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A contest success function for networks☆

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Abstract

This paper models conflict as a contest within a network of friendships and enmities. We assume that each player is either in a friendly or in an antagonistic relation with every other player and players compete for winning by exerting costly efforts. We axiomatically characterize a success function which determines the win probability of each player given the efforts and the network of relations. In an extension, we allow for varying intensities of friendships and enmities. This framework allows for the study of strategic incentives and friendship formation under conflict as well as the application of stability concepts of network theory to contests.

Keywords: conflict, contest, success function, network, pairwise stability
JEL classification: C70, D72, D74, D85

1. Introduction

In many situations of conflict, we often observe that competing parties join forces to fight together against others or refrain from fighting with each other. For instance, lobby groups may cooperate in supporting the same legislation when their interests coincide; political parties may refrain from campaigning against each other when they have a common opportunity; belligerent states may form alliances for joint action if they face a common threat and so on. These parties do not necessarily act in a perfectly coordinated way, especially when their relation is an occasional opportunistic cooperation rather than a long term commitment. Such relations usually rely on informal bilateral agreements and may lead to a complex network.

This paper models conflict as a contest, where players compete for increasing their win probabilities by exerting costly efforts. While doing so, each pair of players

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may be in a *friendly* relation and abstain from competing against each other (their default state being antagonism), and the friendship relations between all pairs define a network. In this setting, we propose and axiomatically characterize a *success function* which determines the probability of winning for each player given all efforts and the network of relations. So far, the axiomatic work in the contest literature has exclusively focused on conflict between groups (e.g., Münster, 2009; Cubel and Sanchez-Pages, 2016) or between individuals (e.g., Skaperdas, 1996). In the former, players are divided into mutually exclusive groups (coalitions) and groups compete with each other, while in the latter each player competes individually against all others. Both approaches are crucial for the study of a broad set of environments. Yet, many competitive situations may lead to networks different than the *all against all* or *groups against groups* types of networks. For instance, in international relations most alliances between states do not mean perfect coordination or long term commitments. The opportunistic nature of tactical alliances[1] does not exclude a partnership between states which are not members of the same coalition, and the friend of a friend can be an enemy at times. Political competition may also lead to a complex network of relations. Despite the zero-sum nature of political gains, we see that a political party does not necessarily target all its opponents during campaigning and it might have a preference for which opponent to damage. In Figure 1, we illustrate a network representing a four party political competition. In this example, the largest opposition party does not campaign against the nationalist party and the minority party hoping that there may be a vote switch from the centrist ruling party to them, which increases the largest opposition's chance of winning the election. However, a coalition of opposition parties fails to emerge given the historical conflict between the nationalist party and the minority party. This 'missing link' should intuitively make the ruling party better off compared to facing a coalition of opponents, while also strengthening the relative power of the largest opposition party within the opposition (being the 'hub' of the opposition). We aim to capture such strategic features of competition in our model. Our paper extends the axiomatic foundations of success functions to contests with any type of network of relations. As a starting point, we propose a class of success functions which we derive through a probabilistic argument following well-known results by McFadden (1973). We simply show that a player's probability of winning is equivalent to the probability that the *effective strength* of the player, which is an additively separable function of the efforts of her friends and enemies, is higher than anyone else's effective strength perturbed by random noise. This class is very general and particularly convenient in a variety of applications. To understand how win probabilities depend on the relations as well as efforts, we provide an axiomatic characterization. Our characterization consists of six axioms, three of which are direct extensions of the well-known anonymity, monotonicity of efforts and exhaustivity axioms in

[1] In his categorization of strategic alliances, Ghez (2011) defines the tactical alliance as a form of state alignment which occurs when states encounter a common immediate threat. For further examples of tactical alliances and definitions of other categories, see Ghez (2011).

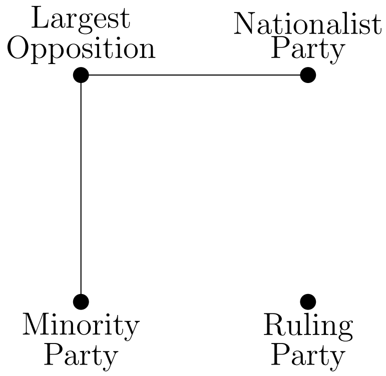

Figure 1: An example network for political competition. We consider an electoral competition with 4 parties: the ruling party, the largest opposition, the nationalist party and the party representing a minority. A link between two parties represents friendship, and the political competition in this country leads to a network where a friend of a friend is an enemy.

the literature. We define three new axioms; namely, the *monotonicity of relations*, the *independence of efforts of commons* (IEC), and the *independence of relations of others* (IRO), all of which incorporate the effect of the network variable on the success function. The monotonicity of relations simply imposes that befriending a player with higher effort than one's own leads to an increase in the probability of winning. We define our two independence axioms on the *relative* win probability of two players; in other words, on the ratio of their win probabilities. IEC states that the relative win probability of two players is independent of the efforts of their common friends and common enemies. In the context of the example in Figure 1, IEC imposes that the relative win probabilities of the minority party and the nationalist party will not change by increased campaigning efforts of the largest opposition party (although each probability will increase). Hence, the effort of the largest opposition affects both parties similarly, so that their relative win probability remains constant. Our final axiom, IRO, allows for making across network comparisons and together with monotonicity of relations, it identifies how probabilities change in response to a changing relation. It implies that the *rate of change* of a relative probability as a result of a new friendship or enmity (the ratio of the new relative probability to the old) remains the same across all pairs of networks which differ only by the new relation. Once we restrict our attention to all against all contests, our functional form belongs to the well-established class characterized by Skaperdas (1996).[2] For contests between groups, our class does not immediately link to the class axiomatized in Münster (2009) as our function determines win probabilities of individual players rather than groups. However, the function obtained by summing up the win probabilities of group members derived by our function belongs to the class axiomatized in Münster (2009).

[2]The success functions characterized in Skaperdas (1996) had been widely applied to represent conflict. For seminal contributions, see, e.g., Haavelmo (1954), Tullock (1975), Rosen (1986).

An immediate extension of our model is to allow for varying intensity of friendship/antagonism across players. We extend both our probabilistic derivation and our axiomatic characterization to the case of *weighted directed networks*, where each player can have different degree of friendliness towards every other player and the degree of friendliness between two players is not necessary mutual. The resulting generalized functional form naturally contains the benchmark case of (unweighted undirected) networks as well as the case of (unweighted) directed networks.

Our framework is useful in connecting two major fields, namely contests and network theory. We consider an application of our benchmark model where players choose both efforts and relations in a two-stage setting where relations are chosen first. We define a solution concept that combines pairwise stability of networks and Nash equilibrium of efforts in line with the idea of subgame perfection. In a contest with symmetric effort costs, our analysis shows that there is always an equilibrium whose outcome is the peace network and the effort profile where all efforts are zero. On the other hand, this would not necessarily be the outcome if players were constrained to be in coalitions or if we introduced cost asymmetries.

The paper develops as follows. Section 2 reviews the literature. We introduce our model formally in Section 3 for the case where all efforts are positive and define our class of success functions. We derive this class probabilistically in Section 3.1, while we provide an axiomatic characterization in Section 3.2. In Section 4.1 we extend the model to the cases where efforts can be zero, and in Section 4.2 we apply this model to network formation problems and contest games. Finally, we extend the model to directed and weighted directed networks in Sections 4.3 and 4.4 respectively. Section 5 concludes. All proofs are in Appendix.

2. Related literature

The contest is the workhorse model for representing conflict and competition over scarce resources. Contest models have been applied in a variety of areas of economics and social sciences, such as rent seeking, industrial organization, incentives within organizations and armed conflict. See Konrad (2009) for an introduction to contest theory and its applications.

A crucial element of a contest model is the success function, which generally defines the mapping from individual efforts to win probabilities. The nature of a contest fundamentally depends on the features of this functional form. Foundational work on success functions is divided into two leading approaches; namely, axiomatic and stochastic approaches. The axiomatic approach to contest success functions started with the seminal work by Skaperdas (1996), which characterizes a class of success functions that includes the well-known ratio-form in Tullock (1975) and the difference-form in Hirshleifer (1989). This characterization has been extended in several directions by relaxing some of the axioms (e.g., Clark and Riis, 1998, Blavatskyy, 2010; see Jia et al., 2013 for a review), by generalizing to multi-dimensional efforts of players (e.g., Rai and Sarin, 2009, Arbatskaya and Mialon, 2010), or by allowing rankings as the outcome of a contest instead of a single winner (e.g., Lu and Wang, 2015, 2016, Vesperoni, 2016). While these contributions are exclusively on all against all contests, Münster (2009) and Cubel and Sanchez-Pages (2016) axiomatically characterize success functions for contests where the competition takes place between mutually exclusive groups of players, i.e., coalitions. Such networks are very important in several contexts; however, they constitute only a small subset of possible networks in conflict. Our paper follows the axiomatic approach in contests, allowing players to compete in every possible network of friendships and enmities. As for the second body of literature on the foundations of success functions, it is concerned with stochastic derivations of success functions following techniques from discrete choice econometric models. The stochastic approach so far has focused on success functions for all against all contests (e.g., Hirshleifer and Riley, 1992, Clark and Riis, 1996, Jia, 2008, Fu and Lu, 2012; see Jia et al., 2013 for a review). Together with the axiomatic characterization, we motivate our function by deriving it following the standard probabilistic arguments in McFadden (1973).

We now selectively review works which study the broad subject of conflict when players compete in networks or are linked to each other in a way that may indirectly define a network. None of these works is axiomatic, nor their focus is on the success function. A recent paper by König et al. (2017) considers conflict within networks of alliances, where they propose a success function that determines each player's win probability. They show that the equilibrium effort of a player is related to an index of her centrality in the network under some restrictions, and they perform an empirical analysis using data from the Second Congo War. This work is an important advance into directions different from our work which is concerned with the axiomatic foundations of the conflict mechanism. A related body of literature is the work on sabotage contests where players direct a specific effort to handicap each particular opponent (e.g., Konrad, 2000, Gürtler, 2008, Gürtler and Münster, 2010; see Amegashie, 2015 for a review). As the efforts are opponent specific, the intensity of competition between each pair of players is different like in our work; however, besides the differences in the theoretical approach (such as multiple efforts, non-axiomatic approach) they do not consider alliances. Other related work with indirectly defined networks is on contests with identity dependent externalities. For example, in some of these papers players may value victory identically, however, the value of defeat depends on the identity of the winner. The identity-based losses of all pairs can be seen as defining a network of relations between players. Applications of this setup are to model ethnic conflict between minorities (e.g., Esteban and Ray, 1999, 2011) or political lobbying between parties on an ideological spectrum (e.g., Klose and Kovenock, 2012, 2013).

Several papers in the subject of coalition formation in conflict study contests between groups. Following the seminal contribution of Olson (1965), many of these works focus on the collective action problem in groups, showing that the power of a group may not increase in its size due to free-riding in the provision of collective effort. As this literature is very broad, we do not attempt a review and refer to Bloch (2012) for a comprehensive survey on endogenous formation of groups in conflict. Other papers consider contests with multiple battlefields; see Kovenock and Roberson (2012b) for an introduction to this literature. In this setting, Kovenock and Roberson (2012a) study incentives for alliance formation in the sense of pooling resource budgets when fighting a common adversary on separate battlefields. A related paper is Rietzke and Roberson (2013), which shows that two players facing a common adversary on separate battlefields have an incentive to enhance each other's strength by transfers that are (not necessarily) pre-committed. Although not strictly related to our paper (as their focus is mostly on network analysis), there are contributions on the broader subject of conflict within networks. Hiller (2017) analyzes a model where there are as many local conflicts as pairs of players and the win probabilities for each pair are determined by the number of their friends. In this model, there is no effort parameter and payoffs are fully determined by the network of relations. Franke and Öztürk (2015) consider players embedded in a network of bilateral conflicts where each pair can choose to fight in each conflict by spending conflict-specific efforts or refrain from fighting. For each bilateral conflict they assume a success function of the ratio-form by Tullock (1975), and the interdependence across conflicts follows from the non-separability of a player's cost function. In this setting, they characterize equilibrium efforts given specific types of conflict networks. Jackson and Nei (2015) define and analyze a new solution concept, called war-stability, for networks where each player is in a friendly or antagonistic relation with every other player. A necessary condition for war-stability is that no coalition of players can successfully attack another coalition. Unlike ours, their success function is deterministic and takes value 0 or 1, and in their model efforts are always exogenous parameters. Goyal and Vigier (2014) consider a two-player game where a designer chooses a network and allocates specific efforts to defend each node, while an adversary allocates specific efforts to attack each node after observing these. They find the optimal network structure for the designer when the probability of destruction of each node is given by the ratio-form success function by Tullock (1975). Dziubinski et al. (2017) analyze a dynamic game of conflict between multiple players on a network of spatial proximities. At each stage a randomly selected player chooses whether to attack one of her neighbors, where in case of attack her win probability is determined by resources cumulated in previous conflicts via a ratio-form success function by Tullock (1975). In this setting, they find that the dynamics of conflict crucially depend on properties of the success function interpreted as rich/poor rewarding.

3. Modeling networks in conflict

We consider a set of players $N = \{1, \ldots, n\}$, where $n \geq 3$. Players compete in a contest for increasing their win probabilities. We assume that a player $i \in N$ is either in a *friendly* relation or in an *antagonistic* relation with every other player in N. We write $F_i \subseteq N$ for the set of friends of i including i herself. In this section, we assume that relations between friends (or enemies) are mutual, so for any pair of players $i, j \in N$, we have $i \in F_j$ if and only if $j \in F_i$. We define a *network* as the profile of sets of friends $F := (F_1, ..., F_n)$ and we denote by $\mathcal{F}$ the set of all networks.[3] Each player $i \in N$ is associated with an effort $x_i > 0$, and we write $x = (x_1, ..., x_n) \in X := \mathbb{R}^n_{++}$ for the profile of efforts. A *success function* defines for each player $i \in N$ a mapping $s_i : X \times \mathcal{F} \to (0, 1)$, which maps any effort profile and network pair (x, F) into player i's win probability $s_i(x, F)$.

We now define a particular class of success functions. For each network $F \in \mathcal{F}$, effort profile $x \in X$ and player $i \in N$, this class is defined by the form

$$s_i^*(x, F) := \frac{\prod_{j \in F_i} f(x_j)}{\sum_{h \in N} \prod_{j \in F_h} f(x_j)}, \tag{1}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is any increasing function. When we restrict our attention to the all against all network, the class of success functions s_i^* belongs to the one characterized in Skaperdas (1996).[4] Conversely, for the peace network where there is no enmity, $s_i^*(x, F) = 1/n$ and the contest degenerates into a fair lottery with equal chances of winning. For contests between groups, s_i^* does not immediately link to the class of success functions in Münster (2009) since it defines win probabilities of individual players rather than groups of players. However, we can easily relate the two concepts by a simple transformation where the win probability of a group is defined by the sum of the win probabilities of its members. The win probability of a group defined by this sum using s_i^* belongs to the class of Münster (2009). We provide a formal derivation in Appendix A. The subclass of group success functions derived from s_i^* corresponds to a low degree of cooperation within coalitions since we conceive friendship as refraining from damaging each other.[5]

3.1. Stochastic derivation

We now show that the form (1) can be derived from basic assumptions via a standard probabilistic argument. For each $x \in X$, $F \in \mathcal{F}$ and $i \in N$, let the *effective strength* of player i be

$$y_i(x, F_i) := \beta \sum_{j \in F_i} g(x_j) - (1 - \beta) \sum_{j \notin F_i} g(x_j),$$

[3] In standard network theory, a network or a *graph* $\mathbf{g}$ is defined as a list of unordered pairs of players $\{i, j\}$ which are linked. Here instead, we define a network as a profile of sets of friends for convenience. We can easily link the two definitions in the following way: for each $F \in \mathcal{F}$ we define $\mathbf{g}$ such that $\{i, j\} \in \mathbf{g}$ if and only if $i \in F_j$.

[4] For the all against all network, our class comprehends the difference-form by Hirshleifer (1989) where $f(x_i) = \exp(x_i)$. Since $f > 1$, the ratio-form by Tullock (1975) where $f(x_i) = x_i^\rho$ with $\rho > 0$ is ruled out; however, the closely related forms $f(x_i) = 1 + x_i^\rho$ and $f(x_i) = (1 + x_i)^\rho$ are included in our class and lead to equivalent predictions in many applications.

[5] This does not constitute a limitation for the purpose of applications. A coalition that combines efforts via a production function can be seen as committed to long-term cooperation and treated as a single player in our model.

where $\beta \in [0,1]$ and $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is any increasing function. Intuitively, a player that competes within a network acquires strength from her friends' efforts (including her own), while her strength may be diminished by the efforts of her enemies. In our specification, the parameter β determines the relative importance of friends and enemies in determining the effective strength of a player, while the additive separability of $y_i(x, F_i)$ in each transformed effort $g(x_j)$ keeps the model simple, imposing perfect substitutability between the transformed efforts.[6] We assume that win probabilities are determined by players' effective strengths, so that the player with the highest effective strength should be the most likely to win. More specifically, for each $x \in X$ and $F \in \mathcal{F}$, let the win probability of player $i \in N$ satisfy

$$s_i(x, F) = \Pr\left(y_i(x, F_i) + \epsilon_i > y_j(x, F_j) + \epsilon_j \text{ for any } j \in N \setminus \{i\}\right), \tag{2}$$

where $(\epsilon_1, \ldots, \epsilon_n)$ are independent and identically distributed Gumbel random shocks. For simplicity let their variance be $\pi^2/6$, so that we get rid of a parameter in what follows. Following McFadden (1973), condition (2) holds if and only if [7]

$$s_i(x, F) = \frac{\prod_{j \in F_i} \exp\left(g(x_j)\right)}{\sum_{h \in N} \prod_{j \in F_h} \exp\left(g(x_j)\right)}. \tag{3}$$

Since $\exp(z) > 1$ for all $z > 0$, $\lim_{z \to 0} \exp(z) = 1$ and the exponential function is continuous and increasing, for any g there is a unique f such that $f(x_i) = \exp\left(g(x_i)\right)$ for all $x_i > 0$ (and for any f there is a unique g with $g(x_i) = \ln\left(f(x_i)\right)$ for all $x_i > 0$). Hence, (1) and (3) define the same class of functions. To summarize, we have shown that any success function from our class (1) is equivalent to the probability that the effective strength of a player is higher than anyone else's, in a context where effective strength takes an additive form and it is perturbed by random noise.[8]

3.2. Axiomatic characterization

In this section we give a characterization of our class of success functions (1) through six axioms. The first three axioms are direct extensions of classical axioms in contest theory and they have similar justifications in our model. The latter three axioms incorporate the concept of friendships in conflict. The first condition, exhaustivity, requires that win probabilities of all players sum up to 1.

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[6] In our axiomatic characterization, the independence axioms IEC and IRO are strictly related to this separability assumption. Alternatively, one may consider different specifications where complementarities between efforts of friends or enemies may arise under particular network structures.

[7] If we add $(1-\beta)\sum_{j \in N} g(x_j)$ to each side of the inequality $y_i(x, F_i) + \epsilon_i > y_j(x, F_j) + \epsilon_j$ in (2), β always cancels out no matter which value it takes in $[0,1]$ since the set of friends and the set of enemies of i are two distinct sets whose union is N. This is why the expression (3) is free of the parameter β.

[8] Clark and Riis (1996) develop a similar derivation of a success function for all against all contests.

Exhaustivity: For any $F \in \mathcal{F}$ and $x \in X$, $\sum_{i \in N} s_i(x, F) = 1$.

Anonymity states that win probabilities are determined by efforts and networks, but not by players' identities. In short, it requires the contest to be a priori fair.

Anonymity: Let α be any permutation of N. For any $F \in \mathcal{F}$ and any $x \in X$, let $\alpha(F) = (F_{\alpha(1)}, ..., F_{\alpha(n)})$ and $\alpha(x) = (x_{\alpha(1)}, ..., x_{\alpha(n)})$. Then, $s_i(x, F) = s_{\alpha(i)}(\alpha(x), \alpha(F))$ for each $i \in N$.

We now impose two monotonicity axioms; namely, monotonicity of efforts and monotonicity of relations. Monotonicity of efforts imposes that the win probability of a player is increasing in her effort. Monotonicity of relations implies that being friends with a player with higher effort increases the win probability. The higher effort requirement is due to the fact that a friendship is a mutual relation and befriending a player has the externality of helping the other player as well as helping oneself. The overall effect on win probabilities may depend on the relative efforts of players but one should always benefit from befriending a player with higher effort.

Monotonicity of efforts: Let $F \in \mathcal{F}$ be any network with some $i, j \in N$ such that $i \notin F_j$ and $x \in X$ be any effort profile. Then, $s_i(x', F) > s_i(x, F)$ for any $x' \in X$ with $x'_i > x_i$ and $x'_k = x_k$ for all $k \neq i$.

Monotonicity of relations: Let $F \in \mathcal{F}$ be any network and $x \in X$ be any effort profile such that there is a pair $i, j \in N$ with $i \notin F_j$. Consider $F' \in \mathcal{F}$ such that $i \in F'_j$ and $F'_h = F_h$ for all $h \notin \{i, j\}$. Then, $s_i(x, F') > s_i(x, F)$ if $x_j > x_i$.

We finally introduce two axioms of independence. The first one, independence of efforts of commons, imposes that the ratio of the win probabilities of two players (their *relative* win probability) is independent of the efforts of their common friends and common enemies. We focus on the ratio of win probabilities rather than, for instance, the difference of win probabilities following a long tradition in probabilistic choice theory which dates back to the seminal work of Luce (1959) on independence of irrelevant alternatives.[9] The idea is that the efforts of common friends and common enemies should similarly affect both players' win probabilities; hence, they should not affect their relative win probability.

Independence of efforts of commons (IEC): Take a network $F \in \mathcal{F}$ and two effort profiles $x, x' \in X$. For any pair of players $i, j \in N$, $\frac{s_i(x,F)}{s_j(x,F)} = \frac{s_i(x',F)}{s_j(x',F)}$ if $x_k = x'_k$ for all $k \in (F_i \cup F_j) \setminus (F_i \cap F_j)$.

The axioms above say very little about what should happen when we move from one network to another. How should we expect win probabilities to change when new friendships/enmities are made, besides that win probabilities should increase upon making a friend with higher effort? The second independence axiom, independence

[9] Most axiomatic work in contest theory follows a related approach (see, e.g., Skaperdas, 1996, Clark and Riis, 1998, Blavatskyy, 2010). An alternative approach is to focus on the difference of win probabilities (see, e.g., Cubel and Sanchez-Pages, 2016).

of relations of others, focuses on the relative win probability of two players and identifies how the relative win probability should change when one of these players makes a new friend or enemy. More specifically, the axiom requires that the *rate of change* of this relative win probability (the ratio of the new to the old) remains the same across all pairs of networks which differ only by the new friendship. So, it can also be seen as a consistency requirement imposing the change that results from befriending a player to be consistent across networks. As for IEC, we focus on ratios rather than differences in win probabilities in line with the aforementioned literature in probabilistic choice theory.

Independence of relations of others (IRO): Let $F \in \mathcal{F}$ be a network such that there are at least two players $i, j \in N$ with $j \in F_i$, and $x \in X$ be any effort profile. Let $F' \in \mathcal{F}$ be the network such that $j \notin F'_i$ and $F'_h = F_h$ for all $h \notin \{i, j\}$. Then, $\left(\frac{s_i(x,F')}{s_h(x,F')}\right) / \left(\frac{s_i(x,F)}{s_h(x,F)}\right) = \left(\frac{s_i(x,G')}{s_h(x,G')}\right) / \left(\frac{s_i(x,G)}{s_h(x,G)}\right)$ for all $h \in N \setminus \{i, j\}$ and $G, G' \in \mathcal{F}$ with $j \in G_i$, $j \notin G'_i$ and $G_k = G'_k$ for all $k \notin \{i, j\}$.

Consider the example with $n = 4$ in Figure 2. By IRO, the rate of change of the relative win probability of a pair of players from network F to F' is equal to the corresponding rate of change from network G to G'. For instance, focusing on players 1 and 3, $\left(\frac{s_1(x,F')}{s_3(x,F')}\right) / \left(\frac{s_1(x,F)}{s_3(x,F)}\right)$ and $\left(\frac{s_1(x,G')}{s_3(x,G')}\right) / \left(\frac{s_1(x,G)}{s_3(x,G)}\right)$ must be equal to each other for all $x \in X$. The idea is that, although the networks F and G are very different, the corresponding win probabilities should change in a similar way when the relation between the same pair of players mutates in both networks.

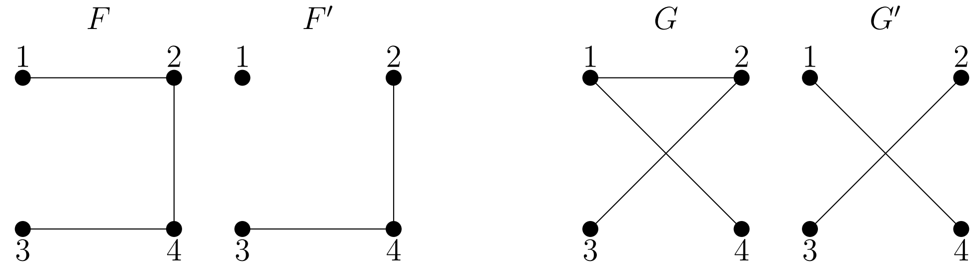


Figure 2: Networks F, F' (G, G') are identical except for the missing friendship between players 1 and 2 in network F' (G').

The following theorem states that our six axioms uniquely characterize the particular class of success functions s_i^* defined by (1).

Theorem 1. *A success function satisfies exhaustivity, anonymity, monotonicity of efforts, monotonicity of relations, IEC and IRO if and only if* $s_i(x, F) = s_i^*(x, F)$ *for each* $i \in N$, $x \in X$ *and* $F \in \mathcal{F}$.

Our characterization in Theorem 1 is tight. In Appendix C, we prove the independence of our axioms by means of examples of success functions satisfying all but one. Our axioms have desirable joint implications. For instance, it can be shown that a player's win probability always increases in her friends' efforts and decreases in her enemies' efforts. Moreover, when a player becomes friend with a player which has much lower effort her win probability may decrease, and a player is always worse off when other players become friends. To see an example, consider three players in all against all network F and the effort profile x with transformed efforts $f(x_1) = 2$, $f(x_2) = 7$ and $f(x_3) = 8$. The win probability of player 2 decreases if player 2 becomes friend with player 1, everything else equal. If $f(x_1) = 6$ instead, befriending player 1 makes player 2 better off in terms of win probability. Player 3, on the other hand, always loses from this friendship. Our first four axioms are natural properties and we believe they should be satisfied by any success function for networks. IEC and IRO treat ratios of win probabilities as a measure of relative performance, which leads to the multiplicative form in the numerator of (1). Alternative approaches to relative performance (e.g., differences of win probabilities) may lead to different functional forms whose analysis we leave for future research.

4. Applications and extensions

Due to its novelty and generality, our framework has potential for many applications. We first extend our success function in Section 4.1 to a setting where we allow for zero efforts. Then, using this extension, in Section 4.2 we consider an application where players choose both their efforts and relations. More specifically, we consider a two-stage setting where relations are chosen in the first stage and efforts are exerted in the second stage.

We have so far assumed that the friendship between two players is mutual and that the degree of friendship is the same across friends. This setting belongs to a broader class of models of conflict on networks where each friendship might have a different weight and is not necessary mutual. In Sections 4.3 and 4.4, we extend our model by introducing directed and weighted directed friendships respectively. We derive a generalized class of success functions via a probabilistic argument and provide an axiomatic characterization of this class.

4.1. The case of zero efforts

So far, we have excluded the cases where efforts are zero or probabilities take value 0 or 1.[10] For this section, we allow that efforts can take zero value, hence, $x_i \geq 0$ for each $i \in N$ and an effort profile is given by $x \in \hat{X} := \mathbb{R}^n_+$. Extending our previous notation, a success function defines for each player $i \in N$ a mapping

[10]In fact, our definitions, probabilistic derivation and axiomatic characterization can be adjusted to incorporate the zero effort case, but at the expense of heavy notation and complications in the proof. We believe for the purpose of the axiomatic characterization these are marginal cases, but for application purposes it is useful to specify how our function extends.

$s_i : \hat{X} \times \mathcal{F} \to [0,1]$ which maps any effort profile and network pair (x, F) into player i's win probability $s_i(x, F)$.

How do we extend our class of functions to incorporate zero efforts? Following the existing assumptions in the literature, a player who exerts positive effort should have positive probability of winning. Moreover, it seems natural that if no player exerts positive effort, then each player should have probability of winning equal to $1/n$ as no player actively participates in the conflict. It is generally assumed that if a player exerts zero effort while some other player exerts positive effort, her win probability is zero; but in our context, we have to distinguish players whose friends exert positive effort from those who have no friends with positive efforts whatsoever. To incorporate all these aspects in our extended success function, we first define some notation. For each network $F \in \mathcal{F}$ and effort profile $x \in \hat{X}$, we define the set of *active* players as $A_{x,F} := \{i \in N : x_j > 0 \text{ for some } j \in F_i\}$; that is, the set of players having at least one friend exerting positive effort. For each network $F \in \mathcal{F}$, effort profile $x \in \hat{X}$ and player $i \in N$, we define our class of success functions by

$$\hat{s}_i(x, F) := \begin{cases} \dfrac{\prod_{j \in F_i} f(x_j)}{\sum_{h \in A_{x,F}} \prod_{j \in F_h} f(x_j)} & \text{if } i \in A_{x,F} \neq \emptyset, \\ 1/n & \text{if } A_{x,F} = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $f : \mathbb{R}_+ \to [1, +\infty)$ is any increasing function with $f(0) = 1$.[11] Note that, in (4), the win probability of a player is zero whenever all friends of the player exert zero effort but there is some player who exerts positive effort; so the player is inactive but the set of active players is non-empty. Conversely, the win probability of a player is positive whenever at least one of the player's friends exerts positive effort (i.e., whenever the player is active), while when all players exert zero effort they are all equally likely to win. We can summarize these points in the following axiom which is satisfied by (4).

Perfect discrimination at zero: For any $F \in \mathcal{F}$ and $x \in \hat{X}$, *(i)* $s_i(x, F) > 0$ if $x_j > 0$ for some $j \in F_i$; *(ii)* $s_i(x, F) = 1/n$ if $x_i = 0$ for all $i \in N$; *(iii)* $s_i(x, F) = 0$ if for all $j \in F_i$, $x_j = 0$ and $x_k > 0$ for some $k \in N$.

4.2. Choosing relations and effort

In this section, we aim to demonstrate the applicability of our model and its potential for future research. We consider a game where players choose their relations as well as their efforts in two stages. In the first stage, players choose their relations and a network $F \in \mathcal{F}$ is formed, while in the second stage each player chooses her effort given the network F. We denote by $x(F)$ the effort profile that results in the second

[11] The restriction $f(0) = 1$ guarantees that becoming friend of a player with zero effort cannot increase the win probability.

stage given F. Following standard assumptions in the literature, we let the payoff of player $i \in N$ take the form

$$\pi_i(x, F) := \hat{s}_i(x, F) - c_i x_i \tag{5}$$

for all $x \in \hat{X}$ and $F \in \mathcal{F}$, where $c_i > 0$ for all $i \in N$. We now define a solution concept that combines pairwise stability of networks and Nash equilibrium of efforts in line with the idea of subgame perfection, assuming relations are formed forecasting how they will affect effort exertion.[12] Given a network $F \in \mathcal{F}$ and a pair of players $i, j \in N$ with $i \in F_j$, let $F - ij$ be the network where players i and j become enemies while all other relations remain as in F. Similarly, given a network $F \in \mathcal{F}$ and a pair of players $i, j \in N$ with $i \notin F_j$, let $F + ij$ be the network where players i and j become friends while all other relations remain as in F. For each network $F \in \mathcal{F}$, let

$$\mathcal{G}_F := \{F' \in \mathcal{F} : F' = F \text{ or } F' = F + ij \text{ or } F' = F - ij \text{ for some } i, j \in N\}$$

be the set of networks that differ from F by at most a pair's relation. We define an equilibrium as follows.

Definition 1. *A network $F^* \in \mathcal{F}$ and an effort profile $x^*(F) \in \hat{X}$ for each $F \in \mathcal{G}_{F^*}$ constitute an equilibrium if, for each player $i \in N$ and network $F \in \mathcal{G}_{F^*}$,*

$$x_i^*(F) = \arg\max_{x_i \geq 0} \pi_i(x_i, x_{\neg i}^*(F), F),$$

and the network F^* satisfies

(i) *for all $i, j \in N$ with $i \in F_j^*$, $\pi_i(x^*(F^*), F^*) \geq \pi_i(x^*(F^* - ij), F^* - ij)$ and $\pi_j(x^*(F^*), F^*) \geq \pi_j(x^*(F^* - ij), F^* - ij)$, and,*

(ii) *for all $i, j \in N$ with $i \notin F_j^*$, $\pi_i(x^*(F^*), F^*) \geq \pi_i(x^*(F^* + ij), F^* + ij)$ or $\pi_j(x^*(F^*), F^*) \geq \pi_j(x^*(F^* + ij), F^* + ij)$.*

Hence, an equilibrium contains a network F^* and an effort profile $x^*(F)$ for each network $F \in \mathcal{G}_{F^*}$. Each effort profile $x^*(F)$ must be a Nash equilibrium given the corresponding network F, and the network F^* must be pairwise stable given that efforts depend on networks according to the aforementioned Nash equilibria.[13]

[12] Pairwise stability is a well-known solution concept in network formation theory (see Jackson, 2005 for a review). In a pairwise stable network, no player can be better off by unilaterally breaking a friendship and no pair of players can both be better off by becoming friends. A refinement of this solution concept is pairwise Nash equilibrium, introduced by Jackson and Wolinsky (1996). Roughly speaking, in a pairwise Nash equilibrium any mutually beneficial friendship is always implemented (as with pairwise stability), while multiple friendships can be broken simultaneously. We refer to Bloch and Jackson (2006) for a discussion of these solution concepts. For simplicity, in our analysis we exclusively focus on pairwise stability, although all our findings are robust to employing pairwise Nash equilibrium as the solution concept.

[13] Goyal and Joshi (2003) employ the same solution concept in a model of oligopolistic competition between firms linked in a network. Belleflamme and Bloch (2004) and Calvó-Armengol and

We refer to F^* and $x^*(F^*)$ as the outcome network and the outcome effort profile respectively.

Proposition 1. *Let payoffs take the form (5) with $f(x_i) = 1 + x_i^\rho$ and $c_i = c$ for all $i \in N$, where $\rho \in (0, 1)$ and $c > 0$ or $\rho = 1$ and $c \in (0, 1/n^2)$. There is an equilibrium where the outcome network F^* is the peace network and the outcome effort profile is $x^*(F^*) = (0, \ldots, 0)$.*

This proposition implies that when effort costs are symmetric 'unarmed peace' is always an equilibrium outcome, where for $\rho = 1$ the restriction $c \in (0, 1/n^2)$ is necessary to guarantee an interior Nash equilibrium $x^*(F)$ for every network $F \in \mathcal{G}_{F^*}$. The intuition is straightforward. To see whether peace is pairwise stable it is sufficient to verify that no player has an incentive to break the friendship with another player while keeping other friendships intact. In this context, when two players become enemies they always end up damaging each other while favoring others, as effort costs are private while benefits are shared with friends within the network.[14]

We illustrate the result in Proposition 1 by a numerical example in Table 1 below, where we assume $c_i = 0.01$ for each $i \in N$. One can see that F^4 in Table 1 (i.e., the peace network) is the unique equilibrium network. To see how robust Proposition 1 is, we introduce asymmetries in effort costs by letting $c_3 = 0.001$ while keeping the rest of the parameters fixed, which gives us Table 2 below. It turns out that once we introduce this asymmetry, we still obtain a unique equilibrium but the outcome network is no longer the peace network but G^4 (up to the permutation of identities of players 1 and 2). Moreover, welfare (defined as the sum of players' payoffs) is monotonic in the number of friendships under symmetric costs as seen in Table 1, but this is no longer the case under asymmetric costs as seen in Table 2. While for the symmetric case the welfare minimizing network is F^1 (all against all network), when we introduce the asymmetry network G^2 (where player 1 and player 2 form a coalition against player 3) minimizes welfare. This is because player 1 and player 2 have higher effort costs than player 3 and an alliance between them balances the contest, fostering competition and leading to higher equilibrium efforts compared to the all against all network where player 3 dominates.

Let us now discuss what type of networks we may obtain in equilibrium if we restrict the type of alliances to coalitions only (as in a standard group contest à la Münster, 2009). More specifically, suppose individuals can leave coalitions unilaterally and pairs of coalitions can merge by unanimous consent of their members. In our numerical example in Table 1 with symmetric effort costs, F^2 is the unique equilibrium of

Zenou (2004) take a similar approach but consider pairwise Nash equilibrium instead of pairwise stability as the solution concept for network formation in the first stage.

[14]If we instead employ pairwise Nash equilibrium as the solution concept for the first stage, we should check that no player has an incentive to break *any number* of friendships (not just one friendship). One can show this is true under the conditions in Proposition 1, where the existence of a Nash equilibrium $x^*(F)$ for each $F \in \mathcal{G}_{F^*}$ is generally guaranteed for 'small enough' c.

the coalition game (up to all permutations of identities of players). Intuitively, the members of the coalition 'collude' against the single player, even though this leads to a welfare loss (i.e., the sum of equilibrium payoffs is lower than with unarmed peace) due to the positive equilibrium efforts. This suggests that coalitions can be harmful for peace (and welfare) as members of a coalition can prevent each other from becoming friends with outsiders. (Players 1 and 2 are better off in F^2 than in F^4.) On the other hand, the unique equilibrium of the coalition game in the asymmetric case is G^6, hence, the peace network as seen in Table 2. So, coalitions can also be helpful for peace (and welfare) as coalition members can coordinate in jointly expanding their alliances. (Players 1 and 2 are better off in G^6 than in G^2.) Which effect prevails depends on the extent of the asymmetries in effort costs.

Table 1: Equilibrium efforts, payoffs and welfare for $n = 3$, $f(x_i) = 1 + x_i$, $c_1 = c_2 = c_3 = 0.01$. Equilibrium efforts x_i^* denote $x_i^*(F)$, payoffs π_i^* denote $\pi_i(x^*(F), F)$, and welfare W^* denotes $W(x^*(F), F) := \sum_{i \in N} \pi_i(x^*(F), F)$ for the corresponding network F in each row. We omit networks which lead to equivalent games under permutation of players' identities. All values are rounded to two decimals.

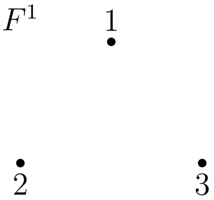
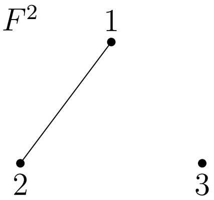
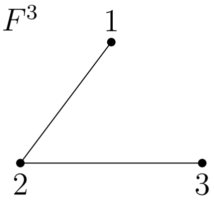
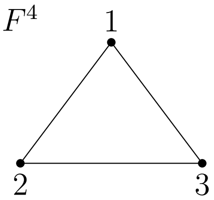
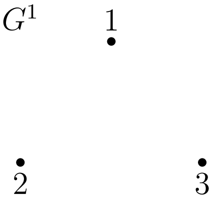
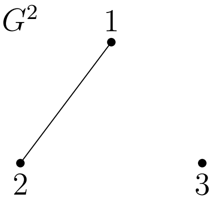
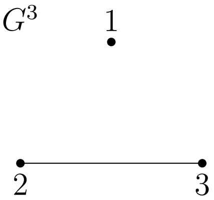
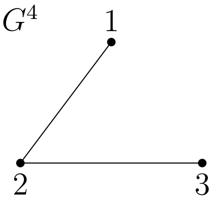
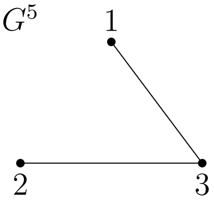
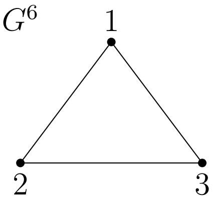
Network	Equilibrium Efforts	Payoffs	Welfare
F^1 	$x_1^* = 21.22$ $x_2^* = 21.22$ $x_3^* = 21.22$	$\pi_1^* = 0.12$ $\pi_2^* = 0.12$ $\pi_3^* = 0.12$	$W^* = 0.36$
F^2 	$x_1^* = 5.07$ $x_2^* = 5.07$ $x_3^* = 11.14$	$\pi_1^* = 0.38$ $\pi_2^* = 0.38$ $\pi_3^* = 0.03$	$W^* = 0.79$
F^3 	$x_1^* = 2.41$ $x_2^* = 0$ $x_3^* = 2.41$	$\pi_1^* = 0.16$ $\pi_2^* = 0.63$ $\pi_3^* = 0.16$	$W^* = 0.95$
F^4 	$x_1^* = 0$ $x_2^* = 0$ $x_3^* = 0$	$\pi_1^* = 0.33$ $\pi_2^* = 0.33$ $\pi_3^* = 0.33$	$W^* = 1$

Table 2: Equilibrium efforts, payoffs and welfare for $n = 3$, $f(x_i) = 1 + x_i$, $c_1 = c_2 = 0.01$, $c_3 = 0.001$. Equilibrium efforts x_i^* denote $x_i^*(G)$, payoffs π_i^* denote $\pi_i(x^*(G), G)$, and welfare W^* denotes $W(x^*(G), G) = \sum_{i \in N} \pi_i(x^*(G), G)$ for the corresponding network G in each row. We omit networks which lead to equivalent games under permutation of players' identities. All values are rounded to two decimals.

Network	Equilibrium Efforts	Payoffs	Welfare
G^1 	$x_1^* = 3.53$ $x_2^* = 3.53$ $x_3^* = 85.17$	$\pi_1^* = 0.01$ $\pi_2^* = 0.01$ $\pi_3^* = 0.82$	$W^* = 0.84$
G^2 	$x_1^* = 11.36$ $x_2^* = 11.36$ $x_3^* = 246.21$	$\pi_1^* = 0.16$ $\pi_2^* = 0.16$ $\pi_3^* = 0.20$	$W^* = 0.52$
G^3 	$x_1^* = 3.27$ $x_2^* = 1.14$ $x_3^* = 20.36$	$\pi_1^* = 0.01$ $\pi_2^* = 0.47$ $\pi_3^* = 0.46$	$W^* = 0.94$
G^4 	$x_1^* = 0.49$ $x_2^* = 0$ $x_3^* = 13.90$	$\pi_1^* = 0.03$ $\pi_2^* = 0.57$ $\pi_3^* = 0.37$	$W^* = 0.97$
G^5 	$x_1^* = 2.41$ $x_2^* = 2.41$ $x_3^* = 0$	$\pi_1^* = 0.16$ $\pi_2^* = 0.16$ $\pi_3^* = 0.63$	$W^* = 0.95$
G^6 	$x_1^* = 0$ $x_2^* = 0$ $x_3^* = 0$	$\pi_1^* = 0.33$ $\pi_2^* = 0.33$ $\pi_3^* = 0.33$	$W^* = 1$

4.3. The case of directed networks

We now extend our model to the case where relations are not necessarily mutual: if player i is a friend of j, this does not necessarily imply the opposite, hence, j may or may not be a friend of i. Throughout this section, we let F_i denote the set of friends of $i \in N$ as before, but $j \in F_i$ does not imply $i \in F_j$. A *directed network* is a profile of all sets of friends $F = (F_1, ..., F_n)$. We denote the set of all directed networks by $\mathcal{F}^d$, and a success function defines for each player $i \in N$ a mapping $s_i : X \times \mathcal{F}^d \to (0, 1)$. For simplicity and to avoid repetition, we use the same notation in Section 3.2, although dropping the mutuality requirement on relations leads to a broader set of networks, $\mathcal{F}^d \supset \mathcal{F}$.

Our class of success functions (1) extends directly and can also be read for directed networks. Our aim is to see whether its characterization extends as well. It turns out that our six axioms do not uniquely pin down (1) in the domain $\mathcal{F}^d$. For a full characterization we need to introduce an additional axiom, *invariance of relative win probabilities (IRWP)*, which we will discuss shortly. Exhaustivity, anonymity, monotonicity of efforts and IEC can be read for directed networks without any modification, while we introduce a slight modification in each of the definitions of monotonicity of relations and IRO. The modified monotonicity of relations, *monotonicity of directed relations (MDR)*, simply requires that a player always benefits from a new friendship towards herself. This is because in the context of directed networks she is not obliged to reciprocate. The need for modifying IRO comes from the fact that the initial data of the axiom is inconclusive under directed networks (since F'_j is unrestricted due to lack of mutuality). The modified IRO, *independence of directed relations of others (IDRO)*, resolves this by requiring friends of j to remain constant in the considered networks.

Monotonicity of directed relations (MDR): Let $F \in \mathcal{F}^d$ be any network and $x \in X$ be any effort profile such that there is a pair $i, j \in N$ with $j \notin F_i$. Consider $F' \in \mathcal{F}^d$ such that $j \in F'_i$ and $F'_h = F_h$ for all $h \neq i$. Then, $s_i(x, F') > s_i(x, F)$.

Independence of directed relations of others (IDRO): Let $F \in \mathcal{F}^d$ be a network such that there are at least two players $i, j \in N$ with $j \in F_i$, and $x \in X$ be any effort profile. Let $F' \in \mathcal{F}^d$ be the network such that $j \notin F'_i$ and $F'_h = F_h$ for all $h \neq i$. Then, $\left(\frac{s_i(x,F')}{s_h(x,F')}\right) / \left(\frac{s_i(x,F)}{s_h(x,F)}\right) = \left(\frac{s_i(x,G')}{s_h(x,G')}\right) / \left(\frac{s_i(x,G)}{s_h(x,G)}\right)$ for all $h \in N \backslash \{i\}$ and $G, G' \in \mathcal{F}^d$ with $j \in G_i$, $j \notin G'_i$ and $G_k = G'_k$ for all $k \neq i$.

Our new axiom, IRWP, imposes that, when player j breaks the friendship towards player i, the relative win probabilities of all pairs of players whose set of friends are unchanged (i.e., all $h, k \neq i$) should remain constant.

Invariance of relative win probabilities (IRWP): Let $F \in \mathcal{F}^d$ be a network such that there are at least two players $i, j \in N$ with $j \in F_i$, and $x \in X$ be any effort profile. Let $F' \in \mathcal{F}^d$ be the network such that $j \notin F'_i$ and $F'_h = F_h$ for all $h \neq i$. Then, $\frac{s_k(x,F')}{s_h(x,F')} = \frac{s_k(x,F)}{s_h(x,F)}$ for all $h, k \neq i$.

IRWP is necessary to rule out the following class of success functions which is compatible with all other axioms. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}^d$,

$$s_i^{\neg IRWP}(x,F) = \frac{\prod_{j \in F_i} f(x_j)\theta^{q_i(F)}(1/\theta)^{r_i(F)}}{\sum_{h \in N} \prod_{j \in F_h} f(x_j)\theta^{q_h(F)}(1/\theta)^{r_h(F)}},$$

where $\theta > 1$, $q_i(F) := |\{k \in N : i \notin F_k \text{ and } k \in F_i\}|$, $r_i(F) := |\{k \in N : i \in F_k \text{ and } k \notin F_i\}|$ and $f : \mathbb{R}_{++} \to (1, +\infty)$ is any increasing function. To see that IDRO and IRWP are independent, we define another class of success functions which satisfy all axioms (including IRWP) but not IDRO. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}^d$,

$$s_i^{\neg IDRO}(x,F) = \frac{\prod_{j \in F_i} f(x_j)\theta^{q_i(F)}}{\sum_{h \in N} \prod_{j \in F_h} f(x_j)\theta^{q_h(F)}},$$

where $\theta > 1$, q_i is as defined above and $f : \mathbb{R}_{++} \to (1, +\infty)$ is any increasing function. We conclude with the following remark. We do not provide a formal proof of this result as it is a particular case of the characterization in the next section, which applies to weighted directed networks.

Remark 1. *A success function defined on the domain $X \times \mathcal{F}^d$ satisfies exhaustivity, anonymity, monotonicity of efforts, MDR, IEC, IDRO and IRWP if and only if $s_i(x,F) = s_i^*(x,F)$ for each $i \in N$, $x \in X$ and $F \in \mathcal{F}^d$.*

4.4. The case of weighted directed networks

We now extend our model to allow both for directionality and varying intensity of friendship across players. For each pair of players $i, j \in N$, we let the intensity of friendship that player i receives from player j be given by $\phi_{i,j} \in (-\infty, 1]$, where $\phi_{i,i} = 1$ so that no player directs a higher friendship intensity to any other player than to herself.[15] We denote by $\phi_i := (\phi_{i,1}, \ldots, \phi_{i,n})$ the vector of intensities of all friendships from all players towards player i (including i herself), and the profile of friendships $\phi = (\phi_1, \ldots, \phi_n)$ defines a weighted directed network of relations. Let Φ denote the space of all weighted directed networks. A *generalized success function* defines for each player $i \in N$ a mapping $p_i : X \times \Phi \to (0,1)$ which determines player i's win probability $p_i(x, \phi)$ for each effort profile $x \in X$ and weighted directed network $\phi \in \Phi$.

We now extend the probabilistic derivation in Section 3 to weighted directed net-

[15]The upper bound to friendship intensity is set to 1 without loss of generality, i.e., all our results hold as long as there is an upper bound to friendship intensity defined as the friendship intensity of a player towards herself (which can be positive or negative). We need friendship intensity to be unbounded from below for technical reasons in the proof of Theorem 2. Alternatively, we could assume $\phi_{i,j}$ to take value in any evenly-spaced finite set of levels of friendship intensity (e.g., all integers between $-k$ and k for some $k \in \mathbb{N}$).

works to define our class of generalized success functions. Let

$$w_i(x, \phi_i) := \sum_{j \in N} \phi_{i,j} g(x_j)$$

be the *generalized effective strength* of each player $i \in N$, where $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is any increasing function. Given this, we require win probabilities to depend on players' strengths, so that the player with the highest strength is the most likely to win. More specifically, for each $x \in X$ and $\phi \in \Phi$ we impose the win probability of player $i \in N$ to satisfy

$$p_i(x, \phi) = \Pr\left(w_i(x, \phi_i) + \epsilon_i > w_j(x, \phi_j) + \epsilon_j \text{ for any } j \in N \setminus \{i\}\right), \tag{6}$$

where $(\epsilon_1, \ldots, \epsilon_n)$ are independent and identically distributed Gumbel random shocks with variance $\pi^2/6$. Following the same arguments in Section 3, it can be shown that (6) holds if and only if

$$p_i(x, \phi) = \frac{\prod_{j \in N} \exp\left(\phi_{i,j} g(x_j)\right)}{\sum_{h \in N} \prod_{j \in N} \exp\left(\phi_{h,j} g(x_j)\right)}, \tag{7}$$

which for generic g defines our class of generalized success functions

$$p_i^*(x, \phi) := \frac{\prod_{j \in N} f(x_j)^{\phi_{i,j}}}{\sum_{h \in N} \prod_{j \in N} f(x_j)^{\phi_{h,j}}}, \tag{8}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is any increasing function. This is because, as argued in Section 3, for each g there is a unique f such that $f(x_j) = \exp(g(x_j))$ for all $x_j > 0$.

We now characterize our class of generalized success functions (8) via straightforward adaptations of our axioms.

Extended exhaustivity (EE): For any $\phi \in \Phi$ and $x \in X$, $\sum_{i \in N} p_i(x, \phi) = 1$.

Extended anonymity (EA): Let α be any permutation of N. For any $\phi \in \Phi$ and $x \in X$, let $\alpha(\phi) = (\phi_{\alpha(1)}, ..., \phi_{\alpha(n)})$ and $\alpha(x) = (x_{\alpha(1)}, ..., x_{\alpha(n)})$. Then, $p_i(x, \phi) = p_{\alpha(i)}(\alpha(x), \alpha(\phi))$ for each $i \in N$.

Extended monotonicity of efforts (EME): Consider any $x \in X$ and $\phi \in \Phi$ with $\phi_{j,i} \neq 1$ for some $i, j \in N$. Then, $p_i(x', \phi) > p_i(x, \phi)$ for any $x' \in X$ with $x'_i > x_i$ and $x'_k = x_k$ for all $k \neq i$.

Extended monotonicity of directed relations (EMDR): Consider any $x \in X$ and $\phi \in \Phi$ such that there is a pair $i, j \in N$ with $\phi_{i,j} \neq 1$. Then, $p_i(x, \phi') > p_i(x, \phi)$ for any $\phi' \in \Phi$ with $\phi'_{i,j} > \phi_{i,j}$ and $\phi'_{h,k} = \phi_{h,k}$ for all $h, k \in N$ such that $(h, k) \neq (i, j)$.

Extended independence of efforts of commons (EIEC): For any $x, x' \in X$, $\phi \in \Phi$ and $i, j \in N$, $\frac{p_i(x,\phi)}{p_j(x,\phi)} = \frac{p_i(x',\phi)}{p_j(x',\phi)}$ if $x'_h = x_h$ for all $h \in N$ such that $\phi_{i,h} \neq \phi_{j,h}$.

Extended independence of directed relations of others (EIDRO): Consider any $x \in X$, $i, j \in N$ and $\phi, \phi' \in \Phi$ with $\phi_{i,j} > \phi'_{i,j}$ and $\phi'_{h,k} = \phi_{h,k}$ for all $h, k \in N$ such that $(h, k) \neq (i, j)$. Then, $\left(\frac{p_i(x,\phi')}{p_h(x,\phi')}\right) / \left(\frac{p_i(x,\phi)}{p_h(x,\phi)}\right) = \left(\frac{p_i(x,\gamma')}{p_h(x,\gamma')}\right) / \left(\frac{p_i(x,\gamma)}{p_h(x,\gamma)}\right)$ for all $h \neq i$ and $\gamma, \gamma' \in \Phi$ with $\gamma'_{i,j} - \gamma_{i,j} = \phi'_{i,j} - \phi_{i,j}$ and $\gamma'_{h,k} = \gamma_{h,k}$ for all $h, k \in N$ such that $(h, k) \neq (i, j)$.

Extended invariance of relative win probabilities (EIRWP): Consider any $x \in X$, $i, j \in N$ and $\phi, \phi' \in \Phi$ with $\phi_{i,j} > \phi'_{i,j}$ and $\phi'_{h,k} = \phi_{h,k}$ for all $h, k \in N$ such that $(h, k) \neq (i, j)$. Then, $\frac{p_k(x,\phi')}{p_h(x,\phi')} = \frac{p_k(x,\phi)}{p_h(x,\phi)}$ for all $k, h \neq i$.

We are now ready to state our result.

Theorem 2. *A generalized success function satisfies EE, EA, EME, EMDR, EIEC, EIDRO and EIRWP if and only if $p_i(x, \phi) = p^*_i(x, \phi)$ for each $i \in N$, $x \in X$ and $\phi \in \Phi$.*

One can easily see the relationship between (1) and (8): if we restrict $\phi_{i,j}$ to take only binary values and $\phi_{i,j} = \phi_{j,i}$, the model reduces to the one in Section 3 and (8) reduces to (1). Similarly, if $\phi_{i,j}$ takes binary values but we drop the restriction $\phi_{i,j} = \phi_{j,i}$, the model reduces to the one in Section 4.3 and (8) reduces to the respective interpretation of (1). There are many more alternative restrictions that can be considered in applications, as the space of weighted directed networks (or equivalently, the rules of network formation) can be restricted in various ways to capture the crucial features of the specific environment. For instance, one may want to restrict friendship intensities to three values in a way that, for each player, each opponent is categorized as friend or enemy but the friendship intensity towards an opponent is never as high as towards herself.[16] Another possibility is to allow for multiple levels of friendship intensities and, for each player, to restrict friendship intensities to take the highest value towards herself, the second highest value towards her friends, the third highest value towards the friends of her friends and so on. Indeed, in certain environments by establishing a friendly relation with somebody one may connect (at some point, perhaps to a lesser degree) to the friends of the new friend. In applications, different restrictions on the space of weighted directed networks can lead to different equilibrium outcomes as we point out in the discussion of Tables 1 and 2.

We now illustrate the applicability of our extended model via a simple example. Suppose players' relations are defined by their location on an ideological spectrum.[17]

[16] We have so far interpreted $\phi_{i,j}$ as the intensity of friendship from j towards i. The degree of enmity can be seen as the lack of friendship on the same scale. How about neutrality? In a zero-sum environment like ours where one's gain is necessarily another's loss, neutrality from a player towards another is impossible. Instead, a player can be neutral towards pairs of other players, in the sense of not favoring one over the other. By EIEC, $\phi_{i,h} = \phi_{j,h}$ implies the relative win probability of i to j is independent of x_h. So, h can be neutral towards the pair i, j in this sense.

[17] See Esteban and Ray (1999, 2011) and Klose and Kovenock (2012, 2013) for contest models

For simplicity we let this spectrum be an interval of the real line and without loss of generality we choose $[0, 1]$. A player's ideological location is a point in this interval, e.g., 0 is extreme left-wing and 1 is extreme right-wing. Suppose the friendship intensity $\phi_{i,j}$ from player j to player i is one minus the distance between their locations, which implies $\phi_{i,j} = \phi_{j,i}$ for all $i, j \in N$. For simplicity, let $n = 3$ and suppose players 2 and 3 are at the opposite extremes of the spectrum, at 0 and 1 respectively (hence, $\phi_{2,3} = \phi_{3,2} = 0$). For a given $x \in X$, player 1 must choose the location to maximize its probability of winning, which we assume takes the form (7).[18] By construction $\phi_{1,2} = \phi_{2,1} = 1 - \phi_{1,3} = 1 - \phi_{3,1}$; hence, player 1's choice reduces to identifying the optimal friendship intensity $\phi^*_{2,1}$. See Figure 3 for a graphical representation.

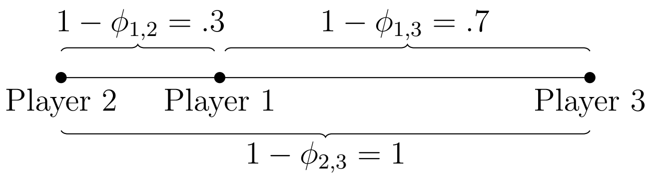


Figure 3: In this example player 1's location is .3, hence, $\phi_{1,2} = .7$ and $\phi_{1,3} = .3$.

Let $x' \in X$ be such that $g(x'_i) = k > 0$ for all $i \in N$, i.e., all efforts are positive and symmetric. By (7),

$$p_1(x', \phi) = \frac{\exp[2k]}{\exp[2k] + \exp[(1 + \phi_{2,1})k] + \exp[(2 - \phi_{2,1})k]},$$

hence, player 1 implements $\phi^*_{2,1}$ that minimizes $\exp[(1 + \phi_{2,1})k] + \exp[(2 - \phi_{2,1})k]$, which gives $\phi^*_{2,1} = 1/2$ for all k. We can conclude that, if efforts are positive and symmetric, player 1 should always take the 'centrist' ideology which is half way between the opponents' locations. Of course, this is just an example, and the result does not necessarily hold if efforts are asymmetric. For instance, consider a marginal increase in the effort of player 2. Should player 1 increase or decrease the intensity of friendship towards this player? One can show that, in a neighborhood of x'_2, the optimal friendship intensity as a function of x_2 is $\phi^*_{2,1}(x_2) = 1 - \frac{g(x_2)}{2k} - \ln\left[\frac{2k}{g(x_2)} - 1\right]$, which increases in x_2 for all k. Then, starting from x', if player 2 marginally increases the effort, player 1's ideology should marginally shift towards player 2's location. It follows that, instead of pursuing a balance of power by taking the side of the weak, player 1 bandwagons by siding with the strong.

where each player's location on an ideological spectrum determines a player's valuation of victory and defeat.

[18] As we leave g generic, the whole exercise can be repeated with the specification (8).

5. Conclusion

We define and axiomatically characterize a class of success functions for contests where each pair of players can be in a friendly or in an antagonistic relation, and these pairwise relations form a network. For all against all contests, our class of success functions belongs to the one characterized in Skaperdas (1996). For contests between groups, the aggregate win probability of group members calculated by our functional form belongs to the class axiomatized in Münster (2009). In our basic setup a success function treats every friend equally. However, it can easily be extended to the setting where each friendship has a weight and relations between players are not necessarily mutual. We define this more general class of models in Section 4.4 following a probabilistic argument in McFadden (1973), and provide an axiomatic characterization.

The model we propose allows to study strategic interaction between parties in conflict who are connected by a complex network of relations. Among many other environments, we commonly see such complex networks in international relations between countries, in political lobbying between interest groups and in electoral campaigning between competing parties. In this paper we consider an application where players choose their relations first, and then they choose their efforts. Our analysis is based on a solution concept that combines pairwise stability of networks and Nash equilibrium of efforts in line with the idea of subgame perfection. In a contest with symmetric effort costs, we show there is an equilibrium whose outcome is the peace network and the profile of zero efforts (i.e., unarmed peace). On the other hand, unarmed peace would not necessarily be the outcome if players were constrained to be in coalitions as in a group contest à la Münster (2009) or if we introduced cost asymmetries. It would be interesting to formulate alternative solution concepts, or consider alternative timings where efforts are chosen before relations or where both relations and efforts are chosen simultaneously. Since a comprehensive analysis of possible games and solution concepts is not the main focus of this paper, we leave these questions open for future research.

Our success function has potential for empirical applications as well. For instance, our model can be used to test for network effects by maximum likelihood or related methods. We refer to Jia et al. (2013) for a review of empirical issues in the estimation of success functions. The success function can also be used as an index of power adjusted for the network of relations. In the context of international relations, a particularly suited collection of datasets is presented by The Correlates of War Project, which spans for about two centuries and provides material for case studies as well as econometric analysis. In this context, the effort of a country can be estimated by its National Material Capabilities (see Singer et al., 1972) while its set of friends can be estimated by its Formal Alliances (see Gibler, 2009). In a recent empirical study on the effects of alliance networks on military conflict between countries, Li et al. (2017) show that outbreaks of conflict are less likely between countries that share common allies even though they are not necessarily allied. It would be interesting to incorporate the aforementioned index of power in their empirical study (to control for countries' bargaining positions in the shadow of conflict), and to study theoretically the problem of conflict outbreak within our contest model to find a rationale for this pacifying effect.

Appendix

A. Relation to the success functions in Münster (2009)

We now show how our class of success functions (1) relates to the one in Münster (2009) by applying a simple transformation. Let $\hat{\mathcal{F}} \subset \mathcal{F}$ be the set of all networks which organize players into mutually exclusive groups, i.e., coalitions. Denote by $\mathcal{C}(F)$ the partition of N defined by network $F \in \hat{\mathcal{F}}$. Münster (2009) shows that under some conditions the win probability of group $C \in \mathcal{C}(F)$ must take the form $\sigma_C(x, F) := \hat{f}_C(x_C)/\left[\sum_{C' \in \mathcal{C}(F)} \hat{f}_{C'}(x_{C'})\right]$ for any $x \in X$ and $F \in \hat{\mathcal{F}}$, where for any group $C \in \mathcal{C}(F)$ the vector x_C defines the efforts of all members of C and the function $\hat{f}_C : \mathbb{R}^{|C|}_{++} \to \mathbb{R}_{++}$ increases in all its arguments. Let $\sigma^*_C(x, F) := \sum_{i \in C} s^*_i(x, F)$. Then, it is straightforward that $\sigma^*_C(x, F)$ belongs to the class of Münster (2009) for $\hat{f}_C(x_C) = |C| \prod_{i \in C} f(x_i)$.

B. Proofs

Proof of Theorem 1

It is easy to verify that (1) satisfies the axioms given in the theorem. To show that the converse holds, we take a success function $s_i : X \times \mathcal{F} \to (0, 1)$ for each $i \in N$ satisfying the axioms. We want to show that for any $i \in N$, $x \in X$ and $F \in \mathcal{F}$, $(*)$ $s_i(x, F) = s^*_i(x, F)$.

Let $F \in \mathcal{F}$ be the network for which $F_i = N$ for each player $i \in N$. Take any effort profile $x \in X$. To show that $(*)$ is true, it suffices to show that $(**)$ $s_i(x, F) = 1/n$. Take any pair $i, j \in N$. Take a permutation α such that $\alpha(i) = j$, $\alpha(j) = i$, $\alpha(k) = k$ for all $k \notin \{i, j\}$. Note that $\alpha(F) = F$. By anonymity, $s_i(x, F) = s_j(\alpha(x), F)$ and $s_j(x, F) = s_i(\alpha(x), F)$. Moreover, by IEC $\frac{s_i(x,F)}{s_j(x,F)} = \frac{s_i(\alpha(x),F)}{s_j(\alpha(x),F)}$ which then is equal to $\frac{s_j(x,F)}{s_i(x,F)}$. This leads to $s_i(x, F) = s_j(x, F)$, and together with exhaustivity this implies $(**)$.

Let $F \in \mathcal{F}$ be any network with at least one pair of players which are enemies, i.e., $F_i \neq N$ for at least one player i. Let $x \in X$ be any effort profile. It is easy to show that there exists a sequence of networks $F^0, \ldots, F^m$ with $m \geq 1$ such that for $t \in \{0, \ldots, m-1\}$ (i) there is a pair of players $i, j \in N$ such that $i \in F^t_j$ and $i \notin F^{t+1}_j$, (ii) for all $k \notin \{i, j\}$, $F^t_k = F^{t+1}_k$ and (iii) $F^0_i = N$ for all $i \in N$ and $F^m = F$. We take any such sequence.

Let $i,j \in N$ be a pair of players such that $i \in F_j^t$ and $i \notin F_j^{t+1}$ for some t. Let h be any player in $N \backslash \{i,j\}$. By IEC, $\frac{s_i(x,F^t)}{s_h(x,F^t)}$ depends only on the efforts of the uncommon friends of i and h in F^t. Similarly, $\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})}$ depends only on the efforts of their uncommon friends in F^{t+1}. Hence, there exists a real valued function $\gamma_{i,j,h}^{F^t}$ such that

$$\gamma_{i,j,h}^{F^t}(x_U) = \left(\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})}\right) / \left(\frac{s_i(x,F^t)}{s_h(x,F^t)}\right)$$

where x_U is the profile of efforts of players $l \in U := [(F_i^t \cup F_h^t) \setminus (F_i^t \cap F_h^t)] \cup [(F_i^{t+1} \cup F_h^{t+1}) \setminus (F_i^{t+1} \cap F_h^{t+1})]$. By IRO, for any pair of networks $G, G' \in \mathcal{F}$ such that $i \in G_j$ and $i \notin G'_j$ and $G'_k = G_k$ for all $k \notin \{i,j\}$,

$$\gamma_{i,j,h}^{F^t}(x_U) = \left(\frac{s_i(x,G')}{s_h(x,G')}\right) / \left(\frac{s_i(x,G)}{s_h(x,G)}\right). \tag{9}$$

By IEC the right hand side of (9) is exclusively a function of $x_{U'}$, where $U' := [(G_i \cup G_h) \setminus (G_i \cap G_h)] \cup [(G'_i \cup G'_h) \setminus (G'_i \cap G'_h)]$. Note that $j \in U$ and $j \in U'$ by construction. As there is no restriction on G except that $i \in G_j$, the function $\gamma_{i,j,h}^{F^t}$ does not depend on the whole network F^t but only on the relation between i,j. Moreover, as (9) must hold for G such that $U' = \{j\}$, the function $\gamma_{i,j,h}^{F^t}$ is constant in all efforts except x_j. A similar set of expressions can be written for the relative win probabilities of players j and h in networks F^t and F^{t+1}. Then, we can define the functions $g_{i,j,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g_{j,i,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ so that

$$g_{i,j,h}(x_j) = \left(\frac{s_i(x,F^{t+1})}{s_h(x,F^{t+1})}\right) / \left(\frac{s_i(x,F^t)}{s_h(x,F^t)}\right), \tag{10}$$

$$g_{j,i,h}(x_i) = \left(\frac{s_j(x,F^{t+1})}{s_h(x,F^{t+1})}\right) / \left(\frac{s_j(x,F^t)}{s_h(x,F^t)}\right). \tag{11}$$

For $n=3$, we can immediately write $g_{i,j,h}(x_j) = g_{i,j}(x_j)$ and $g_{j,i,h}(x_i) = g_{j,i}(x_i)$ for the unique player $h \notin \{i,j\}$. Let $n \geq 4$, so that there are at least two players $h,k \in N \setminus \{i,j\}$. If we write (10) for $h,k \in N \setminus \{i,j\}$ and we take the ratio of the two expressions we obtain

$$\frac{g_{i,j,k}(x_j)}{g_{i,j,h}(x_j)} = \left(\frac{s_h(x,F^{t+1})}{s_k(x,F^{t+1})}\right) / \left(\frac{s_h(x,F^t)}{s_k(x,F^t)}\right).$$

Similarly if we write (11) for $h,k \in N \setminus \{i,j\}$ we obtain

$$\frac{g_{j,i,k}(x_i)}{g_{j,i,h}(x_i)} = \left(\frac{s_h(x,F^{t+1})}{s_k(x,F^{t+1})}\right) / \left(\frac{s_h(x,F^t)}{s_k(x,F^t)}\right).$$

Let $G, G' \in \mathcal{F}$ be such that $i \in G_j$ and $i \notin G'_j$ and $G'_l = G_l$ for all $l \notin \{i,j\}$. Moreover let $G_h = G_k$. Consider the permutation β such that $\beta(k) = h$, $\beta(h) = k$, $\beta(l) = l$ for all $l \notin \{h,k\}$. By anonymity $s_h(x,G) = s_k(\beta(x), \beta(G))$. Note that $\beta(G) = G$, therefore $s_h(x,G) = s_k(\beta(x),G)$. As $G_h = G_k$ implies $G'_h = G'_k$, by anonymity we must also have $s_h(x,G') = s_k(\beta(x),G')$. Moreover, by IEC $\frac{s_h(x,G)}{s_k(x,G)} = \frac{s_h(\beta(x),G)}{s_k(\beta(x),G)}$ which then is equal to $\frac{s_k(x,G)}{s_h(x,G)}$. This implies that $s_h(x,G) = s_k(x,G)$. Similarly, by IEC we also have $s_h(x,G') = s_k(x,G')$. It follows by IRO that

$$1 = \left(\frac{s_h(x,F^{t+1})}{s_k(x,F^{t+1})}\right) \Big/ \left(\frac{s_h(x,F^t)}{s_k(x,F^t)}\right), \tag{12}$$

hence $g_{i,j,h}(x_j)$ does not depend on the identity of h as long as $h \notin \{i,j\}$. Then, we can write $g_{i,j,h}(x_j) = g_{i,j}(x_j)$ and $g_{j,i,h}(x_i) = g_{j,i}(x_i)$ for each $h \in N \setminus \{i,j\}$ also when $n \geq 4$.

Now, let $G' \in \mathcal{F}$ be the network such that $G'_k = \{k\}$ for all $k \in N$. Consider the network $G \in \mathcal{F}$ which differs from G' only by i,j being friends, so $i \in G_j$ and $G_k = G'_k$ for all $k \notin \{i,j\}$. Consider the permutation α defined above. By anonymity $s_i(x,G) = s_j(\alpha(x),\alpha(G))$. Note that $\alpha(G) = G$, therefore $s_i(x,G) = s_j(\alpha(x),G)$. Then, as by IEC $\frac{s_i(x,G)}{s_j(x,G)}$ is constant in x, we must have $s_i(x,G) = s_j(x,G)$. Using IRO, we can write

$$\left(\frac{s_i(x,G')}{s_j(x,G')}\right) \Big/ \left(\frac{s_i(x,G)}{s_j(x,G)}\right) = \frac{s_i(x,G')}{s_j(x,G')} = \frac{g_{i,j}(x_j)}{g_{j,i}(x_i)}.$$

Since $\alpha(G') = G'$, by anonymity $g_{i,j} = g_{j,i}$. Note that any permutation of players besides α also leads to G'. Then, anonymity implies that $g_{i,j}$ and $g_{j,i}$ do not depend on the identities of i and j, hence we can write $g_{i,j} = g_{j,i} = g$ for all $i,j \in N$.

We now determine the win probabilities in network F. For any $i,j \in N$, if we take the product of the rate of change of relative win probabilities for each pair of consecutive networks in the sequence $F^0,\ldots,F^m$, we obtain

$$\left(\frac{s_i(x,F)}{s_j(x,F)}\right) \Big/ \left(\frac{s_i(x,F^0)}{s_j(x,F^0)}\right) = \prod_{t=0}^{m-1} \left(\frac{s_i(x,F^{t+1})}{s_j(x,F^{t+1})}\right) \Big/ \left(\frac{s_i(x,F^t)}{s_j(x,F^t)}\right). \tag{13}$$

Note that $\frac{s_i(x,F^0)}{s_j(x,F^0)} = 1$, so the LHS reduces to $\frac{s_i(x,F)}{s_j(x,F)}$. Conditions (10), (11) and (12) jointly determine the rate of change of relative win probabilities in each step of the sequence. Writing each term on the RHS of (13) in terms of g accordingly,

$$\frac{s_i(x,F)}{s_j(x,F)} = \frac{\prod_{l\in N\setminus F_i} g(x_l)}{\prod_{l\in N\setminus F_j} g(x_l)} = \frac{\prod_{l\in F_j} g(x_l)}{\prod_{l\in F_i} g(x_l)},$$

and defining $f := 1/g$ we obtain

$$\frac{s_i(x,F)}{s_j(x,F)} = \frac{\prod_{l\in F_i} f(x_l)}{\prod_{l\in F_j} f(x_l)}.$$

For any $j \in N$, by exhaustivity,

$$\frac{1}{s_j(x,F)} = \sum_{i \in N} \frac{s_i(x,F)}{s_j(x,F)} = \sum_{i \in N} \left(\frac{\prod_{l \in F_i} f(x_l)}{\prod_{l \in F_j} f(x_l)} \right) = \frac{\sum_{i \in N} \prod_{l \in F_i} f(x_l)}{\prod_{l \in F_j} f(x_l)},$$

hence $s_j(x,F) = \frac{\prod_{l \in F_j} f(x_l)}{\sum_{i \in N} \prod_{l \in F_i} f(x_l)}$. Given this, to prove $(*)$ it is sufficient to show that the function f must be increasing and greater than 1. Consider a pair $i, j \in N$ with $i \notin F_j$. Take any $x, x' \in X$ such that $x'_i > x_i$ and $x'_h = x_h$ for $h \neq i$. By monotonicity of efforts $s_i(x', F) > s_i(x, F)$. Then, it is easy to verify that f must be increasing. To show that $f > 1$, consider the network F' with $i \in F'_j$ and $F'_k = F_k$ for all $k \notin \{i, j\}$. Let $x_j > x_i$. Then, $s_i(x, F') > s_i(x, F)$ by monotonicity of relations, which implies that $f > 1$. So, we achieve the desired result $(*)$. □

Proof of Proposition 1

Let payoffs take the form (5) where $f(x_i) = 1 + x_i^\rho$ and $c_i = c$ for all $i \in N$ with $\rho \in (0,1)$ and $c > 0$, or with $\rho = 1$ and $c \in (0, 1/n^2)$. Our claim is that there is an equilibrium where F^* is the peace network and $x^*(F^*) = (0, \dots, 0)$. Note that, if F^* is peace, the unique Nash equilibrium that follows F^* is $x^*(F^*) = (0, \dots, 0)$, which implies each player i's equilibrium payoff is $\pi_i(x^*(F^*), F^*) = 1/n$. Then, our claim is true if and only if, given any Nash equilibrium $x^*(F)$ for each $F \in \mathcal{G}_{F^*} \backslash F^*$, there is no network $F' \in \mathcal{G}_{F^*} \backslash F^*$ such that $\pi_i(x^*(F'), F') > 1/n$.

Note that any network in $\mathcal{G}_{F^*} \backslash F^*$ consists of all players being friends except for a pair. Without loss of generality, let $F' \in \mathcal{G}_{F^*} \backslash F^*$ be such that player 1 is enemy of player 2 while all other pairs are friends, i.e., $F'_1 = N \backslash \{2\}$ and $F'_i = N$ for all $i \in N \backslash \{1, 2\}$. To identify $x^*(F')$, we solve for all possible Nash equilibria in the game of contest induced by F'. By abuse of notation, we write x_i^* for $x_i^*(F')$ in the rest of the proof. It is immediate that $x_i^* = 0$ for all $i \in N \backslash \{1, 2\}$. Each player $i \in \{1, 2\}$ solves

$$x_i^* = \arg\max_{x_i \geq 0} \left[\frac{f(x_i)}{f(x_i) + f(x_{\neg i}^*) + (n-2) f(x_i) f(x_{\neg i}^*)} - c x_i \right].$$

It is easy to show that if $f(x_i) = 1 + x_i^\rho$ with $\rho \in (0, 1]$ the payoff function is always concave in x_i, and the first order condition is

$$\frac{f(x_{\neg i}^*) f'(x_i^*)}{\left[f(x_i^*) + f(x_{\neg i}^*) + (n-2) f(x_i^*) f(x_{\neg i}^*) \right]^2} = c.$$

Combining the first order conditions of player 1 and player 2, we obtain $f(x_1^*) f'(x_2^*) = f(x_2^*) f'(x_1^*)$. As $f(x_i) = 1 + x_i^\rho$ this implies $x_1^* = x_2^*$ in an interior equilibrium, so that

$$\frac{f'(x_i^*)}{f(x_i^*) \left[2 + (n-2) f(x_i^*) \right]^2} = c.$$

Given $f(x_i) = 1 + x_i^\rho$ with $\rho \in (0, 1]$, the LHS is always decreasing in x_i^*, it has no lower bound, its upper bound is $1/n^2$ if $\rho = 1$ and it has no upper bound if $\rho \in (0, 1)$. Then, the interior equilibrium exists if $\rho \in (0, 1)$, or if $\rho = 1$ and $c \in (0, 1/n^2)$, and it is the unique interior equilibrium. It follows that, within our parameter restrictions, in an interior equilibrium the payoff of each player $i \in \{1, 2\}$ takes value

$$\pi_i(x^*(F'), F') = \frac{1}{2 + (n-2)f(x_i^*)} - cx_i^*$$

which is smaller than $1/n$ for all $x_i^* > 0$, that is the desired result. We now show there is no corner solution. Suppose for a contradiction x^{**} is a corner solution. If $x_2^{**} = 0$, the best reply of player 1 must be positive and satisfy

$$\frac{f'(x_1^{**})}{[1 + (n-1)f(x_1^{**})]^2} = c. \tag{14}$$

To check whether this can hold in equilibrium, we must verify $x_2^{**} = 0$ is a best reply to x_1^{**}, which requires

$$\frac{f(x_1^{**})f'(0)}{[1 + (n-1)f(x_1^{**})]^2} < c.$$

Given $f(x_i) = 1+x_i^\rho$, we can immediately rule out a corner solution for any $\rho \in (0, 1)$. For the remaining case $\rho = 1$ we obtain

$$\frac{f(x_1^{**})}{[1 + (n-1)f(x_1^{**})]^2} < c,$$

which is in contradiction with (14) as $f(x_i^{**}) = 1 + x_i^{**} > 1 = f'(x_i^{**})$ for all $x_i^{**} > 0$. It follows that, within our parameter restrictions, we cannot have an equilibrium where only one player exerts positive effort (the case where only player 2 exerts positive effort is identical). This completes our proof. □

Proof of Theorem 2

It is straightforward to show that (8) satisfies our axioms. To show that the converse holds, we take a generalized success function $p_i : X \times \Phi \to (0, 1)$ for each $i \in N$ satisfying the axioms. We want to show that $(*)$ $p_i(x, \phi) = p_i^*(x, \phi)$ for any $i \in N$, $x \in X$ and $\phi \in \Phi$.

Let $\phi \in \Phi$ be the weighted directed network (network, hereinafter through this proof) where $\phi_{i,j} = 1$ for all $i, j \in N$. Take any effort profile $x \in X$. To show that $(*)$ is true, it suffices to show that $(**)$ $p_i(x, \phi) = 1/n$ for all $i \in N$. Take any pair $i, j \in N$ and a permutation α such that $\alpha(i) = j$, $\alpha(j) = i$, $\alpha(k) = k$ for all $k \notin \{i, j\}$. Note that $\alpha(\phi) = \phi$. By EA, $p_i(x, \phi) = p_j(\alpha(x), \phi)$ and $p_j(x, \phi) = p_i(\alpha(x), \phi)$. Moreover, by EIEC $\frac{p_i(x,\phi)}{p_j(x,\phi)} = \frac{p_i(\alpha(x),\phi)}{p_j(\alpha(x),\phi)}$ which then is equal to $\frac{p_j(x,\phi)}{p_i(x,\phi)}$. This implies that $p_i(x, \phi) = p_j(x, \phi)$, which together with EE leads to $(**)$.

Let $\phi \in \Phi$ be any network with at least one pair of players $i, j \in N$ with $\phi_{i,j} \neq 1$. Let $x \in X$ be any effort profile. It is easy to show that there exists a sequence of networks $\phi^0, \ldots, \phi^m$ with $m \geq 1$ such that for $t \in \{0, \ldots, m-1\}$ (i) there is a pair of players $i, j \in N$ such that $\phi^t_{i,j} = 1$ and $\phi^{t+1}_{i,j} = \phi_{i,j} \neq 1$, (ii) for all $h, k \in N$ with $(h, k) \neq (i, j)$, $\phi^t_{h,k} = \phi^{t+1}_{h,k}$ and (iii) $\phi^0_{i,j} = 1$ for all $i, j \in N$ and $\phi^m = \phi$. We take any such sequence.

Let $i, j \in N$ be a pair of players such that $\phi^t_{i,j} = 1$ and $\phi^{t+1}_{i,j} = \phi_{i,j}$ for some t. Let h be any player in $N \backslash \{i\}$. By EIEC, $\frac{p_i(x,\phi^t)}{p_h(x,\phi^t)}$ depends only on the efforts of players with different friendship intensities towards i and h in ϕ^t. Similarly, $\frac{p_i(x,\phi^{t+1})}{p_h(x,\phi^{t+1})}$ depends only on the efforts of players with different friendship intensities towards i and h in ϕ^{t+1}. Hence, there exists a real valued function $\gamma^{\phi^t,\phi^{t+1}}_{i,j,h}$ such that

$$\gamma^{\phi^t,\phi^{t+1}}_{i,j,h}(x_U) = \left(\frac{p_i(x, \phi^{t+1})}{p_h(x, \phi^{t+1})}\right) / \left(\frac{p_i(x, \phi^t)}{p_h(x, \phi^t)}\right)$$

where x_U is the profile of efforts of players in U, which is the set of players with different friendship intensities towards i and h in ϕ^t or ϕ^{t+1}. By EIDRO, for any pair of networks $\varphi, \varphi' \in \Phi$ such that $\varphi'_{i,j} - \varphi_{i,j} = \phi^{t+1}_{i,j} - \phi^t_{i,j}$ and $\varphi_{k,k'} = \varphi'_{k,k'}$ for all $k, k' \in N$ with $(k, k') \neq (i, j)$,

$$\gamma^{\phi^t,\phi^{t+1}}_{i,j,h}(x_U) = \left(\frac{p_i(x, \varphi')}{p_h(x, \varphi')}\right) / \left(\frac{p_i(x, \varphi)}{p_h(x, \varphi)}\right). \tag{15}$$

By EIEC the right hand side of (15) is exclusively a function of $x_{U'}$, i.e., the profile of efforts of players in U', which is the set of players with different friendship intensities towards i and h in φ' or φ. Note that $j \in U$ and $j \in U'$ by construction. As there are no further restrictions on φ, φ', the function $\gamma^{\phi^t,\phi^{t+1}}_{i,j,h}$ does not depend on the full networks ϕ^t, ϕ^{t+1} but only on the difference $\phi^t_{i,j} - \phi^{t+1}_{i,j}$. Moreover, as (15) must hold for φ, φ' such that $U' = \{j\}$, the function $\gamma^{\phi^t,\phi^{t+1}}_{i,j,h}$ is constant in all efforts except x_j. Then, we can define the function $g^\delta_{i,j,h} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ so that

$$g^\delta_{i,j,h}(x_j) = \left(\frac{p_i(x, \phi^{t+1})}{p_h(x, \phi^{t+1})}\right) / \left(\frac{p_i(x, \phi^t)}{p_h(x, \phi^t)}\right), \tag{16}$$

where $\delta := \phi^t_{i,j} - \phi^{t+1}_{i,j} \in \mathbb{R}_{++}$ and $h \in N \backslash \{i\}$. Note that by EIRWP,

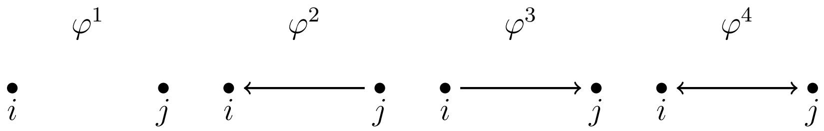
$$\frac{p_k(x, \phi^{t+1})}{p_h(x, \phi^{t+1})} = \frac{p_k(x, \phi^t)}{p_h(x, \phi^t)} \text{ for all } k \in N \backslash \{i\}, \tag{17}$$

hence, $g^\delta_{i,j,h}(x_j) = g^\delta_{i,j,k}(x_j)$ for all $k \in N \backslash \{i\}$, which means we can drop h and write $g^\delta_{i,j}(x_j)$ instead.

Let $\varphi^1, \varphi^2, \varphi^3, \varphi^4 \in \Phi$ be any four networks such that

$$\varphi^1_{i,j} = \varphi^1_{j,i} = \varphi^2_{j,i} = \varphi^2_{i,j} - \delta = \varphi^3_{j,i} - \delta = \varphi^3_{i,j} = \varphi^4_{i,j} - \delta = \varphi^4_{j,i} - \delta,$$

while $\varphi^1_{k,k'} = \varphi^2_{k,k'} = \varphi^3_{k,k'} = \varphi^4_{k,k'}$ for all k, k' with $\{k, k'\} \neq \{i, j\}$. See below for a graphical representation of the relationship between i and j in these networks, where an arrow from i to j (j to i) indicates an increase by δ in the friendship intensity from i to j (j to i) with respect to the intensity in φ^1.



By EIDRO

$$g^\delta_{i,j}(x_j) = \left(\frac{p_i(x, \varphi^1)}{p_j(x, \varphi^1)}\right) / \left(\frac{p_i(x, \varphi^2)}{p_j(x, \varphi^2)}\right). \tag{18}$$

By the same arguments above there also exists an analogously defined function $g^\delta_{j,i}(x_i)$ which can be written using φ^2 and φ^4 as

$$g^\delta_{j,i}(x_i) = \left(\frac{p_j(x, \varphi^2)}{p_i(x, \varphi^2)}\right) / \left(\frac{p_j(x, \varphi^4)}{p_i(x, \varphi^4)}\right).$$

Combining this with (18) we obtain

$$\frac{g^\delta_{i,j}(x_j)}{g^\delta_{j,i}(x_i)} = \left(\frac{p_i(x, \varphi^1)}{p_j(x, \varphi^1)}\right) / \left(\frac{p_i(x, \varphi^4)}{p_j(x, \varphi^4)}\right).$$

Suppose φ^1 is such that $\varphi^1_{h,k} = 1 - \delta$ for all $h, k \in N$, and φ^4 is defined according to the restrictions above. By EIEC and EA $\frac{p_i(x,\varphi^4)}{p_j(x,\varphi^4)} = 1$, which leads to

$$\frac{g^\delta_{i,j}(x_j)}{g^\delta_{j,i}(x_i)} = \frac{p_i(x, \varphi^1)}{p_j(x, \varphi^1)}. \tag{19}$$

Note that $\beta(\varphi^1) = \varphi^1$ for any permutation β. Then, EA implies that the functions $g^\delta_{j,i}$ and $g^\delta_{i,j}$ do not depend on the identities of i and j, hence, we can write $g^\delta_{i,j}(x_j) = g^\delta(x_j)$.

We now show $g^\delta(x_j)$ is an exponential function of δ. To do so, we first prove that $g^\delta(x_j)$ is bounded from above by 1, i.e., $g^\delta(x_j) < 1$. Consider the specific case where φ^4 is such that $\varphi^4_{h,k} = 1$ for all $h, k \in N$, and let φ^3 be defined according to our restrictions above. We have already shown that $p_k(x, \varphi^4) = p_h(x, \varphi^4)$ for all $h, k \in N$. Rewriting $g^\delta(x_j)$ using φ^3 and φ^4, we obtain $g^\delta(x_j) = \frac{p_i(x,\varphi^3)}{p_h(x,\varphi^3)}$ for $h \neq i$. It follows by EIRWP that $p_k(x, \varphi^3) = p_h(x, \varphi^3)$ for all $h, k \in N \backslash \{i\}$, and by EMDR

$p_i(x,\varphi^3) < p_i(x,\varphi^4)$. Then, one can show by EE that $p_h(x,\varphi^3) > p_h(x,\varphi^4)$ for all $h \neq i$. This implies $\frac{p_i(x,\varphi^3)}{p_h(x,\varphi^3)} < 1$, hence $g^\delta(x_j) < 1$. So, $g^\delta(x_j)$ is bounded from above by 1 as desired. By definition $g^\delta(x_j) > 0$, so $g^\delta(x_j) \in (0,1)$. For ϕ^t and ϕ^{t+1} defined above, consider an 'intermediate' network $\tilde{\phi}$ which differs from ϕ^t and ϕ^{t+1} only by the friendship intensity from j to i in such a way that $\tilde{\phi}_{i,j} = \phi^t_{i,j} - \delta'$ for some $\delta' \in (0,\delta)$. Using (16) we can write the functions $g^{\delta'}(x_j)$ and $g^{\delta-\delta'}(x_j)$ for the pairs of networks $\tilde{\phi}, \phi^t$ and $\phi^{t+1}, \tilde{\phi}$ respectively. By construction the multiplication of these two functions is equal to $g^\delta(x_j)$, so $g^\delta(x_j) = g^{\delta'}(x_j)g^{\delta-\delta'}(x_j)$. For any given x_j, this is the Cauchy's exponential equation with respect to δ, where δ can take any value in $\mathbb{R}_{++}$. As $g^\delta(x_j)$ takes value in $(0,1)$, it can be shown by Corollary 2 in Aczel et al. (2000) that there is a unique solution to this equation characterized by a function $c : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $g^\delta(x_j) = \exp(-c(x_j)\delta)$. Then, defining $f(x_j) := \exp(c(x_j))$ we obtain $g^\delta(x_j) = f(x_j)^{-\delta}$. Note that $f(x_j) > 1$ by construction.

We are now ready to determine the win probabilities in network ϕ. For any $i, j \in N$, if we take the product of the rate of change of relative win probabilities for each pair of consecutive networks in the sequence $\phi^0, \ldots, \phi^m$, we obtain

$$\left(\frac{p_i(x,\phi)}{p_j(x,\phi)}\right) \Big/ \left(\frac{p_i(x,\phi^0)}{p_j(x,\phi^0)}\right) = \prod_{t=0}^{m-1} \left(\frac{p_i(x,\phi^{t+1})}{p_j(x,\phi^{t+1})}\right) \Big/ \left(\frac{p_i(x,\phi^t)}{p_j(x,\phi^t)}\right). \tag{20}$$

Note that $\frac{p_i(x,\phi^0)}{p_j(x,\phi^0)} = 1$, so the LHS reduces to $\frac{p_i(x,\phi)}{p_j(x,\phi)}$. Condition (16) and EIRWP jointly determine the rate of change of relative win probabilities in each step of the sequence. Writing each term on the RHS of (20) in terms of f accordingly,

$$\frac{p_i(x,\phi)}{p_j(x,\phi)} = \frac{\prod_{l\in N} f(x_l)^{-(1-\phi_{i,l})}}{\prod_{l\in N} f(x_l)^{-(1-\phi_{j,l})}} = \frac{\prod_{l\in N} f(x_l)^{\phi_{i,l}}}{\prod_{l\in N} f(x_l)^{\phi_{j,l}}}.$$

For any $j \in N$, by EE,

$$\frac{1}{p_j(x,\phi)} = \sum_{i\in N} \frac{p_i(x,\phi)}{p_j(x,\phi)} = \sum_{i\in N} \left(\frac{\prod_{l\in N} f(x_l)^{\phi_{i,l}}}{\prod_{l\in N} f(x_l)^{\phi_{j,l}}}\right) = \frac{\sum_{i\in N}\prod_{l\in N} f(x_l)^{\phi_{i,l}}}{\prod_{l\in N} f(x_l)^{\phi_{j,l}}},$$

hence $p_j(x,\phi) = \frac{\prod_{l\in N} f(x_l)^{\phi_{j,l}}}{\sum_{i\in N}\prod_{l\in N} f(x_l)^{\phi_{i,l}}}$. Given this, to prove $(*)$ it is sufficient to show that the function f must be increasing. Take any $x, x' \in X$ such that $x'_i > x_i$ and $x'_k = x_k$ for $k \neq i$. By EME $p_i(x',\phi) > p_i(x,\phi)$. Then, it is easy to verify that f is increasing. So, we achieve the desired result $(*)$. □

C. Independence of axioms

We demonstrate the independence of the six axioms employed in Theorem 1 by identifying six success functions that violate each of them while satisfying all others. Proofs are omitted as they are straightforward.

1. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^1(x, F)$ as

$$s_i^1(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h) + 1},$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is increasing. This class of success functions satisfies all our axioms except exhaustivity.

2. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^2(x, F)$ as

$$s_i^2(x, F) = \frac{\prod_{h \in F_i} f_h(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f_h(x_h)},$$

where $f_i : \mathbb{R}_{++} \to (1, +\infty)$ is increasing for each $i \in N$ and $f_h \neq f_j$ for some $h, j \in N$. This class of success functions fulfills all our axioms except anonymity.

3. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^3(x, F)$ as

$$s_i^3(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)},$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is decreasing. Then, s_i^3 satisfies all axioms except monotonicity of efforts.

4. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, we define $s_i^4(x, F)$ as

$$s_i^4(x, F) = \frac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)},$$

where $f : \mathbb{R}_{++} \to (0, 1)$ is increasing. Then, s_i^4 fulfills all axioms but monotonicity of relations.

5. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, let us define $s_i^5(x, F)$ as

$$s_i^5(x, F) = \frac{\exp\left(x_i / \sum_{k \in N} x_k\right) \prod_{h \in F_i} \exp(x_h)}{\sum_{j \in N} \exp\left(x_j / \sum_{k \in N} x_k\right) \prod_{h \in F_j} \exp(x_h)}.$$

This success function violates IEC, while it satisfies all other axioms.

6. For each $i \in N$, $x \in X$ and $F \in \mathcal{F}$, let us define $s_i^6(x, F)$ as

$$s_i^6(x, F) = \begin{cases} \dfrac{f(x_i)^2}{\sum_{j \in N} f(x_j)^2} & \text{if } F_k = \{k\} \text{ for all } k \in N, \\[2ex] \dfrac{\prod_{h \in F_i} f(x_h)}{\sum_{j \in N} \prod_{h \in F_j} f(x_h)} & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}_{++} \to (1, +\infty)$ is an increasing function. Then, s_i^6 fulfills all axioms but IRO.

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