \underline{A}} = -2\mathcal{A}_{\underline{I}r'}\mathcal{A}^{r'}_{\underline{J}} \ , \qquad (A.20)$$

where to arrive at this we have made use of $\mathcal{A}_{\underline{I}}{}^{\underline{A}}{}_{\underline{B}} = \mathcal{A}_{\underline{I}r'}(\rho^{r'})^{\underline{A}}{}_{\underline{B}}$. Various other identities can be determined using the relations in (2.5), for example

$$v_{\underline{r}} v^{\underline{r}} = 1 \ , \tag{A.21}$$

gives

$$v_{\underline{r}} \partial_\mu v^{\underline{r}} = 0 \ , \tag{A.22}$$

and similar relations can be obtained by taking further derivatives.

Appendix B

The Integrability Condition of the KSEs

In this appendix we derive the field equations of (1,0) superconformal theory described in chapter 5 from the KSEs using the Bianchi identities. In order to do this we follow a similar discussion to that presented in chapter 3 where the integrability conditions of the (1,0) six dimensional supergravity theory were derived.

Let us start with the KSEs, which are given by

$$\frac{1}{4}\mathcal{F}^r_{\mu\nu}\gamma^{\mu\nu}\epsilon + (Y^r)_a\rho^a\epsilon + \frac{1}{2}h^r_I\phi^I\epsilon = 0 \,, \tag{B.1}$$

$$\frac{1}{12}\mathcal{H}^I_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + D_\mu\phi^I\gamma^\mu\epsilon = 0 \,. \tag{B.2}$$

Using these and the identities that arise from it, along with the Bianchi identities of the theory, we aim to obtain the field equations of the minimal system, which can be written as

$$D^\mu D_\mu\phi^I = -\frac{1}{2}d^I_{rs}(\mathcal{F}^r_{\mu\nu}\mathcal{F}^{\mu\nu s} - 8Y^r_aY^{s,a}) - 3d^I_{rs}h^r_Jh^s_K\phi^J\phi^K \,, \tag{B.3}$$

$$g^{Kr}b_{Irs}Y^s_{ij}\phi^I = 0 \,, \tag{B.4}$$

$$g^{Kr}b_{Irs}\mathcal{F}^s_{\mu\nu}\phi^I = \frac{1}{4!}\epsilon_{\mu\nu\lambda\rho\sigma\tau}g^{Kr}\mathcal{H}^{(4)\lambda\rho\sigma\tau}_r \,. \tag{B.5}$$

Firstly we square the KSE in (B.1) as follows

$$\left(\frac{1}{4}\mathcal{F}^r_{\mu\nu}\gamma^{\mu\nu} - (Y^r)_a\rho^a + \frac{1}{2}h^r_I\phi^I\right)\left(\frac{1}{4}\mathcal{F}^s_{\rho\sigma}\gamma^{\rho\sigma}\epsilon + (Y^s)_b\rho^b\epsilon + \frac{1}{2}h^s_J\phi^J\epsilon\right) = 0 \tag{B.6}$$

then multiplying through with d^I_{rs} and simplifying we find

$$\frac{1}{4}d^I_{rs}\mathcal{F}^r_{\mu\nu}\mathcal{F}^s_{\rho\sigma}\gamma^{\mu\nu\rho\sigma}\epsilon - \frac{1}{2}d^I_{rs}\mathcal{F}^r_{\mu\nu}\mathcal{F}^{s,\mu\nu}\epsilon + 4d^I_{rs}Y^r_aY^{s,a}\epsilon + d^I_{rs}\mathcal{F}^r_{\mu\nu}h^s_J\phi^J\gamma^{\mu\nu}\epsilon + d^I_{rs}h^r_Jh^s_K\phi^J\phi^K\epsilon = 0 \,. \tag{B.7}$$

Furthermore, we make use of the duality that the gamma matrices satisfy in six dimensions, (A.1), for the case of $n = 4$ to find

$$\gamma^{\mu\nu\rho\sigma}\epsilon = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}{}_{\lambda\tau}\gamma^{\lambda\tau}\epsilon \ , \tag{B.8}$$

which means the equation above becomes

$$\begin{aligned} -\frac{1}{8}d^I_{rs}\mathcal{F}^r_{\mu\nu}\mathcal{F}^s_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}{}_{\lambda\tau}\gamma^{\lambda\tau}\epsilon - \frac{1}{2}d^I_{rs}\mathcal{F}^r_{\mu\nu}\mathcal{F}^{s,\mu\nu}\epsilon + 4d^I_{rs}Y^r_a Y^{s,a}\epsilon \quad + \\ d^I_{rs}\mathcal{F}^r_{\mu\nu}h^s_J\phi^J\gamma^{\mu\nu}\epsilon + d^I_{rs}h^r_J h^s_K\phi^J\phi^K\epsilon \quad = \quad 0 \ . \end{aligned} \tag{B.9}$$

Now we act on the KSE in (B.2) with $\gamma^\mu D_\mu$ and this gives

$$\frac{1}{12}D_\mu\mathcal{H}^I_{\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\epsilon + \frac{1}{4}D^\mu\mathcal{H}^I_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon + D_\mu D_\nu\phi^I\gamma^{\mu\nu}\epsilon + D_\mu D^\mu\phi^I\epsilon = 0 \ . \tag{B.10}$$

The third term in this equation can be written as

$$D_\mu D_\nu\phi^I\gamma^{\mu\nu}\epsilon = d^I_{rs}\mathcal{F}^r_{\mu\nu}h^s_J\phi^J\gamma^{\mu\nu}\epsilon - \frac{1}{2}\mathcal{F}^r_{\mu\nu}g^{Is}b_{Jsr}\phi^J\gamma^{\mu\nu}\epsilon \ . \tag{B.11}$$

In addition, the first and second terms in (B.10) can be rewritten using the duality of the gamma matrices and the self-duality of the 3-form field strength. Combining the result of that with (B.11) means the equation in (B.10) becomes

$$\begin{aligned} -\frac{1}{12}D_\mu\mathcal{H}^I_{\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}{}_{\lambda\tau}\gamma^{\lambda\tau}\epsilon + d^I_{rs}\mathcal{F}^r_{\mu\nu}h^s_J\phi^J\gamma^{\mu\nu}\epsilon \\ -\frac{1}{2}\mathcal{F}^r_{\mu\nu}g^{Is}b_{Jsr}\phi^J\gamma^{\mu\nu}\epsilon + D_\mu D^\mu\phi^I\epsilon \quad = \quad 0 \ . \end{aligned} \tag{B.12}$$

Subtracting this equation from (B.9) we get

$$\begin{aligned} \left[D_\mu D^\mu\phi^I + \frac{1}{2}d^I_{rs}\mathcal{F}^r_{\mu\nu}\mathcal{F}^{s,\mu\nu} - 4d^I_{rs}Y^r_a Y^{s,a} - d^I_{rs}h^r_J h^s_K\phi^J\phi^K\right]\epsilon \\ -\frac{1}{2}\mathcal{F}^r_{\mu\nu}g^{Is}b_{Jsr}\phi^J\gamma^{\mu\nu}\epsilon \\ -\frac{1}{12}\left[D_\mu\mathcal{H}^I_{\nu\rho\sigma} - \frac{3}{2}d^I_{rs}\mathcal{F}^r_{\mu\nu}\mathcal{F}^s_{\rho\sigma}\right]\epsilon^{\mu\nu\rho\sigma}{}_{\lambda\tau}\gamma^{\lambda\tau}\epsilon \quad = \quad 0 \ . \end{aligned} \tag{B.13}$$

To proceed we make use of another identity that is obtained when the gaugini KSE (B.1) is contracted with $g^{Ir}b_{Jrs}\phi^J$,

$$g^{Ir}b_{Jrs}\mathcal{F}^s_{\mu\nu}\phi^J\gamma^{\mu\nu}\epsilon + 4g^{Ir}b_{Jrs}Y^s_a\phi^J\rho^a\epsilon + 4d^I_{rs}h^r_J h^s_K\phi^J\phi^K\epsilon = 0 \ . \tag{B.14}$$

Adding this to (B.13) gives

$$\begin{aligned}\left[D_\mu D^\mu \phi^I + \frac{1}{2} d^I_{rs} \mathcal{F}^r_{\mu\nu} \mathcal{F}^{s,\mu\nu} - 4 d^I_{rs} Y^r_a Y^{s,a} + 3 d^I_{rs} h^r_J h^s_K \phi^J \phi^K\right] \epsilon & \\ + \frac{1}{2} \mathcal{F}^r_{\mu\nu} g^{Is} b_{Jsr} \phi^J \gamma^{\mu\nu} \epsilon + 4 g^{Ir} b_{Jrs} Y^s_a \phi^J \rho^a \epsilon & \\ - \frac{1}{12} \left[D_\mu \mathcal{H}^I_{\nu\rho\sigma} - \frac{3}{2} d^I_{rs} \mathcal{F}^r_{\mu\nu} \mathcal{F}^s_{\rho\sigma}\right] \epsilon^{\mu\nu\rho\sigma}{}_{\lambda\tau} \gamma^{\lambda\tau} \epsilon & = 0 \ . \end{aligned} \quad \text{(B.15)}$$

The Bianchi identity for the 3-form field strength is given as

$$D_{[\mu} \mathcal{H}^I_{\nu\rho\sigma]} = \frac{3}{2} d^I_{rs} \mathcal{F}^r_{[\mu\nu} \mathcal{F}^s_{\rho\sigma]} + \frac{1}{4} g^{Ir} \mathcal{H}^{(4)}_{\mu\nu\rho\sigma r} \ , \quad \text{(B.16)}$$

using this (B.15) becomes

$$\begin{aligned}\left[D_\mu D^\mu \phi^I + \frac{1}{2} d^I_{rs} \mathcal{F}^r_{\mu\nu} \mathcal{F}^{s,\mu\nu} - 4 d^I_{rs} Y^r_a Y^{s,a} + 3 d^I_{rs} h^r_J h^s_K \phi^J \phi^K\right] \epsilon & \\ + 4 g^{Ir} b_{Jrs} Y^s_a \phi^J \rho^a \epsilon & \\ + \frac{1}{2} \left[g^{Is} b_{Jsr} \mathcal{F}^r_{\mu\nu} \phi^J - \frac{1}{4!} \epsilon_{\mu\nu\rho\lambda\sigma\tau} g^{Ir} \mathcal{H}^{(4)\rho\lambda\sigma\tau}_r\right] \gamma^{\mu\nu} \epsilon & = 0 \ . \end{aligned} \quad \text{(B.17)}$$

The first line on the lhs gives the scalar field equation, the second line the Y^r equations and the third line gives the relation between the 2-form and the 4-form field strengths.

Bibliography

[1] K. Becker, M. Becker and J. H. Schwarz, “String Theory and M-Theory: A Modern Introduction,” Cambridge University Press, 2007

[2] P. K. Townsend, “The Eleven-Dimensional Supermembrane Revisited,” Phys.Lett. **B350** (1995) 184-187, [arXiv: hep-th/9501068].

[3] E. Witten, “String Theory Dynamics in Various Dimensions,” Nucl.Phys. **B443** (1995) 85-126, [arXiv: hep-th/9503124].

[4] E. Cremmer, B. Julia and J. Scherk, “Supergravity Theory in Eleven-Dimensions”, Phys. Lett. **B76** (1978) 409-412

[5] M. J. Duff and K. S. Stelle, “Multimembrane solutions of D = 11 supergravity,” Phys.Lett. **B253** (1991) 113-118.

[6] R. Gueven, “Black p-brane solutions of D = 11 supergravity theory,” Phys. Lett. **B 276** (1992) 49.

[7] D. Z. Freedman and A. Van Proeyen, “Supergravity”, Cambridge University Press, 2012

[8] Y. Tanii, “Introduction to supergravities in diverse dimensions”, [arXiv:hep-th/9802138].

[9] K. S. Stelle, “BPS branes in supergravity,” Based on the Trieste Lecture Notes, [arXiv: hep-th/9803116].

[10] M. J. Duff, “Ultraviolet Divergences in Extended Supergravity,” Proceedings of the 1981 Trieste Conference “Supergravity 1981”, Editors: Ferrara and Taylor, Cambridge University Press, 1982.

[11] M. Grana, “Flux compactifications in string theory: A Comprehensive review,” Phys.Rept., **423**, 91-158, (2006)

[12] M. Akyol and G. Papadopoulos, “Spinorial geometry and Killing spinor equations of 6-D Supergravity,” Class. Quant. Grav. **28** (2011) 105001, [arXiv: 1010.2632 [hep-th]].

[13] M. Akyol and G. Papadopoulos, "Topology and geometry of 6-dimensional (1,0) supergravity black hole horizons," Class. Quantum Grav. **29** (2012) 055002, [arXiv: 1109.4254 [hep-th]].

[14] M. Akyol and G. Papadopoulos, "(1,0) superconformal theories in six dimensions and Killing spinor equations," JHEP **1207** (2012) 070 , [arXiv: 1204.2167 [hep-th]].

[15] J. Gillard, U. Gran and G. Papadopoulos, "The spinorial geometry of supersymmetric backgrounds", Class. Quant. Grav. **22** (2005) 1033 [arXiv:hep-th/0410155].

[16] U. Gran, J. Gutowski, and G. Papadopoulos, "The spinorial geometry of supersymmetric IIB backgrounds", Class. Quant. Grav., 22 (2005) 2453 [arXiv:hep-th/0501177].

[17] G. Papadopoulos, "Spin Cohomology", J. Geom. Phys. **56**, 1893-1919, 2006, [arXiv:math/0410494].

[18] H. B. Lawson and M-L. Michelson, "Spin Geometry", Princeton University Press, 1989

[19] F. R. Harvey, "Spinors and Calibrations", Academic Press, London, 1990

[20] M. Y. Wang, "Parallel spinors and parallel forms", Ann. Global Anal Geom., Vol 7, No 1 (1989), 59

[21] J. Wess and J. Bagger, "Supersymmetry and Supergravity", 2nd edition Princeton University Press, 1992

[22] P. West, "Introduction to Supersymmetry and Supergravity", 2nd edition World Scientific, 1990

[23] J. M. Figueroa-O'Farrill, "BUSSTEPP Lectures on Supersymmetry," [arXiv: hep-th/0109172].

[24] K. Tod, "All metrics admitting supercovariantly constant spinors," Phys. Lett. **121B** (1983) 241-244

[25] K. Tod, "More on supercovariantly constant spinors," Class. Quant. Grav. **12** (1995) 1801-1820

[26] T. Friedrich and S. Ivanov, "Parallel spinors and connections with skew symmetric torsion in string theory," Asian J.Math **6** (2002) 303-336, [arXiv: math/0102142 [math-dg]].

[27] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram "G-structures and wrapped NS5-branes," Commun.Math.Phys. **247** (2004) 421 [arXiv:hep-th/0205050].

[28] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, "All supersymmetric solutions of minimal supergravity in five- dimensions," Class. Quant. Grav. **20** (2003) 4587 [arXiv:hep-th/0209114].

[29] J. P. Gauntlett, and S. Pakis, "The geometry of D=11 Killing spinors," JHEP **0304** (2003) 039 [arXiv:hep-th/0212008].

[30] M. M. Caldarelli and D. Klemm, "All supersymmetric solutions of N=2, D = 4 gauged supergravity," JHEP **0309** (2003) 019

[31] J. P. Gauntlett and J. B. Gutowski "All supersymmetric solutions of minimal gauged supergravity in five-dimensions," Phys.Rev. **D68** (2003) 105009 [Erratum-ibid. **D70** (2004) 089901][arXiv:hep-th/0304064].

[32] J. Bellorin, P. Messen and T. Ortin, "All the supersymmetric solutions of N=1,d=5 ungauged supergravity," JHEP **0701** (2007) 020, [arXiv: hep-th/0610196].

[33] J. Bellorin, "Supersymmetric solutions of gauged five-dimensional supergravity with general matter couplings," Class. Quant. Grav. **26** (2009) 195012 [arXiv: 0810.0527 [hep-th]].

[34] J. B. Gutowski, D. Martelli and H. S. Reall, "All supersymmetric solutions of minimal supergravity in six dimensions," Class. Quant. Grav. **20** (2003) 5049 [arXiv:hep-th/0306235].

[35] M. Cariglia and O. A. P. Mac Conamhna, "Timelike Killing spinors in seven dimensions," Phys.Rev. **D70** (2004) 125009 [arXiv:hep-th/0407127].

[36] O. A. P. Mac Conamhna, "Refining G-structure classifications," Phys.Rev. **D70** (2004) 105024 [arXiv:hep-th/0408203].

[37] J. P. Gauntlett, D. Martelli and D. Waldram, "Superstrings with intrinsic torsion," Phys.Rev. **D69** (2004) 086002 [arXiv:hep-th/0302158].

[38] J. P. Gauntlett, J. B. Gutowski and S. Pakis, "The geometry of D=11 null Killing spinors," JHEP **0312** (2003) 049 [arXiv:hep-th/0311112].

[39] J. P. Gauntlett, "Classifying supergravity solutions," Fortsch.Phys. **53** (2005) 468-479, [arXiv:hep-th/0501229].

[40] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities," JHEP **0303** (2003) 048 [arXiv:hep-th/0211089].

[41] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Homogeneous fluxes, branes and a maximally supersymmetric solution of M theory," JHEP **0108** (2001) 036, [arXiv: hep-th/0105308].

[42] M. Blau, J. M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "A New maximally supersymmetric background of IIB superstring theory," JHEP **0201** (2002) 047, [arXiv: hep-th/0110242].

[43] M. Blau, J. M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "Penrose limits and maximal supersymmetry," Class. Quant. Grav. **19** (2002) L87-L95, [arXiv: hep-th/0201081].

[44] U. Gran, P. Lohrmann and G. Papadopoulos, "The spinorial geometry of supersymmetric heterotic string backgrounds," JHEP **0602** (2006) 063 [arXiv:hep-th/0510176].

[45] U. Gran, G. Papadopoulos, D. Roest and P. Sloane, "Geometry of all supersymmetric type I backgrounds," JHEP **0708** (2007) 074 [arXiv:hep-th/0703143].

[46] U. Gran, G. Papadopoulos and D. Roest, "Supersymmetric heterotic string backgrounds," Phys. Lett. B **656** (2007) 119 [arXiv:0706.4407 [hep-th]].

[47] U. Gran, J. Gutowski, and G. Papadopoulos, "The G(2) spinorial geometry of supersymmetric IIB backgrounds," Class.Quant.Grav. **23** (2006) 143-206 [arXiv:hep-th/0505074].

[48] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest, "N=31 is not IIB," JHEP **0702** (2007) 044 [arXiv:hep-th/0606049].

[49] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest, "IIB solutions with N $>$ 28 Killing spinors are maximally supersymmetric," JHEP **0712** (2007) 070 [arXiv: 0710.1829[hep-th]].

[50] U. Gran, J. Gutowski, and G. Papadopoulos, "Classification of IIB backgrounds with 28 supersymmetries," JHEP **1001** (2010) 044 [arXiv: 0902.3642[hep-th]].

[51] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest, "N=31, D=11," JHEP **0702** (2007) 043 [arXiv: hep-th/0610331].

[52] U. Gran, J. Gutowski, and G. Papadopoulos, "M-theory backgrounds with 30 Killing spinors are maximally supersymmetric," JHEP **1003** (2010) 112 [arXiv: 1001.1103[hep-th]].

[53] H. Nishino and E. Sezgin, “Matter And Gauge Couplings Of N=2 Supergravity In Six-Dimensions,” Phys. Lett. B **144** (1984) 187.

[54] H. Nishino and E. Sezgin, “The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings,” Nucl. Phys. B **278** (1986) 353.

[55] H. Nishino and E. Sezgin, “New couplings of six-dimensional supergravity,” Nucl. Phys. B **505** (1997) 497 [arXiv:hep-th/9703075].

[56] S. Ferrara, F. Riccioni and A. Sagnotti, “Tensor and vector multiplets in six-dimensional supergravity,” Nucl. Phys. B **519** (1998) 115 [arXiv:hep-th/9711059].

[57] F. Riccioni, “All couplings of minimal six-dimensional supergravity,” Nucl. Phys. B **605** (2001) 245 [arXiv:hep-th/0101074].

[58] A. Chamseddine, J. M. Figueroa-O’Farrill and W. Sabra, “Supergravity vacua and Lorentzian Lie groups,” [arXiv:hep-th/0306278].

[59] M. Cariglia and O. A. P. Mac Conamhna, “The general form of supersymmetric solutions of N = (1,0) U(1) and SU(2) gauged supergravities in six dimensions,” Class. Quant. Grav. **21** (2004) 3171 [arXiv:hep-th/0402055].

[60] D. C. Jong, A. Kaya and E. Sezgin, “6D dyonic string with active hyperscalars,” JHEP **0611** (2006) 047 [arXiv:hep-th/0608034].

[61] R. Gueven, J. T. Liu, C. N. Pope and E. Sezgin, “Fine tuning and six-dimensional gauged N = (1,0) supergravity vacua,” Class. Quant. Grav. **21** (2004) 1001 [arXiv:hep-th/0306201].

[62] R. M. Wald, “General Relativity,” The University of Chicago Press, 1984.

[63] S. M. Carroll, “Spacetime and Geometry: An Introduction to General Relativity,” Addison Wesley, 2004.

[64] W. Israel, “Event Horizons In Static Vacuum Space-Times,” Phys. Rev. **164** (1967) 1776.

[65] B. Carter, “Axisymmetric Black Hole Has Only Two Degrees of Freedom,” Phys. Rev. Lett. **26** (1971) 331.

[66] S. W. Hawking, “Black holes in general relativity,” Commun. Math. Phys. **25** (1972) 152.

[67] D. C. Robinson, “Uniqueness of the Kerr black hole,” Phys. Rev. Lett. **34** (1975) 905.

[68] W. Israel, "Event Horizons in Static, Electrovac Space-Times," Commun. Math. Phys. **8** (1968) 245.

[69] P. O. Mazur, "Proof of Uniqueness of the Kerr-Newman Black Hole Solution," J. Phys. A **15** (1982) 3173.

[70] D. Robinson, "Four decades of black hole uniqueness theorems," appeared in *The Kerr spacetime: Rotating black holes in General Relativity*, eds D. L. Wiltshire, M. Visser and S. M. Scott, pp 115-143, CUP 2009.

[71] R. C. Myers and M. J. Perry, "Black holes in higher dimensional space-times," Annals Phys. **172** (1986) 304.

[72] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, "D-branes and spinning black holes," Phys. Lett. **B391** (1997) 93; [arXiv: hep-th/9602065].

[73] R. Emparan, H. S. Reall, "A rotating black ring in five dimensions," Phys. Rev. Lett., **88**, 101101, (2002), [arXiv:hep-th/0110260].

[74] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "A Supersymmetric black ring," Phys. Rev. Lett. **93** (2004) 211302; [arXiv: hep-th/0407065].

[75] G. W. Gibbons, D. Ida and T. Shiromizu, "Uniqueness and non-uniqueness of static black holes in higher dimensions," Phys. Rev. Lett. **89** (2002) 041101; [arXiv: hep-th/0206049].

[76] M. Rogatko, "Uniqueness theorem of static degenerate and non-degenerate charged black holes in higher dimensions", Phys. Rev. **D67** (2003) 084025; [arXiv: hep-th/0302091].

[77] M. Rogatko, "Classification of static charged black holes in higher dimensions," Phys. Rev. **D73** (2006), 124027; [arXiv: hep-th/0606116].

[78] H. K. Kunduri, J. Lucietti and H. S. Reall, "Near-horizon symmetries of extremal black holes," Class. Quant. Grav. **24** (2007) 4169; [arXiv:0705.4214 [hep-th]].

[79] R. Emparan, T. Harmark, V. Niarchos and N. Obers, "World-Volume Effective Theory for Higher-Dimensional Black Holes," Phys. Rev. Lett. **102** (2009) 191301; [arXiv:0902.0427 [hep-th]];

[80] R. Emparan, T. Harmark, V. Niarchos and N. Obers, "Essentials of Blackfold Dynamics;" [arXiv:0910.1601 [hep-th]].

[81] R. Emparan, H. S. Reall, "Black holes in higher dimensions," Living Rev. Rel. **11** (2008) 6, [arXiv: 0801.3471[hep-th]].

[82] H. S. Reall, "Higher dimensional black holes and supersymmetry," Phys. Rev. **D68** (2003) 024024; [arXiv: hep-th/0211290].

[83] J. B. Gutowski and H. S. Reall, "Supersymmetric AdS(5) black holes," JHEP **0402** (2004) 006, [arXiv: hep-th/0401042].

[84] J. B. Gutowski and H. S. Reall, "General supersymmetric AdS(5) black holes," JHEP **0404** (2004) 048, [arXiv: hep-th/0401129].

[85] J. P. Gauntlett and J. B. Gutowski, "Concentric black rings," Phys.Rev. **D71** (2005) 025013, [arXiv: hep-th/0408010].

[86] J. P. Gauntlett and J. B. Gutowski, "General concentric black rings," Phys.Rev. **D71** (2005) 045002, [arXiv: hep-th/0408122].

[87] J. Gutowski and G. Papadopoulos, "Topology of supersymmetric N=1, D=4 supergravity horizons," JHEP **1011** (2010) 114 [arXiv:1006.4369 [hep-th]].

[88] H. K. Kunduri and J. Lucietti, "A Classification of near-horizon geometries of extremal vacuum black holes," J.Math.Phys. **50** (2009) 082502, [arXiv: 0806.2051 [hep-th]].

[89] J. Gutowski and G. Papadopoulos, "Heterotic Black Horizons," JHEP **07**, 011 (2010); [arXiv:0912.3472 [hep-th]].

[90] J. Gutowski and G. Papadopoulos, "Heterotic horizons, Monge-Ampere equation and del Pezzo surfaces," JHEP **10** (2010) 084; [arXiv:1003.2864 [hep-th]].

[91] U. Gran, J. Gutowski and G. Papadopoulos, "IIB black hole horizons with five-form flux and KT geometry," JHEP **05** (2011) 050; arXiv:1101.1247 [hep-th].

[92] U. Gran, J. Gutowski, G. Papadopoulos, "IIB black hole horizons with five-form flux and extended supersymmetry;" [arXiv:1104.2908 [hep-th]].

[93] J. Gutowski and G. Papadopoulos, "Static M-horizons," JHEP **1201** (2012) 005 [arXiv:1106.3085 [hep-th]].

[94] J. Gutowski and G. Papadopoulos, "M-Horizons," [arXiv:1207.7086 [hep-th]].

[95] S. Ferrara, R. Kallosh and A. Strominger, "N=2 extremal black holes," Phys.Rev. **D52** (1995) 5412-5416, [arXiv: hep-th/9508072].

[96] S. Ferrara and R. Kallosh, "Supersymmetry and attractors," Phys.Rev. **D54** (1996) 1514-1524, [arXiv: hep-th/9602136].

[97] P. Lounesto, "Clifford Algebras and Spinors," Lond.Math.Soc.Lect.Note Ser. **286** (2001) 1-338.

[98] R. L. Bryant, "Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor," [arXiv: math.DG/0004073].

[99] J. M. Figueroa-O'Farrill, "Breaking the M-waves," Class. Quant. Grav. **17** (2000) 2925-2948, [arXiv: hep-th/9904124].

[100] L. J. Romans, "Selfduality For Interacting Fields: Covariant Field Equations For Six-dimensional Chiral Supergravities," Nucl.Phys. **B276** (1986) 71.

[101] M. Nakahara, "Geometry, Topology and Physics," Taylor and Francis Group, Graduate Series in Physics, 2003.

[102] S. Salamon, "Riemannian geometry and holonomy groups," Pitman research notes in mathematics series 201, Longman Scientific and Technical, 1989.

[103] D. D. Joyce, "Riemannian holonomy groups and calibrated geometry," Oxford graduate texts in mathematics 12, Oxford University Press, 2007.

[104] U. Gran, J. Gutowski and G. Papadopoulos, "Geometry of all supersymmetric four-dimensional $\mathcal{N} = 1$ supergravity backgrounds," JHEP **0806** (2008) 102

[105] G. Papadopoulos, "Heterotic supersymmetric backgrounds with compact holonomy revisited," Class. Quant. Grav. **27** (2010) 125008 [arXiv:0909.2870 [hep-th]].

[106] A. Medina and P. Revoy, "Algebres de Lie et produit scalaire invariant", Ann. Scient. Ec. Norm. Sup. **18** (1985) 553.

[107] T. Kawano and S. Yamaguchi, "Dilatonic parallelizable NS-NS backgrounds," Phys. Lett. B **568** (2003) 78 [arXiv:hep-th/0306038].

[108] J. M. Figueroa-O'Farrill, T. Kawano, S. Yamaguchi, "Parallelizable heterotic backgrounds," JHEP **0310** (2003) 012. [hep-th/0308141].

[109] F. Riccioni, A. Sagnotti, "Consistent and covariant anomalies in six-dimensional supergravity," Phys. Lett. **B436** (1998) 298-305 [arXiv: hep-th/9806129].

[110] H. Friedrich, I. Racz and R. M. Wald, "On the rigidity theorem for space-times with a stationary event horizon or a compact Cauchy horizon," Commun. Math. Phys. **204** (1999) 691; [arXiv: gr-qc/9811021].

[111] P. S. Howe and G. Papadopoulos, "Twistor spaces for HKT manifolds," Phys. Lett. B **379** (1996) 80 [arXiv:hep-th/9602108].

[112] G. K. Ambler, "The Maximum Principle in Elliptic Equations," 1995, http://maths.swan.ac.uk/staff/vl/projects/projall.pdf, last retrieved 08/09/12.

[113] J. Gutowski and W. A. Sabra, "Towards Cosmological Black Rings," JHEP **1105** (2011) 020, [arXiv: 1012.2120 [hep-th]].

[114] P. Petersen, "Riemannian Geometry," Graduate Texts in Mathematics, Springer (1998), page 237.

[115] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications," [arXiv:math/0211159];

"Ricci flow with surgery on three-manifolds," [arXiv:math/0303109];

"Finite extinction time for the solutions to the Ricci flow on certain three-manifolds," [arXiv:math/0307245].

[116] L. Andrianopoli, S. Ferrara, A. Marrani and M. Trigiante, "Non-BPS Attractors in 5d and 6d Extended Supergravity," Nucl. Phys. B **795** (2008) 428 [arXiv:0709.3488 [hep-th]].

[117] S. Ferrara, A. Marrani, J. F. Morales and H. Samtleben, "Intersecting Attractors," Phys. Rev. D **79** (2009) 065031 [arXiv:0812.0050 [hep-th]].

[118] G. W. Gibbons and P. K. Townsend, "Vacuum interpolation in supergravity via super p-branes," Phys. Rev. Lett. **71** (1993) 3754 [arXiv: hep-th/9307049].

[119] J. Bagger and N. Lambert, "Gauge symmetry and supersymmetry of multiple M2-branes," Phys. Rev. D **77** (2008) 065008 [arXiv: 0711.0955 [hep-th]].

[120] A. Gustavsson, "Algebraic structures on parallel M2-branes," Nucl. Phys. **B 811** (2009) 66 [arXiv: 0709.1260 [hep-th]].

[121] N. Lambert and C. Papageorgakis, "Nonabelian (2,0) Tensor Multiplets and 3-algebras," JHEP **1008** (2010) 083 [arXiv:1007.2982 [hep-th]].

[122] C. -S. Chu and S. -L. Ko, "Non-abelian Action for Multiple M5-Branes," arXiv:1203.4224 [hep-th].

[123] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Plucker type relations for orthogonal planes," [arXiv: math/0211170 [math-ag]].

[124] G. Papadopoulos, "M2-branes, 3-Lie Algebras and Plucker relations," JHEP **0805** (2008) 054 [arXiv:0804.2662 [hep-th]].

[125] J. P. Gauntlett and J. B. Gutowski, "Constraining Maximally Supersymmetric Membrane Actions," JHEP **0806** (2008) 053 [arXiv:0804.3078 [hep-th]].

[126] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].

[127] H. Samtleben, E. Sezgin and R. Wimmer, "(1,0) superconformal models in six dimensions," JHEP **1112** (2011) 062 [arXiv:1108.4060 [hep-th]].

[128] H. Samtleben, E. Sezgin, R. Wimmer and L. Wulff, "New superconformal models in six dimensions: Gauge group and representation structure," [arXiv:1204.0542 [hep-th]].

[129] C. -S. Chu, "A Theory of Non-Abelian Tensor Gauge Field with Non-Abelian Gauge Symmetry G x G," [arXiv:1108.5131 [hep-th]].

[130] G. Papadopoulos, "New half supersymmetric solutions of the heterotic string," Class. Quant. Grav. **26** (2009) 135001 [arXiv:0809.1156 [hep-th]].

[131] L. S. Brown, R. D. Carlitz, D. B. Creamer and C. -k. Lee, "Propagators in Pseudoparticle Fields," Phys. Lett. B **70** (1977) 180 [Phys. Lett. B **71** (1977) 103].

[132] L. S. Brown, R. D. Carlitz, D. B. Creamer and C. -k. Lee, "Propagation Functions in Pseudoparticle Fields," Phys. Rev. D **17** (1978) 1583.

[133] E. Corrigan and P. Goddard, "Some Aspects Of Instantons," DAMTP-79-18.

[134] H. Osborn, "Solutions Of The Dirac Equation For General Instanton Solutions," Nucl. Phys. B **140** (1978) 45.

[135] E. Corrigan, P. Goddard, H. Osborn and S. Templeton, "Zeta Function Regularization And Multi - Instanton Determinants," Nucl. Phys. B **159** (1979) 469.

[136] A. Strominger, "Open p-branes," Phys. Lett. B **383** (1996) 44 [arXiv: hep-th/9512059].

[137] G. Papadopoulos and P. K. Townsend, "Intersecting M-branes," Phys. Lett. B **380** (1996) 273 [arXiv: hep-th/9603087].