Almost commuting elements of real rank zero C-algebras

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Abstract

The purpose of this thesis is to study the following problem. Suppose that X, Y are bounded self-adjoint operators in a Hilbert space H with their commutator $[X,Y]$ being small. Such operators are called almost commuting. How close is the pair X,Y to a pair of commuting operators X',Y'? In terms of one operator $A = X + iY$, suppose that the self-commutator $[A,A^*]$ is small. How close is A to the set of normal operators?

Our main result is a quantitative analogue of Huaxin Lin's theorem on almost commuting matrices. We prove that for every $(n \times n)$-matrix A with $\|A\| \leqslant 1$ there exists a normal matrix A' such that $\|A - A'\| \leqslant C\|[A,A^*]\|^{1/3}$. We also establish a general version of this result for arbitrary C^*-algebras of real rank zero assuming that A satisfies a certain index-type condition. For operators in Hilbert spaces, we obtain two-sided estimates of the distance to the set of normal operators in terms of $\|[A,A^*]\|$ and the distance from A to the set of invertible operators.

The technique is based on Davidson's results on extensions of almost normal operators, Alexandrov and Peller's results on operator and commutator Lipschitz functions, and a refined version of Filonov and Safarov's results on approximate spectral projections in C^*-algebras of real rank zero.

In Chapter 4 we prove an analogue of Lin's theorem for finite matrices with respect to the normalized Hilbert–Schmidt norm. It is a refinement of a previously known result by Glebsky, and is rather elementary.

In Chapter 5 we construct a calculus of polynomials for almost commuting elements of C^*-algebras and study its spectral mapping properties. Chapters 4 and 5 are based on author's joint results with Nikolay Filonov.

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Contents

Introduction

BDF theory

The study of almost normal operators probably started from the following problem. Assume that A is a bounded operator in a separable Hilbert space H such that its self-commutator $[A, A^*]$ is compact. Operators with this property are called *essentially normal*. Obviously, all compact perturbations of normal operators are essentially normal. Is it true that any essentially normal operator is a sum of a normal operator and a compact operator? The answer was given by Brown, Douglas and Fillmore in [5]. An essentially normal operator A is a compact perturbation of a normal operator if and only if the Fredholm index of $A - \lambda I$ is zero for all $\lambda \notin \sigma_{\mathrm{ess}}(A)$. This condition can be reformulated in terms of the Calkin algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$: the equivalence class of $A - \lambda I$ must belong to $\mathrm{GL}_0(\mathcal{C}(H))$. Here $\mathcal{K}(H)$ is the ideal of all compact operators in H, and GL_0 denotes the connected component containing the unit in the group of invertible elements of $\mathcal{C}(H)$.

The theory in fact goes much further. Two operators A and B are called *compalent* if there exists a unitary operator U such that $UAU^{-1} - B$ is compact. The results of [5] classify all essentially normal operators up to this equivalence. It turns out that there are two invariants: the essential spectrum, i. e. an arbitrary compact subset $X \subset \mathbb{C}$, and an element of an Abelian group $\mathrm{Ext}(X) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z})$. Two operators A and B with the same essential spectra $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B) = X$ are compalent if and only if they have the same index functions, i. e. for each $\lambda \in \mathbb{C} \setminus X$ the operators $A - \lambda I$ and $B - \lambda I$ have the same Fredholm index. The elements of $\mathrm{Ext}(X)$ are in one-to-one correspondence with all possible index functions which are locally constant integer-valued functions on $\mathbb{C} \setminus X$ vanishing at infinity.

Lin's theorem

The problem of approximating almost commuting matrices by commuting ones with respect to the operator norm dates back to Halmos [22]. The original formulation is as follows. By $\mathrm{M}_n(\mathbb{C})$ we denote the set of all complex $(n \times n)$-matrices. Given $X, Y \in \mathrm{M}_n(\mathbb{C})$ such that $X = X^*$, $Y = Y^*$, $\|X\|, \|Y\| \leqslant 1$, and $\|[X, Y]\| \leqslant \delta$, can we find two *commuting* self-adjoint matrices X', Y' satisfying

$$\|X - X'\| + \|Y - Y'\| \leqslant C(\delta), \quad \text{where} \quad C(\delta) \to 0 \quad \text{as} \quad \delta \to 0?$$

In terms of a single matrix, given $\|[A, A^*]\| = \delta$, can we find a normal matrix A' such that $\|A - A'\| \leqslant C(\delta)$?

An obvious positive answer can be given if we allow $C(\delta)$ to depend on n. Indeed, assume the contrary, i. e. that there exists a sequence $T_m \in \mathrm{M}_n(\mathbb{C})$ such that $\|A_m\| \leqslant 1$, $\|[A_m, A_m^*]\| \to 0$, and $\|A_m - N\| \geqslant \varepsilon$ for all $m \in \mathbb{N}$ and all normal N. Since the unit ball in $\mathrm{M}_n(\mathbb{C})$ is compact, this sequence has at least one limit point which must be normal and at the same time be separated from the set of normal matrices. This contradiction proves that for every single n there exists $C(\delta) \to 0$ as $\delta \to 0$. However, the question becomes significantly more challenging if we want $C(\delta) \to 0$ uniformly in n.

An additional evidence of difficulty of this question is that it fails in infinite dimensions. The following example is due to Choi [8]. Let $\{e_k\}_{k \geqslant 1}$ be an orthonormal basis in a separable Hilbert space H, and consider the operator family

$$S_n e_k = \min\{k/n, 1\} e_{k+1}. \tag{1}$$

It can be easily checked that $\|[S_n, S_n^*]\| \to 0$ as $n \to +\infty$. However, the results of [8] show that this operator family is uniformly separated from the set of normal operators. Therefore, the dimension-uniform properties of almost commuting matrices may be different from those of general almost commuting operators, and possible proofs should take this into account. In fact, in the infinite-dimensional case there is an additional index-type obstruction which will be described later.

There were several dimension-dependent results in this context (a review of them can be found in [11]). However, the question of finding or establishing the existence of a uniform $C(\delta)$ remained open until 1995 when a positive answer was given by

Lin [25]. The proof relied on the technique of C^*-algebras. A significantly simpler version of the proof was given in [17]. Let us sketch the main steps here. Assume the contrary, i. e. that there exists a sequence $A_k \in \mathrm{M}_{n_k}(\mathbb{C})$ such that $\|A_k\| \leqslant 1$, $\|[A_k, A_k^*]\| \to 0$, and $\|A_k - A'\| \geqslant \varepsilon$ for any normal A'. We consider this sequence as an element A of a C^*-algebra $\mathcal{M} = \oplus_k \mathrm{M}_{n_k}(\mathbb{C})$. In this algebra, consider an ideal

$$\mathcal{I} = \{\{A_k\} \in \mathcal{M} \colon \|A_k\| \to 0 \text{ as } k \to \infty\}.$$

By $\pi \colon \mathcal{M} \to \mathcal{M}/\mathcal{I}$ we denote the canonical projection onto the quotient algebra. Since $[A, A^*] \in \mathcal{I}$, the element $\pi(A) \in \mathcal{M}/\mathcal{I}$ is normal. We will come to a contradiction if we prove that there exists a normal element B close to $\pi(A)$ that has a normal pre-image in $\mathcal{M}$. Thus, the original question reduces to a lifting problem; in other words, to the problem of finding a certain "approximate inverse" of π.

Note that any self-adjoint element has a self-adjoint pre-image (we can take the real part of any pre-image). The same holds for unitary elements. If we have a normal element with finite spectrum, then we can map its spectrum onto a (finite) subset of $\mathbb{R}$, then lift it, and then map it back. Therefore, any normal element of $\mathcal{M}/\mathcal{I}$ with finite spectrum also has a normal pre-image. We see that the original question can be reduced to approximating normal elements of the C^*-algebra $\mathcal{M}/\mathcal{I}$ by elements with finite spectra.

This approximation is done in two steps. On the first step, it is shown that any element of $\mathcal{M}$ and, as a consequence, any element of $\mathcal{M}/\mathcal{I}$, can be approximated by elements not containing any fixed finite set in their spectra. This follows from the polar decomposition of finite matrices and may fail in infinite dimensions (this is the mentioned index-type obstruction to solving the problem in $\mathcal{B}(H)$). After that, by using continuous functional calculus, it is easy to show that any element can be approximated by elements whose spectra are contained in the following "ε-grid",

$$\Gamma_\varepsilon = \{x + iy \in \mathbb{C} \colon x \in \varepsilon\mathbb{Z} \text{ or } y \in \varepsilon\mathbb{Z}\}.$$

To approximate such elements by elements with finite spectra, it would suffice to remove a small line segment from each segment of Γ_ε. To cut the line segments, we need analogues of spectral projections corresponding to each removed segment. In general, there are no spectral projections in the algebra $\mathcal{M}/\mathcal{I}$. However, we can map Γ_ε onto a unit circle such that this map is a local homeomorphism in a neighbourhood of the needed interval. The corresponding unitary element can then be lifted into $\mathcal{M}$, where it has a spectral projection P onto the image of the line segment. And then $\pi(P)$ can be considered as the desired projection for the original element of $\mathcal{M}/\mathcal{I}$. Applying this procedure to all line segments, we split the spectrum into small disjoint components and, after that, can shrink them into points using a continuous function.

Generalizations of Lin's theorem

Bearing in mind possible generalizations of the result to an arbitrary C^*-algebra $\mathcal{A}$, let us note that it relies on two properties. First, we must be able to remove finite sets from the spectrum of the element. For that, it is sufficient to assume that for any $\lambda \in \mathbb{C}$ the element $A - \lambda I$ lies in the closure $\overline{\mathrm{GL}_0(\mathcal{A})}$. Here $\mathrm{GL}_0(\mathcal{A})$ is the connected component of $\mathrm{GL}(\mathcal{A})$ containing the unit, and $\mathrm{GL}(\mathcal{A})$ is the group of invertible elements of $\mathcal{A}$. Secondly, to be able to cut one-dimensional spectra, we need to assume that $\mathcal{A}$ is an algebra of real rank zero (which means that any self-adjoint element of $\mathcal{A}$ can be approximated by elements with finite spectra). The following is Theorem 3.2 from [18].

Theorem 1. *Let $\mathcal{A}$ be a unital C^*-algebra and let A be a normal element of $\mathcal{A}$. The following conditions are equivalent.*

(i) *$A - \lambda I$ lies in $\overline{\mathrm{GL}_0(\mathcal{A})}$ for every $\lambda \in \mathbb{C}$,*

(ii) *for every $\varepsilon > 0$ there exists a normal element $B \in \mathcal{A}$ such that*

$$\sigma(B) \subset \Gamma_\varepsilon, \quad \|A - B\| \leqslant \varepsilon, \quad \textit{and} \quad B - \lambda I \in \mathrm{GL}_0(\mathcal{A}) \quad \textit{for all} \quad \lambda \in \mathbb{C} \setminus \sigma(B).$$

If the real rank of $\mathcal{A}$ is zero, then (i) *and* (ii) *are equivalent to*

(iii) *For every $\varepsilon > 0$ there exists a normal element $B \in \mathcal{A}$ with finite spectrum and with $\|B - A\| \leqslant \varepsilon$.*

Using the same arguments with a sequence of algebras, a generalization of Lin's theorem can be proved for C^*-algebras of real rank zero provided that A satisfies the condition (i) from Theorem 1.

A more general result is established in [16]. Let $M_{[A,A^*]}$ be the convex hull of the set $\{S_1[A,A^*]S_2\colon \|S_1\|,\|S_2\|\leqslant 1\}$, $B(\varepsilon)=\{A\in\mathcal{A}\colon \|A\|\leqslant\varepsilon\}$, and $\mathcal{N}(\mathcal{A})=\{A\in\mathcal{A}\colon [A,A^*]=0\}$.

Theorem 2. *There exists a nonincreasing function $h(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that*

$$A\in B(\|A\|)\cap\mathcal{N}(\mathcal{A})+h(\varepsilon)M_{[A,A^*]}+B(\varepsilon)$$

for all $\varepsilon\in(0;+\infty)$, all C^*-algebras $\mathcal{A}$ of real rank zero, and all $A\in\mathcal{A}$ satisfying the condition (i) from Theorem 1.

This result implies an analogue of Lin's theorem not only for the operator norm, but also for any continuous seminorm $\|\cdot\|_\star$ satisfying

$$\|UAV\|_\star\leqslant\|A\|_\star\leqslant C_\star\|A\|.$$

for all unitary U,V and some $C_\star>0$. In particular, Theorem 2 implies the BDF theorem and an analogue of Lin's theorem for the normalized Hilbert-Schmidt norm on $M_n(\mathbb{C})$:

$$\|A\|_{2,n}^2=\frac{1}{n}\sum_{i,j=1}^{n}|A_{ij}|^2. \tag{2}$$

The proof of Theorem 2 is based on the following extension of Theorem 1.

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero. Suppose that $A\in\mathcal{A}$ satisfies the condition (i) from Theorem 1. Let $\{\Omega_j\}_{j=1}^m$ be a finite open cover of $\sigma(A)$. Then there exists a family of mutually orthogonal projections $P_j\in\mathcal{A}$ such that*

$$\sum_{j=1}^{m}P_j=I \quad \textit{and} \quad P_jH\subset\Pi_jH \quad \textit{for all } j=1,\dots,m,$$

where $\mathcal{A}\subset\mathcal{B}(H)$ is the Gelfand-Naimark-Segal embedding, and $\Pi_j=E_A(\Omega_j)\in\mathcal{B}(H)$ are spectral projections of A.

This result tells us that the approximation by elements with finite spectra can be made "subordinate" to any finite open cover of the spectrum of A. Theorem 2 follows from Theorem 3 using arguments similar to [17]. Therefore, it still does not give any information on the rate of decay of the function h.

Note also that Lin's theorem fails for triples of self-adjoint operators and for pairs of unitary operators, see [8].

Quantitative results

Due to the abstract character of known proofs of Lin's theorem and its generalizations, they do not give any quantitative information on the behaviour of $C(\delta)$ (other that it tends to zero as $\delta = \|[T, T^*]\| \to 0$). Simple homogeneity arguments show that $C(\delta)$ cannot decay faster that $\delta^{1/2}$ (for any norm).

In [24] it was proved that in finite dimensions $C(\delta)$ can be chosen in the form $C_H G(1/\delta)\delta^{1/5}$, where $G(1/\delta)$ can be explicitly written down and grows slower than any power of $1/\delta$ (as $\delta \to 0$). The proof relied on the fact that X and Y are matrices, and used the fact that $C(\delta)$ can be made sufficiently small. This follows from the original Lin's theorem but, as a consequence, the estimates of the constant C_H rely on non-quantitative arguments. Note that [24] is an up-to-date version of the original paper [23].

Non-quantitative results in the case of the Hilbert-Schmidt norm (2) were first obtained in the abstract context of C^*-algebras with trace, see [20, 21]. However, it turned out that for $\mathrm{M}_n(\mathbb{C})$ simple and relatively elementary quantitative results hold. The following result is proved in [19].

Theorem 4. *Suppose that $X, Y \in \mathrm{M}_n(\mathbb{C})$, $\|X\|, \|Y\| \leqslant 1$, $X = X^*$, $Y = Y^*$. Let $\|[X, Y]\|_{2,n} = \delta$. Then there exist $X', Y' \in \mathrm{M}_n(\mathbb{C})$ such that $\|X'\|, \|Y'\| \leqslant 1$, $X' = X'^*$, $Y' = Y'^*$, and*

$$[X', Y'] = 0, \quad \|X - X'\|_{2,n} \leqslant 12\,\delta^{1/6}, \ \|Y - Y'\|_{2,n} \leqslant 12\,\delta^{1/6}.$$

In addition, $[X, X'] = 0$.

Unlike the case of the operator norm, this result can be extended to m-tuples of normal matrices.

Finally, we would like to mention the result of [9] which states that if we allow the normal approximant to act in $H \oplus H$ instead of H, then the corresponding analogue of Lin's theorem simplifies significantly, and we even have $C(\delta) \leqslant C\delta^{1/2}$, which is the optimal power. This result is discussed in detail in Sections 1.2 and 2.2 and will be actively used in our work.

Chapter 1

Formulations of the results

1.1 Some notational conventions

- By C, we shall always denote various constants whose numerical values can be computed, but are not important. All inequalities containing C should be understood as "there exists a universal constant C such that...".

- In Chapters 2 and 3, the complex plane $\mathbb{C}$ will sometimes be identified with $\mathbb{R}^2$. For example, we may use the notation of the form $\varepsilon\mathbb{Z} \times \varepsilon\mathbb{Z} \subset \mathbb{C}$ without additional comments, meaning $\varepsilon\mathbb{Z} + i\varepsilon\mathbb{Z}$.

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  - $\mathcal{B}(H)$ is the algebra of bounded operators in a Hilbert space H.
  - $\mathcal{K}(H) \subset \mathcal{B}(H)$ is the ideal of compact operators.
  - The quotient algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is called the Calkin algebra.

- By $\mathcal{A}$ we usually denote a unital C^*-algebra. We often assume that it is a sub-algebra of $\mathcal{B}(H)$ for some (not necessarily separable) Hilbert space H.

- $\mathrm{M}_2(\mathcal{A})$ is the set of (2×2)-matrices whose entries belong to $\mathcal{A}$. It is naturally embedded into $\mathcal{B}(H \oplus H)$ as a unital C^*-subalgebra.

- The unit elements of C^*-algebras appearing in our considerations are usually denoted by I. We use the same symbol for the units of $\mathcal{A}$ and $\mathrm{M}_2(\mathcal{A})$, and hope that the meaning is clear from the context.

- – $\mathrm{GL}(\mathcal{A})$ is the group of invertible elements of $\mathcal{A}$.
  – $\mathrm{GL}_0(\mathcal{A})$ is the connected component of $\mathrm{GL}(\mathcal{A})$ containing the unit element. If $A \in \mathcal{A}$ and λ belongs to the unbounded connected component of $\mathbb{C} \setminus \sigma(A)$, then $A - \lambda I \in \mathrm{GL}_0(\mathcal{A})$ (because $\lambda^{-1}A - I \to I$ as $\lambda \to \infty$).

1.2 Extensions of almost normal elements

Let H be a Hilbert space. The following result is due to Davidson. In the original paper [9] it was formulated for finite matrices, but it in fact holds for general Hilbert spaces.

Theorem 1.2.1. *Let $A \in \mathcal{B}(H)$, $\|A\| \leqslant 1$, $\|[A^*, A]\| \leqslant \delta$. There exists a normal element $N \in \mathcal{B}(H)$ and a normal element $T \in \mathcal{B}(H \oplus H)$ such that $\|A \oplus N - T\| \leqslant C\delta^{1/2}$.*

We reformulate Theorem 1.2.1 for the case of a general C^*-algebra.

Theorem 1.2.2. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $A \in \mathcal{A}$, $\|A\| \leqslant 1$, $\|[A, A^*]\| \leqslant \delta$. Suppose that $X = \operatorname{Re} A = (A + A^*)/2$ can be approximated by elements with finite spectra. Then there exists a normal element $N \in \mathcal{A}$ and a normal element $T \in \mathrm{M}_2(\mathcal{A})$ such that*

$$\|N\| \leqslant 1, \quad \|T\| \leqslant 1, \quad \|A \oplus N - T\| \leqslant C\delta^{1/2}.$$

In addition, N can be chosen in such a way that $\mathbb{C} \setminus \sigma(N)$ is connected.

The proof remains essentially the same. For the convenience of the reader, we give a simplified version of it in Chapter 2.

1.3 Quantitative Lin's theorem

Chapter 3 is the central chapter of the thesis. The main results are Theorems 1.3.3 and 1.3.4.

Definition 1.3.1. A unital C^*-algebra is called a *C^*-algebra of real rank zero* if any its self-adjoint element can be approximated by elements with finite spectra.

Proposition 1.3.2. *Let $\mathcal{A}$ be a C^*-algebra. A self-adjoint element $X \in \mathcal{A}$ can be approximated by self-adjoint elements with finite spectra if and only if $X - \lambda I$ can be approximated by invertible self-adjoint elements for all $\lambda \in \mathbb{R}$.*

For the proof, see [16, Remark 5.3]. Hence, Definition 1.3.1 is equivalent to saying that any self-adjoint element can be approximated by invertible self-adjoint elements.

Any von Neumann algebra (for example $\mathrm{M}_n(\mathbb{C})$ or $\mathcal{B}(H)$) has real rank zero, since we can use spectral projections for approximations. If $\mathcal{A}$ is a C^*-algebra of real rank zero and $\mathcal{I} \subset \mathcal{A}$ is a closed two-sided $*$-ideal, then the quotient algebra $\mathcal{A}/\mathcal{I}$ is also of real rank zero. Hence, the Calkin algebra $\mathcal{C}(H)$ has this property. If X is a topological space of positive dimension, then the algebra of continuous functions $C(X)$ is *not* of real rank zero, see [10, Section V.7].

In this chapter we assume that $\mathcal{A}$ is embedded into $\mathcal{B}(H)$, and $\mathrm{M}_2(\mathcal{A})$ is a subset of $\mathcal{B}(H \oplus H)$. Let $P\colon H \oplus H \to H$ be the projection onto the first component so that the following isomorphisms hold:

$$\mathcal{A} \cong P\mathrm{M}_2(\mathcal{A})P \cong (I-P)\mathrm{M}_2(\mathcal{A})(I-P).$$

If $T \in \mathrm{M}_2(\mathcal{A})$, then $[P,T]$ is the "off-diagonal" part of T (up to a sign). Let

$$\mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A}) \stackrel{\text{def}}{=\!=} \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in \mathrm{GL}_0(\mathcal{A}) \right\} \subset \mathrm{GL}_0(\mathrm{M}_2(\mathcal{A})).$$

Note that it is not necessarily the same as the set of all block diagonal elements of $\mathrm{GL}_0(\mathrm{M}_2(\mathcal{A}))$, since there may be no path from the element to the unit within the class of block diagonal elements. A simple example of such behaviour can be found in the Calkin algebra $\mathcal{C}(H)$ for a separable Hilbert space H. An invertible element of $\mathcal{C}(H)$ belongs to $\mathrm{GL}_0(\mathcal{C}(H))$ if and only the Fredholm index of its pre-image in $\mathcal{B}(H)$ is zero. For any invertible $A \in \mathcal{C}(H)$ we have

$$\begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \in \mathrm{GL}_0(\mathrm{M}_2(\mathcal{C}(H)))$$

(since the Fredholm index is additive, and $\mathrm{M}_2(\mathcal{C}(H)) = \mathcal{C}(H \oplus H)$), but, if the index function of A is not trivial, we do not have $A \in \mathrm{GL}_0(\mathcal{C}(H))$.

For $A \in \mathcal{A}$ and $T \in \mathrm{M}_2(\mathcal{A})$, let

$$d_1(A) \stackrel{\text{def}}{=\!=} \sup_{\lambda \in \mathbb{C}} \operatorname{dist}(A - \lambda I, \mathrm{GL}_0(\mathcal{A})),$$

$$d_2(T) \stackrel{\text{def}}{=\!=} \sup_{\lambda \in \mathbb{C}} \operatorname{dist}(T - \lambda I, \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})).$$

Note that for any $\lambda \in \mathbb{C}$

$$\|[P, T]\| = \operatorname{dist}(T - \lambda I, \mathcal{A} \oplus \mathcal{A}) \leqslant d_2(T).$$

For $0 < \delta \leqslant 2$, consider the following function.

$$G(\delta) = \ln(2 + \ln(2\delta^{-1})).$$

It is a slowly growing function of $1/\delta$ which will appear in some statements. The main technical result of this thesis is as follows.

Theorem 1.3.3. *Let $\mathcal{A}$ be a unital real rank zero C^*-algebra. There exist universal constants $C, C_0 > 0$ such that for all ε, δ_P satisfying $0 < \delta_P G(\delta_P) \leqslant C_0 \varepsilon$ and any normal element $T \in \mathrm{M}_2(\mathcal{A})$ satisfying $\|T\| \leqslant 1$, $d_2(T) \leqslant \delta_P$, there exists a normal element $T' \in \mathrm{M}_2(\mathcal{A})$ with*

$$\sigma(T') \subset \varepsilon\mathbb{Z} \times \varepsilon\mathbb{Z}, \quad \|T - T'\| \leqslant C\varepsilon, \quad \|[P, T']\| \leqslant C\delta_P.$$

The proof is given in Section 3.1. Roughly speaking, it can be thought of as an extension of [16, Theorem 2.1] with the additional control of the off-diagonal elements with respect to P.

The main application of Theorem 1.3.3 is the following quantitative version of Huaxin Lin's theorem.

Theorem 1.3.4. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero. There exists a universal constant $C > 0$ such that for any element $A \in \mathcal{A}$ satisfying $\|A\| \leqslant 1$, $\|[A, A^*]\| \leqslant \delta$, and $d_1(A) \leqslant \delta^{1/2}$, there exists a normal element $A' \in \mathcal{A}$ such that $\|A - A'\| \leqslant C\delta^{1/3}$ and $\|A'\| \leqslant \|A\|$.*

The proof consists of three steps. First, with the assistance of Theorem 1.2.2, we construct an approximate normal extension $T \in \mathrm{M}_2(\mathcal{A})$ with $d_2(T) \leqslant C\delta^{1/2}$ and, as a consequence, $\|[P, T]\| \leqslant C\delta^{1/2}$. In the second step, we apply Theorem 1.3.3 to the element T with $\delta_P = \delta^{1/2}$ to approximate it by a normal element T' with finite spectrum and $\|[P, T']\| \leqslant C\delta_P$. Finally, we remove off-diagonal elements in such a way that the element remains normal, see Lemma 3.3.2. This is where we get an additional loss and $\delta^{1/2}$ transforms into $\delta^{1/3}$. Since the new element is normal and commutes with P, we can use its first block with respect to P as the required normal approximation.

Some applications of the results are discussed in Section 3.4.

1.4 The case of the normalized Hilbert-Schmidt norm

The results of Chapter 4 are based on the paper [14] and are independent from Chapter 2 and 3. Recall that for $A \in \mathrm{M}_n(\mathbb{C})$, $A = \{A_{ij}\}_{i,j=1}^n$, we have defined

$$\|A\|_{2,n}^2 = \frac{1}{n}\sum_{i,j=1}^{n} |A_{ij}|^2. \tag{1.4.1}$$

We improve the scheme from [19] to obtain the following result.

Theorem 1.4.1. *Suppose that $X, Y \in \mathrm{M}_n(\mathbb{C})$, $\|X\|, \|Y\| \leqslant 1$, $X = X^*$, $Y = Y^*$. Let $\|[X,Y]\|_{2,n} = \delta \leqslant \frac{1}{16}$. Then there exist $X', Y' \in \mathrm{M}_n(\mathbb{C})$ such that $\|X'\|, \|Y'\| \leqslant 1$, $X' = X'^*$, $Y' = Y'^*$, and*

$$[X', Y'] = 0, \quad \|X - X'\|_{2,n} \leqslant 2\,\delta^{1/4},\ \|Y - Y'\|_{2,n} \leqslant 2\,\delta^{1/4}.$$

In addition, $[X, X'] = 0$.

Theorem 1.4.1 is a particular case of the following theorem regarding m-tuples of self-adjoint operators (which is also a refinement of a result from [19]).

Theorem 1.4.2. *Let $m \geqslant 3$, $X_j = X_j^* \in \mathrm{M}_n(\mathbb{C})$, $\|X_j\| \leqslant 1$ for $j = 1, \ldots, m$. Suppose that $\|[X_i, X_j]\|_{2,n} \leqslant \delta$ for $i, j = 1, \ldots, m$, and let also*

$$\delta \leqslant \frac{1}{16^{2\cdot 4^{m-2}}}. \tag{1.4.2}$$

Then there exist $X_i' \in \mathrm{M}_n(\mathbb{C})$, $i = 1, \ldots, m$, such that

$$\|X_j'\| \leqslant 1, \quad X_j' = X_j'^*, \quad \|X_j - X_j'\|_{2,n} \leqslant 5\delta^{1/4^{m-1}},\ j = 1, \ldots, m,$$

and

$$[X_i', X_j'] = 0, \quad i, j = 1, \ldots, m.$$

In addition, $[X_1, X_1'] = 0$.

Remark 1.4.3. The result of Theorem 1.4.1 is worse than the result for the operator norm. It is likely that Theorem 1.3.4 can be extended to the case of the norm (1.4.1) in the same way as in [16], this may be a subject for future research. Unlike Theorem 1.3.4, the proof of Theorem 1.4.1 is rather elementary.

1.5 Polynomials of almost normal elements in C^*-algebras

The results of Chapter 5 are based on the paper [15] and are independent from the previous chapters. Let $\mathcal{A}$ be an arbitrary unital C^*-algebra, and suppose that $A \in \mathcal{A}$ is normal. It is well known that there exists a unique C^*-algebra homomorphism

$$C(\sigma(A)) \to \mathcal{A}, \quad f \mapsto f(A)$$

from the algebra of continuous functions on the spectrum $\sigma(A)$ into $\mathcal{A}$ such that $f(z) = z$ is mapped into A, $\sigma(f(A)) = f(\sigma(A))$, and

$$\|f(A)\| = \max_{z\in\sigma(A)} |f(z)| \tag{1.5.1}$$

(see, for example, [13]). It is called the continuous functional calculus for normal elements.

The aim of Chapter 5 is to introduce an analogue of functional calculus for "almost normal" elements. More precisely, we shall always be assuming that

$$\|A\| \leqslant 1, \quad \|[A, A^*]\| \leqslant \delta \tag{1.5.2}$$

with a small δ. We restrict the considered class of functions to polynomials in z and $\bar{z}$ and show that some important properties of the functional calculus hold up to an error of order δ.

If $AA^* \neq A^*A$ then the polynomials of A and A^* are, in general, not uniquely defined. We fix the following definition. For a polynomial

$$p(z, \bar{z}) = \sum_{k,l} p_{kl} z^k \bar{z}^l \tag{1.5.3}$$

let

$$p(A, A^*) = \sum_{k,l} p_{kl} A^k (A^*)^l. \tag{1.5.4}$$

It is clear that the map $p \mapsto p(A, A^*)$ is linear and involutive, that is $\overline{p}(A, A^*) = p(A, A^*)^*$ where $\bar{p}(z, \bar{z}) = \sum \bar{p}_{lk} z^k \bar{z}^l$. Using the inequality

$$\|[A, B^m]\| \leqslant m\|B\|^{m-1}\|[A, B]\|$$

and (4.1.3), one can easily show that the map $p \mapsto p(A, A^*)$ is "almost multiplicative",

$$\|p(A, A^*)q(A, A^*) - (pq)(A, A^*)\| \leqslant C(p, q)\,\delta \tag{1.5.5}$$

where

$$C(p, q) = \sum_{k,l,s,t} ls\,|p_{kl}|\,|q_{st}|\,.$$

It takes much more effort to obtain an estimate of the norm $\|p(A, A^*)\|$. Let

$$p_{\max} \stackrel{\text{def}}{=\!=} \max_{|z|\leqslant 1} |p(z, \bar{z})|. \tag{1.5.6}$$

In the case of an analytic polynomial $p(z) = \sum_k p_k z^k$, according to von Neumann's inequality, we have

$$\|p(A)\| \leqslant p_{\max}$$

where it is only assumed that $\|A\| \leqslant 1$ (see, for example, [36, I.9]).

Our main results are as follows.

Theorem 1.5.1. *Let p be a polynomial* (1.5.3). *There exists a constant $C(p)$ such that the estimate*

$$\|p(A, A^*)\| \leqslant p_{\max} + C(p)\delta \tag{1.5.7}$$

holds for all A satisfying (1.5.2)*.*

If A is normal and f is a continuous function then the functional calculus gives the following more precise estimate,

$$\|f(A)\| = \max_{z\in\sigma(A)} |f(z)|. \tag{1.5.8}$$

If $A \in \mathcal{A}$ and $\lambda_j \notin \sigma(A)$, $j = 1, \ldots, m-1$, then there exists $R_j > 0$ such that

$$\|(A - \lambda_j I)^{-1}\| \leqslant R_j^{-1},\ j = 1, \ldots, m-1. \tag{1.5.9}$$

The following theorem gives an analogue of (1.5.8) for almost normal elements A.

Theorem 1.5.2. *Let $A \in \mathcal{A}$ satisfy* (1.5.2) *and* (1.5.9), *and suppose that the set*

$$S = \{z \in \mathbb{C}\colon |z| \leqslant 1,\ |z - \lambda_j| \geqslant R_j,\ j = 1, \ldots, m-1\} \tag{1.5.10}$$

is not empty. For each $\varepsilon > 0$ and each polynomial p defined by (1.5.3) *there exists a constant $C(p, \varepsilon)$ independent of A such that*

$$\|p(A, A^*)\| \leqslant \max_{z \in S} |p(z, \bar{z})| + \varepsilon + C(p, \varepsilon)\delta. \tag{1.5.11}$$

Under the assumptions of Theorem 1.5.2, the set S is a unit disk with $m-1$ "holes" such that $\sigma(A) \subset S$. Note that if $S = \varnothing$, then we can decrease R_j to make it non-empty.

Finally, assume again that A is normal and $\mu \notin f(\sigma(A))$. Then the functional calculus implies that the element $f(A) - \mu I$ is invertible and

$$\left\|(f(A) - \mu I)^{-1}\right\| = \frac{1}{\operatorname{dist}\,(\mu, f(\sigma(A)))}. \tag{1.5.12}$$

The equality (1.5.12) also admits the following approximate analogue with $\sigma(A)$ replaced by S and $f(\sigma(A))$ by $p(S)$, where $p(S)$ is the image of S under p considered as a map from $\mathbb{C}$ to $\mathbb{C}$.

Theorem 1.5.3. *Let S be defined by* (1.5.10), *and let p be a polynomial* (1.5.3). *Then for each $\varepsilon > 0$ and $\varkappa > 0$ there exist constants $C(p, \varkappa, \varepsilon)$, $\delta_0(p, \varkappa, \varepsilon)$ such that for all $\delta < \delta_0(p, \varkappa, \varepsilon)$ and for all $\mu \in \mathbb{C}$ satisfying* $\operatorname{dist}(\mu, p(S)) \geqslant \varkappa$ *the estimate*

$$\|(p(A, A^*) - \mu I)^{-1}\| \leqslant \varkappa^{-1} + \varepsilon + C(p, \varkappa, \varepsilon)\delta \tag{1.5.13}$$

holds for all $A \in \mathcal{A}$ satisfying (1.5.2) *and* (1.5.9).

Remark 1.5.4. The estimates (1.5.11), (1.5.13) only make sense as $\delta \to 0$. The rate of decay of the terms $\varepsilon + C(p, \varepsilon)\delta$ and $\varepsilon + C(p, \varkappa, \varepsilon)\delta$ after choosing an optimal ε depends on the rate of growth of the constants $C(p, \varepsilon)$, $C(p, \varkappa, \varepsilon)$ as $\varepsilon \to 0$, $\varkappa \to 0$, and as the coefficients and the degree of p increase. This rate is rather fast, but the constants are obtained using a certain constructive procedure and can, in principle, be determined.

The situation with Theorem 1.5.1 is different. There is no ε in the right hand side, hence the behaviour in δ is linear. However, for $C(p)$ we are only aware of existence-type results.

The interest to the subject was drawn by its relation with Lin's theorem. An optimal estimate in this theorem (see Introduction) would lead to analogues of Theorems 1.5.1, 1.5.2, 1.5.3 with $\delta^{1/2}$ in the right hand side. The result of Theorem 1.5.1 is better; note that, as mentioned before, the presence of $C(p,\varepsilon)$ in the other theorems destroys the power behaviour in δ. More importantly, these theorems do not require any additional assumptions on $\mathcal{A}$ or A. Initially, this was considered as a possible different approach to the proof of Lin's theorem.

The proofs are based on certain representation theorems for positive polynomials. If a real polynomial of x_1, x_2 is non-negative on the unit disk $\{x : x_1^2 + x_2^2 < 1\}$ then, by a result of [34], it admits a representation

$$\sum_j r_j(x)^2 + \left(1 - x_1^2 - x_2^2\right) \sum_j s_j(x)^2 \tag{1.5.14}$$

with real polynomials r_j and s_j (see Proposition 5.4.2 below). Representations similar to (1.5.14) are usually referred to as *Positivstellensatz*. We also make use of Positivstellensatz for polynomials positive on the sets (1.5.10). The corresponding results for sets bounded by arbitrary algebraic curves were obtained in [7, 32, 33, 34].

In order to prove Theorem 1.5.3, we need uniform with respect to μ estimates for polynomials appearing in Positivstellensatz-type representations. In order to obtain the estimates, we use the scheme introduced in [35, 28].

Chapter 2

Extensions of almost commuting elements

2.1 Operator Lipschitz functions

We will need to introduce two important classes of functions (see [1, 2, 29]). Let H be an infinite-dimensional Hilbert space, not necessarily separable. By $\mathrm{OL}(\mathbb{R})$ we denote the space of all continuous complex-valued functions such that the following quantity is finite:

$$\|f\|_{\mathrm{OL}(\mathbb{R})} = \sup_{A_1, A_2} \frac{\|f(A_1) - f(A_2)\|}{\|A_1 - A_2\|},$$

where the supremum is taken over all *self-adjoint* operators $A_1, A_2 \in \mathcal{B}(H)$, $A_1 \neq A_2$. Note that the elements of $\mathrm{OL}(\mathbb{R})$ are automatically Lipschitz. The converse, however, does not always hold.

For continuous functions $g\colon \mathbb{C} \to \mathbb{C}$, consider also the space $\mathrm{OL}(\mathbb{C})$ with

$$\|g\|_{\mathrm{OL}(\mathbb{C})} = \sup_{N_1, N_2} \frac{\|g(N_1) - g(N_2)\|}{\|N_1 - N_2\|},$$

where the supremum is now taken over all *normal* operators $N_1, N_2 \in \mathcal{B}(H)$, $N_1 \neq N_2$. Both spaces $\mathrm{OL}(\mathbb{R})$, $\mathrm{OL}(\mathbb{C})$ are linear complex quasi-Banach spaces. Only constant functions have zero quasi-norms.

Proposition 2.1.1. *Suppose that the Hilbert space H is infinite-dimensional. Then, for continuous f and g, the norms $\|f\|_{\mathrm{OL}(\mathbb{R})}$, $\|g\|_{\mathrm{OL}(\mathbb{C})}$ do not depend on H.*

Proof. Let us consider the case of $\mathrm{OL}(\mathbb{R})$; the case of $\mathrm{OL}(\mathbb{C})$ is similar. Let $H' \subset H$ be a separable infinite-dimensional subspace of H, and $A_1, A_2 \in \mathcal{B}(H)$, $A_1 = A_1^*$,

$A_2 = A_2^*$. Suppose that f is a continuous function on $\mathbb{R}$. Consider the C^*-algebra generated by A_1, A_2, and the identity operator. This algebra is separable. Hence, it is isomorphic to a subalgebra of $\mathcal{B}(H_0)$ for some separable Hilbert space H_0, and, as a consequence, to a subalgebra of $\mathcal{B}(H')$ (see [10, Theorem I.9.12]). Therefore, there exist operators $A_1', A_2' \in \mathcal{B}(H')$ such that

$$\|A_1 - A_2\| = \|A_1' - A_2'\|, \quad \|f(A_1) - f(A_2)\| = \|f(A_1') - f(A_2')\|.$$

Thus, in the definition of $\|f\|_{\mathrm{OL}(\mathbb{R})}$ it suffices to take the supremum over operators acting in the separable subspace H'. ∎

The following important proposition is proved by Alexandrov and Peller in [3, Theorem 3.1].

Proposition 2.1.2. *Let $P \in \mathcal{B}(H)$ be an orthogonal projection. If $f \in \mathrm{OL}(\mathbb{R})$, then for any $A = A^* \in \mathcal{B}(H)$*

$$\|[P, f(A)]\| \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})} \|[P, A]\|.$$

If $g \in \mathrm{OL}(\mathbb{C})$, then for any normal $N \in \mathcal{B}(H)$

$$\|[P, g(N)]\| \leqslant \|g\|_{\mathrm{OL}(\mathbb{C})} \|[P, N]\|.$$

The following two simple properties will be important in later considerations.

Lemma 2.1.3. *1. Let $f \in \mathrm{OL}(\mathbb{R}^n)$, $n = 1, 2$. Then $f_1(x) = f(\lambda x + \mu)$, where $\lambda > 0$, $\mu \in \mathbb{R}^n$, also belong to $\mathrm{OL}(\mathbb{R}^n)$, and*

$$\|f_1\|_{\mathrm{OL}(\mathbb{R}^n)} = \lambda \|f\|_{\mathrm{OL}(\mathbb{R}^n)}.$$

2. Let $f, g \in \mathrm{OL}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, $n = 1, 2$. Then

$$\|fg\|_{\mathrm{OL}(\mathbb{R}^n)} \leqslant \|f\|_{L_\infty(\mathbb{R}^n)} \|g\|_{\mathrm{OL}(\mathbb{R}^n)} + \|g\|_{L_\infty(\mathbb{R}^n)} \|f\|_{\mathrm{OL}(\mathbb{R}^n)}$$

Proof. Part 1 follows from the definition if we replace A_i, N_i by $\lambda A_i + \mu I$, $\lambda N_i + \mu I$. Part 2 follows from the estimate

$$\begin{aligned} &\|f(A_1)g(A_1) - f(A_2)g(A_2)\| = \\ &\quad = \|f(A_1)g(A_1) - f(A_1)g(A_2) + f(A_1)g(A_2) - f(A_2)g(A_2)\| \\ &\qquad \leqslant \|f(A_1)\| \|g(A_1) - g(A_2)\| + \|g(A_2)\| \|f(A_1) - f(A_2)\|. \end{aligned}$$ ∎

It is clear that linear functions belong to OL. A wider class of operator Lipschitz functions can be described as follows.

Definition 2.1.4. A complex-valued function $f = f(x)$ belongs to the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$ if the following norm is finite:

$$\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)} = \|f\|_{L_\infty(\mathbb{R}^n)} + \int\limits_{\mathbb{R}^n} \frac{\sup_{x\in\mathbb{R}^n} |f(x+2h) - 2f(x+h) + f(x)|}{|h|^{n+1}}\, dh < +\infty. \tag{2.1.1}$$

The following proposition was proved in [2, 29, 1].

Proposition 2.1.5. *For $n = 1, 2$ we have $B^1_{\infty,1}(\mathbb{R}^n) \subset \mathrm{OL}(\mathbb{R}^n)$, and*

$$\|f\|_{\mathrm{OL}(\mathbb{R}^n)} \leqslant C\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)}, \quad \forall f \in B^1_{\infty,1}(\mathbb{R}^n).$$

Lemma 2.1.6. *The space $B^1_{\infty,1}$ has the following properties.*

1. *Assume that*

$$\sup_{x\in\mathbb{R}^n} |f(x)| \leqslant C_1, \qquad \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|(\nabla f)(x) - (\nabla f)(y)|}{|x-y|} \leqslant C_2.$$

Then $f \in B^1_{\infty,1}(\mathbb{R}^n)$ and $\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)} \leqslant C(n)(C_1 + C_2)$. As a corollary, $f \in \mathrm{OL}(\mathbb{R}^n)$ with the same quasi-norm estimate (for $n = 1, 2$).

2. *Suppose that $f \in C_0^\infty(\mathbb{R}^n)$. Let $f_i(x) = f(x - \lambda_i)$, $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$. Assume also that $\operatorname{dist}(\operatorname{supp} f_i, \operatorname{supp} f_j) \geqslant \varepsilon$ for $i \neq j$. Then*

$$\left\| \sum_{i=1}^N f_i \right\|_{B^1_{\infty,1}(\mathbb{R}^n)} \leqslant C(n)\varepsilon^{-1}\|f\|_{L_\infty(\mathbb{R}^n)} + \|f\|_{B^1_{\infty,1}(\mathbb{R}^n)}.$$

Proof. Property 1 is proved by splitting the integral in the norm into two parts. In the integral over $|h| \geqslant 1$, f is estimated by C_1, and the estimate for the integral over $|h| \leqslant 1$ is a direct corollary of

$$|f(x+2h) - 2f(x+h) + f(x)| \leqslant 2C_2|h|^2.$$

To prove Property 2, let us split the integral (2.1.1) into integrals over $|h| \leqslant \varepsilon/3$ and over $|h| \geqslant \varepsilon/3$. The first integral is bounded by $\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)}$ since it only contains expressions of the form $f_i(x+2h) - 2f_i(x+h) + f_i(x)$ for some i. The remaining integral is bounded by

$$\int\limits_{|h|\geqslant\varepsilon/3} \frac{3\|f\|_{L_\infty(\mathbb{R}^n)}\, dh}{|h|^{n+1}} \leqslant C(n)\varepsilon^{-1}\|f\|_{L_\infty(\mathbb{R}^n)}. \blacksquare$$

The function $|z|$ is an example of a Lipschitz, but not operator Lipschitz function. Still, it is "almost" operator Lipschitz with an additional logarithmic factor. The following is Theorem 6.7 from [2].

Proposition 2.1.7. *Let S, T be bounded operators in a Hilbert space. Then*

$$\||S| - |T|\| \leqslant C\|S - T\| \ln \left(2 + \ln \frac{\|S\| + \|T\|}{\|S - T\|}\right).$$

We will also need some statements regarding "diagonal truncations". Let $P \in \mathcal{B}(H)$ be an orthogonal projection. We will often use the following truncation operation.

$$\operatorname{diag}_P A \stackrel{\text{def}}{=\!=} PAP + (I - P)A(I - P) = A - [P, [P, A]]. \tag{2.1.2}$$

Note that $\|[P, [P, A]]\| \leqslant \|[P, A]\|$, hence $\|A - \operatorname{diag}_P A\| \leqslant \|[P, A]\|$.

Proposition 2.1.8. *Let $f \in \mathrm{OL}(\mathbb{R})$, and assume that $A = A^* \in \mathcal{B}(H)$. Let P be an orthogonal projection. Then*

$$\| \operatorname{diag}_P f(A) - f(\operatorname{diag}_P A)\| \leqslant 2\|f\|_{\mathrm{OL}(\mathbb{R})}\|[P, A]\|.$$

Proof. The estimate follows from two inequalities,

$$\|f(A) - f(\operatorname{diag}_P A)\| \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})}\|A - \operatorname{diag}_P A\| \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})}\|[P, A]\|,$$

$$\| \operatorname{diag}_P f(A) - f(A)\| \leqslant \|[P, f(A)]\| \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})}\|[P, A]\|. \blacksquare$$

The following auxiliary lemma gives the precise value of $\|e^{itx}\|_{\mathrm{OL}(\mathbb{R})}$.

Lemma 2.1.9. *Let $A, B \in \mathcal{B}(H)$ be self-adjoint. Then $\|[A, e^{itB}]\| \leqslant t\|[A, B]\|$.*

Proof. Let

$$G(t) = e^{-itB}Ae^{itB} - A.$$

We have $\|G(t)\| = \|[A, e^{itB}]\|$. Hence,

$$\|G(t)\| \leqslant \int_0^t \|G'(s)\|\, ds = \int_0^t \| - ie^{-isB}[A, B]e^{isB}\|\, ds = t\|[A, B]\|. \blacksquare$$

The next lemma is an "improved" version of Proposition 2.1.8 for $f(x) = e^{itx}$.

Lemma 2.1.10. *Suppose $A = A^* \in \mathcal{B}(H)$, and let P be an orthogonal projection. Then*

$$\| \operatorname{diag}_P e^{itA} - e^{it \operatorname{diag}_P A}\| \leqslant \|[P, A]\|^2 t^2/2. \tag{2.1.3}$$

Proof. Since the statement is symmetric under interchanging P and $I - P$, it would suffice to prove

$$\|Pe^{itA}P - Pe^{it(PAP)}P\| \leqslant \|[P, A]\|^2 t^2/2.$$

Similarly to the previous lemma, let

$$G(t) = Pe^{itA}Pe^{-itPAP} - P = (Pe^{itA}P - Pe^{itPAP}P)e^{-itPAP}.$$

Then, since e^{itPAP} is unitary,

$$\begin{aligned}\|Pe^{itA}P - Pe^{it(PAP)}P\| &= \|G(t)\| \leqslant \int_0^t \|G'(s)\|\, ds \\ &= \int_0^t \|iPe^{isA}APe^{-isPAP} - iPe^{isA}PAPe^{-isPAP}\|\, ds \\ &= \int_0^t \|iPe^{isA}(AP - PAP)e^{-isPAP}\|\, ds \\ &= \int_0^t \|[P, e^{isA}](AP - PA)P\|\, ds \leqslant \|[P, A]\|^2 t^2/2\end{aligned}$$

by Lemma 2.1.9. ∎

Remark 2.1.11. It is likely that Lemma 2.1.10 can be generalized to arbitrary functions from $B^2_{\infty,1}(\mathbb{R})$ using triple operator integrals, see [30]. The conjecture is that the following estimate holds,

$$\|\operatorname{diag}_P f(A) - f(\operatorname{diag}_P A)\| \leqslant C\|f\|_{B^2_{\infty,1}(\mathbb{R})}\|[P, A]\|^2.$$

This extension lies beyond the scope of the thesis.

2.2 Proof of Theorem 1.2.2

By Gelfand-Naimark-Segal theorem, we may assume that $\mathcal{A}$ is a sub-algebra of $\mathcal{B}(H)$ for some (not necessarily separable) Hilbert space H. Since X can be approximated by elements with finite spectra with any precision, we can assume that X has finite spectrum and all its spectral projections of X belong to $\mathcal{A}$, and then apply approximation arguments. Since the original construction of [9] relies only on spectral projections of X, it can be extended to our case with minimal changes. Still, we give a complete proof for the convenience of the reader. Instead of estimating the commutators directly (as in [9]), we apply the results of Section 2.1. This simplifies the corresponding steps in the proof.

We fix the following notation for the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx, \quad \check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi) e^{i\xi x}\, d\xi. \tag{2.2.1}$$

Lemma 2.2.1. *For any $\varepsilon > 0$ there exists a function $f_\varepsilon \in L_1(\mathbb{R})$ such that f_ε is continuous everywhere except at 0, $\hat{f}_\varepsilon(x) = x^{-1}$ for $|x| \geqslant \varepsilon$, $\|f_\varepsilon\|_{L_1(\mathbb{R})} \leqslant C\varepsilon^{-1}$, and the values of f are pure imaginary.*

Proof. Let $g \in C^\infty(\mathbb{R})$ be a non-negative even function such that $g(\xi) = 1$ for $|\xi| \geqslant 1$ and $g(\xi) = 0$ for $|\xi| \leqslant \frac{1}{2}$. Let f_ε be the inverse Fourier transform of $g(\xi/\varepsilon)/\xi$ (in the sense of distributions). Note that the inverse Fourier transform of $g(\xi/\varepsilon)$ is the sum of $\delta(x)$ and of a smooth real-valued function. Therefore, f_ε is smooth everywhere except zero and its values are pure imaginary. It is also easy to see that $f_1 \in L_1(\mathbb{R})$. To complete the proof, we observe that $f_\varepsilon(x) = f_1(\varepsilon x)$. ∎

By $E_X(\Delta)$ we denote the spectral projection of a self-adjoint operator X corresponding to a Borel set $\Delta \subset \mathbb{R}$.

Lemma 2.2.2. *Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that $X, Y \in \mathcal{A}$ are self-adjoint elements such that $\|[X,Y]\| \leqslant \delta$ and that X has finite spectrum. Let $\varepsilon > 0$,*

$$a_0 = -\|X\| < a_1 < \ldots < a_n = \|X\|, \quad a_{i+1} - a_i \geqslant \varepsilon \ \textit{for}\ i = 1, \ldots, n.$$

Let also $I_i = [a_i; a_{i+1})$. Then there exists a self-adjoint element $\widetilde{Y} \in \mathcal{A}$ such that

$$\|[X, \widetilde{Y}]\| \leqslant 3\delta, \quad \|Y - \widetilde{Y}\| \leqslant C\delta\varepsilon^{-1}, \ \textit{and}$$

$$E_X(I_i)\widetilde{Y}E_X(I_j) = 0 \quad \textit{for} \quad |i-j| \geqslant 2. \tag{2.2.2}$$

Proof. Assume, as always, that $\mathcal{A} \subset \mathcal{B}(H)$ for a Hilbert space H. Let

$$Y' = \sum_{|i-j|\leqslant 1} E_j Y E_j, \quad \text{where} \quad E_j \stackrel{\text{def}}{=} E_X(I_j),$$

be the "block tri-diagonal truncation" of Y. Note that $E_i \in \mathcal{A}$ because of our assumption on the spectrum of X. We have

$$[X, E_j Y E_j] = E_j [X, Y] E_j.$$

Hence, $\|[X,Y']\| \leqslant 3\delta$ and

$$\|[X,Y-Y']\| \leqslant \|[X,Y]\| + \|[X,Y']\| \leqslant 4\delta.$$

Let

$$Q = \int\limits_{\mathbb{R}} e^{-isX}[X,Y-Y']e^{isX} f_\varepsilon(s)\,ds,$$

where f_ε is the function from Lemma 2.2.1. It is clear that $\|Q\| \leqslant C\delta\varepsilon^{-1}$. Since $f_\varepsilon(s)$ is pure imaginary for all s, we have $Q = Q^*$.

We claim that $[X, Y-Y'-Q] = 0$. Indeed, let u, v be two eigenvectors of X, $Xu = \lambda u$, $Xv = \mu v$. Then

$$\begin{aligned}
([X,Q]u,v) &= \\
&= \int\limits_{\mathbb{R}} \left\{([X,Y-Y']e^{isX}u, Xe^{isX}v) - ([X,Y-Y']Xe^{isX}u, e^{isX}v)\right\} f_\varepsilon(s)ds \\
&= \int\limits_{\mathbb{R}} e^{is(\lambda-\mu)}(\mu-\lambda)([X,Y-Y']u,v) f_\varepsilon(s)ds \\
&= (\mu-\lambda)\hat{f}_\varepsilon(\mu-\lambda)([X,Y-Y']u,v). \quad (2.2.3)
\end{aligned}$$

If $|\lambda-\mu| < \varepsilon$, then

$$([X,Y-Y']u,v) = (\mu-\lambda)((Y-Y')u,v) = 0$$

since Y' is block tri-diagonal. From (2.2.3), this implies $([X,Q]u,v) = 0$. If $|\lambda-\mu| \geqslant \varepsilon$, then $(\mu-\lambda)\hat{f}_\varepsilon(\mu-\lambda) = 1$, and (2.2.3) gives

$$([X,Q]u,v) = ([X,Y-Y']u,v).$$

Therefore, the last equality holds for *all* pairs of eigenvectors of X, and thus $[X, Y-Y'-Q] = 0$.

Let $\widetilde{Y} = Y - Q$. Suppose, as before, $Xu = \lambda u$, $Xv = \mu v$, and $|\lambda-\mu| > \varepsilon$. Then, similarly to (2.2.3),

$$\begin{aligned}
((Y-Q)u,v) &= (Yu,v) - \hat{f}_\varepsilon(\mu-\lambda)([X,Y-Y']u,v) \\
&= (Yu,v) - (\mu-\lambda)\hat{f}_\varepsilon(\mu-\lambda)\,((Y-Y')u,v) = (Y'u,v) = 0.
\end{aligned}$$

We have established (2.2.2), and thus $\widetilde{Y}$ satisfies the statement of the lemma, since $\|[X,\widetilde{Y}]\| = \|[X,Y']\| \leqslant 3\delta$. ∎

Proof of Theorem 1.2.2. Let $A = X + iY$, and suppose that $\|X\| \leqslant 1$, $\|Y\| \leqslant 1$, $X = X^*$, $Y = Y^*$, $\|[X, Y]\| \leqslant \delta$, and that X has finite spectrum. Hence, we have $f(X) \in \mathcal{A}$ for any Borel function f. Without loss of generality, we can assume that $\delta \leqslant 1$.

By $[x]$, let us denote the integer part of a real number x. Let

$$\delta = \varepsilon^2, \quad n = [2/\varepsilon], \quad a_j = -1 + 2j/n, \quad 0 \leqslant j \leqslant n.$$

Then $\varepsilon \leqslant a_{j+1} - a_j = 2/n \leqslant 2\varepsilon$. Let $I_j = [a_j; a_{j+1})$ for $0 \leqslant j \leqslant n-2$, and $I_{n-1} = [a_{n-1}; a_n]$. For $i = 0, \ldots, n-1$ consider the following functions

$$f_i(t) = \begin{cases} 0, & t \leqslant a_i; \\ \sin^2\left(\frac{\pi n}{4}(t - a_i)\right), & a_i < t \leqslant a_{i+1}; \\ 1, & t > a_{i+1}. \end{cases}$$

Let also

$$g_1(s) = \cos^2(\pi s/2),$$

$$g_2(s) = \sin^2(\pi s/2),$$

$$g_3(s) = \sin(\pi s/2)\cos(\pi s/2).$$

The element

$$P_i = \begin{pmatrix} g_1(f_i(X)) & g_3(f_i(X)) \\ g_3(f_i(X)) & g_2(f_i(X)) \end{pmatrix} \in \mathrm{M}_2(\mathcal{A})$$

is an orthogonal projection. Let $E_j = E_X(I_j)$ and

$$L_i = P_i + \sum_{j<i} (0 \oplus E_j) - \sum_{j>i} (0 \oplus E_j), \quad i = 0, \ldots, n-1. \tag{2.2.4}$$

Let also $L_n \stackrel{\text{def}}{=\!=} I \oplus I$ be the identity operator. The elements L_i are orthogonal projections satisfying

$$\sum_{j<i} (E_j \oplus E_j) \leqslant L_i \leqslant \sum_{j \leqslant i} (E_j \oplus E_j). \tag{2.2.5}$$

Let us informally describe the structure of L_j. We can consider P_i as a matrix-valued function of X. On the spectral intervals $[a_{j-1}; a_j)$ for $j \leqslant i$ it equals to $E_{j-1} \oplus 0$. On the interval $[a_i; a_{i+1})$ it is a certain average between $E_i \oplus 0$ and $0 \oplus E_i$, continuously depending on X. And on $[a_j; a_{j+1})$ for $j > i$ it equals to $0 \oplus E_j$. The projection

L_i is a modified version of P_i in such a way that it equals to $E_i \oplus E_i$ before a_i and to $0 \oplus 0$ after a_{i+1}; hence, we have (2.2.5). Note that L_j is no longer a continuous function of X, but the discontinuities are in some sense "concentrated" only in the second component of $H \oplus H$. We will take advantage of it in (2.2.7).

Let us apply Lemma 2.2.2 to X, Y and obtain an element $\widetilde{Y}$ satisfying $\|Y - \widetilde{Y}\| \leqslant C\delta\varepsilon^{-1} = C\delta^{1/2}$. Consider

$$W = \sum_{i=0}^{n-1} E_i \widetilde{Y} E_i.$$

Since $[X, W]$ is a block diagonal truncation of $[X, \widetilde{Y}]$, we have

$$\|[X, W]\| \leqslant \|[X, \widetilde{Y}]\| \leqslant 3\delta. \tag{2.2.6}$$

The element $\widetilde{Y} \oplus W$ commutes with the second and third terms of (2.2.4). We claim that

$$\begin{aligned} &\|[L_j, \widetilde{Y} \oplus W]\| = \|[P_j, \widetilde{Y} \oplus W]\| = \\ &= \left\| \begin{pmatrix} [g_1(f_j(X)), \widetilde{Y}] & g_3(f_j(X))W - \widetilde{Y} g_3(f_j(X)) \\ g_3(f_j(X))\widetilde{Y} - W g_3(f_j(X)) & [g_2(f_j(X)), W] \end{pmatrix} \right\| \leqslant Cn\delta. \end{aligned} \tag{2.2.7}$$

It is easy to check that the derivative of $g_1(f_j(\cdot))$ is Lipschitz, and the function itself is bounded. Therefore, from Lemma 2.1.6, it belongs to $\mathrm{OL}(\mathbb{R})$. For different n, these functions are obtained from each other by scaling. Therefore,

$$\|g_1(f_j(\cdot))\|_{\mathrm{OL}(\mathbb{R})} \leqslant Cn,$$

and the estimate for the top left entry now follows from Proposition 2.1.2. Using (2.2.6), we can apply the same arguments for the bottom right element.

Let us estimate the top right element. The function $g_3(f_j(\cdot))$ belongs to $\mathrm{OL}(\mathbb{R})$ (again from Lemma 2.1.6), and the same scaling arguments imply

$$\|[W, g_3(f_j(X))]\| \leqslant Cn\delta.$$

It now suffices to estimate $(W - \widetilde{Y})g_3(f_j(X))$. We have

$$g_3(f_j(X)) = E_j g_3(f_j(X)) E_j.$$

Hence,

$$(W - \widetilde{Y})g_3(f_j(X)) = [(W - \widetilde{Y}), g_3(f_j(X))]E_j,$$

and the estimate again follows from Proposition 2.1.2. The bottom left element is a conjugate to the top right one (up to a sign). This completes the proof of (2.2.7).

Let us finally construct the normal approximant. Take

$$Z = \sum_{i=0}^{n-1} \lambda_i E_i, \quad X_1 = \sum_{j=0}^{n-1} a_j (L_{j+1} - L_j),$$

where $\lambda_i = (a_i + a_{i+1})/2$. For $i = 0, \ldots, n-1$, let

$$F_{2i} = L_i - \sum_{j<i} (E_j \oplus E_j) = (E_i \oplus E_i) L_i,$$

$$F_{2i+1} = \sum_{j \leqslant i} (E_i \oplus E_i) - L_i = E_i \oplus E_i - F_{2i}.$$

From (2.2.5), F_i are mutually orthogonal projections, and

$$\sum_{i=0}^{2n-1} F_i = I \oplus I.$$

Note also that

$$L_{j+1} - L_j = F_{2j+2} + F_{2j+1} \tag{2.2.8}$$

are also mutually orthogonal projections.

We have $F_i X_1 F_j = F_i (X \oplus Z) F_j = 0$ for $|i - j| \geqslant 2$. For other i, j,

$$\|F_i(X_1 - (X \oplus Z))F_j\| \leqslant \|F_i(X_1 - a_i)F_j\| + \|F_i((X \oplus Z) - a_i)F_j\| \leqslant 4\varepsilon.$$

This implies

$$\|X_1 - (X \oplus Z)\| \leqslant 3 \max_{i,j} \|F_i(X_1 - (X \oplus Z))F_j\| \leqslant 12\varepsilon.$$

Let also

$$Y_1 = \sum_{j=0}^{n-1} (L_{j+1} - L_j)(\widetilde{Y} \oplus W)(L_{j+1} - L_j).$$

Since $E_{j+2} \widetilde{Y} E_j = E_j \widetilde{Y} E_{j+2} = 0$ from (2.2.2), and $[W, E_j] = 0$ from (2.2.5), we have

$$(L_{j+1} - L_j)(\widetilde{Y} \oplus W)(L_{i+1} - L_i) = 0 \quad \text{for} \quad |i - j| \geqslant 3.$$

Hence,

$$Y_1 - (\widetilde{Y} \oplus W) = \sum_{1 \leqslant |i-j| \leqslant 3} (L_{i+1} - L_i)(\widetilde{Y} \oplus W)(L_{j+1} - L_j). \tag{2.2.9}$$

Since $L_{j+1} - L_j$ are orthogonal projections, (2.2.9) implies

$$\|Y_1 - (\widetilde{Y} \oplus W)\| \leqslant 3 \max_{i,j\,:\,|i-j|\geqslant 1} \|(L_{i+1} - L_i)(\widetilde{Y} \oplus W)(L_{j+1} - L_j)\|$$
$$\leqslant 6 \max_j \|[L_j, \widetilde{Y} \oplus W]\| \leqslant C\delta\varepsilon^{-1}.$$

By construction, $[X_1, Y_1] = 0$. Therefore, the element $T = X_1 + iY_1$ is normal and

$$\|(X + iY) \oplus N - T\| \leqslant C\delta^{1/2},$$

where $N = Z + iW$ is also normal. Moreover, $\sigma(Z + iW) \subset \{\lambda_0, \lambda_1, \ldots, \lambda_n\} + i[-1; 1]$, and so its complement is connected. Moreover,

$$\|N\| = \max_j \|E_j(Z + i\widetilde{Y})E_j\| \leqslant C\delta^{1/2} + \max_j \|E_j(X + iY)E_j\| \leqslant 1 + C\delta^{1/2}.$$

Hence, if $\|N\| > 1$ or $\|T\| > 1$, we can replace N and T by $N/\|N\|$ and $T/\|T\|$. The new elements will have the same properties and satisfy $\|N\| \leqslant 1$ and $\|T\| \leqslant 1$.

Chapter 3

Quantitative Lin's Theorem in real rank zero C^*-algebras

In this chapter we prove Theorems 1.3.3 and 1.3.4. For $T \in \mathrm{M}_2(\mathcal{A})$, $\lambda \in \mathbb{C}$, let

$$d_2(T, \lambda) \stackrel{\text{def}}{=\!=} \operatorname{dist}(T - \lambda I, \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A}))$$

so that $d_2(T) = \sup\limits_{\lambda \in \mathbb{C}} d_2(T, \lambda)$. Note that, for every single λ, $d_2(T, \lambda) \leqslant \delta$ implies $\|[P, T]\| \leqslant \delta$, where $P\colon H \oplus H \to H$ is the projection onto the first component.

Let us briefly describe the structure of the proof. It consists of several steps. On each step we reduce the case of a general element T satisfying the assumptions of Theorem 1.3.3 to the case of an element with some additional spectral properties. We usually refer to this process as "removing certain subsets from the spectrum of T", in the same sense as in [16, Section 4]. The first step (Lemma 3.1.2) allows us to remove a small disk from the spectrum of T, preserving the estimate $d_2(T, \lambda) \leqslant C\delta_P$ where λ is the centre of the disk. The second step (Lemma 3.1.4) is a refined version of the previous result. We remove a finite set of disks simultaneously, with the same estimates of $d_2(T, \lambda_j)$ uniform in the number of holes. This reduces the general case to the case where the spectrum of T looks like the left part of Figure 3.1 (see page 42). Next, taking a simple continuous function of T, we transform the left part of Figure 3.1 into the right part, which is the grid Γ_ε (Theorem 3.1.6).

Section 3.3 deals with elements whose spectra are subsets of Γ_ε. We want to remove small portions from all the segments forming Γ_ε. In order to do that, first we show how to remove a point from a simple closed curve (Lemma 3.2.4), then explain how to exclude a line segment from a simple curve which contains a straight part (Lemma 3.2.5) and, finally, prove Lemma 3.2.6 which allows us to remove a line segment from an arbitrary set containing a straight part. After that, the proof of Theorem 1.3.3 is obtained by simultaneously removing centres of all the line segments in Γ_ε (as in Theorem 3.1.4), and shrinking the resulting connected components into points.

3.1 Proof of Theorem 1.3.3: reduction to a grid

We shall need some basic facts regarding polar decompositions. For $A \in \mathcal{B}(H)$, let $|A| = \sqrt{A^*A}$. If A is normal, then there exists a unitary operator U such that

$$U|A| = |A|U = A.$$

If $A \in \mathcal{A} \subset \mathcal{B}(H)$ is not invertible, then U may not belong to $\mathcal{A}$.

If $A \in \mathcal{A}$ is invertible (but not necessarily normal), then there exists a unique unitary $U \in \mathcal{A}$ such that $A = U|A|$. The element U can be defined as

$$U = A(A^*A)^{-1/2}.$$

It satisfies the important relation

$$U|A| = |A^*|U. \tag{3.1.1}$$

Moreover, for invertible A, the condition $A \in \mathrm{GL}_0(\mathcal{A})$ is equivalent to $U \in \mathrm{GL}_0(\mathcal{A})$. An analogue of (3.1.1) holds for general bounded operators (if A is not invertible, then U is only a partial isometry), but we will not use it.

Recall that $\operatorname{diag}_P T = PTP + (1-P)T(1-P)$ for $T \in \mathrm{M}_2(\mathcal{A})$. The following simple lemma will be very helpful in establishing that certain elements belong to $\mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

Lemma 3.1.1. *Suppose that $t \mapsto G_t$ is a continuous map from $[0;1]$ to $\mathrm{M}_2(\mathcal{A})$ such that G_t is invertible for all t, and*

$$\operatorname{diag}_P G_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A}), \quad \|[P, G_t]\| < \|G_t^{-1}\|^{-1}, \quad \forall t \in [0;1].$$

Then $\operatorname{diag}_P G_1 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

Proof. Since $\|G_t - \operatorname{diag}_P G_t\| = \|[P, G_t]\|$, simple perturbation theory arguments imply that $\operatorname{diag}_P G_t$ is also invertible. Hence, the path $t \mapsto \operatorname{diag}_P G_t$ connects G_0 and G_1 within $\mathrm{GL}(\mathcal{A} \oplus \mathcal{A})$. As $G_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$, so does G_1. ∎

For a normal element $T \in \mathrm{M}_2(\mathcal{A})$, we denote its spectral projection onto the set $\{z \in \mathbb{C} \colon |z| < \varepsilon\}$ by $\Pi_\varepsilon \in \mathcal{B}(H \oplus H)$. In general, Π_ε may not belong to $\mathrm{M}_2(\mathcal{A})$.

The following lemma is an analogue of [16, Lemma 4.1], but with the additional control of $\|[P, T]\|$ (i.e. the magnitude of "off-diagonal" elements with respect to P). We will not use it directly, but the proofs of Corollary 3.1.3 and Lemma 3.1.4 are based on it.

Lemma 3.1.2. *Let $\mathcal{A}$ be a unital C^*-algebra. There exists a constant $C_0 > 0$ such that for all $\varepsilon, \delta_P > 0$ satisfying $0 < \delta_P G(\delta_P) \leqslant C_0 \varepsilon$ and any normal $T \in \mathrm{M}_2(A)$ with $\|T\| \leqslant 2$, $d_2(T, 0) \leqslant \delta_P$, we can find an invertible normal element $T_\varepsilon \in \mathrm{M}_2(\mathcal{A})$ with the following properties:*

1. *$\|T_\varepsilon^{-1}\| \leqslant \varepsilon^{-1}$, so the ε-neighbourhood of 0 is contained in $\mathbb{C} \setminus \sigma(T_\varepsilon)$.*
2. *$[T_\varepsilon, \Pi_{\varepsilon'}] = 0$ for all $\varepsilon' \geqslant \varepsilon$, and $T_\varepsilon|_{\mathrm{Ran}(I - \Pi_{2\varepsilon})} = T|_{\mathrm{Ran}(I - \Pi_{2\varepsilon})}$, where Π_ε is the spectral projection of T onto $\{z \in \mathbb{C} \colon |z| < \varepsilon\}$.*
3. *$\|T - T_\varepsilon\| \leqslant 2\varepsilon$.*
4. *$\|[P, T_\varepsilon]\| \leqslant C\delta_P$.*
5. *$\operatorname{diag}_P T_\varepsilon \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ and, as a consequence, $d_2(T_\varepsilon, 0) \leqslant C\delta_P$.*

Proof. There exists an element $T_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ such that $\|T - T_0\| \leqslant 2\delta_P$. The element T_0 admits a unitary polar decomposition $T_0 = V_0|T_0|$, where $|T_0|, V_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$. Let also $T = V|T|$, where $V \in \mathcal{B}(H \oplus H)$ is unitary. Note that V may not belong to $\mathrm{M}_2(\mathcal{A})$, but it is always true that $|T| \in \mathrm{M}_2(\mathcal{A})$. The element V commutes with all functions of T.

Let $\rho_1 \in C^\infty(\mathbb{R}_+)$ be a nonincreasing function such that $\rho_1(t) = 1$ for $0 \leqslant t \leqslant 1/2$ and $\rho_1(t) = 0$ for $t \geqslant 1$. Let $\rho_2 \in C^\infty(\mathbb{R}_+)$ satisfy $\rho_1^2 + \rho_2^2 = 1$. Consider

$$S_\varepsilon = \rho_1(|T|/\varepsilon) V_0 \rho_1(|T|/\varepsilon) + V\rho_2^2(|T|/\varepsilon). \tag{3.1.2}$$

Let us study the properties S_ε. We have $V\rho_2(|T|/\varepsilon) = \tilde{\rho}_2(T/\varepsilon)$, where $\tilde{\rho}_2(z) = z|z|^{-1}\rho_2(|z|)$ is a smooth function. Hence, $S_\varepsilon \in \mathrm{M}_2(\mathcal{A})$. Since ρ_1, ρ_2, $\tilde{\rho}_2$ are smooth and bounded, from Proposition 2.1.5 and Lemmas 2.1.3, 2.1.6 we have

$$\|\rho_1(|\cdot|/\varepsilon)\|_{\mathrm{OL}(\mathbb{R}^2)} \leqslant C\varepsilon^{-1}, \quad \|\rho_2(|\cdot|/\varepsilon)\|_{\mathrm{OL}(\mathbb{R}^2)} \leqslant C\varepsilon^{-1} \quad \|\tilde{\rho}_2(\cdot/\varepsilon)\|_{\mathrm{OL}(\mathbb{R}^2)} \leqslant C\varepsilon^{-1}.$$

Hence, since $[P, V_0] = 0$, from Proposition 2.1.2 we get

$$\|[P, S_\varepsilon]\| \leqslant 2\|[P, \rho_1(|T|/\varepsilon)]\| + \|[P, \tilde{\rho}_2(T/\varepsilon)]\| + \|[P, \rho_2(|T|/\varepsilon)]\| \leqslant C\delta_P\varepsilon^{-1}.$$

The element S_ε may not be normal. We claim that S_ε is close to the unitary element V_0. To establish this, let us estimate their difference,

$$\begin{aligned} S_\varepsilon - V_0 &= (V - V_0)(I - \rho_1(|T|/\varepsilon)) - (I - \rho_1(|T|/\varepsilon))(V_0 - V)\rho_1(|T|/\varepsilon) \\ &= (V - V_0)|T|h_\varepsilon(|T|/\varepsilon) - h_\varepsilon(|T|/\varepsilon)|T|(V_0 - V)\rho_1(|T|/\varepsilon), \end{aligned}$$

where

$$h_\varepsilon(t) = (\varepsilon t)^{-1}(1 - \rho_1(t)).$$

Since $\|h_\varepsilon(|T|/\varepsilon)\| \leqslant 2\varepsilon^{-1}$, we get

$$\|S_\varepsilon - V_0\| \leqslant 2\varepsilon^{-1}\left(\|(V - V_0)|T|\| + \||T|(V - V_0)\|\right). \tag{3.1.3}$$

To estimate the right hand side, let us rewrite

$$(V - V_0)|T| = (T - T_0) + V_0(|T_0| - |T|), \tag{3.1.4}$$

$$|T|(V_0 - V) = (|T^*| - |T_0^*|)V_0 + T_0 - T. \tag{3.1.5}$$

We have $\|T - T_0\| \leqslant 2\delta_P$. From Proposition 2.1.7 it follows that

$$\||T| - |T_0|\| \leqslant C\delta_P G(\delta_P), \quad \||T^*| - |T_0^*|\| \leqslant C\delta_P G(\delta_P).$$

Hence, estimating the right hand sides of (3.1.4), (3.1.5) and using (3.1.3), we obtain

$$\|S_\varepsilon - V_0\| \leqslant C\varepsilon^{-1}\delta_P G(\delta_P).$$

By choosing a sufficiently small C_0 in the statement of the lemma, we can make this difference as small as needed. In addition,

$$\|U_\varepsilon - V_0\| \leqslant \|U_\varepsilon - S_\varepsilon\| + \|S_\varepsilon - V_0\| \leqslant C\varepsilon^{-1}\delta_P G(\delta_P).$$

Let us now choose C_0 in such a way that $\|U_\varepsilon - V_0\| \leqslant 1/6$.

Since V_0 is unitary, the difference $\|S_\varepsilon^* S_\varepsilon - I\|$ can also be made smaller than any fixed constant. Let

$$U_\varepsilon = S_\varepsilon (S_\varepsilon^* S_\varepsilon)^{-1/2}, \quad \text{so that} \quad S_\varepsilon = U_\varepsilon |S_\varepsilon|.$$

The spectrum of $S_\varepsilon^* S_\varepsilon$ is contained in the interval $[1 - \gamma(C_0); 1 + \gamma(C_0)]$, where $\gamma(C_0) \to 0$ as $C_0 \to 0$. For sufficiently small C_0, the element $(S_\varepsilon^* S_\varepsilon)^{-1/2}$ is a smooth function of $S_\varepsilon^* S_\varepsilon$ supported in a neighbourhood of this interval. This function belongs to $\mathrm{OL}(\mathbb{R})$ with the quasi-norm depending only on C_0. As $\|[P, S_\varepsilon]\| \leqslant C\delta_P \varepsilon^{-1}$, we have $\|[P, (S_\varepsilon^* S_\varepsilon)^{-1/2}]\| \leqslant C\delta_P \varepsilon^{-1}$ and

$$\|[P, U_\varepsilon]\| \leqslant C' \delta_P \varepsilon^{-1}. \tag{3.1.6}$$

From (3.1.2), we also get that $[S_\varepsilon, \Pi_\varepsilon] = 0$, so S_ε has block structure with respect to the spectral projection of T. Therefore, U_ε has the same property. Moreover,

$$[S_\varepsilon, \Pi_{\varepsilon'}] = [U_\varepsilon, \Pi_{\varepsilon'}] = 0, \quad (I - \Pi_{\varepsilon'}) S_\varepsilon = (I - \Pi_{\varepsilon'}) U_\varepsilon = (I - \Pi_{\varepsilon'}) V, \quad \forall \varepsilon' \geqslant \varepsilon. \tag{3.1.7}$$

Let $f_1 \in C^\infty(\mathbb{R}_+)$ be a nonincreasing function such that $f_1(t) = 1$ for $0 \leqslant t \leqslant 1$ and $f_1(t) = 0$ for $t \geqslant 2$. Let $f_2(t) = 1 - f_1(t)$. We now construct the element T_ε by taking

$$T_\varepsilon = \varepsilon U_\varepsilon f_1(|T|/\varepsilon) + T f_2(|T|/\varepsilon) = U_\varepsilon(\varepsilon f_1(|T|/\varepsilon) + |T| f_2(|T|/\varepsilon)). \tag{3.1.8}$$

From (3.1.7), U_ε commutes with the expression in brackets in the right hand side, and the element T_ε is normal. We also have $[T_\varepsilon, \Pi_{\varepsilon'}] = 0$ for $\varepsilon' \geqslant \varepsilon$. In other words, T_ε has the same block structure with respect to Π_ε. Note that $\Pi_\varepsilon T_\varepsilon \Pi_\varepsilon = \varepsilon U_\varepsilon|_{\mathrm{Ran}\,\Pi_\varepsilon}$. Therefore, the corresponding block of T_ε is an ε-multiple of a unitary operator (in the subspace $\mathrm{Ran}\,\Pi_\varepsilon$). We also have

$$(I - \Pi_\varepsilon) T_\varepsilon (I - \Pi_\varepsilon) = g_\varepsilon(T),$$

where

$$g_\varepsilon(z) = (1 - \chi_\varepsilon(z)) \left(\varepsilon(z/|z|) f_1(|z|/\varepsilon) + z f_2(|z|/\varepsilon)\right),$$

and χ_ε is the characteristic function of $\{z \in \mathbb{C} \colon |z| < \varepsilon\}$. Note that $|g_\varepsilon(z)| \geqslant \varepsilon$ for $|z| \geqslant \varepsilon$, which implies Property 1. Since $g_\varepsilon(z) = z$ for $|z| \geqslant 2\varepsilon$, we get Properties 2 and 3.

Let us estimate $\|[P, T_\varepsilon]\|$. From (3.1.8), we have

$$[P, T_\varepsilon] = [P, \varepsilon U_\varepsilon f_1(|T|/\varepsilon)] + [P, T f_2(|T|/\varepsilon)]. \tag{3.1.9}$$

For the first term, we have

$$\|[P, \varepsilon U_\varepsilon f_1(|T|/\varepsilon)]\| \leqslant \varepsilon \|[P, U_\varepsilon]\| \|f_1(|T|/\varepsilon)\| + \varepsilon \|[P, f_1(|T|/\varepsilon)]\|.$$

Using (3.1.6) and the scaling argument from Lemma 2.1.3, we get that both terms in the right hand side are bounded by $C\delta_P$. Since $f_1(t) + f_2(t) = 1$, for the second term of (3.1.9) we have

$$\|[P, T f_2(|T|/\varepsilon)]\| \leqslant \|[P, T]\| + \|[P, T f_1(|T|/\varepsilon)]\| \leqslant \delta_P + \varepsilon \|[P, g_1(T/\varepsilon)]\| \leqslant C\delta_P,$$

where $g_1(z) = z f_1(|z|)$. Together with (3.1.8), this implies Property 4.

Finally, let us prove Property 5. From (3.1.6) and since U_ε and V_0 are unitary, $[V_0, P] = 0$, and $\|U_\varepsilon - V_0\| \leqslant 1/6$, the continuous path

$$\varepsilon U_\varepsilon^{(s)} = s\varepsilon U_\varepsilon + (1-s)\varepsilon V_0$$

connects $\varepsilon U_\varepsilon$ and εV_0 and satisfies the assumptions of Lemma 3.1.1, assuming that C_0 is sufficiently small. Hence, $U_\varepsilon \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$. Next, there exists a smooth function $h\colon \mathbb{C} \to \mathbb{C}$ such that $h(z) = z/|z|$ for $|z| \geqslant 1$, $|h(z)| < 1$ for $|z| < 1$, and $h(z)/z > 0$ for all $z \neq 0$. From (3.1.8), we have $|T_\varepsilon| \geqslant \varepsilon$ and hence $h(T_\varepsilon/\varepsilon) = U_\varepsilon$. Let

$$h_t(z) = (1-t)\varepsilon h(z/\varepsilon) + tz.$$

Then $h_t(T_\varepsilon) = U_\varepsilon h_t(|T_\varepsilon|)$, and $\|h_t(T_\varepsilon)^{-1}\| \leqslant \varepsilon^{-1}$ for $0 \leqslant t \leqslant 1$, $\|[P, h_t(T_\varepsilon)]\| \leqslant C\delta_P$ where C is an absolute constant. Moreover, $h_0(T_\varepsilon) = \varepsilon U_\varepsilon$ and $h_1(T_\varepsilon) = T_\varepsilon$. Since $\delta_P G(\delta_P) \leqslant C_0\varepsilon$, by choosing an appropriate C_0, we can also guarantee that h_t satisfies the assumptions of Lemma 3.1.1. This completes the proofs of Property 5 and the lemma. ∎

The operator U_ε appearing in the proof will be important in latter considerations. Let us summarize its properties.

Corollary 3.1.3. *Under the assumptions of Lemma* 3.1.2*, there exists a unitary element* $U_\varepsilon \in \mathrm{M}_2(\mathcal{A})$ *such that:*

1. $\|[P, U_\varepsilon]\| \leqslant C\varepsilon^{-1}\delta_P$.

2. $[U_\varepsilon, \Pi_{\varepsilon'}] = 0$ *for all* $\varepsilon' \geqslant \varepsilon$.

3. $U_\varepsilon(I - \Pi_\varepsilon) = V(I - \Pi_\varepsilon)$ *where* $V \in \mathcal{B}(H \oplus H)$ *is the polar part of* T.

4. $T_\varepsilon = U_\varepsilon |T_\varepsilon|$.

5. $\operatorname{diag}_P U_\varepsilon \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

The next lemma is an extension of Lemma 3.1.2 to the case of multiple holes. It can be done simply by applying Lemma 3.1.2 several times, but then the norms $\|[P, T_\varepsilon]\|$ will increase by a factor C each time and, therefore, will grow exponentially with the number of holes. It turns out that the construction can be improved, and the holes can be created "simultaneously", assuming that they are separated from each other.

Let $\mathcal{O}_\varepsilon(\lambda_0) = \{\lambda \in \mathbb{C}\colon |\lambda - \lambda_0| < \varepsilon\}$. Suppose that $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. Let $\Pi_\varepsilon^j = E_T(\mathcal{O}_\varepsilon(\lambda_j))$ denote the spectral projection of T onto the ε-neighbourhood of λ_j.

Lemma 3.1.4. *Suppose that* $|\lambda_j| \leqslant 1$, $\operatorname{dist}(\lambda_i, \lambda_j) \geqslant 4\varepsilon$ *for* $i \neq j$. *Assume that* $T \in \mathrm{M}_2(\mathcal{A})$ *is normal,* $\|T\| \leqslant 1$, $d_2(T, \lambda_j) \leqslant \delta_P$ *for* $j = 1, \ldots, k$, *and* $0 < \delta_P G(\delta_P) \leqslant C_0\varepsilon$. *Then there exists a normal element* $T_\varepsilon \in \mathrm{M}_2(\mathcal{A})$ *with the following properties:*

1. $\|(T_\varepsilon - \lambda_j I)^{-1}\| \leqslant \varepsilon^{-1}$, *so the* ε*-neighbourhoods of* λ_j *are contained in* $\mathbb{C} \setminus \sigma(T_\varepsilon)$.

2. $[T_\varepsilon, \Pi_\varepsilon^j] = [T_\varepsilon, \Pi_{2\varepsilon}^j] = 0$, $T_\varepsilon|_{\mathrm{Ran}(I - \sum_j \Pi_{2\varepsilon}^j)} = T|_{\mathrm{Ran}(I - \sum_j \Pi_{2\varepsilon}^j)}$, $\forall j = 1, \ldots, k$.

3. $\|T - T_\varepsilon\| \leqslant 2\varepsilon$.

4. $\|[P, T_\varepsilon]\| \leqslant C\delta_P$, *where* C *does not depend on* k.

5. $\operatorname{diag}_P(T_\varepsilon - \lambda_j I) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ *for* $j = 1, \ldots, k$. *As a consequence,* $d_2(T_\varepsilon, \lambda_j) \leqslant C\delta_P$.

Proof. For each j, the element $T - \lambda_j I$ satisfies the assumptions of Lemma 3.1.2 and Corollary 3.1.3. Let us obtain the corresponding unitary element and denote it by U_ε^j. As in Lemma 3.1.2, let $f_1 \in C^\infty(\mathbb{R}_+)$ be a nonincreasing function such that $f_1(t) = 1$ for $0 \leqslant t \leqslant 1$ and $f_1(t) = 0$ for $t \geqslant 2$. Let us impose an additional condition that $g_1(t) \stackrel{\text{def}}{=} \sqrt{f_1(t)}$ is smooth. We claim that the following element satisfies the statement of the lemma.

$$T_\varepsilon = \sum_{j=1}^{k} (\lambda_j + \varepsilon U_\varepsilon^j) f_1\left(|T - \lambda_j I|/\varepsilon\right) + T\left(1 - \sum_{j=1}^{k} f_1\left(|T - \lambda_j I|/\varepsilon\right)\right). \qquad (3.1.10)$$

Corollary 3.1.3 implies

$$[U_\varepsilon^i, \Pi_\varepsilon^j] = [U_\varepsilon^i, \Pi_{2\varepsilon}^j] = 0.$$

Indeed, for $i = j$ it follows directly from Property 2, and for $i \neq j$ it is a consequence of Property 3 and $\operatorname{dist}(\lambda_i, \lambda_j) \geqslant 4\varepsilon$. The element U_ε^j commutes with $f_1(|T - \lambda_j|/\varepsilon)$ because in the block Π_ε the operator $f_1(|T - \lambda_j|/\varepsilon)$ is scalar, and in the block $\Pi_{2\varepsilon} - \Pi_\varepsilon$ both are functions of T. Since everything else in (3.1.10) can be expressed in terms of functions of T, we get that the element T_ε has block structure with respect to the system of projections Π_ε^j, $\Pi_{2\varepsilon}^j - \Pi_\varepsilon^j$ for $j = 1, \ldots, k$, and $I - \sum_{j=1}^k \Pi_{2\varepsilon}^j$. We have

$$\Pi_\varepsilon^j T_\varepsilon = \Pi_\varepsilon^j (\lambda_j + \varepsilon U_\varepsilon^j), \qquad \left(I - \sum_{j=1}^{k} \Pi_{2\varepsilon}^j\right) T_\varepsilon = \left(I - \sum_{j=1}^{k} \Pi_{2\varepsilon}^j\right) T.$$

Hence, the "small" blocks corresponding to Π_ε^j are ε-multiples of unitaries shifted by λ_j. The "largest" block coincides with the corresponding block of T. Similarly to Lemma 3.1.2, let χ_ε be the characteristic function of $\{z \in \mathbb{C} \colon |z| < \varepsilon\}$, and

$$g_\varepsilon(z) = \left(\chi_{2\varepsilon}(z) - \chi_\varepsilon(z)\right)\left(\varepsilon(z/|z|) f_1(|z|/\varepsilon) + z f_2(|z|/\varepsilon)\right).$$

Then

$$(\Pi_{2\varepsilon} - \Pi_\varepsilon) T_\varepsilon = \lambda_i + g_\varepsilon(T - \lambda_i),$$

so that the "intermediate" blocks corresponding to $\Pi_{2\varepsilon}^j - \Pi_\varepsilon^j$ are functions of T. This implies Properties 1 and 2. Since all blocks are normal, the element T_ε is also normal.

Let us estimate the difference between T_ε and T,

$$\begin{aligned} T_\varepsilon - T &= \sum_{j=1}^{k} \left(\lambda_j + \varepsilon U_\varepsilon^j - T\right) f_1(|T - \lambda_j I|/\varepsilon) \\ &= \varepsilon \sum_{j=1}^{k} U_\varepsilon^j f_1(|T - \lambda_j I|/\varepsilon) + \sum_{j=1}^{k} (\lambda_j I - T) f_1(|T - \lambda_j I|/\varepsilon). \end{aligned} \qquad (3.1.11)$$

Since the terms in the first sum act in orthogonal subspaces, we get Property 3. To establish Property 4, let

$$M_\varepsilon = \sum_{j=1}^{k} U_\varepsilon^j f_1(|T - \lambda_j I|/\varepsilon) = \sum_{j=1}^{k} g_1(|T - \lambda_j I|/\varepsilon) U_\varepsilon^j g_1(|T - \lambda_j I|/\varepsilon).$$

The equality holds since $[U_\varepsilon^j, f_1(|T-\lambda_j I|/\varepsilon)] = [U_\varepsilon^j, g_1(|T-\lambda_j I|/\varepsilon)] = 0$; recall that $g_1(t) = \sqrt{f_1(t)}$. Let also

$$h_\varepsilon(z) = \sum_{j=1}^{k} (z - \lambda_j) f_1(|z - \lambda_j|/\varepsilon).$$

Then

$$[P, T_\varepsilon] = [P, T] - [P, h_\varepsilon(T)] + \varepsilon [P, M_\varepsilon]. \tag{3.1.12}$$

Note that $z f_1(|z|/\varepsilon) = \varepsilon(z/\varepsilon) f_1(|z|/\varepsilon)$ and, therefore, by Lemma 2.1.3

$$\|z f_1(|z|/\varepsilon)\|_{\mathrm{OL}(\mathbb{C})} \leqslant C.$$

Part 2 of Lemma 2.1.6 implies $\|h_\varepsilon\|_{B^1_{\infty,1}(\mathbb{C})} \leqslant C$, and, from Proposition 2.1.2 and (3.1.12), we obtain

$$\|[P, T_\varepsilon\| \leqslant C \|[P, T]\| + \varepsilon \, \|[P, M_\varepsilon]\| \,. \tag{3.1.13}$$

Let us estimate the last term. We have

$$\begin{aligned}
[P, M_\varepsilon] = \sum_{j=1}^{k} &[P, g_1(|T - \lambda_j I|/\varepsilon)] U_\varepsilon^j g_1(|T - \lambda_j I|/\varepsilon) \\
&+ \sum_{j=1}^{k} g_1(|T - \lambda_j I|/\varepsilon) [P, U_\varepsilon^j] g_1(|T - \lambda_j I|/\varepsilon) \\
&\qquad + \sum_{j=1}^{k} g_1(|T - \lambda_j I|/\varepsilon) U_\varepsilon^j [P, g_1(|T - \lambda_j I|/\varepsilon)].
\end{aligned} \tag{3.1.14}$$

Recall that $\|[P, U_\varepsilon^j]\| \leqslant C\varepsilon^{-1}\delta_P$. The different terms in the middle sum act in mutually orthogonal subspaces of $H \oplus H$. Therefore, the norm of the sum can be estimated by the maximal norm of the terms and hence does not exceed $C\varepsilon^{-1}\delta_P$, where C is an absolute constant. The first and third terms are estimated similarly to each other, and it suffices to estimate the first term of (3.1.14). We have

$$g_1(|z|) = g_1(|x + iy|) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} e^{isx + ity} \hat{g}_1(s, t) \, ds \, dt,$$

where $\hat{g}_1$ is the Fourier transform of $g_1(|x+iy|)$ as a function of two real variables. Let $\lambda_j = x_j + iy_j$. Let also $T = X + iY$, $X = X^*$, $Y = Y^*$, $[X, Y] = 0$. Then

$$g_1(|T - \lambda_j I|/\varepsilon) = \frac{\varepsilon^2}{2\pi} \int\limits_{\mathbb{R}^2} \hat{g}_1(\varepsilon s, \varepsilon t) e^{-isx_j - ity_j} e^{isX + itY} \, ds \, dt. \tag{3.1.15}$$

Let us rewrite the first term in (3.1.14) using (3.1.15),

$$\frac{\varepsilon^2}{2\pi} \int\limits_{\mathbb{R}^2} \hat{g}_1(\varepsilon s, \varepsilon t) [P, e^{isX+itY}] \left\{ \sum_{j=1}^{k} e^{-isx_j - ity_j} U_\varepsilon^j g_1(|T - \lambda_j I|/\varepsilon) \right\} ds \, dt.$$

The terms in curly brackets, as before, act in orthogonal subspaces of $H \oplus H$. Hence, the operator norm of the sum is bounded by 1. From Lemma 2.1.9, the whole expression is bounded by

$$\frac{\varepsilon^2 \delta_P}{2\pi} \int\limits_{\mathbb{R}^2} |\hat{g}_1(\varepsilon s, \varepsilon t)|(|s| + |t|) \, ds \, dt \leqslant C\varepsilon^{-1}\delta_P.$$

Hence, the commutator $[P, M_\varepsilon]$ admits the same bound. Together with (3.1.13), this completes the proof of Property 4.

Let us establish Property 5. By $T_\varepsilon^{(j)}$, denote the operator T for which we applied the statement only for λ_j. Then, by Lemma 3.1.2, $\mathrm{diag}_P(T_\varepsilon - \lambda_j I) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$. To complete the proof, note that the path $tT_\varepsilon + (1-t)T_\varepsilon^{(j)} - \lambda_j I$ satisfies the assumptions of Lemma 3.1.1. ∎

Consider the grid

$$\Gamma_\varepsilon = \{z = x + iy \in \mathbb{C} \colon x \in \varepsilon\mathbb{Z} \text{ or } y \in \varepsilon\mathbb{Z}\}.$$

Let also $\Lambda_\varepsilon = \varepsilon(\mathbb{Z} + 1/2) + i\varepsilon(\mathbb{Z} + 1/2) \subset \mathbb{C}$ be the set of centres of the cells of Γ_ε. By $\mathcal{O}_\varepsilon(X)$ we denote an open ε-neighbourhood of the set $X \subset \mathbb{C}$.

Lemma 3.1.5. *There exists a family of functions $g_{\varepsilon'} \colon \mathbb{C} \to \mathbb{C}$, $\varepsilon' > 0$, such that*

1. $g_{\varepsilon'} \in C^\infty(\mathbb{C})$, $\|g_{\varepsilon'}\|_{\mathrm{OL}(\mathbb{C})} \leqslant C$.

2. $g_{\varepsilon'}$ *maps* $\mathbb{C} \setminus \mathcal{O}_{\varepsilon'/6}(\Lambda_{\varepsilon'})$ *onto* $\Gamma_{\varepsilon'}$.

3. $|g_{\varepsilon'} - z| \leqslant \varepsilon'$ *for* $z \in \mathbb{C} \setminus \mathcal{O}_{\varepsilon'/6}(\Gamma_{\varepsilon'})$, *and g is homotopic to z within the class of functions satisfying* $g(\mathbb{C} \setminus \mathcal{O}_{\varepsilon'/6}(\Lambda_{\varepsilon'})) \subset \mathbb{C} \setminus \mathcal{O}_{\varepsilon'/6}(\Lambda_{\varepsilon'})$.

Proof. For $\varepsilon' = 1$, there exists $g_1 \in C^\infty(\mathbb{C})$ with the required properties such that $g_1(z) - z$ is $(1, i)$-periodic. It can be constructed in two steps. First, we "blow up" the circles until they start touching the edges of the cells. Then we keep blowing them up, (smoothly) straightening the parts that do not fit into the cell. This gives us a function satisfying the desired properties, expect for a small neighbourhood of $\mathbb{Z}+i\mathbb{Z}$. This neighbourhood can be "shrunk" into $\mathbb{Z}+i\mathbb{Z}$ by applying another smooth function which is also a periodic perturbation of z. Since the composition of smooth periodic perturbations of z is a function of the same type, we have constructed g_1. The general case is covered by $g_{\varepsilon'}(z) = \varepsilon' g_1(z/\varepsilon')$. The homotopy for Property 3 can be chosen to be linear. ∎

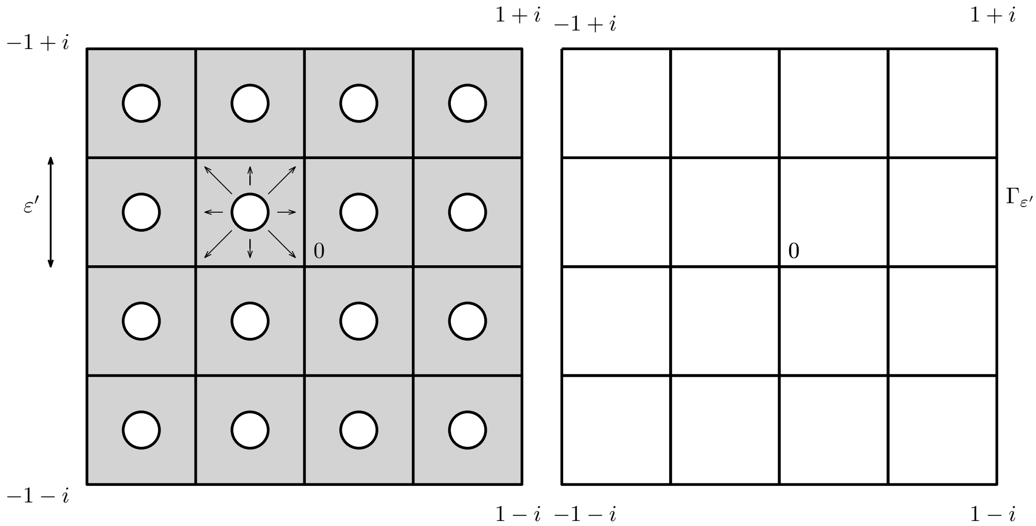

Figure 3.1: The spectra of T_1 and $T_{\varepsilon'} = g_{\varepsilon'}(T_1)$, $\varepsilon' = 1/2$

Using the functions g_ε, we can reduce the initial problem to the case when $\sigma(T) \subset \Gamma_\varepsilon$.

Theorem 3.1.6. *Let $\mathcal{A}$ be a unital C^*-algebra. There exists a universal constant $C_0 > 0$ such that for all ε', δ_P with $0 < \delta_P G(\delta_P) \leqslant C_0\varepsilon'$ and any normal element $T \in \mathrm{M}_2(\mathcal{A})$ satisfying $\|T\| \leqslant 1$, $d_2(T) \leqslant \delta_P$ there exists a normal element $T_{\varepsilon'} \in \mathrm{M}_2(\mathcal{A})$ such that:*

1. $\sigma(T_{\varepsilon'}) \subset \Gamma_{\varepsilon'}$.
2. $\|T - T_{\varepsilon'}\| \leqslant 2\varepsilon'$.
3. $\|[P, T_{\varepsilon'}]\| \leqslant C\delta_P$.

4. $\mathrm{diag}_P(T'_\varepsilon - \lambda I) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ *for* $\lambda \in \Lambda_{\varepsilon'}$.

Proof. We first apply Lemma 3.1.4 with $\varepsilon = \varepsilon'/6$ to remove the $\varepsilon'/6$-neighbourhood of $\Lambda_{\varepsilon'}$ from $\sigma(T)$. Let us denote the resulting element by T_1. Then we consider $T_{\varepsilon'} = g_{\varepsilon'}(T_1)$, where $g_{\varepsilon'}$ is the function from Lemma 3.1.5, see Figure 3.1. Properties 1–3 follow from Lemma 3.1.4 and the properties of $g_{\varepsilon'}$. Property 4 follows from Lemma 3.1.1 similarly to Property 5 of Lemma 3.1.2. ∎

3.2 Proof of Theorem 1.3.3: removal of line segments

The following two lemmas are contained in [27]. We give more elementary proofs from [16, Lemma 1.8] for the convenience of the reader. These lemmas will be used in the proof of Lemma 3.2.4.

Lemma 3.2.1. *Suppose that $\mathcal{A}$ is a C^*-algebra of real rank zero, and let $U, V \in \mathcal{A}$ be unitary elements such that $-1 \notin \sigma(U)$, $-1 \notin \sigma(V)$. Then for any $\varepsilon > 0$ there exists a unitary element W_ε such that $\|UV - W_\varepsilon\| \leqslant \varepsilon$ and $-1 \notin \sigma(W_\varepsilon)$.*

Proof. Recall that the fact that $-1 \notin \sigma(U)$ is equivalent to the existence of a self-adjoint element X such that $U = (iI - X)^{-1}(iI + X)$ (Cayley transform). Let also $V = (iI - Y)^{-1}(iI + Y)$, where $Y = Y^* \in \mathcal{A}$. Since $\mathcal{A}$ is of real rank zero, for each $\varepsilon > 0$ there exists a self-adjoint element $X_\varepsilon \in \mathrm{GL}(\mathcal{A})$ such that $\|X - X_\varepsilon\| \leqslant \varepsilon$. There also exists an element Y_ε such that $\|Y - Y_\varepsilon\| \leqslant \varepsilon$ and $Y_\varepsilon - X_\varepsilon^{-1} \in \mathrm{GL}(\mathcal{A})$. Let $U_\varepsilon = (iI - X_\varepsilon)^{-1}(iI + X_\varepsilon)$, $V_\varepsilon = (iI - Y_\varepsilon)^{-1}(iI + Y_\varepsilon)$. We have $\|U_\varepsilon V_\varepsilon - UV\| \to 0$ as $\varepsilon \to 0$, and

$$(iI - X_\varepsilon)(U_\varepsilon V_\varepsilon + I)(iI - Y_\varepsilon) = 2(X_\varepsilon Y_\varepsilon - I) = 2X_\varepsilon(Y_\varepsilon - X_\varepsilon^{-1}) \in \mathrm{GL}(\mathcal{A}),$$

which gives $-1 \notin \sigma(U_\varepsilon V_\varepsilon)$. ∎

Lemma 3.2.2. *Suppose that $\mathcal{A}$ is a C^*-algebra of real rank zero, and let $U \in \mathrm{GL}_0(\mathcal{A})$ be unitary. Then for any $\varepsilon > 0$ there exists a unitary element $U' \in \mathrm{GL}_0(\mathcal{A})$ such that $-1 \notin \sigma(U')$ and $\|U - U'\| \leqslant \varepsilon$.*

Proof. Let $Z(t)$, $0 \leqslant t \leqslant 1$, be a path in $\mathrm{GL}_0(\mathcal{A})$ connecting U and I. Let $\widetilde{Z}(t) = Z(t)|Z(t)|^{-1}$ be the "normalized" path (in the sense that its elements are unitary). There exists a finite set of points $0 \leqslant t_0 < t_1 < \ldots < t_m = 1$ such that $\|\widetilde{Z}(t_{i+1}) - \widetilde{Z}(t_i))\| < 1$ for $0 \leqslant i \leqslant m-1$. We have $U = V_m V_{m-1} \ldots V_1$, where $V_j = \widetilde{Z}(t_j)\widetilde{Z}(t_{j-1})^{-1}$, and therefore $\|V_j - I\| \leqslant 1$ and $-1 \notin \sigma(V_j)$. We now can apply Lemma 3.2.1 and, by induction, obtain that the product of V_j can be approximated by operators with the same property. Note that we automatically have $U' \in \mathrm{GL}_0(\mathcal{A})$ since $\mathbb{C} \setminus \sigma(U')$ is connected (see Section 1.1). ∎

By $\mathbb{T}$ we denote the unit circle in $\mathbb{C}$. By $\operatorname{int}\Gamma$ we denote the interior of a (closed) Jordan curve Γ.

Definition 3.2.3. We say that a simple C^2-smooth closed curve Γ parametrized by a map $\varphi\colon \mathbb{T} \to \mathbb{C}$ is *admissible* if there exists a homotopy of $\varphi_t\colon \mathbb{C} \to \mathbb{C}$ and a continuous family $\lambda_t \in \operatorname{int}\varphi_t(\mathbb{T})$, $t \in [0;1]$, $\lambda_0 = 0$, such that

1. $\varphi_0(z) = z, \ \varphi_1|_{\mathbb{T}} = \varphi.$

2. φ_t is a diffeomorphism of $\mathbb{C}$ such that $\varphi_t(z) = z$ for $|z| \geqslant C_1$, where C_1 does not depend on t.

3. $\|\varphi_t\|_{\mathrm{OL}(\mathbb{C})} \leqslant C_2$ uniformly in t.

4. $\operatorname{dist}(\lambda_t, \varphi_t(\mathbb{T})) \geqslant C_3$ uniformly in t.

Any sufficiently smooth curve is admissible, but we do not need it in such generality. In fact, we will only use this definition for two explicitly described curves, see right parts of Figures 3.3, 3.4 below. Note that the family $\{\lambda_t\}$ is a part of the definition of admissibility and is not unique. We shall usually consider curves together with points $\lambda = \lambda_1 \in \operatorname{int}\Gamma$. The notation $C(\lambda, \Gamma)$ means that the constant depends on Γ and λ, and may also depend on the path $\{\lambda_t\}$.

Starting from the next lemma, we assume that $\mathcal{A}$ is of real rank zero, since we are performing operations with one-dimensional spectra. Theorem 3.1.6, similarly to Theorem 1 from Introduction, holds without this assumption.

The following lemma allows us to remove a point from the spectrum of an element whose spectrum lies on an admissible curve. We need to keep track of the off-diagonal elements of T. In fact, we can make it in such a way that off-diagonal elements become zero.

Lemma 3.2.4. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero. Suppose that $\Gamma \subset \mathbb{C}$ is an admissible curve such that $0 \in \Gamma$. Let $T \in \mathrm{M}_2(\mathcal{A})$ be a normal element with $\sigma(T) \subset \Gamma$ and $\|[P,T]\| \leqslant \delta_P$. Assume that*

$$\mathrm{diag}_P(T - \lambda_1 I) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A}),$$

where $\lambda_1 \in \mathrm{int}\,\Gamma$ is from Defininition 3.2.3. Then there exists a normal element $T_0 \in \mathrm{M}_2(\mathcal{A})$ such that

1. *$\sigma(T_0) \subset \Gamma$, $0 \notin \sigma(T_0)$.*
2. *$\|T_0 - T\| \leqslant C(\lambda, \Gamma)\delta_P$.*
3. *$T_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ and, as a consequence, $[T_0, P] = 0$.*

Proof. Let $\Gamma = \varphi(\mathbb{T})$ in the notation of Definition 3.2.3. We can always assume that δ_P is small enough, as we can choose the constant in Property 3 to be large so that the statement becomes trivial for all other δ_P. The idea of the proof is to reduce the statement to the case of $\Gamma = \mathbb{T}$ (using the homotopy φ from the definition of Γ). In this case, we remove the off-diagonal elements with a small perturbation such that the element remains unitary. Then we apply Lemma 3.2.2 to each (unitary) block and remove a small arc from the spectrum. Finally, we map everything back to Γ. The properties of φ allow us to control off-diagonal elements on all steps.

The formal proof is as follows. Let $U_t = \varphi_t(\varphi^{-1}(T))$. By Proposition 2.1.2, $\|[P, \varphi^{-1}(T)]\| \leqslant C(\lambda, \Gamma)\delta_P$ and $\|[U_t, P]\| \leqslant C(\lambda, \Gamma)\delta_P$ (note that φ^{-1} is a smooth compactly supported perturbation of z and therefore belongs to $\mathrm{OL}(\mathbb{C})$). Consider now

$$V_t = \mathrm{diag}_P\, U_t.$$

We have $[V_t, P] = 0$, $\|V_t - U_t\| \leqslant C(\lambda, \Gamma)\delta_P$. For δ_P small enough, $V_t - \lambda_t I$ is a path in $\mathrm{GL}(\mathcal{A} \oplus \mathcal{A})$ connecting V_0 and $\mathrm{diag}_P(T - \lambda_1 I)$, which implies $V_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ for sufficiently small δ_P.

Let

$$U' = V_0(V_0^* V_0)^{-1/2}. \tag{3.2.1}$$

The element U' is unitary and $[U', P] = 0$. Since $\|V_0 - U_0\| \leqslant C(\lambda, \Gamma)\delta_P$, from (3.2.1) we have $\|U' - U_0\| \leqslant C(\lambda, \Gamma)\delta_P$ and $\|U' - V_0\| \leqslant C(\lambda, \Gamma)\delta_P$. Thus, possibly after choosing a smaller δ_P, we can guarantee $U' \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$. By Lemma 3.2.2, there exists a unitary element $\widetilde{U} \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ such that $\varphi^{-1}(0) \notin \sigma(\widetilde{U})$ and $\|U' - \widetilde{U}\| \leqslant \delta_P$. It is now easy to see that the element $T_0 = \varphi(\widetilde{U})$ satisfies Properties 1 and 2. Property 3 follows from the fact that $\mathbb{C} \setminus \sigma(T_0)$ is connected. ∎

The next lemma uses this to construct a unitary operator similarly to Corollary 3.1.3. This allows us to remove a line segment from the spectrum of T, but we still need to assume that the spectrum lies on an admissible curve.

Lemma 3.2.5. *Let Γ, $\lambda = \lambda_1 \in \operatorname{int}\Gamma$ satisfy the assumptions of Lemma 3.2.4, and suppose that $\Gamma \cap \mathcal{O}_1(0) = (-1; 1)$. Then there exists $\delta_P^0(\lambda, \Gamma) > 0$ such that for all $\mathcal{A}$, T satisfying the assumptions of Lemma 3.2.4 with $\delta_P \in [0; \delta_P^0]$ there exists a unitary element $U \in \mathrm{M}_2(\mathcal{A})$ with the following properties:*

1. $[P, U] \leqslant C\delta_P$.
2. $[\Pi, U] = 0$, *where Π is the spectral projection of T onto $(-1; 1)$.*
3. $(I - \Pi)U = (I - \Pi)V$, *where $V \in \mathcal{B}(H \oplus H)$ is the polar part of T.*
4. ΠU *is self-adjoint.*
5. $\operatorname{diag}_P U \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

Proof. From Lemma 3.2.4, there exists a normal element $T_0 \in \mathrm{GL}_0(\mathcal{A}\oplus\mathcal{A})$ such that $\sigma(T_0) \subset \Gamma \setminus \{0\}$, $\|T - T_0\| \leqslant C(\lambda, \Gamma)\delta_P$. Let $T_0 = V_0|T_0|$. Then $V_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

As in Lemma 3.1.2, let $\rho_1 \in C^\infty(\mathbb{R}_+)$ be a nondecreasing function such that $\rho_1(t) = 1$ for $0 \leqslant t \leqslant 1/2$ and $\rho_1(t) = 0$ for $t \geqslant 1$. Let $\rho_2 \in C^\infty(\mathbb{R})$ satisfy $\rho_1^2 + \rho_2^2 = 1$. Consider

$$S = \rho_1(|T|)(\operatorname{Re} V_0)\rho_1(|T|) + V\rho_2^2(|T|). \tag{3.2.2}$$

The remaining part of the proof is very similar to Lemma 3.1.2. The principal difference is that the block of S corresponding to Π is self-adjoint, as well as the corresponding block of U. The main steps of the proof are as follows. We have

$$S - V_0 = (V - V_0)\rho_2^2(|T|) + \left\{\rho_1(|T|)(\operatorname{Re} V_0)\rho_1(|T|) - V_0\rho_1^2(|T|)\right\}. \tag{3.2.3}$$

It is easy to see that $\rho_2^2(t) = th_2(t)$ for some smooth bounded function h_2. Hence,

$$(V - V_0)\rho_2^2(|T|) = (V|T| - V_0|T|)h_2(|T|) = (T - T_0)h_2(|T|) + V_0(|T_0| - |T|)h_2(|T|). \tag{3.2.4}$$

By construction, $\sigma(T_0) \cap \mathcal{O}_1(0) \subset (-1; 1)$, and

$$\rho_1(|T_0|)(\operatorname{Re} V_0)\rho_1(|T_0|) = (\operatorname{Re} V_0)\rho_1^2(|T_0|) = V_0\rho_1^2(|T_0|)$$

since the functions $z\rho_1^2(|z|)/|z|$ and $(\operatorname{Re} z)\rho_1^2(|z|)/|z|$ coincide on $\sigma(T_0)$. This implies

$$\rho_1(|T|)(\operatorname{Re} V_0)\rho_1(|T|) - V_0\rho_1^2(|T|) = (\rho_1(|T|) - \rho_1(|T_0|))(\operatorname{Re} V_0)\rho_1(|T|)+ \\ + \rho_1(T_0)(\operatorname{Re} V_0)\left(\rho_1(|T|) - \rho_1(|T_0|)\right) - V_0(\rho_1^2(|T|) - \rho_1^2(|T_0|)). \tag{3.2.5}$$

From Proposition 2.1.7 it follows that

$$\||T| - |T_0|\| \leqslant C(\lambda, \Gamma)\delta_P G(\delta_P)$$

and, as a consequence,

$$\|\rho_1(|T|) - \rho_1(|T_0|)\| \leqslant C(\lambda, \Gamma)\delta_P G(\delta_P), \quad \|\rho_1^2(|T|) - \rho_1^2(|T_0|)\| \leqslant C(\lambda, \Gamma)\delta_P G(\delta_P).$$

From (3.2.4), (3.2.5), and (3.2.3) we get

$$\|S - V_0\| \leqslant C_1(\lambda, \Gamma)\delta_P G(\delta_P). \tag{3.2.6}$$

Therefore, for $\delta_P < \delta_P^0(\lambda, \Gamma)$ with sufficiently small $\delta_P^0(\lambda, \Gamma)$, the element S will be invertible and have $S = U|S|$ for some unitary U. In addition, since the element

$$\Pi S \Pi = \Pi\rho_1(|T|) \operatorname{Re} V_0\rho_1(|T|)\Pi + \Pi V\rho_2(|T|)\Pi$$

is self-adjoint and $[\Pi, S] = 0$, the element ΠU will also be self-adjoint. Therefore, we have Properties 2–4.

Similarly to Lemma 3.1.2, let $\tilde{\rho}_2(z) := z|z|^{-1}\rho_2^2(|z|)$. Since $[P, \operatorname{Re} V_0] = 0$, we have

$$\|[P, U]\| \leqslant C_1\|[P, S]\| \leqslant 2C_1\left(\|[P, \rho_1(|T|)]\| + \|[P, \tilde{\rho}_2(|T|)]\|\right) \leqslant C\delta_P,$$

which yields Property 1. Finally, Property 1 implies that $\operatorname{diag}_P U$ is close to U. From (3.2.6), U is close to V_0. Since $V_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$, we get Property 5. ■

The following lemma is the key step of the proof. It extends Lemma 3.2.5 to the case of general normal elements whose spectra contain line segments. Recall that $\mathcal{O}_r(0) = \{z \in \mathbb{C} \colon |z| < r\}$.

Lemma 3.2.6. *There exists $\delta_P^0 > 0$ such that for every unital C^*-algebra $\mathcal{A}$ of real rank zero and every normal $T \in \mathrm{M}_2(\mathcal{A})$ with*

$$\|[P,T]\| = \delta_P \leqslant \delta_P^0, \quad \sigma(T) \cap \mathcal{O}_3(0) \subset (-3;3),$$

and

$$\mathrm{diag}_P(T \pm iI) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A}), \tag{3.2.7}$$

there exists an element U with the properties from Lemma 3.2.5.

Proof. Let us describe the general idea first. We need to remove a part of the line segment on Figure 3.2. The spectrum of our element does not lie on a simple closed curve, so we cannot apply our previous lemmas directly. However, we can construct an auxiliary element (in our notation T_5) with this property such that in a neighbourhood of this line segment it looks the same as T. Then the element U obtained for T_5 can then be used for T. The construction of T_5 consists of several steps in which we remove the unneeded parts from the spectrum of T without affecting the segment $[-1;1]$.

The formal proof is as follows. There exists a smooth function $g_1 \colon \mathbb{C} \to \mathbb{C}$ such that $g_1 \in \mathrm{OL}(\mathbb{C})$, $g_1(z) = 3z/|z|$ for $|z| \geqslant 3$, $g_1(z) = z$ for $|z| \leqslant 2$, and $g_1(z)/z > 0$ for all $z \neq 0$. Let $T_1 = g_1(T)$. Then $\sigma(T_1) \subset \Theta = (-3;3) \cup \{z \in \mathbb{C} \colon |z| = 3\}$, see Figure 3.2.

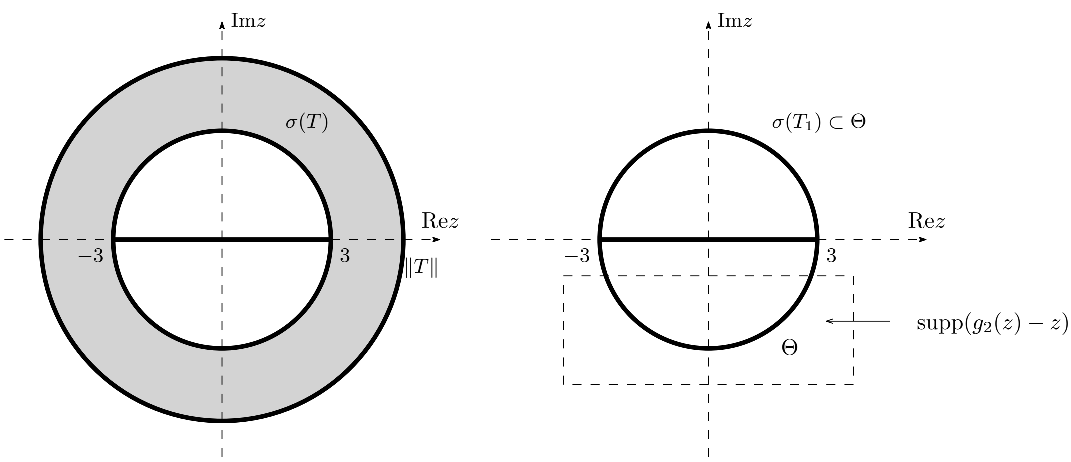

Figure 3.2: The spectra of T and $T_1 = g_1(T)$

The element T_1 satisfies (3.2.7) because for the operator family $T_t = tg_1(T) + (1-t)T$, the path

$$\operatorname{diag}_P(T_t \pm iI)$$

satisfies the assumptions of Lemma 3.1.1 if δ_P is sufficiently small.

There exists a diffeomorphism $g_2 \colon \mathbb{C} \to \mathbb{C}$ mapping the arc of Θ between $(-3-3i)/\sqrt{2}$ and $(3-3i)/\sqrt{2}$ into the line segment $[-2-2i; 2-2i]$ such that $g_2(z) = z$ outside the lower rectangle at the right of Figure 3.2. We have $g_2 \in \mathrm{OL}(\mathbb{C})$ since it is a smooth compactly supported perturbation of z.

Let $\Theta_2 = g_2(\Theta)$. There exists a map $g_3 \colon \mathbb{C} \to \mathbb{C}$ such that $g_3(z) = z$ outside the upper rectangle of Figure 3.3 and that $g_3(\Theta_2)$ is an admissible curve. Again, $g_3 \in \mathrm{OL}(\mathbb{C})$. Note that g_3 is not a diffeomorphism: it maps two top arcs of Θ_2 between -3 and 3 into one.

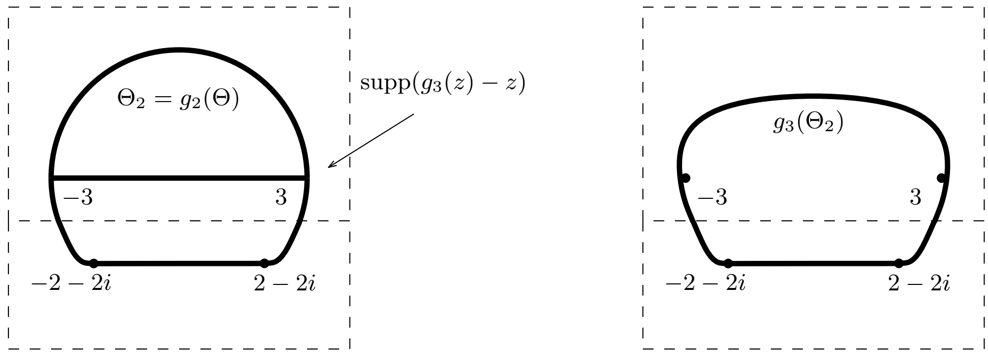

Figure 3.3: The spectra of $T_2 = g_2(T_1)$ and $T_3 = g_3(T_2)$

Let $T_2 = g_2(T_1)$, $T_3 = g_3(T_2)$. The element T_2 will satisfy (3.2.7) by the same arguments as for T_1: we can consider a linear homotopy between $g_2(z)$ and z. The same holds for T_3; note that we only need to consider $T_3 - iI$ since there is only one bounded connected component now. The element $T_3 + 2iI$ satisfies the assumptions of Lemma 3.2.5. Let U_3 be the unitary element obtained from that lemma. Now, as in Lemma 3.1.2, let $f_1 \in C^\infty(\mathbb{R}_+)$ be a nonincreasing function such that $f_1(t) = 1$ for $0 \leqslant t \leqslant 1$ and $f_1(t) = 0$ for $t \geqslant 2$. Let $f_2(t) = 1 - f_1(t)$. Consider

$$T_4 = U_3 f_1(|T_2 + 2iI|) + (T_2 + 2iI) f_2(|T_2 + 2iI|) - 2iI.$$

In this construction the unitary element U_3 is generated from T_3, and then is "attached" to T_2. It is possible because T_2 and T_3 coincide in the lower rectangle of

Figure 3.3 (in the sense that $f(T_2) = f(T_3)$ for any function f supported in the lower rectangle). Let $\widetilde{\Pi}_1$ be the spectral projection of T_2 onto $[-i; i]$; note that it coincides with the same projection for T_3. Let also $\widetilde{\Pi}_2$ be the similar projection for $[-2i; 2i]$. The elements T_2, T_3, U_3 and T_4 have block structure with respect to $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$. Similarly to the proof of Theorem 3.1.4, the element T_4 is normal. In addition, $\widetilde{\Pi}_2(T_4 + 2iI)$ is self-adjoint, $\widetilde{\Pi}_1(T_4 + 2iI)\Big|_{\operatorname{Ran}\widetilde{\Pi}_1}$ is unitary, and the spectrum of $(\widetilde{\Pi}_2 - \widetilde{\Pi}_1)T_4\Big|_{\operatorname{Ran}(\widetilde{\Pi}_2 - \widetilde{\Pi}_1)}$ is contained in $[-2i; 2i] \setminus (-i; i)$. Hence, $\sigma(T_4) \subset \Theta_4$ which is Θ_2 with part of the lower arc removed (see Figure 3.4). In addition, $\Pi T_4 = \Pi T$ (the middle part of Θ is left untouched), and $\|[P, T_4]\| \leqslant C_3\delta_P$. The element T_4 satisfies (3.2.7), because the linear homotopy between $T_4 + iI$ and $T_2 + iI$ satisfies the assumptions of Lemma 3.1.1.

There exists a smooth map $g_4 \colon \mathbb{C} \to \mathbb{C}$ such that $g_4(z) = z$ outside the oval-shaped areas on Figure 3.4 and that $g_4(\Theta_4)$ is an admissible curve (i. e. g_4 maps the remaining parts of the lower arc into the ends of central line segment, and does not affect the rest of Θ_3).

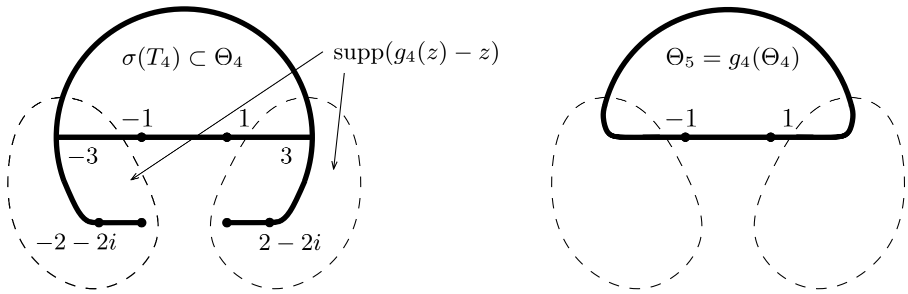

Figure 3.4: The spectra of T_4 and $T_5 = g_4(T_4)$

We finally get an element $T_5 = g_4(T_4)$ satisfying the assumptions of Lemma 3.2.6. This lemma gives an element U_5. Using the same idea as in constructing T_4, let

$$T_6 = U_5 f_1(|T|) + T_5 f_2(|T|).$$

The element T_6 is the operator T from the spectrum of which we have removed the segment $(-1; 1)$.

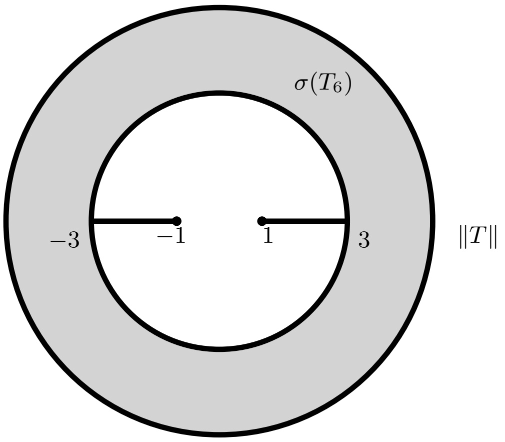

Figure 3.5: The spectrum of T_6

The element U can now be taken as the polar part of T_6. ∎

Proof of Theorem 1.3.3. Let us first apply Theorem 3.1.6. It reduces the result to the case of an element T_ε such that $\sigma(T_\varepsilon) \subset \Gamma_\varepsilon$. We also have

$$\operatorname{diag}_P(T_\varepsilon - \lambda I) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A}), \quad \forall \lambda \in \Lambda_\varepsilon,$$

and $\|[P, T_\varepsilon]\| \leqslant C\delta_P$ (provided that C_0 is sufficiently small). Let us assume that $\varepsilon = 1/2N$ for some $N \in \mathbb{N}$. It is clear that this will not affect the generality.

Consider the set $\Gamma_\varepsilon \cap [-1;1] \times [-1;1]$. This set consists of $4N(4N+1)$ horizontal and vertical line segments. By Δ denote the set of centres of all horizontal segments, and by Δ' the set of centres of vertical segments. Let $\lambda_j \in \Delta$. Consider the element $T_j = 6(T_\varepsilon - \lambda_j I)/\varepsilon$. This element satisfies $\|[P, T_j]\| \leqslant 6\delta_P\varepsilon^{-1} \leqslant 6C_0 G(\delta_P)^{-1}$ and the other assumptions of Lemma 3.2.6. Let U_j be the corresponding unitary element obtained from that lemma.

Similarly, for $\lambda'_j \in \Delta'$, the element $6i(T_\varepsilon - \lambda'_j I)/\varepsilon$ also satisfies the assumptions of Lemma 3.2.6. Let U'_j be the element obtained from Lemma 3.2.6 multiplied by $-i$.

Finally, let $f_1 = g_1^2$, $g_1 \in C^\infty(\mathbb{R}_+)$ be a nonincreasing function such that $f_1(t) = 1$ for $0 \leqslant t \leqslant 1$ and $f_1(t) = 0$ for $t \geqslant 2$. As in Lemma 3.1.4, consider

$$T'_\varepsilon = \sum_{j=1}^{4N(4N+1)} (\lambda_j + \varepsilon U_j/6) f_1\left(6|T_\varepsilon - \lambda_j I|/\varepsilon\right) + \sum_{j=1}^{4N(4N+1)} (\lambda'_j + \varepsilon U'_j/6) f_1\left(6|T_\varepsilon - \lambda'_j I|/\varepsilon\right)$$
$$+ T\left(1 - \sum_{j=1}^{4N(4N+1)} \left(f_1\left(6|T_\varepsilon - \lambda_j I|/\varepsilon\right) + f_1\left(6|T_\varepsilon - \lambda'_j I|/\varepsilon\right)\right)\right). \quad (3.2.8)$$

By the same arguments as in the proof of Property 4 from Lemma 3.1.4, we have $\|[P, T'_\varepsilon]\| \leqslant C\delta_P$. The spectrum of T'_ε is contained in the $5\varepsilon/12$-neighbourhood of $\varepsilon\mathbb{Z} + i\varepsilon\mathbb{Z}$ and therefore splits into disjoint connected components of diameters $5\varepsilon/12$.

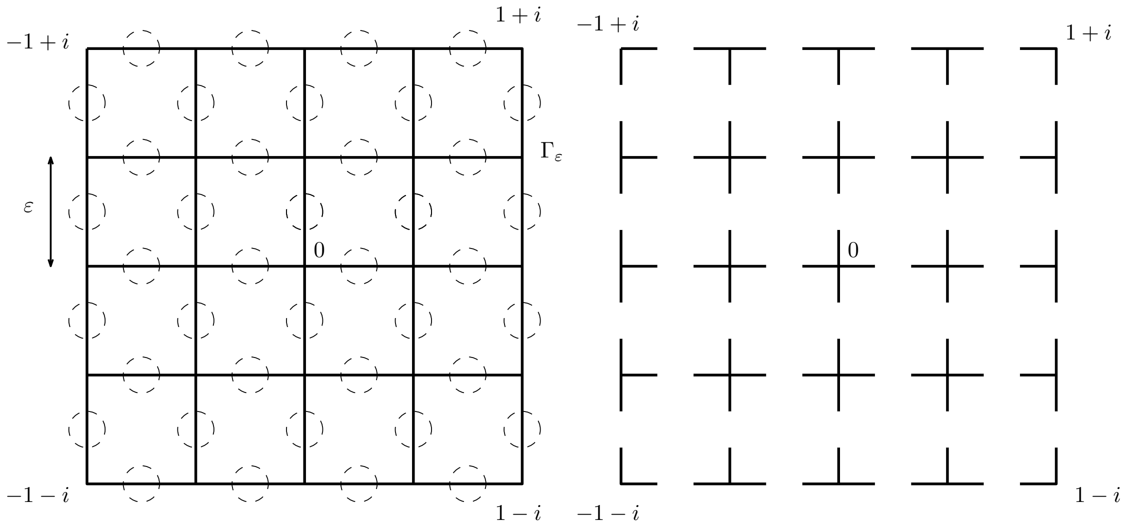

Figure 3.6: The spectra of T_ε and T'_ε (for $\varepsilon = 1/2$)

Finally, there exists a smooth function $h\colon \mathbb{C} \to \mathbb{C}$ such that $h(z) - z$ is $(1, i)$-periodic and that h maps the $5/12$-neighbourhood of every point of $\mathbb{Z} + i\mathbb{Z}$ into this point. The element $T' = \varepsilon h(T'_\varepsilon/\varepsilon)$ has finite spectrum and satisfies the assertions of the theorem. ∎

Remark 3.2.7. If $\mathcal{A} = \mathcal{B}(H)$ for some Hilbert space H, then the proofs can be simplified. The group $\mathrm{GL}(B(H))$ is connected; hence, we never need to check that the elements belong to GL_0. Moreover, we no longer need the smooth maps in Lemma 3.2.6 to be homotopic to the identity. Therefore, we can use a simpler construction which dates back to [17]: to map the line segment $(-1; 1)$ into $\mathbb{T} \setminus \{-1\}$ and map the rest of the spectrum to -1. Then we can flatten the circle and use Lemma 3.2.5; note that Lemma 3.2.2 becomes obvious since we have spectral projections for unitary operators. If $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$, then, in addition, $d_1(A) = 0$, $d_2(T) = \|[P, T]\|$.

3.3 Proof of Theorem 1.3.4

The results of the previous sections reduce the general case to the case of elements whose spectra are contained in sets of the form

$$\Sigma_\varepsilon \stackrel{\text{def}}{=\!=} (\varepsilon\mathbb{Z} \times \varepsilon\mathbb{Z}) \cap ([-1;1] \times [-1;1]).$$

It turns out that if $\sigma(T) \subset \Sigma_\varepsilon$, then we can remove its off-diagonal elements with respect to P in such a way that the element remains normal. The idea is to map the spectrum onto a line segment, then remove off-diagonal elements from the resulting self-adjoint element (it will remain self-adjoint), and then map it back. The choice of particular maps is important since it is the only place where we get a loss in the power of δ. The following lemma describes these maps.

Lemma 3.3.1. *Let $\varepsilon = 1/2N$, $N \in \mathbb{N}$. There exist two functions $f_N : \mathbb{C} \to \mathbb{R}$, $g_N : \mathbb{R} \to \mathbb{C}$ such that*

$$g_N(f_N(z)) = z, \quad \forall z \in \Sigma_\varepsilon, \tag{3.3.1}$$

and $\|g_N\|_{\mathrm{OL}(\mathbb{R})} \leqslant CN$, $\|f_N\|_{\mathrm{OL}(\mathbb{C})} \leqslant C$.

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be a 2-periodic function $\varphi(2k) = 1$, $\varphi(2k+1) = -1$ for $k \in \mathbb{Z}$. Let $\alpha \in C_0^\infty(\mathbb{R})$ satisfy $\alpha(x) = \arcsin(x/2)$ for $|x| \leqslant 1$ (i. e. α is an arbitrary smooth compactly supported extension of the arcsine). Let

$$f_N(x+iy) = y + \frac{\alpha(x)}{2\pi N}\varphi(2Ny).$$

Let also $\eta \in C^\infty(\mathbb{R})$ be a function satisfying

$$\eta(x) = k \text{ for } x \in [k - 1/4; k + 1/4], \quad \forall k \in \mathbb{Z},$$

and $\|\eta\|_{\mathrm{OL}(\mathbb{C}))} \leqslant C$. The function η can be constructed as a suitable periodic perturbation of x. Finally, consider

$$g_N(x) = 2\sin(2\pi Nx) + i\frac{\eta(2Nx)}{2N}. \tag{3.3.2}$$

The property (3.3.1) is verified by direct computation for $z = k/2N + il/2N$, $k, l = -2N, \dots, 2N$. The estimates in $\mathrm{OL}(\mathbb{R})$ and $\mathrm{OL}(\mathbb{C})$ follow from Lemma 2.1.3, since these functions are obtained from fixed smooth functions by scaling and multiplication by α. ∎

The following lemma is the concluding technical step of the proof. We use the functions f_N and g_N obtained in Lemma 3.3.1 to remove off-diagonal elements from an element with finite spectrum in such a way that it remains normal.

Lemma 3.3.2. *Suppose $T \in \mathrm{M}_2(\mathcal{A})$ is normal, $\|[P,T]\| = \delta_P$, and $\sigma(T) \subset \Sigma_\varepsilon$ for $\varepsilon > 0$. Then there exists a normal element $T' \in \mathrm{M}_2(\mathcal{A})$ such that $\|T - T'\| \leqslant C(\delta_P + \delta_P^2\varepsilon^{-2})$ and $[P,T'] = 0$.*

Proof. Without loss of generality, we may assume that $\varepsilon = 1/2N$ for some $N \in \mathbb{N}$. Otherwise, we can apply the statement to γT, where $1/2 < \gamma < 1$, and then multiply the result by γ^{-1}. The self-adjoint element $T_1 = f_N(T)$ satisfies $\|[P,T_1]\| \leqslant C\delta_P$ since the functions f_N are uniformly bounded in $\mathrm{OL}(\mathbb{C})$. We have $\|T_1 - \operatorname{diag}_P T_1\| \leqslant C\delta_P$. As T_1 is close to $\operatorname{diag}_P T_1$, we might expect that $T = g_N(T_1)$ is close to $T' = g_N(\operatorname{diag}_P T_1)$. Since $\operatorname{diag}_P T_1$ is self-adjoint and commutes with P, the element T' is normal and also commutes with P. Hence, T' is a normal approximation of T commuting with P.

Let us estimate their difference. We have

$$T = \operatorname{diag}_P T + [P,[P,T]],$$

and $T = g_N(T_1)$, $T' = g_N(\operatorname{diag}_P T_1)$. Therefore,

$$\begin{aligned}\|T - T'\| &= \|[P,[P,T]] + \operatorname{diag}_P g_N(T_1) - g_N(\operatorname{diag}_P T_1)\| \\ &\leqslant \|[P,T]\| + \|\operatorname{diag}_P g_N(T_1) - g_N(\operatorname{diag}_P T_1)\|.\end{aligned}$$

The first term of the right hand side is bounded by $C\delta_P$ because $\|[P,T]\| \leqslant C\delta_P$. The second term is of the form from Corollary 2.1.8. Recall (3.3.2) and split g_N into a sum of two functions. The function $2\sin(2\pi N x)$ is estimated by $CN^2\delta_P^2$ using Lemma 2.1.10. The remaining term is bounded by $C\delta_P$ using Corollary 2.1.8 since the family $\eta(2Nx)/2N$ is uniformly bounded in $\mathrm{OL}(\mathbb{R})$. Therefore, we have

$$\|T - T'\| \leqslant C(\delta_P + \delta_P^2\varepsilon^{-2}),$$

and $[P,T'] = 0$. ∎

With all the preparations made, we can now complete the proof of the main result.

Proof of Theorem 1.3.4. Using Theorem 1.2.2, construct normal elements $T \in \mathrm{M}_2(\mathcal{A})$ and $N \in \mathcal{A}$ satisfying $\|T - A \oplus N\| \leqslant C\delta^{1/2}$ and (as a corollary) $\|[P,T]\| \leqslant C\delta^{1/2}$. From the assumptions of Theorem 1.3.4 and since $\mathbb{C} \setminus \sigma(N)$ is connected, we have

$$\operatorname{dist}(A - \lambda I, \mathrm{GL}_0(\mathcal{A})) \leqslant \delta^{1/2}, \quad \operatorname{dist}(N - \lambda I, \mathrm{GL}_0(\mathcal{A})) = 0$$

Hence, $\operatorname{dist}(T - \lambda I, \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})) \leqslant C\delta^{1/2}$, which implies $d_2(T) \leqslant C\delta^{1/2}$.

There exists $C_1 > 0$ such that the element T satisfies the assumptions of Theorem 1.3.3 with $\varepsilon = C_1\delta^{1/3}$, $\delta_P = \delta^{1/2}$. The theorem gives a normal element T_1 with $\sigma(T_1) \subset \Sigma_\varepsilon$, $\|T - T_1\| \leqslant C\varepsilon$, and $\|[P, T_1]\| \leqslant C_2\delta_P = C_2\delta^{1/2}$. Lemma 3.3.2 applied to T_1 gives a normal element T' such that $\|T' - T_1\| \leqslant C_3(\delta_P + \delta_P^2\varepsilon^{-2}) \leqslant C\delta^{1/3}$ and $[P, T'] = 0$. Therefore, we have

$$\|T' - A \oplus N\| \leqslant C\delta^{1/3}.$$

Since T' commutes with P, we get that $PT'P$ is normal and

$$\|PT'P - A\| \leqslant C\delta^{1/3}.$$

Finally, it is easy to see that the element $A' = PT'P\frac{\|A\|}{\|PT'P\|}$ has the same properties and satisfies $\|A'\| \leqslant \|A\|$.

Remark 3.3.3. The element T' from Lemma 3.3.2 has the following special property: it is the image of the self-adjoint element $\operatorname{diag}_P T_1$ under the map g_N. Hence, its spectrum lies on a curve which is the one-to-one image of $[-1; 1]$. The same holds for $A' = PT'P$. Normal elements of this type are important since they admit normal liftings from quotient algebras. More precisely, if $A_\mathcal{I} = A_\mathcal{I}^* \in \mathcal{A}/\mathcal{I}$, where $\mathcal{I} \subset \mathcal{A}$ is a $*$-ideal, then it has a self-adjoint pre-image $A \in \mathcal{A}$ (since we can take the real part of any pre-image). Hence, $g_N(A)$ will be a normal pre-image of $g_N(A_\mathcal{I})$.

In addition, since the values of g_N belong to $[-1; 1] + i[-1; 1]$, the normal pre-image can be chosen to have norm not greater than $\sqrt{2}$.

3.4 Some applications

3.4.1 Two-sided estimate in $\mathcal{B}(H)$

Let $\mathcal{A} = \mathcal{B}(H)$ for a Hilbert space H. If $A \in \mathcal{A}$ is normal, then $\operatorname{dist}(A, \mathrm{GL}(\mathcal{B}(H))) = 0$. Indeed, if $A = U|A|$, then $U(\varepsilon I + |A|) \in \mathrm{GL}(\mathcal{B}(H))$ for every $\varepsilon > 0$. Moreover,

$\mathrm{GL}(\mathcal{B}(H)) = \mathrm{GL}_0(\mathcal{B}(H))$ since any unitary element can be continuously deformed into the identity (using spectral projections). Let us denote the set of all normal elements by $\mathcal{N} \subset \mathcal{B}(H)$.

Theorem 3.4.1. *For all $A \in \mathcal{B}(H)$, $\|A\| \leqslant 1$, $\|[A, A^*]\| = \delta$ we have*

$$\max\{d_1(A), \delta/9\} \leqslant \operatorname{dist}(A, \mathcal{N}) \leqslant C \max\{d_1(A)^{2/3}, \delta^{1/3}\}. \tag{3.4.1}$$

Proof. The right inequality follows from Theorem 1.3.4. To prove the left one, assume that $A = N + X$, where N is normal and $\|X\| \leqslant \|A\|$ (this is always possible). Then $\|N\| \leqslant 2\|A\| \leqslant 2$, and

$$\|[A, A^*]\| = \|[N, X^*] + [N^*, X] + [X, X^*]\| \leqslant 8\|X\| + \|X\|^2 \leqslant 9\|X\|.$$

Taking the infimum over all possible X (we can obviously consider only $\|X\| \leqslant \|A\|$), we get

$$\|[A, A^*]\| \leqslant 9 \operatorname{dist}(A, \mathcal{N});$$

Together with $\operatorname{dist}(N, \mathrm{GL}_0(\mathcal{B}(H))) = 0$ for all normal N, this implies the left inequality of (3.4.1). ∎

Note that in [6] it is shown that if $\dim\ker T \neq \dim\ker T^*$, then $\operatorname{dist}(T, \mathcal{N}) = \max\{m_e(T), m_e(T^*)\}$, where

$$m_e(T) = \inf_{\lambda \in \sigma_{\mathrm{ess}}(T)} |\lambda|,$$

and $\operatorname{dist}(T, \mathcal{N}) = 0$ if $\dim\ker T = \dim\ker T^*$.

In the case of a general C^*-algebra of real rank zero, Theorem 3.4.1 holds if we replace $\mathcal{N}$ by the set $\mathcal{N}_f$ of normal elements with finite spectra. It is known that some normal elements may not belong to $\overline{\mathcal{N}_f}$. Indeed, if $A \in \overline{\mathcal{N}_f}$, then $d_1(A) = 0$ because it is true for all elements with finite spectra (see Section 1.1). The converse is also true, see Theorem 1 or [16]. In the Calkin algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$, the condition $d_1(A) = 0$ is equivalent to A having trivial index function, see [16, Lemma 3.4] and references therein. The equivalence class of the operator (1) from Introduction is an example of a normal element of $\mathcal{C}(H)$ with non-trivial index function, and hence it cannot be approximated by elements with finite spectra.

3.4.2 Quasidiagonal operators and the BDF theorem

The results of [16, Section 3.3] admit quantitative versions. Let H be a separable Hilbert space. Suppose that $H = \oplus_k H_k$, where H_k are finite-dimensional Hilbert subspaces. Operators of the form $\oplus_k S_k$, where $\{S_k\}$ is a uniformly bounded system of operators acting in H_k each, are called *block diagonal* (with respect to the system $\{H_k\}$). If H_k can be chosen in such a way that $\dim H_k = 1$ for all k, then the operator is called *diagonal*. Equivalently, diagonal operators are block diagonal with normal blocks S_k. An operator $A \in \mathcal{B}(H)$ is called *quasidiagonal* if it is a compact perturbation of a block diagonal operator. The following result is well known and can be found in [18, Proposition 2.8] or [5, Corollaries 11.4 and 11.12].

Proposition 3.4.2. *The set of compact perturbations of normal operators in a separable Hilbert space H is norm closed and coincides with the set of all quasidiagonal operators $S \in \mathcal{B}(H)$ such that $[S, S^*] \in \mathcal{K}(H)$.*

The following is Lemma 3.7 from [16].

Proposition 3.4.3. *Let H be separable. For each $r > 0$, the set*

$$\{A \in \mathcal{B}(H)\colon A \text{ is normal and } \|A\| \leqslant r\} + \mathcal{K}(H)$$

is norm closed and coincides with the set of all quasidiagonal operators $S = \oplus_k S_k + K$ such that all S_k are normal, $\|S_k\| \leqslant r$, and $K \in \mathcal{K}(H)$.

For $A \in \mathcal{B}(H)$, let $\|A\|_{\mathrm{ess}} \stackrel{\mathrm{def}}{=\!=} \|A + \mathcal{K}(H)\|_{\mathcal{C}(H)}$, where $\mathcal{C}(H)$ is the Calkin algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ (it is usually called the essential norm). Let also

$$d_1^{\mathrm{ess}}(A) \stackrel{\mathrm{def}}{=\!=} \sup_{\lambda \in \mathbb{C}} \operatorname{dist}(A - \lambda I + \mathcal{K}(H), \mathrm{GL}_0(\mathcal{C}(H))) \leqslant d_1(A).$$

Theorem 3.4.4. *Let H be a separable Hilbert space, and assume that $A \in \mathcal{B}(H)$ satisfies $\|A\| \leqslant 1$, $\|[A, A^*]\| = \delta$, $d_1(A) \leqslant \delta^{1/2}$. Then the following holds.*

1. *Suppose that A is a compact perturbation of a normal operator. Then there exists a diagonal operator A_d such that*

$$A - A_d \in \mathcal{K}(H), \quad \|A_d\| \leqslant \|A\|, \quad \text{and} \quad \|A - A_d\| \leqslant C\delta^{1/3}.$$

2. Let $\delta_{\mathrm{ess}} \stackrel{\mathrm{def}}{=\!=} \|[A,A^*]\|_{\mathrm{ess}} > 0$*, so that* $[A,A^*] \notin \mathcal{K}(H)$*, and assume that* $d_1^{\mathrm{ess}}(A) \leqslant \delta_{\mathrm{ess}}^{1/2}$*. Then there exists a diagonal operator* A_d *such that*

$$\|A - A_d\|_{\mathrm{ess}} \leqslant C\delta_{\mathrm{ess}}^{1/3}, \quad \textit{and} \quad \|A - A_d\| \leqslant C(\delta^{1/3} + \delta_{\mathrm{ess}}^{1/9}).$$

Proof. For Part 1, let us apply Proposition 3.4.3. Suppose that $A = S + K$, where $S = \oplus_k S_k$ is normal with S_k acting in H_k, $\|S_k\| \leqslant \|A\|$, and K is compact. Let E_n be the orthogonal projection onto $\oplus_{k=1}^n H_k$. Let $\delta_n = \|K - E_n K E_n\|$. A simple computation shows that

$$\|[(E_nAE_n)^*, E_nAE_n]\| \leqslant \|[A,A^*]\| + 2\delta_n.$$

By Theorem 1.3.4, since the spaces E_nH are finite-dimensional, there exist normal operators A_n acting in E_nH such that

$$\|E_nAE_n - A_n\| \leqslant C(\|[A,A^*]\| + 2\delta_n)^{1/3}, \quad \|A_n\| \leqslant \|A\|.$$

Then the operators $B_n = A_n \oplus S_{n+1} \oplus S_{n+2} \oplus \ldots$ are normal and satisfy

$$\|B_n - A\| \leqslant (\|[A^*,A]\| + 2\delta_n)^{1/3} + \delta_n, \quad \|B_n\| \leqslant \|A\|.$$

The operator K is compact, hence $\delta_n \to 0$. Taking a sufficiently large n, we can choose $A_d = B_n$.

Assume now that $[A,A^*] \notin \mathcal{K}(H)$, i.e. $\delta_{\mathrm{ess}} > 0$. Since $d_1^{\mathrm{ess}}(A) \leqslant \delta_{\mathrm{ess}}^{1/2}$, we can apply Theorem 1.3.4 to the equivalence class $A + \mathcal{K}(H) \in \mathcal{C}(H)$. We obtain that there exists a normal element $A'_{\mathcal{C}} \in \mathcal{C}(H)$ with $\|(A + \mathcal{K}(H)) - A'_{\mathcal{C}}\|_{\mathcal{C}(H)} \leqslant C\delta_{\mathrm{ess}}^{1/3}$. By Remark 3.3.3, this element has spectrum lying on a curve and admits a normal pre-image $A' \in \mathcal{B}(H)$ with $\|A'\| \leqslant \sqrt{2}$. Hence, there exist a normal operator A', a compact operator K, and a bounded operator R such that

$$A = A' + K + R \tag{3.4.2}$$

with $\|A'\| \leqslant \sqrt{2}$ and $\|R\| \leqslant C\delta_{\mathrm{ess}}^{1/3}$. We have $d_1(A' + K) = 0$ because $A' + K$ is a compact perturbation of a normal operator. By Part 1, since

$$\|[A' + K, (A' + K)^*]\| \leqslant \|[A,A^*]\| + 3\|R\| \leqslant C(\delta + \delta_{\mathrm{ess}}^{1/3}),$$

there exists a diagonal normal operator A_d and a compact operator L such that $A' + K = A_d + L$ and

$$\|L\| \leqslant C\|[A' + K, (A' + K)^*]\|^{1/3} \leqslant C_1(\delta + \delta_{\mathrm{ess}}^{1/3})^{1/3}.$$

Since $A - A_d = L + R$, this implies Part 2 of the theorem. ∎

Similarly to [16], we can obtain the classical BDF theorem as a corollary.

Corollary 3.4.5. *Suppose that $A \in \mathcal{B}(H)$, $[A, A^*] \in \mathcal{K}(H)$, and $d_1^{\mathrm{ess}}(A) = 0$. Then A is a compact perturbation of a normal operator.*

Proof. Let us repeat the proof of Part 2 of Theorem 3.4.4 for $d_1^{\mathrm{ess}}(A) = 0$, $\delta_{\mathrm{ess}} = 0$. We get that

$$A = A' + K + R,$$

where K is compact, A' is normal, $\|A'\| \leqslant \sqrt{2}$, and $\|R\|$ can be made arbitrarily small. Hence, A belongs to the closure of the set from Proposition 3.4.3 with $r = \sqrt{2}$. Since the last set is closed, A is a compact perturbation of a normal operator. ∎

Chapter 4

The case of the normalized Hilbert-Schmidt norm

4.1 Proof of Theorem 1.4.1

Lemma 4.1.1. *Let* $-1 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n \leqslant 1$*. Then for any* $k, m \in \mathbb{N}$ *there exists a partition*

$$\{1, \ldots, n\} = J \cup \bigcup_{a=-m}^{m} L_a$$

such that

1. $\# J \leqslant \frac{n}{k}$.
2. $|\lambda_i - \lambda_j| < \frac{1}{m}$, $i, j \in L_a$.
3. $|\lambda_i - \lambda_j| \geqslant \frac{1}{km}$, $i \in L_a$, $j \in L_b$, $a \neq b$.

Proof. Consider the following partition $\{1, \ldots, n\} = \bigcup_{-km}^{km-1} I_j$:

$$I_j = \left\{ l \colon \lambda_l \in \left(\frac{j}{km}; \frac{j+1}{km} \right] \right\}, \; j = -km+1, \ldots, km-1;$$

$$I_{-km} = \left\{ l \colon \lambda_l \in \left[-1; -1 + \frac{1}{km} \right] \right\}.$$

Let us merge I_j with $j \equiv r \pmod{k}$ into J_r:

$$J_r = \bigcup_{a=-m}^{m-1} I_{ak+r}, \quad r = 0, 1, \ldots, k-1.$$

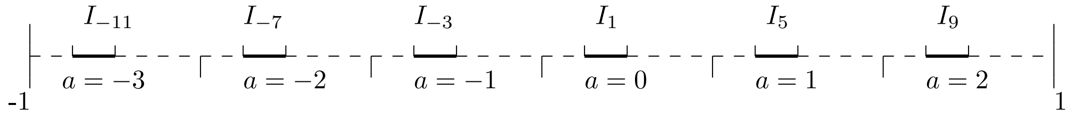

Figure 4.1: The subset of $[-1;1]$ corresponding to J_1 for $m=3$, $k=4$.

Obviously, $\bigcup_{r=0}^{k-1} J_r = \{1,\ldots,n\}$. By the pigeonhole principle, there exists an r_0 such that $\#J_{r_0} \leqslant \frac{n}{k}$. Let

$$J = J_{r_0}, \quad L_a = \bigcup_{(a-1)k+r_0<j<ak+r_0} I_j, \quad a=-m,\ldots,m.$$

Property 1 follows from the definition of J. Furthermore, every sub-interval of $[-1;1]$ corresponding J_r consists of $k-1$ subsequent intervals corresponding to I_l, and we have

$$|\lambda_i - \lambda_j| \leqslant \frac{k-1}{km} < \frac{1}{m}, \quad \forall i,j \in L_a,$$

which implies Property 2. Finally, two intervals corresponding to L_a and L_b with $a \neq b$ are separated by one of the intervals corresponding to I_{ak+r_0}, and hence Property 3 is true. ∎

Proof of theorem 1.4.1. We can choose a basis in $\mathbb{C}^n$ such that

$$X = \mathrm{diag}(\lambda_1,\ldots,\lambda_n), \quad -1 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n \leqslant 1.$$

Let us apply Lemma 4.1.1 to X for some $k,m \in \mathbb{N}$ (we shall fix their choice later). We obtain a partition $\{1,\ldots,n\} = J \cup \bigcup_{a=-m}^{m} L_a$. Let

$$X' = \mathrm{diag}(\mu_1,\ldots,\mu_n),$$

where

$$\mu_j = \begin{cases} \lambda_j, & j \in J \\ \frac{1}{2}\left(\min\limits_{k\in L_a} \lambda_k + \max\limits_{k\in L_a} \lambda_k\right), & k \in L_a. \end{cases}$$

Obviously, $\|X'\| \leqslant 1$. Property 2 from Lemma 4.1.1 implies $|\lambda_j - \mu_j| \leqslant \frac{1}{2m}$ for all j. Hence,

$$\|X - X'\|_{2,n}^2 = \frac{1}{n}\sum_{j=1}^{n} |\mu_j - \lambda_j|^2 \leqslant \frac{1}{4m^2}. \tag{4.1.1}$$

In the same basis, let $Y = \{Y_{ij}\}_{i,j=1}^{n}$, so that $[X,Y]_{ij} = (\lambda_i - \lambda_j)Y_{ij}$. We have

$$\sum_{i,j=1}^{n} |\lambda_i - \lambda_j|^2 |Y_{ij}|^2 = n\delta^2. \tag{4.1.2}$$

Let us define $Y' = \{Y'_{ij}\}_{i,j=1}^n$ by

$$Y'_{ij} = \begin{cases} Y_{ij}, & \exists b\colon i,j \in L_b; \\ 0, & \text{otherwise.} \end{cases}$$

The matrix Y' is self-adjoint and block diagonal. The blocks of Y' are sub-matrices of Y, the norm of each one does not exceed $\|Y\|$, which yields $\|Y'\| \leqslant \|Y\| \leqslant 1$. Since each block of X' is a scalar matrix, we have $[X', Y'] = 0$. Let us estimate the difference between Y and Y'.

$$n\|Y - Y'\|_{2,n}^2 \leqslant \sum_{a\neq b}\sum_{i\in L_a}\sum_{j\in L_b} |Y_{ij}|^2 + 2\sum_{i\in J}\sum_{j=1}^n |Y_{ij}|^2. \tag{4.1.3}$$

In the second sum we used the fact that $Y_{ij} = \overline{Y}_{ji}$. The first sum can be estimated using (4.1.2) and Property 3 from Lemma 4.1.1:

$$\sum_{a\neq b}\sum_{i\in L_a}\sum_{j\in L_b} |Y_{ij}|^2 \leqslant k^2m^2 \sum_{a\neq b}\sum_{i\in L_a}\sum_{j\in L_b} |\lambda_i - \lambda_j|^2|Y_{ij}|^2 \leqslant n\delta^2k^2m^2. \tag{4.1.4}$$

To estimate the second sum, consider two matrices $\widetilde{Y}$ and P,

$$\widetilde{Y}_{ij} = \begin{cases} Y_{ij}, & i \in J; \\ 0, & i \notin J; \end{cases}$$

$$P = \operatorname{diag}(p_1, \dots, p_n), \quad \text{where} \quad p_j = \begin{cases} 1, & j \in J, \\ 0, & j \notin J. \end{cases}$$

Clearly, $\widetilde{Y} = PY$ and $\|\widetilde{Y}\| \leqslant \|Y\| \leqslant 1$. Moreover,

$$\sum_{i\in J}\sum_{j=1}^n |Y_{ij}|^2 = \operatorname{tr}(PY^2P) \leqslant \operatorname{tr} P\|Y\|^2 \leqslant \#J \leqslant \frac{n}{k}. \tag{4.1.5}$$

Combining the inequalities (4.1.3)–(4.1.5), we obtain

$$\|Y - Y'\|_{2,n}^2 \leqslant \delta^2k^2m^2 + \frac{2}{k}.$$

Finally, let us fix the choice of k and m mentioned in the beginning of the proof,

$$k = \left[\frac{2}{\delta^{1/2}}\right], \quad m = \left[\frac{1}{2\delta^{1/4}}\right].$$

Then

$$\|X - X'\|_{2,n} \leqslant \frac{1}{2m} \leqslant 2\delta^{1/4}, \tag{4.1.6}$$

and

$$\|Y - Y'\|_{2,n} \leqslant \sqrt{\delta^{1/2} + \frac{2}{k}} \leqslant \sqrt{3}\,\delta^{1/4}, \tag{4.1.7}$$

where we used (4.1.1), the fact that $2\delta^{1/4} \leqslant 1$, and the inequality $[x]^{-1} \leqslant 2x^{-1}$ for $x \geqslant 1$. ∎

4.2 Proof of Theorem 1.4.2

The scheme from Theorem 1.4.1 can be applied simultaneously to the pairs (X_1, X_j), $j = 2, \ldots, m$. We denote the resulting operators by $\widetilde{X}_i$, $i = 1, \ldots, m$. If $\delta \leqslant 1/16$, then, by (4.1.6) and (4.1.7),

$$\|X - \widetilde{X}_1\|_{2,n} \leqslant 2\delta^{1/4}; \quad \|X_i - \widetilde{X}_i\|_{2,n} \leqslant \sqrt{3}\delta^{1/4}, \quad i = 2, \ldots, m.$$

Let us estimate the commutators of $\widetilde{X}_i$:

$$\begin{aligned} \|[\widetilde{X}_i, \widetilde{X}_j] - [X_i, X_j]\|_{2,n} &\leqslant \|(\widetilde{X}_i - X_i)\widetilde{X}_j\|_{2,n} + \\ &+ \|X_i(\widetilde{X}_j - X_j)\|_{2,n} + \|(X_j - \widetilde{X}_j)X_i\|_{2,n} + \|\widetilde{X}_j(X_i - \widetilde{X}_i)\|_{2,n} \leqslant 4\sqrt{3}\delta^{1/4}, \end{aligned}$$

where we again used (4.1.7) and the fact that $\|AB\|_{2,n} \leqslant \|A\|\|B\|_{2,n}$. This gives

$$\|[\widetilde{X}_i, \widetilde{X}_j]\|_{2,n} \leqslant (4\sqrt{3} + \delta^{3/4})\delta^{1/4} \leqslant 8\delta^{1/4}$$

and

$$[\widetilde{X}_1, \widetilde{X}_i] = 0, \quad i = 2, \ldots, m. \tag{4.2.1}$$

Let us again apply the scheme from Theorem 1.4.1 to the pairs $(\widetilde{X}_2, \widetilde{X}_j)$, $j = 3, \ldots, m$. Note that, since the construction preserves common invariant subspaces, it will also preserve the relations (4.2.1). Hence, we can repeat this $m-1$ times and obtain a set of m commuting operators $X'_1, \ldots, X'_m$. Let us find the conditions on δ and estimate the differences between X_i and X'_i.

We denote δ from the statement of the theorem by δ_1. On i-th step, δ_i is replaced by $\delta_{i+1} = 8\delta_i^{1/4}$. This gives

$$\delta_i = 8^{1+1/4+1/16+\ldots+1/4^{i-1}}\delta^{1/4^{i-1}} \leqslant 16\delta^{1/4^{i-1}}.$$

The sequence $\{\delta_i\}$ is increasing. Condition (1.4.2) implies $\delta_{m-1} \leqslant 1/16$ and, consequently, $\delta_i \leqslant 1/16$ for all $i = 1, \ldots, m-1$. Hence, the assumptions of Theorem 1.4.1 are met on every step.

Finally, let us estimate the differences between X_i and X_i'. On the i-th step, the matrices X are perturbed by matrices whose norms do not exceed

$$2\delta_i^{1/4} = \frac{1}{4}\delta_{i+1} \leqslant 4\delta^{1/4^i}.$$

Adding up the perturbations, we finally obtain

$$\|X_i - X_i'\|_{2,n} \leqslant 2(\delta_1^{1/4} + \delta_2^{1/4} + \ldots + \delta_{m-1}^{1/4}) \leqslant$$

$$\leqslant 4(\delta^{1/4^{m-1}} + \delta^{1/4^{m-2}} + \ldots + \delta^{1/4}) \leqslant 4\gamma(1 + \gamma^4 + \gamma^{16} + \ldots) \leqslant 5\gamma,$$

where $\gamma = \delta^{1/4^{m-1}} \leqslant 1/4$. ∎

Chapter 5

Polynomials of almost normal arguments in C^*-algebras

The proofs of Theorems 1.5.1–1.5.3 consist of two parts. Sections 5.1–5.3 are devoted to the "operator-theoretic" part, which is essentially based on Lemma 5.1.2. The "algebraic" part is the existence of representations (5.1.2) for the polynomials (5.2.1), (5.2.2), (5.2.5) which is discussed in Sections 5.4–5.6.

5.1 Positive elements of C^*-algebras

Recall that a Hermitian element $B \in \mathcal{A}$ is called *positive* ($B \geqslant 0$) if one of the following two equivalent conditions holds (see, for example, [13, §1.6]):

1. $\sigma(B) \subset [0, +\infty)$.

2. $B = H^*H$ for some $H \in \mathcal{A}$.

The set of all positive elements in $\mathcal{A}$ is a cone: if $A, A \geqslant 0$, then $\alpha A + \beta B \geqslant 0$ for all real $\alpha, \beta \geqslant 0$. There exists a partial ordering on the set of Hermitian elements of $\mathcal{A}$: $A \leqslant B$ iff $B - A \geqslant 0$. For $B = B^*$, we have

$$-\|B\|I \leqslant B \leqslant \|B\|I \tag{5.1.1}$$

and, moreover, if $0 \leqslant B \leqslant \beta I$, $\beta \in \mathbb{R}$, then $\|B\| \leqslant \beta$. The following fact is also well known.

Proposition 5.1.1. *Let $H \in \mathcal{A}$, $\rho > 0$. Then $H^*H \geqslant \rho^2 I$ if and only if the element H is invertible and $\|H^{-1}\| \leqslant \rho^{-1}$.*

Our proofs use the following simple lemma.

Lemma 5.1.2. *Let $A \in \mathcal{A}$ satisfy* (1.5.2), *and let*

$$q = \sum_{j=0}^{N} r_j^2 + \sum_{i=0}^{m-1} \left(\sum_{j=0}^{N} r_{ij}^2 \right) g_i, \tag{5.1.2}$$

where r_j, r_{ij}, g_i are real-valued polynomials of the form (1.5.3). *Assume that $g_i(A, A^*) \geqslant 0$, $i = 0, \dots, m-1$. Then*

$$q(A, A^*) \geqslant -C\delta I$$

with some non-negative constant C depending on r_j, r_{ij}, g_j.

Proof. Note that q is real-valued, so that $q(A, A^*)$ is self-adjoint. Since $g_i(A, A^*) \geqslant 0$, we have $g_i(A, A^*) = B_i^* B_i$ for some $B_i \in \mathcal{A}$. Then

$$r_{ij}(A, A^*) g_i(A, A^*) r_{ij}(A, A^*) = (B_i r_{ij}(A, A^*))^* (B_i r_{ij}(A, A^*)) \geqslant 0.$$

We also have $r_j(A, A^*)^2 \geqslant 0$. From (1.5.5), we have

$$\|q(A, A^*) - \sum_j r_j(A, A^*)^2 - \sum_{i,j} r_{ij}(A, A^*) g_i(A, A^*) r_{ij}(A, A^*)\| \leqslant C'\delta,$$

and now the proof is completed by using (5.1.1). ∎

5.2 Proofs of Theorems 1.5.1–1.5.3

Proof of Theorem 1.5.1. Proposition 5.4.2 below implies that the polynomial

$$q(z, \bar{z}) = p_{\max}^2 - |p(z, \bar{z})|^2 \tag{5.2.1}$$

admits a representation (5.1.2) with $m = 1$, $g_0(z, \bar{z}) = 1 - |z|^2$ because, by the definition of $p_{\max}$, the polynomial q is non-negative on the unit disk.

Let us apply Lemma 5.1.2 to q. By (1.5.5), we have $g_0(A, A^*) = I - AA^* \geqslant 0$. Therefore

$$q(A, A^*) \geqslant -C_1(p)\delta I$$

from which, using (5.2.1) and (1.5.5), we get

$$p_{\max}^2 I - p(A,A^*)^* p(A,A^*) \geqslant -C_2(p)\delta I,$$

$$p(A,A^*)^* p(A,A^*) \leqslant \left(p_{\max}^2 + C_2(p)\delta\right) I$$

and

$$\|p(A,A^*)\| \leqslant p_{\max} + \frac{C_2(p)\delta}{2p_{\max}}. \blacksquare$$

Proof of Theorem 1.5.2. Let now $p_{\max} := \max_{z\in S} |p(z,\bar z)|$. By Theorem 5.4.1, the polynomial

$$q(z,\bar z) = p_{\max}^2 + \varepsilon p_{\max} - |p(z,\bar z)|^2 \tag{5.2.2}$$

admits a representation (5.1.2) with

$$g_0(z,\bar z) = 1 - |z|^2, \quad g_i(z,\bar z) = |z-\lambda_i|^2 - R_i^2, \quad i = 1,\dots,m-1, \tag{5.2.3}$$

because it is strictly positive on the set S. Note that

$$S = \{z \in \mathbb{C}\colon\, g_i(z,\bar z) \geqslant 0,\ i = 0,\dots,m-1\}. \tag{5.2.4}$$

Proposition 5.1.1 and (1.5.9) imply

$$g_i(A,A^*) = (A-\lambda_i I)(A-\lambda_i I)^* - R_i^2 I \geqslant 0,$$

so we can again apply Lemma 5.1.2. Using (1.5.5), we obtain

$$q(A,A^*) \geqslant -C_1\delta I, \quad C_1 > 0,$$

$$p(A,A^*)p(A,A^*)^* \leqslant \left(p_{\max}^2 + \varepsilon p_{\max} + C_2(p,\varepsilon)\delta\right) I,$$

and

$$\|p(A,A^*)\| \leqslant p_{\max}\sqrt{1 + \frac{\varepsilon}{p_{\max}} + \frac{C_2(p,\varepsilon)\delta}{p_{\max}^2}} \leqslant p_{\max} + \varepsilon + \frac{C_2(p,\varepsilon)\delta}{p_{\max}}. \blacksquare$$

Proof of Theorem 1.5.3. Fix $\gamma > 0$. By Theorem 5.4.1, the polynomial

$$q(z,\bar z) = |p(z,\bar z) - \mu|^2 - \varkappa^2 + \gamma. \tag{5.2.5}$$

also admits a representation (5.1.2) with the same g_i given by (5.2.3). This is because, by the definitions of μ and $\varkappa$, we have $q(z,\bar z) > 0$ for all $z \in S$. Since $g_i(A,A^*) \geqslant 0$, Lemma 5.1.2 implies

$$q(A,A^*) \geqslant -C\delta I, \quad C > 0.$$

Using (5.2.5) and (1.5.5), we obtain

$$(p(A, A^*) - \mu I)^*(p(A, A^*) - \mu I) \geqslant \left(\varkappa^2 - \gamma - C'\delta\right) I. \tag{5.2.6}$$

Let us choose γ and δ_0 such that $\gamma + C'\delta \leqslant \varkappa^2/2$. Now, (5.2.6) and Proposition 5.1.1 yield

$$\|(p(A, A^*) - \mu I)^{-1}\| \leqslant \left(\varkappa^2 - \gamma - C'\delta\right)^{-1/2} \leqslant \varkappa^{-1} + \frac{\gamma}{\varkappa^3} + \frac{C'\delta}{\varkappa^3}.$$

Choosing $\gamma \leqslant \varepsilon\varkappa^3$, we obtain the required inequality with $\varkappa^{-3}C'$ instead of C.

The constant C', in general, depends on $p, \varkappa, \gamma$, and μ. Let us show that the theorem holds with C independent of μ. For $|\mu| \geqslant \|p(A, A^*)\| + \varkappa$ it is obvious as

$$\left\|(p(A, A^*) - \mu I)^{-1}\right\| \leqslant \frac{1}{|\mu| - \|p(A, A^*)\|} \leqslant \varkappa^{-1}.$$

Thus we can restrict the consideration to the compact set

$$M = \{\mu \in \mathbb{C} \colon |\mu| \leqslant \|p(A, A^*)\| + \varkappa,\ \operatorname{dist}(\mu, p(S)) \geqslant \varkappa\}.$$

The estimate $q(z, \bar{z}) \geqslant \gamma$ holds for all $\mu \in M$. The number N of the polynomials r_j and r_{ij} as well as their powers and coefficients are bounded uniformly on M by Remark 5.6.1. Since C' depends only on these parameters, C may be chosen independent of μ. ∎

5.3 Corollaries and remarks

Remark 5.3.1. As mentioned in the beginning of the section, the proofs rely on the existence of representations of the form (5.1.2) for certain polynomials. In addition, we need continuity of such a representation with respect to the parameter μ to establish Theorem 1.5.3. We are also interested in the possibility of explicitly computing the constants C and δ_0, which may be important in applications. It is clearly possible if we have explicit formulae for the polynomials in (5.1.2). We show below that this can be done in Theorems 1.5.2 and 1.5.3 (see Remark 5.6.1).

Remark 5.3.2. In general, it is not possible to find a constant C in Theorem 1.5.1 which would work for all polynomials p. As an example, consider $\mathcal{A} = \mathrm{M}_2(\mathbb{C})$,

$$A = \begin{pmatrix} 0 & \sqrt{\delta} \\ 0 & 0 \end{pmatrix}, \quad 0 < \delta < 1.$$

It is clear that the element A satisfies (1.5.9). Let $\varepsilon < 1$. There exists a continuous function f such that $f(z) = -1/z$ whenever $|z| \geqslant \varepsilon$ and $|f(z)| \leqslant 1/\varepsilon$ for $|z| \leqslant 1/\varepsilon$. There also exists a polynomial $q(z, \bar{z})$ such that $|q(z, \bar{z}) - f(z)| \leqslant \varepsilon$ for $|z| \leqslant 1$. Now, let

$$p(z, \bar{z}) = \frac{1}{\varepsilon}\left(z + z^2 q(z, \bar{z})\right).$$

We have $p_{\max} \leqslant 2 + \varepsilon^2$, but, since $A^2 = 0$, $p(A, A^*) = A/\varepsilon$ and $\|p(A, A^*)\| = \sqrt{\delta}/\varepsilon$. Taking ε small, we see that (1.5.7) cannot hold with a C independent of p.

Proposition 5.3.3. *Under the assumptions of Theorem* 1.5.2*, there exists a constant $C(p, \varepsilon)$ such that*

$$\|\operatorname{Im} p(A, A^*)\| \leqslant \max_{z\in S} |\operatorname{Im} p(z, \bar{z})| + \varepsilon + C(p, \varepsilon)\delta.$$

Proof. It suffices to apply Theorem 1.5.2 to the polynomial $q(z, \bar{z}) = \frac{p(z,\bar{z}) - \overline{p(z,\bar{z})}}{2i}$. ∎

In other words, if the values of p on S are almost real, then the element $p(A, A^*)$ itself is almost self-adjoint.

Proposition 5.3.4. *Under the assumptions of Theorem* 1.5.2*, there exists a constant $C(p, \varepsilon)$ such that*

$$\|p(A, A^*)p(A, A^*)^* - I\| \leqslant \max_{z\in S} \left||p(z, \bar{z})|^2 - 1\right| + \varepsilon + C(p, \varepsilon)\delta, \tag{5.3.1}$$

$$\|p(A, A^*)^* p(A, A^*) - I\| \leqslant \max_{z\in S} \left||p(z, \bar{z})|^2 - 1\right| + \varepsilon + C(p, \varepsilon)\delta. \tag{5.3.2}$$

Proof. It is sufficient to apply Theorem 1.5.2 to the polynomial $q(z, \bar{z}) = |p(z, \bar{z})|^2 - 1$ and use (1.5.5). ∎

Remark 5.3.5. Denote the right hand side of (5.3.1), (5.3.2) by γ. If $\gamma < 1$ then

$$(1 - \gamma)I \leqslant p(A, A^*)^* p(A, A^*) \leqslant (1 + \gamma)I$$

and

$$(1 - \gamma)I \leqslant p(A, A^*)p(A, A^*)^* \leqslant (1 + \gamma)I,$$

which implies that $p(A, A^*)$ and $p(A, A^*)^* p(A, A^*)$ are invertible. The element

$$U = p(A, A^*)\left(p(A, A^*)^* p(A, A^*)\right)^{-1/2}$$

is unitary (because it is invertible and $uu^* = 1$) and close to u,

$$\|p(A,A^*) - U\| \leqslant \sqrt{1+\gamma}\left(\frac{1}{\sqrt{1-\gamma}} - 1\right) \to 0 \quad \text{as} \quad \gamma \to 0.$$

Thus if the absolute values of p on S are close to 1 then $p(A,A^*)$ is close to a unitary element.

Definition 5.3.6. The set

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C}\colon \|(A-\lambda I)^{-1}\| > 1/\varepsilon\} \cup \sigma(A)$$

is called the *ε-pseudospectrum* of the element $A \in \mathcal{A}$.

Its main properties are discussed, for example, in [12, Ch. 9]. Note that, under the assumptions of Theorem 1.5.3, $\sigma_\varepsilon(A) \subset \mathcal{O}_\varepsilon(S)$ for all $\varepsilon > 0$, where $\mathcal{O}_\varepsilon(S)$ is the ε-neighbourhood of S. If A is normal then

$$\sigma_\varkappa(p(A,A^*)) = \mathcal{O}_\varkappa\left(p(\sigma(A))\right), \quad \varkappa > 0.$$

The following statement is Theorem 1.5.3 reformulated in these terms.

Proposition 5.3.7. *Under the assumptions of Theorem* 1.5.3*, for all $\varepsilon > 0$ and $\varkappa > 0$ there exist $C(p,\varkappa,\varepsilon)$ and $\delta_0(p,\varkappa,\varepsilon)$ such that*

$$\sigma_{\varkappa'}(p(A,A^*)) \subset \mathcal{O}_\varkappa(p(S)), \quad \forall \delta < \delta_0(p,\varkappa,\varepsilon),$$

where $(\varkappa')^{-1} = \varkappa^{-1} + \varepsilon + C(p,\varkappa,\varepsilon)\delta$.

Proof. Assume that $\operatorname{dist}(\mu, p(S)) \geqslant \varkappa$. By Theorem 1.5.3, $\|(p(A,A^*) - \mu I)^{-1}\| \leqslant (\varkappa')^{-1}$ and, consequently, $\mu \notin \sigma_{\varkappa'}\left(p(A,A^*)\right)$. ∎

5.4 Representations of non-negative polynomials

This section is devoted to a special case of the following theorem, which is often called *Putinar's Positivestellensatz.* As usual, we denote the ring of real polynomials in n variables by $\mathbb{R}[x_1,\ldots,x_n]$.

Theorem 5.4.1. [32] *Let $g_0,\ldots,g_{m-1} \in \mathbb{R}[x_1,\ldots,x_n]$. Let the set*

$$S = \{x \in \mathbb{R}^n \colon g_i(x) \geqslant 0,\ i = 0,\ldots,m-1\}$$

be compact and nonempty. If a polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ *is positive on* S *then there exist an integer* N *and polynomials*

$$r_i, r_{ij} \in \mathbb{R}[x_1, \dots, x_n], \quad i = 0, \dots, m-1, \quad j = 0, \dots, N,$$

such that

$$p = \sum_{j=0}^{N} r_j^2 + \sum_{i=0}^{m-1} \left(\sum_{j=0}^{N} r_{ij}^2 \right) g_i. \tag{5.4.1}$$

The first result of this type was proved in [7] for the case $m = 1$ with S being a disk. The proof was not constructive and involved Zorn's Lemma. In [32], Theorem 5.4.1 was proved in a similar way. In [35] and [28], an alternative proof of Theorem 5.4.1 was presented with its major part being constructive and based on the results of [31].

In Section 5.2, we have used Theorem 5.4.1 with the polynomials

$$g_0(x) = 1 - |x|^2, \quad g_i(x) = |x - \lambda_i|^2 - R_i^2, \quad i = 1, \dots, m-1, \tag{5.4.2}$$

where $x = (x_1, x_2)$, $|x|^2 = x_1^2 + x_2^2$, $\lambda_i \in \mathbb{R}^2$, and $R_i \in \mathbb{R}$. Let

$$S = \{x \in \mathbb{R}^2 : g_i(x) \geqslant 0,\ i = 0, \dots, m-1\}. \tag{5.4.3}$$

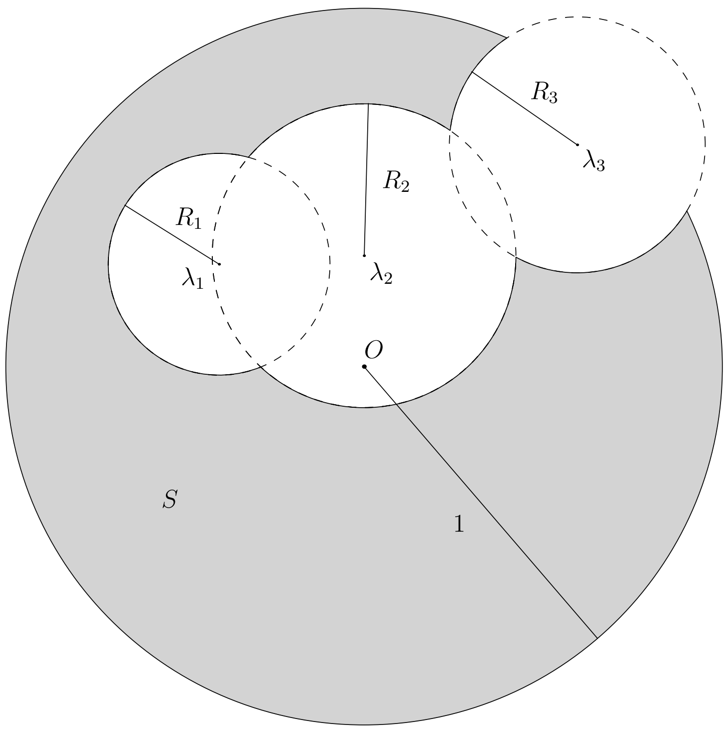

Figure 5.1: An example of the set S

As before, the set S is a unit disk with several "holes" centred at λ_i and of radii R_i, see Figure 5.1.

In the next section, we give a constructive proof of Theorem 5.4.1 for the polynomials (5.4.2). It turns out that in this case the proof simplifies and can be made completely explicit.

If we replace positivity of p with non-negativity, then for $m = 1$ the result still holds.

Proposition 5.4.2. *Let $p \in \mathbb{R}[x_1, x_2]$ be non-negative on the unit disk $\{x \in \mathbb{R}^2 : |x| \leqslant 1\}$. Then for some N it admits a representation*

$$p = \sum_{j=0}^{N} r_j^2 + \left(\sum_{j=0}^{N} s_j^2 \right) \left(1 - |x|^2\right),$$

where $r_j, s_j \in \mathbb{R}[x_1, x_2]$, $j = 0, \ldots, N$.

Proposition 5.4.2 is a particular case of [34, Corollary 3.3]. We have used it to obtain the representation (5.1.2) for the polynomial (5.2.1) in Theorem 1.5.1. Note that, in contrast with Proposition 5.4.2, the condition $p > 0$ on S in Theorem 5.4.1 cannot be replaced by $p \geqslant 0$ (see Remark 5.6.2 below).

5.5 Constructive proof for the polynomials (5.4.2)

The proposed proof relies on the general scheme introduced in [35] and [28] for the purposes of proving Theorem 5.4.1. In the special case (5.4.2), we make all the constants "computable" and also added a slight variation, the possibility of which was mentioned in [28]. Namely, instead of referring to results of [35] which use [31], we directly apply the results from [31] (see Proposition 5.5.4 and Lemma 5.5.6 below).

We need the following explicit version of the Lojasiewicz inequality (see, e.g., [4]). Recall that the angle between intersecting circles is the minimal angle between their tangents in the intersection points.

Lemma 5.5.1. *Let $g_0, \ldots, g_{m-1}$ be the polynomials* (5.4.2). *Assume that $S \neq \varnothing$ and none of the disks $\{x : g_i(x) > 0\}$ with $i > 0$ is contained in the union of the others. Then for any $x \in [-1, 1]^2 \setminus S$ the following estimate holds:*

$$\operatorname{dist}(x, S) \leqslant -c_0 \min\{g_0(x), \ldots, g_{m-1}(x)\}.$$

If the circles $S_i = \{x : g_i(x) = 0\}$ are pairwise disjoint or tangent, then $c_0 = R_{\min}^{-1}$ where $R_{\min} = \min_{i=0,\dots,m-1} R_i$ with $R_0 = 1$. Otherwise, c_0 can be chosen as

$$c_0 = \frac{\sqrt{2}+1}{R_{\min}^2 \sin(\varphi_{\min}/2)},$$

where $\varphi_{\min}$ is the minimal angle between the pairs of intersecting non-tangent circles S_i.

The proof relies on the following simple "high-school geometry" lemma. By $\angle BAC$ we denote the angle between the line segments AB and AC.

Lemma 5.5.2. *Let S_1, S_2 be a pair of intersecting circles with centers at λ_1, λ_2 and of radii R_1, R_2. Let y, y' be the intersection points of S_1 and S_2, and let $\varphi = \angle(S_1, S_2)$ be the angle between the circles S_1 and S_2. Assume that x lies inside of the first circle, so that $|x-\lambda_1| < R_1$, and suppose also that the points x and λ_2 are in the same half-plane with respect to the line $\lambda_1 y$. Finally, let $|x-y| \leqslant \min(R_1, R_2)\sin\varphi/2$. Then*

$$|x-y| \leqslant \frac{2}{\sin\varphi/2} \max_{i=1,2} \left(R_i - |x-\lambda_i|\right). \tag{5.5.1}$$

Proof. It is easy to see that

$$\angle y\lambda_1\lambda_2 + \angle y\lambda_2\lambda_1 = \varphi \text{ or } \pi - \varphi.$$

Therefore, $\max\left(\angle y\lambda_1\lambda_2, \angle y\lambda_2\lambda_1\right) \geqslant \varphi/2$, which gives

$$\frac{|yy'|}{2} = R_1 \sin\angle y\lambda_1\lambda_2 = R_2 \sin\angle y\lambda_2\lambda_1 \geqslant \min(R_1, R_2)\sin\varphi/2 \geqslant |x-y|. \tag{5.5.2}$$

Denote the intersection points of the line $\lambda_1\lambda_2$ with the circles S_1 and S_2 by z' and z respectively (the distance between z and z' is chosen to be smallest possible). From (5.5.2) it follows that x lies inside the sector $\lambda_1 y z'$.

Let us show that at least one of the following conditions holds:

1) $\angle(xy, S_1) \geqslant \varphi/2$;

2) $|x-\lambda_2| < R_2$ and $\angle(xy, S_2) \geqslant \varphi/2$.

Indeed, $\angle zyz' = \varphi/2$ or $(\pi-\varphi)/2$. If x does not belong to the intersection of the disks, then $\angle(xy, S_1) \geqslant \angle zyz' \geqslant \varphi/2$, and the first condition holds. If x belongs to the intersection, then $\max\left(\angle(xy, S_1), \angle(xy, S_2)\right) \geqslant \varphi/2$, and either 1) or 2) is true.

The cases 1) and 2) can be treated in a similar way. Let us restrict ourselves to the first one.

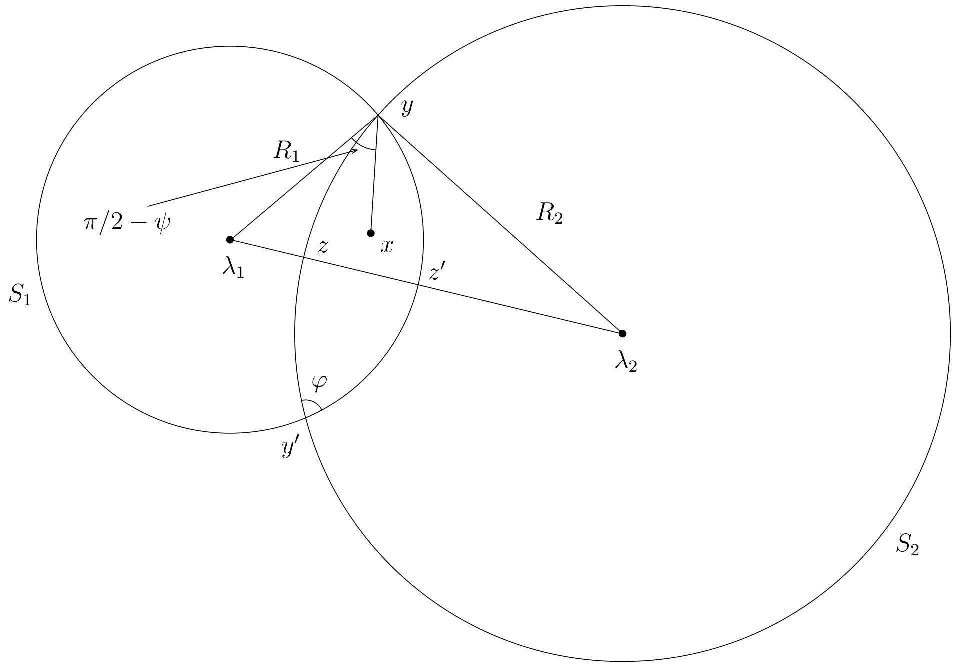

Figure 5.2: To the proof of Lemma 5.5.2

Denote $\psi = \angle(xy, S_1)$. By the cosine theorem for the triangle $xy\lambda_1$, we have

$$|x-\lambda_1| = \sqrt{R_1^2 + |x-y|^2 - 2R_1|x-y|\sin\psi} \leqslant \sqrt{R_1^2 - R_1|x-y|\sin\psi},$$

because, by assumption, $|x-y| \leqslant R_1\sin\varphi/2 \leqslant R_1\sin\psi$. Consequently,

$$R_1 - |x-\lambda_1| \geqslant R_1\left(1-\sqrt{1-\frac{|x-y|\sin\psi}{R_1}}\right) \geqslant \frac{|x-y|\sin\psi}{2} \geqslant \frac{|x-y|\sin\varphi/2}{2},$$

and this implies (5.5.1). ∎

Proof of Lemma 5.5.1. Let $x \notin S$. Then there exists i such that $g_i(x) < 0$. Let y be the closest to x point of S, $\operatorname{dist}(x,S) = |x-y|$. It is clear that $y \in S_i$, where $S_i = \{x \in \mathbb{R}^2 \colon g_i(x) = 0\}$. If y belongs to S_i only for a single i, or if it is a tangent point of S_i and S_j (but not an intersection point), then

$$\operatorname{dist}(x,S) = |x-y| = R_i - |x-\lambda_i| = \frac{R_i^2 - |x-\lambda_i|^2}{R_i + |x-\lambda_i|} \leqslant \frac{-g_i(x)}{R_{\min}}, \quad i \neq 0, \qquad (5.5.3)$$

$$\operatorname{dist}(x,S) = |x-y| = \frac{|x|^2-1}{|x|+1} \leqslant \frac{-g_0(x)}{R_0} \text{ for } i = 0, \qquad (5.5.4)$$

and there is nothing more to prove.

Let $\varepsilon = R_{\min} \sin(\varphi_{\min}/2)$, and consider the case $|x - y| \geqslant \varepsilon$. Then, from (5.5.3), (5.5.4), it follows that $-g_i(x) \geqslant R_{\min}\varepsilon$. However, $\operatorname{dist}(x, S) \leqslant \sqrt{2} + 1$ for all $x \in [-1, 1]^2$. Therefore,

$$\operatorname{dist}(x, S) \leqslant -\frac{\sqrt{2}+1}{\varepsilon R_{\min}}\, g_i(x),$$

which completes the proof in the case $|x - y| \geqslant \varepsilon$.

Suppose now that $|x - y| < \varepsilon$ and y is an intersection point of multiple circles. First assume that none of these circles is S_0. Then there exists S_j such that it contains y and its centre λ_j lies in the same half-plane as x with respect to $\lambda_i y$ (otherwise, the point y would not be the closest to x point of S). By Lemma 5.5.2,

$$|x - y| \leqslant \frac{-2 \min g_i(x)}{R_{\min} \sin(\varphi_{\min}/2)} \leqslant \frac{-(\sqrt{2}+1) \min g_i(x)}{R_{\min}^2 \sin(\varphi_{\min}/2)}.$$

If one of the circles is S_0, then the proof is essentially the same. There are several possibilities. There may exist a pair of circles S_i, S_j, $i, j > 0$, satisfying the conditions of Lemma 5.5.2. Or, alternatively, one of the circles may satisfy Condition 1) from the proof of Lemma 5.5.2. These two cases are in fact covered by previous considerations. The third possibility is when the point x lies outside of S_0 and the angle between xy and S_0 is greater than or equal to $\varphi/2$. This case is considered in the same way as the last part of Lemma 5.5.2 using the cosine theorem. We omit further details. ∎

For the polynomials

$$q(x) = \sum_{|\alpha| \leqslant d} q_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n],$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, consider the norm

$$\|q\| = \max_{\alpha} |q_\alpha| \frac{\alpha_1! \ldots \alpha_n!}{(\alpha_1 + \ldots + \alpha_n)!}. \tag{5.5.5}$$

The following proposition is also elementary and is proved in [28]:

Proposition 5.5.3. *Let $x, y \in [-1, 1]^n$, $q \in \mathbb{R}[x_1, \ldots, x_n]$, and $\deg q = d$. Then*

$$|q(x) - q(y)| \leqslant d^2 n^{d-1/2} \|q\| |x - y|.$$

The next proposition, which is a quantitative version of Pòlya's inequality, is proved in [31].

Proposition 5.5.4. *Let $f \in \mathbb{R}[y_1, \ldots, y_n]$ be a homogeneous polynomial of degree d. Assume that f is strictly positive on the simplex*

$$\Delta_n = \{y \in \mathbb{R}^n \colon y_i \geqslant 0, \ \sum_i y_i = 1\}. \tag{5.5.6}$$

Let $f_* = \min_{y \in \Delta_n} f(y) > 0$. Then, for any $N > \frac{d(d-1)\|f\|}{2f_*} - d$, all the coefficients of the polynomial $(y_1 + \ldots + y_n)^N f(y_1, \ldots, y_n)$ are positive.

Further on, without loss of generality, we shall be assuming that $0 \leqslant g_i(x) \leqslant 1$ for all $x \in S$ (if not, we normalize g_i multiplying them by positive constants).

Lemma 5.5.5. *Under the conditions of Theorem* 5.4.1 *with g given by* (5.4.2)*, let $p^* = \min_{x \in S} p(x) > 0$. Then*

$$p(x) - c_0 d^2 2^{d-1/2} \|p\| \sum_{i=0}^{m-1} (1 - g_i(x))^{2k} g_i(x) \geqslant \frac{p^*}{2}, \quad \forall x \in [-1,1]^2, \tag{5.5.7}$$

where an integer k is chosen in such a way that

$$(2k+1)p^* \geqslant m c_0 d^2 2^{d+1/2} \|p\|,$$

and c_0 is the constant from Lemma 5.5.1.

Proof. Let $x \in S$. Then $p(x) \geqslant p^*$. Due to our choice of k, the elementary inequality

$$(1-t)^{2k} t < \frac{1}{2k+1}, \quad 0 \leqslant t \leqslant 1, \quad k \geqslant 0, \tag{5.5.8}$$

implies that the absolute value of the second term in the left hand side of (5.5.7) does not exceed $\frac{p^*}{2}$.

Assume now that $x \in [-1,1]^2 \setminus S$. Let $y \in S$ be such that $\operatorname{dist}(x,y) = \operatorname{dist}(x,S)$. Then Proposition (5.5.3) and Lemma 5.5.1 yield

$$\begin{aligned} p(x) \geqslant p(y) - |p(x) - p(y)| &\geqslant p^* - d^2 2^{d-1/2} \|p\| \operatorname{dist}(x,S) \\ &\geqslant p^* + c_0 d^2 2^{d-1/2} \|p\| g_{\min}(x), \end{aligned} \tag{5.5.9}$$

where $g_{\min}(x)$ is the (negative) minimum of the values of $g_i(x)$. Note that $(1 - g_{\min}(x))^{2k} > 1$. From (5.5.9), we get

$$\begin{aligned} p(x) &- c_0 d^2 2^{d-1/2} \|p\| (1 - g_{\min}(x))^{2k} g_{\min}(x) \\ &\geqslant p(x) - c_0 d^2 2^{d-1/2} \|p\| g_{\min}(x) \geqslant p^*. \end{aligned}$$

On the other hand, (5.5.8) and the choice of k imply that the terms with $g_i(x) > 0$ contribute no more than

$$\frac{(m-1)c_0 d^2 2^{d-1/2}\|p\|}{2k+1} \leqslant \frac{p^*}{2}$$

to the sum (5.5.7). The remaining terms in (5.5.7) with $g_i(x) < 0$ may only increase the left hand side. ∎

Lemma 5.5.6. *Let $p \in \mathbb{R}[x_1, x_2]$ and $p_* = \min_{x\in[-1;1]^2} p(x) > 0$. Then, for some $M \in \mathbb{N}$,*

$$p = \sum_{|\alpha|\leqslant M} b_\alpha \gamma_1^{\alpha_1}\gamma_2^{\alpha_2}\gamma_3^{\alpha_3}\gamma_4^{\alpha_4} \tag{5.5.10}$$

where $b_\alpha \geqslant 0$,

$$\gamma_1(x) = \frac{1+x_1}{4}, \quad \gamma_2(x) = \frac{1-x_1}{4}, \quad \gamma_3(x) = \frac{1+x_2}{4}, \quad \gamma_4(x) = \frac{1-x_2}{4}. \tag{5.5.11}$$

This lemma was obtained in [31] for arbitrary convex polyhedra and associated linear functions γ_k. Below we prove it for the square $[-1,1]^2$, because in this particular case the formulae are considerably simpler.

Proof. Consider the following $\mathbb{R}$-algebra homomorphism

$$\varphi\colon \mathbb{R}[y_1, y_2, y_3, y_4] \to \mathbb{R}[x_1, x_2], \quad y_i \mapsto \gamma_i(x).$$

In order to prove the lemma, it suffices to find a polynomial $\tilde{p} \in \mathbb{R}[y_1, y_2, y_3, y_4]$ with positive coefficients such that $\varphi(\tilde{p}) = p$. If $p = \sum_{i+j\leqslant d} p_{ij}x_1^i x_2^j$ and

$$\tilde{p}_1(y) = \sum_{i+j\leqslant d} 2^{i+j} p_{ij}(y_1 - y_2)^i(y_3 - y_4)^j(y_1 + y_2 + y_3 + y_4)^{d-i-j},$$

then $\varphi(\tilde{p}_1) = p$ because

$$\varphi(y_1 + y_2 + y_3 + y_4) = 1, \quad 2\varphi(y_1 - y_2) = x_1, \quad 2\varphi(y_3 - y_4) = x_2.$$

Let

$$V = \{y \in \Delta_4\colon 2y_1 + 2y_2 = 2y_3 + 2y_4 = 1\},$$

where Δ_4 is the simplex (5.5.6). If $y \in V$ then $\tilde{p}_1(y) = p(4y_1 - 1, 4y_3 - 1) \geqslant p_*$, as $(4y_1 - 1, 4y_3 - 1) \in [-1,1]^2$. For an arbitrary y, let $y_0 \in V$ be such that $\operatorname{dist}(y, y_0) = \operatorname{dist}(y, V)$. Then, from Proposition 5.5.3,

$$\tilde{p}_1(y) \geqslant \tilde{p}_1(y_0) - |\tilde{p}_1(y) - \tilde{p}_1(y_0)| \geqslant p_* - d^2 2^{2d-1}\|\tilde{p}_1\| \operatorname{dist}(y, V). \tag{5.5.12}$$

Let

$$r(y) = 2(y_1 + y_2 - y_3 - y_4)^2.$$

It is easy to see that $\varphi(r) = 0$ and

$$r(y) = (2y_1 + 2y_2 - 1)^2 + (2y_3 + 2y_4 - 1)^2, \quad \forall y \in \Delta_4.$$

If we rewrite the last expression in the coordinates $\frac{y_1+y_2}{\sqrt{2}}, \frac{y_1-y_2}{\sqrt{2}}, \frac{y_3+y_4}{\sqrt{2}}, \frac{y_3-y_4}{\sqrt{2}}$ (obtained by two rotations by the angle $\pi/4$), then we get

$$r(y) \geqslant 8\,\mathrm{dist}(y, V)^2, \quad \forall y \in \Delta_4. \tag{5.5.13}$$

Let

$$\tilde{p}_2(y) = \tilde{p}_1(y) + \frac{2^{4d-6}d^4\|\tilde{p}_1\|^2}{p_*}(y_1 + y_2 + y_3 + y_4)^{d-2}r(y).$$

We still have $\varphi(\tilde{p}_2) = p$. The inequalities (5.5.12) and (5.5.13) imply that

$$\tilde{p}_2(y) \geqslant p_* - d^2 2^{2d-1}\|\tilde{p}_1\|\,\mathrm{dist}(y, V) + \frac{2^{4d-3}d^4\|\tilde{p}_1\|^2}{p_*}\,\mathrm{dist}(y, V)^2 =$$
$$\frac{2^{4d-3}d^4\|\tilde{p}_1\|^2}{p_*}\left(\mathrm{dist}(y, V) - \frac{p_*}{d^2 2^{2d-1}\|\tilde{p}_1\|}\right)^2 + \frac{p_*}{2} \geqslant \frac{p_*}{2}, \quad \forall y \in \Delta_4.$$

Finally, since $\tilde{p}_2$ is homogeneous, Proposition 5.5.4 with $N > \frac{d(d-1)\|\tilde{p}_2\|}{p_*} - d$ shows that all the coefficients of

$$\tilde{p}(y) = (y_1 + y_2 + y_3 + y_4)^N \tilde{p}_2(y)$$

are positive. Applying the homomorphism φ to $\tilde{p}$, we obtain the desired representation of p. ∎

End of the proof of Theorem 5.4.1. Let us apply Lemma 5.5.5 to p. It is sufficient to find a representation of the left hand side of (5.5.7), because the second term is already of the form (5.4.1). By Lemma 5.5.6, the left hand side of (5.5.7) can be represented in the form (5.5.10). Note that γ_i can be rewritten as

$$\frac{1}{4}(1 \pm x_{1,2}) = \frac{1}{8}\left((1 \pm x_{1,2})^2 + g_0(x) + x_{2,1}^2\right). \tag{5.5.14}$$

Substituting the last equality into (5.5.10), we obtain the desired representation for (5.5.7) and, therefore, for p. ∎

5.6 Some remarks

Remark 5.6.1. If g_i are given by (5.4.2) then, in principle, it is possible to write down explicit formulae for the polynomials appearing in (5.4.1). Indeed, assume that we have a polynomial p such that $p(x) \geqslant p^* > 0$ for all $x \in S$. Then

$$p(x) = \hat{p}(x) + c_0 d^2 2^{d-1/2} \|p\| \sum_{i=0}^{m-1} (1 - g_i(x))^{2k} g_i(x), \tag{5.6.1}$$

where k is chosen in such a way that $(2k+1)p^* \geqslant mc_0 d^2 2^{d+1/2} \|p\|$. The second term in the right hand side of (5.6.1) is an explicit expression of the form (5.4.1), and the coefficients of $\hat{p}$ can be found from (5.6.1). From Lemma 5.5.5, we know that $\hat{p}(x) \geqslant p^*/2$ for all $x \in [-1; 1]^2$. Now it suffices to represent

$$\hat{p}(x) = \sum_{k+l \leqslant \hat{d}} \hat{p}_{kl} \, x_1^k x_2^l$$

in the form (5.4.1). Consider the following polynomials

$$\tilde{p}_1(y) = \sum_{i+j \leqslant \hat{d}} 2^{i+j} \hat{p}_{ij} (y_1 - y_2)^i (y_3 - y_4)^j (y_1 + y_2 + y_3 + y_4)^{\hat{d}-i-j},$$

$$\tilde{p}_2(y) = \tilde{p}_1(y) + \frac{2^{4\hat{d}-4} \hat{d}^4 \|\tilde{p}_1\|^2}{p^*} (y_1 + y_2 + y_3 + y_4)^{\hat{d}-2} (y_1 + y_2 - y_3 - y_4)^2,$$

and

$$\tilde{p}(y) = (y_1 + y_2 + y_3 + y_4)^N \tilde{p}_2(y) \quad \text{where} \quad N > \frac{2\hat{d}(\hat{d}-1)\|\tilde{p}_2\|}{p^*} - \hat{d}.$$

If we replace y_i, $i = 1, 2, 3, 4$, with $\gamma_i(x)$ given by (5.5.11) in the definition of $\tilde{p}$, then we get $\hat{p}(x)$. The coefficients of $\tilde{p}$ are positive. Therefore, if we substitute y_i with γ_i and then apply (5.5.14), we obtain an expression of the form (5.4.1) for $\hat{p}(x)$. Combining it with (5.6.1), we get the desired expression for p. As a consequence, if we have a continuous family of positive polynomials with a uniform lower bound on S and uniformly bounded degrees, then the polynomials in the representation (5.4.1) may also be chosen to be continuously depending on this parameter, and also with uniformly bounded degrees.

Remark 5.6.2. In [33], an analogue of Theorem 5.4.1 for a non-negative polynomial p and $m > 1$ was established under some additional assumptions on the zeros of p. The next theorem shows that, in general, Theorem 5.4.1 may not be true if $p \geqslant 0$.

Theorem 5.6.3. *Let g_i be defined by* (5.4.2), *and assume that $\lambda_i \neq \lambda_j$ for some i and j. Then the polynomial $g_i g_j$ cannot be represented in the form* (5.4.1).

This result is probably well known to specialists, although we could not find it in the literature. For reader's convenience, we prove it below. Let g_i be defined by (5.4.2), and let

$$S_i = \{x \in \mathbb{R}^2 : g_i(x) = 0\}, \quad S_i(\mathbb{C}) = \{x \in \mathbb{C}^2 : g_i(x) = 0\}. \tag{5.6.2}$$

Lemma 5.6.4. *Let $q \in \mathbb{R}[x_1, x_2]$ be a polynomial such that $q(x) = 0$ on an open arc of S_i. Then $g_i \mid q$ (that is, q is divisible by g_i).*

Proof. Consider q as an analytic function on $S_i(\mathbb{C})$. Since the set $S_i(\mathbb{C})$ is connected, $q \equiv 0$ on the whole $S_i(\mathbb{C})$. Hilbert's Nullstellensatz (see, for example, [37, Section 16.3]) gives that $g_i \mid q^k$ for some integer k (in $\mathbb{C}[x_1, x_2]$ and, consequently, in $\mathbb{R}[x_1, x_2]$). As the polynomial g_i is irreducible, we have $g_i \mid q$. ∎

Lemma 5.6.5. *Let $\lambda_i \neq \lambda_j$. Then $S_i(\mathbb{C}) \cap S_j(\mathbb{C}) \neq \varnothing$.*

Proof. Let the circles S_i and S_j be given by the equations

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 = R_1^2, \quad (x_1 - b_1)^2 + (x_2 - b_2)^2 = R_2^2.$$

Subtracting one from the other, we get a system of a linear and a quadratic equation. The linear one is solvable because $\lambda_i \neq \lambda_j$. Substituting the solution into the quadratic equation, we reduce it to a non-degenerate quadratic equation in one complex variable, which also has a solution. ∎

Proof of Theorem 5.6.3. Assume that $p = g_i g_j$ satisfies (5.4.1). The left hand side of (5.4.1) vanishes on the set S_i. All the terms r_k^2 and $r_{kl}^2 g_k$ in the right hand side of (5.4.1) are non-negative on $S_i \cap \partial S$, and therefore are equal to zero on this set. By Lemma 5.6.4, they all are multiples of g_i. Similarly, all the terms in the right hand side are multiples of g_j. Therefore, $g_i \mid r_k$, $g_j \mid r_k$, and $g_i^2 g_j^2 \mid r_k^2$.

Since the polynomials g_k and g_i are coprime for all $k \neq i$, we have $g_i^2 \mid r_{kl}^2$ for $k \neq i$ and $g_j^2 \mid r_{kl}^2$ for $k \neq j$. Thus any term in the right hand side of (5.4.1) is a multiple of either $g_i^2 g_j$ or $g_i g_j^2$. Dividing (5.4.1) by $g_i g_j$, we see that the left hand side is identically equal to 1, and the right hand side vanishes on the intersection $S_i(\mathbb{C}) \cap S_j(\mathbb{C})$ which is nonempty by Lemma 5.6.5. This contradiction proves the theorem. ∎

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