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On the essential norms of Toeplitz operators with continuous symbols

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Abstract

It is well known that the essential norm of a Toeplitz operator on the Hardy space $H^p(\mathbb{T})$, $1 < p < \infty$ is greater than or equal to the $L^\infty(\mathbb{T})$ norm of its symbol. In 1988, A. Böttcher, N. Krupnik, and B. Silbermann posed the question on whether or not equality holds in the case of continuous symbols. We answer this question in the negative. On the other hand, we show that the essential norm of a Toeplitz operator $T(a)$ with a continuous symbol a is less than or equal to $2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty}$.

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1. Introduction

For Banach spaces X and Y, let $\mathcal{B}(X,Y)$ and $\mathcal{K}(X,Y)$ denote the sets of bounded linear and compact linear operators from X to Y, respectively.

For $A \in \mathcal{B}(X,Y)$, let

$$\operatorname{Ker} A := \{x \in X|\ Ax = 0\}, \quad \operatorname{Ran} A := \{Ax|\ x \in X\}.$$

The operator A is called Fredholm if

$$\dim \operatorname{Ker} A < +\infty, \quad \dim (X/\operatorname{Ran} A) < +\infty.$$

The essential spectrum of $A \in \mathcal{B}(X) := \mathcal{B}(X,X)$ is the set

$$\operatorname{Spec}_{\mathrm{e}}(A) := \{\lambda \in \mathbb{C} :\ A - \lambda I\ \text{is not Fredholm}\}.$$

The essential norm of $A \in \mathcal{B}(X, Y)$ is defined as follows:

$$\|A\|_{\mathrm{e}} := \inf\{\|A - K\| : K \in \mathcal{K}(X, Y)\}.$$

For any $A \in \mathcal{B}(X)$, $\mathrm{Spec}_{\mathrm{e}}(A)$ and $\|A\|_{\mathrm{e}}$ are equal to the spectrum and the norm of the corresponding element $[A]$ of the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ (see, e.g., [5, Sect. 4.3]). Hence the essential spectral radius of $A \in \mathcal{B}(X)$ is less than or equal to its essential norm:

$$r_{\mathrm{e}}(A) := \sup\{|\lambda| : \lambda \in \mathrm{Spec}_{\mathrm{e}}(A)\} \le \|A\|_{\mathrm{e}}. \tag{1}$$

Let $\mathbb{T}$ be the unit circle: $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For a function $f \in L^1(\mathbb{T})$, let

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-ik\theta}\, d\theta, \quad k \in \mathbb{Z}$$

be the Fourier coefficients of f. The Hardy spaces, the Riesz projection, and the Toeplitz operator with the symbol $a \in L^\infty(\mathbb{T})$ are defined in the usual way:

$$H^p(\mathbb{T}) := \left\{f \in L^p(\mathbb{T}) : \widehat{f}(k) = 0 \text{ for all } k < 0\right\}, \quad 1 \le p \le \infty,$$

$$(Pf)\left(e^{i\theta}\right) := \sum_{k \ge 0} \widehat{f}(k) e^{ik\theta} \quad \text{for} \quad f\left(e^{i\theta}\right) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ik\theta},$$

$$T(a)f := P(af), \quad f \in L^1(\mathbb{T}).$$

If $1 < p < \infty$, the Riesz projection $P : L^p(\mathbb{T}) \to H^p(\mathbb{T})$ is bounded (see, e.g., [11, Ch. 9]) and hence the Toeplitz operator

$$T(a) = PaI : H^p(\mathbb{T}) \to H^p(\mathbb{T}), \ 1 < p < \infty, \quad a \in L^\infty(\mathbb{T}) \tag{2}$$

is bounded. Everywhere in the paper, $T(a)$ denotes operator (2).

Since

$$a(\mathbb{T})_{\mathrm{e}} := \left\{\lambda \in \mathbb{C} : \frac{1}{a - \lambda} \notin L^\infty(\mathbb{T})\right\} \subseteq \mathrm{Spec}_{\mathrm{e}}(T(a))$$

(see, e.g., [4, Theorem 2.30]), inequality (1) implies

$$\|a\|_{L^\infty} \le r_{\mathrm{e}}(T(a)) \le \|T(a)\|_{\mathrm{e}}. \tag{3}$$

On the other hand,

$$\|T(a)\|_{\mathrm{e}} \le \|T(a)\| = \|Pa\,\mathrm{I}\| \le \|P\|\|a\|_{L^\infty}.$$

Since

$$\|P\|_{L^p \to L^p} = \frac{1}{\sin \frac{\pi}{p}}$$

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(see [12]), one gets

$$\|a\|_{L^\infty} \leq \|T(a)\|_{\mathrm{e}} \leq \frac{1}{\sin\frac{\pi}{p}}\|a\|_{L^\infty}. \tag{4}$$

If $p = 2$, inequality (4) turns into the equality $\|T(a)\|_{\mathrm{e}} = \|a\|_{L^\infty}$. If $a \equiv 1$, then $\|a\|_{L^\infty} = 1 = \|T(a)\|_{\mathrm{e}}$, so the first inequality in (4) is sharp. If

$$a\left(e^{i\vartheta}\right) := \sin\frac{\pi}{p} \pm i\cos\frac{\pi}{p}, \quad \pm\vartheta \in (0, \pi),$$

then $\|a\|_{L^\infty} = 1$, and it follows from the Gohberg-Krupnik theory of Toeplitz operators with piecewise continuous symbols that

$$\frac{1}{\sin\frac{\pi}{p}} \in \mathrm{Spec}_{\mathrm{e}}(T(a))$$

(see, e.g., [4, Theorem 5.39]). Hence

$$\|T(a)\|_{\mathrm{e}} \geq \frac{1}{\sin\frac{\pi}{p}}$$

(see (1)), and the second inequality in (4) is also sharp if one considers Toeplitz operators with discontinuous symbols.

The situation is different in the case of continuous symbols. If $a \in C(\mathbb{T})$, then $\mathrm{Spec}_{\mathrm{e}}(T(a)) = a(\mathbb{T})$ (see, e.g., [4, Theorem 2.42]). In particular, $\mathrm{Spec}_{\mathrm{e}}(T(a))$ does not depend on p. It is natural to ask whether $\|T(a)\|_{\mathrm{e}}$ depends on p for $a \in C(\mathbb{T})$. Since $\|T(a)\|_{\mathrm{e}} = \|a\|_{L^\infty}$ for $p = 2$, this question can be rephrased as follows: does the equality $\|T(a)\|_{\mathrm{e}} = \|a\|_{L^\infty}$ hold for all $p \in (1, \infty)$ and all $a \in C(\mathbb{T})$? This question was posed in [3, Sect. 7.6], where it was proved that

$$\|T(a)\|_{\mathrm{e}} = \|a\|_{L^\infty} \text{ for all } a \in (C + H^\infty)(\mathbb{T}) \iff \|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = 1.$$

Here and below,
$$\mathbf{e}_m(z) := z^m, \quad z \in \mathbb{C}, \ m \in \mathbb{Z}.$$
Note that for every $f \in H^p(\mathbb{T})$,

$$f\left(e^{i\theta}\right) = \sum_{n=0}^\infty \widehat{f}(n)e^{in\theta}, \quad \theta \in [-\pi, \pi],$$

one has

$$\mathbf{e}_{-1}\left(e^{i\theta}\right)f\left(e^{i\theta}\right) = \widehat{f}(0)e^{-i\theta} + \sum_{n=1}^\infty \widehat{f}(n)e^{i(n-1)\theta},$$

and hence

$$\left(T(\mathbf{e}_{-1})f\right)\left(e^{i\theta}\right) = \sum_{n=1}^\infty \widehat{f}(n)e^{i(n-1)\theta}.$$

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If $\widehat{f}(0) = 0$, then

$$T(\mathbf{e}_{-1})f = \mathbf{e}_{-1}f \quad \Longrightarrow \quad |T(\mathbf{e}_{-1})f| = |f| \quad \text{a.e. on} \ \ \mathbb{T}.$$

So, the equality $\|T(\mathbf{e}_{-1})f\|_{H^p} = \|f\|_{H^p}$ holds on a co-dimension one subspace of $H^p(\mathbb{T})$, and the equality $\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = 1$ looks plausible. Nevertheless, we show that the answer to the above question is negative and $\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} > 1$ for every $p \neq 2$ (see Section 5).

The constant $\frac{1}{\sin\frac{\pi}{p}}$ in the right-hand side of (4) tends to infinity as $p \to 1$ or ∞. It turns out that a better estimate holds for $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ if $a \in (C + H^\infty)(\mathbb{T})$. Namely,

$$\|T(a)\|_{\mathrm{e}} \leq 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty} \leq 2\|a\|_{L^\infty}$$

(see Section 4).

The proof of our main results relies upon the use of measures of noncompactness (see Section 2) and approximation properties of Hardy spaces (see Section 3).

2. Measures of noncompactness of a linear operator

For a bounded subset Ω of a Banach space Y, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r.

For $A \in \mathcal{B}(X, Y)$, set

$$\|A\|_\chi := \chi\left(A(B_X)\right),$$

where B_X denotes the unit ball in X. Let

$$\|A\|_m := \inf_{\substack{M \subseteq X \ \text{closed linear subspace} \\ \dim(X/M) < \infty}} \left\|A|_M\right\|,$$

where $A|_M$ denotes the restriction of A to M.

The measures of noncompactness $\|\cdot\|_\chi$ and $\|\cdot\|_m$ have the following properties

$$\frac{1}{2}\|A\|_\chi \leq \|A\|_m \leq 2\|A\|_\chi \tag{5}$$

and

$$\|A\|_\chi \leq \|A\|_{\mathrm{e}}, \quad \|A\|_m \leq \|A\|_{\mathrm{e}} \tag{6}$$

(see [13, (3.7) and (3.29)]; note that there is a typo in [13, (3.7)], where the factor 2 is missing in the right-hand side). The constants $\frac{1}{2}$ and 2 in (5) are optimal (see [1, 2.5.2 and 2.5.6]).

In general, $\|A\|_{\mathrm{e}}$ cannot be estimated above by $\|A\|_\chi$ or $\|A\|_m$ (see [2]). Some restrictions on the geometry of X or Y are needed for such estimates.

Definition 2.1. *A Banach space Z is said to have the* bounded compact approximation property (BCAP) *if there exists a constant $M \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $F \subset Z$, there exists an operator $T \in \mathcal{K}(Z)$ such that*

$$\|I - T\| \leq M \quad and \quad \|y - Ty\| < \varepsilon, \quad \forall y \in F. \tag{7}$$

Here I is the identity map from Z to itself.

We say that Z has the dual compact approximation property (DCAP) *if there exists a constant $M^* \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $G \subset Z^*$, there exists an operator $T \in \mathcal{K}(Z)$ such that*

$$\|I - T\| \leq M^* \quad and \quad \|z - T^*z\| < \varepsilon, \quad \forall z \in G. \tag{8}$$

The greatest lower bound of the constants M (constants M^*) for which (7) ((8), respectively) holds will be denoted by $M(Z)$ (by $M^*(Z)$).

It is easy to see that

$$Z \text{ has the DCAP} \implies Z^* \text{ has the BCAP and } M(Z^*) \leq M^*(Z),$$

and that if Z is reflexive, then

$$Z \text{ has the DCAP} \iff Z^* \text{ has the BCAP and } M(Z^*) = M^*(Z).$$

Although we will apply the results of this Section only to reflexive spaces, we consider here the general (non-reflexive) case.

A comprehensive study of various approximation properties can be found in [14] (see also [8] for examples of function spaces that have the BCAP).

If Y has the BCAP, then

$$\|A\|_{\mathrm{e}} \leq M(Y)\|A\|_{\chi}, \quad \forall A \in \mathcal{B}(X, Y)$$

(see [13, Theorem 3.6]) and hence

$$\|A\|_{\mathrm{e}} \leq 2M(Y)\|A\|_m, \quad \forall A \in \mathcal{B}(X, Y)$$

(see (5)).

Theorem 2.2. *If X has the DCAP, then*

$$\|A\|_{\mathrm{e}} \leq M^*(X)\|A\|_m, \quad \forall A \in \mathcal{B}(X, Y). \tag{9}$$

Proof. Take any $\varepsilon > 0$. According to the definition of $\|A\|_m$, there exists a subspace M of X having finite codimension and such that

$$\|Ax\| \leq (\|A\|_m + \varepsilon)\|x\|, \quad \forall x \in M. \tag{10}$$

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Let $Q : X \to M$ be a bounded projection onto M. Then $I - Q$ is a finite rank operator. Since $(I - Q)^* \in \mathcal{K}(X^*)$, there exist $z_1, \dots, z_n \in X^*$ such that

$$\min_{k=1,\dots,n} \|(I - Q)^* z - z_k\| < \varepsilon \tag{11}$$

for every $z \in X^*$ with $\|z\| \leq 1$. Since X has the DCAP, there exists $T \in \mathcal{K}(X)$ such that $\|I - T\| \leq M^*(X) + \varepsilon$ and

$$\|z_k - T^* z_k\| < \varepsilon, \quad k = 1, \dots, n.$$

Take any $z \in X^*$ with $\|z\| \leq 1$ and choose k for which the minimum in (11) is achieved. Then

$$\|(I - T)^* (I - Q)^* z\| \leq \|(I - T)^* ((I - Q)^* z - z_k)\| + \|(I - T)^* z_k\|$$
$$< \|(I - T)^*\| \varepsilon + \varepsilon = \|I - T\| \varepsilon + \varepsilon \leq (M^*(X) + \varepsilon + 1)\varepsilon.$$

Hence

$$\|(I - Q)(I - T)\| = \|(I - T)^* (I - Q)^*\| < (M^*(X) + \varepsilon + 1)\varepsilon.$$

Then using (10), one gets

$$\|AQ(I - T)x\| \leq (\|A\|_m + \varepsilon)\|Q(I - T)x\|$$
$$\leq (\|A\|_m + \varepsilon)\Big(\|(I - T)x\| + \|(I - Q)(I - T)x\|\Big)$$
$$\leq (\|A\|_m + \varepsilon)(M^*(X) + \varepsilon + (M^*(X) + \varepsilon + 1)\varepsilon)$$

for every $x \in X$ with $\|x\| \leq 1$. Since

$$A - AQ(I - T) = A(I - Q) + AQT \in \mathcal{K}(X, Y),$$

the above implies

$$\|A\|_e \leq (\|A\|_m + \varepsilon)(M^*(X) + \varepsilon + (M^*(X) + \varepsilon + 1)\varepsilon)$$

for any $\varepsilon > 0$. Passing to the limit as $\varepsilon \to 0$, one arrives at (9). $\qquad \square$

3. Approximation properties of Hardy spaces

Theorem 3.1. *The Hardy space $H^p = H^p(\mathbb{T})$, $1 < p < \infty$ has the bounded compact approximation and the dual compact approximation properties with*

$$M(H^p), M^*(H^p) \leq 2^{\left|1 - \frac{2}{p}\right|}.$$

Proof. Let

$$K_n\left(e^{i\theta}\right) := \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} = \frac{1}{2\pi(n+1)} \left(\frac{\sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}\right)^2,$$
$$\theta \in [-\pi, \pi], \quad n = 0, 1, 2, \dots$$

be the n-th Fejér kernel, and let

$$(\mathbf{K}_n f)\left(e^{i\vartheta}\right) := (K_n * f)\left(e^{i\vartheta}\right) = \int_{-\pi}^{\pi} K_n\left(e^{i\vartheta - i\theta}\right) f\left(e^{i\theta}\right) d\theta, \quad \vartheta \in [-\pi, \pi],$$

where $f \in L^1(\mathbb{T})$. It is well known that $\|K_n\|_{L^1(\mathbb{T})} = 1$,

$$(\mathbf{K}_n f)\left(e^{i\vartheta}\right) = \sum_{k=-n}^{n} \widehat{f}(k) \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta}, \tag{12}$$

where $\widehat{f}(k)$ is the k-th Fourier coefficient of f, $\|\mathbf{K}_n\|_{L^p \to L^p} = 1$ for $1 \le p \le \infty$, $\mathbf{K}_n$ converge strongly to the identity operator on $L^p(\mathbb{T})$, $1 \le p < \infty$ as $n \to \infty$, and $\mathbf{K}_n$ map $H^p(\mathbb{T})$ into itself (see, e.g., [11, Ch. 2]). It follows from (12) and Parseval's theorem that $\|I - \mathbf{K}_n\|_{L^2 \to L^2} = 1$. Since $\|I - \mathbf{K}_n\|_{L^p \to L^p} \le 1 + \|\mathbf{K}_n\|_{L^p \to L^p} = 2$, the Riesz-Thorin interpolation theorem (see, e.g., [14, vol. II, Theorem 2.b.14]) applied to $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$ implies

$$\|I - \mathbf{K}_n\|_{L^p \to L^p} \le 2^{1 - \frac{2}{p}}, \quad 2 \le p \le \infty.$$

Similarly, interpolating between $L^2(\mathbb{T})$ and $L^1(\mathbb{T})$, one gets

$$\|I - \mathbf{K}_n\|_{L^p \to L^p} \le 2^{\frac{2}{p} - 1}, \quad 1 \le p \le 2.$$

The above inequalities imply that

$$\|I - \mathbf{K}_n\|_{H^p \to H^p} \le 2^{\left|1 - \frac{2}{p}\right|}, \quad 1 \le p \le \infty.$$

It is easy to see that the adjoint to $\mathbf{K}_n : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ operator can be identified with $\mathbf{K}_n : H^{p'}(\mathbb{T}) \to H^{p'}(\mathbb{T})$, $p' = \frac{p}{p-1}$ (see [6, §7.2]). Hence the conditions in Definition 2.1 are satisfied for $T = \mathbf{K}_n$ with a sufficiently large n. $\square$

If $1 < p < \infty$, the Hardy space $H^p(\mathbb{T})$ is reflexive. Although $(H^p(\mathbb{T}))^*$ is isomorphic to $H^{p'}(\mathbb{T})$, these two spaces are not isometrically isomorphic, and it is not clear whether or not $M^*(H^p) = M(H^{p'})$. Unfortunately, the exact values of $M(H^p)$, $M^*(H^p)$ do not seem to be known.

4. An upper estimate for the essential norm of a Toeplitz operator

Theorem 4.1. *Let $a \in (C+H^\infty)(\mathbb{T})$. Then the following holds for the Toeplitz operator $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$,*

$$\|T(a)\|_m = \|a\|_{L^\infty}, \quad \|T(a)\|_e \leq 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty}. \tag{13}$$

Proof. It is sufficient to prove the equality in (13) as the inequality then follows from Theorems 2.2 and 3.1. Since $\|T(b)\| \leq \|P\|\|b\|_{L^\infty}$ for any $b \in L^\infty(\mathbb{T})$ and functions of the form $a = \mathbf{e}_{-n}h$, $h \in H^\infty(\mathbb{T})$, $n \in \mathbb{N}$ are dense in $(C + H^\infty)(\mathbb{T})$ (see, e.g., [10, Ch. IX, Theorem 2.2]), it is sufficient to prove (13) for such a function. Let $H_n^p(\mathbb{T})$ be the subspace of $H^p(\mathbb{T})$ consisting of all functions with the first n Fourier coefficients equal to 0. Then $H_n^p(\mathbb{T})$ has codimension n and

$$\|T(a)f\|_{H^p} = \|af\|_{H^p} \leq \|a\|_{L^\infty}\|f\|_{H^p}, \quad \forall f \in H_n^p(\mathbb{T}).$$

Hence $\|T(a)\|_m \leq \|a\|_{L^\infty}$. Since $\|T(a)\|_m$ is greater than or equal to the essential spectral radius of $T(a)$ (see [13, §6]) and the latter is greater than or equal to $\|a\|_{L^\infty}$ (see (3)), one has the opposite inequality $\|T(a)\|_m \geq \|a\|_{L^\infty}$. $\square$

5. The essential norm of the backward shift operator

Theorem 5.1. *The following equalities hold for the Toeplitz operator $T(\mathbf{e}_{-1})$: $H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$,*

$$\|T(\mathbf{e}_{-1})\|_\chi = \|T(\mathbf{e}_{-1})\|_e = \|T(\mathbf{e}_{-1})\|_{H^p \to H^p}.$$

Proof. Since

$$\|T(\mathbf{e}_{-1})\|_\chi \leq \|T(\mathbf{e}_{-1})\|_e \leq \|T(\mathbf{e}_{-1})\|_{H^p \to H^p}$$

(see (6)), it is sufficient to prove that $\|T(\mathbf{e}_{-1})\|_\chi \geq \|T(\mathbf{e}_{-1})\|_{H^p \to H^p} =: C_p$. For any $\varepsilon > 0$, there exists $q \in H^p(\mathbb{T})$,

$$q\left(e^{i\theta}\right) = \sum_{k=0}^\infty c_k e^{ik\theta} = c_0 + \sum_{k=1}^\infty c_k e^{ik\theta} =: c_0 + q_0\left(e^{i\theta}\right), \quad \theta \in [-\pi, \pi],$$

such that $\|q\|_{H^p} = 1$ and

$$\|q_0\|_{H^p} = \|\mathbf{e}_{-1}q_0\|_{H^p} = \|T(\mathbf{e}_{-1})q\|_{H^p} \geq C_p - \varepsilon.$$

Since $\mathbf{e}_N$, $N \in \mathbb{N}$ is an inner function and $\mathbf{e}_N(0) = 0$, one has $\|f \circ \mathbf{e}_N\|_{H^p} = \|f\|_{H^p}$ for any $f \in H^p(\mathbb{T})$ (see [7, Theorem 5.5]). Hence $\|q \circ \mathbf{e}_N\|_{H^p} = 1$ and

$$\|T(\mathbf{e}_{-1})(q \circ \mathbf{e}_N)\|_{H^p} = \|\mathbf{e}_{-1}(q_0 \circ \mathbf{e}_N)\|_{H^p} = \|q_0 \circ \mathbf{e}_N\|_{H^p} \geq C_p - \varepsilon.$$

Take any finite set $\{\varphi_1, \ldots, \varphi_m\} \subset H^p(\mathbb{T})$ and choose polynomials

$$\psi_j(z) := \sum_{k=0}^{n_j} \psi_{j,k} z^k, \quad z \in \mathbb{C}, \quad j = 1, \ldots, m$$

such that $\|\varphi_j - \psi_j\|_{H^p} \leq \varepsilon$. Then choose $N \in \mathbb{N}$ such that

$$N > \max\{n_1, \ldots, n_m\} + 1 \tag{14}$$

and set $h := \|f\|_{H^p}^{1-p} |f|^{p-2} \overline{f}$, where $f = q_0 \circ \mathbf{e}_N$. A standard calculation gives $\|h\|_{L^{p'}} = 1$ and

$$\int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_N) \left(e^{i\theta}\right) h \left(e^{i\theta}\right) d\theta = \|q_0 \circ \mathbf{e}_N\|_{H^p}.$$

The Fourier series of h has the form

$$\sum_{k \in \mathbb{Z}} h_k e^{ikN\theta}, \quad h_k \in \mathbb{C}.$$

It follows from (14) that

$$\{kN | \ k \in \mathbb{Z}\} \cap \{1, \ldots, n_j + 1\} = \emptyset, \quad j = 1, \ldots, m.$$

Hence

$$\int_{-\pi}^{\pi} (\mathbf{e}_1 \psi_j) \left(e^{i\theta}\right) h \left(e^{i\theta}\right) d\theta = 0.$$

So,

$$\int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_N - \mathbf{e}_1 \psi_j) \left(e^{i\theta}\right) h \left(e^{i\theta}\right) d\theta = \|q_0 \circ \mathbf{e}_N\|_{H^p}, \quad j = 1, \ldots, m.$$

On the other hand, Hölder's inequality implies

$$\left| \int_{-\pi}^{\pi} (q_0 \circ \mathbf{e}_N - \mathbf{e}_1 \psi_j) \left(e^{i\theta}\right) h \left(e^{i\theta}\right) d\theta \right| \leq \|q_0 \circ \mathbf{e}_N - \mathbf{e}_1 \psi_j\|_{H^p},$$

since $\|h\|_{L^{p'}} = 1$. Hence

$$\|q_0 \circ \mathbf{e}_N - \mathbf{e}_1 \psi_j\|_{H^p} \geq \|q_0 \circ \mathbf{e}_N\|_{H^p}$$

and

$$\|T(\mathbf{e}_{-1})(q \circ \mathbf{e}_N) - \varphi_j\|_{H^p} = \|\mathbf{e}_{-1}(q_0 \circ \mathbf{e}_N) - \varphi_j\|_{H^p} = \|q_0 \circ \mathbf{e}_N - \mathbf{e}_1 \varphi_j\|_{H^p}$$
$$\geq \|q_0 \circ \mathbf{e}_N - \mathbf{e}_1 \psi_j\|_{H^p} - \varepsilon \geq \|q_0 \circ \mathbf{e}_N\|_{H^p} - \varepsilon \geq C_p - 2\varepsilon, \quad j = 1, \ldots, m.$$

So, for every finite set $\{\varphi_1, \ldots, \varphi_m\} \subset H^p(\mathbb{T})$, there exist an element of the image of the unit ball $T(\mathbf{e}_{-1})(B_{H^p})$ that lies at a distance at least $C_p - 2\varepsilon$ from every element of $\{\varphi_1, \ldots, \varphi_m\}$. This means that $T(\mathbf{e}_{-1})(B_{H^p})$ cannot be covered by a finite family of open balls of radius $C_p - 2\varepsilon$. Hence

$$\|T(\mathbf{e}_{-1})\|_\chi \geq C_p - 2\varepsilon = \|T(\mathbf{e}_{-1})\|_{H^p \to H^p} - 2\varepsilon, \quad \forall \varepsilon > 0,$$

i.e. $\|T(\mathbf{e}_{-1})\|_\chi \geq \|T(\mathbf{e}_{-1})\|_{H^p \to H^p}$. $\qquad \square$

It is known that $\|T(\mathbf{e}_{-1})\|_{H^p \to H^p} > 1$ for $p \neq 2$ (see [3, § 7]). So, it follows from Theorem 5.1 that

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} > 1 = \|\mathbf{e}_{-1}\|_{L^\infty}, \quad p \neq 2. \tag{15}$$

The exact value of $\|T(\mathbf{e}_{-1})\|_{H^p \to H^p}$ does not seem to be known (see [9]), but it follows from the proof of Theorem 3.1 that

$$\|T(\mathbf{e}_{-1})\|_{H^p \to H^p} = \|\mathbf{e}_{-1}(I - \mathbf{K}_0)\|_{H^p \to H^p} = \|I - \mathbf{K}_0\|_{H^p \to H^p} \leq 2^{\left|1 - \frac{2}{p}\right|}$$

(see [3, 7.8] and [9]).

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