Spectral properties of Hankel operators and related topics

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*This work would not have been possible without
the many people who shared a bit of their life
paths, academic or otherwise, with me.*

I am extremely grateful to all.

Preface

The purpose of this thesis is to describe some aspects related to the spectral theory of Hankel operators. Such linear transformations form probably one of the most important and versatile classes of operators to study spaces of analytic functions. They were introduced by Hermann Hankel in the form of finite matrices whose entries only depend on the sum of their coordinates. In the sense that given a finite sequence of complex numbers $\{\alpha(j)\}_{j=0}^{2N-2}$, he considered the $N \times N$ matrix with constant anti-diagonals:

$$\varGamma_N(\alpha) = \begin{pmatrix} \alpha(0) & \alpha(1) & \alpha(2) & \dots & \alpha(N-1) \\ \alpha(1) & \alpha(2) & \iddots & \iddots & \alpha(N) \\ \alpha(2) & \iddots & \iddots & \iddots & \alpha(N+1) \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \alpha(N-1) & \alpha(N) & \alpha(N+1) & \dots & \alpha(2N-2) \end{pmatrix}.$$

He studied them mainly for the algebraic properties of their determinants. Later on, D. Hilbert showed that the now called "finite Hilbert matrix", $\varGamma_N(\beta)$, a finite $N \times N$ Hankel matrix obtained by considering the sequence

$$\beta(j) = \frac{1}{\pi(j+1)}, \quad j \geq 0, \tag{P.1}$$

can be used to give a full answer to a problem of approximation theory. In particular, he studied the asymptotics of its determinants as $N \to \infty$, see [30]. From then on, infinite Hankel matrices and Hankel integral operators have been introduced. Infinite Hankel matrices are defined as operators on $\ell^2(\mathbb{Z}_+)$ as

$$\varGamma(\widehat{\omega})c(j) = \sum_{k=0}^{\infty} \widehat{\omega}(j+k)c(k), \quad c \in \ell^2(\mathbb{Z}_+), \tag{P.2}$$

where $\widehat{\omega}$ denotes the sequence of Fourier coefficients of the symbol ω, a bounded function on the unit circle $\mathbb{T}$. Similarly, integral Hankel operators are defined as integral operators on $L^2(\mathbb{R}_+)$ of the form

$$\boldsymbol{\varGamma}(\widehat{\boldsymbol{\omega}})f(t) = \int_0^{\infty} \widehat{\boldsymbol{\omega}}(t+s)f(s)ds, \tag{P.3}$$

where $\widehat{\boldsymbol{\omega}}$ is the Fourier transform of the symbol $\boldsymbol{\omega}$, a bounded function on $\mathbb{R}$. One of the first results in the theory of infinite Hankel matrices matrices is due to Kronecker. He characterised infinite Hankel matrices of the finite rank as those matrices whose entries are given by the Fourier coefficients of a rational function on the unit circle with poles outside the unit disc. A vast amount of literature is dedicated to the study of Hankel matrices and operators, because of their many applications in a wide variety of problems ranging from classical problems of analysis, such as moment problems, orthogonal polynomials and approximation theory, to more modern problems, such as stationary processes, control theory and mathematical physics.

To the reader's convenience, we divide the text into three main Parts:

- Part I. introduces the notation for some of the most useful function spaces and fixes the main concepts that are used in the subsequent parts;
- Part II. is the core of the text. We find the spectral density of Hankel matrices and integral operators with symbols belonging to the class of piece-wise continuous functions;
- Part III. is a "bonus" section, in which the reader can find two articles that are still related to Hankel operators, but deal with slightly different topics than the ones treated in Part 2.

More in detail, our exposition starts with the introductory Chapter 1, where we fix the notation for the subsequent sections. We introduce the relevant function spaces (L^p, Hardy, BMO, VMO) as well as operator algebras ($\mathfrak{B}(\mathcal{H}), \mathfrak{S}_p(\mathcal{H})$) on a separable Hilbert space $\mathcal{H}$. In Chapter 2 we introduce, in fairly general terms, the Schur-Hadamard multipliers. These are a sub-class of linear transformations of the spaces $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{S}_p(\mathcal{H})$, where $\mathcal{H}$ denotes either $\ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{R}_+)$. We describe some of their general properties and discuss some examples that will be relevant to the later chapters. In particular, we look at "truncations" of operators, for example the mapping that takes a fixed $N \times N$ matrix to its upper triangular part as follows:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,N} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & a_{N,3} & \dots & a_{N,N} \end{pmatrix} \mapsto \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,N} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,N} \\ 0 & 0 & a_{3,3} & \dots & a_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{N,N} \end{pmatrix}.$$

Using Schur-Hadamard multipliers, we introduce, from an abstract point of view, the logarithmic spectral density of a bounded operator with respect to a sequence of multipliers. In particular, we study some of their properties relevant to the later chapters and present some new abstract results which are useful in the later Chapters. We conclude the Chapter by proving a Theorem contained in Chapter 1, which highlights the connection between the subject of the Chapter and estimates for the norms of operators.

In Chapter 3, we move on to introducing Hankel operators. We do so by presenting four unitarily equivalent representations: as "compressed multiplication" operators acting on the Hardy class of the unit circle, as infinite matrices with constant anti-diagonals, as "compressed multiplication" operators on the real line and, finally, as integral operators on the positive half-line. In particular, it will be clear how the Hankel matrix in (P.2) and the Hankel integral operator in (P.3) are unitarily equivalent. The presentation is kept very informal so that the reader can focus more on the fact that these four different representations are all unitarily equivalent, as it is seen on the commutative diagram at the end of the Chapter. Of course, en passant, we will briefly touch upon the matter of boundedness and compactness of these objects by mentioning both Nehari and Fefferman's results.

Part II, with its two Chapters, is at the core of the text and uses the concepts introduced in the earlier Part. Specifically, in Chapter 4 and 5, we study the logarithmic spectral densities of bounded Hankel matrices and integral operators whose symbols are piecewise continuous functions, i.e. symbols that are continuous outside an at most countable set of jump discontinuities. Our results bring together results from two distinct fields of study within the theory of Hankel operators: spectral theory of non-compact Hankel operators and spectral asymptotics of compact ones.

In the non-compact case, our starting point is the formula, proved by S. Power, [50], characterising the essential spectrum of a Hankel matrix with piecewise continuous symbol in terms of the size of the jumps of the latter. It can be simply stated for the infinite Hilbert matrix, $\mathit{\Gamma}(\widehat{\gamma}) = \{\beta(j+k)\}_{j,k\geq 0}$, where β is the sequence in (P.1). In this case, its symbol is the function on $\mathbb{T}$ given by

$$\gamma(e^{i\vartheta}) = \pi^{-1}e^{-i\vartheta}(\pi-\vartheta), \quad \vartheta \in [0, 2\pi).$$

It is easy to see that γ has a unique jump-discontinuity on the unit circle $\mathbb{T}$ at $v = 1$. Moreover, the height of the jump is $2i$ and so, Power's formula, stated in its full generality in Chapter 4, shows that

$$\operatorname{spec}_{ess}(\mathit{\Gamma}(\widehat{\gamma})) = [0, 1]. \tag{P.4}$$

Its $N \times N$ truncation, $\mathit{\Gamma}_N(\widehat{\gamma})$, is a compact Hankel matrix. In this case, Widom, [61, Theorem4.3], showed that its singular values distribute inside $\operatorname{spec}_{ess}(\mathit{\Gamma}(\widehat{\gamma}))$ at a rate that is proportional to $\log(N)$ for large values of N. In other words, he showed that the singular-value counting function, defined as

$$\mathsf{n}(t; \mathit{\Gamma}_N(\widehat{\gamma})) := \#\{n : s_n(\mathit{\Gamma}_N(\widehat{\gamma})) > t\}, \quad t > 0,$$

where $s_n(\mathit{\Gamma}_N(\widehat{\gamma}))$ denote the singular values of $\mathit{\Gamma}_N(\widehat{\gamma}$, exhibits the following asymptotic behaviour:

$$\mathsf{n}(t; \mathit{\Gamma}_N(\widehat{\gamma})) = \log(N)(\mathsf{c}(t) + o(1)), \quad N \to \infty, \tag{P.5}$$

where c is a function supported on $\text{spec}_{ess}(\Gamma(\widehat{\gamma}))$, see Chapter 4 for its definition, and we refer to it as the *logarithmic spectral density* of $\Gamma(\widehat{\omega})$.

Similarly, in the case of Hankel integral operators, Power's formula, given in Chapter 5, shows that the Hankel operator

$$\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})f(t) = \int_0^\infty \frac{(1-e^{-(s+t)})}{\pi(s+t)} f(s)ds,$$

has essential spectrum given by the interval $[0,1]$. For the compact Hankel integral operator

$$\boldsymbol{\Gamma}_N(\widehat{\boldsymbol{\omega}})f(t) = \int_0^\infty \frac{(1-e^{-(s+t)})e^{-(s+t)/N}}{\pi(s+t)} f(s)ds,$$

the result of [**22**, Lemma 4.1] shows that the singular values of $\boldsymbol{\Gamma}_N(\widehat{\boldsymbol{\omega}})$ accumulate inside the essential spectrum of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ and furthermore:

$$\mathsf{n}(t;\boldsymbol{\Gamma}_N(\widehat{\boldsymbol{\omega}})) = \log(N)(\mathsf{c}(t)+o(1)), \quad N\to\infty, \tag{P.6}$$

with c being the same function appearing in (P.5). These examples provide the motivation for the results of Chapter 4 and 5 where we answer the following:

QUESTIONS.

(a) Can we extend (P.5) and (P.6) to include the case of a general piecewise continuous symbol?
(b) For which classes of truncations do (P.5) and (P.6) and their generalisations hold?
(c) For a self-adoint Hankel operator, can we derive similar formulae for the eigenvalue counting functions?

The answer to these questions is a combination of the material already given in Chapters 2 and 3. Indeed, the concept of a Schur-Hadamard multiplier provides the right framework to generalise truncations of an operator and helps us make sense of the universality of the logarithmic spectral density. On the other hand, the problem of generalising (P.5) and (P.6) to any piecewise continuous symbol requires one to delve into the theory of Hankel operators. The results of both Chapter 4 and 5 answer all of the three questions above in the positive. Moreover, the answers both in the non-selfadjoint and selfadjoint cases can be understood as some sort of superposition principle, in the sense that each jump-discontinuity of the symbol contributes independently to the logarithmic spectral density of the associated Hankel operator. This fact is often seen in the literature on the subject and it is referred to as "Localisation principle". We will discuss this in the relevant sections of both chapters. The universality of the logarithmic spectral density of a Hankel operator should not be surprising. In fact, one can draw a parallel with what is already known in the literature for Schrödinger operators and their dentity of states, which is invariant

under a change of boundary conditions and on the class of regions on which they are truncated, see for instance the results contained in [**14, 35**].
Even though the assumptions necessary for (P.5) and (P.6) to hold look very similar, the non-discrete nature of $(0, \infty)$ makes the process of truncating an integral operator a much more delicate matter and our assumptions reflect this. Indeed, in the case of a Hankel matrix, one only needs to truncate the "region at ∞" to obtain a compact operator for which it is possible to count singular values. In contrast, for a Hankel integral operator, we are forced to truncate both the "region at 0" as well as the "region at ∞" to ensure compactness. This is because, in the case of a matrix only the behaviour at ∞ of its entries may cause the matrix to be non-compact, while, for an integral operator, the picture becomes more complicated in so far as the behaviour of its integral kernel in a vicinity of 0 also has a role. The style and the organization of the exposition in both Chapters is kept as parallel as possible so that the reader can immediately spot the similarities between the two cases. We note that a version of Chapter 4 has already appeared on the Journal of Integral Equations and Operator Theory, [**18**].

The last Part contains two papers published in collaboration with Dr. Gebert and Prof. Pushnitski respectively. In Chapter 6, we analyse the asymptotics of the determinants of matrices of the form $I_N - \beta H_N$, where I_N is the $N \times N$ identity matrix, $|\beta| < 1$ and H_N is the $N \times N$ truncation of a Hankel matrix with a symbol which has finitely many jump discontinuities and some degree of smoothness away from them. The article in question has been published in the Bulletin of the London Mathematical Society, [**19**], and fits into the broader theory of asymptotics of Toeplitz+Hankel determinants, see [**6, 4, 5, 16, 17**]. However, the methods we used differ substantially from those used in most of the literature cited. Indeed, most of the authors in the field use the approach of solving a particular Riemann-Hilbert problem, while ours is more direct as we look at a power series decomposition and study the rate of decay of the traces of finite Hankel matrices.

Finally, Chapter 7 is the candidate's first paper, published in collaboration with Prof. A. Pushnitski, [**20**], and studies weighted Hankel operators with continuous spectrum. In particular, using methods of scattering theory with a trace-class condition, some formulae for the absolutely continuous part of the spectrum are explicitly derived. Furthermore, the result shows that these not only depend on the asymptotic behaviour of the kernel, as in the unweighted case, but also depend on the asymptotics of the weights. This work is partly based on the works of Howland, [**31, 32**], even though our approach is far simpler and does not make use of Mourre's inequality. As a consequence, we are not able to say anything about the presence of singular continuous spectrum, unlike Howland's results. However, as a byproduct, our methods can also be applied to weighted Hankel matrices to show that similar formulae hold, generalising the results in [**34**].

Even though the text is divided into Parts and Chapters, with their sections and subsections, only the latter are relevant in the ordering of the items appearing. The numbering of the statements reflects their position within the text. So for instance, Theorem 3.4.5 means the fifth statement of section 4 of Chapter 3. Displayed formulae are numbered independently, but follow a similar ordering. For example, equation (7.1.2) indicates the second formula of section 1 of Chapter 7.

Contents

Part I

Function Spaces, Hankel Operators and Schur-Hadamard Multipliers.

CHAPTER 1

Introduction and basic concepts

1. Spaces of Functions

This brief section serves only to introduce the notation and terminology for some of the function spaces that are used later on. We use the customary notation for the more common ones, such as spaces of k-times continuously-differentiable functions. We begin by introducing L^p-spaces for notational purposes.

DEFINITION 1.1.1. Let $\mathbb{X}$ denote either $\mathbb{T} := \{x \in \mathbb{C} \mid |x| = 1\}$ or $\mathbb{R}$ and $1 \leq p \leq \infty$. We define the space $L^p(\mathbb{X})$ as the space of all functions $f : \mathbb{X} \to \mathbb{C}$ such that

$$\|f\|^p_{L^p(\mathbb{X})} = \int_{\mathbb{X}} |f(x)|^p \, d\boldsymbol{m}(x) < \infty, \quad 1 \leq p < \infty$$
$$\|f\|_{L^\infty(\mathbb{X})} = \operatorname{ess\,sup}_{x\in\mathbb{X}} |f(x)| \,, \quad p = \infty.$$

where $d\boldsymbol{m}$ is either the normalised Lebesgue measure of $\mathbb{T}$ (i.e. $d\boldsymbol{m}(x) = (2\pi i x)^{-1} dx$, $x \in \mathbb{T}$) or the usual Lebesgue measure on $\mathbb{R}$.

In the case that $p = 2$, we introduce two unitary transformations:

(i) the Fourier transform on $L^2(\mathbb{T})$, i.e. the operator

$$\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}_+)$$
$$(\mathcal{F}f)(j) = \widehat{f}(j) := \int_{\mathbb{T}} f(z)\overline{z}^j d\boldsymbol{m}(z), \quad j \in \mathbb{Z}_+.$$

(ii) the Fourier Transform on $L^2(\mathbb{R})$, defined formally as

$$(\boldsymbol{\Phi} f)(\xi) = \widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\xi x} dx, \quad \xi \in \mathbb{R}.$$

With these at hand, we define the Hardy spaces $H^2_+(\mathbb{X})$.

DEFINITION 1.1.2. We define the **Hardy Spaces** on $\mathbb{T}$ and $\mathbb{R}$ as follows

(i) on $\mathbb{T}$, we define it as the space

$$H^2_+(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) \mid (\mathcal{F}f)(j) = 0 \ \forall j \leq -1 \right\},$$
$$\|f\|^2_{H^2_+(\mathbb{T})} = \sum_{j\geq 0} |(\mathcal{F}f)(j)|^2;$$

(ii) on $\mathbb{R}$, we define it as the space

$$H^2_+(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \mid (\boldsymbol{\Phi} f)(\xi) = 0 \ \forall \xi < 0 \right\},$$
$$\|f\|^2_{H^2_+(\mathbb{R})} = \int_{\mathbb{R}_+} |(\boldsymbol{\Phi} f)(\xi)|^2 \, d\xi.$$

REMARK 1.1.3. As a matter of fact, one can define a whole scale of Hardy spaces, denoted by $H^p(\mathbb{X})$, for $1 \le p \le \infty$. We do not define them here, as we do not make use of these spaces. For more on this, see [**37, 44**]. It is also important to note that the space $H^2_+(\mathbb{T})$ can be canonically identified with the space of functions of the form

$$\widetilde{g}(v) = \lim_{r \to 1-} g(rv), \quad v \in \mathbb{T},$$

where g is an analytic function on the unit disc, $\mathbb{D}$, and moreover

$$\|\widetilde{g}\|_{H^2_+(\mathbb{T})} = \sup_{r<1} \left(\int_{\mathbb{T}} |g(rv)|^2 \, d\boldsymbol{m}(v) \right)^{1/2}.$$

Similarly, $H^2_+(\mathbb{R})$ can be identified as the space of functions of the form

$$\widetilde{f}(x) = \lim_{\varepsilon \to 0+} f(x + i\varepsilon), \quad x \in \mathbb{R}$$

where f is an analytic function on the upper half plane and moreover we have

$$\|\widetilde{f}\|_{H^2_+(\mathbb{R})} = \sup_{\varepsilon > 0} \left(\int_{\mathbb{R}} |f(x + i\varepsilon)|^2 \, dx \right)^{1/2}.$$

For more on this, one can check [44]. Finally, it is worth mentioning that $H^2_+(\mathbb{X})$ is a Hilbert space when it is equipped with the usual inner product:

$$(f,\, g)_{H^2_+(\mathbb{X})} = \int_{\mathbb{X}} f(x)\overline{g(x)} d\boldsymbol{m}(x).$$

We will also mention the spaces of functions of Bounded Mean Oscillation, abbr. BMO, and Vanishing Mean Oscillation, abbr. VMO, on both $\mathbb{T}$ and $\mathbb{R}$. Let us start by BMO($\mathbb{T}$). A similar construction defines BMO($\mathbb{R}$).

Let $f \in L^1(\mathbb{T})$, and $I \subset \mathbb{T}$ be an arc. Put

$$\langle f \rangle_I = \frac{1}{m(I)} \int_I f(v) d\boldsymbol{m}(v).$$

The space of functions of *Bounded Mean Oscillation* BMO($\mathbb{T}$) is

$$f \in \mathrm{BMO}(\mathbb{T}) \iff f \in L^1(\mathbb{T}) \ \text{ and } \ \sup_I \langle |f - \langle f \rangle_I| \rangle_I < \infty.$$

Similarly, the space of functions of *Vanishing Mean Oscillation* VMO($\mathbb{T}$) is

$$f \in \mathrm{VMO}(\mathbb{T}) \iff f \in L^1(\mathbb{T}) \ \text{ and } \ \lim_{\boldsymbol{m}(I) \to 0} \langle |f - \langle f \rangle_I| \rangle_I = 0.$$

The construction of these spaces on the real line is identical, once one replaces the arc I with an interval and, in the case of $VMO(\mathbb{R})$, imposes the extra condition that the limit vanishes as $\boldsymbol{m}(I) \to \infty$.

The works of Fefferman and Sarason, [**21, 60**], show not only how interesting these spaces are, but also their connection with the theory of Hankel operators, as we shall mention later on.

2. Basic notions of Operator Theory

Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces with norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ respectively. We also denote by $\mathcal{X}^*$ the dual space of $\mathcal{X}$, i.e. the space of all bounded linear functionals on $\mathcal{X}$.

Definition 1.2.1. A linear transformation $A : \mathcal{X} \to \mathcal{Y}$ is said to be **bounded** if there is a constant $C > 0$ such that

$$\|Ax\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}.$$

Let $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ be the set of all such linear transformations, then we define the *operator norm* on $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = \sup_{\|x\|_{\mathcal{X}}=1} \|Ax\|_{\mathcal{X}}, \quad A \in \mathfrak{B}(\mathcal{X}, \mathcal{Y}).$$

If $\mathcal{X} = \mathcal{Y}$, for simplicity we set $\mathfrak{B}(\mathcal{X}, \mathcal{X}) = \mathfrak{B}(\mathcal{X})$.

Remark 1.2.2. If $\mathcal{Y}$ is a Banach space, i.e. it is a complete normed vector space, the space $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ equipped with the operator norm is a non-separable Banach algebra, i.e. it is a complete normed algebra, with no countable dense subset.

Definition 1.2.3. Let $A : \mathcal{X} \to \mathcal{Y}$ be a bounded linear transformation. Then its **adjoint** is the unique linear map $A^* : \mathcal{Y}^* \to \mathcal{X}^*$ satisfying the equality:

$$(A^*\ell)(x) = \ell(Ax),$$

for any $\ell \in \mathcal{Y}^*$ and any $x \in \mathcal{X}$. If $\mathcal{X} = \mathcal{Y}$, we say that A is **formally selfadjoint** if $A^* = A|_{\mathcal{X}^*}$. If $\mathcal{X}$ is a Hilbert space, we say A is **selfadjoint** if $A = A^*$.

A standard result in operator theory, see [**56**, Theorem VI.2], gives us that

PROPOSITION 1.2.4. *Let $\mathcal{X}, \mathcal{Y}$ be Banach Spaces and $A \in \mathfrak{B}(\mathcal{X}, \mathcal{Y})$. Then*

$$\|A\| = \|A^*\|.$$

We will also need the following useful theorem, see [**56**, Theorem I.7]

THEOREM 1.2.5 (BLT Theorem). *Let $\mathcal{X}$ be a normed linear space and let $\mathcal{Y}$ be a Banach space. Suppose $A : \mathcal{X} \to \mathcal{Y}$ is a bounded linear transformation. Then A can be uniquely extended to a bounded linear transformation, $\tilde{A}$, from the completion of $\mathcal{X}$ with respect to the norm $\| \cdot \|_{\mathcal{X}}$ to $\mathcal{Y}$. Furthermore $\|\tilde{A}\| = \|A\|$.*

From now on, we will be interested in the case where $\mathcal{X} = \mathcal{Y} = \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space equipped with inner product $(\cdot,\cdot)_{\mathcal{H}}$. In this case, we have the following

Definition 1.2.6. Let $A \in \mathcal{B}(\mathcal{H})$, then its **spectrum** is defined as the set

$$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} \mid \nexists (A - \lambda\mathbb{I})^{-1} \in \mathfrak{B}(\mathcal{H})\},$$

where $\mathbb{I}$ is the identity operator. We define its **resolvent set** as $\rho(A) = \mathbb{C} \setminus \operatorname{spec}(A)$.

With this at hand we have the following simple identities

PROPOSITION 1.2.7 (Resolvent identity). *Let $A, B \in \mathfrak{B}(\mathcal{H})$. If $\lambda \in \rho(A) \cap \rho(B)$, then*

$$(A - \lambda\mathbb{I})^{-1} - (B - \lambda\mathbb{I})^{-1} = (A - \lambda\mathbb{I})^{-1}(B - A)(B - \lambda\mathbb{I})^{-1}.$$

In the next chapters, we will often talk about the essential spectrum of an operator, this is defined as follows

Definition 1.2.8. Let $A \in \mathfrak{B}(\mathcal{H})$, then its **essential spectrum** is defined as the set of points $\lambda \in \operatorname{spec}(A)$ such that

(i) either $\dim \operatorname{Ker}(A - \lambda\mathbb{I}) = \infty$,
(ii) or $\operatorname{Ran}(A - \lambda\mathbb{I})$ is not closed.

Among the bounded operators, we distinguish an important class of them.

Definition 1.2.9. A bounded linear operator A is said to be **compact** if for any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that $x_n \to 0$ as $n \to 0$ weakly, one has that the sequence $\|Ax_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. We denote by $\mathfrak{S}_\infty(\mathcal{H})$ the set of all compact operators.

If A is compact, then we can find a non-increasing sequence $\{s_n(A)\}_{n=1}^{\infty} \subset \mathbb{R}_+$ and two ortho-normal systems $\{\xi_n\}, \{\eta_n\} \subset \mathcal{H}$ such that we can write

$$A = \sum_{n=1}^{\infty} s_n(A)(\cdot, \xi_n)_{\mathcal{H}}\eta_n,$$

where the sum converges absolutely to A. We call the series the **Schmidt expansion** of A and the $s_n(A)$ the **singular values** of A, which can be defined in two equivalent ways:

(1) $s_n(A) = \lambda_n(|A|)$, the eigenvalues of $|A| = \sqrt{A^*A}$ (A^* denotes the adjoint of A) in decreasing order with multiplicities taken into account;

(2) $s_n(A) = \inf\{\|A - B\| \mid rank(B) \leq n-1\}$, where $rank(B) = \dim \mathrm{Ran}(B)$.

See [44] for more on the above discussion. Among the compact operators, it is possible to distinguish a few sub-classes that are characterised by the rate of decay of the singular values. These are sometimes regarded as the non-commutative analog of ℓ^p-spaces. We define them below

DEFINITION 1.2.10. Let $0 < p \leq \infty$, then we define the p-th Schatten class $\mathfrak{S}_p(\mathcal{H})$ as the (sub-)set of compact operators A whose singluar values $s_n(A)$ satisfy:

$$\|A\|_p^p = \sum_{n=1}^{\infty} s_n(A)^p < \infty, \qquad p < \infty,$$
$$\|A\|_\infty = \sup_{n\in\mathbb{N}} s_n(A) < \infty, \qquad p = \infty.$$

Elements of $\mathfrak{S}_1(\mathcal{H})$ are often called **trace-class operators**, while elements of $\mathfrak{S}_2(\mathcal{H})$ are called **Hilbert-Schmidt operators**.

REMARK 1.2.11. The second definition of $s_n(A)$ immediately implies

$$\|A\|_\infty = \sup_n s_n(A) = s_1(A) = \|A\|,$$

so from now on, we will drop the subscript for the norm on $\mathfrak{S}_\infty(\mathcal{H})$.

For trace-class operators one can define an infinite-dimensional analog of the trace of a finite matrix as the following theorem shows, see [56, Chapter IV]:

PROPOSITION 1.2.12. *Let $A \in \mathfrak{S}_1(\mathcal{H})$, then for any orthonormal basis $\{\xi_n\}_{n=1}^\infty \subset \mathcal{H}$ the quantity:*

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} (A\xi_n, \xi_n)_{\mathcal{H}}$$

is finite and independent of the orthonormal system chosen. Furthermore, for any two $A, B \in \mathfrak{S}_1(\mathcal{H})$ one has that

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \quad \textit{and} \quad |\mathrm{Tr}(A)| \leq \|A\|_1,$$

*in other words $\mathrm{Tr}(\cdot)$ is a continuous linear functional on $\mathfrak{S}_1(\mathcal{H})$. We call $\mathrm{Tr}(A)$ the **trace** of A.*

We have a number of useful statements, see [11], collected in the following proposition:

PROPOSITION 1.2.13. *Let $\mathcal{H}$ be any separable Hilbert space and let $\mathfrak{S}_p = \mathfrak{S}_p(\mathcal{H})$. Then:*

(i) *for all* $1 \leq p \leq \infty$, $\mathfrak{S}_p$ *is a separable Banach space and the set of finite rank operators, which we shall denote by* $\mathfrak{F}$, *is* $\|\cdot\|_p$*-dense in* $\mathfrak{S}_p$. *Furthermore one has the equality:*

$$\|A\|_p = \mathrm{Tr}(|A|^p)^{1/p}, \qquad 1 \leq p < \infty;$$

(ii) *for any* $1 < p < \infty$, $\mathfrak{S}_p^* = \mathfrak{S}_q$, *where* $\mathfrak{S}_p^*$ *is the dual space of* $\mathfrak{S}_p$ *and* q *is such that* $q^{-1} + p^{-1} = 1$. *For* $p = 1$, *one has* $\mathfrak{S}_1^* = \mathfrak{B}(\mathcal{H})$ *and* $\mathfrak{S}_\infty^* = \mathfrak{S}_1$. *In particular, any linear functional* $\lambda \in \mathfrak{S}_p^*$ *can be written as:*

$$\lambda(A) = \mathrm{Tr}(AB^*),$$

for a unique $B \in \mathfrak{S}_q$;

(iii) *(Hölder inequality for* $\mathfrak{S}_p$*) let* $1 \leq r, p, q \leq \infty$ *be such that* $r^{-1} = p^{-1} + q^{-1}$. *For* $A \in \mathfrak{S}_p$ *and* $B \in \mathfrak{S}_q$, *one has that* $AB \in \mathfrak{S}_r$ *and:*

$$\|AB\|_r \leq \|A\|_p \|B\|_q.$$

REMARK 1.2.14.

(i) If $p \in (0, 1)$, the estimate (see [11, Chapter 11])

$$\|A + B\|_p^p \leq \|A\|_p^p + \|B\|_p^p,$$

shows that $\|\cdot\|_p$ is not a true norm as it does not satisfy the triangle inequality. So $\mathfrak{S}_p$ is not Banach space when $p \in (0,\, 1)$. However, it is a complete metric space when equipped with the metric $\rho(A; B) = \|A - B\|_p^p$.

(ii) The Proposition above shows that the pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{S}_p \times \mathfrak{S}_q &\to \mathbb{C} \\ (A, B) &\mapsto \langle A, B \rangle = \mathrm{Tr}(AB^*), \end{aligned} \tag{1.2.1}$$

with the usual understanding on the indices p and q, has the following properties:

(a) for all p one has: $\|A\|_p = \sup_{\|B\|_q = 1} |\mathrm{Tr}(AB^*)|$;

(b) for $p = 2$, the functional $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{S}_2$ and also it is easy to check that $\|A\|_2 = \sqrt{\langle A, A \rangle}$, and so, since $\mathfrak{S}_2$ is complete, it is a separable Hilbert space.

Hilbert-Schmidt operators can be easily characterised, as the following theorem shows, see [56, Theorem VI.23]

PROPOSITION 1.2.15. *Let* $(\mathcal{M}, \mu)$ *be a measure space and let* $\mathcal{H} = L^2(\mathcal{M}, \mu)$. *An operator* $A \in \mathfrak{B}(\mathcal{H})$ *is Hilbert-Schmidt if and only if there exists a function* $k \in L^2(\mathcal{M}^2, \mu \cdot \mu)$ *such that*

$$Af(t) = \int_{\mathcal{M}} k(t, s) a(s) d\mu(s).$$

Moreover,

$$\|A\|_2^2 = \int_{\mathcal{M}} \int_{\mathcal{M}} |k(t,s)|^2 \, d\mu(s) d\mu(t).$$

Before ending this introductory Section on operator theory, we give two well-known results, but first we need a definition.

Definition 1.2.16. Let $\{A_n\} \subset \mathfrak{B}(\mathcal{H})$ then we say that $A_n \to A$ as $n \to \infty$ in weak-* sense if and only if for any $B \in \mathfrak{S}_1(\mathcal{H})$, one has $\mathrm{Tr}(BA_n^*) \to \mathrm{Tr}(BA^*)$ as $n \to \infty$. We will say that a set $\mathfrak{R}$ is weak-$*$ dense in $\mathfrak{B}(\mathcal{H})$ if any $A \in \mathfrak{B}(\mathcal{H})$ is a weak-$*$ limit of a sequence in $\mathfrak{R}$.

THEOREM 1.2.17.

(i) Let $N \geq 1$, and let $\mathfrak{B}_N$ be the set of $N \times N$ matrices with complex entries. Then the set

$$\bigcup_{N \in \mathbb{N}} \mathfrak{B}_N \subset \mathfrak{S}_p(\ell^2(\mathbb{Z}_+))$$

is $\|\cdot\|_p$-dense for any $1 \leq p \leq \infty$ and it is weak-* dense in $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$.

(ii) Let $\xi > 0$, then the set

$$\bigcup_{\xi > 0} \mathfrak{B}(L^2(\xi^{-1}, \xi))$$

is weak-* dense in $\mathfrak{B}(L^2(\mathbb{R}))$. Furthermore, the set

$$\bigcup_{\xi > 0} \mathfrak{S}_p(L^2(\xi^{-1}, \xi))$$

is $\|\cdot\|_p$-dense in $\mathfrak{S}_p$ for any $1 \leq p \leq \infty$.

2.1. Positive operators. An important sub-space of $\mathfrak{B}(\mathcal{H})$ is the space of selfadjoint operators. Their spectral theory tells us that the spectrum of a self-adjoint operator is a subset of the real line. For this reason, we define a partial order, similar to the one of real numbers. This is done as follows:

Definition 1.2.18. Let $A \in \mathfrak{B}(\mathcal{H})$. Then A is said to be **non-negative**, $A \geq 0$, if and only if $(Af, f)_{\mathcal{H}} \geq 0$, for any $f \in \mathcal{H}$. Similarly, we say A is **positive**, $A > 0$, if and only if $(Af, f) > 0$.

Moreover, if $A, B \in \mathfrak{B}(\mathcal{H})$ are selfadjoint, then we write $A \leq B$ (resp. $A < B$) if and only if $B - A \geq 0$ (resp. $B - A > 0$).

Remark 1.2.19. It is useful to note that any non-negative operator is selfadjoint. To see this, first notice that for any $A \geq 0$ and any $x \in \mathcal{H}$, $(Ax, x)_{\mathcal{H}} = \overline{(Ax, x)_{\mathcal{H}}} =$

$(x, Ax)_{\mathcal{H}}$. Now, let $x, y \in \mathcal{H}$, then by the polarization identity we have

$$\begin{aligned} 4(Ax, y)_{\mathcal{H}} &= (A(x+y), x+y)_{\mathcal{H}} - (A(x-y), x-y)_{\mathcal{H}} \\ &- i(A(x+iy), x+iy)_{\mathcal{H}} + i(A(x-iy), x-iy)_{\mathcal{H}} \\ &= (x+y, A(x+y))_{\mathcal{H}} - (x-y, A(x-y))_{\mathcal{H}} \\ &- i(x+iy, A(x+iy))_{\mathcal{H}} + i(x-iy, A(x-iy))_{\mathcal{H}} \\ &= 4(A^*x, y)_{\mathcal{H}} \end{aligned}$$

and so A is selfadjoint. From this, it also follows that if $A \leq B$, then $\|A\| \leq \|B\|$.

A well-known result is the following

PROPOSITION 1.2.20. *An operator $A \geq 0$ if and only if* $\operatorname{spec}(A) \subset [0, \infty)$*.*

We note that any selfadjoint operator can be expressed as the linear combination of at most two non-negative operators. Indeed to see this, let $|A| = \sqrt{A^2}$ and let $A_{\pm} = (|A| \pm A)/2$, then using the previous proposition, it is easy to see that $A_{\pm} \geq 0$ and furthermore

$$A = A_+ - A_-.$$

This decompositon allows us to study selfadjoint operators through positive operators. For instance, we can see that since $A_+ A_- = \mathbb{O}$, then $\operatorname{Ran}(A_+) \perp \operatorname{Ran}(A_-)$ and so we can decompose the spectrum of A into the union of the spectra of A_+ and A_- as follows

$$\operatorname{spec}(A) = \operatorname{spec}(A_+) \cup \operatorname{spec}(-A_-).$$

In particular, for a compact operator A, the eigenvalues of $\pm A$, denoted by $\lambda_n^{\pm}(A)$ respectively, coincide with the singular values of $A_{\pm}$, i.e. we have

$$s_n(A_{\pm}) = \lambda_n^{\pm}(A), \quad n \geq 1.$$

This fact will turn out to be useful in Chapter 2.

Just as for real numbers, fractional powers of operators are only well-defined for non-negative operators. For instance, to make sense of A^{α}, for $\alpha \in (0, 1)$, one uses the representation

$$s^{\alpha} = c_{\alpha} \int_0^{\infty} t^{\alpha-1}(t+s)^{-1} s dt, \quad c_{\alpha} = \frac{sin(\pi\alpha)}{\pi},$$

which is valid only for $s \geq 0$. To be able to write A^{α} in terms of the integral above we need to force $A \geq 0$ so that $(t + A)^{-1}$ is well-defined almost everywhere. Of course, one would also like to understand the sense in which the above integral converges for a bounded operator A. In this case, the convergence is in the strong operator topology. We will return to this fact later at the end of Chapter 2.

The following theorem shows that fractional powers respect the partial order defined earlier, see [**11**, Theorem 2, Chp. 10.4]:

THEOREM 1.2.21 (Heinz Inequality)**.** *Let $A, B \geq 0$ and $0 < \alpha < 1$. If $A \leq B$, then $A^\alpha \leq B^\alpha$ and*

$$\|A^\alpha\| \leq \|B^\alpha\|.$$

Moreover if $B \in \mathfrak{S}_p$ for some $p \geq 1$, then $A^\alpha \in \mathfrak{S}_{p/\alpha}$ and we have a similar estimate for $\|A^\alpha\|_{p/\alpha}$.

The Heinz inequality is useful in proving the following theorem, [8, Theorem 1]:

THEOREM 1.2.22. *Let $A, B \geq 0$ be such that $T = B - A \in \mathfrak{S}_p$ for some $1 \leq p \leq \infty$. Then for any $0 < \alpha < 1$ one has $B^\alpha - A^\alpha \in \mathfrak{S}_{p/\alpha}$ and moreover:*

$$\|B^\alpha - A^\alpha\|_{\mathfrak{S}_{p/\alpha}} \leq \|T\|_{\mathfrak{S}_p}^\alpha.$$

The original paper, [8], containing this result and its proof is in Russian, so we will provide a translation of the latter at the end of the next Chapter. We choose to postpone the proof, as the argument refers to the theory of linear maps acting on $\mathfrak{B}(\mathcal{H})$, which we are going to discuss in the next Chapter.

CHAPTER 2

Schur-Hadamard Multipliers

1. General setting

In this Chapter, we introduce and discuss Schur-Hadamard multipliers. They play a central role in the study of spectral densities of Hankel operators discussed in Part II. These are a special type of linear transformations acting either on $\mathfrak{B}(\mathcal{H})$, where $\mathcal{H}$ denotes either $\ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{R}_+)$, or on any of its normed subspaces, such as the $\mathfrak{S}_p$ classes. Following [25], we use the following terminology:

DEFINITION 2.1.1. Let $(\mathfrak{R}, \|\cdot\|_{\mathfrak{R}})$ be any normed subspace of $\mathfrak{B}(\mathcal{H})$. Then we call any linear operator $\mathscr{T}_{\mathfrak{R}} : \mathfrak{R} \to \mathfrak{R}$ a **transformator** on $\mathfrak{R}$. We also define the norm of $\mathscr{T}_{\mathfrak{R}}$ as:

$$|||\mathscr{T}|||_{\mathfrak{R}} = \sup_{\|A\|_{\mathfrak{R}}=1} \|\mathscr{T}(A)\|_{\mathfrak{R}}.$$

We simply write $\mathscr{T}$ when it is clear on which subspace of $\mathfrak{B}(\mathcal{H})$ it acts, and denote by $\mathscr{I}$ and $\mathscr{O}$ the identity and the zero transformators respectively.

EXAMPLE 2.1.2. Let $\mathcal{H}$ be any separable Hilbert space and let $A \in \mathfrak{B}(\mathcal{H})$, then we can define the transformators of left and right multiplication by A

$$\mathscr{L}_A(B) = AB, \ \mathscr{R}_A(B) = BA, \quad B \in \mathfrak{B}(\mathcal{H}).$$

It is easy to verify that $|||\mathscr{L}_A|||_{\mathfrak{B}(\mathcal{H})} = |||\mathscr{R}_A|||_{\mathfrak{B}(\mathcal{H})} = \|A\|$. The same holds for any $\mathfrak{S}_p$-class, $p \geq 1$.

EXAMPLE 2.1.3 (**Main Example and Definition**). Let us now define Schur-Hadamard multipliers first on $\ell^2(\mathbb{Z}_+)$, then on $L^2(\mathbb{R}_+)$. Any bounded operator A on $\ell^2(\mathbb{Z}_+)$ can be regarded as an "infinite matrix" whose (i,j)-th entry with respect to the standard basis $\{e_j\}_{j\geq 0}$ of $\ell^2(\mathbb{Z}_+)$ is:

$$A_{i,j} = (Ae_i, e_j), \qquad i, j \geq 0.$$

Let $\tau = \{\tau(j,k)\}_{j,k\geq 0}$ be a bounded sequence, then the "*Schur-Hadamard multiplication*" of τ and A, denoted by $\tau \star A$, is the (possibly unbounded) operator on $\ell^2(\mathbb{Z}_+)$ whose (i,j)-th entry is

$$(\tau \star A)_{i,j} = \tau(i,j) A_{i,j}.$$

In particular, any sequence τ acts as a transformator, $\mathscr{S}_\tau$, on $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$, via the identity:

$$\mathscr{S}_\tau(A) = \tau \star A.$$

We call τ a "*Schur-Hadamard multiplier*". A vast amount of literature is dedicated to determining the main properties of τ necessary for $\mathscr{S}_\tau$ to be bounded on $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$ as well as on $\mathfrak{S}_p$, as we shall soon see below.

To begin with, suppose that $\mathscr{S}_\tau$ acts on $\mathfrak{S}_2$ rather than on $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$, then it is easy to show that $|||\mathscr{S}_\tau|||_{\mathfrak{S}_2}$ is finite if and only if τ is a bounded sequence. Indeed, for any $A \in \mathfrak{S}_2$ one has

$$\|\mathscr{S}_\tau(A)\|_2^2 = \sum_{i,j\geq 0} |\tau(i,j)A_{i,j}|^2 \leq \|\tau\|_\infty^2 \|A\|_2^2 \Rightarrow |||\mathscr{S}_\tau|||_{\mathfrak{S}_2} \leq \|\tau\|_\infty.$$

Let $\varepsilon > 0$ be given, by the definition of sup we can find i_0, j_0 so that $|\tau(i_0, j_0)| \geq \|\tau\|_\infty - \varepsilon$. Thus, if B is the operator whose matrix $\{b_{i,j}\}_{i,j\geq 0}$ is 0 everywhere except for the (i_0, j_0)-th position, where it is 1, we see that:

$$\|\mathscr{S}_\tau(B)\|_{\mathfrak{S}_2} = |\tau(i_0, j_0)| \geq \|\tau\|_\infty - \varepsilon,$$

and so we obtain

$$|||\mathscr{S}_\tau|||_{\mathfrak{S}_2} = \|\tau\|_\infty.$$

The above example can be considered a starting point to define Schur-Hadamard multipliers for bounded integral operators acting on $L^2(\mathbb{R}_+)$. To do this, let k be a measurable function on $\mathbb{R}_+^2$ and let $\mathrm{Op}(k) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be the (possibly unbounded) operator given by:

$$\mathrm{Op}(k)f(t) = \int_{\mathbb{R}_+} k(t,s)f(s)ds, \qquad f \in L^2(\mathbb{R}_+).$$

Of course, in the next few pages we will only be dealing with bounded integral operators and we will refer to those measurable functions k for which the induced integral operator $\mathrm{Op}(k)$ is bounded as an *integral kernel*.

Then, for a measurable function τ on $\mathbb{R}_+^2$ and any bounded integral operator $\mathrm{Op}(k)$, the Schur-Hadamard multiplication of τ and $\mathrm{Op}(k)$ is defined through the formal relation:

$$\tau \star \mathrm{Op}(k) = \mathrm{Op}(\tau \cdot k).$$

So any fixed τ induces a transformator, $\mathscr{S}_\tau$, on the set of bounded integral operator, via the equality

$$\mathscr{S}_\tau(\mathrm{Op}(k)) = \tau \star \mathrm{Op}(k). \tag{2.1.1}$$

However, this definition only makes sense for integral operators. To extend Schur-Hadamard multiplication to bounded, non-integral operators one can use limits. Put simply, let τ be a bounded function on $\mathbb{R}_+^2$. Then for any Hilbert-Schmidt operator

K, Proposition 1.2.15 shows that $K = \mathrm{Op}(k)$, for some $k \in L^2(\mathbb{R}^2_+)$ and so we can immediately define

$$\tau \star K = \mathrm{Op}(\tau \cdot k).$$

Suppose now that we have the estimate

$$\|\mathscr{S}_\tau(K)\| \leq C\|K\|, \quad \forall\ K \in \mathfrak{S}_2.$$

Since $\mathfrak{S}_2$ in dense in $\mathfrak{S}_\infty$, by the BLT Theorem 1.2.5 we obtain that $\mathscr{S}_\tau$ extends uniquely to the whole of $\mathfrak{S}_\infty$. Once we have defined $\mathscr{S}_\tau$ on $\mathfrak{S}_\infty$, the duality form in (1.2.1) can be used to extend the action of the transformator $\mathscr{S}_\tau$ to $\mathfrak{S}_1$ and $\mathfrak{B}(L^2(\mathbb{R}_+))$. The argument using the BLT Theorem can also be adapted to properly define the Schur-Hadamard multiplication on the space of bounded operators on $\ell^2(\mathbb{Z}_+)$.

In both the case of operators on $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{R}_+)$, extensive studies have been carried out, see [**7, 12, 45**] and references therein, to understand the interplay between the analytical properties of the multiplier τ and the boundedness of the transformator $\mathscr{S}_\tau$ acting on $\mathfrak{B}(\mathcal{H})$ as well as the Schatten classes $\mathfrak{S}_p$, $1 \leq p \leq \infty$. Let us briefly mention, and in one case partly prove, some of these.

We start by setting up some notation. Let $\mathfrak{M}(\mathcal{H})$ denote the set of all functions τ such that the induced transformator $\mathscr{S}_\tau$ is bounded on $\mathfrak{B}(\mathcal{H})$. It is clear that $\mathfrak{M}(\mathcal{H})$ is a normed algebra with respect to the norm

$$\|\tau\|_{\mathfrak{M}(\mathcal{H})} = |||\mathscr{S}_\tau|||_{\mathfrak{B}(\mathcal{H})}$$

Similarly, we denote by $\mathfrak{M}_p(\mathcal{H})$, $1 \leq p \leq \infty$, the set of all multipliers τ so that the induced $\mathscr{S}_\tau$ is bounded on $\mathfrak{S}_p$ and we set

$$\|\tau\|_{\mathfrak{M}_p(\mathcal{H})} = |||\mathscr{S}_\tau|||_{\mathfrak{S}_p}.$$

It is also useful to point out that the map $\tau \mapsto \overline{\tau}$ induces an involution on $\mathfrak{M}(\mathcal{H})$ as well as on any $\mathfrak{M}_p(\mathcal{H})$.

The limiting argument we presented before shows that $\mathfrak{M}(\mathcal{H}), \mathfrak{M}_1(\mathcal{H})$ and $\mathfrak{M}_\infty(\mathcal{H})$ are all complete. An adaptation of the same, in fact, shows that for $1 < p < \infty$, the classes $\mathfrak{M}_p(\mathcal{H})$ are all complete. Moreover, it implies that $\mathfrak{M}_p(\mathcal{H})$ and $\mathfrak{M}(\mathcal{H})$ are non-separable C*-algebras. From [**12**], we have the following useful result:

THEOREM 2.1.4 (Duality Principle). *Let $1 \leq p \leq \infty$ and $\mathscr{S}_\tau : \mathfrak{S}_p \to \mathfrak{S}_p$. Then its adjoint transformator $\mathscr{S}_\tau^* : \mathfrak{S}_p^* \to \mathfrak{S}_p^*$ satisfies the following identity:*

$$\mathscr{S}_\tau^* = \mathscr{S}_{\overline{\tau}}|_{\mathfrak{S}_p^*}.$$

In particular, we have the following identities:

$$\|\tau\|_{\mathfrak{M}_p(\mathcal{H})} = \|\tau\|_{\mathfrak{M}_q(\mathcal{H})}, \quad 1 < p < \infty,\ p^{-1} + q^{-1} = 1, \tag{2.1.2}$$

$$\|\tau\|_{\mathfrak{M}_1(\mathcal{H})} = \|\tau\|_{\mathfrak{M}_\infty(\mathcal{H})} = \|\tau\|_{\mathfrak{M}(\mathcal{H})}. \tag{2.1.3}$$

As the last chain of equalities holds, we denote $\|\tau\|_{\mathfrak{M}(\mathcal{H})}$ either of the three quantities.

Remark 2.1.5. The equality of norms in (2.1.2) and (2.1.3) above immediately imply that

$$\mathfrak{M}_1(\mathcal{H}) = \mathfrak{M}_\infty(\mathcal{H}) = \mathfrak{M}(\mathcal{H}), \tag{2.1.4}$$

$$\mathfrak{M}_p(\mathcal{H}) = \mathfrak{M}_q(\mathcal{H}), \quad 1 < p < \infty,\ p^{-1} + q^{-1} = 1. \tag{2.1.5}$$

Moreover, it also leads to the chain of continuous embeddings

$$\mathfrak{M}(\mathcal{H}) \subsetneq \mathfrak{M}_p(\mathcal{H}) \subsetneq \mathfrak{M}_2(\mathcal{H}) = L^\infty, \quad 1 \le p \le \infty,\ p \ne 2,$$

from which it immediately follows that

$$\|\tau\|_{\mathfrak{M}(\mathcal{H})} \ge \|\tau\|_{\mathfrak{M}_p(\mathcal{H})} \ge \|\tau\|_{L^\infty},$$

where L^∞ denotes either $\ell^\infty(\mathbb{Z}_+^2)$ or $L^\infty(\mathbb{R}_+^2)$ depending on whether $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ or $\mathcal{H} = L^2(\mathbb{R}_+)$.

Example 2.1.6. From the previous discussion, it emerged that $\mathfrak{M}_2$ is nothing but $L^\infty(\mathbb{R}_+^2)$ or $\ell^\infty(\mathbb{Z}_+^2)$ in disguise. Let us now discuss some basic examples of multipliers $\tau \in \mathfrak{M}(\ell^2(\mathbb{Z}_+))$. More examples will be presented in the relevant Sections of Chapters 4 and 5. An immediate example of a bounded multiplier is a factorisable multiplier of the form

$$\tau(j,k) = f(j)g(k)$$

where both f and g are bounded sequences. To see this, let M_f and M_g be the operators of multiplication by f and g respectively on $\ell^2(\mathbb{Z}_+)$ and let $\mathscr{L}_f$, $\mathscr{R}_g$ be the transformators of left and right multiplication induced by them. Then

$$\mathscr{S}_\tau(A) = M_f A M_g = \mathscr{L}_f \mathscr{R}_g(A)$$

and so, from Example 2.1.2, it follows $\|\tau\|_{\mathfrak{M}} = \|\mathscr{L}_f \mathscr{R}_g\|_{\mathfrak{B}(\mathcal{H})} = \|M_f\|\|M_g\| = \|f\|_\infty \|g\|_\infty$. In particular, from the Duality Principle 2.1.4 it follows that for any $p \ge 1$

$$\|\tau\|_{\mathfrak{M}_p} = \|f\|_\infty \|g\|_\infty.$$

We can extend this argument simply by taking $\{f_n\}_{n=0}^\infty$, $\{g_n\}_{n=0}^\infty \subset \ell^\infty(\mathbb{Z}_+)$, such that

$$\sum_{n=0}^\infty \|f_n\|_\infty \|g_n\|_\infty$$

is finite. Then the multiplier

$$\tau(j,k)=\sum_{n=0}^{\infty} f_n(j)g_n(k)$$

is bounded on the space of bounded operators on $\ell^2(\mathbb{Z}_+)$ because we have that

$$\mathscr{S}_\tau(A)=\sum_{n=0}^{\infty}\mathscr{L}_{f_n}\mathscr{R}_{g_n}(A)$$

and so the triangle inequality shows that

$$\|\tau\|_{\mathfrak{M}}\le\sum_{n=0}^{\infty}\|f_n\|_\infty\|g_n\|_\infty<\infty.$$

The above example can be taken as the motivation behind the important result, stated below, that fully characterises the algebra $\mathfrak{M}(\mathcal{H})$ in both the case of $\mathcal{H}=\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$. It was proved in its entirety by the authors of [**9**, **45**].

THEOREM 2.1.7. *Let $\mathcal{H}$ be either $\ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{R}_+)$ and Λ denote either $\mathbb{Z}_+$ or $\mathbb{R}_+$. For a function $\tau\in L^\infty(\Lambda^2)$, the following are equivalent:*

(i) $\tau\in\mathfrak{M}(\mathcal{H})$;

(ii) there exist auxiliary measure space $(\mathcal{M},\nu)$ and ν-measurable functions $a:\Lambda\times\mathcal{M}\to\mathbb{C}$, $b:\Lambda\times\mathcal{M}\to\mathbb{C}$ such that:

$$\tau(\lambda,\mu)=\int_{\mathcal{M}} a(\lambda,t)b(\mu,t)d\nu(t), \tag{2.1.6}$$

and such that

$$A^2:=\sup_\lambda\int_{\mathcal{M}}|a(\lambda,t)|^2\,d\nu(t)<\infty, \tag{2.1.7}$$

$$B^2:=\sup_\lambda\int_{\mathcal{M}}|b(\lambda,t)|^2\,d\nu(t)<\infty. \tag{2.1.8}$$

Furthermore one has that

$$\|\tau\|_{\mathfrak{M}(\mathcal{H})}\le AB.$$

PROOF. The proof of *(i)*⇒*(ii)* is rather involved and includes elements of function and operator theory and it can be found in [**45**].
Let us concentrate on showing *(ii)*⇒*(i)* in the case of $\mathcal{H}=L^2(\mathbb{R}_+)$. Let us start by showing that for any Hilbert-Schmidt operator K we have the estimate

$$\|\mathscr{S}_\tau(K)\|\le AB\|K\|.$$

To this end, let $K\in\mathfrak{S}_2$ and recall that any Hilbert-Schmidt operator is an integral operator with integral kernel $k\in L^2(\mathbb{R}_+^2)$ and so we can write

$$\mathscr{S}_\tau(K)=\mathrm{Op}(\tau\cdot k).$$

Now, for $f, g \in L^2(\mathbb{R}_+)$,we have

$$
\begin{aligned}
(\mathscr{S}_\tau(K)f, g)_{L^2(\mathbb{R}_+)} &= \int_{\mathbb{R}_+} \mathscr{S}_\tau(K)f(\lambda)\overline{g(\lambda)}d\lambda \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tau(\lambda,\, \mu)k(\lambda,\mu)f(\mu)\overline{g(\lambda)}d\mu d\lambda \\
&= \int_{\mathcal{M}} \left(\int_{\mathbb{R}_+} (K\boldsymbol{B}(t)f(\lambda))\overline{(\boldsymbol{A}^*(t)g)(\lambda)}d\lambda\right) d\nu(t) \\
&= \int_{\mathbb{R}_+} (K\boldsymbol{B}(t)x, \boldsymbol{A}^*(t)y)_{L^2(\mathbb{R}_+)}\, d\nu(t). \qquad (2.1.9)
\end{aligned}
$$

where in the second equality we exchanged summation and integration because of Fubini's theorem. Also in the last equality we wrote for any $f \in L^2(\mathbb{R}_+)$

$$
\begin{aligned}
(\boldsymbol{A}(t)f)(\cdot) &:= a(\cdot, t)f(\cdot), \\
(\boldsymbol{B}(t)f)(\cdot) &:= b(\cdot, t)f(\cdot).
\end{aligned}
$$

Notice that for any function $f \in L^2(\mathbb{R}_+)$, $\boldsymbol{A}(t)f$ and $\boldsymbol{B}(t)f$ may not make sense for some values of t, since they are defined up to sets of measure 0 with respect to the measure ν, however we have the estimates:

$$
\int_{\mathcal{M}} \|\boldsymbol{A}(t)f\|^2_{L^2(\mathbb{R}_+)} d\nu(t) \le \int_{\mathcal{M}}\int_{\mathbb{R}_+} |a(\lambda,\, t)|^2\, |f(\lambda)|^2\, d\lambda\, d\nu(t) \le A^2\|f\|^2_{L^2(\mathbb{R}_+)},
$$

$$
\int_{\mathcal{M}} \|\boldsymbol{B}(t)f\|^2_{L^2(\mathbb{R}_+)} d\nu(t) \le B^2\|f\|^2_{L^2(\mathbb{R}_+)}.
$$

Using these, an upper bound for the quadratic form $(\mathscr{S}_\tau(\mathrm{Op}(k))f, g)_{L^2(\mathbb{R}_+)}$ is easily obtained as follows

$$
\begin{aligned}
\left|(\mathscr{S}_\tau(K)f, g)_{L^2(\mathbb{R}_+)}\right| &\le \int_{\mathcal{M}} \left|(K\boldsymbol{B}(t)f, \boldsymbol{A}^*(t)g)_{L^2(\mathbb{R}_+)}\right| d\nu(t) \\
&\le \int_{\mathcal{M}} \|K\|\|\boldsymbol{B}(t)f\|_{L^2(\mathbb{R}_+)}\|\boldsymbol{A}^*(t)g)\|_{L^2(\mathbb{R}_+)} d\nu(t) \\
&\le AB\|K\|\|f\|_{L^2(\mathbb{R}_+)}\|g\|_{L^2(\mathbb{R}_+)},
\end{aligned}
$$

where both the second and third inequalities are a consequence of the Cauchy-Schwarz inequality. In particular, it follows that for any Hilbert-Schmidt operator K, we have the estimate

$$
\|\mathscr{S}_\tau(K)\| \le AB\|K\|.
$$

By the BLT Theorem 1.2.5, $\mathscr{S}_\tau$ extends to $\mathfrak{S}_\infty$ by continuity without increasing its norm. In particular, for any compact operator K, we have the estimate

$$
\|\mathscr{S}_\tau(K)\| \le AB\|K\| \Rightarrow \|\mathscr{S}_\tau\|_{\mathfrak{M}_\infty} \le AB.
$$

The Duality Principle 2.1.4 finally concludes the argument. □

EXAMPLE 2.1.8 (**Hankel multipliers**). A useful sub-class of multipliers is that of Hankel multipliers, i.e. multipliers τ of the form

$$\tau(j,k) = \alpha(j+k), \quad j,k \geq 0$$

where $\{\alpha(j)\}_{j\geq 0}$ is a fixed sequence of complex numbers. Multipliers of this form have been studied in the literature. For instance, in [7, Section 9] we find sufficient conditions that guarantee $\tau \in \mathfrak{M}$. In [48, Chapter 6], the author gives an exhaustive answer to the problem and gives a necessary and sufficient condition for $\tau \in \mathfrak{M}$. In particular, it is shown that $\tau \in \mathfrak{M}$ if and only if α is a Fourier multiplier of $H^1(\mathfrak{S}_1)$, the Hardy class of $\mathfrak{S}_1$-valued functions on the unit circle $\mathbb{T}$. In other words $\tau \in \mathfrak{M}$ if and only if one has that

$$\sum_{j\in\mathbb{Z}_+} m(j)v^j \in H^1(\mathfrak{S}_1) \implies \sum_{j\in\mathbb{Z}_+} \alpha(j)m(j)v^j \in H^1(\mathfrak{S}_1),$$

where $\{m(j)\}_{j\in\mathbb{Z}}$ is a sequence of trace-class operators. Finally, in [1, Section 4] the authors also address the boundedness of the transformator $\mathscr{S}_\tau$ when acting on Schatten classes $\mathfrak{S}_p$ for p in the range (0, 1).

Let us now give a concrete example of how one can use Theorem 2.1.7 to show $\tau \in \mathfrak{M}$. Suppose that the sequence α is of the form

$$\alpha(j) = \widehat{\varphi}(j) = \int_0^1 \varphi(e^{2\pi i\vartheta})e^{-2\pi ij\vartheta}d\vartheta,$$

for some $\varphi \in L^p(0,1), p \geq 1$. Our goal is to show that $\|\alpha\|_{\mathfrak{M}} \leq \|\varphi\|_{L^p(0,1)}$.

Note that the case of $p > 1$ follows immediately because of the estimate

$$\|\varphi\|_{L^1(0,1)} \leq \|\varphi\|_{L^p(0,1)}.$$

So it suffices for us to show this for $p = 1$. In this case, it is easy to see that

$$\int_0^1 a(j,\vartheta)b(k,\vartheta)d\vartheta = \alpha(j+k),$$

where a, b are the functions

$$a(j,\vartheta) = \left|\varphi(e^{2\pi i\vartheta})\right|^{1/2} e^{-2\pi ij\vartheta},$$
$$b(k,\vartheta) = \varphi(e^{2\pi i\vartheta}) \left|\varphi(e^{2\pi i\vartheta})\right|^{-1/2} e^{-2\pi ik\vartheta}.$$

By Theorem 2.1.7, it is sufficient to show the finiteness of the quantities

$$A^2 := \sup_{j\geq 0} \int_0^1 |a(j,\vartheta)|^2 \, d\vartheta,$$
$$B^2 := \sup_{k\geq 0} \int_0^1 |b(k,\vartheta)|^2 \, d\vartheta.$$

Now, it is easy to see that for A we have

$$A^2 = \int_0^1 \left|\varphi(e^{2\pi i\vartheta})\right| d\vartheta = \|\varphi\|_{L^1(0,1)}$$

Similarly, for B we have

$$B^2 = \int_0^1 \left|\varphi(e^{2\pi i\vartheta})\right| d\vartheta = \|\varphi\|_{L^1(0,1)}.$$

Thus Theorem 2.1.7 shows that

$$\|\tau\|_{\mathfrak{M}} \leq AB = \|\varphi\|_{L^1(0,1)}.$$

A similar argument shows that the same holds when $\mathscr{S}_\tau$ acts on $\mathfrak{B}(L^2(\mathbb{R}_+))$ and

$$\tau(t,s) = \boldsymbol{\Phi}(\varphi)(t+s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(\xi) e^{-i\xi(t+s)} d\xi,$$

for any function $\varphi \in L^p(\mathbb{R})$ with $1 \leq p \leq 2$. Note that in this case, we restrict the range of p to be sure that $\boldsymbol{\Phi}(\varphi)$ is well-defined.

Example 2.1.9 (**Toeplitz multipliers**). Another useful class of multipliers is that of Toeplitz multipliers, i.e. multipliers τ of the form

$$\tau(j,k) = \beta(j-k),$$

where β is a double-sided sequence of complex numbers. Even though we do not know who did it first, it is a well-known fact that the following are equivalent:

(a) $\tau \in \mathfrak{M}$;

(b) there exists a finite, complex Borel measure μ on $\mathbb{T}$ such that $\beta(j) = \widehat{\mu}(j)$, where $\widehat{\mu}$ is the sequence

$$\widehat{\mu}(j) = \int_{\mathbb{T}} \overline{v}^j d\mu(v), \quad j \in \mathbb{Z};$$

(c) β is a Fourier Multiplier of $L^\infty(\mathbb{T})$, i.e. one has that

$$\sum_{j\in\mathbb{Z}} a(j)v^j \in L^\infty(\mathbb{T}) \Longrightarrow \sum_{j\in\mathbb{Z}} \beta(j)a(j)v^j \in L^\infty(\mathbb{T}).$$

In [28, Theorem 3.6.4] it is proved that (b)⇔(c) and so it is sufficient to show that (b)⇒(a)⇒(c).

(b)⇒(a): Suppose that for some finite, complex Borel measure μ, we have $\beta = \widehat{\mu}$, then

$$\tau(j,k) = \int_{\mathbb{T}} \overline{v}^{j-k} d\mu(v).$$

By Theorem 2.1.7, we immediately obtain that $\tau \in \mathfrak{M}$ and, moreover $\|\tau\|_{\mathfrak{M}} \leq \|\mu\|$, where $\|\mu\|$ is the total variation of the measure μ.

(a)⇒(c): Suppose $\tau \in \mathfrak{M}$ and let $T = \{t(j-k)\}_{j,k\geq 0}$ be a bounded Toeplitz matrix. Recall that T is a bounded Toeplitz matrix if and only if $t(j) = \widehat{f}(j)$ for some $f \in L^\infty(\mathbb{T})$ (see [46, Chapter 3]). Since

$$\mathscr{S}_\tau(T) = \{\beta(j-k)t(j-k)\}_{j,k\geq 0}$$

is also a bounded Toeplitz matrix, then $\beta(j)t(j) = \widehat{g}(j)$, for some bounded function g on $\mathbb{T}$. In other words we have shown that

$$\sum_{j\in\mathbb{Z}} t(j)v^j \in L^\infty(\mathbb{T}) \Longrightarrow \sum_{j\in\mathbb{Z}} \beta(j)t(j)v^j \in L^\infty(\mathbb{T}),$$

i.e. β is a Fourier multiplier of $L^\infty(\mathbb{T})$.

Similar results can be obtained when $\tau(x,y) = \beta(x-y)$ is a Toeplitz multiplier on $\mathfrak{B}(L^2(\mathbb{R}_+))$. In this case, $\tau \in \mathfrak{M}$ if and only if there exists a finite, complex Borel measure on $\mathbb{R}$ such that $\beta = \widehat{\mu}$, where $\widehat{\mu}$ is the Fourier transform of the measure μ.

Although Theorem 2.1.7 gives a full description of the algebra $\mathfrak{M}(\mathcal{H})$, its hypotheses are not so easy to check since they rely on finding a factorization of the multiplier τ. Unfortunately, there are no known necessary and sufficient conditions relying solely on the smoothness of the multiplier, however some partial results do exist in the literature, see for instance [12, Section 5.2]. Most of the results given there rely on τ having some degree of smoothness, say at least one derivative, or belonging to some Sobolev space. In the following pages we look at two specific examples of multipliers which have a discontinuity at some points of their domain and we shall see how this affects the boundedness of the induced transformator. We will first study the "discrete" case of $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ and then move on to the "continuous" case of $\mathcal{H} = L^2(\mathbb{R}_+)$, in both cases we will only write $\mathfrak{M}$ and $\mathfrak{M}_p$, since the underlying Hilbert spaces are fixed.

2. Discrete Case

2.1. Upper triangular truncation. Let us begin by taking φ to be such that

$$\varphi(j) = \begin{cases} 0 & j < 0, \\ 1 & j \geq 0, \end{cases}$$

and define $\Phi(i,j) = \varphi(i-j)$. Then $\mathscr{S}_\Phi$ is formally defined as the transformator such that for any bounded operator A one has:

$$\mathscr{S}_\Phi(A)_{i,j} = (\Phi \star A)_{i,j} = \begin{cases} 0 & i < j, \\ A_{i,j} & i \geq j, \end{cases}$$

where $A_{i,j}$ are the matrix entries of A. Thus $\mathscr{S}_\Phi$ sends A to its upper triangular part and so it is often called **upper triangular truncation** and is an example of a Toeplitz multiplier. We have the following result:

THEOREM 2.2.1. *Let $1 < p < \infty$ and consider the operator $\mathscr{S}_\Phi : \mathfrak{S}_p \to \mathfrak{S}_p$, then:*

$$\|\Phi\|_{\mathfrak{M}_p} = C(p), \tag{2.2.10}$$

where $C(p) \geq 1$ is such that:

(i) $C(2) = 1$;

(ii) $C(p) = C(q)$, *where* $p^{-1} + q^{-1} = 1$.

Furthermore, $C(p) \to \infty$ as $p \to 1^+$ and as $p \to \infty$.

The proof of this result can be found in [25], Chapter 3, Section 6. The authors prove the result by showing that:

(1) for $p = 2$, $C(2) = 1$;
(2) by letting $p = 2^r, r \in \mathbb{N}$ and using an identity which holds for $\mathscr{S}_\Phi$ acting on the set of finite-rank, selfadjoint operators they show:
$$\|\Phi\|_{\mathfrak{M}_{2^r}} = C(2^r),$$
and furthermore $C(2^r) \to \infty$ as $r \to \infty$.
(3) using interpolation between $\mathfrak{S}_{2^r}$ and $\mathfrak{S}_{2^{r+1}}$, they show the result for all $p \geq 2$. Finally, by an application of the Duality Principle 2.1.4 above, they conclude the proof.

REMARK 2.2.2. As a matter of fact, the constant $C(2^r)$ is explicitly found by the authors of [26], where they show that:

$$C(2^r) = \cot(2^{-r-1}\pi), \qquad r \in \mathbb{N}.$$

They also conjecture that this also holds for any other $p \geq 2$.

EXAMPLE 2.2.3. The behaviour of the constant $C(p)$ in the result above suggests that, in fact, $\mathscr{S}_\Phi$ is unbounded on $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$. This fact is well-known, see for instance [3], and can be easily proved.

Indeed, notice that Φ is a Toeplitz multiplier and so, by Example 2.1.9, to show that $\Phi \notin \mathfrak{M}$, it is sufficient to show that the sequence

$$\varphi(j) = \begin{cases} 0 & j < 0, \\ 1 & j \geq 0, \end{cases}$$

is not a Fourier multiplier of $L^\infty(\mathbb{T})$. To do this consider the function

$$f_N(v) = \sum_{0<|j|\leq N} \frac{v^j}{\pi j}, \quad v \in \mathbb{T}.$$

It is easy to see that $\|f_N\|_{L^\infty(\mathbb{T})} \leq 1$, for any $N \geq 1$. Now, we have

$$\widetilde{f}_N(v) = \sum_{0<|j|\leq N} \frac{\varphi(j)v^j}{\pi j} = \sum_{j=1}^{N} \frac{v^j}{\pi j}.$$

And so it is easy to see that

$$\|\widetilde{f}_N\|_{L^\infty(\mathbb{T})} \geq \left|\widetilde{f}_N(1)\right| = \sum_{j=1}^{N} \frac{1}{\pi j} \geq \frac{1}{\pi}\log(N+1).$$

Thus showing that φ is not a Fourier multiplier of $L^\infty(\mathbb{T})$.

Remark 2.2.4. The authors of [3] show, by means of an adaptation of the above example together with some abstract results borrowed from the theory of Haagerup-type factorisation of matrices, see [3] and references therein, that the transformator $\mathscr{S}_\Phi$ acting on the space of $N \times N$ matrices on $\mathbb{C}^N$, $\mathfrak{B}_N$, is so that

$$|||\mathscr{S}_\Phi|||_{\mathfrak{B}_N} = \frac{1}{\pi}\log(N) + o(\log(N)), \qquad N \to \infty.$$

2.2. The main triangle truncation. Let us now consider another type of truncation. For any $N \geq 1$, consider the sequence:

$$\psi_N(j) = \begin{cases} 1 & 0 \leq j \leq N-1, \\ 0 & j \geq N, \end{cases}$$

and let $\Psi_N(i,j) = \psi_N(i+j)$. Let us define the transformator $\mathscr{S}_{\Psi_N}$ via the identity:

$$\mathscr{S}_N(A) = \Psi_N \star A, \quad A \in \mathfrak{B}(\mathcal{H}).$$

In other words, $\mathscr{S}_{\Psi_N}$ associates to a matrix A the matrix whose entries are set to 0 after the N-th cross-diagonal. We call this transformator the **main triangle truncation**, following the terminology of [39].

As for the case of $\mathscr{S}_\Phi$, we wish to study the properties of the transformator $\mathscr{S}_{\Psi_N}$. To do so, let us introduce some notation. For $N \geq 1$, let J_N be the $N \times N$ matrix:

$$J_N = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We can consider J_N as the operator acting on $\ell^2(\mathbb{Z}_+)$, by considering it as the block matrix:

$$\begin{pmatrix} J_N & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix},$$

where $\mathbb{O}$ is the zero operator. From now on, we will use the notation J_N without distinction in both cases, its meaning will be clear from the context.

It is easy to see that, for any N, the operator J_N is a partial isometry onto $\mathbb{C}^N$ and therefore $\|J_N\| = 1$. We now have the following straightforward

LEMMA 2.2.5. *Let $\mathscr{J}_N$ be the transformator such that $\mathscr{J}_N(A) = AJ_N$. Then, for any $N \in \mathbb{N}$, the transformator $\mathscr{S}_{\Psi_N}$ satisfies the following identity:*

$$\mathscr{S}_{\Psi_N} = \mathscr{J}_N \mathscr{S}_\Phi \mathscr{J}_N,$$

where $\mathscr{S}_\Phi$ is the upper triangular truncation.

With this at hand and together with the observation made in Example 2.1.2, we have that for any $1 < p < \infty$ and any N:

$$\|\Psi_N\|_{\mathfrak{M}_p} = \|\|\mathscr{S}_{\Psi_N}\|\|_{\mathfrak{S}_p} = \|\|\mathscr{J}_N \mathscr{S}_\Phi \mathscr{J}_N\|\|_{\mathfrak{S}_p} \leq \|\|\mathscr{J}_N\|\|^2_{\mathfrak{S}_p} \|\|\mathscr{S}_\Phi\|\|_{\mathfrak{S}_p} = \|\Phi\|_{\mathfrak{M}_p} = C(p),$$

where $C(p)$ is as given in (2.2.10). This proves part of the following:

THEOREM 2.2.6. *For any $p \in (1, \infty)$ and any $N \in \mathbb{N}$, one has that:*

$$\|\Psi_N\|_{\mathfrak{M}_p} \leq C(p). \tag{2.2.11}$$

Furthermore, for any $A \in \mathfrak{S}_p$, we have that $\mathscr{S}_{\Psi_N}(A) \to A$ as $N \to \infty$, in other words the transformator $\mathscr{S}_{\Psi_N} \to \mathscr{I}$ strongly as $N \to \infty$, where $\mathscr{I}$ is the identity transformator.

On the other hand, for $\mathscr{S}_{\Psi_N}$ acting on $\mathfrak{S}_1, \mathfrak{S}_\infty$ and on $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$, we have that:

$$\|\Psi_N\|_{\mathfrak{M}} = \frac{1}{\pi}\log(N) + o(\log(N)), \qquad N \to \infty.$$

REMARK 2.2.7. We remark that in [**39**], only a lower bound was given for the norm of Ψ_N. In particular, they only showed that

$$\|\|\mathscr{S}_{\Psi_N}\|\|_{\mathfrak{B}_N} \geq C \log N,$$

where $C > 0$ is not explicitly given.

PROOF OF THEOREM. The identity presented in Lemma 2.2.5 actually shows that, when restricted to the space of finite $N \times N$ matrices, $\mathfrak{B}_N$, the transformators $\mathscr{S}_{\Psi_N}$ and $\mathscr{S}_\Phi$ are unitarily equivalent, since the transformator $\mathscr{J}_N$ is invertible on $\mathfrak{B}_N$ and $\mathscr{J}_N^{-1} = \mathscr{J}_N$. Thus, by the results of [**3**]:

$$\|\|\mathscr{S}_{\Psi_N}\|\|_{\mathfrak{B}_N} = \|\|\mathscr{S}_\Phi\|\|_{\mathfrak{B}_N} = \frac{1}{\pi}\log(N) + o(\log(N)), \qquad N \to \infty.$$

Also, note that if $\mathscr{P}_N$ is the transformator such that $\mathscr{P}_N(A) = P_N A P_N$, with P_N the projection from $\ell^2(\mathbb{Z}_+)$ onto $\mathbb{C}^N$, then

$$\mathscr{S}_{\Psi_N}\mathscr{P}_N = \mathscr{S}_{\Psi_N}.$$

Therefore one has that

$$\|\Psi_N\|_{\mathfrak{M}} = \|\|\mathscr{S}_{\Psi_N}\|\|_{\mathfrak{B}_N}, \quad \forall N.$$

By the Duality Principle 2.1.4, the equality also holds for $\mathfrak{S}_1$ as well as $\mathfrak{S}_\infty$.

Let us now concentrate on proving the result on the convergence in the strong sense to $\mathscr{I}$. First of all, it is easy to see that for any matrix $B \in \mathfrak{M}_K$ we have that

$$\mathscr{S}_{\Psi_N}(B) = B$$

whenever $N \geq 2K + 1$.

Let us now consider the case a general $A \in \mathfrak{S}_p$, for $1 < p < \infty$. For any given $\varepsilon > 0$, by Theorem 1.2.17-(i), we can find a $B_\varepsilon \in \mathfrak{M}_{K_\varepsilon}$ so that $\|A - B_\varepsilon\|_p < \varepsilon$. Now, using (2.2.11) and the fact that $\mathscr{S}_{\Psi_N}(B_\varepsilon) = B_\varepsilon$, we have:

$$\begin{aligned}\|\mathscr{S}_{\Psi_N}(A) - A\|_p &\leq \|\mathscr{S}_{\Psi_N}(A) - \mathscr{S}_{\Psi_N}(B_\varepsilon)\|_p + \|\mathscr{S}_{\Psi_N}(B_\varepsilon) - B_\varepsilon\|_p + \|B_\varepsilon - A\|_p \\ &\leq (C(p) + 1)\|A - B_\varepsilon\|_p < (C(p) + 1)\varepsilon\end{aligned}$$

for any $N \geq 2K_\varepsilon + 1$, thus the result holds. $\square$

3. The Continuous case

We wish now to study the action of the continuous analogues of the transformators $\mathscr{S}_\Phi$ and $\mathscr{S}_{\Psi_N}$ studied earlier. In this case, and unlike $\ell^2(\mathbb{Z}_+)$, we restrict ourselves to the set of integral operators and we will use the definition of multipliers given in (2.1.1). Of course, our discussion will work in general for any bounded operator A on $L^2(\mathbb{R}_+)$ since one can work with the "implicit" definition of a multiplier discussed earlier using limits and the duality form (1.2.1).

3.1. The upper triangular truncation. Let $\mathbb{1}_+$ denote the characteristic function of $\mathbb{R}_+$ and let $\Theta(t, s) = \mathbb{1}_+(t - s)$, then we we define the transformator, $\mathscr{S}_\Theta$, formally as

$$\mathscr{S}_\Theta(\mathrm{Op}(k)) = \Theta \star \mathrm{Op}(k)$$

as the **upper triangular truncation**. We have an analogous statement to Theorem 2.2.1:

THEOREM 2.3.1. *Let $1 < p < \infty$, then $\mathscr{S}_\Theta : \mathfrak{S}_p \to \mathfrak{S}_p$ is such that:*

$$\|\Theta\|_{\mathfrak{M}_p} = C(p), \tag{2.3.12}$$

where $C(p)$ is as in Theorem 2.2.1 and has the same properties.

Remark 2.3.2. Theorems 2.2.1 and 2.3.1 are just two instances of Matsaev Theorem, see e.g. [**25**]. This result finds the norm of a specific chain of projections and it fits in the broader theory of nest algebras and chains of projections. We will not go into any detail here, see [**15**] and [**25**] for more.

EXAMPLE 2.3.3. Theorem 2.3.1 does not tell us what happens in the case when $\mathscr{S}_\Theta$ acts on $\mathfrak{B}(L^2(\mathbb{R}_+))$. As before, in all of these cases the transformator in question is unbounded.

To see this, consider the transformator $\mathscr{P} : \mathfrak{B}(L^2(\mathbb{R}_+)) \to \mathfrak{B}(L^2(0,1))$ so that for any operator A and any $f \in L^2(0,1)$ we have:

$$\mathscr{P}(A)f = \mathbb{1}A\mathbb{1}f$$

where $\mathbb{1}$ is the indicator function of $(0,1)$. Example 2.1.2 gives $|||\mathscr{P}|||_{\mathfrak{M}} = 1$ and so

$$|||\mathscr{P}\mathscr{S}_\Theta\mathscr{P}|||_{\mathfrak{M}} \leq |||\mathscr{S}_\Theta|||_{\mathfrak{M}}.$$

From this, it is sufficient to show the unboundedness of $\mathscr{S}_\Theta$ when acting on the space $\mathfrak{B}(L^2(0,1))$.

To this end, let $\varepsilon > 0$ be fixed and let $H^{(\varepsilon)}$ be the "pre-Hilbert Transform" defined as

$$H^{(\varepsilon)} : L^2(0,1) \to L^2(0,1),$$
$$H^{(\varepsilon)}f(t) = \frac{1}{\pi}\int_{\Omega_\varepsilon} \frac{f(s)}{t-s}ds,$$

where $\Omega_\varepsilon = \{(t,s) \in (0,\tau)\times(0,\tau) : \ |t-s| > \varepsilon\}$. It is known in the literature, see [26, 47], that we have the estimate

$$\|H^{(\varepsilon)}\| \leq 1, \qquad \forall \varepsilon > 0.$$

Let us now consider $\widetilde{H}^{(\varepsilon)} = \mathscr{S}_\Theta(H^{(\varepsilon)})$ and let $f \equiv 1$. Then:

$$\widetilde{H}^{(\varepsilon)}f(t) = \int_0^{t-\varepsilon} \frac{1}{t-s}ds = \frac{\log(\varepsilon) - \log(t)}{\pi}.$$

Whereby we have the lower bound:

$$|||\mathscr{S}_\Theta|||_{\mathfrak{M}} \geq \|\widetilde{H}^{(\varepsilon)}\| \geq \frac{1}{\pi}\|\log(\varepsilon) - \log(t)\|_2$$
$$\geq \frac{1}{\pi}\left|\log(\varepsilon) - 1\right|, \qquad \forall \tau, \varepsilon > 0$$

In particular this shows that $\mathscr{S}_\Theta$ is not bounded on $\mathfrak{B}(L^2(\mathbb{R}_+))$, and by duality on $\mathfrak{S}_1$ as well as $\mathfrak{S}_\infty$. Furthermore, an adaptation of the above shows that this is the case on $\mathfrak{B}(L^2(0,\tau))$, for any $\tau > 0$. This is in contrast with the situation we had for discrete analogue of this type of truncation. In that case we only had that the norm of the operator restricted to the set of $N \times N$ matrices depended on N only, as illustrated in Remark 2.2.4.

3.2. The Main triangle truncation. Using the approach illustrated in (2.1.1), we can also define a continuous analogue of the transformator $\mathscr{S}_{\Psi_N}$ in the following way.

Let $\mathbb{1}_\tau$ denote the characteristic function of the interval $(0,\tau)$ and let us consider

$$\Delta_\tau(t,s) = \mathbb{1}_\tau(t+s), \qquad t,s>0.$$

Define $\mathscr{S}_{\Delta_\tau}$ to be the transformator defined formally by:

$$\mathscr{S}_{\Delta_\tau}(\mathrm{Op}(k)) = \Delta_\tau \star \mathrm{Op}(k).$$

Let us also define the operator $J_\tau : L^2(\mathbb{R}_+) \to L^2(0,\tau)$ to be the operator such that for any $f \in L^2(\mathbb{R}_+)$ one has:

$$J_\tau f(t) = \mathbb{1}_\tau(t) f(\tau - t), \qquad 0<t<\tau.$$

It is immediate to see that for any fixed $\tau > 0$, J_τ is a partial isometry onto $L^2(0,\tau)$ and so $\|J_\tau\| = 1$.

Let $\mathscr{J}_\tau$ be the transformator defined as

$$\mathscr{J}_\tau(A) = J_\tau A, \qquad A \in \mathfrak{B}(L^2(\mathbb{R}_+)),$$

then we see that $\|\mathscr{J}_\tau\|$ is a partial isometry onto $\mathfrak{B}(L^2(0,\tau))$. The importance of such a transformator can be seen in the following:

LEMMA 2.3.4. *For any $\tau > 0$, the transformator $\mathscr{S}_{\Delta_\tau}$ satisfies:*

$$\mathscr{S}_{\Delta_\tau} = \mathscr{J}_\tau \mathscr{S}_\Theta \mathscr{J}_\tau,$$

where $\mathscr{S}_\Theta$ is the upper triangular truncation. Also note that the above equality implies that on $\mathfrak{B}(L^2(0,\tau))$ the operators $\mathscr{S}_{\Delta_\tau}$ and $\mathscr{S}_\Theta$ are similar.

PROOF. Let k be a measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ and let $\mathrm{Op}(k)$ be the associated integral operator. Then:

$$\mathscr{J}_\tau \,\mathrm{Op}(k) = \mathrm{Op}(\mathbb{1}_\tau(t) k(\tau - t, \cdot))$$

whence:

$$\mathscr{S}_\Theta \mathscr{J}_\tau \,\mathrm{Op}(k) = \mathrm{Op}(\mathbb{1}_\tau(t)\mathbb{1}_+(t-s) k(\tau - t, s)).$$

From this we get:

$$\begin{aligned}\mathscr{J}_\tau \mathscr{S}_\Theta \mathscr{J}_\tau \,\mathrm{Op}(k) &= \mathrm{Op}(\mathbb{1}_\tau(t)\mathbb{1}_\tau(\tau-t)\mathbb{1}_+(\tau-(t+s))k(t,s)) \\ &= \mathrm{Op}(\mathbb{1}_\tau(t+s)k(t,s)) = \mathrm{Op}(\Lambda_\tau \cdot k) \\ &= \mathscr{S}_{\Delta_\tau}(\mathrm{Op}(k)).\end{aligned}$$

Note now that, when acting on the space of operators on $L^2(0,\tau)$, we have $\mathscr{J}_\tau^2 = \mathscr{I}$ and so the last claim follows immediately. □

From the above we can conclude that $\mathscr{S}_{\Delta_\tau}$ is bounded if and only if $\mathscr{S}_\Theta$ is. In particular, we obtain that on $\mathfrak{B}(L^2(\mathbb{R}_+))$ the norm of $\mathscr{S}_{\Delta_\tau}$ is not well-behaved since

$$\|\Delta_\tau\|_{\mathfrak{M}} \geq \frac{1}{\pi}\left|\log(\varepsilon)\right|\left|1 - \frac{(\log(\tau)-1)^2}{\log(\varepsilon)}\right|, \qquad \forall \tau, \varepsilon > 0.$$

A simple duality argument, following that presented in the proof of Theorem 2.2.1, also shows that the following is true:

THEOREM 2.3.5. *Let $p \in (1, \infty)$ and let $\mathscr{S}_{\Delta_\tau} : \mathfrak{S}_p \to \mathfrak{S}_p$. Then we have:*

$$\|\Delta_\tau\|_{\mathfrak{M}_p} \leq C(p), \qquad \forall \tau > 0.$$

Furthermore, $\mathscr{S}_{\Delta_\tau} \to \mathscr{I}$ as $\tau \to \infty$ and $\mathscr{S}_{\Delta_\tau} \to \mathscr{O}$ as $\tau \to 0$. In both cases the convergence is intended in the strong sense.

PROOF. Let k be a function supported on the square $[\xi^{-1}, \xi]^2$, with $\xi > 0$ and such that $\operatorname{Op}(k) \in \mathfrak{S}_p$. Then we have that

$$\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k)) = \mathbb{O}$$

if $\tau < \xi^{-1}$ and if $\tau > 2\xi$:

$$\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k)) = \operatorname{Op}(k).$$

In the general case, given $\varepsilon > 0$ and k so that $\operatorname{Op}(k) \in \mathfrak{S}_p$, by Theorem 1.2.17-(ii), we can find a function k_ε supported on the square $[\xi_\varepsilon^{-1}, \xi_\varepsilon]$, with $\xi_\varepsilon > 0$, so that

$$\|\operatorname{Op}(k) - \operatorname{Op}(k_\varepsilon)\|_p < \varepsilon.$$

Whence, we obtain:

$$\begin{aligned}\|\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k)) - \operatorname{Op}(k)\|_p &\leq \|\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k)) - \mathscr{S}_\tau(\operatorname{Op}(k_\varepsilon))\|_p + \|\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k_\varepsilon)) - \operatorname{Op}(k_\varepsilon)\|_p \\ &\quad + \|\operatorname{Op}(k_\varepsilon) - \operatorname{Op}(k)\|_p \\ &\leq (C(p)+1)\|\operatorname{Op}(k) - \operatorname{Op}(k_\varepsilon)\|_p < (C(p)+1)\varepsilon\end{aligned}$$

whenever $\tau > 2\xi_\varepsilon$. Similarly we have:

$$\begin{aligned}\|\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k))\|_p &\leq \|\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k)) - \mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k_\varepsilon))\|_p + \|\mathscr{S}_{\Delta_\tau}(\operatorname{Op}(k_\varepsilon))\|_p \\ &\leq C(p)\|\operatorname{Op}(k) - \operatorname{Op}(k_\varepsilon)\|_p < C(p)\varepsilon\end{aligned}$$

whenever $\tau < \xi_\varepsilon^{-1}$. □

4. Schur-Hadamard multipliers and spectral densities

In this section, we study one of the many applications of Schur-Hadamard multipliers. In particular, through these objects, we will see how one can measure the"density" of the spectrum of an operator $A \in \mathfrak{B}(\mathcal{H})$, for $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{R}_+)$.

To this end, let $\underline{\tau} = \{\tau_N\}_{N\geq 1}$ be a sequence of bounded functions on $\mathbb{R}^2_+$, then $\underline{\tau}$ induces a sequence of Schur-Hadamard multipliers on $\mathfrak{B}(\mathcal{H})$. If $\mathcal{H} = \ell^2(\mathbb{Z}_+)$, $\underline{\tau}$ induces a sequence of multipliers by considering the restriction to $\mathbb{Z}^2_+$ of the function τ_N, this operation is well-defined since we are considering bounded functions, not essentially bounded ones.

To measure the "density" of $A \in \mathfrak{B}(\mathcal{H})$ we only consider only those sequences $\underline{\tau}$ such that $\tau_N \star A \in \mathfrak{S}_\infty$ for all N. Then the singular values of $\tau_N \star A$ exist and we can study their distribution for large values of N. In other words, we study the asymptotic behaviour of the counting function of $\tau_N \star A$ defined as

$$\mathsf{n}(t; \tau_N \star A) = \#\{n : \; s_n(\tau_N \star A) > t\}, \quad t > 0,$$

where $s_n(\tau_N \star A)$ are the singular values of the operator $\tau_N \star A$.

Similarly, for a selfadjoint operator A and τ_N so that $\tau_N(\lambda, \mu) = \overline{\tau_N(\mu, \lambda)}$, one can study the same problem via the functions

$$\mathsf{n}_\pm(t; \tau_N \star A) = \#\{n : \; \lambda_n^\pm(\tau_N \star A) > t\}, \quad t > 0,$$

where $\lambda_n^\pm(\tau_N \star A)$ are the positive eigenvalues of $\pm\tau_N \star A$.

In particular, we want to study from an abstract point of view the functionals

$$\overline{\mathsf{LD}}_{\underline{\tau}}(t; A) := \limsup_{N\to\infty} \frac{\mathsf{n}(t; \tau_N \star A)}{\log(N)}, \tag{2.4.13}$$

$$\underline{\mathsf{LD}}_{\underline{\tau}}(t; A) := \liminf_{N\to\infty} \frac{\mathsf{n}(t; \tau_N \star A)}{\log(N)}. \tag{2.4.14}$$

It is clear that the functionals above measure the asymptotic distribution of the singular values of $\tau_N \star A$ on a logarithmic scale. If for a given $\underline{\tau}$ both limits exist and coincide, we denote their common value by $\mathsf{LD}_{\underline{\tau}}(t; A)$ and we call it the *logarithmic spectral density* of $|A|$ with respect to $\underline{\tau}$.

Remark 2.4.1. Of course, one can replace the logarithm in (2.4.13) and (2.4.14), with, say, N^α, for some $\alpha > 0$. In this way, one can measure the power-like distribution of the eigenvalues. The motivation behind the definition of the functions $\overline{\mathsf{LD}}_{\underline{\tau}}, \underline{\mathsf{LD}}_{\underline{\tau}}$ comes from the results of Widom, [**61**, Theorem 4.3], where he showed that the logarithmic spectral density of the Hilbert matrix exists and is finite. We will get back to this fact in Chapter 4.

Similarly, in the selfadjoint case, we do the same for the functions $\overline{\mathsf{LD}}^\pm_{\underline{\tau}}(t; A)$, $\underline{\mathsf{LD}}^\pm_{\underline{\tau}}(t; A)$ defined just as in (2.4.13) and (2.4.14) with the functions $\mathsf{n}_\pm$ replacing n.

We denote their common value, if it exists $\mathsf{LD}^{\pm}_{\underline{\tau}}(t;A)$. We refer to $\mathsf{LD}^{+}_{\underline{\tau}}(t;A)$ (resp. $\mathsf{LD}^{-}_{\underline{\tau}}(t;A)$) it as the *positive* (resp. *negative*) *logarithmic spectral density* of A with respect to $\underline{\tau}$.

These functionals can be useful in studying the spectral properties of A. In our approach to studying the logarithmic spectral density of a Hankel operator, we need to address some issues related to the abstract properties of the functionals $\overline{\mathsf{LD}}_{\underline{\tau}}$, $\underline{\mathsf{LD}}_{\underline{\tau}}$. In particular, we study:

(a) their behaviour with respect to the sequence $\underline{\tau}$, most notably their invariance under a change of sequence;
(b) their behaviour with respect to sums of operators;
(c) their behaviour when a selfadjoint operator is almost symmetric with respect to $\underline{\tau}$.

Of particular interest will be sequences of multipliers $\underline{\tau} = \{\tau_N\}_{N\geq 1} \in \ell^\infty(\mathfrak{M})$, i.e. sequences of multipliers such that

$$\|\underline{\tau}\|_{\ell^\infty(\mathfrak{M})} := \sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}} < \infty. \tag{2.4.15}$$

In this case we shall say that $\underline{\tau}$ *induces uniformly bounded multipliers.*

In order to answer (a)-(c), we need a couple of facts regarding the counting functions first. For a compact operator K, it is clear that

$$\mathsf{n}(t;K) = \mathsf{n}(t;K^*), \tag{2.4.16}$$

as the non-zero singular values of K and K^* coincide and, furthermore one has

$$\mathsf{n}(t;K) = \mathsf{n}(t^2;K^*K), \quad t>0. \tag{2.4.17}$$

Moreover, if K is selfadjoint we have the identity:

$$\mathsf{n}(t;K) = \mathsf{n}_+(t;K) + \mathsf{n}_-(t;K), \quad t>0.$$

We also need the following simple estimate:

Lemma 2.4.2. *Let $K \in \mathfrak{S}_p$, $1 \leq p < \infty$, then, for any $t>0$ one has*

$$\mathsf{n}(t;K) \leq \frac{\|K\|_p^p}{t^p}.$$

The same is true for the functions $\mathsf{n}_\pm(t;K)$ if K is selfadjoint.

An important set of estimates is collected in the following lemma, see [11, Thm. 9, Ch. 9]:

LEMMA 2.4.3 (**Weyl Inequality**). *Let A, B be compact operators and $0 < s < t$, then*

$$\mathsf{n}(t; A + B) \leq \mathsf{n}(t - s; A) + \mathsf{n}(s; B), \tag{2.4.18}$$

$$\mathsf{n}_{\pm}(t; A + B) \leq \mathsf{n}_{\pm}(t - s; A) + \mathsf{n}_{\pm}(s; B), \tag{2.4.19}$$

with the last inequality holding for selfadjoint operators.

Before moving on to the next lemma, we introduce the following notation for a sequence of functions $\{a_N(t)\}_{N\geq 1}$:

$$a_N(t) = O_t(1), \quad N \to \infty \iff c_1(t) \leq |a_N(t)| \leq c_2(t), \quad \forall N$$

for some functions $c_1(t)$, $c_2(t) > 0$. We now have

LEMMA 2.4.4. *Let $\underline{\tau} = \{\tau_N\}_{N\geq 1} \in \ell^\infty(\mathfrak{M})$ be such that $\tau_N \to c \in \mathbb{C}$ as $N \to \infty$ pointwise. Then for any compact operator K on $\mathcal{H}$ we have*

$$\mathsf{n}(t; \tau_N \star K) = O_t(1), \quad N \to \infty.$$

If K is selfadjoint, the same holds for $\mathsf{n}_{\pm}(t; \tau_N \star K)$.

To prove this result, we need the following

LEMMA 2.4.5. *Let $\underline{\tau}$ as in Lemma 2.4.4. For any $K \in \mathfrak{S}_\infty$, $\tau_N \star K \to cK$ as $N \to \infty$ in the operator norm. If A is bounded, then $\tau_N \star A \to cA$ in the strong operator topology.*

PROOF OF LEMMA 2.4.5. We will show the result for $\mathcal{H} = \ell^2(\mathbb{Z}_+)$. The proof for $L^2(\mathbb{R}_+)$ remains the same once one replaces sums with integrals.

First we note that for any N, $\tau_N \star K - cK = (\tau_N - c) \star K$ and so, without any loss of generality, we can assume that $c = 0$.

Let us begin by showing that if $K \in \mathfrak{S}_2$, one has $(\tau_N \star K) \to \mathbb{O}$ in the operator norm as $N \to \infty$. To this end, recall that $K \in \mathfrak{S}_2$ if and only if there exists a sequence $k \in \ell^2(\mathbb{Z}_+^2)$ such that the matrix entries of K coincide with k and furthermore $\|K\|_2 = \|k\|_{\ell^2(\mathbb{Z}_+^2)}$, see Theorem 1.2.15 in Chapter 1. Using the estimate

$$\|\tau_N \star K\| \leq \|\tau_N \star K\|_2, \tag{2.4.20}$$

we only need to show that

$$\|\tau_N \star K\|_2^2 = \sum_{j,k\geq 0} |\tau_N(j,k)|^2 \, |k(j,k)|^2 \to 0. \tag{2.4.21}$$

However, an application of the Dominated Convergence Theorem immediately yields that $\|\tau_N \star K\|_2 \to 0$ as $N \to \infty$, and so our claim holds.

Since Hilbert-Schmidt operators are dense inside the set of compact operators, then for any $K \in \mathfrak{S}_\infty$, we can find $K_\varepsilon \in \mathfrak{S}_2$ so that $\|K - K_\varepsilon\| < \varepsilon$ for any given $\varepsilon > 0$. Since $\underline{\tau} \in \ell^\infty(\mathfrak{M})$, the triangle inequality implies that for N sufficiently large

$$\begin{aligned}\|\tau_N \star K\| &\leq \|\tau_N \star (K - B)\| + \|\tau_N \star B\| \\ &\leq \|\underline{\tau}\|_{\ell^\infty(\mathfrak{M})}\|K - B\| + \|\tau_N \star B\| \\ &< (1 + \|\underline{\tau}\|_{\ell^\infty(\mathfrak{M})})\varepsilon\end{aligned}$$

For the second part of the statement, suppose A is bounded. Let e_j, $j \geq 0$ be the standard basis vectors of $\ell^2(\mathbb{Z}_+)$. Since $\tau_N \to 0$ pointwise, a simple calculation shows $(\tau_N \star A)e_j \to 0$ in $\ell^2(\mathbb{Z}_+)$ and so we obtain that $(\tau_N \star A)x \to 0$ as $N \to \infty$ for any finite sequence $x \in \ell^2(\mathbb{Z}_+)$.

For any $x \in \ell^2(\mathbb{Z}_+)$, the result follows from a standard $\varepsilon/3$ argument. In particular, for $\varepsilon > 0$, we find a finite sequence x_ε so that $\|x - x_\varepsilon\|_{\ell^2(\mathbb{Z}_+)} < \varepsilon/3$. The triangle inequality and the assumption $\underline{\tau} \in \ell^\infty(\mathfrak{M})$ finally prove the assertion. □

Proof Lemma 2.4.4. By Lemma 2.4.5, for a given $\varepsilon > 0$, we can find N so large that $\|\tau_N \star K - cK\| < \varepsilon$. Whereby it follows that $\mathsf{n}(\varepsilon; \tau_N \star K - cK) = 0$ and so Weyl's inequality (2.4.18) gives the assertion since

$$\mathsf{n}(t + \varepsilon; cK) \leq \mathsf{n}(t; \tau_N \star K) \leq \mathsf{n}(t - \varepsilon; cK). \qquad \square$$

The first of our results addresses the universality of $\overline{\mathsf{LD}}_{\underline{\tau}}, \underline{\mathsf{LD}}_{\underline{\tau}}$. In other words, we show that the functions $\overline{\mathsf{LD}}_{\underline{\tau}}, \underline{\mathsf{LD}}_{\underline{\tau}}$ are almost invariant with respect to the sequence of multipliers chosen.

THEOREM 2.4.6 (Invariance Principle). *Let $\underline{\tau}^{(1)}, \underline{\tau}^{(2)}$ be sequences of Schur-Hadamard multipliers and $A \in \mathfrak{B}(\mathcal{H})$. If for some finite $p \geq 1$ one has that*

$$\sup_{N \geq 1} \|(\tau_N^{(1)} - \tau_N^{(2)}) \star A\|_p < \infty, \tag{2.4.22}$$

then

$$\begin{aligned}\overline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t+0; A) \leq \overline{\mathsf{LD}}_{\underline{\tau}^{(2)}}(t; A) \leq \overline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t-0; A), \\ \underline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t+0; A) \leq \underline{\mathsf{LD}}_{\underline{\tau}^{(2)}}(t; A) \leq \underline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t-0; A).\end{aligned}$$

Similarly, if $A \in \mathfrak{B}(\mathcal{H})$ and $\tau_N^{(i)}(\lambda, \mu) = \overline{\tau_N^{(i)}(\mu, \lambda)}$ for $i = 1, 2.$, then one has that

$$\begin{aligned}\overline{\mathsf{LD}}^{\pm}_{\underline{\tau}^{(1)}}(t+0; A) \leq \overline{\mathsf{LD}}^{\pm}_{\underline{\tau}^{(2)}}(t; A) \leq \overline{\mathsf{LD}}^{\pm}_{\underline{\tau}^{(1)}}(t-0; A), \\ \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}^{(1)}}(t+0; A) \leq \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}^{(2)}}(t; A) \leq \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}^{(1)}}(t-0; A).\end{aligned}$$

With the understanding that if one of the quantities is infinite, then so are the remaining ones.

Proof of Theorem 2.4.6. By Weyl's inequality (2.4.18) we have that for any $0 < s < t$:

$$\mathsf{n}(t+s; \tau_N^{(1)} \star A) \leq \mathsf{n}(t; \tau_N^{(2)} \star A) + \mathsf{n}(s; (\tau_N^{(1)} - \tau_N^{(2)}) \star A), \tag{2.4.23}$$

$$\mathsf{n}(t; \tau_N^{(2)} \star A) \leq \mathsf{n}(t-s; \tau_N^{(1)} \star A) + \mathsf{n}(s; (\tau_N^{(2)} - \tau_N^{(1)}) \star A). \tag{2.4.24}$$

Using Lemma 2.4.2 as well as out assumption (2.4.22), we obtain that

$$\mathsf{n}(s; (\tau_N^{(2)} - \tau_N^{(1)}) \star A) \leq s^{-p} \sup_{N \geq 1} \|(\tau_N^{(1)} - \tau_N^{(2)}) \star A\|_p$$

and so dividing through by $\log(N)$ and sending $N \to \infty$ in (2.4.23) and (2.4.24) gives that

$$\overline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t+s; A) \leq \overline{\mathsf{LD}}_{\underline{\tau}^{(2)}}(t; A) \leq \overline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t-s; A),$$

$$\underline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t+s; A) \leq \underline{\mathsf{LD}}_{\underline{\tau}^{(2)}}(t; A) \leq \underline{\mathsf{LD}}_{\underline{\tau}^{(1)}}(t-s; A).$$

The rest now follows by sending $s \to 0$. □

Next, we study the behaviour of the functionals with respect to sums of operators. In particular, we show that for any $\underline{\tau}$, if the function $\overline{\mathsf{LD}}_{\underline{\tau}}(t; A+B)$ exists, it is almost equal to the sum $\overline{\mathsf{LD}}_{\underline{\tau}}(t; A) + \overline{\mathsf{LD}}_{\underline{\tau}}(t; B)$, if both summands are finite, provided A, B are almost orthogonal in a sense better specified below. The same, of course, can be said for $\underline{\mathsf{LD}}_{\underline{\tau}}(t; A+B)$.

THEOREM 2.4.7 (Asymptotic orthogonality). *Let A_i, with $1 \leq i \leq L$, be a family of operators on $\mathcal{H}$ such that for some $p \in [1, \infty)$ one has*

$$\sup_{N \geq 1} \|(\tau_N \star A_j)^*(\tau_N \star A_k)\|_p < \infty, \quad j \neq k, \tag{2.4.25}$$

$$\sup_{N \geq 1} \|(\tau_N \star A_j)(\tau_N \star A_k)^*\|_p < \infty, \quad j \neq k. \tag{2.4.26}$$

Then, for $S = \sum_{j=1}^L A_j$ and for any $t > 0$ we have:

$$\sum_{j=1}^{L} \overline{\mathsf{LD}}_{\underline{\tau}}(t+0; A_j) \leq \overline{\mathsf{LD}}_{\underline{\tau}}(t; S) \leq \sum_{j=1}^{L} \overline{\mathsf{LD}}_{\underline{\tau}}(t-0; A_j), \tag{2.4.27}$$

$$\sum_{j=1}^{L} \underline{\mathsf{LD}}_{\underline{\tau}}(t+0; A_j) \leq \underline{\mathsf{LD}}_{\underline{\tau}}(t; S) \leq \sum_{j=1}^{L} \underline{\mathsf{LD}}_{\underline{\tau}}(t-0; A_j). \tag{2.4.28}$$

Provided all of terms exist and are finite. If all A_j are selfadjoint and $\tau_N(\lambda, \mu) = \overline{\tau_N(\mu, \lambda)}$, then we have

$$\sum_{j=1}^{L} \overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t+0; A_j) \leq \overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t; S) \leq \sum_{j=1}^{L} \overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t-0; A_j), \tag{2.4.29}$$

$$\sum_{j=1}^{L} \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t+0; A_j) \leq \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t; S) \leq \sum_{j=1}^{L} \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t-0; A_j). \tag{2.4.30}$$

REMARK 2.4.8. Even though the result above is new in the literature, it is easily comparable to some results available in the literature, especially in [54], see Theorem 2.2, as well as in [55], see Thorem 2.3. In fact, our terminology is borrowed from them. Both in this thesis and in the cited articles, the basic idea underlying their proofs date back to [10] and are all collected in [11].

PROOF OF THEOREM 2.4.7. By induction and without any loss of generality, it is suffient to prove the inequalities (2.4.27)-(2.4.30) for $L = 2$. In this case, we put $A_1 = A$, $A_2 = B$, moreover, for brevity, we write $A^{(N)}$ and $B^{(N)}$ instead of $\tau_N \star A$, $\tau_N \star B$ respectively. In this framework, the assumptions (2.4.25) and (2.4.26) can be easily restated as

$$\sup_{N\geq 1} \|A^{(N)*}B^{(N)}\|_p < \infty, \quad \sup_{N\geq 1} \|B^{(N)*}A^{(N)}\|_p < \infty. \tag{2.4.31}$$

With these simplifications, we first focus on the non-selfadjoint case and, in particular, on the proof of (2.4.27), as (2.4.28) follows the same reasoning.
Put $\mathcal{X} = \mathcal{H} \oplus \mathcal{H}$ and define the block diagonal operator $\mathcal{A}_N = \operatorname{diag}\left\{A^{(N)}, B^{(N)}\right\}$, such that

$$\mathcal{A}_N(f, g) = (A^{(N)}f, B^{(N)}g).$$

Similarly, let $\mathcal{A} = \operatorname{diag}\{A, B\}$. Since $\mathcal{A}_N$ is diagonal, we have the following useful identity:

$$\mathsf{n}(t; \mathcal{A}_N) = \mathsf{n}(t; A^{(N)}) + \mathsf{n}(t; B^{(N)}). \tag{2.4.32}$$

Define the operator $\mathcal{J} : \mathcal{X} \to \mathcal{H}$ as

$$\mathcal{J}(f, g) = f + g.$$

Then, the operator $(\mathcal{J}\mathcal{A}_N)^*(\mathcal{J}\mathcal{A}_N)$ can be written as an 2×2 block-matrix of the form:

$$\begin{pmatrix} A^{(N)*}A^{(N)} & A^{(N)*}B^{(N)} \\ B^{(N)*}A^{(N)} & B^{(N)*}B^{(N)} \end{pmatrix}.$$

Since the operator $\mathcal{A}_N^*\mathcal{A}_N$ is the block diagonal 2×2 matrix

$$\begin{pmatrix} A^{(N)*}A^{(N)} & \mathbb{O} \\ \mathbb{O} & B^{(N)*}B^{(N)} \end{pmatrix},$$

it is easy to see that the difference $(\mathcal{J}\mathcal{A}_N)^*(\mathcal{J}\mathcal{A}_N) - \mathcal{A}_N^*\mathcal{A}_N$ is the 2×2 matrix

$$\mathcal{K}_N = \begin{pmatrix} \mathbb{O} & A^{(N)*}B^{(N)} \\ B^{(N)*}A^{(N)} & \mathbb{O} \end{pmatrix}.$$

From (2.4.31), we immediately get that

$$\sup_{N\geq 1} \|\mathcal{K}_N\|_p < \infty,$$

and so, the Weyl inequality (2.4.18) and Lemma 2.4.2 gives as $N \to \infty$

$$\mathsf{n}(t; \mathcal{J}\mathcal{A}_N) = \mathsf{n}(t^2; (\mathcal{J}\mathcal{A}_N)^*(\mathcal{J}\mathcal{A}_N)) \leq \mathsf{n}(t^2 - s; \mathcal{A}_N^*\mathcal{A}_N) + O_s(1). \tag{2.4.33}$$

Just as in the proof of Theorem 2.4.6, we swap the roles of $\mathcal{J}\mathcal{A}_N$ and $\mathcal{A}_N$ and use (2.4.17) to obtain

$$\mathsf{n}(\sqrt{t^2 + s}; \mathcal{A}_N) - O_s(1) \leq \mathsf{n}(t; \mathcal{J}\mathcal{A}_N) \leq \mathsf{n}(\sqrt{t^2 - s}; \mathcal{A}_N) + O_s(1).$$

Recall now that we set $S = \sum_{j=1}^{L} A_j$. We have

$$S^{(N)}S^{(N)*} = A^{(N)}A^{(N)*} + A^{(N)}B^{(N)*} + B^{(N)}A^{(N)*} + B^{(N)}B^{(N)*},$$
$$(\mathcal{J}\mathcal{A}_N)(\mathcal{J}\mathcal{A}_N)^* = A^{(N)}A^{(N)*} + B^{(N)}B^{(N)*}.$$

Write $D_N = A^{(N)}A^{(N)*} - (\mathcal{J}\mathcal{A}_N)(\mathcal{J}\mathcal{A}_N)^*$, then from (2.4.31) it follows that $\sup_{N\geq 1} \|D_N\|_p < \infty$, and so (2.4.17) in conjunction with(2.4.16) and (2.4.18) gives for $N \to \infty$:

$$\mathsf{n}(\sqrt{t^2 + s}; \mathcal{J}\mathcal{A}_N) - O_s(1) \leq \mathsf{n}(t; S^{(N)}) \leq \mathsf{n}(\sqrt{t^2 - s}; \mathcal{J}\mathcal{A}_N) + O_s(1). \tag{2.4.34}$$

Thus, putting together (2.4.33) and (2.4.34) we obtain that

$$\mathsf{n}(\sqrt{t^2 + s}; \mathcal{A}_N) - O_s(1) \leq \mathsf{n}(t; S^{(N)}) \leq \mathsf{n}(\sqrt{t^2 - s}; \mathcal{A}_N) + O_s(1). \tag{2.4.35}$$

Therefore, diving through by $\log N$, sending $N \to \infty$ and using (2.4.32) yields

$$\overline{\mathsf{LD}}(t; S) \leq \overline{\mathsf{LD}}(\sqrt{t^2 - s}; A) + \overline{\mathsf{LD}}(\sqrt{t^2 - s}; B),$$
$$\underline{\mathsf{LD}}(t; S) \geq \underline{\mathsf{LD}}(\sqrt{t^2 + s}; A) + \underline{\mathsf{LD}}(\sqrt{t^2 + s}; B),$$

from which (2.4.27) follows immediately once we send $s \to 0$.

Let us now move to the selfadjoint case. Just as before, we will only prove (2.4.29), as (2.4.30) follows a similar approach. Our assumption on the multipliers τ_N, namely that $\tau_N(\lambda, \mu) = \overline{\tau_N(\mu, \lambda)}$, ensures that the operators $A^{(N)}, B^{(N)}$ are selfadjoint and thus so is $A^{(N)} + B^{(N)}$. Note that in this case (2.4.31) reduces to the following

$$\sup_{N\geq 1} \|A^{(N)}B^{(N)}\|_p = \sup_{N\geq 1} \|B^{(N)}A^{(N)}\|_p < \infty,$$

because we have $(A^{(N)}B^{(N)})^* = B^{(N)}A^{(N)}$.
As before, for a selfadjoint operator T, we define

$$T_\pm = \frac{1}{2}\left(|T| \pm T\right).$$

It is easy to see that $\lambda_n^\pm(T) = s_n(T_\pm)$ from which we obtain that for any $t > 0$

$$\mathsf{n}_\pm(t; T) = \mathsf{n}(t; T_\pm). \tag{2.4.36}$$

We will use this fact extensively. Put $K_N^+ = (A^{(N)}+B^{(N)})_+ - (A_+^{(N)}+B_+^{(N)})$, similarly define $K_N^- = (A^{(N)}+B^{(N)})_- - (A_-^{(N)}+B_-^{(N)})$. Then by the Weyl's inequality (2.4.18) and (2.4.36), we obtain

$$\mathsf{n}_\pm(t; A^{(N)} + B^{(N)}) \leq \mathsf{n}(t-s; A_\pm^{(N)} + B_\pm^{(N)}) + \mathsf{n}(s; K_\pm^{(N)}), \tag{2.4.37}$$

$$\mathsf{n}_\pm(t; A^{(N)} + B^{(N)}) \geq \mathsf{n}(t+s; A_\pm^{(N)} + B_\pm^{(N)}) - \mathsf{n}(s; K_\pm^{(N)}), \tag{2.4.38}$$

If for the operators $K_N^\pm$ we can prove that we have the following estimate

$$\sup_{N\geq 1} \|K_N^\pm\|_{2p} = \frac{1}{2}\sup_{N\geq 1} \big\| |A^{(N)} + B^{(N)}| - (|A^{(N)}| + |B^{(N)}|) \big\|_{2p} < \infty, \tag{2.4.39}$$

then from Lemma 2.4.4 and (2.4.36) it follows that as $N \to \infty$

$$\mathsf{n}_\pm(t; A^{(N)} + B^{(N)}) \leq \mathsf{n}(t-s; A_\pm^{(N)} + B_\pm^{(N)}) + O_s(1), \tag{2.4.40}$$

$$\mathsf{n}_\pm(t; A^{(N)} + B^{(N)}) \geq \mathsf{n}(t+s; A_\pm^{(N)} + B_\pm^{(N)}) - O_s(1). \tag{2.4.41}$$

We will get back to showing (2.4.39) later. Now, it is easy to see that the products $A_+^{(N)}B_+^{(N)}$, $A_-^{(N)}B_-^{(N)}$ and their adjoints are so that

$$\sup_{N\geq 1} \|A_\pm^{(N)}B_\pm^{(N)}\|_p = \sup_{N\geq 1} \|B_\pm^{(N)}A_\pm^{(N)}\|_p < \infty,$$

therefore re-running the same argument presented in the non-selfadjoint case, with the due modifications in place, we obtain for any $0 < s' < t'$

$$\begin{aligned} \mathsf{n}(t'; A_\pm^{(N)} + B_\pm^{(N)}) &\leq \mathsf{n}(t'-s'; A_\pm^{(N)}) + \mathsf{n}(t'-s'; B_\pm^{(N)}) + O_{s'}(1) \\ &= \mathsf{n}_\pm(t'-s'; A^{(N)}) + \mathsf{n}_\pm(t'-s'; B^{(N)}) + O_{s'}(1), \end{aligned} \tag{2.4.42}$$

$$\begin{aligned} \mathsf{n}(t'; A_\pm^{(N)} + B_\pm^{(N)}) &\geq \mathsf{n}(t'+s'; A_\pm^{(N)}) + \mathsf{n}(t'+s'; B_\pm^{(N)}) - O_{s'}(1) \\ &= \mathsf{n}_\pm(t'+s'; A^{(N)}) + \mathsf{n}_\pm(t'+s'; B^{(N)}) - O_{s'}(1). \end{aligned} \tag{2.4.43}$$

Putting everything back together, dividing through by $\log(N)$ and sending $N \to \infty$ we finally obtain

$$\overline{\mathsf{LD}}_{\mathcal{T}}^\pm(t; A+B) \leq \overline{\mathsf{LD}}_{\mathcal{T}}^\pm(t-s-s'; A) + \overline{\mathsf{LD}}_{\mathcal{T}}^\pm(t-s-s'; B) \tag{2.4.44}$$

$$\overline{\mathsf{LD}}_{\mathcal{T}}^\pm(t; A+B) \geq \overline{\mathsf{LD}}_{\mathcal{T}}^\pm(t+s+s'; A) + \overline{\mathsf{LD}}_{\mathcal{T}}^\pm(t+s+s'; B). \tag{2.4.45}$$

Finally sending both $s, s' \to 0$ yields (2.4.29).

Let us now show that (2.4.39) is indeed true. We have

$$\begin{aligned} 2K_N^\pm &= |A^{(N)} + B^{(N)}| - (|A^{(N)}| + |B^{(N)}|) \\ &= T_N^{1/2} - S_N^{1/2} \end{aligned}$$

where $T_N = |A^{(N)} + B^{(N)}|^2$, $S_N = (|A^{(N)}| + |B^{(N)}|)^2$. From Theorem 1.2.22, with $\alpha = 1/2$, it is sufficient to show that

$$\sup_{N\geq 1} \|T_N - S_N\|_p$$

is finite. To this end, notice that

$$T_N - S_N = A^{(N)}B^{(N)} + B^{(N)}A^{(N)} - \left|A^{(N)}\right|\left|B^{(N)}\right| - \left|B^{(N)}\right|\left|A^{(N)}\right|.$$

Since we can write $\left|A^{(N)}\right| = U_N A^{(N)}$, $\left|B^{(N)}\right| = V_N B^{(N)}$, where U_N, V_N are projections that commute with $A^{(N)}$, $B^{(N)}$ respectively, it follows that

$$\sup_{N\geq 1} \|T_N - S_N\|_p \leq 4 \sup_{N\geq 1} \|A^{(N)}B^{(N)}\|_p,$$

whereby we obtain the desired estimate

$$\sup_{N\geq 1} \|K_N^{\pm}\|_{2p} \leq \sup_{N\geq 1} \|A^{(N)}B^{(N)}\|_p^{1/2} < \infty.$$

This concludes the proof of the assertion. □

Finally, the result below describes the behaviour of the functionals $\overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t;A)$, $\underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t;A)$ when the selfadjoint operator A has an asymptotically symmetric spectrum, in a sense we specify below.

THEOREM 2.4.9 (**Asymptotic symmetry**). *Let $A \in \mathfrak{B}(\mathcal{H})$ be a selfadjoint operator and $\underline{\tau} = \{\tau_N\}_{N\geq 1}$ be such that $\tau_N(x,y) = \overline{\tau_N(y,x)}$. Suppose there exists a unitary operator U for which*

$$\sup_{N\geq 1} \|U(\tau_N \star A) + (\tau_N \star A)U\|_p < \infty$$

for some $p \geq 1$. Then for $t > 0$

$$\overline{\mathsf{LD}}^{-}_{\underline{\tau}}(t+0;A) \leq \overline{\mathsf{LD}}^{+}_{\underline{\tau}}(t;A) \leq \overline{\mathsf{LD}}^{-}_{\underline{\tau}}(t-0;A),$$
$$\underline{\mathsf{LD}}^{-}_{\underline{\tau}}(t+0;A) \leq \underline{\mathsf{LD}}^{+}_{\underline{\tau}}(t;A) \leq \underline{\mathsf{LD}}^{-}_{\underline{\tau}}(t-0;A),$$

provided all of the quantities are finite. Moreover, if all limit exists

$$\overline{\mathsf{LD}}_{\underline{\tau}}(t+0;A) \leq 2\overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t;A) \leq \overline{\mathsf{LD}}_{\underline{\tau}}(t-0;A),$$
$$\underline{\mathsf{LD}}_{\underline{\tau}}(t+0;A) \leq 2\underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t;A) \leq \underline{\mathsf{LD}}_{\underline{\tau}}(t-0;A).$$

Proof of Theorem 2.4.9. Write

$$K_N = \tau_N \star A + U^*(\tau_N \star A)U.$$

By assumption $\sup_{N\geq 1} \|K_N\|_p$ is finite. Furthermore, by Weyl inequality (2.4.19)

$$\begin{aligned} \mathsf{n}_{\pm}(t;\tau_N \star A) &= \mathsf{n}_{\pm}(t; -U^*(\tau_N \star A)U + K_N) \\ &\leq \mathsf{n}_{\pm}(t-s; -U^*(\tau_N \star A)U) + \mathsf{n}_{\pm}(s;K_N) \\ &= \mathsf{n}_{\mp}(t-s;\tau_N \star A) + \mathsf{n}_{\pm}(s;K_N), \end{aligned}$$

where $0 < s < t$. In particular, this gives that

$$\mathsf{n}_+(t;\tau_N \star A) \geq \mathsf{n}_-(t+s;\tau_N \star A) - \mathsf{n}_-(s, K_N),$$
$$\mathsf{n}_+(t;\tau_N \star A) \leq \mathsf{n}_-(t-s;\tau_N \star A) + \mathsf{n}_+(s, K_N).$$

The result follows once we divide by $\log(N)$, send $N \to \infty$ and use Lemma 2.4.2. ☐

Although Theorems 2.4.6-2.4.9 are quite abstract, they play a fundamental role in the study of spectral densities of Hankel operators. We will define such operators in the next Chapter and study their spectral densities in Chapter 4 and 5.

5. A proof of Theorem 1.2.22

We conclude this Chapter on transformators and Schur-hadamard multipliers by reproducing the proof, found in [8], of Theorem 1.2.22. We recall its statement below

THEOREM (1.2.22)**.** *Let $A, B \geq 0$ be such that $T = B - A \in \mathfrak{S}_p$ for some $p \geq 1$. Then $B^\alpha - A^\alpha \in \mathfrak{S}_{p/\alpha}$ for any $0 < \alpha < 1$ and moreover*

$$\|B^\alpha - A^\alpha\|_{p/\alpha} \leq \|T\|_p^\alpha.$$

The main idea of the proof is to write

$$B^\alpha - A^\alpha = \int_0^1 \frac{d}{d\varepsilon}(A_\varepsilon^\alpha - A^\alpha)d\varepsilon$$

where $A_\varepsilon = A + \varepsilon T$, for $0 \leq \varepsilon \leq 1$. Using the above, we would be done once we find a suitable estimate for

$$\left\| \frac{d}{d\varepsilon}(A_\varepsilon^\alpha - A^\alpha) \right\|_{p/\alpha}.$$

Of course, in the process we will also have to make sense of the derivative of the operator-valued function $A_\varepsilon^\alpha - A^\alpha$. We chose to give a full proof of this result as it is an interesting example of the interplay between the theory of transformators and the theory of bounded operators on a Hilbert space $\mathcal{H}$. The result that links Theorem 1.2.22 to the theory of transformators is an interpolation Theorem by Mitjiagin, see [**25**, pg.135] and [**42**, Theorem 1'], which, in fact, can be considered a generalisation of the Duality Principle 2.1.4 to any transformator.

THEOREM 2.5.1. *Let $\mathscr{S}$ be a transformator such that both $\mathscr{S}$ is bounded when acting on $\mathfrak{S}_\infty$ and on $\mathfrak{S}_1$. Then $\mathscr{S}$ is bounded on $\mathfrak{S}_p$ for any $1 < p < \infty$. Moreover*

$$|||\mathscr{S}|||_{\mathfrak{S}_p} \leq C \max\{|||\mathscr{S}|||_{\mathfrak{S}_1}, |||\mathscr{S}|||_{\mathfrak{S}_\infty}\}.$$

For a bounded operator $H > 0$ and $\alpha \in (0,1)$, we can represent H^α, $H^{\alpha-1}$ in the form of an integral, namely:

$$H^\alpha = c_\alpha \int_0^\infty t^{\alpha-1}(t+H)^{-1}H dt, \quad c_\alpha = \frac{\sin(\pi\alpha)}{\pi}, \tag{2.5.46}$$

$$H^{\alpha-1} = d_\alpha \int_0^\infty t^{\alpha}(t+H)^{-2} dt, \quad d_\alpha = \alpha^{-1}c_\alpha. \tag{2.5.47}$$

The first integral converges in the strong operator topology, while the second in the operator norm. We can use these representations to prove the following lemma:

LEMMA 2.5.2. *Let $H > 0$, $0 < \alpha < 1$ and $2\delta = 1 - \alpha$. The transformator $\mathscr{T}_\alpha$ defined as*

$$\mathscr{T}_\alpha(X) = c_\alpha \int_0^\infty t^\alpha (t+H)^{-1}H^{-\delta}XH^{-\delta}(t+H)^{-1}dt. \tag{2.5.48}$$

is bounded on $\mathfrak{B}(\mathcal{H})$ as well as on $\mathfrak{S}_p$ for any $p \geq 1$ and moreover we have

$$|||\mathscr{T}_\alpha|||_{\mathfrak{B}(\mathcal{H})} \leq \alpha, \quad |||\mathscr{T}_\alpha|||_{\mathfrak{S}_p} \leq \alpha. \tag{2.5.49}$$

PROOF OF LEMMA. Notice that the transformator $\mathscr{T}_\alpha$ is formally selfadjoint, so since $\mathfrak{B}(\mathcal{H})$ is the dual space of $\mathfrak{S}_1$, then the boundedness of $\mathscr{T}_\alpha$ on $\mathfrak{B}(\mathcal{H})$ implies its boundedness on $\mathfrak{S}_1$. Similarly, since $\mathfrak{S}_1$ is the dual space of $\mathfrak{S}_\infty$, then we deduce that $\mathscr{T}_\alpha$ is bounded on $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ if and only if it is bounded on $\mathfrak{B}(\mathcal{H})$. So from Theorem 2.5.1, we obtain that $\mathscr{T}_\alpha$ is bounded on $\mathfrak{S}_p$ for any $p \in (1,\infty)$.
So it remains to verify $|||\mathscr{T}_\alpha|||_{\mathfrak{B}(\mathcal{H})} \leq \alpha$. To this end, for $g, h \in \mathcal{H}$, we want to estimate the quadratic form

$$(\mathscr{T}_\alpha(X)g,\, h) = c_\alpha \int_0^\infty t^\alpha \left((t+H)^{-1}H^{-\delta}XH^{-\delta}(t+H)^{-1}g, h\right) dt \tag{2.5.50}$$

$$= c_\alpha \int_0^\infty t^\alpha \left(XH^{-\delta}(t+H)^{-1}g, H^{-\delta}(t+H)^{-1}h\right) dt \tag{2.5.51}$$

where in the first equality we used the fact that the integral converges strongly to put the integral outside the inner product, while in the second we used the fact that H is selfadjoint. From the Cauchy-Schwartz and the boundedness of X it follows that:

$$|(\mathscr{T}_\alpha(X)g, h)| \leq c_\alpha \|X\| J(g)J(h). \tag{2.5.52}$$

where J is the quantity

$$J(f)^2 = \int_0^\infty t^\alpha \|H^{-\delta}(t+H)^{-1}f\|^2 dt. \tag{2.5.53}$$

for any $f \in \mathcal{H}$. So it remains to find an estimate for $J(f)$. Note that as $2\delta = \alpha - 1$ and as H is selfadjoint, we have that

$$\|H^{-\delta}(t+H)^{-1}f\|^2 = \left(H^{-\delta}(t+H)^{-1}f, H^{-\delta}(t+H)^{-1}f\right)$$
$$= \left(H^{1-\alpha}(t+H)^{-2}f, f\right).$$

Using the projection-valued measure version of the Spectral Theorem for selfadjoint operators, see [56, Theorem VII.8], we can find a finite positive measure μ_f supported on $\operatorname{spec}(H)$ such that

$$\big(H^{1-\alpha}(t+H)^{-2}f, f\big) = \int_0^\infty \frac{s^{1-\alpha}}{(t+s)^2} d\mu_f(s). \tag{2.5.54}$$

Using Fubini's Theorem, we finally obtain that

$$J(f)^2 = \int_0^\infty \int_0^\infty \frac{t^\alpha s^{1-\alpha}}{(t+s)^2} dt d\mu_f(s) = d_\alpha^{-1}\|f\|^2. \tag{2.5.55}$$

Taking into account the latter representation and (2.5.52), we obtain

$$|(\mathscr{T}_\alpha(X)g,\, h)| \leq c_\alpha d_\alpha^{-1}\|g\|\|h\|\|X\|$$

From which the desired result immediately follows. □

Let us now concentrate on finding an expression for the derivative of the operator-valued function $A_\varepsilon^\alpha - A^\alpha$.

Lemma 2.5.3. *Let $A, B > 0$ be bounded operators, $T = B - A$ and $A_\varepsilon = A + \varepsilon T$ for $\varepsilon \in (0, 1)$. Then for any $0 < \alpha < 1$, the derivative of $A_\varepsilon^\alpha - A^\alpha$ exists in the operator norm and*

$$\frac{d}{d\varepsilon}(A_\varepsilon^\alpha - A^\alpha) = c_\alpha \int_0^\infty t^\alpha (t + A_\varepsilon)^{-1} T (t + A_\varepsilon)^{-1} dt. \tag{2.5.56}$$

Moreover, if $T \in \mathfrak{S}_p$, $p \geq 1$, the integral converges in the $\mathfrak{S}_p$ norm.

Proof of Lemma. Let us show the identity in the operator norm. In this case, we need to show that

$$\lim_{h\to 0} \frac{(A_{\varepsilon+h}^\alpha - A^\alpha) - (A_\varepsilon^\alpha - A^\alpha)}{h} = c_\alpha \int_0^\infty t^\alpha (t + A_\varepsilon)^{-1} T (t + A_\varepsilon)^{-1} dt. \tag{2.5.57}$$

To do this, notice that in the strong operator topology, the Resolvent Identity in Proposition 1.2.7 and (2.5.46) gives

$$\begin{aligned}\mathscr{D}_{\varepsilon+h}(T) := \frac{(A_{\varepsilon+h}^\alpha - A^\alpha) - (A_\varepsilon^\alpha - A^\alpha)}{h} &= \frac{A_{\varepsilon+h}^\alpha - A_\varepsilon^\alpha}{h} \\ &= c_\alpha \int_0^\infty t^\alpha (t + A_{\varepsilon+h})^{-1} T (t + A_\varepsilon)^{-1} dt.\end{aligned}$$

So, at least in the strong operator topology we immediately have

$$\mathscr{D}_{\varepsilon+h}(T) \to c_\alpha \int_0^\infty t^\alpha (t + A_\varepsilon)^{-1} T (t + A_\varepsilon)^{-1} dt =: \mathscr{D}_\varepsilon(T)$$

as $h \to 0$. We are done once we show that the same holds in the operator norm. Applying the Resolvent identity, Proposition 1.2.7, once more gives

$$\mathscr{D}_{\varepsilon+h}(T) - \mathscr{D}_\varepsilon(T) = c_\alpha h \int_0^\infty t^\alpha (t + A_{\varepsilon+h})^{-1} T (t + A_\varepsilon)^{-1} T (t + A_\varepsilon)^{-1} dt$$

Let $\mathscr{F}_h(T) = \mathscr{D}_{\varepsilon+h}(T) - \mathscr{D}_\varepsilon(T)$, then for any $f, g \in \mathcal{H}$ we have that

$$|(\mathscr{F}_h(T)f, g)| \leq c_\alpha |h| \int_0^\infty t^\alpha \left|((t + A_{\varepsilon+h})^{-1}T(t + A_\varepsilon)^{-1}T(t + A_\varepsilon)^{-1}f, g)\right| dt$$

Since $A, B > 0$ then $(t + A_{\varepsilon+h})^{-1} \to (t + A_\varepsilon)^{-1}$ as $h \to 0$ for any $t > 0$ and so $\|(t + A_{\varepsilon+h})^{-1}\| \leq C_\varepsilon$, for some $C_\varepsilon > 0$ independent of h, therefore the Cauchy-Schwartz inequality yields the estimate

$$|(\mathscr{F}_h(T)f, g)| \leq Cc_\alpha |h| \|T\|^2 J_\varepsilon(f)J_\varepsilon(g)$$

where for any $f \in \mathcal{H}$

$$J_\varepsilon(f)^2 = \int_0^\infty t^\alpha \|(t + A_\varepsilon)^{-1}f\|^2 dt.$$

Using the projection-valued measure version of the Spectral Theorem, [56, Theorem VII.8], we can find a family of positive measures $\mu_f^{(\varepsilon)}$ each supported on the spectrum of A_ε, so that

$$\begin{aligned} J_\varepsilon(f)^2 &= \int_0^\infty t^\alpha \left|((t + A_\varepsilon)^{-2}f, f)\right| dt \\ &= \int_0^\infty \int_0^\infty \frac{t^\alpha}{(t+s)^2} d\mu_f^{(\varepsilon)}(s)dt \\ &= \int_0^\infty \int_0^\infty \frac{t^\alpha}{(t+s)^2} dt d\mu_f^{(\varepsilon)}(s) \\ &= d_\alpha^{-1} \int_0^\infty s^{\alpha-1} d\mu_f^{(\varepsilon)}(s) = d_\alpha^{-1} \left(A_\varepsilon^{\alpha-1}f, f\right) \end{aligned}$$

As $A, B > 0$, so is A_ε and therefore $A_\varepsilon^{\alpha-1}$ is a bounded operator on $\mathcal{H}$. From this we obtain that $J_\varepsilon(f) \leq \|A_\varepsilon^{\alpha-1}\|\|f\|$ from which we see that

$$\|\mathscr{F}_h(T)\| \leq C\alpha\|T\|^2 |h| \|A_\varepsilon^{\alpha-1}\| \tag{2.5.58}$$

and so the result follows immediately in the case of the operator norm.
Note now that the transformator $\mathscr{F}_h$ is formally selfadjoint, since it is a linear combinaton of formally selfadjoint transformators, therefore, by the interpolation Theorem of Mitjagin, Theorem 2.5.1, we see that the result holds for $T \in \mathfrak{S}_p$ for any $p \geq 1$. ☐

We are now eady to finalise the proof of the theorem.

Proof of Theorem 1.2.22. Let us assume for the moment that both $A, B > 0$, and let $T = B - A > 0$. Then we have that

$$B^\alpha - A^\alpha = \int_0^1 \frac{d}{d\varepsilon}(A_\varepsilon^\alpha - A^\alpha) d\varepsilon$$

From Lemma 2.5.3, we see that with $2\delta = 1 - \alpha$

$$\frac{d}{d\varepsilon}(A_\varepsilon^\alpha - A^\alpha) = c_\alpha \int_0^\infty t^\alpha (t + A_\varepsilon)^{-1} T (t + A_\varepsilon)^{-1} dt = \mathscr{T}_\alpha (A_\varepsilon^{-\delta} T A_\varepsilon^{-\delta})$$

where $\mathscr{T}_\alpha$ is the transformator defined in Lemma 2.5.2 with $H = A_\varepsilon$. From the same lemma, it follows that for any $p \geq 1$, we have

$$\|\mathscr{T}_\alpha (A_\varepsilon^{-\delta} T A_\varepsilon^{-\delta})\|_{p/\alpha} \leq \alpha \|A_\varepsilon^{-\delta} T A_\varepsilon^{-\delta}\|_{p/\alpha}$$

Note that we have

$$A_\varepsilon^{-\delta} T A_\varepsilon^{-\delta} = (T^\delta A_\varepsilon^{-\delta})^* T^\alpha (T^\delta A_\varepsilon^{-\delta}).$$

Since $A_\varepsilon \geq \varepsilon T$, the Heinz Inequality 1.2.21, shows that $\|T^\delta A_\varepsilon^{-\delta}\| \leq \varepsilon^{-\delta}$, therefore we have that

$$\|\mathscr{T}_\alpha (A_\varepsilon^{-\delta} T A_\varepsilon^{-\delta})\|_{p/\alpha} \leq \alpha \varepsilon^{\alpha - 1} \|T^\alpha\|_{p/\alpha} = \alpha \varepsilon^{\alpha-1} \|T\|_p^\alpha.$$

The last inequality leads to the the desired inequality after integration over ε.
To weaken the assumptions of $A, B > 0$, we can replace A and B with $A + \sigma\mathbb{I}$ and $B + \sigma\mathbb{I}$, $\sigma > 0$ respectively and then pass to the limit as $\sigma \to 0$ in the estimates above.
Finally, let us remove the assuption $T \geq 0$. To do this, recall that $T = T_+ - T_-$, where $T_\pm = (|T| \pm T)/2$ are non-negative operators. Put $S_\pm = B + T_\pm$, then the Heinz inequality 1.2.21 immediately implies that $B^\alpha - A^\alpha \leq S_\pm^\alpha - A^\alpha$ and therefore we have

$$\|B^\alpha - A^\alpha\|_{p/\alpha} \leq \|S_\pm^\alpha - A^\alpha\|_{p/\alpha} \leq \|T_\pm\|_p^\alpha \leq \left(\|T_+\|_p^p + \|T_-\|_p^p\right)^{\alpha/p} = \|T\|_p^\alpha.$$

This concludes the proof. □

CHAPTER 3

Hankel Operators

The next Chapters will focus on Hankel operators and their spectral properties. For this reason, in this Chapter we define them, give four unitarily equivalent representations and review some of their spectral properties. We start with the usual definition as "compressed multiplication operators" on $H^2_+(\mathbb{T})$ and then move on to introducing Hankel matrices, Hankel operators on $H^2_+(\mathbb{R})$ and, finally, Hankel integral operators.

1. Hankel operators on the unit circle and Hankel matrices

In this setting, we have an orthogonal projection, P_+, sometimes called the *Riesz projection*, defined as

$$\begin{aligned} P_+ &: L^2(\mathbb{T}) \to H^2_+(\mathbb{T}) \\ (P_+f)(v) &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{T}} \frac{f(z)}{z-(1-\varepsilon)v} z dm(z). \end{aligned} \tag{3.1.1}$$

Let $\omega \in L^\infty(\mathbb{T})$ be fixed. We define

$$\begin{aligned} H(\omega) &: H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}) \\ H(\omega)f &= P_+\Omega J P_+ f, \end{aligned} \tag{3.1.2}$$

where $(\Omega f)(v) = \omega(v)f(v)$, while J is so that $Jf(v) = f(\overline{v})$.

Since $\{v^n\}_{n\geq 0}$ is an orthonormal basis of $H^2_+(\mathbb{T})$, then the operator $H(\omega)$ can be represented as an "infinite matrix" as follows

$$H(\omega)_{j,k} = \big(H(\omega)v^j, v^k\big)_{L^2(\mathbb{T})} = \big(\Omega(v^{-j}), v^k\big)_{L^2(\mathbb{T})} = (\omega, v^{j+k})_{L^2(\mathbb{T})} = \widehat{\omega}(j+k).$$

This shows that $H(\omega)$ is unitarily equivalent to the Hankel matrix $\mathit{\Gamma}(\widehat{\omega})$ given by

$$\begin{aligned} \mathit{\Gamma}(\widehat{\omega}) &: \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+) \\ (\mathit{\Gamma}(\widehat{\omega}))a(k) &= \sum_{j=0}^{\infty} \widehat{\omega}(j+k)a(j), \quad k \geq 0. \end{aligned}$$

In other words, $\Gamma(\widehat{\omega})$ is the "infinite matrix" with constant anti-diagonals below

$$\Gamma(\widehat{\omega}) = \begin{pmatrix} \widehat{\omega}(0) & \widehat{\omega}(1) & \widehat{\omega}(2) & \widehat{\omega}(3) \\ \widehat{\omega}(1) & \widehat{\omega}(2) & \widehat{\omega}(3) & \iddots \\ \widehat{\omega}(2) & \widehat{\omega}(3) & \iddots & \iddots \\ \widehat{\omega}(3) & \iddots & \iddots & \iddots \end{pmatrix}. \tag{3.1.3}$$

Such a representation shows that Hankel operators depend only on the analytic part of the function ω. Therefore they are not uniquely determined, in the sense that $H(\omega) = H(\psi)$ if (and only if) $\omega - \psi = \eta$ is an anti-analytic function of the unit disc $\mathbb{D}$, i.e. if (and only if) in $L^\infty(\mathbb{D})$ we can write

$$\eta(v) = \sum_{j=-\infty}^{-1} \widehat{\eta}(j) v^j, \quad v \in \mathbb{D}.$$

They are, however, uniquely determined once we restrict the function ω to be analytic inside $\mathbb{D}$.

2. Transfer to the line and Integral Hankel Operators

Let $\hat{\mathbb{R}}$ be the one-point compactification of the real line, $\mathbb{R}$. The map

$$\mu : \mathbb{T} \to \hat{\mathbb{R}}$$

$$\mu(v) = \frac{i}{2}\frac{1+v}{1-v} \tag{3.2.4}$$

is a bijection and induces the following unitary operator:

$$\mathcal{U} : L^2(\mathbb{R}) \to L^2(\mathbb{T})$$

$$(\mathcal{U}g)(v) = \frac{i\sqrt{2\pi}}{1-z} g(\mu(v)), \quad v \in \mathbb{T}.$$

whose adjoint is given by the operator

$$\mathcal{U}^* : L^2(\mathbb{T}) \to L^2(\mathbb{R})$$

$$(\mathcal{U}^* f)(x) = \frac{1}{\sqrt{2\pi}(x+i/2)} f(\mu^{-1}(x)), \quad x \in \mathbb{R}.$$

With this at hand, we see that $H(\omega)$ is unitarily equivalent to the Hankel operator on $H^2_+(\mathbb{R})$ given by:

$$\boldsymbol{H}(\boldsymbol{\omega}) = \boldsymbol{P}_+ \, \boldsymbol{\Omega} \, \boldsymbol{J}$$

where $\boldsymbol{P}_+ = \mathcal{U}^* P_+ \mathcal{U}$, $\boldsymbol{J} = \mathcal{U}^* J \mathcal{U}$ is so that $(\boldsymbol{J} f)(x) = f(-x)$ and $\boldsymbol{\Omega} = \mathcal{U}^* \Omega \mathcal{U}$ is such that $(\boldsymbol{\Omega}\, f)(x) = \boldsymbol{\omega}(x) f(x)$ and $\boldsymbol{\omega} = \omega \circ \mu^{-1}$.

The last representation, as an integral operator is achieved by applying the Fourier transform $\boldsymbol{\Phi}$, defined formally as:

$$(\boldsymbol{\Phi}\, f)(t) = \widehat{f}(t) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(x) e^{-ixt} dx.$$

Then $\widehat{\boldsymbol{P}}_+ = \boldsymbol{\Phi}\, \boldsymbol{P}_+\, \boldsymbol{\Phi}^*$ act as multiplication by the function $\mathbb{1}_+$, the characteristic function of the half-axis. Furthermore, one has that:

$$\widehat{\boldsymbol{J}} f(t) = (\boldsymbol{\Phi}\, \boldsymbol{J}\, \boldsymbol{\Phi}^*\, f)(t) = \widehat{f}(-t).$$

Finally one has that

$$\widehat{\boldsymbol{\Omega}} f(t) = \boldsymbol{\Phi}\, \boldsymbol{\Omega}\, \boldsymbol{\Phi}^*\, f = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{\boldsymbol{\omega}}(t-s)\widehat{f}(s) ds.$$

Thus, we see that $\boldsymbol{H}(\boldsymbol{\omega})$ is unitarily equivalent to the integral operator on $L^2(\mathbb{R}_+)$ given by

$$\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}) f)(t) = (2\pi)^{-1/2} \int_{\mathbb{R}_+} \widehat{\boldsymbol{\omega}}(t+s)\widehat{f}(s) ds.$$

Note that all formulae should be, of course, understood in the sense of distributions. From now on, we call the functions ω and $\boldsymbol{\omega}$ the *symbols* of the respective operators.

REMARK 3.2.1. It is clear that from this construction one has that $\Gamma(\widehat{\omega})$ and $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ are unitarily equivalent, with respect to the operator $\mathcal{L} = \boldsymbol{\Phi}\mathcal{U}^*\mathcal{F}^*$. In other words, the operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ can be represented as a Hankel matrix on $\ell^2(\mathbb{Z}_+)$ using the Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x}x^n), \quad n \geq 0,$$

see [46, Chapter 1] for more on this.

3. Discussion and basic results

We can summarise the above discussion via the following commutative diagrams:

$$\begin{array}{ccc} L^2(\mathbb{T}) & \xrightarrow{\mathcal{U}^*} & L^2(\mathbb{R}; dx) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \boldsymbol{\Phi}} \\ \ell^2(\mathbb{Z}) & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}; dt) \end{array} \qquad \begin{array}{ccc} H^2_+(\mathbb{T}) & \xrightarrow{\mathcal{U}^*} & H^2_+(\mathbb{R}) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \boldsymbol{\Phi}} \\ \ell^2(\mathbb{Z}_+) & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}_+) \end{array}$$

and

$$\begin{array}{ccc} H(\omega) & \xrightarrow{\mathcal{U}^*} & \boldsymbol{H}(\boldsymbol{\omega}) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \boldsymbol{\Phi}} \\ \Gamma(\widehat{\omega}) & \xrightarrow{\mathcal{L}} & \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}) \end{array} \qquad \begin{array}{ccc} \omega(v) & \xrightarrow{\mathcal{U}^*} & \boldsymbol{\omega}(x) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \boldsymbol{\Phi}} \\ \widehat{\omega}(j) & \xrightarrow{\mathcal{L}} & \widehat{\boldsymbol{\omega}}(t) \end{array}$$

Each of the representations above has its advantages. In particular, (3.1.2) implies

(1) $H(\omega)$ is bounded if the symbol is bounded and

$$\|H(\omega)\| \leq \|\omega\|_{L^\infty(\mathbb{T})}.$$

Furthermore, it is selfadjoint if $\omega(v) = \overline{\omega(\overline{v})}$.

(2) If ω is continuous on the circle, the associated Hankel operator $H(\omega)$ is compact. Indeed, by Weierstrass Theorem, the symbol ω is the uniform limit of trigonometric polynomials p_n. Since $H(p_n)$ are finite rank, the estimate

$$\|H(\omega) - H(p_n)\| \le \|\omega - p_n\|_{L^\infty(\mathbb{T})},$$

implies that $H(\omega) \in \mathfrak{S}_\infty$.

Translated into the language of Hankel operators on $H^2_+(\mathbb{R})$, the first condition implies that $\boldsymbol{H}(\boldsymbol{\omega})$ is bounded if $\boldsymbol{\omega}$ is bounded on $\hat{\mathbb{R}}$ and it is selfadjoint if $\boldsymbol{\omega}(x) = \overline{\boldsymbol{\omega}(-x)}$. Similarly, the second can be re-phrased to say that $\boldsymbol{H}(\boldsymbol{\omega})$ is compact if the symbol $\boldsymbol{\omega}$ is continuous on $\hat{\mathbb{R}}$, i.e. it is continuous on $\mathbb{R}$ and it has equal limits at $\pm\infty$.

Necessary and sufficient conditions for boundedness and compactness of Hankel operators as operators on the Hardy classes are known in the literature and are summarised in the following

THEOREM 3.3.1 (Nehari-Fefferman Theorem). *With the notation as above, one has*

(i) $H(\omega)$ *(resp.* $\boldsymbol{H}(\boldsymbol{\omega})$*) is a bounded operator if and only if* $\omega \in \mathrm{BMO}(\mathbb{T})$ *(resp.* $\boldsymbol{\omega} \in \mathrm{BMO}(\mathbb{R})$*);*

(ii) $H(\omega)$ *(resp.* $\boldsymbol{H}(\boldsymbol{\omega})$*) is compact if and only if* $\omega \in \mathrm{VMO}(\mathbb{T})$ *(resp.* $\boldsymbol{\omega} \in \mathrm{VMO}(\mathbb{R})$*).*

On the other hand, the realizations as operators on $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$ give a simple necessary and sufficient condition for $H(\omega), \boldsymbol{H}(\boldsymbol{\omega})$ to be Hilbert-Schimdt operators in terms of $\widehat{\omega}$ and $\widehat{\boldsymbol{\omega}}$, i.e. one has:

$$H(\omega) \in \mathfrak{S}_2 \iff \{\sqrt{(j+1)}\,\widehat{\omega}(j)\}_{j\ge 0} \in \ell^2(\mathbb{Z}_+),$$
$$\boldsymbol{H}(\boldsymbol{\omega}) \in \mathfrak{S}_2 \iff \sqrt{t}\,\widehat{\boldsymbol{\omega}}(t) \in L^2(\mathbb{R}_+).$$

Necessary and sufficient conditions for $H(\omega), \boldsymbol{H}(\boldsymbol{\omega}) \in \mathfrak{S}_p$ exist in the literature, see [46, Chapter 6], but they are expressed in terms of the symbols belonging to some Besov classes, which are not defined here. The following lemma gives a simple sufficient condition for $H(\omega)$ and $\boldsymbol{H}(\boldsymbol{\omega})$ to be trace-class in terms of the smoothness of their symbols:

LEMMA 3.3.2.

(i) If $\omega \in C^2(\mathbb{T})$*, then* $H(\omega) \in \mathfrak{S}_1$*. Furthermore, there exists* $C > 0$ *such that:*

$$\|H(\omega)\|_1 \le \|\omega\|_\infty + C\|\omega''\|_{L^2(\mathbb{T})}.$$

(ii) If $\boldsymbol{\omega} \in C^2(\hat{\mathbb{R}})$*, then* $\boldsymbol{H}(\boldsymbol{\omega}) \in \mathfrak{S}_1$ *and we have the estimate*

$$\|\boldsymbol{H}(\boldsymbol{\omega})\|_{\mathfrak{S}_1} \le \|\,\boldsymbol{\omega}\,\|_\infty + C\left\|(1+t^2)^{-1/2}\,\boldsymbol{\omega}''\right\|_{L^2(\mathbb{R})}.$$

Proof of Lemma 3.3.2. Let us prove (i). First, recall two facts:

(a) any $\omega \in C^2(\mathbb{T})$ is the uniform limit of the sequence

$$\omega_N(v) = \sum_{|j| \le N-1} \widehat{\omega}(j) v^j, \quad v \in \mathbb{T}.$$

Thus for any N and $v \in \mathbb{T}$, the Cauchy-Schwarz inequality together with Plancherel Identity give:

$$|\omega(v) - \omega_N(v)| \le \|\omega''\|_2 \left(2 \sum_{j \ge N} j^{-4}\right)^{1/2} \le C \|\omega''\|_{L^2(\mathbb{T})} N^{-3/2}.$$

(b) for $A \in \mathfrak{S}_\infty$ one has $s_N(A) = \inf\{\|A - B\| \mid rank(B) \le N-1\}$, $N \ge 1$, and, in particular, $s_1(A) = \|A\|$.

Since $rank\,(H(\omega_N)) \le N$, the above facts imply for $N \ge 2$:

$$\begin{aligned} s_N(H(\omega)) &= \inf\{\|H(\omega) - B\| \mid rank(B) \le N-1\} \\ &\le \|H(\omega) - H(\omega_{N-1})\| \\ &\le \|\omega - \omega_{N-1}\|_\infty \\ &\le C \frac{\|\omega''\|_{L^2(\mathbb{T})}}{(N-1)^{3/2}}. \end{aligned}$$

Thus $H(\omega) \in \mathfrak{S}_1$ and, furthermore,

$$\|H(\omega)\|_1 = s_1(H(\omega)) + \sum_{n \ge 2} s_n(H(\omega)) \le \|\omega\|_\infty + C\|\omega''\|_{L^2(\mathbb{T})}.$$

To prove (ii), recall that $\boldsymbol{H}(\boldsymbol{\omega})$ is unitarily equivalent to $H(\omega)$, where $\omega = \boldsymbol{\omega} \circ \mu$, where μ is the bijection defined in (3.2.4). Thus $\boldsymbol{H}(\boldsymbol{\omega})$ is trace-class if and only if $H(\omega)$ is and furthermore $\|\boldsymbol{H}(\boldsymbol{\omega})\|_{\mathfrak{S}_1} = \|H(\omega)\|_{\mathfrak{S}_1}$. Since μ is a smooth bijection and $\boldsymbol{\omega} \in C^2(\hat{\mathbb{R}})$, then $\omega \in C^2(\mathbb{T})$, thus (i) shows that $\boldsymbol{H}(\boldsymbol{\omega})$ is indeed a trace-class operator and we have:

$$\|\boldsymbol{H}(\boldsymbol{\omega})\|_{\mathfrak{S}_1} \le \|\omega\|_{L^\infty(\mathbb{T})} + C\|\omega''\|_{L^2(\mathbb{T})}.$$

Now it is not so hard to see that

$$\| \boldsymbol{\omega} \|_{L^\infty(\mathbb{R})} = \|\omega\|_{L^\infty(\mathbb{T})},$$

and furthermore the change of variables $2t = \cot(\vartheta/2)$ yields

$$\begin{aligned} \|\omega''\|^2_{L^2(\mathbb{T})} &= \frac{1}{2\pi} \int_0^{2\pi} \left|\boldsymbol{\omega}''\left(\mu(e^{i\vartheta})\right)\right|^2 d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} \left|\boldsymbol{\omega}''\left(\frac{1}{2}\cot\left(\frac{\vartheta}{2}\right)\right)\right|^2 d\vartheta \\ &= \frac{2}{\pi} \int_{\mathbb{R}} \frac{|\boldsymbol{\omega}''(t)|^2}{1+4t^2} dt \le C \int_{\mathbb{R}} \frac{|\boldsymbol{\omega}''(t)|^2}{1+t^2} dt = C \|(1+t^2)^{-1/2}\, \boldsymbol{\omega}''\, \|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Putting all of this together finally gives the desired inequality. ☐

One of the many applications of Hankel operators on $H^2_+(\mathbb{T})$ and on $H^2_+(\mathbb{R})$ is to study the properties of commutators of multiplication operators and the Riesz projection, i.e. commutators of the form

$$[P_+, \omega] = P_+\omega - \omega P_+,$$
$$[\boldsymbol{P}_+, \boldsymbol{\omega}] = \boldsymbol{P}_+ \boldsymbol{\omega} - \boldsymbol{\omega} \boldsymbol{P}_+$$

where ω and $\boldsymbol{\omega}$ denote both the functions and the associated operator of multiplication acting on $H^2_+(\mathbb{T})$ and on $H^2_+(\mathbb{R})$ respectively. One such application is the following useful

LEMMA 3.3.3.

(i) If $\omega \in C^2(\mathbb{T})$, the commutator $[P_+, \omega]$ is trace-class;

(ii) Similarly, if $\boldsymbol{\omega} \in C^2(\hat{\mathbb{R}})$, then the commutator $[\boldsymbol{P}_+, \boldsymbol{\omega}]$ is trace-class.

PROOF OF LEMMA. We will only prove (i), since the proof of (ii) can be easily reproduced by switching to boldface characters.
Write $P_- = I - P_+$, where I is the identity operator. Since P_+ is a projection and $P_+P_- = P_-P_+ = 0$, one has

$$[P_+, \omega] = [P_+, (P_+ + P_-)\omega](P_+ + P_-) = P_+\omega P_- - P_-\omega P_+.$$

Therefore, the identity $P_- = JP_+J - P_+JP_+$ leads to

$$[P_+, \omega] = H(\omega)J - JH(\overline{\omega})^* - P_+\omega P_+ JP_+ + P_+JP_+\omega P_+.$$

Since P_+JP_+ is a rank-one operator (projection onto constants), $[P_+, \omega]$ is trace-class if and only if $H(\omega)J - JH(\overline{\omega})^*$ is, which follows immediately from Lemma 3.3.2-(i). □

Part II

Spectral Density of Hankel operators with piecewise continuous symbols.

CHAPTER 4

The spectral density of Hankel matrices with piecewise continuous symbols

1. Introduction

1.1. General setting and first results. As we saw in the previous Chapter, any (essentially) bounded function ω, called a *symbol*, on $\mathbb{T}$ induces a bounded Hankel matrix, $\Gamma(\widehat{\omega})$, on $\ell^2(\mathbb{Z}_+)$, and its "matrix" entries are

$$\Gamma(\widehat{\omega})_{j,k} = \widehat{\omega}(j+k), \quad j,k \geq 0,$$

where $\widehat{\omega}$ denotes the sequence of Fourier coefficients

$$\widehat{\omega}(j) = \int_0^{2\pi} \omega(e^{i\vartheta})e^{-ij\vartheta}\frac{d\vartheta}{2\pi}, \quad j \in \mathbb{Z}.$$

The matrix $\Gamma(\widehat{\omega})$ is always symmetric. In particular, it is selfadjoint if and only if $\widehat{\omega}$ is real-valued. For instance, this is the case when ω satisfies the following symmetry condition

$$\omega(v) = \overline{\omega(\overline{v})}, \quad v \in \mathbb{T}. \tag{4.1.1}$$

In this Chapter, we shall consider symbols in the class of the piece-wise continuous functions on $\mathbb{T}$, denoted by $PC(\mathbb{T})$, i.e. those symbols ω for which the limits

$$\omega(z+) = \lim_{\varepsilon\to 0+} \omega(ze^{i\varepsilon}), \quad \omega(z-) = \lim_{\varepsilon\to 0+} \omega(ze^{-i\varepsilon}), \tag{4.1.2}$$

exist and are finite for all $z \in \mathbb{T}$. The points $z \in \mathbb{T}$ for which the quantity

$$\varkappa_z(\omega) = \frac{\omega(z+) - \omega(z-)}{2} \neq 0$$

are called the *jump discontinuities* of ω and $\varkappa_z(\omega)$ is the *half-height of the jump* of the symbol at z. Due to the presence of jump discontinuities, Hankel matrices with these symbols are non-compact. The compactness of $\mathbb{T}$ and the existence of the limits in (4.1.2) can be used to show that the sets

$$\Omega_s = \{z \in \mathbb{T} | \, |\varkappa_z(\omega)| > s\}, \quad s > 0,$$

are finite. To see this, one can argue by contradiction. Indeed, assuming Ω_s is infinite, then the compactness of $\mathbb{T}$ implies we can find $z \in \mathbb{T}$ and a sequence $\{z_n\}_{n\geq 1} \in \Omega_s$ such that $z_n \to z$. The boundedness of the symbol ω shows that either of the limits $\omega(z+)$ or $\omega(z-)$ does not exists, thus reaching a contradiction. Therefore, the set of jump-discontinuities of ω, denoted by Ω, is at most countable, as one can write $\Omega = \cup_{n\geq 1}\Omega_{1/n}$. If the symbol satisfies (4.1.1), Ω is symmetric with respect to the real axis and for any $z \in \Omega$

$$\varkappa_{\overline{z}}(\omega) = -\overline{\varkappa_z(\omega)},$$

whereby we obtain that $|\varkappa_z(\omega)| = |\varkappa_{\overline{z}}(\omega)|$, and if $z = \pm 1 \in \Omega$, $\varkappa_z(\omega)$ is purely imaginary.

Hankel matrices with piece-wise continuous symbols still attract attention in both the operator theory and spectral theory community, see for instance [**51, 53**] and references therein. S. Power, [**49**], showed that the essential spectrum of such matrices consists of bands depending only on the heights of the jumps of the symbol and gave the following identity:

$$\text{spec}_{ess}\left(\Gamma(\widehat{\omega})\right) = [0, -i\varkappa_1(\omega)] \cup [0, -i\varkappa_{-1}(\omega)] \cup \bigcup_{z\in\Omega\setminus\{\pm 1\}} \left[-i(\varkappa_z(\omega)\varkappa_{\overline{z}}(\omega))^{1/2}, i(\varkappa_z(\omega)\varkappa_{\overline{z}}(\omega))^{1/2}\right], \tag{4.1.3}$$

where the notation $[a, b], a, b \in \mathbb{C}$ denotes the line segment joining a and b. Assuming that the symbol has finitely many jumps and, say, it is Lipschitz continuous on the left and on the right of the jumps, in [**51**], a more detailed picture is obtained for the absolutely continuous (a.c.) spectrum of $|\Gamma(\widehat{\omega})| = \sqrt{\Gamma(\widehat{\omega})^*\Gamma(\widehat{\omega})}$, where the following formula is obtained

$$\text{spec}_{ac}\left(|\Gamma(\widehat{\omega})|\right) = \bigcup_{z\in\Omega} [0, |\varkappa_z(\omega)|]\,.$$

Furthermore, it is shown that each band contributes 1 to the multiplicity of the a.c. spectrum.

Example. First examples of symbols fitting in this scheme are the following

$$\gamma(e^{i\vartheta}) = i\pi^{-1}e^{-i\vartheta}(\pi - \vartheta), \quad \rho(e^{i\vartheta}) = 2\mathbb{1}_E(e^{i\vartheta}), \quad \vartheta \in [0, 2\pi). \tag{4.1.4}$$

where $\mathbb{1}_E$ is the characteristic function of the set $E = \{\vartheta \in [0, 2\pi) : \cos\vartheta > 0\}$. It is clear that both $\gamma, \rho \in PC(\mathbb{T})$, and their jumps occur at $z = 1$ and $z = \pm i$ respectively and $\varkappa_1(\gamma) = i$, $\varkappa_{\pm i}(\rho) = \mp 1$. Simple integration by parts shows that

$$\widehat{\gamma}(j) = \frac{1}{\pi(j+1)}, \quad \widehat{\rho}(j) = \frac{2\sin(\pi j/2)}{\pi j}, \quad j \geq 0,$$

with the understanding that $\widehat{\rho}(0) = 1$. Power's result in (4.1.3) in these cases gives

$$\text{spec}_{ess}\left(\Gamma(\widehat{\gamma})\right) = [0,\, 1]\,, \quad \text{spec}_{ess}\left(\Gamma(\widehat{\rho})\right) = [-1,\, 1]\,. \tag{4.1.5}$$

The matrix $\Gamma(\widehat{\gamma})$, known as the Hilbert matrix, has simple a.c. spectrum coinciding with the interval $[0, 1]$ and a full spectral decomposition was exhibited in [**59**]. In

[38], the authors perform a more detailed spectral analysis of $\Gamma(\widehat{\rho})$ and show that its spectrum is purely a.c. of multiplicity one and coincides with the interval $[-1, 1]$.

For $N \geq 1$, let $\Gamma^{(N)}(\widehat{\omega})$ be the $N \times N$ Hankel matrix

$$\Gamma^{(N)}(\widehat{\omega}) = \{\widehat{\omega}(j+k)\}_{j,k=0}^{N-1}.$$

We wish to give a description of the relationship between the spectrum of the infinite matrix $\Gamma(\widehat{\omega})$ and that of its truncation $\Gamma^{(N)}(\widehat{\omega})$. More specifically:

(i) for a non-selfadjoint Hankel matrix, we study the distribution of the singular values of $\Gamma^{(N)}(\widehat{\omega})$ inside the spectrum of $|\Gamma(\widehat{\omega})|$;

(ii) in the selfadjoint setting, we study the distribution of the eigenvalues of $\Gamma^{(N)}(\widehat{\omega})$ inside the spectrum of $\Gamma(\widehat{\omega})$.

To do so, for a non-selfadjoint Hankel matrix $\Gamma(\widehat{\omega})$ we study the asymptotic behaviour of the *singular-value counting function*

$$\mathsf{n}(t; \Gamma^{(N)}(\widehat{\omega})) = \#\{n : \ s_n(\Gamma^{(N)}(\widehat{\omega})) > t\}, \quad t > 0,$$

as $N \to \infty$. Here $\{s_n(\Gamma^{(N)}(\widehat{\omega}))\}_{n\geq 1}$ is the sequence of singular values of $\Gamma^{(N)}(\widehat{\omega})$. In particular, we study the *logarithmic spectral density* of $|\Gamma(\widehat{\omega})|$, defined as

$$\mathsf{LD}_{\square}(t; \Gamma(\widehat{\omega})) := \lim_{N\to\infty} \frac{\mathsf{n}(t; \Gamma^{(N)}(\widehat{\omega}))}{\log(N)}. \tag{4.1.6}$$

For a selfadjoint $\Gamma(\widehat{\omega})$, its spectrum, $\mathrm{spec}(\Gamma(\widehat{\omega}))$, is a subset of the real line and so we look at how the positive and negative eigenvalues of $\Gamma^{(N)}(\widehat{\omega})$ distribute inside $\mathrm{spec}(\Gamma(\widehat{\omega}))$. To this end, we analyze the behaviour of the *eigenvalue counting functions*

$$\mathsf{n}_{\pm}(t; \Gamma^{(N)}(\widehat{\omega})) = \#\{n : \ \lambda_n^{\pm}(\Gamma^{(N)}(\widehat{\omega})) > t\}, \quad t > 0,$$

as $N \to \infty$. Here $\{\lambda_n^{\pm}(\Gamma^{(N)}(\widehat{\omega}))\}_{n\geq 1}$ are the sequences of positive eigenvalues of $\pm\Gamma^{(N)}(\widehat{\omega})$ respectively. In this setting, we study the functions

$$\mathsf{LD}_{\square}^{\pm}(t; \Gamma(\widehat{\omega})) := \lim_{N\to\infty} \frac{\mathsf{n}_{\pm}(t; \Gamma^{(N)}(\widehat{\omega}))}{\log(N)}. \tag{4.1.7}$$

Similarly to the non-selfadjoint setting, we call the function $\mathsf{LD}_{\square}^{+}$ (resp. $\mathsf{LD}_{\square}^{-}$) in (4.1.7) the *positive* (resp. *negative*) *logarithmic spectral density* of $\Gamma(\omega)$.

The $\square$ appearing as an index in the definitions of the logarithmic spectral densities in (4.1.6) and (4.1.7) has been chosen to stress the fact that, a priori, these quantities depend on our choice to truncate the infinite matrix $\Gamma(\widehat{\omega})$ to its upper $N \times N$ square. Furthermore, the terminology we use for the functions $\mathsf{LD}_{\square}$ and $\mathsf{LD}_{\square}^{\pm}$ comes from the fact that we are only studying a logarithmically-small portion of the singular values (or eigenvalues) of the matrix $\Gamma^{(N)}(\widehat{\omega})$. Their definitions are motivated by the results obtained by Widom (see [61, Theorem 4.3]) for the Hilbert matrix $\Gamma(\widehat{\gamma})$, where he showed that

$$\mathsf{LD}_{\square}(t;\Gamma(\widehat{\gamma}))=\mathsf{c}(t), \tag{4.1.8}$$

$$\mathsf{LD}_{\square}^{-}(t;\Gamma(\widehat{\gamma}))=0, \quad \mathsf{LD}_{\square}^{+}(t;\Gamma(\widehat{\gamma}))=\mathsf{c}(t). \tag{4.1.9}$$

Here $\mathsf{c}(t):=0$ whenever $t\notin(0,1)$ and

$$\mathsf{c}(t):=\frac{1}{\pi^2}\operatorname{arcsech}(t)=\frac{1}{\pi^2}\log\left(\frac{1+\sqrt{1-t^2}}{t}\right), \quad t\in(0,1]. \tag{4.1.10}$$

We note that a factor of 2π is missing in the statement of [61, Theorem 4.3]. The aim of this Chapter is to extend (4.1.8) to a general symbol $\omega\in PC(\mathbb{T})$. In particular, for a non-selfadjoint Hankel matrix, we aim to show that

$$\mathsf{LD}_{\square}(t;\Gamma(\widehat{\omega}))=\sum_{z\in\Omega}\mathsf{c}\left(t\,|\varkappa_z(\omega)|^{-1}\right), \tag{4.1.11}$$

where c is the function defined in (4.1.10). Recall that the symbol ρ defined in (4.1.4) has jumps at ±1 whose half-height is $\varkappa_{\pm i}(\rho)=\mp1$, so for the Hankel matrix $\Gamma(\widehat{\rho})$ the formula (4.1.11) yields

$$\mathsf{LD}_{\square}(t;\Gamma(\widehat{\rho}))=2\mathsf{c}\,(t)\,.$$

For selfadjoint Hankel matrices, we extend the result in (4.1.9) to symbols $\omega\in PC(\mathbb{T})$ satisfying (4.1.1) and obtain

$$\begin{aligned}\mathsf{LD}_{\square}^{\pm}(t;\Gamma(\widehat{\omega}))=\mathsf{c}\left(t\,|\varkappa_1(\omega)|^{-1}\right)\mathbb{1}_{\pm}(-i\varkappa_1(\omega))+\mathsf{c}\left(t\,|\varkappa_{-1}(\omega)|^{-1}\right)\mathbb{1}_{\pm}(-i\varkappa_{-1}(\omega))\\+\sum_{z\in\Omega^+}\mathsf{c}\left(t\,|\varkappa_z(\omega)|^{-1}\right),\end{aligned} \tag{4.1.12}$$

where $\Omega^+=\{z\in\Omega\mid\operatorname{Im}z>0\}$, and $\mathbb{1}_{\pm}$ is the indicator function of the half-line $(0,\pm\infty)$. Again, the function c has been defined in (4.1.10). In particular, for the symbol ρ in (4.1.4), we obtain that

$$\mathsf{LD}_{\square}^{\pm}(t;\Gamma(\widehat{\rho}))=\mathsf{c}\,(t)\,.$$

In this Chapter, we not only show that (4.1.11) and (4.1.12) are indeed true, but we also use Theorem 2.4.6 of Chapter 2 as a starting point for our investigation into which class of matrix "truncations" leave these identities unchanged. For instance, the main results of this Chapter, Theorems 4.1.2 and 4.1.3 below, tell us that the singular values of the matrix $\Gamma^{(N)}(\widehat{\omega})$ and of the regularised matrix

$$\Gamma_N(\widehat{\omega})=\left\{e^{-\frac{j+k}{N}}\widehat{\omega}(j+k)\right\}_{j,k\geq0}, \quad N\geq1, \tag{4.1.13}$$

have the same distribution for large values of N.

1.2. Schur-Hadamard multipliers and spectral densities. To address these questions, we use the tools we already developped in Chapter 2, where we introduced both Schur-Hadamard multipliers and the logarithmic spectral density of a bounded operator with respect to a sequence, $\underline{\tau} = \{\tau_N\}_{N\geq 1}$, of multipliers.
Recall that for a bounded sequence $(\tau(j,k))_{j,k\geq 0}$, called a *multiplier*, and a bounded operator A on $\ell^2(\mathbb{Z}_+)$, the Schur-Hadamard multiplication of τ and A is the operator on $\ell^2(\mathbb{Z}_+)$, $\tau \star A$, whose (i,j)-th entry with respect to the standard basis of $\ell^2(\mathbb{Z}_+)$ is

$$(\tau \star A)_{j,k} = \tau(j,k)A_{j,k}, \quad j,k \geq 0, \tag{4.1.14}$$

We already studied the boundedness of this operation on the space of bounded operators and the Schatten classes $\mathfrak{S}_p$ in Chapter 2 where we introduced the operator norms

$$\|\tau\|_{\mathfrak{M}} = \sup_{\|A\|=1} \|\tau \star A\|, \tag{4.1.15}$$

$$\|\tau\|_{\mathfrak{M}_p} = \sup_{\|A\|_{\mathfrak{S}_p}=1} \|\tau \star A\|_{\mathfrak{S}_p}, \quad 1 \leq p \leq \infty. \tag{4.1.16}$$

For the purposes of this Chapter, take τ as the restriction to $\mathbb{Z}_+^2$ of a bounded function defined on $[0, \infty)^2$. Then τ induces a sequence of multipliers $\underline{\tau} = \{\tau_N\}_{N\geq 1}$ given by

$$\tau_N(j,k) = \tau(jN^{-1}, kN^{-1}).$$

If the function τ is such that the sequence $\underline{\tau} = \{\tau_N\}_{N\geq 1} \in \ell^\infty(\mathfrak{M})$, i.e. if $\underline{\tau}$ satisfies the following

$$\|\underline{\tau}\|_{\ell^\infty(\mathfrak{M})} = \sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}} < \infty, \tag{4.1.17}$$

we say that τ *induces a uniformly bounded sequence of multipliers*. From now on, we denote by τ both the function and the induced sequence of multipliers.

An easy example of such a sequence is the $N \times N$ truncation of an infinite matrix. To see this take the function

$$\tau_\square(x,y) = \mathbb{1}_\square(x,y), \tag{4.1.18}$$

where $\mathbb{1}_\square$ is the characteristic function of the half-open unit square $[0,1)^2$. For any bounded operator A, $\tau_N \star A$ is the truncation to its upper $N \times N$ block and so for any $N \geq 1$

$$\|\tau_N\|_{\mathfrak{M}} = 1.$$

Some more examples of Schur-Hadamard multipliers of this form will be given below.

Then, in a totally analogous way to (4.1.8), we define the logarithmic spectral density of $|\Gamma(\widehat{\omega})|$ with respect to τ to be

$$\mathsf{LD}_\tau(t; \Gamma(\widehat{\omega})) := \lim_{N\to\infty} \frac{\mathsf{n}(t; \tau_N \star \Gamma(\widehat{\omega}))}{\log N}, \quad t > 0, \tag{4.1.19}$$

Similarly, for a selfadjoint Hankel matrix and a multiplier τ such that $\tau(x,y) = \overline{\tau(y,x)}$, we define the positive and negative logarithmic spectral densities of $\Gamma(\widehat{\omega})$ as

$$\mathsf{LD}^{\pm}_{\tau}(t;\Gamma(\widehat{\omega})) := \lim_{N\to\infty} \frac{\mathsf{n}_{\pm}(t;\tau_N \star \Gamma(\widehat{\omega}))}{\log N}, \quad t>0. \tag{4.1.20}$$

Note that when $\tau = \tau_{\square}$ as in (4.1.18), the functions $\mathsf{LD}_{\tau}(t;\Gamma(\widehat{\omega}))$ and $\mathsf{LD}^{\pm}_{\tau}(t;\Gamma(\widehat{\omega}))$ are precisely those defined in (4.1.8) and (4.1.9).

1.3. Statement of the main results. As anticipated, our main results are not only concerned with the existence of the limits in (4.1.8) and (4.1.9), but also with their universality. In other words, for a Hankel matrix $\Gamma(\widehat{\omega})$ and a given multiplier τ, we show that under some mild assumptions on τ, see (A)-(C) below, the functions $\mathsf{LD}_{\tau}(t;\Gamma(\widehat{\omega}))$ and $\mathsf{LD}^{\pm}_{\tau}(t;\Gamma(\widehat{\omega}))$ are independent of the choice of τ.

Assumptions 4.1.1. Let us state the following assumptions on τ:

(A) τ induces a uniformly bounded Schur-Hadamard multiplier, i.e. (4.1.17) holds;

(B) $\tau(0,0)=1$ and for some $\varepsilon>0$ and some $\beta>1/2$, there exists $C_\beta>0$, so that

$$|\tau(x,y)-1| \le C_\beta \left|\log(x+y)\right|^{-\beta}, \quad \forall\, 0\le x,\, y\le\varepsilon;$$

(C) for some $\alpha>1/2$ one can find C_α so that

$$|\tau(x,y)| \le C_\alpha \log(x+y+2)^{-\alpha}, \quad \forall\, x,\, y\ge 0.$$

Then (4.1.11) is a particular case of the following:

THEOREM 4.1.2. *Let τ be a multiplier satisfying Assumptions 4.1.1-(A)-(C). Let $\omega\in PC(\mathbb{T})$ and Ω be the set of its discontinuities. Then*

$$\mathsf{LD}_{\tau}(t;\Gamma(\widehat{\omega})) = \sum_{z\in\Omega} \mathsf{c}\left(t\left|\varkappa_z(\omega)\right|^{-1}\right) \tag{4.1.21}$$

where $\mathsf{c}(t)$ is the function defined in (4.1.10).

Analogously for the selfadjoint case, (4.1.12) is a particular case of the Theorem below:

THEOREM 4.1.3. *Let τ satisfy Assumptions 4.1.1-(A)-(C) and such that $\tau(x,y)=\overline{\tau(y,x)}$. Suppose $\omega\in PC(\mathbb{T})$ satisfies (4.1.1) and let $\Omega^+ = \{z\in\Omega \mid \operatorname{Im} z > 0\}$. Then*

$$\begin{aligned}\mathsf{LD}^{\pm}_{\tau}(t;\Gamma(\widehat{\omega})) = &\sum_{z\in\Omega^+} \mathsf{c}\left(t\left|\varkappa_z(\omega)\right|^{-1}\right)\\ &+ \mathsf{c}\left(t\left|\varkappa_1(\omega)\right|^{-1}\right)\mathbb{1}_{\pm}(-i\varkappa_1(\omega))\\ &+ \mathsf{c}\left(t\left|\varkappa_{-1}(\omega)\right|^{-1}\right)\mathbb{1}_{\pm}(-i\varkappa_{-1}(\omega)),\end{aligned} \tag{4.1.22}$$

where $\mathsf{c}(t)$ *is the function defined in* (4.1.10) *and* $\mathbb{1}_{\pm}$ *is the characteristic function of the half-line* $(0, \pm\infty)$.

1.4. Remarks.

(A) It is clear that Theorems 4.1.2 and 4.1.3 generalise the result of Widom in [61] mentioned earlier in (4.1.10) to any multiplier τ and, in both instances, we only describe the behaviour of a logarithmically small portion of the spectrum of $\tau_N \star \Gamma(\widehat{\omega})$ as most of the points lie in a vicinity of 0.

(B) Both Theorems 4.1.2 and 4.1.3 deal with a rather general class of symbols and for this reason we cannot say more about the error term in the asymptotic expansion of the functions $\mathsf{n}, \mathsf{n}_{\pm}$. In fact, we can only write

$$\mathsf{n}(t; \tau_N \star \Gamma(\widehat{\omega})) = \log(N) \sum_{z \in \Omega} \mathsf{c}(t\, |\varkappa_z(\omega)|^{-1}) + o(\log(N)), \quad N \to \infty.$$

If, however, we were to restrict our attention to those symbols with finitely many jumps and some degree of smoothness away from them (say Lipschitz continuity), we would obtain a more precise estimate, see Chapter 6, however the trade-off would be that of making our results less general.

(C) Studying the spectral density of operators is common to many areas of spectral theory. In particular, our results can be put in parallel to well-known results in the spectral theory of Schrödinger operators, where the existence and universality of the density of states is a well-studied problem for a wide class of potentials, see [14] and [35, Section 5] for an introduction and references therein for more on this subject.

(D) Both Theorems 4.1.2 and 4.1.3 assume that the multiplier τ induces a uniformly bounded multiplier on the space of bounded operators. However, this condition can be substantially weakened in two different ways.
Firstly, we can weaken Assumption 4.1.1-(A) on the multiplier τ by assuming that for some finite $p > 1$, $\tau \in \ell^\infty(\mathfrak{M}_p)$, i.e. we have

$$\sup_{N \geq 1} \|\tau_N\|_{\mathfrak{M}_p} = \sup_{N \geq 1} \left(\sup_{\|A\|_{\mathfrak{S}_p} = 1} \|\tau_N \star A\|_{\mathfrak{S}_p} \right) < \infty. \tag{4.1.23}$$

However, as a trade-off, we need to impose more stringent conditions on the symbol, as the following statement shows:

PROPOSITION 4.1.4. *Suppose* $\tau \in \ell^\infty(\mathfrak{M}_p)$ *satisfies* (B) *and* (C) *in Assumptions* 4.1.1. *If the symbol* ω *can be written as*

$$\omega(v) = -i \sum_{z \in \Omega} \varkappa_z(\omega) \gamma(\overline{z} v) + \eta(v), \quad v \in \mathbb{T}, \tag{4.1.24}$$

where Ω is a finite subset of $\mathbb{T}$, γ is the symbol in (4.1.4) and η is a symbol for which $\Gamma(\widehat{\eta}) \in \mathfrak{S}_p$, then (4.1.21) holds. Furthermore, if $\tau(x,y) = \overline{\tau(y,x)}$ and ω satisfies (4.1.1), then (4.1.22) holds.

Note that we restricted our range of p to the interval $(1,\infty)$, because the Duality Principle 2.1.4 of Chapter 2 implies that

$$\ell^\infty(\mathfrak{M}_1) = \ell^\infty(\mathfrak{M}_\infty) = \ell^\infty(\mathfrak{M}).$$

Secondly, we can assume that τ only induces a uniformly bounded Schur-Hadamard multiplier on the space bounded Hankel matrices, i.e. that

$$\sup_{N\geq 1}\left(\sup_{\|\Gamma(\widehat{\omega})\|=1} \|\tau_N \star \Gamma(\widehat{\omega})\|\right) < \infty. \tag{4.1.25}$$

In this case, Theorems 4.1.2 and 4.1.3 still hold in their generality and we have the following

PROPOSITION 4.1.5. *Let $\omega \in PC(\mathbb{T})$ and let τ satisfy (4.1.25) as well as (B) and (C) in Assumptions 4.1.1. Then (4.1.21) holds. Furthermore, if ω satisfies the symmetry condition (4.1.1) and $\tau(x,y) = \overline{\tau(y,x)}$, then (4.1.22) holds.*

We chose to make use of Assumption (A) instead of (4.1.23) and (4.1.25), because there are no known necessary and sufficient conditions for a multiplier to satisfy either of them. We give specific examples of multipliers that satisfy these conditions below.

1.5. Some Examples of Schur-Hadamard multipliers.

EXAMPLE 4.1.6 (Factorisable multipliers). If the function τ can be factorised as

$$\tau(x,y) = f(x)g(y), \quad x,y \geq 0,$$

for some bounded function f, g, then it is easy to see that it induces a uniformly bounded Schur-Hadamard multiplier in the sense of (4.1.17), and furthermore

$$\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}} \leq \|f\|_\infty \|g\|_\infty.$$

As it was pointed out earlier in (4.1.18), the truncation to the upper $N \times N$ square is an example of such a multiplier. Another example is given by choosing the function $\tau_1(x,y) = e^{-(x+y)} = e^{-x}e^{-y}$. This induces the regularisation in (4.1.13) and it is immediate to see that

$$\sup_{N\geq 1} \|(\tau_1)_N\|_{\mathfrak{M}} = 1.$$

Furthermore, τ_1 satisfies the assumptions (B) and (C) in Assumptions 4.1.1, and so Theorems 4.1.2 and 4.1.3 hold.

EXAMPLE 4.1.7 (Non-examples). In stark contrast to the square truncation in (4.1.18), the so-called "main triangle projection" induced by the function

$$\tau_2(x, y) = \mathbb{1}_{[0,1)}(x + y) \tag{4.1.26}$$

is not uniformly bounded on the bounded operators. Indeed, in Chapter 2 we saw that

$$\sup_{\|A\|=1} \|(\tau_2)_N \star A\| = \pi^{-1}\log(N) + o(\log(N)), \quad N \to \infty.$$

However, Theorem 2.2.6 shows that τ_2 is uniformly bounded on any Schatten class $\mathfrak{S}_p$, for $1 < p < \infty$ and so Proposition 4.1.4 holds.
Proposition 4.1.5 shows that (4.1.21) and (4.1.22) still hold in the case that the Schur-Hadamard multiplier is only uniformly bounded on the set of bounded Hankel matrices. An example of such a multiplier is given by the indicator function, $\tau_{\beta,\gamma}$, of the region

$$\Xi_{\beta,\gamma} = \{(x, y) \in [0, 1]^2 \mid x \leq -\beta y + \gamma\}, \quad \beta,\, \gamma \in \mathbb{R}.$$

Even though $\tau_{\beta,\gamma}$ does not induce, in general, a uniformly bounded Schur-Hadamard multiplier, it has been shown in [**13**, Theorem 1(a)] that this is the case on the set of bounded Hankel matrices for $\beta \neq 1, 0$ and any γ (at $\beta = 1$ and $\gamma = 1$, $\tau_{1,1}$ reduces to the multiplier τ_2 considered above). With this at hand, an appropriate choice of the parameters β and γ gives (4.1.21) and (4.1.22).

EXAMPLE 4.1.8 (Hankel-type multipliers). Let $\Sigma \subset \mathbb{R}$ and suppose that we can write

$$\tau_N(j, k) = \tau\left(\frac{j}{N}, \frac{k}{N}\right) = \int_\Sigma e^{-it(j+k)} f_N(t)dt, \quad \forall j,\, k \geq 0$$

for some functions $f_N \in L^1(\Sigma, m)$ so that $\sup_{N\geq 1} \|f_N\|_{L^1(\Sigma)} < \infty$. An adaptation of Example 2.1.8 shows that τ induces a uniformly bounded Schur-Hadamard multiplier and moreover

$$\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}} \leq \sup_{N\geq 1} \|f_N\|_{L^1(\Sigma)}.$$

With this at hand, it is possible to show that the function

$$\tau_3(x, y) = (1 - (x + y))\, \mathbb{1}_{[0,1)}(x + y), \tag{4.1.27}$$

induces a uniformly bounded Schur-Hadamard multiplier, since one has the following representation

$$\tau_3(jN^{-1}, kN^{-1}) = \int_0^{2\pi} e^{-it(j+k)} F_N(e^{it})dt, \quad j,\, k \geq 0, \tag{4.1.28}$$

where F_N in (4.1.28) denotes the N-th Fejér kernel and $\|F_N\|_{L^1(0,2\pi)} = 2\pi$ for all $N \geq 1$.

Other Hankel-type multipliers are induced by the functions

$$\tau_1(x,y) = e^{-(x+y)},$$
$$\tau_2(x,y) = \mathbb{1}_{[0,1)}(x+y),$$
$$\tau_3(x,y) = (1-(x+y))\,\mathbb{1}_{[0,1)}\,(x+y)$$

and are related to the Abel-Poisson, Dirichlet and Cesaro summation methods respectively and share some of the properties of the operators of convolution with the respective kernels, see Section 3 for more on the Poisson kernel.

1.6. Outline of the proofs. The proof of Theorems 4.1.2 and 4.1.3 consist of a combination of the abstract results concerned with the general properties of the functions $\mathsf{LD}, \mathsf{LD}^{\pm}$ (see Chapter 2) and more hands-on function theoretic ones that are specific to the theory of Hankel matrices, see Section 3.

To prove Theorem 4.1.2, we firstly assume that the set of jump-discontinuities, Ω, of the symbol ω is finite and we write

$$\omega(v) = -i\sum_{z\in\Omega} \varkappa_z(\omega)\gamma(\overline{z}v) + \eta(v), \quad v \in \mathbb{T}, \tag{4.1.29}$$

where γ is the symbol in (4.1.4) and η is a continuous function on $\mathbb{T}$. The analysis of $\mathsf{LD}_\tau(t; \Gamma(\widehat{\omega}))$ then proceeds with the study of each summand appearing in (4.1.29) and the interactions this has with all the others. In particular, Assumption 4.1.1-(A) allows us to disregard the contribution coming from the matrix $\Gamma(\widehat{\eta})$, i.e. it gives that

$$\mathsf{LD}_\tau(t; \Gamma(\widehat{\omega})) = \mathsf{LD}_\tau\left(t; \sum_{z\in\Omega} \varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)\right),$$

where $\gamma_z(v) = -i\gamma(\overline{z}v)$, $v \in \mathbb{T}$. The invariance of the functions LD_τ with respect to the choice of multiplier, proved in the Invariance Principle Theorem 2.4.6, gives that

$$\mathsf{LD}_\tau\left(t; \sum_{z\in\Omega} \varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)\right) = \mathsf{LD}_{\tau_1}\left(t; \sum_{z\in\Omega} \varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)\right)$$

where the multiplier $\tau_1(x,y) = e^{-(x+y)}$ is given in the Example 4.1.6 above, and it is shown to induce the regularisation in (4.1.13), i.e. $(\tau_1)_N \star \Gamma(\widehat{\omega}) = \Gamma_N(\widehat{\omega})$. For the multiplier τ_1, we explicitly show that the operators $\Gamma(\widehat{\gamma}_z)$ are mutually "almost orthogonal" in the sense that if $z \neq w \in \Omega$, then both

$$\Gamma_N(\widehat{\gamma}_z)^*\Gamma_N(\widehat{\gamma}_w), \quad \Gamma_N(\widehat{\gamma}_z)\Gamma_N(\widehat{\gamma}_w)^*$$

are trace-class. From here, the Almost orthogonality Theorem 2.4.7, gives that each jump contributes independently, or in other words that we can write

$$\mathsf{LD}_{\tau_1}\left(t; \sum_{z\in\Omega} \varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)\right) = \sum_{z\in\Omega} \mathsf{LD}_{\tau_1}(t; \varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)).$$

We note here that the above is another instance of the general fact that jumps occurring at different points of the unit circle contribute independently to the spectral properties of the operator $\mathit{\Gamma}(\widehat{\omega})$. For this reason, we follow the terminology used by the authors of [**54**] and we refer to this fact as the *"Localisation Principle"*.
Finally, using once again the Invariance Principle Theorem 2.4.6, and the result of Widom in (4.1.10), we obtain the identity (4.1.11) for a symbol ω with finitely many jumps.

The proof of Theorem 4.1.3 roughly follows the same outline. However, instead of writing the symbol ω as in (4.1.29), we make use of the symmetry of the set of jump-discontinuities, Ω, to decompose it as follows

$$\omega(v) = \varkappa_1\gamma_1(v) + \varkappa_{-1}\gamma_{-1}(v) + \sum_{z\in\Omega^+}\omega_z(v) + \eta(v), \quad v \in \mathbb{T}, \tag{4.1.30}$$

where $\Omega^+ = \{z \in \Omega \mid \operatorname{Im} z > 0\}$, η is a continuous symbol on $\mathbb{T}$ and with $\gamma_z(v) = -i\gamma(\overline{z}v)$ we set

$$\omega_z(v) = \varkappa_z(\omega)\gamma_v(z) + \overline{\varkappa_z(\omega)}\gamma_{\overline{v}}(z).$$

The same strategy used in the proof of Theorem 4.1.2 leads to the following identity

$$\begin{aligned}\mathsf{LD}^{\pm}_{\tau}(t;\mathit{\Gamma}(\widehat{\omega})) = \mathsf{LD}^{\pm}_{\tau_1}(t;\varkappa_1(\omega)\mathit{\Gamma}(\widehat{\gamma}_1)) + \mathsf{LD}^{\pm}_{\tau_1}(t;\varkappa_{-1}(\omega)\mathit{\Gamma}(\widehat{\gamma}_{-1}))\\ + \sum_{z\in\Omega^+}\mathsf{LD}^{\pm}_{\tau_1}(t;\mathit{\Gamma}(\widehat{\omega}_z)).\end{aligned}$$

The fact that the jumps of ω are arranged symmetrically around $\mathbb{T}$ can be used to show that the positive and negative eigenvalues of the compact operator

$$\mathit{\Gamma}_N(\widehat{\omega}_z) = (\tau_1)_N \star \mathit{\Gamma}(\widehat{\omega}_z)$$

are arranged almost symmetrically around 0, in a sense that we will specify in Lemma 4.3.2-(ii). Using the Almost Symmetry Theorem 2.4.9, we conclude that

$$\mathsf{LD}^{\pm}_{\tau_1}(t;\mathit{\Gamma}(\widehat{\omega}_z)) = \frac{1}{2}\mathsf{LD}_{\tau_1}(t;\mathit{\Gamma}(\widehat{\omega}_z)). \tag{4.1.31}$$

Using once again the result of Widom in (4.1.10), we arrive at (4.1.22). It is worth noting here that (4.1.31) shows that if ω has jumps occurring at a pair of complex conjugate points, then the upper and lower logarithmic spectral densities, $\mathsf{LD}^{\pm}(t,\mathit{\Gamma}(\widehat{\omega}))$, contribute equally to the logarithmic spectral density of $|\mathit{\Gamma}(\widehat{\omega})|$, we refer to this as the *"Symmetry Principle"*, following the terminology used by the authors of [**55**].

Both Theorem 4.1.2 and 4.1.3 are then extended to the case of a symbol with infinitely-many jump-discontinuities using an approximation argument first presented by Power in [**49**] and subsequently in [**46**, Ch. 10, Thm. 1.10], see Section 4 below.

2. An abstract result for spectral densities

2.1. Setup. As before, $\mathfrak{S}_\infty$ denotes the ideal of compact operators. For any $p > 0$, $\mathfrak{S}_p$ denotes the ideal of compact operators whose singular values are p-summable, $\|\cdot\|_p$ denotes the Schatten p-norm and we set $\mathfrak{S}_0 = \cap_{p>0}\mathfrak{S}_p$. All operators in this section are bounded operators acting on the space of square summable sequence $\ell^2(\mathbb{Z}_+)$.

For a bounded τ on $[0,\infty)^2$ we have already defined in the Introduction the meaning of $\tau_N \star A$. In this section, we study the action that the sequence τ induces on $\mathfrak{B}_0$, defined as the subspace of $\mathfrak{B}(\ell^2(\mathbb{Z}_+))$:

$$A \in \mathfrak{B}_0 \qquad \Longleftrightarrow \qquad A_{j,k} = O\left(\frac{1}{j+k}\right), \quad j,k \to \infty. \tag{4.2.32}$$

Clearly $A \in \mathfrak{B}_0$ if and only if there exists a sequence $a \in \ell^\infty(\mathbb{N}^2)$ so that

$$A_{j,k} = \frac{a_{j,k}}{\pi(j+k+1)}, \qquad \forall j,k \geq 0.$$

From the Hilbert inequality one obtains the estimate $\|A\| \leq \|a\|_{\ell^\infty(\mathbb{N}^2)}$, and so the boundedness of A. To see this, consider the quadratic form (Af,g) for f, $g \in \ell^2(\mathbb{Z}_+)$, then

$$\begin{aligned}
\left|(Af,g)_{\ell^2(\mathbb{Z}_+)}\right| &\leq \sum_{j,k\geq 0} \left|A_{j,k} f(j)\overline{g(k)}\right| \\
&\leq \sum_{j,k\geq 0} \frac{\left|a_{j,k} f(j)\overline{g(k)}\right|}{\pi(j+k+1)} \\
&\leq \|a\|_{\ell^\infty(\mathbb{N}^2)} \left(\Gamma(\widehat{\gamma})\,|f|\,,|g|\right)_{\ell^2(\mathbb{Z}_+)} \\
&\leq \|a\|_{\ell^\infty(\mathbb{N}^2)} \|f\|_{\ell^2(\mathbb{Z}_+)} \|g\|_{\ell^2(\mathbb{Z}_+)}
\end{aligned}$$

with the last inequality following from the boundedness of the Hilbert matrix $\Gamma(\widehat{\gamma})$ discussed in the introduction.

If the multiplier τ satisfies Assumption 4.1.1-(C), i.e. if for some $\alpha > 1/2$, one has

$$|\tau(x,y)| \leq \frac{C_\alpha}{\log(x+y+2)^\alpha}, \quad \forall x,\, y,$$

it is not difficult to see that when $A \in \mathfrak{B}_0$ one has that $\tau_N \star A \in \mathfrak{S}_2$, since we have the following estimate

$$\begin{aligned}
\|\tau_N \star A\|_2^2 &= \sum_{j,k,\geq 0} \left|\tau\left(\frac{j}{N}, \frac{k}{N}\right) A_{j,k}\right|^2 \\
&\leq C_\alpha \sum_{j,k\geq 0} \frac{1}{\log\left(\frac{j+k}{N}+2\right)^{2\alpha} (j+k+1)^2} < \infty.
\end{aligned}$$

In particular, $\tau_N \star A$ is a compact operator for any given N and so the functionals $\overline{\mathsf{LD}}_\tau(t;A)$ and $\underline{\mathsf{LD}}_\tau(t;A)$, see (2.4.13) and (2.4.14), are well-defined. If $\overline{\mathsf{LD}}_\tau(t;A) = \underline{\mathsf{LD}}_\tau(t;A)$, we denote by $\mathsf{LD}_\tau(t;A)$ their common value. Similarly, for a selfadjoint operator $A \in \mathfrak{B}_0$ and for τ such that $\tau(x,y) = \overline{\tau(y,x)}$, the functionals $\overline{\mathsf{LD}}^{\pm}_\tau(t;A)$, $\underline{\mathsf{LD}}^{\pm}_\tau(t;A)$ are also well-defined and we denote by $\mathsf{LD}^{\pm}_\tau(t;A)$ their common value, if it exists.

2.2. Invariance of spectral densities. For a fixed operator $A \in \mathfrak{B}_0$, we wish to study the relation between $\overline{\mathsf{LD}}_\tau(t;A), \underline{\mathsf{LD}}_\tau(t;A)$ and the Schur-Hadamard multiplier τ.

THEOREM 4.2.1. *Suppose τ_1, τ_2 are multipliers satisfying (B) and (C) in Assumptions 4.1.1. Then for any $A \in \mathfrak{B}_0$,*

$$\sup_{N\geq 1} \|(\tau_1 - \tau_2)_N \star A\|_2$$

is finite and so the Invariance Principle Theorem 2.4.6 holds.

The assertion follows once we prove the following

LEMMA 4.2.2. *Let σ satisfy Assumption 4.1.1-(C) and be such that $\sigma(0,0) = 0$ and such that for some $\varepsilon > 0$ and some $\beta > 1/2$, there exists $C_\beta > 0$, so that*

$$|\sigma(x,y)| \leq C_\beta \left|\log(x+y)\right|^{-\beta}, \quad \forall\, 0 \leq x,\, y \leq \varepsilon. \tag{4.2.33}$$

Then for any $A \in \mathfrak{B}_0$, one has $\sigma_N \star A \in \mathfrak{S}_2$ and furthermore there exists $C > 0$, independent of N, such that

$$\|\sigma_N \star A\|_2 \leq C.$$

PROOF OF LEMMA. We need to estimate the following quantity

$$\|\sigma_N \star A\|_2^2 = \sum_{j,k\geq 0} \left|\sigma\left(\frac{j}{N}, \frac{k}{N}\right)\right|^2 |A_{j,k}|^2 .$$

A modification of the integral test and the assumption that $A \in \mathfrak{B}_0$, shows that one can find $C > 0$ so that

$$\begin{aligned} \|\sigma_N \star A\|_2^2 &\leq C \iint_{\mathbb{R}_+^2} \frac{\left|\sigma\left(\frac{x}{N}, \frac{y}{N}\right)\right|^2}{(x+y+1)^2} dxdy \\ &= C \iint_{\mathbb{R}_+^2} \frac{\left|\sigma\left(s,t\right)\right|^2}{(s+t+1/N)^2} dsdt \quad (:= I_N), \end{aligned}$$

the last inequality follows from the change of variables $x = Ns$, $y = Nt$. Let $\Omega_\varepsilon = \{(s,t) \in \mathbb{R}^2_+ \mid s^2+t^2 < \varepsilon\}$ and $\Omega^c_\varepsilon = \mathbb{R}^2_+ \setminus \Omega_\varepsilon$, then:

$$I_N = \iint_{\Omega_\varepsilon} \frac{|\sigma(s,t)|^2}{(s+t+1/N)^2} dsdt \qquad (:= J_1)$$
$$+ \iint_{\Omega^c_\varepsilon} \frac{|\sigma(s,t)|^2}{(s+t+1/N)^2} dsdt \qquad (:= J_2)$$

We will show that each summand is uniformly bounded. Since σ satisfies (4.2.33), it follows

$$J_1 \leq \frac{C_\beta}{\log(2)^2} \iint_{\Omega_\varepsilon} \frac{1}{\log(s^2+t^2)^{2\beta}(s^2+t^2)} dsdt$$
$$\leq C \int_0^\varepsilon \frac{1}{r\log(r)^{2\beta}} dr < \infty.$$

The second inequality is a consequence of writing the integral in polar coordinates and, since $\beta > 1/2$, the last integral is finite. Using Assumption 4.1.1-(C), it follows that

$$J_2 \leq C \iint_{\Omega^c_\varepsilon} \frac{dsdt}{(s+t)^2 \log(s+t+2)^{2\alpha}}$$
$$\leq C \int_\varepsilon^\infty \frac{dx}{x\log(x+2)^{2\alpha}} < \infty.$$

We have thus obtained that I_N is uniformly bounded in N, whereby the assertion follows. □

Proof of Theorem 4.2.1. Note that if τ_1 and τ_2 satisfy the hypotheses, then $\sigma = \tau_1 - \tau_2$ satisfies the hypotheses of Lemma 4.2.2 and so the result follows immediately. □

3. Hankel operators and the Abel summation method

3.1. Hankel operators. In the Introduction, we defined Hankel matrices acting on $\ell^2(\mathbb{Z}_+)$, and in Chapter 3 we saw that they are unitarily equivalent to a Hankel operator on the Hardy class $H^2_+(\mathbb{T})$. However, by adding the eigenvalue 0 of infinite multiplicity, we can equivalently define a Hankel operator as an integral operator acting on $L^2(\mathbb{T})$, making our computations slightly easier.
Let $\mathbb{T}$ be the unit circle in the complex plane, and $\boldsymbol{m}$ the Lebesgue measure normalised to 1, i.e $d\boldsymbol{m}(z) = (2\pi i z)^{-1}dz$. In Chapter 3, we defined the Riesz projection as

$$P_+ : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$$
$$(P_+ f)(v) = \lim_{\varepsilon\to 0} \int_{\mathbb{T}} \frac{f(z)z}{z-(1-\varepsilon)v} d\boldsymbol{m}(z), \quad v \in \mathbb{T}. \tag{4.3.34}$$

For a symbol ω, the *Hankel operator* $H(\omega)$ is:

$$\begin{aligned} H(\omega) : L^2(\mathbb{T}) &\to L^2(\mathbb{T}) \\ H(\omega)f &= P_+ \omega J P_+ f \end{aligned} \tag{4.3.35}$$

where J is the involution $Jf(v) = f(\overline{v})$ and, by a slight abuse of notation, ω denotes both the symbol and the induced operator of multiplication on $L^2(\mathbb{T})$. We can immediately see that if ω satisfies (4.1.1), $H(\omega)$ is selfadjoint. Furthermore, it is easy to see that

$$\|H(\omega)\| \le \|\omega\|_{L^\infty(\mathbb{T})}. \tag{4.3.36}$$

For any non-negative integers j, k, one has

$$\begin{aligned} \left(H(\omega)z^j, z^k\right)_{L^2(\mathbb{T})} &= \left(P_+ \omega J P_+ z^j, z^k\right)_{L^2(\mathbb{T})} \\ &= \left(\omega \cdot z^{-j}, z^k\right)_{L^2(\mathbb{T})} = \widehat{\omega}(j+k), \end{aligned}$$

and so the matrix representation of $H(\omega)$ in the basis $\{z^n\}_{n\in\mathbb{Z}}$ is the block-matrix

$$\begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathit{\Gamma}(\widehat{\omega}) \end{pmatrix},$$

with respect to the orthogonal decomposition $L^2(\mathbb{T}) = H^2_+(\mathbb{T}) \oplus (H^2_+(\mathbb{T}))^\perp$, where $H^2_+(\mathbb{T})$ is the Hardy space defined earlier in Chapter 1. In other words, $H(\omega)$ and $\mathit{\Gamma}(\widehat{\omega})$ are unitarily equivalent (modulo kernels) under the Fourier transform. It is clear that all the results of Chapter 3 on the boundedness and compactness of Hankel operators on the Hardy class $H^2_+(\mathbb{T})$ hold in this case also.

For $0 < r < 1$, let P_r be the Poisson kernel, defined as

$$P_r(v) = \sum_{j=-\infty}^{\infty} r^{|n|} v^n = \frac{1-r^2}{|1-rv|^2}, \quad v \in \mathbb{T}.$$

For $\omega_r = P_r * \omega$, we have the identity

$$H(\omega_r) = C_r H(\omega) C_r, \tag{4.3.37}$$

where C_r is the operator of convolution by P_r on $L^2(\mathbb{T})$. Furthermore, it is unitarily equivalent (modulo kernels) to the Hankel matrix

$$\mathit{\Gamma}^{(r)}(\widehat{\omega}) = \left\{r^{j+k}\widehat{\omega}(j+k)\right\}_{j,k\ge 0}. \tag{4.3.38}$$

Note that for $r = e^{-1/N}$, the above reduces to the truncation considered in (4.1.13). We collect some easy-to-prove facts in the following

LEMMA 4.3.1. *For $0 < r < 1$, the map $H(\omega) \mapsto H(\omega_r)$ has the following properties*

(i) *for any bounded Hankel operator* $H(\omega)$, $H(\omega_r) \in \mathfrak{S}_0$. *Furthermore,* (4.3.37) *and Hölder's inequality for Schatten classes (see Proposition 1.2.13-(iii)) give for* $1 \le p \le \infty$
$$\|H(\omega_r)\|_p \le \frac{1}{(1-r^{2p})^{1/p}}\|H(\omega)\|;$$
(ii) *if* $H(\omega) \in \mathfrak{S}_p$ *for some* $1 \le p \le \infty$, *then* (4.3.37) *implies* $\|H(\omega_r)\|_p \le \|H(\omega)\|_p$.

3.2. Almost Orthogonal and Almost Symmetric Hankel operators. Recall that for a function $\eta : \mathbb{T} \to \mathbb{C}$, its *singular support*, denoted $\operatorname{sing\,supp}\eta$, is defined as the smallest closed subset, M, of $\mathbb{T}$ such that $\eta \in C^\infty(\mathbb{T}\setminus \mathrm{M})$.

LEMMA 4.3.2. *The following statements hold*

(i) *Let* $\omega_1, \omega_2 \in L^\infty(\mathbb{T})$ *have disjoint singular supports. Set* $(\omega_i)_r = P_r * \omega_i, i = 1, 2$. *Then*
$$\sup_{r<1} \|H((\omega_1)_r)^* H((\omega_2)_r)\|_1 < \infty, \quad \sup_{r<1} \|H((\omega_1)_r) H((\omega_2)_r)^*\|_1 < \infty;$$
(ii) *Suppose* $\omega \in L^\infty(\mathbb{T})$ *be such that* $\pm 1 \notin \operatorname{sing\,supp}(\omega)$. *Let* $\mathfrak{s}(v) = \operatorname{sign}(\operatorname{Im}(v))$, $v \in \mathbb{T}$. *Then*
$$\sup_{r<1} \|\mathfrak{s}H(\omega_r) + H(\omega_r)\mathfrak{s}\|_1 < \infty.$$

REMARK 4.3.3. Similar results are already known in the literature, but only from a qualitative standpoint. Under the same assumptions of (i), it is already known that both $H(\omega_1)^*H(\omega_2)$ and $H(\omega_2)^*H(\omega_1) \in \mathfrak{S}_0$. Similarly for (ii), it is also known that $\mathfrak{s}H(\omega)+H(\omega)\mathfrak{s} \in \mathfrak{S}_0$. For a proof of both facts see [**54**, Lemma 2.5] and [**55**, Lemma 4.2] respectively, even though both facts are already mentioned in [**50**]. In our case, we need the uniform bounds to be able to localise the logarithmic spectral density of a Hankel operator on the jump of its symbol.

To prove the statements in Lemma 4.3.2, we need the following lemma:

LEMMA 4.3.4. *If* K *is an operator on* $L^2(\mathbb{T})$ *with integral kernel* $k \in C^\infty(\mathbb{T}^2)$, *then* $K \in \mathfrak{S}_1$.

PROOF OF LEMMA 4.3.4. The proof of this fact is folklore. It can be proved by approximating the kernel k by trigonometric polynomials, see for instance [**41**, Chapter 30, Thm. 13]. □

We are now ready to prove Lemma 4.3.2.

PROOF OF LEMMA 4.3.2. (i): we will only show the first inequality, as the second can be proved in the same way. From the assumptions on ω_1, ω_2, we can find

$\zeta_1, \zeta_2 \in C^\infty(\mathbb{T})$ such that $\operatorname{supp}\zeta_1 \cap \operatorname{supp}\zeta_2 = \varnothing$ and such that $(1-\zeta_i)\omega_i$ vanishes identically in a neighbourhood of $\operatorname{sing\,supp}\omega_i$. We will repeatedly use the following two facts:

(a) for any $\varphi \in L^\infty(\mathbb{T})$, Young's inequality holds, see [28, Theorem 1.2.12], i.e one has the estimate:

$$\|P_r * \varphi\|_{L^\infty(\mathbb{T})} \le \|\varphi\|_{L^\infty(\mathbb{T})}; \tag{4.3.39}$$

(b) one has that $P_r * \omega \in C^\infty(\mathbb{T})$ and furthermore $(P_r * \omega) \to \omega$ as $r \to 1-$ locally uniformly on $\mathbb{T} \setminus \operatorname{sing\,supp}\omega$. The same is true for its derivatives $(P_r * \omega)^{(n)}$. See [28, Theorem 1.2.19].

We set $\widetilde{\zeta}_i = 1 - \zeta_i$, $i = 1, 2$ and use the triangle inequality to obtain

$$\begin{aligned} \|H((\omega_1)_r)^* H((\omega_2)_r)\|_1 \le \|H(\widetilde{\zeta}_1(\omega_1)_r)^* H(\widetilde{\zeta}_2(\omega_2)_r)\|_1 + \|H(\widetilde{\zeta}_1(\omega_1)_r)^* H(\zeta_2(\omega_2)_r)\|_1 \\ + \|H(\zeta_1(\omega_1)_r)^* H(\widetilde{\zeta}_2(\omega_2)_r)\|_1 + \|H(\zeta_1(\omega_1)_r)^* H(\zeta_2(\omega_2)_r)\|_1, \end{aligned}$$

from which we see that it is sufficient to find uniform bounds for each summand above.

Recall that $H(\omega_i) = P_+\omega_i J P_+$ and P_+ is a projection, thus:

$$H(\zeta_1(\omega_1)_r)^* H(\zeta_2(\omega_2)_r) = P_+ J \overline{\zeta_1(\omega_1)_r} P_+ \zeta_2(\omega_2)_r J P_+.$$

Since ζ_1 and ζ_2 have disjoint supports, the operator $\overline{\zeta}_1 P_+ \zeta_2$ has a $C^\infty(\mathbb{T}^2)$ integral kernel given by

$$\frac{\overline{\zeta}_1(z)\zeta_2(v)}{v - z} v, \quad v,\, z \in \mathbb{T}.$$

Lemma 4.3.4 shows that $\overline{\zeta}_1 P_+ \zeta_2 \in \mathfrak{S}_1$. Furthermore, using Hölder inequality for the Schatten classes and (4.3.39), we deduce that

$$\sup_{r<1} \|H(\zeta_1(\omega_1)_r)^* H(\zeta_2(\omega_2)_r)\|_1 \le \|\omega_1\|_{L^\infty(\mathbb{T})} \|\omega_2\|_{L^\infty(\mathbb{T})} \|\overline{\zeta}_1 P_+ \zeta_2\|_1 < \infty.$$

Lemma 3.3.2-(i) shows that $H(\widetilde{\zeta}_1(\omega_1)_r) \in \mathfrak{S}_1$ and furthermore:

$$\begin{aligned} \|H(\widetilde{\zeta}_1(\omega_1)_r)^* H(\zeta_2(\omega_2)_r)\|_1 &\le \|H(\widetilde{\zeta}_1(\omega_1)_r)\|_1 \|H(\zeta_2(\omega_2)_r)\| \\ &\le \|\zeta_2\|_{L^\infty(\mathbb{T})} \|\omega_2\|_{L^\infty(\mathbb{T})} (C\|(\widetilde{\zeta}_1(\omega_1)_r)''\|_{L^2(\mathbb{T})} + \|\omega_1\|_{L^\infty(\mathbb{T})}), \end{aligned} \tag{4.3.40}$$

for some $C > 0$ independent of r. In (4.3.40) we used once more the Hölder inequality for Schatten classes together with the estimates (4.3.36) and (4.3.39).

From (b) and the fact that $\widetilde{\zeta}_i\omega_i$ vanishes identically on $\operatorname{sing\,supp}\zeta_i$, we conclude that $(\widetilde{\zeta}_i(\omega_i)_r)'' \to (\widetilde{\zeta}_i\omega_i)''$ uniformly on the whole of $\mathbb{T}$, and so

$$\sup_{r<1} \|(\widetilde{\zeta}_i(\omega_i)_r)''\|_{L^2(\mathbb{T})} < \infty. \tag{4.3.41}$$

Using (4.3.41) in (4.3.40) finally gives

$$\sup_{r<1} \|H(\widetilde{\zeta_1}(\omega_1)_r)^* H(\zeta_2(\omega_2)_r)\|_1 < \infty.$$

Similarly one can show that

$$\sup_{r<1} \|H(\widetilde{\zeta_1}(\omega_1)_r)^* H(\widetilde{\zeta_2}(\omega_2)_r)\|_1 < \infty, \quad \sup_{r<1} \|H(\zeta_1(\omega_1)_r)^* H(\widetilde{\zeta_2}(\omega_2)_r)\|_1 < \infty.$$

(ii) Since $\pm 1 \notin \operatorname{sing\,supp} \omega$, then we can write $\omega = \varphi + \eta$ for some $\eta \in C^\infty(\mathbb{T})$ and some φ vanishing identically in a neighbourhood U of ± 1. With this decomposition of ω, we can see that

$$H(\omega_r) = H(\varphi_r) + H(\eta_r).$$

Since η is smooth, then $H(\eta) \in \mathfrak{S}_1$ and so the triangle inequality and Hölder inequality for Schatten classes imply that

$$\sup_{r<1} \|\mathfrak{s} H(\omega_r) + H(\omega_r)\mathfrak{s}\|_1 \le 2\|H(\eta)\|_1 + \sup_{r<1} \|\mathfrak{s} H(\varphi_r) + H(\varphi_r)\mathfrak{s}\|_1.$$

So it is sufficient to consider those symbols ω vanishing on a neighbourhood, U, of ± 1.

Fix a smooth function ζ such that $0 \le \zeta \le 1$, it vanishes identically on some open $V \subset U$ so that $\pm 1 \in V$, $\zeta \equiv 1$ on $\mathbb{T}\backslash U$ and such that $\zeta(v) = \zeta(\overline{v})$, $v \in \mathbb{T}$. We can write:

$$\begin{aligned} \mathfrak{s} H(\omega_r) + H(\omega_r)\mathfrak{s} &= \mathfrak{s} H((1-\zeta)\omega_r) + H((1-\zeta)\omega_r)\mathfrak{s} \\ &\quad + \mathfrak{s} H(\zeta\omega_r) + H(\zeta\omega_r)\mathfrak{s} \end{aligned} \tag{4.3.42}$$

Let us study these operators more closely. Using the triangle inequality, we obtain that

$$\sup_{r<1} \|\mathfrak{s} H((1-\zeta)\omega_r) + H((1-\zeta)\omega_r)\mathfrak{s}\|_1 \le 2 \sup_{r<1} \|H((1-\zeta)\omega_r)\|_1. \tag{4.3.43}$$

Using (b) and the fact that $(1-\zeta)\omega \equiv 0$ on $\mathbb{T}$, we conclude that $((1-\zeta)\omega_r)'' \to 0$ on $\mathbb{T}$. Lemma 3.3.2-(i) once more gives

$$\sup_{r<1} \|H((1-\zeta)\omega_r)\|_1 \le \sup_{r<1} (\|\omega\|_{L^\infty(\mathbb{T})} + C\|((1-\zeta)\omega_r)''\|_{L^2(\mathbb{T})}) < \infty. \tag{4.3.44}$$

For the operators appearing in the second line of (4.3.42), write

$$\mathfrak{s} H(\zeta\omega_r) + H(\zeta\omega_r)\mathfrak{s} = ([\mathfrak{s}, P_+]\,\zeta)\,\omega_r J P_+ + P_+ \omega_r J\,(\zeta\,[P_+, \mathfrak{s}])\,.$$

Let us now prove that the commutators $[\mathfrak{s}, P_+]\,\zeta$, $\zeta\,[\mathfrak{s}, P_+] \in \mathfrak{S}_1$. By our choice of $\mathfrak{s}$ and ζ, we have $J\mathfrak{s} = -\mathfrak{s} J$ and $J\zeta = \zeta J$, whence it follows that

$$\begin{aligned} [\mathfrak{s}, P_+]\,\zeta &= \mathfrak{s} P_+ \zeta - \mathfrak{s}\zeta P_+ + \mathfrak{s}\zeta P_+ - P_+ \mathfrak{s}\zeta = \mathfrak{s}\,[P_+, \zeta] + [\mathfrak{s}\zeta, P_+]\,, \\ \zeta\,[\mathfrak{s}, P_+] &= \zeta\mathfrak{s} P_+ - P_+\mathfrak{s}\zeta + P_+\mathfrak{s}\zeta - \zeta P_+ \mathfrak{s} = [\mathfrak{s}\zeta, P_+] + [P_+, \zeta]\,\mathfrak{s}. \end{aligned}$$

Furthermore, our choice of ζ gives that the product $\mathfrak{s}\zeta \in C^\infty(\mathbb{T})$, and Lemma 3.3.3-(i) together with (4.3.39) implies that

$$\sup_{r<1} \|\mathfrak{s}H(\zeta\omega_r) + H(\zeta\omega_r)\mathfrak{s}\|_1 \le \|\omega\|_{L^\infty(\mathbb{T})}(\| [\mathfrak{s}, P_+]\,\zeta\|_1 + \|\zeta\,[P_+, \mathfrak{s}]\,\|_1) < \infty. \quad (4.3.45)$$

Putting together (4.3.43), (4.3.44) and (4.3.45) and using the triangle inequality on (4.3.42) gives the assertion. □

3.3. Spectral density of our model operator: the Hilbert matrix. An important ingredient to the proof of all our results is the model operator for which it is possible to explicitly compute the spectral density. Following the ideas of previous works, [**49**, **51**], a natural candidate is the Hilbert matrix, given by the symbol γ defined in (4.1.4). Putting together the result of Widom, see [**61**, Theorem 5.1] and Theorem 4.2.1, we obtain

PROPOSITION 4.3.5. *Let τ satisfy Assumptions 4.1.1-(B) and (C), then one has that*

$$\overline{\mathsf{LD}}_\tau(t; \Gamma(\widehat{\gamma})) = \underline{\mathsf{LD}}_\tau(t; \Gamma(\widehat{\gamma})) = \mathsf{c}(t), \quad t > 0,$$

where c has been defined in (4.1.10). If $\tau(x,y) = \overline{\tau(y,x)}$, then we also have

$$\overline{\mathsf{LD}}^+_\tau(t; \Gamma(\widehat{\gamma})) = \underline{\mathsf{LD}}^+_\tau(t; \Gamma(\widehat{\gamma})) = \mathsf{c}(t),$$
$$\overline{\mathsf{LD}}^-_\tau(t; \Gamma(\widehat{\gamma})) = \underline{\mathsf{LD}}^-_\tau(t; \Gamma(\widehat{\gamma})) = 0.$$

As an immediate consequence of the above, we obtain

COROLLARY 4.3.6. *Let $z \in \mathbb{T}$ be fixed and let $\gamma_z(v) = -i\gamma(\overline{z}v)$. Then the same result of Proposition 4.3.5 holds for the operator $\Gamma(\widehat{\gamma}_z)$.*

PROOF OF PROPOSITION. Theorem 4.2.1, shows that it is sufficient for the statement to hold for $\tau_\square(x,y) = \mathbb{1}_\square(x,y)$ defined in (4.1.18). This has already been done in [**61**, Theorem 5.1] and it has already been discussed in the Introduction in (4.1.10). Since the Hilbert matrix is a positive-definite operator, it is easy to see that $\tau_N \star \Gamma(\widehat{\gamma})$ is positive-definite and so

$$\overline{\mathsf{LD}}_\tau(t; \Gamma(\widehat{\gamma})) = \overline{\mathsf{LD}}^+_\tau(t; \Gamma(\widehat{\gamma})), \quad \overline{\mathsf{LD}}^-_\tau(t; \Gamma(\widehat{\gamma})) = 0.$$

The statement can be independently proved using the function $\tau_1(x,y) = e^{-(x+y)}$ discussed in the Introduction, however we postpone this to the Appendix. □

PROOF OF COROLLARY 4.3.6. Indeed, note that $\widehat{\gamma}_z(j) = -i\,\overline{z}^j\,\widehat{\gamma}(j)$, $j \ge 0$. Hence, for any function τ one has:

$$\tau_N \star \Gamma(\widehat{\gamma}_z) = -iU_{\overline{z}}(\tau_N \star \Gamma(\widehat{\gamma}))U_{\overline{z}}$$

where $U_{\overline{z}}$ is the unitary operator of multiplication by $\overline{z}^j$, $j \geq 0$. From this, we immediately see that

$$s_n(\tau_N \star \varGamma(\widehat{\psi}_z)) = s_n(\tau_N \star \varGamma(\widehat{\gamma})), \quad \forall n \geq 1$$

and so the statement follows immediately from Proposition 4.3.5. □

4. Proof of Theorem 4.1.2

The proof of the result will be broken down in two Steps. For brevity, we denote by $\varGamma^{(N)}(\widehat{\omega})$ the operator $\tau_N \star \varGamma(\widehat{\omega})$. We also recall that Ω is the set of jump-discontinuities of the symbol ω and c is the function in (4.1.10).

Step 1. Finitely many jumps. Suppose that Ω is finite. Setting $\gamma_z(v) = -i\gamma(\overline{z}v)$, with γ being the symbol defined in (4.1.4), write

$$\omega(v) = \sum_{z\in\Omega} \varkappa_z(\omega)\gamma_z(v) + \eta(v) \tag{4.4.46}$$

where η is continuous on $\mathbb{T}$ and let Φ denote the symbol

$$\Phi(v) = \sum_{z\in\Omega} \varkappa_z(\omega)\gamma_z(v).$$

Weyl's inequality (2.4.18) shows that for $0 < s < t$ one has

$$\begin{aligned} \mathsf{n}(t; \varGamma^{(N)}(\widehat{\omega})) &\leq \mathsf{n}(t-s; \varGamma^{(N)}(\widehat{\Phi})) + \mathsf{n}(s; \varGamma^{(N)}(\widehat{\eta})), \\ \mathsf{n}(t; \varGamma^{(N)}(\widehat{\omega})) &\geq \mathsf{n}(t+s; \varGamma^{(N)}(\widehat{\Phi})) - \mathsf{n}(s; \varGamma^{(N)}(\widehat{\eta})). \end{aligned}$$

Since $\varGamma(\widehat{\eta})$ is compact, Lemma 2.4.4 shows that

$$\mathsf{n}(s; \varGamma^{(N)}(\widehat{\eta})) = O_s(1), \quad N \to \infty,$$

and so, using the definition of the functionals $\underline{\mathsf{LD}}_\tau, \overline{\mathsf{LD}}_\tau$ we deduce that for any $t > 0$

$$\overline{\mathsf{LD}}_\tau(t; \varGamma(\widehat{\omega})) \leq \overline{\mathsf{LD}}_\tau(t-0; \varGamma(\widehat{\Phi})), \tag{4.4.47}$$

$$\underline{\mathsf{LD}}_\tau(t; \varGamma(\widehat{\omega})) \geq \underline{\mathsf{LD}}_\tau(t+0; \varGamma(\widehat{\Phi})). \tag{4.4.48}$$

Integration by parts shows that

$$\begin{aligned} \widehat{\Phi}(j) &= \sum_{z\in\Omega} \varkappa_z(\omega)\widehat{\gamma}_z(j) \\ &= \frac{-i}{\pi(j+1)} \sum_{z\in\Omega} \varkappa_z(\omega) z^j = O\left(\frac{1}{j+1}\right), \quad j \to \infty \end{aligned} \tag{4.4.49}$$

thus $\varGamma(\widehat{\Phi}) \in \mathfrak{B}_0$. Now, Theorem 4.2.1 gives that for any multiplier τ satisfiyng assumptions (B) and (C)

$$\overline{\mathsf{LD}}_\tau(t; \varGamma(\widehat{\Phi})) \leq \overline{\mathsf{LD}}_{\tau_1}(t-0; \varGamma(\widehat{\Phi})), \tag{4.4.50}$$

$$\underline{\mathsf{LD}}_\tau(t; \varGamma(\widehat{\Phi})) \geq \underline{\mathsf{LD}}_{\tau_1}(t+0; \varGamma(\widehat{\Phi})). \tag{4.4.51}$$

where $\tau_1(x,y) = e^{-(x+y)}$ induces the regularisation in (4.1.13). The Fourier transform $\mathcal{F}$ on $L^2(\mathbb{T})$, defined as

$$(\mathcal{F}f)(j) = \int_{\mathbb{T}} f(z)\overline{z}^j d\boldsymbol{m}(z), \quad f \in L^2(\mathbb{T}),\, j \geq 0,$$

implies that modulo kernels, see (4.3.38), we have

$$\Gamma^{(N)}(\widehat{\Phi}) = \sum_{z\in\Omega} \varkappa_z(\omega)\Gamma^{(N)}(\widehat{\gamma}_z) = \sum_{z\in\Omega} \varkappa_z(\omega)\mathcal{F}H((\gamma_z)_N)\mathcal{F}^*,$$

where $(\gamma_z)_N = P_r * \gamma_z$, with P_r being the Poisson kernel with $r = e^{-1/N}$. By Lemma 4.3.2-(i) and unitary equivalence, we have that whenever $z \neq w$

$$\sup_{N\geq 1} \|\Gamma^{(N)}(\widehat{\gamma}_z)^*\Gamma^{(N)}(\widehat{\gamma}_w)\|_{\mathfrak{S}_1} = \sup_{N\geq 1} \|H((\gamma_z)_N)^*H((\gamma_w)_N)\|_{\mathfrak{S}_1} < \infty.$$

Using the Almost Orthogonality Theorem 2.4.7 2, it then follows that for $t > 0$

$$\overline{\mathsf{LD}}_{\tau_1}(t;\Gamma(\widehat{\Phi})) \leq \sum_{z\in\Omega} \overline{\mathsf{LD}}_{\tau_1}(t-0;\varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)), \tag{4.4.52}$$

$$\underline{\mathsf{LD}}_{\tau_1}(t;\Gamma(\widehat{\Phi})) \geq \sum_{z\in\Omega} \underline{\mathsf{LD}}_{\tau_1}(t+0;\varkappa_z(\omega)\Gamma(\widehat{\gamma}_z)). \tag{4.4.53}$$

Finally, Corollary 4.3.6 together with (4.4.47), (4.4.48), (4.4.50), (4.4.51), (4.4.52) and (4.4.53) and the continuity of c at $t \neq 0$ gives that

$$\overline{\mathsf{LD}}_{\tau}(t;\Gamma(\widehat{\omega})) \leq \sum_{z\in\Omega} \mathsf{c}\left(\frac{t}{|\varkappa_z(\omega)|}\right),$$

$$\underline{\mathsf{LD}}_{\tau}(t;\Gamma(\widehat{\omega})) \geq \sum_{z\in\Omega} \mathsf{c}\left(\frac{t}{|\varkappa_z(\omega)|}\right).$$

The obvious inequality $\underline{\mathsf{LD}}_{\tau}(t;H(\omega)) \leq \overline{\mathsf{LD}}_{\tau}(t;H(\omega))$ proves the assertion.

Remark 4.4.1. We note that (4.4.52) and (4.4.53) hold if we consider any symbol ω which is smooth except for a finite set of jumps discontinuities. These two together are yet another instance of the Localisation principle we referred to in the Introduction.

Step 2. From finitely many to infinitely many jumps. Suppose now that Ω is infinite. Define the sets:

$$\Omega_0 = \{z \in \mathbb{T} \mid |\varkappa_z(\omega)| \geq 2^{-1}\},$$
$$\Omega_n = \{z \in \mathbb{T} \mid 2^{-n-1} \leq |\varkappa_z(\omega)| < 2^{-n}\}, \quad n \geq 1.$$

As we mentioned earlier, these are finite. Let φ_n be functions such that $\operatorname{sing\,supp}\varphi_n = \Omega_n$, $\varkappa_z(\varphi_n) = \varkappa_z(\omega)$ for any $z \in \Omega_n$ and such that

$$\|\varphi_n\|_\infty = \max_{z\in\Omega_n} |\varkappa_z(\omega)|.$$

Let $\Phi = \sum_{n\geq 0} \varphi_n \in L^\infty(\mathbb{T})$. Since $\omega - \Phi \in C(\mathbb{T})$, the operator $\Gamma(\widehat{\omega} - \widehat{\Phi})$ is compact and so, by Lemma 2.4.4 once again we obtain

$$\overline{\mathsf{LD}}_\tau(t; \Gamma(\widehat{\omega})) \leq \overline{\mathsf{LD}}_\tau(t-0; \Gamma(\widehat{\Phi})),$$
$$\underline{\mathsf{LD}}_\tau(t; \Gamma(\widehat{\omega})) \geq \underline{\mathsf{LD}}_\tau(t+0; \Gamma(\widehat{\Phi})).$$

For a fixed $s > 0$, let M be so that $\|\Phi - \Phi_M\|_\infty < s$, where $\Phi_M = \sum_{n=0}^M \varphi_n$. The uniform boundedness of τ then gives

$$\|\tau_N \star (\Gamma(\widehat{\Phi}) - \Gamma(\widehat{\Phi}_M))\| \leq \left(\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}}\right) \|\Phi - \Phi_M\|_\infty < \left(\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}}\right) s := s'.$$

Letting $\widetilde{\Omega}_M = \cup_{n=0}^M \Omega_n$, we then obtain that:

$$\overline{\mathsf{LD}}_\tau(t; H(\omega)) \leq \overline{\mathsf{LD}}_\tau(t-s'; H(\Phi_M)) = \sum_{z\in\widetilde{\Omega}_M} \mathsf{c}\left(\frac{t-s'}{|\varkappa_z(\omega)|}\right),$$
$$\underline{\mathsf{LD}}_\tau(t; H(\omega)) \geq \underline{\mathsf{LD}}_\tau(t+s'; H(\Phi_M)) = \sum_{z\in\widetilde{\Omega}_M} \mathsf{c}\left(\frac{t+s'}{|\varkappa_z(\omega)|}\right).$$

The equalities above follow from the *Step 1.*, since Φ_M has finitely many jumps. Finally, sending $s \to 0$ and noting that there are only finitely many $z \in \Omega$ such that $t \leq |\varkappa_z(\omega)|$, one obtains

$$\underline{\mathsf{LD}}_\tau(t; H(\omega)) = \overline{\mathsf{LD}}_\tau(t; H(\omega)).$$

5. Proof of Theorem 4.1.3

Just as in the proof of Theorem 4.1.2, we break the argument into two steps, and use the same notation as before for the operator $\tau_N \star \Gamma(\widehat{\omega})$ and for the symbols γ_z. We also set $\Omega^+ = \{z \in \Omega \mid \operatorname{Im} z > 0\}$.

Step 1. Finitely many jumps. Just as before, suppose that the symbol ω has finitely-many jump-discontinuities. Write

$$\omega(v) = \left(\varkappa_1(\omega)\gamma_1(v) + \varkappa_{-1}(\omega)\gamma_{-1}(v) + \sum_{z\in\Omega^+} \omega_z(v)\right) + \eta(v), \tag{4.5.54}$$

where η is continuous on $\mathbb{T}$ and

$$\omega_z(v) = \varkappa_z(\omega)\gamma_z(v) + \overline{\varkappa_z(\omega)}\gamma_{\overline{z}}(v).$$

If ω has no jump at ± 1, the corresponding quantities do not appear in the above. Denoting by Φ the sum in the brackets, Weyl inequality (2.4.19) gives for $0 < s < t$

$$\mathsf{n}_\pm(t; \Gamma^{(N)}(\widehat{\omega})) \leq \mathsf{n}_\pm(t-s; \Gamma^{(N)}(\widehat{\Phi})) + \mathsf{n}_\pm(s; \Gamma^{(N)}(\widehat{\eta})),$$
$$\mathsf{n}_\pm(t; \Gamma^{(N)}(\widehat{\omega})) \geq \mathsf{n}_\pm(t+s; \Gamma^{(N)}(\widehat{\Phi})) - \mathsf{n}_\pm(s; \Gamma^{(N)}(\widehat{\eta})).$$

Therefore, using Lemma 2.4.4, we obtain that $\mathsf{n}_\pm(s;\Gamma^{(N)}(\widehat{\eta})) = O_s(1)$, and so, for any $t>0$, it follows that

$$\overline{\mathsf{LD}}^\pm_\tau(t;\Gamma(\widehat{\omega})) \le \overline{\mathsf{LD}}^\pm_\tau(t-0;\Gamma(\widehat{\Phi})),$$
$$\underline{\mathsf{LD}}^\pm_\tau(t;\Gamma(\widehat{\omega})) \ge \underline{\mathsf{LD}}^\pm_\tau(t+0;\Gamma(\widehat{\Phi})).$$

Integration by parts once again shows that

$$\widehat{\Phi}(j) = O\left(\frac{1}{j+1}\right), \quad j\to\infty.$$

Thus $\Gamma(\widehat{\Phi}) \in \mathfrak{B}_0$ and so Theorem 4.2.1 shows that it is sufficient to prove the result for the multiplier $\tau(x,y) = e^{-(x+y)}$. Since the symbols

$$\varkappa_1(\omega)\gamma_1,\ \varkappa_{-1}(\omega)\gamma_{-1},\ \omega_z$$

have mutually disjoint singular supports for $z\in\Omega^+$, Lemma 4.3.2-(ii) and the Almost Orthogonality Theorem 2.4.7 imply that for $t>0$

$$\begin{aligned}\overline{\mathsf{LD}}^\pm_\tau(t;\Gamma(\widehat{\Phi})) \le \sum_{z\in\Omega^+} &\overline{\mathsf{LD}}^\pm_\tau(t-0;\Gamma(\widehat{\omega}_z)) + \overline{\mathsf{LD}}^\pm_\tau(t-0;\varkappa_1(\omega)\Gamma(\widehat{\gamma}_1))\\ &+\overline{\mathsf{LD}}^\pm_\tau(t-0;\varkappa_{-1}(\omega)\Gamma(\widehat{\gamma}_{-1}))\end{aligned} \tag{4.5.55}$$

$$\begin{aligned}\underline{\mathsf{LD}}^\pm_\tau(t;\Gamma(\widehat{\Phi})) \ge \sum_{z\in\Omega^+} &\underline{\mathsf{LD}}^\pm_\tau(t+0;\Gamma(\widehat{\omega}_z)) + \underline{\mathsf{LD}}^\pm_\tau(t+0;\varkappa_1(\omega)\Gamma(\widehat{\gamma}_1))\\ &+\underline{\mathsf{LD}}^\pm_\tau(t+0;\varkappa_{-1}(\omega)\Gamma(\widehat{\gamma}_{-1})).\end{aligned} \tag{4.5.56}$$

The operators $\varkappa_{\pm1}(\omega)\Gamma(\widehat{\gamma}_{\pm1})$ are sign definite, and furthermore one has that

$$\varkappa_{\pm1}(\omega)\Gamma(\widehat{\gamma}_{\pm1}) \ge 0 \text{ (resp. } \le 0) \text{ if } -i\varkappa_{\pm1}(\omega)\ge 0 \text{ (resp. } \le 0).$$

In either case, Proposition 4.3.5 gives that

$$\begin{aligned}\overline{\mathsf{LD}}^\pm_\tau(t;\varkappa_{\pm1}(\omega)\Gamma(\widehat{\gamma}_{\pm1})) &= \mathbb{1}_\pm(-i\varkappa_{\pm1}(\omega))\overline{\mathsf{LD}}_\tau(t;\varkappa_{\pm1}(\omega)\Gamma(\widehat{\gamma}_{\pm1}))\\ &= \mathbb{1}_\pm(-i\varkappa_{\pm1}(\omega))\mathsf{c}\left(t\,|\varkappa_{\pm1}(\omega)|^{-1}\right)\end{aligned} \tag{4.5.57}$$

where $\mathbb{1}_\pm$ is the indicator function of $\mathbb{R}_\pm = (0,\pm\infty)$.
From Lemma 4.3.2-(ii), the Almost Orthogonality Theorem 2.4.9 and Theorem 4.1.2 above, we get that for any $z\in\Omega^+$

$$\begin{aligned}\overline{\mathsf{LD}}^\pm_\tau\left(t;\Gamma(\widehat{\omega}_z)\right)) &= \frac{1}{2}\overline{\mathsf{LD}}_\tau\left(t;\Gamma(\widehat{\omega}_z)\right))\\ &= \mathsf{c}\left(t\,|\varkappa_z(\omega)|^{-1}\right).\end{aligned} \tag{4.5.58}$$

Using (4.5.57) and (4.5.58) in (4.5.55) and (4.5.56), the continuity of c at $t\neq 0$ gives that

$$\underline{\mathsf{LD}}^\pm_\tau(t;\Gamma(\widehat{\omega})) = \overline{\mathsf{LD}}^\pm_\tau(t;\Gamma(\widehat{\Phi}))$$

and so we arrive at (4.1.7).

REMARK 4.5.1. As we wrote earlier in the Introduction, if the symbol has a pair of complex conjugate jumps, then (4.5.58) shows that the upper and lower logarithmic spectral density of $\Gamma(\widehat{\omega})$ contribute equally to the logarithmic spectral density of $|\Gamma(\widehat{\omega})|$. This is an effect of the Symmetry Principle we referred to in the Introduction.

Step 2. From finitely many to infinitely many jumps. For fixed $s > 0$, define the set

$$\Omega_s^+ = \{z \in \Omega \mid |\varkappa_z(\omega)| > s \text{ and } \operatorname{Im} z > 0\}.$$

Just as in *Step 2.* in the proof of Theorem 4.1.2, we can find a symbol $\omega_s \in PC(\mathbb{T})$ so that $\|\omega - \omega_s\|_\infty < s$, the set of its discontinuities is precisely $\Omega_s^+ \cup \{\pm 1\}$ and

$$\varkappa_z(\omega) = \varkappa_z(\omega_s), \quad \forall z \in \Omega_s^+ \cup \{\pm 1\}.$$

The set $\Omega_s^+ \cup \{\pm 1\}$ is finite, thus from Weyl inequality (2.4.19) and *Step 1.* we obtain

$$\overline{\mathsf{LD}}_\tau^\pm(t; \Gamma(\widehat{\omega})) \le \overline{\mathsf{LD}}_\tau^\pm(t - s'; \Gamma(\widehat{\omega}_s))$$
$$\underline{\mathsf{LD}}_\tau^\pm(t; \Gamma(\widehat{\omega})) \ge \underline{\mathsf{LD}}_\tau^\pm(t + s'; \Gamma(\widehat{\omega}_s)),$$

where $s' = \left(\sup_{N\ge 1} \|\tau_N\|_{\mathfrak{M}}\right) s$. Finally, sending $s \to 0$ and using the continuity of c at $t \neq 0$ establishes the result in its generality. ☐

PROOF OF PROPOSITION 4.1.4. The same reasoning of *Step 1.* in both proofs above applies in this case, with only one minor change. Since we assume that τ induces a uniformly bounded multiplier on $\mathfrak{S}_p$, $p > 1$, i.e. that (4.1.23) holds, in (4.4.46) and (4.5.54) we need to assume that η is a symbol so that $\Gamma(\widehat{\eta}) \in \mathfrak{S}_p$. Then Lemma 2.4.4 shows that $\mathsf{n}(s; \Gamma^{(N)}(\widehat{\eta})) = O_s(1)$ and, in the selfadjoint case $\mathsf{n}_\pm(s; \Gamma^{(N)}(\widehat{\eta})) = O_s(1)$. The rest follows immediately. ☐

PROOF OF PROPOSITION 4.1.5. Exactly the same reasoning of the proofs of Theorems 4.1.2 and 4.1.3 above applies in this case, with the only difference being that in this case τ is no longer inducing a uniformly bounded multiplier on the whole space of bounded operators, just on Hankel matrices. However, all of the terms appearing in the arguments just presented are bounded Hankel operators and so the same arguments apply in this case. ☐

6. An independent proof of Proposition 4.3.5

By virtue of Theorem 4.2.1, choose the function $\tau_1(x, y) = e^{-(x+y)}$, which yields

$$((\tau_1)_N \star \Gamma(\widehat{\omega}))_{j,k} = e^{-\frac{j+k}{N}} \widehat{\omega}(j + k) = \Gamma^{(r)}(\widehat{\omega})_{j,k}, \quad r = e^{-1/N},$$

where $\Gamma^{(r)}(\widehat{\omega})$ is the Poisson truncation in (4.3.38). We start our proof with the following Lemma, similar to [22, Lemma 4.1]:

Lemma 4.6.1. *For any $m \in \mathbb{N}$ one has that:*

$$\mathrm{Tr}\, \Gamma^{(r)}(\widehat{\gamma})^m = \frac{|\log(1-r)|}{2\pi} \int_{\mathbb{R}} \left(\frac{1}{r \cosh(\pi\eta)} \right)^m d\eta + o(|\log(1-r)|), \quad r \to 1^-.$$

Proof of Lemma. Let us define the operator $L : L^2(0,1) \to \ell^2(\mathbb{Z}_+)$ as follows

$$(Lf)(j) = \frac{1}{\sqrt{\pi}} \int_0^1 f(s) s^j ds, \quad j \geq 0.$$

Its boundedness can be established using the Schur test. A simple calculation yields the identity $\Gamma(\widehat{\gamma}) = LL^*$, from which if follows that, with $\Gamma^{(r)}(\widehat{\gamma}) = \Gamma(\widehat{\gamma}_r)$

$$\begin{aligned} \Gamma^{(r)}(\widehat{\gamma}) &= \frac{1}{r} L \mathbb{1}_r L^* \\ &= \frac{1}{r} (L\mathbb{1}_r)(L\mathbb{1}_r)^* \end{aligned}$$

where $\mathbb{1}_r$ is the characteristic function of the interval $(0, r)$ and so one obtains

$$r^m \,\mathrm{Tr}\, \Gamma^{(r)}(\widehat{\gamma})^m = \mathrm{Tr}\, (\mathbb{1}_r L^* L \mathbb{1}_r)^m, \tag{4.6.59}$$

therefore we only need to compute the latter trace. Recall now that for any bounded operator X, there is a unitary equivalence between $XX^*|_{\ker(XX^*)^\perp}$ and $X^*X|_{\ker(X^*X)^\perp}$. Hence, the trace of $(\mathbb{1}_r L^* L \mathbb{1}_r)^m$ and that of $(\mathbb{1}_r L^* L \mathbb{1}_r)^m$ coincide. Note however that the operator L^*L is an operator acting on $L^2(0,1)$ whose integral kernel is:

$$k(t,s) = \frac{1}{\pi(1-ts)}, \quad t,\, s \in (0,1).$$

Following the procedure described in [61], define the unitary transformation:

$$\begin{gathered} U : L^2(0,1) \to L^2(\mathbb{R}_+) \\ (Uf)(x) = \frac{1}{\cosh(x)} f(\tanh(x)), \quad x > 0. \end{gathered}$$

Then we have $B = UL^*LU^* : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the convolution operator

$$(Bf)(x) = \int_{\mathbb{R}_+} \frac{f(y)}{\pi \cosh(x-y)} dy, \quad x > 0.$$

In this way, we have reduced our problem to evaluating the trace of the integral operator $(\widetilde{\mathbb{1}}_r B \widetilde{\mathbb{1}}_r)^m$, where $\widetilde{\mathbb{1}}_r$ is the characteristic function of the interval $(0, \mathrm{arctanh}(r))$. By adding 0 to its spectrum, we also consider $\widetilde{\mathbb{1}}_r B \widetilde{\mathbb{1}}_r$ as an integral operator acting on $L^2(\mathbb{R})$, with integral kernel

$$\frac{\widetilde{\mathbb{1}}_r(s)\widetilde{\mathbb{1}}_r(t)}{\pi \cosh(s-t)}, \quad s,t \in \mathbb{R}.$$

We now use the following result:

THEOREM 4.6.2 ([40]). *Let P be an orthogonal projection and B be a bounded operator such that $PB \in \mathfrak{S}_2$. Let φ be such that $\varphi(0) = 0$ and $\varphi'' \in L^\infty(\operatorname{spec}(B))$, then:*

$$|\operatorname{Tr}\varphi(PBP) - \operatorname{Tr} P\varphi(B)P| \leq \|\varphi''\|_{L^\infty(\operatorname{spec}(B))}\|PB(I-P)\|_2^2. \tag{4.6.60}$$

Note that $\mathbb{1}_r L^* L \in \mathfrak{S}_2$ for any $r < 1$ so by unitary equivalence $\widetilde{\mathbb{1}}_r B \in \mathfrak{S}_2$. Furthermore, the operator B is unitarily equivalent, under the Fourier Transform, to the operator of multiplication on $L^2(\mathbb{R})$ by the function

$$\frac{1}{\cosh(\pi\xi/2)}, \quad \xi \in \mathbb{R}.$$

Whence we can estimate $\operatorname{Tr}(\widetilde{\mathbb{1}}_r B \widetilde{\mathbb{1}}_r)^m$ by:

$$\begin{aligned}
\operatorname{Tr}\widetilde{\mathbb{1}}_r B^m \widetilde{\mathbb{1}}_r &= \frac{1}{2\pi}\int_{\mathbb{R}} \widetilde{\mathbb{1}}_r(x)dx \int_{\mathbb{R}} \left(\frac{1}{\cosh(\pi\xi/2)}\right)^m d\xi \\
&= \frac{\operatorname{arctanh}(r)}{\pi}\int_{\mathbb{R}} \left(\frac{1}{\cosh(\pi\xi)}\right)^m d\xi \\
&= \frac{|\log(1-r)|}{2\pi}\int_{\mathbb{R}} \left(\frac{1}{\cosh(\pi\xi)}\right)^m d\xi + o(|\log(1-r)|), \quad r \to 1^-.
\end{aligned}$$

We also have that:

$$\|\widetilde{\mathbb{1}}_r B(1-\widetilde{\mathbb{1}}_r)\|_2^2 = \|(\widetilde{\mathbb{1}}_r B - B\widetilde{\mathbb{1}}_r)(1-\widetilde{\mathbb{1}}_r)\|_2^2 \leq \|[\widetilde{\mathbb{1}}_r, B]\|_2^2,$$

thus we need to find an estimate for the Hilbert-Schmidt norm of the integral operator $[\widetilde{\mathbb{1}}_r, B]$, which has integral kernel given by:

$$k(t,s) = \frac{\widetilde{\mathbb{1}}_r(t) - \widetilde{\mathbb{1}}_r(s)}{\pi\cosh(\pi(t-s))}, \quad t, s \in \mathbb{R}.$$

It follows that

$$\|[\widetilde{\mathbb{1}}_r, B]\|_2^2 = \iint_{\mathbb{R}^2} k^2(t,s)dtds = \int_{\mathbb{R}} \frac{\varphi(z)}{\pi^2\cosh^2(\pi z)}dz,$$

with

$$\begin{aligned}
\varphi(z) &= \int_{\mathbb{R}} (\widetilde{\mathbb{1}}_r(z+y) - \widetilde{\mathbb{1}}_r(y))^2 dy \\
&= 2\min\{|z|, \operatorname{arctanh} r\} \leq 2|z|.
\end{aligned}$$

Whereby obtaining that

$$\|[\widetilde{\mathbb{1}}_r, B]\|_2^2 \leq C\int_{\mathbb{R}} \frac{|z|}{\cosh^2(z)}dz < \infty.$$

Using (4.6.59) and (4.6.60), we obtain:

$$\operatorname{Tr}(\Gamma^{(r)}(\widehat{\gamma}))^m = \frac{|\log(1-r)|}{2\pi r^m}\int_{\mathbb{R}} \left(\frac{1}{r\cosh(\pi\xi)}\right)^m d\xi + o(|\log(1-r)|),$$

as $r \to 1^-$. □

Proposition 4.3.5 now follows from a two-step approximation argument. In the first stage, using the Weierstrass Approximation theorem and the Lemma 4.6.1, we prove that for any function $\varphi \in C_c^\infty(\mathbb{R}_+)$ one has that

$$\lim_{r\to 1^-} \frac{\operatorname{Tr}\varphi(\Gamma^{(r)}(\widehat{\gamma}))}{|\log(1-r)|} = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi\left(\frac{1}{\cosh(\pi\eta)}\right) d\eta. \tag{4.6.61}$$

In the second, we set $r = e^{-1/N}$ and we note that we can replace $\left|\log(1-e^{-1/N})\right|$ with $\log(N)$ in the limits above and that we can write

$$\mathsf{n}(t;\Gamma^{(N)}(\widehat{\gamma})) = \operatorname{Tr}\mathbb{1}_{(t,1)}(\Gamma^{(N)}(\widehat{\gamma})).$$

Choose sequences $\varphi_n^\pm \in C_c^\infty(\mathbb{R}_+)$ for which we have

$$0 \le \varphi_n^-(x) \le \mathbb{1}_{(t,1)}(x) \le \varphi_n^+(x) \le 1, \quad \forall x,$$

and $\varphi_n^\pm(x) \to \mathbb{1}_{(t,1)}(x)$ pointwise in x as $n \to \infty$. From the properties of Tr and (4.6.61) it follows

$$\overline{\mathsf{LD}}_\tau(t;\Gamma(\widehat{\gamma})) \le \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_n^+\left(\frac{1}{\cosh(\pi\eta)}\right) d\eta,$$
$$\overline{\mathsf{LD}}_\tau(t;\Gamma(\widehat{\gamma})) \ge \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_n^-\left(\frac{1}{\cosh(\pi\eta)}\right) d\eta.$$

Finally, an application of the Dominated Convergence Theorem gives the result.

CHAPTER 5

The spectral density of integral Hankel operators with piecewise continuous symbols

1. Introduction

1.1. General setting and discussion. Recall that for an essentially bounded function $\boldsymbol{\omega}$ on $\mathbb{R}$, called a *symbol*, the associated integral Hankel operator is the (bounded) operator

$$\begin{aligned} \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}) &: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \\ \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})f(t) &= \int_{\mathbb{R}_+} \widehat{\boldsymbol{\omega}}(s+t)f(s)ds, \end{aligned} \tag{5.1.1}$$

where $\widehat{\boldsymbol{\omega}}$ denotes the Fourier transform of $\boldsymbol{\omega}$

$$\widehat{\boldsymbol{\omega}}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \boldsymbol{\omega}(x)e^{-ixt}dx, \quad t \in \mathbb{R}. \tag{5.1.2}$$

Of course, the above identity is to be understood in the distributional sense.
It is easy to see that $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ depends linearly on the symbol $\boldsymbol{\omega}$ and that it is always a symmetric operator. In particular, it is selfadjoint when $\widehat{\boldsymbol{\omega}}$ is real-valued. This is the case, for instance, when the symbol satisfies the following symmetry condition

$$\boldsymbol{\omega}(x) = \overline{\boldsymbol{\omega}(-x)}, \quad \forall x \in \mathbb{R}. \tag{5.1.3}$$

In this Chapter, we focus on *piecewise continuous symbols*, i.e. symbols, $\boldsymbol{\omega}$, for which the following limits

$$\boldsymbol{\omega}(v+) = \lim_{\varepsilon\to 0+} \boldsymbol{\omega}(v+\varepsilon), \quad \boldsymbol{\omega}(v-) = \lim_{\varepsilon\to 0+} \boldsymbol{\omega}(v-\varepsilon), \quad |v| < \infty, \tag{5.1.4}$$

$$\boldsymbol{\omega}(\infty+) = \lim_{x\to+\infty} \boldsymbol{\omega}(x), \quad \boldsymbol{\omega}(\infty-) = \lim_{x\to-\infty} \boldsymbol{\omega}(x), \tag{5.1.5}$$

exist and are finite. We denote the class of all such symbols by $PC(\hat{\mathbb{R}})$, where $\hat{\mathbb{R}}$ denotes the extended real line $\mathbb{R} \cup \{\pm\infty\}$ with the points $\pm\infty$ identified. Those points $v \in \hat{\mathbb{R}}$ for which the quantities

$$\varkappa_v(\boldsymbol{\omega}) = \frac{\boldsymbol{\omega}(v+) - \boldsymbol{\omega}(v-)}{2} \neq 0, \quad v \in \hat{\mathbb{R}},$$

are called the *jump discontinuities* of the symbol, while the associated $\varkappa_v(\boldsymbol{\omega})$ is called the *half-height of the jump*.

Since $\hat{\mathbb{R}}$ is a compact set and the limits in (5.1.4) and (5.1.5) exist everywhere on the line, the sets

$$\Omega_s = \{v \in \hat{\mathbb{R}} \mid \ |\varkappa_v(\boldsymbol{\omega})| > s\}$$

are finite for any $s > 0$ and so $\boldsymbol{\omega}$ cannot have more than countably-many jump-discontinuities. We denote the set of its jump-discontinuities by Ω. Moreover, if the symbol satisfies the symmetry condition (5.1.3), then its jump-discontinuities are symmetric around 0, i.e. $v \in \Omega$ if and only if $-v \in \Omega$ and, furthermore, we have

$$\varkappa_v(\boldsymbol{\omega}) = -\overline{\varkappa_{-v}(\boldsymbol{\omega})}.$$

Consequently, it follows that $|\varkappa_v(\boldsymbol{\omega})| = |\varkappa_{-v}(\boldsymbol{\omega})|$ and, also, at $v = 0, \infty$ it follows that $\varkappa_v(\boldsymbol{\omega})$ is purely imaginary. Because of the presence of jump discontinuities in the symbol, the operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ in non-compact and has non-zero essential spectrum depending on the half-heights of the jumps as it was proved by S. Power, [**49**]:

$$\begin{aligned}\operatorname{spec}_{ess}(\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = [0, -i\varkappa_0(\boldsymbol{\omega})] \cup [0, -i\varkappa_\infty(\boldsymbol{\omega})] \cup \\ \bigcup_{v \in \Omega \setminus \{0,\infty\}} \left[-i(\varkappa_v(\boldsymbol{\omega})\varkappa_{-v}(\boldsymbol{\omega}))^{1/2}, i(\varkappa_v(\boldsymbol{\omega})\varkappa_{-v}(\boldsymbol{\omega}))^{1/2}\right],\end{aligned} \tag{5.1.6}$$

where the notation $[a, b], a, b \in \mathbb{C}$ denotes the line segment joining a and b. Moreover, should one assume that the symbol has only finitely many jumps and, say, Lipschitz continuity on the left and on the right of the jumps, it is shown in [**51**] that the absolutely continuous (a.c.) spectrum of $|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})| = \sqrt{\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})^* \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})}$ has the same banded structure and

$$\operatorname{spec}_{ac}(|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})|) = \bigcup_{v \in \Omega} [0, |\varkappa_v(\boldsymbol{\omega})|],$$

where each band contributes 1 to the multiplicity of the a.c. spectrum.

For $N \geq 1$, define the "truncation" of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ as the operator $L^2(\mathbb{R}_+)$

$$\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}})f(t) = \int_{\mathbb{R}_+} \mathbb{1}_N(t)\widehat{\boldsymbol{\omega}}(s+t)\mathbb{1}_N(s)f(s)ds, \tag{5.1.7}$$

where $\mathbb{1}_N$ is the characteristic function of the interval (N^{-1}, N). Just as in the previous Chapter, our aim is to understand:

(i) the distribution of the singular values of $\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}})$, when $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ is a non-selfadjoint operator;

(ii) the distribution of the eigenvalues of $\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}})$, when $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ is selfadjoint.

To this end, in the first case, we study the logarithmic spectral density of $|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})|$ with respect to the square truncation, defined as

$$\mathsf{LD}_\square(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) := \lim_{N\to\infty} \frac{\mathsf{n}(t; \boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}}))}{\log(N)}, \quad t > 0. \tag{5.1.8}$$

In the self-adjoint case, we study the positive and negative logarithmici spectral densities of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ with respect to the square truncation:

$$\mathsf{LD}_{\square}^{\pm}(t;\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) := \lim_{N\to\infty}\frac{\mathsf{n}_{\pm}(t;\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}}))}{\log(N)}, \quad t>0. \tag{5.1.9}$$

The functions $\mathsf{n}(t;\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}})), \mathsf{n}_{\pm}(t;\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}}))$ are the counting functions and are defined in Chapter 2.

As before, the $\square$ appearing as an index in the definitions (5.1.8) and (5.1.9) is there to stress the fact that, a priori, the functions may depend on our choice of truncating the integral kernels of our operators to the square (N^{-1}, N). Our aim in this Chapter is to show that these quantities are, in fact, universal, i.e. they do not depend on the "truncation" chosen for their definitions. Let us illustrate the results we obtain by means of the following example:

EXAMPLE 5.1.1. Consider the symbols

$$\boldsymbol{\gamma}_0(x) = \frac{(\pi\,\mathrm{sign}(x) - 2\tan^{-1}(x))}{\pi i}, \quad \boldsymbol{\gamma}_\infty(x) = \frac{2}{\pi i}\tan^{-1}(x), \quad x\in\mathbb{R}, \tag{5.1.10}$$

here $\mathrm{sign}(x) := x/|x|, x\neq 0$ and $\mathrm{sign}(0) := 0$. It is clear that both symbols belong to $PC(\hat{\mathbb{R}})$ and are such that $\boldsymbol{\gamma}_0 \in C(\hat{\mathbb{R}}\setminus\{0\})$ and $\boldsymbol{\gamma}_\infty \in C(\mathbb{R})$. It is also very easy to see that

$$\varkappa_0(\boldsymbol{\gamma}_0) = \varkappa_\infty(\boldsymbol{\gamma}_\infty) = -i.$$

The associated integral Hankel operators are well-known in the literature, see [46, Ch.1], and have the following explicit form:

$$\boldsymbol{\Gamma}(\widehat{\boldsymbol{\gamma}}_0)f(t) = \int_{\mathbb{R}_+}\frac{(1-e^{-(s+t)})f(s)}{\pi(s+t)}ds, \tag{5.1.11}$$

$$\boldsymbol{\Gamma}(\widehat{\boldsymbol{\gamma}}_\infty)f(t) = \int_{\mathbb{R}_+}\frac{e^{-(s+t)}f(s)}{\pi(s+t)}ds. \tag{5.1.12}$$

In fact, in [46, Ch. 1] it is shown that the operators $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\gamma}}_0)$ and $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\gamma}}_\infty)$ are unitarily equivalent to the Hankel matrices:

$$\Gamma_0 = \left\{\frac{(-1)^{j+k}}{\pi(j+k+1)}\right\}_{j,k\geq 0}, \quad \Gamma_\infty = \left\{\frac{1}{\pi(j+k+1)}\right\}_{j,k\geq 0}. \tag{5.1.13}$$

We saw in the previous Chapter that the distribution of the eigenvalues of the $N\times N$ truncation of the matrices $\Gamma_0, \Gamma_\infty, \Gamma_0^{(N)}$ and $\Gamma_\infty^{(N)}$ respectively, behave asymptotically as follows:

$$\mathsf{n}(t;\Gamma_0^{(N)}) = \mathsf{n}(t;\Gamma_\infty^{(N)}) = \log(N)(\mathsf{c}(t) + o(1)), \quad N\to\infty, \tag{5.1.14}$$

where $\mathsf{c}(t) := 0$ for $t\notin(0,1]$ and

$$\mathsf{c}(t) := \frac{1}{\pi^2}\mathrm{arcsech}(t) = \frac{1}{\pi^2}\log\left(\frac{1+\sqrt{1-t^2}}{t}\right), \quad t\in(0,1]. \tag{5.1.15}$$

Since the operators $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ and $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ are unitarily equivalent to the matrices Γ_0 and Γ_∞ respectively, it is only reasonable to expect that the distribution of eigenvalues of their "truncations" $\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}}_0)$, $\boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\omega}}_\infty)$ exhibits the same asymptotic behaviour for large values of N, or, in other words, using the definitions in (5.1.8) that

$$\mathsf{LD}_\square(t;\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_0)) = \mathsf{LD}_\square(t;\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_\infty)) = \mathsf{c}(t). \tag{5.1.16}$$

Furthermore, since (5.1.14) holds for other types of "truncations" of the matrix, we expect to see the same in the setting of integral Hankel operators.

1.2. Spectral densities and Schur-Hadamard multipliers. For us to be able to state our results in their full generality, we use the concept of Schur-Hadamard multipliers discussed in Chapter 2. Let us briefly recall this below.
Let $k:\mathbb{R}_+^2\to\mathbb{C}$ be a measurable function, called an *integral kernel*, the integral operator $\mathrm{Op}(k)$ is defined as the operator on $L^2(\mathbb{R}_+)$

$$\mathrm{Op}(k)f(t) = \int_{\mathbb{R}_+} k(t,s)f(s)ds.$$

For a fixed bounded measurable function $\tau:[0,\infty)^2\to\mathbb{C}$, called a *multiplier*, the Schur-Hadamard multiplication of τ and $\mathrm{Op}(k)$ is the (possibly unbounded) integral operator on $L^2(\mathbb{R}_+)$, $\tau\star\mathrm{Op}(k)$, whose integral kernel is given by the pointwise product

$$\tau(t,s)k(t,s),\quad t,s\in\mathbb{R}_+. \tag{5.1.17}$$

In Chapter 2, we have extended the concept of Schur-Hadamard multiplication to include the case of a general bounded operator on $L^2(\mathbb{R}_+)$ via a limiting argument. In the next few sections, we will be referring to such an extension whenever we talk of the Schur-Hadamard multiplication of τ and any bounded operator A. We also studied the matter of finiteness of the operator norms

$$\|\tau\|_{\mathfrak{M}} = \sup_{\|A\|=1}\|\tau\star A\|, \tag{5.1.18}$$

$$\|\tau\|_{\mathfrak{M}_p} = \sup_{\|A\|_{\mathfrak{S}_p}=1}\|\tau\star A\|_p,\quad 1\le p\le\infty. \tag{5.1.19}$$

In this Chapter, however, we restrict our attention to multipliers of a specific form. More precisely, let $\varphi_0,\varphi_\infty\in L^\infty(\mathbb{R}_+^2)$ and let $N\ge 1$ be integer. Then we define the sequence of multipliers $\underline{\tau}=\{\tau_N\}_{N\ge1}$, where for each N we have

$$\tau_N(s,t) = \varphi_0(t/N,\, s/N)\varphi_\infty(Nt,\, Ns),\quad t,s>0. \tag{5.1.20}$$

If $\underline{\tau}\in\ell^\infty(\mathfrak{M})$, i.e. if it satisfies the following condition

$$\|\underline{\tau}\|_{\ell^\infty(\mathfrak{M})} = \sup_{N\ge1}\|\tau_N\|_{\mathfrak{M}} < \infty, \tag{5.1.21}$$

we say that $\underline{\tau}$ *induces a uniformly bounded sequence of multipliers.* Since $\underline{\tau}$ is completely determined by the couple $(\varphi_0, \varphi_\infty)$, we will call $\underline{\tau}$ a *(multiplier) couple.* An easy example of a sequence of multipliers of this form is given by the square truncation of an integral operator. Indeed, let

$$\varphi_0(t,s) = \mathbb{1}_{(0,1)}(t)\mathbb{1}_{(0,1)}(s), \quad \varphi_\infty(t,s) = \mathbb{1}_{(1,\infty)}(t)\mathbb{1}_{(1,\infty)}(s), \tag{5.1.22}$$

where $\mathbb{1}_{(0,1)}$ and $\mathbb{1}_{(1,\infty)}$ are the characteristic functions of the intervals $(0,1)$ and $(1,\infty)$. Let $\underline{\tau}_\square := (\varphi_0, \varphi_\infty)$. It is easy to check that $\underline{\tau}$ gives rise to the multipliers

$$\tau_N(t,s) = \mathbb{1}_{(N^{-1},N)}(t)\mathbb{1}_{(N^{-1},N)}(s). \tag{5.1.23}$$

From which the square truncation in (5.1.7) is obtained. More concrete examples of multiplier couples of this form will be discussed below.

Now, for a Hankel operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ we define the logarithmic spectral density of $|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})|$ with respect to $\underline{\tau}$ as

$$\mathsf{LD}_{\underline{\tau}}(t;\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) := \lim_{N\to\infty} \frac{\mathsf{n}(t;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}))}{\log N}, \quad t > 0. \tag{5.1.24}$$

Similarly, for a self-adjoint Hankel integral operator and $\underline{\tau}$ such that $\tau_N(x,y) = \overline{\tau_N(y,x)}$, we define the positive and negative logarithmic spectral densities of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ to be

$$\mathsf{LD}^{\pm}_{\underline{\tau}}(t;\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) := \lim_{N\to\infty} \frac{\mathsf{n}_{\pm}(t;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}))}{\log N}, \quad t > 0. \tag{5.1.25}$$

Of course, when $\underline{\tau} = \underline{\tau}_\square$ as in (5.1.23), the functions $\mathsf{LD}_{\underline{\tau}}(t;\Gamma(\widehat{\omega}))$ and $\mathsf{LD}^{\pm}_{\underline{\tau}}(t;\Gamma(\widehat{\omega}))$ are precisely those defined in (5.1.8) and (5.1.9).

1.3. Statement of the main results. Just as in Chapter 4, the main results of this Chapter not only show the existence of the limits in (5.1.8) and (5.1.9), but also, given some mild assumptions on the couple $\underline{\tau}$, their universality. Let us now state the following assumptions on the $\underline{\tau}$:

ASSUMPTIONS 5.1.2.

(A) $\underline{\tau}$ induces a uniformly bounded sequence of Schur-Hadamard multipliers, i.e. (5.1.21) holds;

(B) the limits

$$a_0 = \lim_{t,s\to 0} \varphi_0(t,s), \qquad b_0 = \lim_{t,s\to 0} \varphi_\infty(t,s),$$
$$a_\infty = \lim_{t,s\to \infty} \varphi_0(t,s), \qquad b_\infty = \lim_{t,s\to \infty} \varphi_\infty(t,s),$$

exist and are finite;

(C) there exist $\alpha > 1/2$ and some positive constants C_α, such that for some $\varepsilon > 0$ one has

$$|\varphi_0(t,s) - a_0| \le C_\alpha \left|\log(s+t)\right|^{-\alpha}, \tag{5.1.26}$$

$$|\varphi_\infty(t,s) - b_0| \le C_\alpha \left|\log(s+t)\right|^{-\alpha}, \tag{5.1.27}$$

for all $s, t \in [0, \varepsilon]$;

(D) we can find $\beta > 1/2$ and some positive constants C_β, such that for all $s, t > 0$ one has

$$|\varphi_0(t,s) - a_\infty| \le C_\beta \left|\log(s+t+2)\right|^{-\beta}, \tag{5.1.28}$$

$$|\varphi_\infty(t,s) - b_\infty| \le C_\beta \left|\log(s+t+2)\right|^{-\beta}; \tag{5.1.29}$$

Then (5.1.16) is a particular case of the following:

THEOREM 5.1.3. *Suppose $\underline{\tau} = (\varphi_0, \varphi_\infty)$ satisfies Assumptions 5.1.2 (A)-(D) with $a_0 = b_\infty = 1$ and $a_\infty = b_0 = 0$. Let $\boldsymbol{\omega} \in PC(\hat{\mathbb{R}})$ and Ω be the set of its discontinuities. Then*

$$\mathsf{LD}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = \sum_{v\in\Omega} \mathsf{c}\left(t\,|\varkappa_v(\boldsymbol{\omega})|^{-1}\right) \tag{5.1.30}$$

where $\mathsf{c}(t)$ is the function defined in (5.1.15).

Analogously, for the self-adjoint case we have the Theorem below:

THEOREM 5.1.4. *Let $\underline{\tau} = (\varphi_0, \varphi_\infty)$ satisfy Assumptions 5.1.2 (A)-(D) with $a_0 = b_\infty = 1$ and $a_\infty = b_0 = 0$ and suppose that $\tau_N(x,y) = \overline{\tau_N(y,x)}$, for all N. Suppose $\boldsymbol{\omega} \in PC(\hat{\mathbb{R}})$ satisfies (5.1.3) and let $\Omega^+ = \{v \in \Omega \,|\, v > 0\}$. Then*

$$\begin{aligned} \mathsf{LD}^{\pm}_{\tau}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = &\sum_{v\in\Omega^+} \mathsf{c}\left(t\,|\varkappa_v(\boldsymbol{\omega})|^{-1}\right) \\ &+ \mathsf{c}\left(t\,|\varkappa_0(\boldsymbol{\omega})|^{-1}\right) \mathbb{1}_{\pm}(-i\varkappa_0(\boldsymbol{\omega})) \\ &+ \mathsf{c}\left(t\,|\varkappa_\infty(\boldsymbol{\omega})|^{-1}\right) \mathbb{1}_{\pm}(-i\varkappa_\infty(\boldsymbol{\omega})), \end{aligned} \tag{5.1.31}$$

where $\mathsf{c}(t)$ is the function defined in (5.1.15) and $\mathbb{1}_\pm$ is the characteristic function of the half-line $(0, \pm\infty)$.

1.4. Remarks.

(A) Theorems 5.1.3 and 5.1.4 are virtually the same as Theorems 4.1.2 and 4.1.3 of Chapter 4 and both generalise to integral Hankel operator the results of Widom, [**61**, Theorem 4.3]. However, upon a closer inspection, the reader may notice that they rely on a different construction and set of assumptions. This is because we need to truncate both the regions of $\mathbb{R}^2_+$ close to 0 and "at infinity", unlike the case of matrices where we only truncated the region at "infinity". This is done to ensure that all Hankel operators with piece-wise continuous symbols are compact. For instance, if we consider $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ and only truncate the region at infinity, say by considering the couple $\underline{\tau} = (e^{-(s+t)}, 1)$, then the operator $(\tau_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ with integral kernel

$$\frac{e^{-(1+1/N)(s+t)}}{\pi(s+t)}$$

is not compact and so we cannot proceed onto studying the behaviour of its eigenvalues for large values of N. However, there are a few cases for which the results of Theorem 5.1.3 and 5.1.4 hold without truncating both 0 and ∞ at the same time. The first of these cases is when we only truncate the region at infinity. However, the trade-off is that we cannot admit symbols with a jump at infinity, as the following Proposition states

PROPOSITION 5.1.5. *Suppose $\boldsymbol{\omega} \in PC(\hat{\mathbb{R}})$ is so that $\infty \notin \Omega$ and suppose the couple $\underline{\tau} = (\varphi_0, 1)$ satisfies Assumptions 5.1.2 (A)-(D) with $a_0 = 1$ and $a_\infty = 0$. Then (5.1.30) holds. If $\varphi_0(t,s) = \overline{\varphi_0(s,t)}$, then also (5.1.31) holds.*

In particular, it shows that for the couple $\underline{\tau} = (e^{-(s+t)}, 1)$ one has that

$$\mathsf{n}(t; \tau_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = \log(N)(\mathsf{c}(t) + o(1)), \quad N \to \infty.$$

It is noteworthy that, except for the fact that $\infty \notin \Omega$, we did not make any assumptions on the finiteness of Ω.

Similarly, if we only wish to truncate the region around the origin, the trade-off is that we can only admit a symbol with a single jump at infinity, as the following proposition shows:

PROPOSITION 5.1.6. *Suppose $\boldsymbol{\omega} \in PC(\hat{\mathbb{R}})$ is so that $\Omega = \{\infty\}$ and suppose the couple $\underline{\tau} = (1, \varphi_\infty)$ satisfies Assumptions 5.1.2 (A)-(D) with $b_0 = 0$ and $b_\infty = 1$. Then (5.1.30) holds. Moreover, if $\varphi_\infty(t,s) = \overline{\varphi_\infty(s,t)}$, then also (5.1.31) holds.*

In particular, for the couple $\underline{\tau} = (1, 1 - e^{-(s+t)})$, we have that

$$\mathsf{n}(t; \tau_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = \log(N)(\mathsf{c}(t) + o(1)), \quad N \to \infty.$$

Both $(e^{-(s+t)}, 1)$ and $(1, 1 - e^{-(s+t)})$ will be useful later to compute the spectral density of the model operators.

(B) In a totally analogous way to the case of Hankel matrices, Assumption 5.1.2-(A) can be weakened by assuming that $\underline{\tau}$ does not induce a uniformly bounded sequence of multipliers on the space of bounded operators, but rather on a smaller subspace. For instance, we can assume that for some $p > 1$ we have

$$\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}_p} = \sup_{N\geq 1} \sup_{\|A\|_p=1} \|\tau_N \star A\|_p < \infty. \tag{5.1.32}$$

In this case we need to make a compromise on the generality of the symbol, as the following proposition shows

PROPOSITION 5.1.7. *Suppose $\boldsymbol{\omega} \in PC(\hat{\mathbb{R}})$ has only finitely many jumps and*

$$\boldsymbol{\omega}(t) = \sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\gamma_0(t-v) + \varkappa_\infty(\boldsymbol{\omega})\boldsymbol{\gamma}_\infty(t) + \boldsymbol{\eta}(t)$$

for some $\boldsymbol{\eta}$ such that $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}}) \in \mathfrak{S}_p$. If $\underline{\tau}$ satisfies (5.1.32) and Assumptions 5.1.2 (B)-(D) with $a_0 = b_\infty = 1$ and $a_\infty = b_0 = 0$, then (5.1.30) holds. Moreover, (5.1.31) holds if $\tau_N(s,t) = \overline{\tau_N(t,s)}$ and $\boldsymbol{\omega}(t) = \overline{\boldsymbol{\omega}(-t)}$.

Another way to weaken the uniform boundedness condition of (A) is to assume that the couple $\underline{\tau}$ induces a uniformly bounded multiplier on the space of bounded Hankel integral operators, i.e. that the following holds:

$$\sup_{N\geq 1} \sup_{\|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})\|=1} \|\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})\| < \infty. \tag{5.1.33}$$

In this, case we have the following

PROPOSITION 5.1.8. *Suppose the couple $\underline{\tau} = (\varphi_0, \varphi_\infty)$ satisfies (5.1.33) and Assumptions 5.1.2 (B)-(D). Then (5.1.30) holds. Furthermore, if $\tau_N(s,t) = \overline{\tau_N(t,s)}$ and $\boldsymbol{\omega}(t) = \overline{\boldsymbol{\omega}(-t)}$, then (5.1.31) holds too.*

We will give some examples of multipliers satisfying (5.1.32) and (5.1.33) below.

1.5. More on Schur-Hadamard multipliers. As we said earlier, we are considering multipliers of the form

$$\tau_N(t,s) = \varphi_0(t/N, s/N)\varphi_\infty(Nt, Ns),$$

where φ_0 and φ_∞ are two bounded functions on $\mathbb{R}_+^2$. However, the most relevant examples that the reader should bear in mind are a sub-class of such objects. Let us describe them. Let $\varphi_0 \in L^\infty(\mathbb{R}_+^2)$ be so that

$$\lim_{t,s\to 0+} \varphi_0(t,s) = 1 \text{ and } \lim_{t,s\to\infty} \varphi_0(t,s) = 0.$$

Then, the couple $\underline{\tau} = (\varphi_0,\, 1 - \varphi_0)$, induces a sequence of multipliers of the form

$$\tau_N(t,s) = \varphi_0(t/N, s/N)(1 - \varphi_0(Nt, Ns)).$$

Within this class fall a number of important "truncations" which appear numerous times on the literature on Schur-Hadamard multipliers. It is useful to group these in two main groups depending on whether φ_0 is factorisable or not.

EXAMPLE 5.1.9 (Factorisable φ_0). This is perhaps the most straightforward class of multipliers to deal with since we can immediately establish the boundedness of the couple $\underline{\tau}$. To this class belong multipliers for which φ_0 can be written as

$$\varphi_0(t,s) = f(t)g(s)$$

where f and g are bounded functions of a real variable. Then, denoting by M_{f_N} and M_{g_N} the operators of multiplication by $f_N(t) = f(t/N)$ and $g_N(t) = g(t/N)$ respectively, it is easy to see that for any integral operator $\mathrm{Op}(k)$ we have

$$\tau_N \star \mathrm{Op}(k) = M_{f_N}\,\mathrm{Op}(k)M_{g_N} - M_{f_N}M_{f_{1/N}}\,\mathrm{Op}(k)M_{g_{1/N}}M_{g_N}.$$

From which, we immediately have that the couple $\underline{\tau}$ induces a uniformly bounded sequence of multipliers. Moreover, the triangle inequality immediately yields the estimate

$$\|\tau_N\|_{\mathfrak{M}} \leq \|f\|_\infty\|g\|_\infty + \|f\|_\infty^2\|g\|_\infty^2.$$

A first example of a multiplier belonging to this class is given by

$$\varphi_0(t,s) = \mathbb{1}_{(0,2)}(t)\mathbb{1}_{(0,2)}(s).$$

It is easy to see that the couple $\underline{\tau} = (\varphi_0, 1-\varphi_0)$ is just a disguised version of the couple $\underline{\tau}_\square$ inducing the square truncation in (5.1.23).
Another, easy example of such a multiplier is obtained by taking

$$\varphi_0(t,s) = e^{-(t+s)} = e^{-t}e^{-s}.$$

Then, by our previous remarks, the couple $\underline{\tau} = (\varphi_0, 1-\varphi_0)$ induces a uniformly bounded sequence of multipliers. Since it also satisfies all of the hypotheses of Theorems 5.1.3 and 5.1.4, then (5.1.30) and (5.1.31) hold.

EXAMPLE 5.1.10 (Non-factorisable φ_0). In contrast to the earlier case, this class is far richer since it contains multipliers that can be

(i) uniformly bounded on $\mathfrak{S}_p$, $p > 1$ while being unbounded on the space of bounded operators;
(ii) uniformly bounded on the space of bounded Hankel integral operators, and unbounded on the bounded operators;
(iii) uniformly bounded on the space of bounded operators.

For the first case, let

$$\varphi_0(t,s) = \mathbb{1}_{(0,1)}(t+s).$$

The couple $\underline{\tau} = (\varphi_0, 1-\varphi_0)$ induces the *main triangle projection*, discussed in Chapter 2, where we showed that the induced multipliers are not uniformly bounded on the space of bounded operators on $L^2(\mathbb{R}_+)$ since we have

$$\|\tau_N\|_{\mathfrak{M}} \geq \frac{1}{\pi}\log(N)\left(1+\frac{|\log(\delta)-1|^2}{\log(N)}\right), \quad \forall\ N,\ \delta > 0.$$

Thus Theorems 5.1.3 and 5.1.4 do not hold. However, Theorem 2.3.5 shows that τ_N is uniformly bounded on the Schatten classes $\mathfrak{S}_p$ whenever $p \in (1,\infty)$ and so Proposition 5.1.7 holds.

In the second case, the results of [**13**] show that if we choose φ_0 to be the characteristic function of the region

$$\Xi_{\varepsilon,\delta} = \{(t,s) \in (0,1)^2 \mid y + \varepsilon x \leq \delta\},$$

then for $\varepsilon \neq 1$ and any δ, the couple $\underline{\tau} = (\varphi_0, 1-\varphi_0)$ induces a uniformly bounded sequence of multipliers on the space of bounded Hankel integral operators, albeit being unbounded when acting on the whole space of bounded operators. Note that when $\varepsilon = 1$ and $\delta = 1$, the couple $(\varphi_0, 1-\varphi_0)$ induces the main triangle projection discussed above. Therefore an appropriate choice of parameters δ, ε provides an example of a couple that satisfies the estimate (5.1.33) and Assumptions 5.1.2 (B)-(D), and so Proposition 5.1.8 holds.

Finally, choosing φ_0 to be the function

$$\varphi_0(t,s) = (1-(t+s))\,\mathbb{1}_{(0,1)}(t+s),$$

we obtain an example of a couple that induces uniformly bounded multipliers, τ_N. Indeed, observe that $\varphi_0(t,s)$ is the restriction to $\mathbb{R}^2_+$ of the function

$$(1-|t+s|)\,\mathbb{1}_{(-1,1)}(t+s) = \int_{\mathbb{R}} \frac{4}{\xi^2}\sin^2\left(\frac{\xi}{2}\right) e^{-i\xi(t+s)}d\xi,$$

and so Theorem 2.1.7 together with the triangle inequality shows that

$$\begin{aligned}\|\tau_N\|_{\mathfrak{M}} &\leq \|(\varphi_0)_N\|_{\mathfrak{M}} + \|(\varphi_0)_N\|_{\mathfrak{M}}\|(\varphi_0)_{1/N}\|_{\mathfrak{M}} \\ &\leq \frac{\pi}{2N} + \frac{\pi^2}{4},\end{aligned}$$

thus $(\varphi_0, 1-\varphi_0)$ induces a uniformly bounded sequence of multipliers. In addition, we note that the couple $(\varphi_0, 1-\varphi_0)$ satisfies the hypotheses of both Theorem 5.1.3 and 5.1.4, thus (5.1.30) and (5.1.31) hold.

1.6. How the proof works: basic ideas. The proof of the Theorems 5.1.3 and 5.1.4 follows the same strategy used to prove the main Theorems 4.1.2 and 4.1.3.

Let us explain their main points, starting with the proof of Theorem 5.1.3. As a first step, we decompose the symbol $\boldsymbol{\omega}$ with finitely many steps into the sum

$$\boldsymbol{\omega}(t) = \sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\gamma_v(t) + \boldsymbol{\eta}(t), \quad t \in \mathbb{R},$$

where $\gamma_v(t) = -i\gamma_0(t-v)$ if $|v| < \infty$ and $\gamma_v(t) = \gamma_\infty(t)$ otherwise. Both γ_0, γ_∞ are the symbols defined in (5.1.10) and $\boldsymbol{\eta}$ is some continuous function on $\hat{\mathbb{R}}$. Once we have the decomposed the symbol, we proceed onto a term-by-term analysis of all of the summands and their interactions with one another. The first step in this direction is to eliminate the contribution coming from the compact operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})$. To do this, we use the fact that for any couple $\underline{\tau}$ satisfying (A) and (B), Lemma 2.4.4 shows that

$$\mathsf{n}(s; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})) = O_s(1), \quad N \to \infty,$$

from which the Weyl inequality (2.4.18) gives that

$$\mathsf{LD}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = \mathsf{LD}_{\underline{\tau}}\left(t; \sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)\right).$$

In this manner we have reduced our problem to studying the logarithmic spectral density of a finite sum of very well-understood integral Hankel operators. The Invariance Principle Theorem 2.4.6 can now be used to deduce that for any couple $\underline{\tau}$ satisfying Assumptions 5.1.2 (B)-(D)

$$\mathsf{LD}_{\underline{\tau}}\left(t; \sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)\right) = \mathsf{LD}_{\square}\left(t; \sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)\right).$$

The latter identity now allows us to use a hands-on approach to show that the products of the square truncations of the operators $\boldsymbol{\Gamma}(\widehat{\gamma}_v)$ are almost orthogonal, in the sense that

$$\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v)^*\, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_w), \quad \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v)\, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_w)^*$$

are Hilbert-Schmidt operators uniformly in N whenever $v \neq w$, see Theorems 5.3.2 and 5.3.3 below. Finally, using the Asymptotic Orthogonality Theorem 2.4.7, we arrive at the conclusion that the logarithmic spectral density of the sum

$$\sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)$$

is localised around the jumps, in the sense that we can write:

$$\mathsf{LD}_{\square}\left(t; \sum_{v\in\Omega} \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)\right) = \sum_{v\in\Omega} \mathsf{LD}_{\square}\left(t; \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)\right).$$

This is yet another instance of what is known in the literature as *Localisation Principle*, i.e. the fact that the jumps of $\boldsymbol{\omega}$ located at different places on the real line contribute independently to the spectral properties of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$.

Following this reasoning, we have now arrived at computing the spectral density of the operators $\varkappa_v(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_v)$ with respect to the couple $\mathcal{T}_\square$. Using the results of [22] and, once more, the Invariance Principle Theorem 2.4.6, we easily obtain that for any $v \in \hat{\mathbb{R}}$

$$\mathsf{LD}_\square(t; \varkappa_v(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_v)) = \mathsf{c}(t\,|\varkappa_v(\boldsymbol{\omega})|^{-1}), \quad t > 0,$$

where c is the function defined in (5.1.15). Putting all the pieces together, we finally obtain the equality in (5.1.30).

The proof of Theorem 5.1.4 roughly follows the same ideas. However, at some key points of the proof we use the fact that $\boldsymbol{\omega}$ satisfies the symmetry condition (5.1.3) to decompose is as follows

$$\boldsymbol{\omega}(t) = \sum_{v \in \Omega^+} \boldsymbol{\omega}_v(t) + \varkappa_0(\boldsymbol{\omega})\gamma_0(t) + \varkappa_\infty(\boldsymbol{\omega})\gamma_\infty(t) + \boldsymbol{\eta}(t),$$

where $\boldsymbol{\eta}$ is some continuous function on $\hat{\mathbb{R}}$, $\Omega^+ = \{v \in \Omega \,|\, v \in (0, \infty)\}$, γ_0, γ_∞ are the symbols in (5.1.10) and

$$\boldsymbol{\omega}_v(t) = \varkappa_v(\boldsymbol{\omega})\gamma_v(t) + \overline{\varkappa_{-v}(\boldsymbol{\omega})}\gamma_{-v}(t).$$

With such a decomposition at hand, and following the same steps as before, we arrive at the following identity:

$$\begin{aligned}\mathsf{LD}^\pm_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = \mathsf{LD}^\pm_\square(t; \varkappa_0(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_0)) + \mathsf{LD}^\pm_\square(t; \varkappa_\infty(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_\infty))\\ + \sum_{v \in \Omega^+} \mathsf{LD}^\pm_\square(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)).\end{aligned}$$

From this, we now need to compute the upper and lower logarithmic spectral densities of the operators $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)$ as well as $\varkappa_\infty(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ and $\varkappa_0(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ with respect to the square truncation $\mathcal{T}_\square$. However, an argument based on the fact that the operators $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ and $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ are positive definite operators, immediately shows that

$$\begin{aligned}\mathsf{LD}^\pm_\square(t; \varkappa_0(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_0)) = \mathbb{1}_\pm(-i\varkappa_0(\boldsymbol{\omega}))\mathsf{c}(t\,|\varkappa_0(\boldsymbol{\omega})|^{-1}),\\ \mathsf{LD}^\pm_\square(t; \varkappa_\infty(\boldsymbol{\omega})\,\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = \mathbb{1}_\pm(-i\varkappa_\infty(\boldsymbol{\omega}))\mathsf{c}(t\,|\varkappa_\infty(\boldsymbol{\omega})|^{-1}),\end{aligned}$$

where $\mathbb{1}_\pm$ is the characteristic function of the half-line $(0, \pm\infty)$. Moreover, since the symbol $\boldsymbol{\omega}_v \in C^\infty(\hat{\mathbb{R}} \setminus \{\pm v\})$ and has jumps symmetrically located at $\pm v$, Theorem 5.3.6 together with the Asymptotic Symmetry Theorem 2.4.9 shows that the upper and lower logarithmic spectral densities of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)$ equally contribute to the logarithmic spectral densities of $|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)|$, or in other words

$$\mathsf{LD}^\pm_\square(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)) = \frac{1}{2}\mathsf{LD}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)) = \mathsf{c}(t\,|\varkappa_v(\boldsymbol{\omega})|^{-1}).$$

The last equality follows directly from Theorem 5.1.3. The equality above is, in fact, another instance of the general philosophy that symmetrically located jumps of $\boldsymbol{\omega}$ should contribute equally to the spectral properties of the associated Hankel integral operator. This phenomenon has been referred to as the *Symmetry Principle*, by the authors of [54]. Putting all of the pieces together, we finally obtain the sought after equality (5.1.31).

Extending the results to symbols with infinitely many jumps requires a simple limiting argument, relying on Assumption (A), which reprises the one originally presented in [50] and, subesequently, presented in [46, Chapter 10].

2. An abstract property of spectral densities of integral operators

2.1. Setup. In this section, all operators are bounded integral operators on $L^2(\mathbb{R}_+)$ and $\|\cdot\|_p$ denotes the norm of the $\mathfrak{S}_p$ Schatten class, $p \geq 1$. We will make use of the following subspace of the bounded integral operators:

$$\operatorname{Op}(k) \in \mathfrak{B}_0 \quad \Longleftrightarrow \quad |k(t,s)| \leq C(t+s)^{-1}, \quad \forall\, t,s \in \mathbb{R}_+. \tag{5.2.34}$$

It is fairly easy to chech that if $\operatorname{Op}(k) \in \mathfrak{B}_0$, then $\|\operatorname{Op}(k)\| \leq C\pi$. We also need the following two subspaces of $\mathfrak{B}_0$:

$$\operatorname{Op}(k) \in \mathfrak{B}_0^{(0)} \Longleftrightarrow \operatorname{Op}(k) \in \mathfrak{B} \text{ and } |k(t,s)| = O\left((t+s)^{-\varkappa}\right), \quad \varkappa < 1,\ t,s \to 0, \tag{5.2.35}$$

$$\operatorname{Op}(k) \in \mathfrak{B}_0^{(\infty)} \Longleftrightarrow \operatorname{Op}(k) \in \mathfrak{B} \text{ and } |k(t,s)| = O\left((t+s)^{-\varkappa}\right), \quad \varkappa > 1,\ t,s \to \infty. \tag{5.2.36}$$

For the couple $\underline{\tau} = (\varphi_0, \varphi_\infty)$, we have already defined the meaning of $\tau_N \star \operatorname{Op}(k)$. Suppose now that $\underline{\tau}$ satisfies both (C) and (D) in Assumptions 5.1.2 with $a_0 = b_\infty = 1$ and $b_0 = a_\infty = 0$, then $\tau_N \star \operatorname{Op}(k) \in \mathfrak{S}_2$ for any $\operatorname{Op}(k) \in \mathfrak{B}_0$ because of the immediate estimate:

$$\begin{aligned}\|\tau_N \star \operatorname{Op}(k)\|_2^2 &= \iint_{\mathbb{R}_+^2} |\tau_N(t,s)k(t,s)|^2\, dtds \\ &\leq C \iint_{\mathbb{R}_+^2} \frac{1}{\log\left(\frac{s+t}{N}+2\right)^{2\gamma}(s+t)^2} dtds.\end{aligned}$$

We remark that $\|\tau_N \star \operatorname{Op}(k)\|_2$ may not be uniformly bounded in N. We shall see below when this is the case. Similar estimates (at least in spirit) show that $\tau_N \star \operatorname{Op}(k)$ is a Hilbert-Schmidt operator if either $b_0 \neq 0$ and $\operatorname{Op}(k) \in \mathfrak{B}_0^{(0)}$, or $a_\infty \neq 0$ and $\operatorname{Op}(k) \in \mathfrak{B}_0^{(\infty)}$.

With this observation at hand, it is evident that for $\operatorname{Op}(k) \in \mathfrak{B}_0$, the functionals $\overline{\mathsf{LD}}_{\underline{\tau}}(t; \operatorname{Op}(k))$, $\underline{\mathsf{LD}}_{\underline{\tau}}(t; \operatorname{Op}(k))$ defined in (2.4.13) and (2.4.14) respectively are well-defined. Similarly, if $\operatorname{Op}(k) \in \mathfrak{B}_0$ is self-adjoint and $\tau_N(t,s) = \overline{\tau_N(s,t)}$ for all N, then the functionals $\overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t; \operatorname{Op}(k)), \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t; \operatorname{Op}(k))$ are well-defined too.

2.2. Invariance of spectral densities. We are now interested in studying how these functions depend on the sequence of multipliers τ_N and, ultimately, on our choice of functions $\varphi_0, \varphi_\infty$. To this end, we have the following theorem:

THEOREM 5.2.1. *Let $\underline{\tau}^{(1)}$ and $\underline{\tau}^{(2)}$ satisfy (B)-(D) in Assumptions 5.1.2 with $b_0^{(i)} = a_\infty^{(i)} = 0$ and $b_\infty^{(i)} = a_0^{(i)} = 1$. Then, for any $\mathrm{Op}(k) \in \mathfrak{B}_0$, we can find a constant C such that*

$$\left\| \left(\underline{\tau}^{(1)} - \underline{\tau}^{(2)}\right)_N \star \mathrm{Op}(k) \right\|_2 \leq C,$$

thus the estimates of the Invariance Principle Theorem 2.4.6 hold. Moreover, the same is true if

(i) $\mathrm{Op}(k) \in \mathfrak{B}_0^{(0)}$ *and* $a_0^{(i)} = b_\infty^{(i)} = 1$, $a_\infty^{(i)} = 1$ *and, if non-zero, we have* $b_0^{(i)} = 1$;
(ii) $\mathrm{Op}(k) \in \mathfrak{B}_0^{(\infty)}$ *and* $a_0^{(i)} = b_\infty^{(i)} = 1$, $b_0^{(i)} = 0$, *and, if non-zero, we have* $a_0^{(i)} = 1$.

REMARK 5.2.2. Note that the values of the limits $b_0^{(i)}$ and $a_\infty^{(i)}$ in (i) and (ii), when non-zero, need not coincide. We can have, for instance, that $b_0^{(1)} = 0$ and $b_0^{(2)} = 1$.

To this end, we need the following

LEMMA 5.2.3. *Let $\mathrm{Op}(k) \in \mathfrak{B}_0$. Suppose $\underline{\sigma} = (\varphi_0, \varphi_\infty)$ satisfies (B)-(D) in Assumptions 5.1.2 with $a_\infty = b_0 = 0$ and either $a_0 = 0$ or $b_\infty = 0$, then $\sigma_N \star \mathrm{Op}(k) \in \mathfrak{S}_2$ and furthermore*

$$\sup_{N \geq 1} \|\sigma_N \star \mathrm{Op}(k)\|_2 < \infty. \tag{5.2.37}$$

Moreover, (5.2.37) holds when:

(i) $\mathrm{Op}(k) \in \mathfrak{B}_0^{(0)}$ *and we have* $a_0 = a_\infty = b_\infty = 0$ *and* $b_0 \neq 0$;
(ii) $\mathrm{Op}(k) \in \mathfrak{B}_0^{(\infty)}$ *and we have* $a_0 = b_0 = b_\infty = 0$ *and* $a_\infty \neq 0$.

PROOF OF LEMMA 5.2.3. For brevity, set $\sigma_N \star \mathrm{Op}(k) = \mathrm{Op}(k_N)$, where $k_N = \tau_N \cdot k$. We need to estimate the following

$$\begin{aligned} \|\mathrm{Op}(k_N)\|_2^2 &= \iint_{\mathbb{R}_+^2} |k(t,s)\sigma_N(t,s)|^2 \, dtds \\ &\leq C \iint_{\mathbb{R}_+^2} \frac{|\sigma_N(t,s)|^2}{(t+s)^2} dtds. \end{aligned}$$

We will show that the latter integral is finite. To do so, define the set $\Omega := \{(t,s) \in \mathbb{R}_+^2 \mid t+s < \varepsilon\}$ and let $\Omega^c := \mathbb{R}_+^2 \setminus \Omega$. Let us assume, to begin with, that $a_0 =$

$0, b_\infty \neq 0$. Then

$$\begin{aligned}\iint_{\mathbb{R}^2_+} \frac{|\sigma_N(t,s)|^2}{(t+s)^2} dtds &= \iint_{\mathbb{R}^2_+} \frac{|\varphi_0(t/N,s/N)\varphi_\infty(Nt,Ns)|^2}{(t+s)^2} dtds \\ &\leq C \iint_{\mathbb{R}^2_+} \frac{|\varphi_0(t/N,s/N)|^2}{(t+s)^2} dtds \\ &= C \iint_{\Omega} \frac{|\varphi_0(t,s)|^2}{(t+s)^2} dtds \qquad (:= I_1) \\ &+ C \iint_{\Omega^c} \frac{|\varphi_0(t,s)|^2}{(t+s)^2} dtds, \qquad (:= I_2).\end{aligned}$$

The inequality in the second line is obtained from the fact that $\varphi_\infty \in L^\infty(\mathbb{R}^2_+)$, while the subsequent equality comes from the change of variables $t = Nx$, $s = Ny$ and from writing $\mathbb{R}^2_+ = \Omega \cup \Omega^c$. Since $\underline{\sigma}$ satisfies Assumption 5.1.2 (C) with $a_0 = 0$, then

$$\begin{aligned}I_1 &\leq C'_\alpha \int_0^\varepsilon \int_0^\varepsilon \frac{dtds}{|\log((t+s))|^{2\alpha}(t+s)^2} \\ &\leq C'_\alpha \int_0^\varepsilon \frac{d\lambda}{\lambda|\log(\lambda)|^{2\alpha}} = \frac{C'_\alpha}{(2\alpha-1)|\log(\varepsilon)|^{2\alpha-1}}\end{aligned}$$

where the equality on the second line comes from the change of variables $\lambda = e^{-t}$. Now, since $\underline{\sigma}$ satisfies Assumption 5.1.2 (D) with $a_\infty = 0$, we also have

$$\begin{aligned}I_2 &\leq C'_\beta \iint_{\Omega^c} \frac{dtds}{\log(t+s+2)^{2\beta}(t+s)^2} \\ &\leq C'_\beta \int_\varepsilon^\infty \frac{1}{t\log(t+2)^{2\beta}} dt < \infty.\end{aligned}$$

Since the integrals I_1 and I_2 are uniformly bounded, the result holds. A similar argument shows (5.2.37) when $a_0 \neq 0, b_\infty = 0$.

Suppose that (i) holds, then we can write

$$\begin{aligned}\|\operatorname{Op}(k_N)\|_2^2 &= \iint_{\mathbb{R}^2_+} |\varphi_0(t/N,s/N)\varphi_\infty(Nt,Ns)k(t,s)|^2 dtds \\ &\leq C \iint_{\mathbb{R}^2_+} |\varphi_\infty(Nt,Ns)k(t,s)|^2 dtds \\ &\leq C \iint_{\Omega} \frac{|\varphi_\infty(Nt,Ns)|^2}{(t+s)^{2\varkappa}} dtds \qquad (:= J_1) \\ &+ C \iint_{\Omega^c} \frac{|\varphi_\infty(Nt,Ns)|^2}{(t+s)^2} dtds, \qquad (:= J_2).\end{aligned}$$

Using the fact that φ_∞ is bounded, we have that

$$\begin{aligned} J_1 &\le \|\varphi_\infty\|_{L^\infty(\mathbb{R}_+^2)} \iint_\Omega \frac{dsdt}{(t+s)^{2\varkappa}} \\ &\le C \int_0^\varepsilon \frac{dt}{t^{2\varkappa-1}} < \infty. \end{aligned}$$

The finiteness of the latter integral is guaranteed by the fact that $\varkappa < 1$. Also, since $\underline{\sigma}$ satisfies Assumption 5.1.2 (D) with $b_\infty = 0$, we have

$$\begin{aligned} J_2 &\le C_\beta \iint_{\Omega^c} \frac{dsdt}{(s+t)^2 \log(N(s+t)+2)^{2\beta}} \\ &= C_\beta \iint_{N\Omega^c} \frac{dsdt}{(s+t)^2 \log(s+t+2)^{2\beta}} \\ &\le C_\beta \iint_{\Omega^c} \frac{dsdt}{(s+t)^2 \log(s+t+2)^{2\beta}} \\ &\le C_\beta \int_\varepsilon^\infty \frac{dt}{t \log(t+2)^{2\beta}} < \infty. \end{aligned}$$

Since both J_1 and J_2 are uniformly bounded in N, estimate (5.2.37) holds in this case.

A similar argument to the one just presented shows that (5.2.37) holds when (ii) holds. □

Proof of Theorem 5.2.1. The estimates contained in the Invariance Principle Theorem 2.4.6 holds when for the couples $\underline{\tau}^{(i)}$ as above one has:

$$\sup_{N\ge 1} \|(\underline{\tau}^{(1)} - \underline{\tau}^{(2)})_N \star \mathrm{Op}(k)\|_p < \infty,$$

for some finite $p \ge 1$. However, note that we can write

$$(\underline{\tau}^{(1)} - \underline{\tau}^{(2)})_N \star \mathrm{Op}(k) = (\underline{\sigma}^{(1)} - \underline{\sigma}^{(2)})_N \star \mathrm{Op}(k),$$

where $\sigma_N^{(1)}$ and $\sigma_N^{(2)}$ are multipliers induced by the couples

$$\underline{\sigma}^{(1)} = \left(\varphi_0^{(1)} - \varphi_0^{(2)}, \varphi_\infty^{(1)}\right), \quad \underline{\sigma}^{(2)} = \left(\varphi_0^{(2)}, \varphi_\infty^{(1)} - \varphi_\infty^{(2)}\right).$$

Therefore, by the triangle inequality it is sufficient to show that both the quantities

$$\sup_{N\ge 1} \|\underline{\sigma}_N^{(1)} \star \mathrm{Op}(k)\|_p, \quad \sup_{N\ge 1} \|\underline{\sigma}_N^{(2)} \star \mathrm{Op}(k)\|_p$$

are finite for some $p \ge 1$. Notice, however, that the couples $\underline{\sigma}^{(1)}, \underline{\sigma}^{(2)}$ satisfy the hypotheses of Lemma 5.2.3, and so for $p = 2$ we obtain

$$\sup_{N\ge 1} \|\underline{\sigma}_N^{(1)} \star \mathrm{Op}(k)\|_2 < \infty.$$

The assertion now follows immediately. □

3. Asymptotic orthogonality and densities of model operators

3.1. Model operators and their factorization. We now move on to studying two simple model operators which will be useful later on in the proof of our main results. As a matter of fact, we already mentioned in the Introduction two different symbols which have a single jump at 0 and at ∞ respectively, see (5.1.10). Let us recall them here:

$$\gamma_0(x) = \frac{(\pi \operatorname{sign}(x) - 2\arctan(x))}{\pi i}, \quad \gamma_\infty(x) = \frac{2\arctan(x)}{\pi i}, \; x \in \mathbb{R}.$$

As we mentioned at the beginnning ot the Chapter, both γ_0 and γ_∞ have a single jump at 0 and ∞ respectively, and $\varkappa_0(\gamma_0) = \varkappa_\infty(\gamma_\infty) = -i$. Furthermore, their Fourier transforms are well-known to be:

$$\widehat{\gamma}_0(t) = \frac{1-e^{-t}}{\pi t}, \quad \widehat{\gamma}_\infty(t) = \frac{e^{-t}}{\pi t}, \quad t > 0. \tag{5.3.38}$$

The associated integral Hankel operators $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ and $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ can be used to model two different situations: that of a jump at a finite point for the former, that of a jump at infinity for the latter. To model a jump at $v \in \mathbb{R}$, however, we need to slightly modify the symbol by defining the following:

$$\gamma_v(x) = \gamma_0(x - v), \quad x \in \mathbb{R}. \tag{5.3.39}$$

It is easy to check that for any v one has that $\widehat{\gamma}_v(t) = e^{-ivt}\widehat{\gamma}_0(t)$ and so

$$\boldsymbol{\Gamma}(\widehat{\gamma}_v) = U_v \boldsymbol{\Gamma}(\widehat{\gamma}_0) U_v, \tag{5.3.40}$$

where U_v is the unitary operator of multiplication by the exponential e^{-ivt} on $L^2(\mathbb{R}_+)$. Let now $\mathcal{L}$ be the operator on $L^2(\mathbb{R}_+)$ given by

$$\mathcal{L}f(t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} e^{-st} f(s) ds.$$

The Schur test, [**29**, Theorem 5.2] shows that $\|\mathcal{L}\| \le 1$ and a simple computation shows that

Lemma 5.3.1. *The operators $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$, $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ have the following factorization:*

$$\boldsymbol{\Gamma}(\widehat{\gamma}_0) = \mathcal{L}\mathbb{1}_0\mathcal{L}, \qquad \boldsymbol{\Gamma}(\widehat{\gamma}_\infty) = \mathcal{L}\mathbb{1}_\infty\mathcal{L}, \tag{5.3.41}$$

where $\mathbb{1}_0$ and $\mathbb{1}_\infty$ are the characteristic functions of the intervals $(0,1)$ and $(1,\infty)$ respectively.

3.2. Model symbols with jumps at finite points. The representations (5.3.40) and (5.3.41) can be used to show that the jumps at two distincts finite points do not interact with each other. In other words, that the following theorem holds

THEOREM 5.3.2. *Let v_1, v_2 be distinct points on the real line. Then*

$$\sup_{N\geq 1} \| \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_1})^* \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_2})\|_2 < \infty, \tag{5.3.42}$$

$$\sup_{N\geq 1} \| \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_1}) \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_2})^*\|_2 < \infty. \tag{5.3.43}$$

PROOF. We will only show (5.3.42) as (5.3.43) is proved in exactly the same way. Note that using (5.3.40) and (5.3.41), we have that

$$\begin{aligned}\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_1})^* \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_2}) &= \mathbb{1}_N U_{v_1}^* \, \boldsymbol{\Gamma}(\widehat{\gamma}_0) U_{v_1}^* \mathbb{1}_N U_{v_2} \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_0) U_{v_2} \\ &= \mathbb{1}_N U_{v_1}^* \mathcal{L} \mathbb{1}_0 \mathcal{L} U_{v_1}^* \mathbb{1}_N U_{v_2} \mathcal{L} \mathbb{1}_0 \mathcal{L} U_{v_2} \mathbb{1}_N,\end{aligned}$$

and so, it is sufficient to show that

$$\sup_{N\geq 1} \|\mathbb{1}_0 \mathcal{L} U_{v_1}^* \mathbb{1}_N U_{v_2} \mathcal{L} \mathbb{1}_0\|_2 < \infty.$$

Note now that the operator $A_N := \mathbb{1}_0 \mathcal{L} U_{v_1}^* \mathbb{1}_N U_{v_2} \mathcal{L} \mathbb{1}_0$ has integral kernel given by

$$a_N(t,s) = \mathbb{1}_0(t) \frac{e^{-\frac{1}{2N}(t+s-i(v_1-v_2))} - e^{-2N(t+s-i(v_1-v_2))}}{\pi(t+s-i(v_1-v_2))} \mathbb{1}_0(s). \tag{5.3.44}$$

And so we have

$$\begin{aligned}\|A_N\|_2^2 &= \int_0^1 \int_0^1 |a_N(t,s)|^2 \, dsdt \\ &\leq \frac{4}{\pi^2} \int_0^1 \int_0^1 \frac{dsdt}{(t+s)^2 + (v_1-v_2)^2} \\ &= \frac{4}{\pi^2} \int_0^1 \frac{1}{|v_1-v_2|} \left(\tan^{-1}\left(\frac{s+1}{|v_1-v_2|}\right) - \tan^{-1}\left(\frac{s}{|v_1-v_2|}\right)\right) ds \\ &\leq \frac{4}{\pi^2 \, |v_1-v_2|} \tan^{-1}\left(\frac{2}{|v_1-v_2|}\right).\end{aligned}$$

Thus we have that

$$\sup_{N\geq 1} \| \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_1})^* \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_{v_2})\|_2 \leq \frac{4}{\pi^2 \, |v_1-v_2|} \tan^{-1}\left(\frac{2}{|v_1-v_2|}\right) < \infty.$$

□

3.3. Model symbols with jumps at a finite point and infinity. The representations (5.3.40) and (5.3.41) can also be put to use in showing the symbol with a jump at a finite point and one with jump at infinity are asymptotically orthogonal in the sense specified below:

THEOREM 5.3.3. *For any $v \in \mathbb{R}$, we have*

$$\sup_{N\geq 1} \| \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v) \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_\infty)\|_2 < \infty.$$

Proof of Theorem. Using the representations (5.3.40) and (5.3.41), we have that

$$\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v)\,\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_\infty) = \mathbb{1}_N U_v\,\boldsymbol{\Gamma}(\widehat{\gamma}_0)U_v\mathbb{1}_N\,\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_\infty)$$
$$= \mathbb{1}_N U_v\mathcal{L}\mathbb{1}_0\mathcal{L}U_v\mathbb{1}_N\mathcal{L}\mathbb{1}_\infty\mathcal{L}\mathbb{1}_N,$$

and so we will have proved the statement once we manage to show that

$$\sup_{N\geq 1}\|A_N\|_{\mathfrak{S}_2} < \infty,$$

where $A_N := \mathbb{1}_0\mathcal{L}U_v\mathbb{1}_N\mathcal{L}\mathbb{1}_\infty$. As before, this follows from the fact that $\|\mathcal{L}\| \leq 1$ as well as $\|\mathbb{1}_N\| = 1$ and the estimate

$$\|\,\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v)\,\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_\infty)\|_2 \leq \|A_N\|_2. \tag{5.3.45}$$

The operator A_N has integral kernel given by

$$a_N(t,s) = \mathbb{1}_0(t)\frac{e^{-\frac{s+t+iv}{2N}} - e^{-2N(s+t+iv)}}{\pi(s+t+iv)}\mathbb{1}_\infty(s), \quad t,s>0. \tag{5.3.46}$$

So, explicitly we have

$$\|A_N\|_2^2 = \int_0^1\int_1^\infty\left|\frac{e^{-\frac{s+t+iv}{2N}} - e^{-2N(s+t+iv)}}{\pi(s+t+iv)}\right|^2 dsdt$$
$$\leq \frac{4}{\pi^2}\int_0^1\int_1^\infty\frac{1}{(s+t)^2}dsdt$$
$$= \frac{4\log(2)}{\pi^2}.$$

This, together with (5.3.45) gives the result. □

3.4. Asymptotically symmetric symbols. In this section we study Hankel integral operators whose symbol satisfies the symmetry condition (5.1.3). Our aim is to show that under certain condition on the symbols, see below, $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ is almost symmetric in the sense of the Almost Symmetry Theorem 2.4.9. To begin with, let

$$\mathfrak{s}(t) = \begin{cases}\operatorname{sign}(t), & t\neq 0,\\ 1, & t=0.\end{cases} \tag{5.3.47}$$

It is easy to see that the operator of multiplication by $\mathfrak{s}$ is unitary on $L^2(\mathbb{R})$ and so the operator of convolution by its Fourier transform, $C_{\widehat{\mathfrak{s}}}$, is unitary on $L^2(\mathbb{R}_+)$.

Let us now consider the couple $\underline{\tau} = (1, e^{-(t+s)})$. With such a choice of multiplier couple, we have that

$$\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}) = e^{-t/N}\,\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})e^{-t/N} = \boldsymbol{\Gamma}(\widehat{\mathfrak{p}_{1/N} * \boldsymbol{\omega}}), \tag{5.3.48}$$

where $e^{-t/N}$ denotes the operator of multiplication by $e^{-t/N}$ and $\mathfrak{p}_{1/N}$ is the Poisson kernel $\mathfrak{p}_y$ with $y = 1/N$, defined as

$$\mathfrak{p}_y(t) = \frac{y}{\pi(t^2 + y^2)}, \quad y > 0,\, t \in \mathbb{R}.$$

It is easy to see that from the first equality in (5.3.48), it follows that if $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}) \in \mathfrak{S}_1$ then

$$\|\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})\|_{\mathfrak{S}_1} \le \| \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})\|_{\mathfrak{S}_1}. \tag{5.3.49}$$

Before proving the main result, let us recall the following definition

DEFINITION 5.3.4. For a symbol $\boldsymbol{\omega}$, its *singular support*, $\operatorname{sing\,supp} \boldsymbol{\omega}$, is the smallest closed set, S such that $\boldsymbol{\omega} \in C^\infty(\hat{\mathbb{R}} \setminus S)$.

We will also need the following lemma:

LEMMA 5.3.5. *Let $\boldsymbol{\omega} \in L^\infty(\mathbb{R})$. The following estimate holds*

$$\|\mathfrak{p}_{1/N} * \boldsymbol{\omega}\,\|_{L^\infty(\mathbb{R})} \le \|\,\boldsymbol{\omega}\,\|_{L^\infty(\mathbb{R})}.$$

Furthermore, $\mathfrak{p}_{1/N} * \boldsymbol{\omega} \to \boldsymbol{\omega}$ locally uniformly on $\operatorname{sing\,supp} \boldsymbol{\omega}$ as $N \to \infty$. The same is true for all of its derivatives $(\mathfrak{p}_{1/N} * \boldsymbol{\omega})^{(n)}$.

With this at hand, we are now ready to prove the following general

THEOREM 5.3.6. *Let $\boldsymbol{\omega} \in PC(\hat{\mathbb{R}})$ be such that $\boldsymbol{\omega}(x) = \overline{\boldsymbol{\omega}(-x)}$ and such that $0, \infty \notin \operatorname{sing\,supp} \boldsymbol{\omega}$. Then, with $C_{\widehat{\mathfrak{s}}}$ being the unitary operator of convolution by $\widehat{\mathfrak{s}}$, we have*

$$\sup_N \|C_{\widehat{\mathfrak{s}}}\, \boldsymbol{\Gamma}(\widehat{\mathfrak{p}_{1/N} * \boldsymbol{\omega}}) + \boldsymbol{\Gamma}(\widehat{\mathfrak{p}_{1/N} * \boldsymbol{\omega}}) C_{\widehat{\mathfrak{s}}}\|_{\mathfrak{S}_1} < \infty. \tag{5.3.50}$$

PROOF. For simplicity, let us write $\boldsymbol{\omega}_N = \mathfrak{p}_{1/N} * \boldsymbol{\omega}$. As discussed in Chapter 3, the operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_N)$ is unitarily equivalent, under the Fourier transform $\boldsymbol{\Phi}$ to the operator

$$\boldsymbol{H}(\boldsymbol{\omega}_N) = \boldsymbol{P}_+ \boldsymbol{\omega}_N\, \boldsymbol{J} \boldsymbol{P}_+,$$

where $\boldsymbol{P}_+$ is the orthogonal projection from $L^2(\mathbb{R}) \to H^2_+(\mathbb{R})$ and $\boldsymbol{J} f(t) = f(-t)$. Thus, showing (5.3.50) is equivalent to proving the finiteness of

$$\sup_N \|\mathfrak{s} \boldsymbol{H}(\boldsymbol{\omega}_N) + \boldsymbol{H}(\boldsymbol{\omega}_N)\mathfrak{s}\|_{\mathfrak{S}_1}.$$

Since $0, \infty \notin \operatorname{sing\,supp} \boldsymbol{\omega}$, we can write $\boldsymbol{\omega} = \boldsymbol{\varphi} + \boldsymbol{\eta}$ for some $\boldsymbol{\eta} \in C^\infty(\hat{\mathbb{R}})$ and some $\boldsymbol{\varphi}$ vanishing identically in a neighbourhood U of 0 and ∞. With this decomposition of $\boldsymbol{\omega}$, we can see that

$$H(\boldsymbol{\omega}_N) = H(\boldsymbol{\varphi}_N) + H(\boldsymbol{\eta}_N).$$

Since $\boldsymbol{\eta}$ is smooth, $H(\boldsymbol{\eta}) \in \mathfrak{S}_1$ and so the triangle inequality together with (5.3.49) imply that

$$\sup_N \|\mathfrak{s}H(\boldsymbol{\omega}_N) + H(\boldsymbol{\omega}_N)\mathfrak{s}\|_{\mathfrak{S}_1} \leq 2\|H(\boldsymbol{\eta})\|_{\mathfrak{S}_1} + \sup_N \|\mathfrak{s}H(\boldsymbol{\varphi}_N) + H(\boldsymbol{\varphi}_N)\mathfrak{s}\|_{\mathfrak{S}_1}.$$

So, without any loss of generality, we can assume $\boldsymbol{\omega}$ vanishes identically on a neighbourhood, U, of 0 and ∞.

Fix now a smooth, even function $\boldsymbol{\zeta}$ such that $0 \leq \boldsymbol{\zeta} \leq 1$, it vanishes identically on some open $V \subset U$ so that $0, \infty \in V$ and $\boldsymbol{\zeta} \equiv 1$ on $\hat{\mathbb{R}} \setminus U$.Then, we can write:

$$\begin{aligned}\mathfrak{s}\boldsymbol{H}(\boldsymbol{\omega}_N) + \boldsymbol{H}(\boldsymbol{\omega}_N)\mathfrak{s} = \mathfrak{s}\boldsymbol{H}((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N) + \boldsymbol{H}((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N)\mathfrak{s} \\ + \mathfrak{s}\boldsymbol{H}(\boldsymbol{\zeta}\,\boldsymbol{\omega}_N) + \boldsymbol{H}(\boldsymbol{\zeta}\,\boldsymbol{\omega}_N)\mathfrak{s}\end{aligned} \tag{5.3.51}$$

Let us study this decomposition more carefully. In the first line, the triangle inequality yields

$$\sup_N \|\mathfrak{s}\boldsymbol{H}((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N) + \boldsymbol{H}((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N)\mathfrak{s}\|_{\mathfrak{S}_1} \leq 2\sup_N \|\boldsymbol{H}((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N)\|_{\mathfrak{S}_1}. \tag{5.3.52}$$

Using Lemma 5.3.5 and the fact that $(1-\boldsymbol{\zeta})\,\boldsymbol{\omega} \equiv 0$ on $\hat{\mathbb{R}}$, we conclude that $((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N)'' \to 0$ uniformly on $\hat{\mathbb{R}}$. Therefore Lemma 3.3.2-(ii) gives

$$\sup_N \|H((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N)\|_{\mathfrak{S}_1} \leq \sup_N(\|\,\boldsymbol{\omega}\,\|_{L^\infty(\mathbb{R})} + C\|(1+t^2)^{-1/2}((1-\boldsymbol{\zeta})\,\boldsymbol{\omega}_N)''\|_{L^2(\mathbb{R})}) < \infty. \tag{5.3.53}$$

For the operators appearing in the second line of (5.3.51), write

$$\mathfrak{s}\boldsymbol{H}(\boldsymbol{\zeta}\,\boldsymbol{\omega}_N) + \boldsymbol{H}(\boldsymbol{\zeta}\,\boldsymbol{\omega}_N)\mathfrak{s} = ([\mathfrak{s}, \boldsymbol{P}_+]\,\boldsymbol{\zeta})\,\boldsymbol{\omega}_N\,\boldsymbol{J}\boldsymbol{P}_+ + \boldsymbol{P}_+\,\boldsymbol{\omega}_N\,\boldsymbol{J}\,(\boldsymbol{\zeta}\,[\boldsymbol{P}_+, \mathfrak{s}])\,.$$

Let us now prove that the commutators $[\mathfrak{s}, \boldsymbol{P}_+]\,\boldsymbol{\zeta}$, $\boldsymbol{\zeta}\,[\mathfrak{s}, \boldsymbol{P}_+] \in \mathfrak{S}_1$. By our choice of $\mathfrak{s}$ and $\boldsymbol{\zeta}$, we have $\boldsymbol{J}\mathfrak{s} = -\mathfrak{s}\boldsymbol{J}$ and $\boldsymbol{J}\boldsymbol{\zeta} = \boldsymbol{\zeta}\boldsymbol{J}$, whereby it follows

$$\begin{aligned}[\mathfrak{s}, \boldsymbol{P}_+]\,\boldsymbol{\zeta} &= \mathfrak{s}\boldsymbol{P}_+\boldsymbol{\zeta} - \mathfrak{s}\boldsymbol{\zeta}\boldsymbol{P}_+ + \mathfrak{s}\boldsymbol{\zeta}\boldsymbol{P}_+ - \boldsymbol{P}_+\mathfrak{s}\boldsymbol{\zeta} = \mathfrak{s}\,[\boldsymbol{P}_+, \boldsymbol{\zeta}] + [\mathfrak{s}\boldsymbol{\zeta}, \boldsymbol{P}_+]\,,\\ \boldsymbol{\zeta}\,[\mathfrak{s}, \boldsymbol{P}_+] &= \boldsymbol{\zeta}\mathfrak{s}\boldsymbol{P}_+ - \boldsymbol{P}_+\mathfrak{s}\boldsymbol{\zeta} + \boldsymbol{P}_+\mathfrak{s}\boldsymbol{\zeta} - \boldsymbol{\zeta}\boldsymbol{P}_+\mathfrak{s} = [\mathfrak{s}\boldsymbol{\zeta}, \boldsymbol{P}_+] + [\boldsymbol{P}_+, \boldsymbol{\zeta}]\,\mathfrak{s}.\end{aligned}$$

Moreover, our choice of $\boldsymbol{\zeta}$ gives that the product $\mathfrak{s}\boldsymbol{\zeta} \in C^\infty(\hat{\mathbb{R}})$, thus Lemma 3.3.3-(ii) imply that $[\mathfrak{s}, \boldsymbol{P}_+]\,\boldsymbol{\zeta}$, $\boldsymbol{\zeta}\,[\mathfrak{s}, \boldsymbol{P}_+] \in \mathfrak{S}_1$. Finally Lemma 5.3.5 and the triangle inequality give

$$\sup_N \|\mathfrak{s}\boldsymbol{H}(\boldsymbol{\zeta}\,\boldsymbol{\omega}_N) + \boldsymbol{H}(\boldsymbol{\zeta}\,\boldsymbol{\omega}_N)\mathfrak{s}\|_{\mathfrak{S}_1} \leq \|\,\boldsymbol{\omega}\,\|_{L^\infty(\mathbb{R})}(\|\,[\mathfrak{s}, \boldsymbol{P}_+]\,\boldsymbol{\zeta}\|_{\mathfrak{S}_1} + \|\boldsymbol{\zeta}\,[\boldsymbol{P}_+, \mathfrak{s}]\,\|_{\mathfrak{S}_1}) < \infty. \tag{5.3.54}$$

Putting together (5.3.52), (5.3.53) and (5.3.54) and using the triangle inequality on (5.3.51) gives the assertion. □

3.5. Spectral density of the model operators. Before moving on to finding an explicit formula for the spectral densities of the model operators, we introduce the following terminology:

DEFINITION 5.3.7. We say $\underline{\tau} = (\varphi_0, \varphi_\infty)$ is an *admissible* couple for $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ if it satisfies (B)-(D) in Assumptions 5.1.2 with $a_0 = b_\infty = 1$, $a_\infty = 0$ and when non-zero, $b_0 = 1$.

Similarly, we say $\underline{\tau} = (\varphi_0, \varphi_\infty)$ is an *admissible* couple for $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$ if it satisfies (B)-(D) in Assumptions 5.1.2 with $a_0 = b_\infty = 1$, $b_0 = 0$ and when non-zero, $a_\infty = 1$.

PROPOSITION 5.3.8. *Let $\underline{\tau}$ be an admissible couple for $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$, then for $t > 0$*

$$\mathsf{LD}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = \frac{1}{\pi^2}\operatorname{sech}^{-1}(t). \tag{5.3.55}$$

Furthermore, if $\tau_N(t,s) = \overline{\tau_N(s,t)}$ for all N, we have for $t > 0$:

$$\mathsf{LD}^+_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = \frac{1}{\pi^2}\operatorname{sech}^{-1}(t), \quad \mathsf{LD}^-_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = 0, t > 0. \tag{5.3.56}$$

Similarly, if $\underline{\tau}$ is an admissible couple for $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$, then

$$\mathsf{LD}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = \frac{1}{\pi^2}\operatorname{sech}^{-1}(t), \quad t > 0. \tag{5.3.57}$$

If $\tau_N(t,s) = \overline{\tau_N(s,t)}$ for all N, we have:

$$\mathsf{LD}^+_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = \frac{1}{\pi^2}\operatorname{sech}^{-1}(t), \quad \mathsf{LD}^-_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = 0, \quad t > 0. \tag{5.3.58}$$

PROOF. Let us begin with (5.3.55). The couple $\underline{\sigma} = (e^{-(s+t)}, 1)$ is admissible for $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$. Thus, for any other admissible couple $\underline{\tau}$, Theorem 5.2.1-(i) gives that

$$\overline{\mathsf{LD}}_{\underline{\sigma}}(t+0; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) \le \overline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) \le \overline{\mathsf{LD}}_{\underline{\sigma}}(t-0; \boldsymbol{\Gamma}(\widehat{\gamma}_0)), \tag{5.3.59}$$

$$\underline{\mathsf{LD}}_{\underline{\sigma}}(t+0; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) \le \underline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) \le \underline{\mathsf{LD}}_{\underline{\sigma}}(t-0; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) \tag{5.3.60}$$

Similar inequalities hold for the upper and lower logarithmic spectral densities of $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ if $\tau_N(t,s) = \overline{\tau_N(s,t)}$. It is therefore sufficient to find the logarithmic spectral density of $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ with respect to $\underline{\sigma}$. An argument similar to the one presented at the end of Chapter 4, exploiting [**22**, Lemma 4.1], shows that

$$\mathsf{LD}_{\underline{\sigma}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = \mathsf{LD}^+_{\underline{\sigma}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = \frac{1}{\pi^2}\operatorname{sech}^{-1}(t), \tag{5.3.61}$$

$$\mathsf{LD}^-_{\underline{\sigma}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_0)) = 0. \tag{5.3.62}$$

Similarly, to prove (5.3.57), we choose the couple $\underline{\sigma} = (1, 1 - e^{-(s+t)})$. So for any other admissible couple for $\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)$, Theorem 5.2.1-(ii) gives that

$$\overline{\mathsf{LD}}_{\underline{\sigma}}(t+0; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) \le \overline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) \le \overline{\mathsf{LD}}_{\underline{\sigma}}(t-0; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)), \tag{5.3.63}$$

$$\underline{\mathsf{LD}}_{\underline{\sigma}}(t+0; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) \le \underline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) \le \underline{\mathsf{LD}}_{\underline{\sigma}}(t-0; \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)). \tag{5.3.64}$$

The same inequalities also hold for the upper and lower inequalities if $\tau_N(t,s) = \overline{\tau_N(s,t)}$. Therefore, analogously to the case of $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$, an application of [22, Lemma 4.1] together with the arguments in the Appendix of Chapter 4 gives

$$\mathsf{LD}_{\underline{\sigma}}(t;\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = \mathsf{LD}^+_{\underline{\sigma}}(t;\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = \frac{1}{\pi^2}\operatorname{sech}^{-1}(t), \tag{5.3.65}$$

$$\mathsf{LD}^-_{\underline{\sigma}}(t;\boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) = 0. \tag{5.3.66}$$

□

Corollary 5.3.9. *Let $v \in \mathbb{R}$ and let $\boldsymbol{\gamma}_v(t) = \boldsymbol{\gamma}_0(t-v)$ and let $\underline{\tau}$ be admissible for $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$. Then identities (5.3.55) and (5.3.56) also hold for $\boldsymbol{\Gamma}(\widehat{\gamma}_v)$.*

Proof. If we choose $\underline{\sigma} = (e^{-(s+t)}, 1)$, (5.3.40) yields

$$\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_v) = U_v(\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_0))U_v, \tag{5.3.67}$$

where U_v is the unitary operator of multiplication by the function e^{-ivt}. Consequently, for the singular values of $\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_v)$ we have

$$s_n(\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_v)) = s_n(\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_0)), \quad n \geq 1. \tag{5.3.68}$$

Thus we obtain that $\mathsf{n}(t;\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_v)) = \mathsf{n}(t;\sigma_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_0))$. So for any admissible couple $\underline{\tau}$, the result follows at once from the definition of the functions $\mathsf{LD}_{\underline{\tau}}, \mathsf{LD}^{\pm}_{\underline{\tau}}$, Theorem 5.2.1-(i) and Proposition 5.3.8. □

4. Proof of Theorem 5.1.3

The proof of the result will be broken down in two Steps. We also recall that Ω is the set of jump-discontinuities of the symbol $\boldsymbol{\omega}$ and c is the function in (5.1.15).
Step 1. Finitely many jumps. Suppose that Ω is finite. Setting $\boldsymbol{\gamma}_v(x) = -i\gamma_0(x-v)$, with $\boldsymbol{\gamma}_0$ being the symbol defined in (5.1.10), write

$$\boldsymbol{\omega}(x) = \sum_{v\in\Omega\setminus\{\infty\}} \varkappa_z(\boldsymbol{\omega})\gamma_v(x) + \varkappa_\infty(\boldsymbol{\omega})\gamma_\infty(x) + \boldsymbol{\eta}(x), \tag{5.4.69}$$

where $\boldsymbol{\eta}$ is continuous on $\hat{\mathbb{R}}$. Of course, $\varkappa_\infty(\boldsymbol{\omega})$ may be zero, in which case the corresponding quantity does not appear in (5.4.69). Let $\boldsymbol{\Psi}$ denote the symbol

$$\boldsymbol{\Psi}(x) = \sum_{v\in\Omega\setminus\{\infty\}} \varkappa_v(\boldsymbol{\omega})\gamma_v(x) + \varkappa_\infty(\boldsymbol{\omega})\gamma_\infty(x),$$

then using Weyl's inequality (2.4.18) one obtains the inequalities

$$\mathsf{n}(t+s;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) - \mathsf{n}(s;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})) \leq \mathsf{n}(t;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})),$$

$$\mathsf{n}(t;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \leq \mathsf{n}(t-s;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) + \mathsf{n}(s;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})),$$

where $0 < s < t$. Since $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})$ is compact, Lemma 2.4.4 shows that

$$\mathsf{n}(s;\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})) = O_s(1), \quad N \to \infty$$

and so, using the definition of the functionals $\underline{\mathsf{LD}}_{\mathcal{T}}, \overline{\mathsf{LD}}_{\mathcal{T}}$ we deduce that for any $t > 0$

$$\overline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \le \overline{\mathsf{LD}}_{\mathcal{T}}(t-0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})), \tag{5.4.70}$$

$$\underline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \ge \underline{\mathsf{LD}}_{\mathcal{T}}(t+0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})). \tag{5.4.71}$$

By the linearity of the Fourier Transform, one has that

$$\begin{aligned}\widehat{\boldsymbol{\Psi}}(t) &= \sum_{v\in\Omega\setminus\{\infty\}} \varkappa_v(\boldsymbol{\omega})\widehat{\gamma}_v(t) + \varkappa_\infty(\boldsymbol{\omega})\widehat{\gamma}_\infty(t) \\ &= \frac{1-e^{-t}}{\pi t} \sum_{v\in\Omega\setminus\{\infty\}} \varkappa_v(\boldsymbol{\omega})e^{-ivt} + \varkappa_\infty(\boldsymbol{\omega})\frac{e^{-t}}{\pi t}.\end{aligned} \tag{5.4.72}$$

Thus $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}}) \in \mathfrak{B}_0$. Thus, using the couple $\mathcal{T}_\square = (\varphi_0, \varphi_\infty)$ defined in (5.1.22), which induces the square trancution discussed in the Introduction, and Theorem 5.2.1, applied to the operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})$, gives

$$\overline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) \le \overline{\mathsf{LD}}_\square(t-0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})), \tag{5.4.73}$$

$$\underline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) \ge \underline{\mathsf{LD}}_\square(t+0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})). \tag{5.4.74}$$

For brevity, set $\boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v) = (\tau_\square)_N \star \boldsymbol{\Gamma}(\widehat{\gamma}_v)$. Now Theorem 5.3.2 and Theorem 5.3.3, give that whenever $v \ne w$

$$\sup_{N\ge1} \| \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v)^* \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_w)\|_{\mathfrak{S}_2} < \infty,$$

$$\sup_{N\ge1} \| \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_v) \, \boldsymbol{\Gamma}^{(N)}(\widehat{\gamma}_w)^*\|_{\mathfrak{S}_2} < \infty.$$

Thus the Asymptotic Orthogonality Theorem 2.4.7, implies that for $t > 0$

$$\begin{aligned}\overline{\mathsf{LD}}_\square(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) \le &\sum_{v\in\Omega\setminus\{\infty\}} \overline{\mathsf{LD}}_\square(t-0; \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)) \\ &+ \overline{\mathsf{LD}}_\square(t-0; \varkappa_\infty(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)),\end{aligned} \tag{5.4.75}$$

$$\begin{aligned}\underline{\mathsf{LD}}_\square(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) \ge &\sum_{v\in\Omega\setminus\{\infty\}} \underline{\mathsf{LD}}_\square(t+0; \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)) \\ &+ \underline{\mathsf{LD}}_\square(t+0; \varkappa_\infty(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)).\end{aligned} \tag{5.4.76}$$

Finally, since the couple $\tau_\square$ is admissible for both $\boldsymbol{\Gamma}(\widehat{\gamma}_0)$ and $\boldsymbol{\Gamma}(\gamma_\infty)$, Proposition 5.3.8 and its Corollary 5.3.9 together with (5.4.70), (5.4.71), (5.4.73), (5.4.74), (5.4.75) and (5.4.76) and the continuity of c at $t \ne 0$ give that

$$\overline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \le \sum_{v\in\Omega} \mathsf{c}\left(\frac{t}{|\varkappa_v(\boldsymbol{\omega})|}\right),$$

$$\underline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \ge \sum_{v\in\Omega} \mathsf{c}\left(\frac{t}{|\varkappa_v(\boldsymbol{\omega})|}\right).$$

The obvious inequality $\underline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \le \overline{\mathsf{LD}}_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}))$ proves the assertion.

REMARK 5.4.1. We note that (5.4.75) and (5.4.76) hold if we consider any symbol $\boldsymbol{\omega}$ which is smooth except for a finite set of jumps discontinuities. These two together are yet another instance of the Localisation principle we referred to in the Introduction.

Step 2. From finitely many to infinitely many jumps. Suppose now that Ω is infinite. Define the sets:

$$\Omega_0 = \{v \in \hat{\mathbb{R}} \mid |\varkappa_z(\boldsymbol{\omega})| \geq 2^{-1}\},$$
$$\Omega_n = \{v \in \hat{\mathbb{R}} \mid 2^{-n-1} \leq |\varkappa_v(\boldsymbol{\omega})| < 2^{-n}\}, \quad n \geq 1.$$

As we mentioned earlier, these are finite. Let $\boldsymbol{\psi}_n$ be bounded functions such that $\boldsymbol{\psi}_n \in C^\infty(\hat{\mathbb{R}} \setminus \Omega_n)$, $\varkappa_v(\boldsymbol{\psi}_n) = \varkappa_v(\boldsymbol{\omega})$ for any $v \in \Omega_n$ and such that

$$\|\boldsymbol{\psi}_n\|_\infty = \max_{v\in\Omega_n} |\varkappa_v(\boldsymbol{\omega})| .$$

Let $\boldsymbol{\Psi} = \sum_{n\geq 0} \boldsymbol{\psi}_n \in L^\infty(\hat{\mathbb{R}})$. Since $\boldsymbol{\omega} - \boldsymbol{\Psi} \in C(\hat{\mathbb{R}})$, the operator $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}} - \widehat{\boldsymbol{\Psi}}) \in \mathfrak{S}_\infty$ and so, by Lemma 2.4.4 once again we obtain

$$\overline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \leq \overline{\mathsf{LD}}_{\underline{\tau}}(t-0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})),$$
$$\underline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \geq \underline{\mathsf{LD}}_{\underline{\tau}}(t+0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})).$$

For a fixed $s > 0$, let M be so that $\|\boldsymbol{\Psi} - \boldsymbol{\Psi}_M\|_\infty < s$, where $\boldsymbol{\Psi}_M = \sum_{n=0}^M \boldsymbol{\varphi}_n$. The uniform boundedness of $\underline{\tau}$ then gives

$$\|\tau_N \star (\Gamma(\widehat{\boldsymbol{\Psi}}) - \Gamma(\widehat{\boldsymbol{\Psi}}_M))\| \leq \left(\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}}\right) \|\boldsymbol{\Psi} - \boldsymbol{\Psi}_M\|_\infty < \left(\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}}\right) s := s'.$$

Letting $\widetilde{\Omega}_M = \cup_{n=0}^M \Omega_n$, we then obtain that:

$$\overline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \leq \overline{\mathsf{LD}}_{\underline{\tau}}(t-s'; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}}_M)) = \sum_{v\in\widetilde{\Omega}_M} \mathsf{c}\left(\frac{t-s'}{|\varkappa_v(\boldsymbol{\omega})|}\right),$$
$$\underline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \geq \underline{\mathsf{LD}}_{\underline{\tau}}(t+s'; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}}_M)) = \sum_{v\in\widetilde{\Omega}_M} \mathsf{c}\left(\frac{t+s'}{|\varkappa_v(\boldsymbol{\omega})|}\right).$$

Since the symbol $\boldsymbol{\Phi}_M$ has finitely many jumps, *Step 1.* gives the equalities above. Finally, sending $s \to 0$ and noting that there are only finitely many $v \in \Omega$ such that $t \leq |\varkappa_v(\boldsymbol{\omega})|$, one obtains

$$\underline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = \overline{\mathsf{LD}}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) = \sum_{v\in\Omega} \mathsf{c}\left(\frac{t}{|\varkappa_v(\boldsymbol{\omega})|}\right).$$

5. Proof of Theorem 5.1.4

Just as in the proof of Theorem 5.1.3, we break the argument into two steps, and use the same notation as before for the operator $\tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ and for the symbols $\boldsymbol{\gamma}_v$. We also set $\Omega^+ = \{v \in \Omega \mid v > 0,\ |v| < \infty\}$.
Step 1. Finitely many jumps. Just as before, suppose that the symbol $\boldsymbol{\omega}$ has finitely-many jump-discontinuities. Define for $v \in \Omega^+$ the function

$$\boldsymbol{\omega}_v(x) = \varkappa_v(\boldsymbol{\omega})\gamma_v(x) + \overline{\varkappa_v(\boldsymbol{\omega})}\gamma_{-v}(x),$$

then for some $\boldsymbol{\eta} \in C(\hat{\mathbb{R}})$, we can decompose ω as

$$\boldsymbol{\omega}(x) = \left(\varkappa_0(\boldsymbol{\omega})\gamma_0(x) + \varkappa_\infty(\boldsymbol{\omega})\gamma_\infty(x) + \sum_{v\in\Omega^+} \boldsymbol{\omega}_v(x)\right) + \boldsymbol{\eta}(x). \tag{5.5.77}$$

If $\boldsymbol{\omega}$ has no jump at 0 and ∞, the corresponding quantities do not appear in the above. Denoting by $\boldsymbol{\Psi}$ the sum in the brackets, Weyl inequality (2.4.19) gives for $0 < s < t$

$$\begin{aligned}\mathsf{n}_\pm(t+s; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) - \mathsf{n}(s; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})) &\le \mathsf{n}_\pm(t; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})),\\ \mathsf{n}_\pm(t; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) &\le \mathsf{n}_\pm(t-s; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) + \mathsf{n}_\pm(s; \tau_N \star \boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}})).\end{aligned}$$

By Lemma 2.4.4, we obtain that $\mathsf{n}_\pm(s; \tau_N \star \boldsymbol{\Gamma}(\widehat{\eta})) = O_s(1)$, and so, for any $t > 0$, it follows that

$$\overline{\mathsf{LD}}^\pm_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \le \overline{\mathsf{LD}}^\pm_{\mathcal{T}}(t-0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})), \tag{5.5.78}$$

$$\underline{\mathsf{LD}}^\pm_{\mathcal{T}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \ge \underline{\mathsf{LD}}^\pm_{\mathcal{T}}(t+0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})). \tag{5.5.79}$$

Since the symbol $\boldsymbol{\Psi}$ is so that

$$\left|\widehat{\boldsymbol{\Psi}}(t)\right| \le \frac{C}{t}, \quad t > 0,$$

then $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}}) \in \mathfrak{B}_0$ and so, by Theorem 5.2.1, it suffices to prove our result for the couple $\mathcal{T}_\square = (\varphi_0, \varphi_\infty)$ defined in (5.1.22), which, as we said before, induces the square truncation. Theorems 5.3.2 and 5.3.3 give that the operators

$$\varkappa_0(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_0),\ \ \varkappa_\infty(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_\infty),\ \ \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)$$

satisfy the conditions of the Asymptotic Orthogonality Theorem 2.4.7 and therefore we obtain that for $t > 0$

$$\begin{aligned}\overline{\mathsf{LD}}^{\pm}_{\square}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) \leq \sum_{v \in \Omega^+} \overline{\mathsf{LD}}^{\pm}_{\square}(t-0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)) \\ + \overline{\mathsf{LD}}^{\pm}_{\square}(t-0; \varkappa_0(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_0)) \\ + \overline{\mathsf{LD}}^{\pm}_{\square}(t-0; \varkappa_\infty(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_\infty))\end{aligned} \tag{5.5.80}$$

$$\begin{aligned}\underline{\mathsf{LD}}^{\pm}_{\square}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\Psi}})) \geq \sum_{v \in \Omega^+} \underline{\mathsf{LD}}^{\pm}_{\square}(t+0; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)) \\ + \underline{\mathsf{LD}}^{\pm}_{\square}(t+0; \varkappa_\infty(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_\infty)) \\ + \underline{\mathsf{LD}}^{\pm}_{\square}(t+0; \varkappa_0(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_0))\end{aligned} \tag{5.5.81}$$

Now, for $v = 0, \infty$, the operators $\varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)$ are sign definite, and furthermore one has that

$$\varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v) \geq 0 \text{ (resp. } \leq 0) \text{ if } -i\varkappa_v(\boldsymbol{\omega}) \geq 0 \text{ (resp. } \leq 0).$$

In either case, Proposition 4.3.5 gives that

$$\begin{aligned}\overline{\mathsf{LD}}^{\pm}_{\square}(t; \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)) &= \mathbb{1}_{\pm}(-i\varkappa_v(\boldsymbol{\omega}))\overline{\mathsf{LD}}_{\square}(t; \varkappa_v(\boldsymbol{\omega})\, \boldsymbol{\Gamma}(\widehat{\gamma}_v)) \\ &= \mathbb{1}_{\pm}(-i\varkappa_v(\boldsymbol{\omega}))\mathsf{c}\left(t\, |\varkappa_v(\boldsymbol{\omega})|^{-1}\right)\end{aligned} \tag{5.5.82}$$

where $\mathbb{1}_{\pm}$ is the indicator function of $\mathbb{R}_{\pm} = (0, \pm\infty)$.
Puttin together the results of Theorem 5.3.6, the Almost Symmetry Theorem 2.4.9 and Theorem 5.1.3 just proved, we get that for any $v \in \Omega^+$

$$\begin{aligned}\overline{\mathsf{LD}}^{\pm}_{\square}\left(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)\right) &= \frac{1}{2}\overline{\mathsf{LD}}_{\square}\left(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_v)\right) \\ &= \mathsf{c}\left(t\, |\varkappa_v(\boldsymbol{\omega})|^{-1}\right).\end{aligned} \tag{5.5.83}$$

Using (5.5.82) and (5.5.83) in (5.5.78), (5.5.79), (5.5.80) and (5.5.81), and by the continuity of the function c at $t \neq 0$ we finally arrive at the sought-after equality (5.1.31).

Remark 5.5.1. As we wrote earlier in the Introduction, if the symbol satisfies (5.1.3) and has a pair of jumps at $\pm v$, then (5.5.83) shows that the upper and lower logarithmic spectral density of $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})$ contribute equally to the logarithmic spectral density of $|\boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})|$. This is an effect of what is known in the literature as the Symmetry Principle, mentioned in the Introduction.

Step 2. From finitely many to infinitely many jumps. For fixed $s > 0$, define the set

$$\Omega_s^+ = \{v \in \Omega \mid |\varkappa_v(\boldsymbol{\omega})| > s \text{ and } v > 0\}.$$

Just as in *Step 2.* in the proof of Theorem 5.1.3, we can find a symbol $\boldsymbol{\omega}_s \in PC(\hat{\mathbb{R}})$ so that $\| \boldsymbol{\omega} - \boldsymbol{\omega}_s \|_\infty < s$, the set of its discontinuities is precisely $\Omega_s^+ \cup \{0, \infty\}$ and

$$\varkappa_v(\boldsymbol{\omega}) = \varkappa_v(\boldsymbol{\omega}_s), \quad \forall v \in \Omega_s^+ \cup \{0, \infty\}.$$

The set $\Omega_s^+ \cup \{0, \infty\}$ is finite, thus from Weyl inequality (2.4.19) and *Step 1.* we obtain

$$\overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \leq \overline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t - s'; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_s))$$
$$\underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}})) \geq \underline{\mathsf{LD}}^{\pm}_{\underline{\tau}}(t + s'; \boldsymbol{\Gamma}(\widehat{\boldsymbol{\omega}}_s)),$$

where $s' = \left(\sup_{N\geq 1} \|\tau_N\|_{\mathfrak{M}}\right) s$. Finally, sending $s \to 0$ and using the continuity of c at $t \neq 0$ establishes the result in its generality. □

Proof of Proposition 5.1.7. The same reasoning of *Step 1.* in both proofs above applies in this case, with only one minor change. Since we assume that the couple $\underline{\tau} = (\varphi_0, \varphi_\infty)$ induces a uniformly bounded multiplier on $\mathfrak{S}_p$, $p > 1$, i.e. that (5.1.32) holds, in (5.4.69) and (5.5.77) we need to assume that $\boldsymbol{\eta}$ is a symbol so that $\boldsymbol{\Gamma}(\widehat{\boldsymbol{\eta}}) \in \mathfrak{S}_p$. Then Lemma 2.4.4 shows that $\mathsf{n}(s; \boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\eta}})) = O_s(1)$ and, in the self-adjoint case $\mathsf{n}_\pm(s; \boldsymbol{\Gamma}^{(N)}(\widehat{\boldsymbol{\eta}})) = O_s(1)$. The rest follows immediately. □

Proof of Proposition 5.1.8. Exactly the same reasoning of the proofs of Theorems 5.1.3 and 5.1.4 above applies in this case, with the only difference being that in this case the couple $\underline{\tau}$ is no longer inducing a uniformly bounded multiplier on the whole space of bounded operators, just on the space of bounded Hankel integral operators. However, all of the terms appearing in the arguments just presented are bounded Hankel operators and so the same arguments apply in this case. □

Part III

Asymptotics of determinants of Identity minus Hankel matrices and weighted integral Hankel operators.

CHAPTER 6

On determinants of the Identity minus a Hankel matrix

1. Introduction and results

Given a bounded function f on the unit circle $\mathbb{T} := \{v \in \mathbb{C} : |v| = 1\}$, the associated Hankel matrix $H(f) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, is given by its matrix elements

$$(H(f))(j,k) = \widehat{f}(j+k), \qquad j+k \in \mathbb{Z}_+, \tag{6.1.1}$$

where for $k \in \mathbb{Z}_+$

$$\widehat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-ikt}dt. \tag{6.1.2}$$

The function f is called the *symbol* of the matrix $H(f)$.

Throughout this Chapter, we restrict our attention to symbols which have finitely-many jump discontinuities and satisfy $\sup_{z\in\mathbb{T}} |f(z)| \le 1$. We will make more precise assumptions later on, see conditions (A) - (C). Hankel matrices with jump discontinuities in their symbols are well-studied [**46, 49, 50, 61**] and still attract attention in the operator theory community, see e.g. [**51**].

Our goal here is to study the large N behaviour of $\det(\mathbb{I}_N - \beta H_N(f))$, where $H_N(f)$ is the $N \times N$ restriction of the Hankel matrix $H(f)$, $\beta \in \mathbb{C}$ so that $|\beta| < 1$ and $\mathbb{I}_N$ is the identity matrix. Our assumption on the boundedness of the symbol ensures that $\|H(f)\| \le 1$ and so $\|\beta H(f)\| \le |\beta| < 1$. We will compute the first order term in its asymptotic expansion for large N and show that

$$\det(\mathbb{I}_N - \beta H_N(f)) = N^{-\gamma_f(\beta)+o(1)} \tag{6.1.3}$$

as $N \to \infty$, where the exponent $\gamma_f(\beta)$ depends explicitly on the location of the jumps as well as their height, see our main result Theorem 6.1.3 for the precise formulation.

To illustrate our result, consider the following two explicit Hankel matrices

$$\mathbf{H} := \left\{\frac{1}{\pi(j+k+1)}\right\}_{j,k\ge 0}, \qquad \mathbf{S} := \left\{\frac{\sin(\pi(j+k)/2)}{\pi(j+k)}\right\}_{j,k\ge 0}, \tag{6.1.4}$$

with the convention that $\mathbf{S}(0,0) = 1/2$, where the symbols ψ and η are given in Example 6.1.2 later on. The matrix $\mathbf{H}$ is the *Hilbert matrix* and is well-known in the literature. For these Hankel matrices Theorem 6.1.3 states that the asymptotic formula (6.1.3) holds with

$$\gamma_\psi(\beta) = \frac{1}{2\pi^2}\left(\pi \arcsin(\beta) + \arcsin^2(\beta)\right) \quad \text{and} \quad \gamma_\eta(\beta) = \frac{1}{\pi^2}\arcsin^2\left(\frac{\beta}{2}\right). \qquad (6.1.5)$$

The different expressions for $\gamma_\psi(\beta)$ and $\gamma_\eta(\beta)$ in the two cases are related to their symbols having jumps located differently on $\mathbb{T}$. In the case of $\mathbf{H}$, the symbol has only one jump located at 1, causing the appearance of both the linear and the quadratic arcsin term in (6.1.5). In the case of $\mathbf{S}$, however, the symbol has jumps at the conjugate points $\pm i$ and the linear arcsin term does not appear. In general, only jumps at $\pm 1 \in \mathbb{T}$ will cause a linear arcsin term whereas an $\arcsin^2$ term will always occur if there are jumps at conjugate points on $\mathbb{T}$. We also note that $\gamma_\psi(\beta) < 0$ for $\beta \in (-1, 0)$ and therefore we have in this case power-like growth in (6.1.3).

The problem which we study here fits into the more general framework of asymptotics of determinants of Hankel, Toeplitz and Hankel plus Toeplitz matrices. These are well-studied objects, see for example [**4**, **5**, **6**, **16**, **17**] and references therein. Exhaustive answers to various questions related to the asymptotics of Toeplitz and Hankel determinants have been found, however the behaviour of completely general Hankel plus Toeplitz determinants is not entirely understood yet. In most known results, the Hankel and Toeplitz matrix are related to the same symbol. We prove here a first order asymptotic formula for a simple class of Hankel plus Toeplitz determinants which, to the best of our knowledge, does not fall directly in the cases considered before.

We end the introduction with a word about the proof. The first step in studying our problem is to make use of the series expansion of the logarithm $\log(1-v) = -\sum_{n\in\mathbb{N}} v^n/n$ valid for $|v| < 1$ which implies

$$\log\det\left(\mathbb{I}_N - \beta H_N(f)\right) = \operatorname{Tr}\log\left(\mathbb{I}_N - \beta H_N(f)\right) = \sum_{n\in\mathbb{N}} \beta^n \operatorname{Tr} H_N(f)^n/n. \qquad (6.1.6)$$

The fact that the series expansion is only valid for $|v| < 1$ is the reason why we take $|\beta| < 1$. The asymptotic behaviour of $\operatorname{Tr} H_N(f)^n$ is found in Lemma 6.3.1, and it partially follows from [**61**, Theorem 4.3], where this was obtained for the simpler case of the Hilbert matrix with only one jump in its symbol. Surprisingly, the first order contributions in the asymptotic expansion of $\operatorname{Tr} H_N(f)^n$ are the coefficients of the power series of arcsin and $\arcsin^2$ times $\log N$, see Proposition 6.2.3.

1.1. Model and results. As we saw earlier on, the Hankel matrix $H(f) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is determined by its matrix elements

$$(H(f))(j,k) = \widehat{f}(j+k), \quad j,k \in \mathbb{Z}_+, \qquad (6.1.7)$$

where $\widehat{f}(k)$ is defined in (6.1.2) for $k \in \mathbb{Z}_+$. It is clear that $H(f)$ depends linearly on f. Throughout the Chapter, we make the following assumptions on the symbol f:

(A) for all $z \in \mathbb{T}$ the following limits exist

$$f(z^+) := \lim_{\varepsilon \to 0+} f(e^{i\varepsilon}z) \qquad \text{and} \qquad f(z^-) := \lim_{\varepsilon \to 0+} f(e^{-i\varepsilon}z). \tag{6.1.8}$$

The points where the limits do not coincide are called *jump-discontinuities* and we only assume a finite number of them.

(B) with Ω denoting the set of all discontinuities of f, we assume that $f \in C^\gamma(\mathbb{T}\backslash\Omega)$, for some $1/2 < \gamma \leq 1$ and some $C > 0$ that for all $\delta > 0$ and all $z \in \Omega$

$$|f(z^+) - f(e^{i\delta}z)| \leq C\delta^\gamma \quad \text{and} \quad |f(z^-) - f(e^{-i\delta}z)| \leq C\delta^\gamma; \tag{6.1.9}$$

(C) it satisfies the bound

$$\sup_{v \in \mathbb{T}} |f(v)| \leq 1. \tag{6.1.10}$$

We write $f \in PD(\mathbb{T})$ if it satisfies all of the above assumptions. For future reference we define for $z \in \mathbb{T}$

$$\varkappa_z(f) := \frac{f(z^+) - f(z^-)}{2} \tag{6.1.11}$$

and refer to this as the *half-height of the jump*.

Remark 6.1.1.

(i) Assumption (B) is only of technical nature. It simplifies the proofs and for most relevant examples these assumptions are satisfied.

(ii) The bound (6.1.10) guarantees $|\varkappa_z(f)| \leq 1$ for all $z \in \mathbb{T}$ and that the operator $H(f)$ satisfies $\|H(f)\| \leq 1$, see [46].

Example 6.1.2. The most important example of a symbol fitting in our framework is given by

$$\psi(e^{it}) = i\pi^{-1}e^{-it}(\pi - t), \quad t \in [0, 2\pi). \tag{6.1.12}$$

Integration by parts shows that this is a symbol for the Hilbert matrix given in (6.1.4), i.e. that $\mathbf{H} = H(\psi)$. Another example of a function in this class is given by

$$\eta(e^{it}) = 1_{\{\cos t > 0\}}, \quad t \in [0, 2\pi) \tag{6.1.13}$$

with jump-discontinuities at the points $\pm i$. This is a symbol for the Hankel matrix $\mathbf{S} = H(\eta)$ given in (6.1.4).

As before, let $H_N(f)$ denote the $N \times N$ restriction of the infinite matrix $H(f)$, i.e. let $H_N(f) := \mathbb{1}_N H(f) \mathbb{1}_N$, where $\mathbb{1}_N$ is the orthogonal projection onto the span of $\{e_j\}_{j=0}^{N-1}$, where e_j, $j \in \mathbb{Z}_+$, are the standard basis vectors of $\ell^2(\mathbb{Z}_+)$. Setting

$$D_N^\beta(f) := \det\left(\mathbb{I}_N - \beta H_N(f)\right), \tag{6.1.14}$$

our result can be stated as follows

THEOREM 6.1.3. *Suppose $f \in PD(\mathbb{T})$. Let $\Omega \subset \mathbb{T}$ be the set of its jump discontinuities. For $\beta \in \mathbb{C}$ with $|\beta| < 1$ we have*

$$\log D_N^\beta(f) = -\frac{\gamma_f(\beta)}{2\pi^2} \log N + o(\log N), \tag{6.1.15}$$

as $N \to \infty$, where

$$\begin{aligned} \gamma_f(\beta) :=& \pi\big(\arcsin\big(-i\beta\varkappa_1(f)\big) + \arcsin(-i\beta\varkappa_{-1}(f))\big) \\ &+ \sum_{z\in\Omega} \arcsin^2(-i\beta(\varkappa_z(f)\varkappa_{\overline{z}}(f))^{1/2}). \end{aligned} \tag{6.1.16}$$

REMARK 6.1.4. (i) The expression in (6.1.16) is independent of the choice of the analytic branch of the square root. This follows from the power series

$$\arcsin^2(v) = \frac{1}{2}\sum_{m=1}^{\infty} \frac{(m!)^2\, 4^m v^{2m}}{(2m)!m^2}, \quad |v| \le 1, \tag{6.1.17}$$

which implies that $\arcsin^2$ is a function of v^2.

(ii) It is evident that we have a non-zero contribution in (6.1.15) only if both z and $\overline{z}$ are jump-discontinuities of the symbol f. For example, in the case of selfadjoint Hankel matrices the jump discontinuities only appear in pairs $z, \overline{z}$ and there is always a non-trivial contribution. Moreover, the terms $\arcsin\left(-i\beta\varkappa_{\pm1}(f)\right)$ only appear for jumps at $z = \pm1$. This is yet another manifestation of the subtle differences between jumps at $z = \pm1$ compared to those located at $z \in \mathbb{T} \setminus \{\pm1\}$, see for example [**50, 54**].

(iii) Even though the proof of the theorem relies on $|\beta| < 1$, we believe that the above asymptotic formula holds for $|\beta| = 1$. Indeed, using different methods, this can be achieved for the special case of the Hilbert matrix **H** given in (6.1.4). In this case one can prove

$$\log\det\left(\mathbb{I}_N - \mathbf{H}_N\right) = -\frac{\gamma_\psi(1)}{2\pi^2}\log N + o\left(\log N\right) \tag{6.1.18}$$

as $N \to \infty$, see [**24**], where

$$\frac{\gamma_\psi(1)}{2\pi^2} = \frac{1}{2\pi^2}\left(\pi \arcsin(1) + \arcsin^2(1)\right) = \frac{3}{8}. \tag{6.1.19}$$

The authors use the explicit diagonalization of the Hilbert matrix and their methods cannot immediately be generalized to arbitrary Hankel matrices with jump discontinuities in the symbol considered here.

Using our methods, one can also consider asymptotics of determinants related to powers of Hankel matrices. For instance, one can prove the following

COROLLARY 6.1.5. *Let $0 \le \beta < 1$ and, as before, denote by **H** the Hilbert matrix. Then as $N \to \infty$*

$$\log\det\left(\mathbb{I}_N - \beta^2 \mathbb{1}_N \mathbf{H}^2 \mathbb{1}_N\right) = -\frac{1}{\pi^2}\arcsin^2(\beta)\log N + o(\log N). \tag{6.1.20}$$

REMARK 6.1.6. Determinants of the above form appear in the study of the asymptotic behaviour of ground-state overlaps of many-body fermionic systems. In this context Corollary 6.1.5 gives a partial answers to a question asked in [36, Rmk. 2.7]. We will not explain the problem here and refer to [22, 23] for a precise formulation and further reading about the relation of the problem to determinants of Hankel operators.

2. Proof of Theorem 6.1.3

In the following we denote by $\mathfrak{S}_p$ the standard Schatten-p-class and by $\|\cdot\|_p$ its norm for $p \geq 1$.

Let $f \in PD(\mathbb{T})$ and we write for brevity $\varkappa_z = \varkappa_z(f)$. Since the operator norm of Hankel operators satisfies $\|H(f)\| \leq \sup_{v\in\mathbb{T}} |f(v)|$ and we assumed $\sup_{v\in\mathbb{T}} |f(v)| \leq 1$, we obtain for all $N \in \mathbb{N}$ that

$$\|\beta H_N(f)\| \leq |\beta| < 1. \tag{6.2.21}$$

Hence using the series expansion

$$\log(1-v) = -\sum_{k\in\mathbb{N}} \frac{v^k}{k} \tag{6.2.22}$$

valid for all $|v| < 1$, we obtain that

$$\log\det(\mathbb{I}_N - \beta H_N(f)) = -\sum_{k=1}^{\infty} \beta^k \frac{\operatorname{Tr} H_N(f)^k}{k}. \tag{6.2.23}$$

In the next step we compute the asymptotics of $\operatorname{Tr} H_N(f)^k$ when $N \to \infty$. This is the main part of the proof.

THEOREM 6.2.1. *Let $f \in PD(\mathbb{T})$ and Ω the set of its jumps discontinuities. We denote by B the Beta function. Then, for $k \in \mathbb{N}$ odd, we obtain*

$$\operatorname{Tr} H_N(f)^k = \frac{\varkappa_1^k + \varkappa_{-1}^k}{2\pi^2}(-i)^k B\Big(\frac{k}{2}, \frac{1}{2}\Big) \log N + o(\log N) \tag{6.2.24}$$

as $N \to \infty$ and for $k \in \mathbb{N}$ even we obtain

$$\operatorname{Tr} H_N(f)^k = \sum_{z\in\Omega} \frac{(\varkappa_z \varkappa_{\overline{z}})^{k/2}}{2\pi^2}(-i)^k B\Big(\frac{k}{2}, \frac{1}{2}\Big) \log N + o(\log N) \tag{6.2.25}$$

as $N \to \infty$.

In particular, it follows that for any $k \in \mathbb{N}$ the following limits exists

$$\lim_{N\to\infty} \frac{\operatorname{Tr} H_N(f)^k}{\log N} =: \mu_k(f) \tag{6.2.26}$$

where $\mu_k(f) \in \mathbb{C}$ is given in (6.2.24), respectively (6.2.25). Moreover, we need the following proposition.

PROPOSITION 6.2.2. *Let $f \in PD(\mathbb{T})$. Then*

$$\limsup_{N\to\infty} \frac{\|H_N(f)\|_2^2}{\log N} < \infty. \tag{6.2.27}$$

We prove Theorem 6.2.1 and Proposition 6.2.2 in Section 4. Theorem 6.2.1 might be of independent interest. We also need one more proposition to prove Theorem 6.1.3.

PROPOSITION 6.2.3. *Let $|v| \leq 1$, then the series*

$$S(v) := \sum_{m=1}^{\infty} v^m \frac{B\left(\frac{m}{2}, \frac{1}{2}\right)}{2\pi^2 m} \qquad \text{and} \qquad T(v) := \sum_{m=1}^{\infty} v^{2m} \frac{B\left(m, \frac{1}{2}\right)}{4\pi^2 m}.$$

are absolutely convergent and the following identities hold

$$S(v) = \frac{1}{2\pi}\arcsin(v) + \frac{1}{2\pi^2}\arcsin^2(v) \quad \text{and} \quad T(v) = \frac{1}{2\pi^2}\arcsin^2(v). \tag{6.2.28}$$

REMARK 6.2.4. The series $S(v)$ can also be written as an integral. A computation shows that

$$\frac{1}{\pi}\int_0^\infty \operatorname{sech}^m(u\pi)\, du = \frac{1}{2\pi^2} B\left(\frac{m}{2}, \frac{1}{2}\right). \tag{6.2.29}$$

Hence, Fubini's theorem implies for $|v| \leq 1$ that

$$-S(v) = -\frac{1}{\pi}\sum_{m=1}^{\infty}\int_0^\infty \frac{(v\operatorname{sech}(\pi u))^m}{m} du = \frac{1}{\pi}\int_0^\infty \log\left(1 - v\operatorname{sech}(\pi u)\right) du. \tag{6.2.30}$$

PROOF OF PROPOSITION 6.2.3. We split the sum $S(v)$, $|v| \leq 1$, in two parts, one corresponding to the odd terms and the even ones. The odd contribution is

$$\begin{aligned} I^{(\text{odd})}(v) = \frac{1}{2\pi^2}\sum_{m=0}^{\infty}\frac{B\left(m+\frac{1}{2}, \frac{1}{2}\right) v^{2m+1}}{2m+1} &= \frac{1}{2\pi}\sum_{m=0}^{\infty}\frac{(2m)! v^{2m+1}}{4^m (m!)^2 (2m+1)} \\ &= \frac{1}{2\pi}\arcsin(v). \end{aligned} \tag{6.2.31}$$

The even contribution to the sum is

$$\begin{aligned} I^{(\text{even})}(v) = \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{B\left(m, \frac{1}{2}\right) v^{2m}}{2m} &= \frac{1}{4\pi^2}\sum_{m=1}^{\infty}\frac{(m!)^2 4^m v^{2m}}{(2m)! m^2} \\ &= \frac{1}{2\pi^2}\arcsin^2(v). \end{aligned} \tag{6.2.32}$$

Here, we used the power series expansions for arcsin and $\arcsin^2$ stated in [27, (1.641) and (1.645)] which is absolutely convergent for $|v| \leq 1$. This gives the result where we note that $T(v)$ is the same as $I^{(\text{even})}(v)$. □

Given Theorem 6.2.1, Proposition 6.2.2 and Proposition 6.2.3, we are in position to prove Theorem 6.1.3.

Proof of Theorem 6.1.3. Since by assumption $\|\beta H_N(f)\| < 1$, we use the series expansion (6.2.23) and obtain for any $M \in \mathbb{N}$ that

$$\log\det(\mathbb{I}_N - \beta H_N(f)) = -\sum_{k=1}^{M} \beta^k \frac{\operatorname{Tr} H_N(f)^k}{k} - \sum_{k=M+1}^{\infty} \beta^k \frac{\operatorname{Tr} H_N(f)^k}{k}. \tag{6.2.33}$$

First, we focus on

$$\limsup_{N\to\infty} \Big| \frac{1}{\log N} \sum_{k=M+1}^{\infty} \beta^k \frac{\operatorname{Tr} H_N(f)^k}{k} \Big|. \tag{6.2.34}$$

To do so, we use $\|H_N(f)\| \leq 1$ to obtain the inequality

$$|\operatorname{Tr} H_N(f)^k| \leq \|H_N(f)\|^{k-2} \|H_N(f)\|_2^2 \leq \|H_N(f)\|_2^2 \tag{6.2.35}$$

valid for $k \in \mathbb{N}$ and $k \geq 2$ which yields for $M > 1$ that

$$\Big| \sum_{k=M+1}^{\infty} \beta^m \frac{\operatorname{Tr} H_N(f)^k}{k} \Big| \leq \sum_{k=M+1}^{\infty} \beta^k \frac{\|H_N(f)\|_2^2}{k} \leq \|H_N(f)\|_2^2 \frac{\beta^{M+2}}{1-\beta}. \tag{6.2.36}$$

Proposition 6.2.2 implies that $\limsup_{N\to\infty} \|H_N(f)\|_2^2/\log N = \mu$ for some $\mu \in \mathbb{R}$ and therefore since $|\beta| < 1$ we have that

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \Big| \frac{1}{\log N} \sum_{k=M+1}^{\infty} \beta^k \frac{\operatorname{Tr} H_N(f)^k}{k} \Big| \leq \mu \limsup_{M\to\infty} \frac{\beta^{M+2}}{1-\beta} = 0. \tag{6.2.37}$$

Plugging this into (6.2.33) and recalling that $\lim_{N\to\infty} \operatorname{Tr} H_N(f)^k/\log N = \mu_k(f)$, $k \in \mathbb{N}$, by Theorem 6.2.1, we obtain that

$$\begin{aligned} \limsup_{N\to\infty} \frac{\log\det(\mathbb{I}_N - \beta H_N(f))}{\log N} &\leq \limsup_{M\to\infty} \Big(-\sum_{k=1}^{M} \frac{\beta^k}{k} \mu_k(f) \Big) \\ &\quad + \limsup_{M\to\infty} \limsup_{N\to\infty} \Big| \frac{1}{\log N} \sum_{k=M+1}^{\infty} \beta^k \frac{\operatorname{Tr} H_N(f)^k}{k} \Big| \\ &= \limsup_{M\to\infty} \Big(-\sum_{k=1}^{M} \frac{\beta^k}{k} \mu_k(f) \Big). \end{aligned} \tag{6.2.38}$$

Since $|\varkappa_z \varkappa_{\bar{z}}| \leq 1$ for all $z \in \Omega$ by assumption, the sum $\sum_{k=1}^{\infty} \frac{\beta^k}{k} \mu_k(f)$ for $|\beta| \leq 1$ is absolutely convergent, see Proposition 6.2.3. This implies that

$$\limsup_{M\to\infty} -\sum_{k=1}^{M} \frac{\beta^k}{k} \mu_k(f) = -\sum_{k=1}^{\infty} \frac{\beta^k}{k} \mu_k(f). \tag{6.2.39}$$

Along the very same lines we also obtain that

$$\liminf_{N\to\infty} \frac{\log\det(\mathbb{I}_N - \beta H_N(f))}{\log N} \geq -\sum_{k=1}^{\infty} \frac{\beta^k}{k} \mu_k(f) \tag{6.2.40}$$

and therefore we end up with

$$\lim_{N\to\infty} \frac{\log\det(\mathbb{I}_N - \beta H_N(f))}{\log N} = -\sum_{k=1}^{\infty} \frac{\beta^k}{k}\mu_k(f) \tag{6.2.41}$$

and the power series in Proposition 6.2.3 below give the result. □

Proof of Corollary 6.1.5. To prove Corollary 6.1.5 one uses the expansion (6.2.23) and obtains for $0 \le \beta < 1$

$$\log\det\left(\mathbb{I}_N - \beta^2 \mathbb{1}_N \mathbf{H}^2 \mathbb{1}_N\right) = -\sum_{k\in\mathbb{N}} \beta^{2k} \frac{\operatorname{Tr}(\mathbb{1}_N\mathbf{H}^2\mathbb{1}_N)^k}{k}. \tag{6.2.42}$$

For the rest of the proof we use the abbreviations $A := 1_N \mathbf{H}^2 1_N$ and $B := \mathbf{H}_N^2$. Then

$$\operatorname{Tr} A^k - \operatorname{Tr} B^k = \sum_{j=0}^{k-1} \operatorname{Tr} A^j (A-B) B^{k-1-j} \tag{6.2.43}$$

and Hölder's inequality implies

$$\left|\operatorname{Tr} A^j (A-B) B^{k-1-j}\right| \le \|A\|^j \|A-B\|_1 \|B\|^{k-1-j}. \tag{6.2.44}$$

From the definition of $\mathbf{H}$ we obtain $\|A\|, \|B\| \le 1$. Moreover, by the positivity of $A - B$, we obtain

$$\|A-B\|_1 = \operatorname{Tr} \mathbb{1}_N \mathbf{H}(\mathbb{I} - \mathbb{1}_N)\mathbf{H}\mathbb{1}_N = \|\mathbb{1}_N \mathbf{H}(\mathbb{I} - \mathbb{1}_N)\|_2^2 = O(1) \tag{6.2.45}$$

as $N \to \infty$, where the last inequality follows easily from the explicit matrix elements of $\mathbf{H}$. Equations (6.2.43)–(6.2.45) imply

$$\operatorname{Tr}(\mathbb{1}_N \mathbf{H}^2 \mathbb{1}_N)^n = \operatorname{Tr} \mathbf{H}_N^{2n} + O(1). \tag{6.2.46}$$

For the latter we computed the first order asymptotics as $N \to \infty$ in Lemma 6.3.1. Then the assertion follows from Propostition 6.2.3 along the very same lines as Theorem 6.1.3. □

3. Analysis of the model operator

To prove Theorem 6.2.1, we first investigate a family of model operators related to the Hilbert matrix introduced in [46, Chap. 10.1]. We recall the Hilbert matrix $\mathbf{H} := H(\psi)$ introduced in (6.1.4) with symbol

$$\psi(e^{it}) = \frac{1}{\pi} i e^{-it}(\pi - t), \quad t \in [0, 2\pi). \tag{6.3.47}$$

In particular, one has $\|\mathbf{H}\| = 1$. We define the following model symbols for $z \in \mathbb{T}$

$$\psi_z(e^{it}) := \frac{1}{i}\psi(\bar{z}e^{it}), \quad t \in [0,\, 2\pi). \tag{6.3.48}$$

For any $z \in \mathbb{T}$ this function satisfies

$$\psi_z(z^+) - \psi_z(z^-) = 2. \tag{6.3.49}$$

Furthermore, the corresponding Hankel matrix $H(\psi_z)$ admits the representation

$$H(\psi_z) = \frac{1}{i} U_z \mathbf{H} U_z \tag{6.3.50}$$

where $U_z : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is the unitary operator given by $(U_z x)(j) := z^j x(j)$ for $x \in \ell^2(\mathbb{Z}_+)$, $z \in \mathbb{T}$ and $j \in \mathbb{Z}_+$. In particular, one can compute the matrix elements explicitly and one obtains for $z \in \mathbb{T}$

$$(H(\psi_z))(j,k) = \frac{1}{i\pi} \frac{z^{j+k}}{j+k+1}. \tag{6.3.51}$$

The large N asymptotics of traces of powers of the model operators can be computed explicitly:

LEMMA 6.3.1. *We denote by B the Beta-function. Let $k \in \mathbb{N}$. Then, for $a \in \mathbb{C}$ we obtain that*

$$\operatorname{Tr} H_N(a\psi_z)^k = \begin{cases} \frac{(-i)^k (a)^k}{2\pi^2} B\Big(\frac{k}{2}, \frac{1}{2}\Big) \log N + o(\log N), & z \in \pm 1 \\ o(\log N), & z \in \mathbb{T} \setminus \{\pm 1\} \end{cases} \tag{6.3.52}$$

as $N \to \infty$ while for $z \in \mathbb{T} \setminus \{\pm 1\}$ and $a, b \in \mathbb{C}$ we obtain that

$$\operatorname{Tr} H_N(a\psi_z + b\psi_{\overline{z}})^k = \begin{cases} O(1), & k \in \mathbb{N} \text{ odd}, \\ \frac{(-i)^k (ab)^{k/2}}{\pi^2} B\Big(\frac{k}{2}, \frac{1}{2}\Big) \log N + o(\log N), & k \in \mathbb{N} \text{ even} \end{cases} \tag{6.3.53}$$

as $N \to \infty$.

We prove Lemma 6.3.1 in Section 5.

4. Proof of Theorem 6.2.1

Let $f \in PD(\mathbb{T})$ and let Ω be the set of its jump discontinuities and we define

$$\Psi := \sum_{z \in \Omega} \varkappa_z \psi_z. \tag{6.4.54}$$

The definition of ψ_z in (6.3.48), the identity (6.3.49) and the definition of $\varkappa_z = \varkappa_z(f)$ in (6.1.11), imply that the jumps of f and Ψ are located at the same points and the heights of the jumps are the same, i.e. $\varkappa_z(f) = \varkappa_z(\Psi)$ for all $z \in \mathbb{T}$. Moreover, by assumption (B), $f \in C^\gamma(\mathbb{T} \setminus \Omega)$ for some $1/2 < \gamma \leq 1$ and clearly $\Psi \in C^\infty(\mathbb{T} \setminus \Omega)$ which implies

$$f - \Psi \in C^\gamma(\mathbb{T}). \tag{6.4.55}$$

We first prove Proposition 6.2.2.

Proof of Proposition 6.2.2. We first note that the definition of Besov spaces $B_2^{1/2}(\mathbb{T})$ given in [46, eq. (A.2.10)] implies that $C^\gamma(\mathbb{T}) \subset B_2^{1/2}(\mathbb{T})$ for $1/2 < \gamma \leq 1$. Hence, [46, Chap. 6, Thm. 2.1] and (6.4.55) imply that

$$H(f) - H(\Psi) \in \mathfrak{S}_2 \tag{6.4.56}$$

for all $\gamma > 1/2$. Hence, using the above and Jensen's inequality we obtain

$$\|H_N(f)\|_2^2 \leq 2\|H(\Psi)\|_2^2 + O(1) \tag{6.4.57}$$

as $N \to \infty$. The explicit representation of the matrix entries of $H(\Psi)$ in (6.3.51) implies

$$\|H(\Psi)\|_2^2 \leq C \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \frac{1}{(j+k+1)^2}, \tag{6.4.58}$$

for some constant $C > 0$. Estimating the latter double sum by the corresponding integral, we obtain

$$\sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \frac{1}{(j+k+1)^2} = O(\log N) \tag{6.4.59}$$

as $N \to \infty$. This, together with (6.4.57) and (6.4.58), gives the assertion. □

Lemma 6.4.1. *Let $k \in \mathbb{N}$. Then the asymptotic formula*

$$\operatorname{Tr} H_N(f)^k = \operatorname{Tr} H_N(\Psi)^k + o(\log N) \tag{6.4.60}$$

holds as $N \to \infty$.

Proof. As before, by (6.4.55) we obtain $H(f) - H(\Psi) \in \mathfrak{S}_2$ for all $1/2 < \gamma \leq 1$, see (6.4.56). Moreover, (6.4.55) also implies that the Fourier coefficients of $f - \Psi$ are absolutely summable, see [43, Thm. 1.13].
We write $H(f) = H(\Psi) + A$, where $A := H(f - \Psi)$ and set $A_N := \mathbb{1}_N A \mathbb{1}_N$. For $k = 1$, using the absolute summability of the Fourier coefficients of $f - \Psi$, we obtain that

$$\big|\operatorname{Tr} H_N(f) - \operatorname{Tr} H_N(\Psi)\big| \leq \sum_{j=0}^{N-1} \big|(H_N(f-\Psi))(j,j)\big| = O(1), \tag{6.4.61}$$

as $N \to \infty$. For $k \geq 2$, the identity

$$\operatorname{Tr} H_N(f)^k = \operatorname{Tr} H_N(\Psi)^k + \sum_{j=0}^{k-1} \operatorname{Tr} H_N(f)^j A_N H_N(\Psi)^{k-1-j} \tag{6.4.62}$$

holds. To control the error, we use $A \in \mathfrak{S}_2$. The cyclicity of the trace and the Hölder inequality for $\mathfrak{S}_p$ classes, implies for $1 \leq j \leq k-1$

$$\begin{aligned} \big|\operatorname{Tr} H_N(f)^j A_N H_N(\Psi)^{k-1-j}\big| &\leq \|A_N\|_2 \|H_N(\Psi)^{k-1-j} H_N(f)^j\|_2 \\ &\leq C^{k-1-j} \|A\|_2 \|H_N(f)\|_2, \end{aligned} \tag{6.4.63}$$

where we used that $\|H_N(\Psi)\| \le C$ for some constant $C > 0$ independent of N, $\|H_N(f)\| \le 1$ and the standard inequality $\|CD\|_2 \le \|C\|\|D\|_2$ valid for compact operators C and D. Proposition 6.2.2 implies that

$$\|H_N(f)\|_2 = O(\log N)^{1/2} = o(\log N) \tag{6.4.64}$$

as $N \to \infty$. For $j = 0$ we use in (6.4.63) the bound

$$\big| \operatorname{Tr} A_N H_N(\Psi)^{k-1-j} \big| \le C^{k-2-j} \|A\|_2 \|H_N(\Psi)\|_2. \tag{6.4.65}$$

The asymptotic formula $\|H_N(\psi)\|_2 = O(\log N)^{1/2} = o(\log N)$ holds as well for $N \to \infty$, see (6.4.71) and (6.4.59). This together with (6.4.63) gives the result. $\square$

In view of Lemma 6.3.1 we divide the set of discontinuities as follows

$$\Omega = \Omega_1 \cup \Omega_2, \tag{6.4.66}$$

where Ω_1 and Ω_2 are the sets defined below

$$\Omega_1 := \big\{ \pm 1 \big\} \cup \big\{ z \in \Omega : \ \overline{z} \notin \Omega \big\}, \quad \Omega_2 := \big\{ z \in \Omega : \ \operatorname{Im} z > 0, \ \overline{z} \in \Omega \big\}. \tag{6.4.67}$$

With this notation at hand we show

Lemma 6.4.2. *Let $k \in \mathbb{N}$. Then*

$$\operatorname{Tr} H_N(\Psi)^k = \sum_{z \in \Omega_1} \operatorname{Tr} H_N(\varkappa_z \psi_z)^k + \sum_{z \in \Omega_2} \operatorname{Tr} H_N(\varkappa_z \psi_z + \varkappa_{\overline{z}} \psi_{\overline{z}})^k + O(1) \tag{6.4.68}$$

as $N \to \infty$.

Proof. We first prove that

$$\operatorname{Tr} H_N(\Psi)^k = \operatorname{Tr} \mathbb{1}_N H(\Psi)^k \mathbb{1}_N + O(1). \tag{6.4.69}$$

From [40, Thm. 1.2] we infer that for some constant $C_k > 0$ depending on k

$$\big| \operatorname{Tr} H_N(\Psi)^k - \operatorname{Tr} \mathbb{1}_N H(\Psi)^k \mathbb{1}_N \big| \le C_k \|\mathbb{1}_N H(\Psi)(\mathbb{I} - \mathbb{1}_N)\|_2^2. \tag{6.4.70}$$

The explicit representation of the kernel of $H(\psi_z)$ in (6.3.51) implies for some constant $C > 0$ that

$$\|\mathbb{1}_N H(\Psi)(\mathbb{I} - \mathbb{1}_N)\|_2^2 \le C \sum_{j=0}^{N-1} \sum_{k=N}^{\infty} \frac{1}{(j+k+1)^2} < \infty. \tag{6.4.71}$$

To prove the assertion we note that for $z, w \in \Omega$ with $z \neq w$ and $z \neq \overline{w}$

$$H(\psi_z) H(\psi_w) \in S^1, \tag{6.4.72}$$

which is proven in [54, Lem. 2.5]. This implies

$$\operatorname{Tr} \mathbb{1}_N H(\Psi)^k \mathbb{1}_N = \sum_{z \in \Omega_1} \operatorname{Tr} \mathbb{1}_N H(\varkappa_z \psi_z)^k \mathbb{1}_N + \sum_{z \in \Omega_2} \operatorname{Tr} \mathbb{1}_N H(\varkappa_z \psi_z + \varkappa_{\overline{z}} \psi_{\overline{z}})^k \mathbb{1}_N + O(1). \tag{6.4.73}$$

The same argument as in (6.4.70) yields

$$\operatorname{Tr} \mathbb{1}_N H(\psi_z)^k \mathbb{1}_N = \operatorname{Tr} H_N(\psi_z)^k + O(1) \tag{6.4.74}$$

and

$$\operatorname{Tr} \mathbb{1}_N H(\varkappa_z\psi_z + \varkappa_{\overline{z}}\psi_{\overline{z}})^k \mathbb{1}_N = \operatorname{Tr} H_N(\varkappa_z\psi_z + \varkappa_{\overline{z}}\psi_{\overline{z}})^k + O(1). \tag{6.4.75}$$

This gives the assertion together with (6.4.69) and (6.4.73). □

Proof of Theorem 6.2.1. The theorem follows directly from Lemma 6.4.1, Lemma 6.4.2 and the asymptotics deduced in Lemma 6.3.1. □

5. Proof of Lemma 6.3.1

Proof of Lemma 6.3.1. The statement for $z = 1$ follows directly from [**61**, proof of Thm. 4.3], see especially [**61**, eq. (12)], where it is proven that

$$\operatorname{Tr} H_N(\psi_1)^k = \frac{(-i)^k}{2\pi^2} B\Big(\frac{k}{2}, \frac{1}{2}\Big) \log N + o(\log N), \tag{6.5.76}$$

as $N \to \infty$. As before, B denotes the Beta function. We remark that the result in the paper cited above has been corrected to take into account a factor of $1/2\pi$ missing in the computations of [**61**, proof of Thm. 4.3]. Similar results to the above are true in greater generality, see [**18**].
As we mentioned earlier on in (6.3.50) , we have

$$H(\psi_{-1}) = U_{-1}\mathbf{H}U_{-1},$$

where U_{-1} is the unitary and selfadjoint operator of multiplication by the sequence $(-1)^n$ on $\ell^2(\mathbb{Z}_+)$. Therefore, the result of [**61**] gives

$$\operatorname{Tr} H_N(\psi_{-1})^k = \frac{(-i)^k}{2\pi^2} B\Big(\frac{k}{2}, \frac{1}{2}\Big) \log N + o(\log N). \tag{6.5.77}$$

This and (6.5.76) give the first part of (6.3.52).

Next we consider the case $z \in \Omega\setminus\{\pm 1\}$ and note that the second part of (6.3.52) follows from (6.3.53) with $b = 0$. Therefore, we only prove (6.3.53). Let $a, b \in \mathbb{C}$. First we note that the unitary U_z and the projection 1_N commute. Using representation (6.3.50), we expand $H_N(a\psi_z + b\psi_{\overline{z}})^k = (-i)^k\big(aU_z\mathbf{H}_NU_z + bU_{\overline{z}}\mathbf{H}_NU_{\overline{z}}\big)^k$ in 2^k terms and obtain

$$\begin{aligned} H_N(a\psi_z + b\psi_{\overline{z}})^k &= (-i)^k\big(aU_z\mathbf{H}_NU_z + bU_{\overline{z}}\mathbf{H}_NU_{\overline{z}}\big)^k \\ &= \begin{cases} (-i)^k(ab)^{(k-1)/2}a\, U_z\mathbf{H}_N^kU_z + (ab)^{(k-1)/2}b\, U_{\overline{z}}\mathbf{H}_N^kU_{\overline{z}} + A_1, & k \text{ odd} \\ (-i)^k(ab)^{k/2}\big(U_z\mathbf{H}_N^kU_{\overline{z}} + U_{\overline{z}}\mathbf{H}_N^kU_z\big) + A_2, & k \text{ even} \end{cases} \end{aligned} \tag{6.5.78}$$

for some operators A_1 and A_2. We first deal with the errors A_1 and A_2. The operators A_1 and A_2 consist of a sum of 2^k-2 terms and each summand has at least one factor $\mathbf{H}_N U_{z^2}\mathbf{H}_N$ or $\mathbf{H}_N U_{\overline{z}^2}\mathbf{H}_N$. More precisely, any factor of A_1 is either of the form

$$a^r b^s U_z\mathbf{H}_N U_z \cdots U_z\mathbf{H}_N U_{z^2}\mathbf{H}_N U_z \cdots \mathbf{H}_N U_{\overline{z}} \tag{6.5.79}$$

for some $r,s\in\mathbb{N}$ with $s+r=k$ or the adjoint of the latter. Since $\operatorname{Im} z\neq 0$, we have from [46, Chap. 10, Lem. 1.2] that the matrix elements of $\mathbf{H}_N U_{z^2}\mathbf{H}_N$ satisfy

$$\left|\left(\mathbf{H}_N U_{z^2}\mathbf{H}_N(j,k)\right)\right| \leq \frac{2}{|1-z^2|}\frac{1}{(1+j)(1+k)}. \tag{6.5.80}$$

Using this and the pointwise bound on the matrix elements of $U_z\mathbf{H}_N U_z$ of the form $\left|(U_z\mathbf{H}_N U_z)(j,k)\right| \leq \frac{1}{\pi(j+k+1)}$, $j,k\in\mathbb{Z}_+$, we estimate

$$\begin{aligned}
&\left|\operatorname{Tr} 1_N U_z\mathbf{H}U_z\cdots U_z\mathbf{H}_N U_{z^2}\mathbf{H}_N U_z\cdots U_{\overline{z}}\mathbf{H}_N U_{\overline{z}}1_N\right| \\
&\quad= \left|\sum_{j_1,\cdots,j_k=1}^{\infty} \left(U_z\mathbf{H}_N U_z\right)(j_1,j_2)\cdots\left(\mathbf{H}_N U_{z^2}\mathbf{H}_N\right)(j_p,j_{p+1})\cdots\left(U_{\overline{z}}\mathbf{H}_N U_{\overline{z}}\right)(j_k,j_1)\right| \\
&\quad\leq \frac{2}{|1-z|^2}\left|(x,\mathbf{H}_N^{k-1}x)\right| \\
&\quad\leq \frac{2}{|1-z|^2}\|x\|_2^2<\infty
\end{aligned} \tag{6.5.81}$$

for some $p\in\mathbb{N}$, where we defined $x\in\ell^2(\mathbb{Z}_+)$ with $x(j):=1/(j+1)$. Writing out all terms of $\operatorname{Tr} A_i$, $i=1,2$, explicitly in terms of its matrix elements and using a bound of the form (6.5.81) implies for $i=1,2$ that, as $N\to\infty$,

$$\left|\operatorname{Tr} A_i\right| = O(1). \tag{6.5.82}$$

For $k\in\mathbb{N}$ odd we obtain

$$\begin{aligned}
&\left|\operatorname{Tr}\left((ab)^{(k-1)/2}aU_z\mathbf{H}_N^k U_z+(ab)^{(k-1)/2}bU_{\overline{z}}\mathbf{H}_N^k U_{\overline{z}}\right)\right| \\
=&|ab|^{(k-1)/2}\left|\operatorname{Tr}\left(aU_z\mathbf{H}_N^k U_z+bU_{\overline{z}}\mathbf{H}_N^k U_{\overline{z}}\right)^k\right| \\
\leq&|ab|^{(k-1)/2}\left(|a|\left|\sum_{j=0}^{N-1} z^{2j}\mathbf{H}_N^k(j,j)\right|+|b|\left|\sum_{n=0}^{N-1}\overline{z}^{2j}\mathbf{H}_N^k(j,j)\right|\right).
\end{aligned} \tag{6.5.83}$$

From the explicit matrix elements $\mathbf{H}_N(j,k)=\frac{1}{\pi(j+k+1)}$ we obtain for all $j\in\mathbb{Z}_+$ that $0\leq\mathbf{H}_N^k(j+1,j+1)\leq\mathbf{H}_N^k(j,j)$, i.e. the sequence $a_j:=\mathbf{H}_N^k(j,j)$, $j\in\mathbb{Z}_+$, is strictly monotonously decreasing. Now Lemma 6.5.1 below gives, as $N\to\infty$,

$$(6.5.83)=O(1). \tag{6.5.84}$$

In the case $k\in\mathbb{N}$ even the definition of U_z yields

$$(-i)^k(ab)^{k/2}\operatorname{Tr}\left(U_z\mathbf{H}_N^k U_{\overline{z}}+U_{\overline{z}}\mathbf{H}_N^k U_z\right)=2(-i)^k(ab)^{k/2}\operatorname{Tr}\mathbf{H}_N^k \tag{6.5.85}$$

but this is just the asymptotics of the Hilbert matrix which was discussed in the first part of the proof. This gives the assertion. □

LEMMA 6.5.1. *Let $z \in \mathbb{T} \setminus \{1\}$ and $(a_n)_{n\in\mathbb{Z}_+}$ be such that $0 \le a_{n+1} \le a_n$ for all $n \in \mathbb{Z}_+$. Then*

$$\left|\sum_{n=0}^{N} z^n a_n\right| \le a_0 \frac{2}{|1-z|} \tag{6.5.86}$$

and, in particular, $\sum_{n=0}^{N} z^n a_n = O(1)$ as $N \to \infty$.

PROOF. The lemma follows directly from Abel's summation formula

$$\sum_{n=0}^{N} z^n a_n = B_N a_N + \sum_{k=1}^{N-1} B_k(a_k - a_{k-1}) \tag{6.5.87}$$

where $B_k = \sum_{l=0}^{k} z^l$.

□

CHAPTER 7

Weighted integral Hankel operators with continuous spectrum

1. Introduction

The aim of this Chapter is to consider some variants (perturbations) of the following simple integral operator:

$$A_\alpha : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \quad \alpha > -1/2,$$
$$(A_\alpha f)(t) = \int_0^\infty \frac{t^\alpha s^\alpha}{(s+t)^{1+2\alpha}} f(s) ds, \quad f \in L^2(\mathbb{R}_+). \tag{7.1.1}$$

Since the integral kernel of A_α is homogeneous of degree -1, this operator can be explicitly diagonalised by the Mellin transform

$$\mathcal{M}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{-\frac{1}{2}+i\xi} f(t) dt, \quad \xi \in \mathbb{R},$$

which is a unitary map from $L^2(\mathbb{R}_+, dt)$ to $L^2(\mathbb{R}, d\xi)$. Mellin transform effects a unitary transformation of A_α into the operator of multiplication by the function (here Γ is the standard Gamma function)

$$\mathbb{R} \ni \xi \mapsto \frac{\left|\Gamma(\frac{1}{2}+\alpha+i\xi)\right|^2}{\Gamma(1+2\alpha)}$$

in $L^2(\mathbb{R}, d\xi)$. The spectrum of A_α is given by the range of this function. Observe that this function is even in ξ and monotone increasing on $(-\infty, 0)$; we denote its maximum, attained at $\xi = 0$, by

$$\pi_\alpha = \frac{\Gamma(\frac{1}{2}+\alpha)^2}{\Gamma(1+2\alpha)}. \tag{7.1.2}$$

With this notation, we can summarise the above discussion by

PROPOSITION 7.1.1. *For $\alpha > -1/2$, the operator A_α of (7.1.1) in $L^2(\mathbb{R}_+)$ is bounded and selfadjoint, and has a purely absolutely continuous (a.c.) spectrum of multiplicity two given by*

$$\operatorname{spec}_{ac}(A_\alpha) = [0, \pi_\alpha].$$

This includes the well-known case $\alpha = 0$ of the Carleman operator; in this case $\pi_0 = \pi$.

In [32], Howland considered integral Hankel operators on $L^2(\mathbb{R}_+)$ with kernels whose asymptotic behaviour is modelled on that of the Carleman operator. For a real-valued function $a = a(t)$, $t > 0$ (we call it a *kernel*), let us denote by $H(a)$ the Hankel operator in $L^2(\mathbb{R}_+)$ defined by

$$(H(a)f)(t) = \int_0^\infty a(t+s)f(s)ds, \quad t > 0.$$

Howland considered kernels a with the asymptotic behaviour

$$ta(t) \to \begin{cases} a_0 & t \to 0, \\ a_\infty & t \to \infty. \end{cases} \tag{7.1.3}$$

Among other things, in [32] he proved

THEOREM 7.1.2. [32] *Let $a \in C^2(\mathbb{R}_+)$ have the asymptotic behaviour (7.1.3) and satisfy the regularity conditions*

$$(ta(t))''(t) = \begin{cases} O(t^{-2+\varepsilon}), & t \to 0, \\ O(t^{-2-\varepsilon}), & t \to \infty, \end{cases}$$

with some $\varepsilon > 0$. Then the a.c. spectrum of $H(a)$ is given by

$$\operatorname{spec}_{ac}(H(a)) = [0, \pi a_0] \cup [0, \pi a_\infty], \tag{7.1.4}$$

where each interval contributes multiplicity one to the spectrum.

We pause here to explain the convention that is used in (7.1.4) and that will be used in similar relations below. Relation (7.1.4) means that the a.c. part of $H(a)$ is unitarily equivalent to the direct sum of the operators of multiplication by λ in $L^2([0, \pi a_0], d\lambda)$ and in $L^2([0, \pi a_\infty], d\lambda)$. We also assume that if, for example, $a_0 = 0$, then the first term drops out of the union in (7.1.4); and that if, for example, $a_\infty < 0$, then the interval $[0, \pi a_\infty]$ should be understood as $[\pi a_\infty, 0]$.

Theorem 7.1.2 makes precise the intuition that for the Carleman operator A_0, corresponding to the kernel $a(t) = 1/t$, both $t = 0$ and $t = \infty$ are singular points and each of these points contributes multiplicity one to the spectrum. The aim of this Chapter is to show that the above intuition is also valid for operators A_α with all $\alpha > -1/2$. We do this by considering weighted Hankel operators. These operators generalise A_α in the same manner as the operators $H(a)$ with kernels as in Theorem 7.1.2 generalise the Carleman operator A_0.

For a real-valued kernel $a(t)$ and for a complex-valued function (we will call it a *weight*) $w(t)$, $t > 0$, we denote by $wH(a)\overline{w}$ the weighted Hankel operator in $L^2(\mathbb{R}_+)$, given by

$$(wH(a)\overline{w}f)(t) = \int_0^\infty w(t)a(t+s)\overline{w(s)}f(s)ds, \quad f \in L^2(\mathbb{R}_+).$$

Under our assumptions below, this operator will be bounded. Since a is assumed real-valued, the operator $wH(a)\overline{w}$ is selfadjoint. Here and in what follows by a slight abuse of notation we use the same symbol (in this case w) to denote both a function on $\mathbb{R}_+$ and the operator of multiplication by this function in $L^2(\mathbb{R}_+)$.

We fix $\alpha > -1/2$ and consider a, w with the asymptotic behaviour

$$t^{1+2\alpha}a(t) \to \begin{cases} a_0 & t \to 0, \\ a_\infty & t \to \infty, \end{cases} \qquad t^{-\alpha}w(t) \to \begin{cases} b_0 & t \to 0, \\ b_\infty & t \to \infty. \end{cases} \tag{7.1.5}$$

The aim of this Chapter is to prove

THEOREM 7.1.3. *Fix $\alpha > -1/2$. Let $a \in C^2(\mathbb{R}_+)$ be a real-valued kernel such that for some $a_0, a_\infty \in \mathbb{R}$ and for some $\varepsilon > 0$, we have*

$$\frac{d^m}{dt^m}(t^{1+2\alpha}a(t) - a_0) = O(t^{-m+\varepsilon}), \quad t \to 0, \tag{7.1.6}$$

$$\frac{d^m}{dt^m}(t^{1+2\alpha}a(t) - a_\infty) = O(t^{-m-\varepsilon}), \quad t \to \infty, \tag{7.1.7}$$

with $m = 0, 1, 2$. Assume further that the complex valued weight $w(t)$ is such that $t^{-\alpha}w(t)$ is bounded on $\mathbb{R}_+$ and for some $b_0, b_\infty \in \mathbb{C}$,

$$\int_0^1 \left| |w(t)|^2 t^{-2\alpha} - |b_0|^2 \right| t^{-1} dt < \infty, \qquad \int_1^\infty \left| |w(t)|^2 t^{-2\alpha} - |b_\infty|^2 \right| t^{-1} dt < \infty. \tag{7.1.8}$$

Then the a.c. spectrum of $wH(a)\overline{w}$ is given by

$$\operatorname{spec}_{ac}(wH(a)\overline{w}) = [0, \pi_\alpha a_0 |b_0|^2] \cup [0, \pi_\alpha a_\infty |b_\infty|^2],$$

where each interval contributes multiplicity one to the spectrum.

Remark.

(1) Howland in [**32**] uses Mourre's estimate and proves also the absence of singular continuous spectrum in the framework of Theorem 7.1.2. Here we use the trace class method of scattering theory. This method is technically simpler to use but it gives no information on the singular continuous spectrum.

(2) Conditions on a and w in Theorem 7.1.3 are far from being sharp. For example, it is not difficult to relax conditions (7.1.6), (7.1.7) by replacing $t^{\pm\varepsilon}$ by $|\log t|^{-1-\varepsilon}$, see [**52**] for a related calculation.

(3) Howland's results of [**32**] for unweighted Hankel operators were extended in [**52**] to kernels $a(t)$ with more complicated (oscillatory) asymptotic behaviour at $t \to \infty$.

(4) An important precursor to Howland's work [**32**] was Power's analysis [**49, 50**] of the essential spectrum of Hankel operators with piecewise continuous symbols. In this context we note that the essential spectrum of

the weighted Hankel operators considered in Theorem 7.1.3 is easy to describe. By following the method of proof of this theorem and using Weyl's theorem on the preservation of the essential spectrum under compact perturbations instead of the Kato-Rosenblum theorem, one can check that if both $t^{1+2\alpha}a(t)$ and $t^{-\alpha}w(t)$ are bounded and satisfy the asymptotic relation (7.1.5), then the essential spectrum of $wH(a)\overline{w}$ is given by the union of the intervals

$$\mathrm{spec}_{ess}\,(wH(a)\overline{w}) = [0, \pi_\alpha a_0 \left|b_0\right|^2] \cup [0, \pi_\alpha a_\infty \left|b_\infty\right|^2].$$

(5) Boundedness and Schatten class conditions for weighted Hankel operators with the power weights $w_\alpha(t) = t^\alpha$ have been studied by several authors; see e.g. [**58, 33**] and the references in [**2**, Section 2].

(6) In [**34**], interesting non-trivial discrete analogues of the operators A_α are analysed. These operators act in $\ell^2(\mathbb{Z}_+)$ and are formally defined as infinite matrices with entries of the form

$$w(j)a(j+k)w(k), \quad j,k \in \mathbb{Z}_+. \tag{7.1.9}$$

For each $\alpha > -1/2$, the authors of [**34**] describe some families of sequences $\{a(j)\}$ and $\{w(j)\}$ with the asymptotic behaviour

$$j^{1+2\alpha}a(j) \to 1, \quad j^{-\alpha}w(j) \to 1, \quad j \to \infty,$$

for which the operators (7.1.9) are explicitly diagonalised. It turns out that the spectrum of each of these operators is purely a.c., has multiplicity one and coincides with the interval $[0, \pi_\alpha]$, where π_α is the same as in (7.1.2).

2. Proof of main Theorem

2.1. Outline of the proof. Let a, w be as in Theorem 7.1.3. First we identify two suitable "model" kernels φ_0 and φ_∞ in $C^\infty(\mathbb{R}_+)$ such that $\varphi_0(t)+\varphi_\infty(t) = t^{-1-2\alpha}$ and

$$\frac{d^m}{dt^m}\varphi_0(t) = O(e^{-t/2}), \quad t \to \infty, \quad \text{and} \quad \frac{d^m}{dt^m}\varphi_\infty(t) = O(1), \quad t \to 0 \tag{7.2.10}$$

for all $m \geq 0$. Then we write the kernel a as

$$a(t) = a_0\varphi_0(t) + a_\infty\varphi_\infty(t) + \text{error},$$

where the error term is negligible in a suitable sense both as $t \to 0$ and as $t \to \infty$. Similarly, we write

$$\left|w(t)\right|^2 = \left|b_0\right|^2 \mathbb{1}_0(t)t^{2\alpha} + \left|b_\infty\right|^2 \mathbb{1}_\infty(t)t^{2\alpha} + \text{error},$$

where $\mathbb{1}_0$ and $\mathbb{1}_\infty$ are the characteristic functions of the intervals $(0,1)$ and $(1,\infty)$ respectively and the error term is again negligible in a suitable sense. With these representations, denoting $w_\alpha(t) = t^\alpha$, we write

$$wH(a)w = a_0 |b_0|^2 \mathbb{1}_0 w_\alpha H(\varphi_0) w_\alpha \mathbb{1}_0 + a_\infty |b_\infty|^2 \mathbb{1}_\infty w_\alpha H(\varphi_\infty) w_\alpha \mathbb{1}_\infty + \text{error} \quad (7.2.11)$$

and prove that the error term here is a trace class operator. By the Kato-Rosenblum theorem (see e.g. [**57**, Theorem XI.8]), this reduces the problem to the description of the a.c. spectrum of the sum of the first two operators in the right side of (7.2.11). Observe that these two operators act in the orthogonal subspaces $L^2(0,1)$ and $L^2(1,\infty)$. This reduces the problem to identifying the a.c. spectra of

$$\mathbb{1}_0 w_\alpha H(\varphi_0) w_\alpha \mathbb{1}_0 \quad \text{and} \quad \mathbb{1}_\infty w_\alpha H(\varphi_\infty) w_\alpha \mathbb{1}_\infty. \quad (7.2.12)$$

We are unable to identify the spectra of these operators directly and therefore we resort to the following trick. We observe that the operator A_α, whose spectrum is given by Proposition 7.1.1, can also be represented in the form (7.2.11) with $a_0 |b_0|^2 = a_\infty |b_\infty|^2 = 1$. This allows us to conclude that the a.c. spectrum of each of the two operators in (7.2.12) coincides with $[0, \pi_\alpha]$ and has multiplicity one. Now we can go back to (7.2.11) and finish the proof.

2.2. Factorisation of A_α. For $\alpha > -1/2$, let L_α be the integral operator in $L^2(\mathbb{R}_+)$ given by

$$(L_\alpha f)(t) = \frac{1}{\sqrt{\Gamma(1+2\alpha)}} \int_0^\infty t^\alpha s^\alpha e^{-st} f(s) ds, \quad t > 0. \quad (7.2.13)$$

The boundedness of L_α is easy to establish by the Schur test. It is evident that L_α is selfadjoint. A direct calculation gives the identity

$$A_\alpha = L_\alpha^2.$$

This factorisation is an important technical ingredient of the proof.

2.3. Trace class properties of auxiliary operators.

LEMMA 7.2.1. *Let L_α be the operator (7.2.13) and let u be a locally integrable function on $\mathbb{R}_+$. Then the operator uL_α is in the Hilbert-Schmidt class if and only if*

$$\int_0^\infty |u(t)|^2 \frac{dt}{t} < \infty.$$

PROOF. A direct evaluation of the Hilbert-Schmidt norm:

$$\frac{1}{\Gamma(1+2\alpha)} \int_0^\infty \int_0^\infty s^{2\alpha} t^{2\alpha} e^{-2ts} |u(t)|^2 \, dt \, ds = 2^{-1-2\alpha} \int_0^\infty |u(t)|^2 \frac{dt}{t}. \quad \square$$

A necessary and sufficient condition is known (see [**58**]) for $w_\alpha H(g) w_\alpha$ to belong to trace class in terms of g being in a certain Besov class. For our purposes it suffices to use a simple sufficient condition expressed in elementary terms.

LEMMA 7.2.2. *Let $g \in C^2(\mathbb{R}_+)$ be such that for some $\varepsilon > 0$ and for $m = 0, 1, 2$, one has*

$$\frac{d^m}{dt^m}(t^{1+2\alpha}g(t)) = \begin{cases} O(t^{-m+\varepsilon}), & t \to 0, \\ O(t^{-m-\varepsilon}), & t \to \infty. \end{cases}$$

Then $w_\alpha H(g) w_\alpha$ is trace class.

PROOF. Lemma 2 in [58] asserts that $w_\alpha H(g) w_\alpha$ is trace class if the function $k(t) = t^{2+2\alpha}g(t)$ satisfies the condition

$$\int_{-\infty}^{\infty}\int_0^{\infty} \left|\hat{k}(x+iy)\right| dy\, dx < \infty,$$

where

$$\hat{k}(\zeta) = \int_0^\infty k(t) e^{i\zeta t} dt, \quad \zeta = x + iy, \quad y > 0.$$

Let us check that this condition is satisfied under our hypothesis on g. First note that under our hypothesis, we have

$$k^{(m)}(t) = O(t^{1-m+\varepsilon}), \quad t \to 0, \qquad k^{(m)}(t) = O(t^{1-m-\varepsilon}), \quad t \to \infty. \tag{7.2.14}$$

Next, integrating by parts once and twice in the expression for $\hat{k}$, we get

$$\hat{k}(\zeta) = -\frac{1}{i\zeta}\int_0^\infty k'(t) e^{i\zeta t} dt = \frac{1}{(i\zeta)^2}\int_0^\infty k''(t) e^{i\zeta t} dt, \quad \operatorname{Im}\zeta > 0,$$

and therefore we have the estimates

$$\left|\hat{k}(\zeta)\right| \le \frac{1}{|\zeta|}\int_0^\infty |k'(t)|\, e^{-yt} dt, \quad \left|\hat{k}(\zeta)\right| \le \frac{1}{|\zeta|^2}\int_0^\infty |k''(t)|\, e^{-yt} dt \tag{7.2.15}$$

for $\zeta = x + iy$. For $|\zeta| \le 1$ we use the first one of these estimates, which together with (7.2.14) yields

$$\left|\hat{k}(\zeta)\right| \le \frac{C}{|\zeta|}\int_0^1 t^{\varepsilon} e^{-yt} dt + \frac{C}{|\zeta|}\int_1^\infty t^{-\varepsilon} e^{-yt} dt \le C\frac{1 + y^{-1+\varepsilon}}{|\zeta|}.$$

The right side here is integrable in the domain $|\zeta| < 1$, $\operatorname{Im}\zeta > 0$, if $0 < \varepsilon < 1$.

For $|\zeta| > 1$ we use the second estimate in (7.2.15), which yields

$$\left|\hat{k}(\zeta)\right| \le \frac{C}{|\zeta|^2}\int_0^1 t^{-1+\varepsilon} e^{-yt} dt + \frac{C}{|\zeta|^2}\int_1^\infty t^{-1-\varepsilon} e^{-yt} dt \le C\frac{y^{-\varepsilon} + e^{-y}}{|\zeta|^2},$$

and again the right side is integrable in the domain $|\zeta| > 1$, $\operatorname{Im}\zeta > 0$, if $0 < \varepsilon < 1$. □

The following lemma allows us to get rid of the cross terms that are hidden in the error term in (7.2.11).

LEMMA 7.2.3. *The operators $\mathbb{1}_0 L_\alpha \mathbb{1}_0$ and $\mathbb{1}_\infty L_\alpha \mathbb{1}_\infty$ are trace class. Further, the operators $\mathbb{1}_0 A_\alpha \mathbb{1}_\infty$ and $\mathbb{1}_\infty A_\alpha \mathbb{1}_0$ are trace class.*

Proof. Let us prove the first statement. We will regard $\mathbb{1}_0 L_\alpha \mathbb{1}_0$ as acting on $L^2(0,1)$ and $\mathbb{1}_\infty L_\alpha \mathbb{1}_\infty$ as acting on $L^2(1,\infty)$. Consider the unitary operators

$$U_+ : L^2(1,\infty) \to L^2(\mathbb{R}_+), \quad (U_+ f)(x) = e^{x/2} f(e^x), \quad x > 0,$$
$$U_- : L^2(0,1) \to L^2(\mathbb{R}_+), \quad (U_- f)(x) = e^{-x/2} f(e^{-x}), \quad x > 0.$$

A straightforward calculation shows that

$$U_+ \mathbb{1}_\infty L_\alpha \mathbb{1}_\infty U_+^* = H(\psi_+) \quad \text{and} \quad U_- \mathbb{1}_0 L_\alpha \mathbb{1}_0 U_+^* = H(\psi_-),$$

where the kernels $\psi_\pm$ are given by

$$\psi_+(t) = e^{t(\alpha+1/2)} e^{-e^t}, \quad \psi_-(t) = e^{-t(\alpha+1/2)} e^{-e^{-t}}, \quad t > 0.$$

As both functions $\psi_\pm$ are Schwartz class, using Lemma 7.2.2 we find that the unweighted Hankel operators $H(\psi_\pm)$ are trace class.

To prove the second statement of the lemma, we write $1 = \mathbb{1}_0 + \mathbb{1}_\infty$ and use the factorisation $A_\alpha = L_\alpha^2$ to obtain

$$\mathbb{1}_0 A_\alpha \mathbb{1}_\infty = \mathbb{1}_0 L_\alpha^2 \mathbb{1}_\infty = \mathbb{1}_0 L_\alpha (\mathbb{1}_0 + \mathbb{1}_\infty) L_\alpha \mathbb{1}_\infty = (\mathbb{1}_0 L_\alpha \mathbb{1}_0) L_\alpha \mathbb{1}_\infty + \mathbb{1}_0 L_\alpha (\mathbb{1}_\infty L_\alpha \mathbb{1}_\infty).$$

Now observe that both terms in the right side are trace class by the first part of the lemma. Thus, $\mathbb{1}_0 A_\alpha \mathbb{1}_\infty$ is trace class and by a similar reasoning $\mathbb{1}_\infty A_\alpha \mathbb{1}_0$ is also trace class. ☐

We note that a more careful analysis of the kernels $\psi_\pm$ shows that the operators $\mathbb{1}_0 L_\alpha \mathbb{1}_0$ and $\mathbb{1}_\infty L_\alpha \mathbb{1}_\infty$ belong to the Schatten class $\mathfrak{S}_p$ for any $p > 0$.

2.4. Kernels φ_0 and φ_∞. Recall the notation $w_\alpha(t) = t^\alpha$. By a direct calculation of the integral kernels, we have

$$L_\alpha \mathbb{1}_\infty L_\alpha = w_\alpha H(\varphi_0) w_\alpha, \quad L_\alpha \mathbb{1}_0 L_\alpha = w_\alpha H(\varphi_\infty) w_\alpha,$$

with

$$\varphi_0(t) = \frac{1}{\Gamma(1+2\alpha)} \int_1^\infty x^{2\alpha} e^{-xt} dx, \quad \varphi_\infty(t) = \frac{1}{\Gamma(1+2\alpha)} \int_0^1 x^{2\alpha} e^{-xt} dx.$$

Using the integral representation for the Gamma function, we obtain

$$\varphi_0(t) + \varphi_\infty(t) = t^{-1-2\alpha}, \quad t > 0.$$

Further, it is straightforward to see that the estimates (7.2.10) hold true for all $m \geq 0$. The following lemma gives a description of the spectra of the two operators (7.2.12).

Lemma 7.2.4. *We have*

$$\operatorname{spec}_{ac}(\mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0) = \operatorname{spec}_{ac}(\mathbb{1}_\infty L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_\infty) = [0, \pi_\alpha], \tag{7.2.16}$$

with multiplicity one in both cases.

PROOF. First let us consider the operators $\mathbb{1}_0 A_\alpha \mathbb{1}_0$ and $\mathbb{1}_\infty A_\alpha \mathbb{1}_\infty$. We claim that these two operators are unitarily equivalent to each other. Indeed, let

$$U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \qquad (Uf)(t) = (1/t)f(1/t), \quad t > 0.$$

Then it is easy to see that U is unitary and $UA_\alpha U^* = A_\alpha$. It follows that

$$U\mathbb{1}_\infty A_\alpha \mathbb{1}_\infty U^* = \mathbb{1}_0 A_\alpha \mathbb{1}_0. \tag{7.2.17}$$

Next, write

$$A_\alpha = \mathbb{1}_0 A_\alpha \mathbb{1}_0 + \mathbb{1}_\infty A_\alpha \mathbb{1}_\infty + (\mathbb{1}_\infty A_\alpha \mathbb{1}_0 + \mathbb{1}_0 A_\alpha \mathbb{1}_\infty).$$

By Lemma 7.2.3, the two cross terms in brackets here are trace class; thus, we can apply the Kato-Rosenblum theorem. Recalling Proposition 7.1.1, we obtain that the a.c. spectrum of the sum

$$\mathbb{1}_0 A_\alpha \mathbb{1}_0 + \mathbb{1}_\infty A_\alpha \mathbb{1}_\infty \tag{7.2.18}$$

is $[0, \pi_\alpha]$ with multiplicity two. Now observe that the two operators in (7.2.18) act in orthogonal subspaces $L^2(0, 1)$ and $L^2(1, \infty)$ and, by (7.2.17), they are unitarily equivalent to each other. Thus, we obtain

$$\operatorname{spec}_{ac}(\mathbb{1}_0 A_\alpha \mathbb{1}_0) = \operatorname{spec}_{ac}(\mathbb{1}_\infty A_\alpha \mathbb{1}_\infty) = [0, \pi_\alpha],$$

with multiplicity one in both cases.

Finally, write

$$\mathbb{1}_0 A_\alpha \mathbb{1}_0 = \mathbb{1}_0 L_\alpha^2 \mathbb{1}_0 = \mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0 + \mathbb{1}_0 L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_0.$$

By Lemma 7.2.3, the second term in the right side here is trace class. Thus, by the Kato-Rosenblum theorem, we obtain

$$\operatorname{spec}_{ac}(\mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0) = [0, \pi_\alpha],$$

with multliplicity one, which gives the description of the a.c. spectrum of the first operator in (7.2.16). The second operator is considered in the same way. □

2.5. Concluding the proof. First we prove an intermediate statement. We denote $v(t) = w(t)t^{-\alpha}$ and use the notation $\mathfrak{S}_1$ for the trace class.

LEMMA 7.2.5. *Under the hypothesis of Theorem 7.1.3, we have*

$$\operatorname{spec}_{ac}(v\mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0 \overline{v}) = [0, \pi_\alpha |b_0|^2], \tag{7.2.19}$$

$$\operatorname{spec}_{ac}(v\mathbb{1}_\infty L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_\infty \overline{v}) = [0, \pi_\alpha |b_\infty|^2], \tag{7.2.20}$$

with multiplicity one in both cases.

Proof. We prove the first relation (7.2.19); the second relation is proven in a similar way. First we write

$$v\mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0 \overline{v} = TT^*, \quad T = v\mathbb{1}_0 L_\alpha \mathbb{1}_\infty$$

and recall that for any bounded operator T, the operators $(TT^*)|_{(\operatorname{Ker} TT^*)^\perp}$ and $(T^*T)|_{(\operatorname{Ker} T^*T)^\perp}$ are unitarily equivalent. Thus, it suffices to describe the a.c. spectrum of the operator

$$T^*T = \mathbb{1}_\infty L_\alpha |v|^2 \mathbb{1}_0 L_\alpha \mathbb{1}_\infty.$$

Next, by the hypothesis (7.1.8), we can write

$$|v(t)|^2 = |b_0|^2 + q_1(t)q_2(t), \qquad \text{with } \int_0^1 \frac{|q_1(t)|^2 + |q_2(t)|^2}{t} dt < \infty.$$

This yields

$$\mathbb{1}_\infty L_\alpha |v|^2 \mathbb{1}_0 L_\alpha \mathbb{1}_\infty = |b_0|^2 \mathbb{1}_\infty L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_\infty + (\mathbb{1}_\infty L_\alpha \mathbb{1}_0 q_1)(q_2 \mathbb{1}_0 L_\alpha \mathbb{1}_\infty).$$

By Lemma 7.2.1, both operators in brackets here are Hilbert-Schmidt. It follows that the product of these operators is trace class, i.e.

$$\mathbb{1}_\infty L_\alpha |v|^2 \mathbb{1}_0 L_\alpha \mathbb{1}_\infty = |b_0|^2 \mathbb{1}_\infty L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_\infty + T, \quad T \in \mathfrak{S}_1.$$

Lemma 7.2.4 gives the description of the a.c. spectrum of the first term in the right side here. Now an application of the Kato-Rosenblum theorem gives

$$\operatorname{spec}_{ac} \left(\mathbb{1}_\infty L_\alpha |v|^2 \mathbb{1}_0 L_\alpha \mathbb{1}_\infty \right) = [0, \pi_\alpha |b_0|^2],$$

with multiplicity one. This yields (7.2.19). □

Proof of Theorem 7.1.3. We would like to establish the representation

$$wH(a)\overline{w} = a_0 v(\mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0)\overline{v} + a_\infty v(\mathbb{1}_\infty L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_\infty)\overline{v} + T, \quad T \in \mathfrak{S}_1. \quad (7.2.21)$$

Observe that the first two operators in the right side act in orthogonal subspaces and their a.c. spectra are described by Lemma 7.2.5. Thus, applying the Kato-Rosenblum theorem, we will have the required result as soon as the representation (7.2.21) is proven.

As a first step, let us write

$$a(t) = a_0\varphi_0(t) + a_\infty\varphi_\infty(t) + g(t), \quad t > 0,$$

and examine the error term g. We have, using $\varphi_0(t) + \varphi_\infty(t) = t^{-1-2\alpha}$,

$$\begin{aligned} t^{1+2\alpha} g(t) = (t^{1+2\alpha} a(t) - a_0) + a_0(1 - t^{1+2\alpha}\varphi_0(t)) - a_\infty t^{1+2\alpha}\varphi_\infty(t) \\ = (t^{1+2\alpha} a(t) - a_0) + (a_0 - a_\infty) t^{1+2\alpha}\varphi_\infty(t). \end{aligned}$$

Thus, by the hypothesis (7.1.6) and by the second estimate in (7.2.10), we obtain

$$\frac{d^m}{dt^m}(t^{1+2\alpha} g(t)) = O(t^{-m+\varepsilon}) + O(t^{-m+1+2\alpha}) = O(t^{-m+\varepsilon'}), \quad \varepsilon' = \min\{\varepsilon, 1+2\alpha\},$$

as $t \to 0$. Similarly, we have

$$t^{1+2\alpha} g(t) = (t^{1+2\alpha} a(t) - a_\infty) + (a_\infty - a_0) t^{1+2\alpha} \varphi_0(t),$$

and so, by the hypothesis (7.1.7) and by the first estimate in (7.2.10), we get

$$\frac{d^m}{dt^m} (t^{1+2\alpha} g(t)) = O(t^{-m-\varepsilon}), \quad t \to \infty.$$

Thus, g satisfies the hypothesis of Lemma 7.2.2 and so we obtain

$$w_\alpha H(g) w_\alpha \in \mathfrak{S}_1, \quad \text{and so} \quad w H(g) \overline{w} \in \mathfrak{S}_1.$$

This gives the intermediate representation

$$\begin{aligned} wH(a)\overline{w} &= a_0 w H(\varphi_0)\overline{w} + a_\infty w H(\varphi_\infty)\overline{w} + T' \\ &= a_0 v L_\alpha \mathbb{1}_\infty L_\alpha \overline{v} + a_\infty v L_\alpha \mathbb{1}_0 L_\alpha \overline{v} + T', \quad T' \in \mathfrak{S}_1. \end{aligned} \tag{7.2.22}$$

Consider the first term in the right side of (7.2.22). We can write

$$L_\alpha \mathbb{1}_\infty L_\alpha = \mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0 + (\mathbb{1}_\infty L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_\infty + \mathbb{1}_\infty L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0 + \mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_\infty).$$

By Lemma 7.2.3, all terms in brackets here are trace class operators, and so we obtain

$$v L_\alpha \mathbb{1}_\infty L_\alpha \overline{v} - v \mathbb{1}_0 L_\alpha \mathbb{1}_\infty L_\alpha \mathbb{1}_0 \overline{v} \in \mathfrak{S}_1.$$

In a similar way, we obtain

$$v L_\alpha \mathbb{1}_0 L_\alpha \overline{v} - v \mathbb{1}_\infty L_\alpha \mathbb{1}_0 L_\alpha \mathbb{1}_\infty \overline{v} \in \mathfrak{S}_1.$$

Substituting this back into (7.2.22), we arrive at (7.2.21). □

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