



## King's Research Portal

[Link to publication record in King's Research Portal](#)

*Citation for published version (APA):*

Rozenblum, G., & Shargorodsky, E. (2020). *Eigenvalue estimates and asymptotics for weighted pseudodifferential operators with singular measures in the critical case*. <https://arxiv.org/abs/2011.14877>

### **Citing this paper**

Please note that where the full-text provided on King's Research Portal is the Author Accepted Manuscript or Post-Print version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version for pagination, volume/issue, and date of publication details. And where the final published version is provided on the Research Portal, if citing you are again advised to check the publisher's website for any subsequent corrections.

### **General rights**

Copyright and moral rights for the publications made accessible in the Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognize and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Research Portal

### **Take down policy**

If you believe that this document breaches copyright please contact [librarypure@kcl.ac.uk](mailto:librarypure@kcl.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.

# Eigenvalue estimates and asymptotics for weighted pseudodifferential operators with singular measures in the critical case

Grigori Rozenblum

Eugene Shargorodsky

*To Ari Laptev, with best wishes, on the occasion of his 70th birthday*

## 1 Introduction

We study the eigenvalue distribution for self-adjoint compact operators of the type  $\mathfrak{A}^*P\mathfrak{A}$ , where  $\mathfrak{A}$  is a pseudodifferential operator of negative order  $-l$  in a domain  $\Omega \subset \mathbb{R}^N$  and  $P$  is a signed measure in  $\Omega$ . A number of spectral problems can be reduced to this one, the most important one, probably, being  $\lambda(-\Delta)^l u = Pu$ , closely related to the Schrödinger operator. In particular, if  $P$  is an absolutely continuous measure, the spectral asymptotics for such operators has been justified under rather mild conditions imposed on  $P$ , see [3]. The case of a singular measure on  $\Omega$  is not that well studied. In 1951, M.G.Krein discovered that for the ‘singular string’ described by the equation  $-\lambda u'' = Pu$  with the Dirichlet boundary conditions at the endpoints of an interval and with  $P$  being a Borel measure, the leading term in the asymptotics of the eigenvalues is determined by the absolutely continuous part  $P_{ac}$  of  $P$  only, while the singular part  $P_{sing}$  makes a weaker contribution. Further on, in papers by M.Sh. Birman, M.Z. Solomyak, and V.V. Borzov, see, e.g. [3], this property of singular measures was proved for a wide class of ‘high order’ spectral problems, in particular for  $\lambda(-\Delta)^l u = Pu$  in a domain  $\Omega \subset \mathbb{R}^N$ , provided  $2l > N$ . For ‘low order problems’,  $2l < N$ , the influence of the singular part of the measure is different. The known cases concern  $P_{sing}$  concentrated on a surface of codimension 1, and here, in the opposite, it makes a leading order contribution to the spectral asymptotics, see [7]. The intermediate, ‘critical’ case  $2l = N$  has been studied even less (it is the common wisdom that for many questions in spectral theory this case is the hardest one). Until

---

G. Rozenblum: Chalmers Univ. of Technology; EIMI, and St.Petersburg State University; email: [grigori@chalmers.se](mailto:grigori@chalmers.se)

E.Shargorodsky: King’s College London and Technische Universität Dresden; email: [eugene.shargorodsky@kcl.edu](mailto:eugene.shargorodsky@kcl.edu)

very recently, there were very few results here. In dimension  $\mathbf{N} = 2$ , if a measure  $P$  is concentrated at the boundary of  $\Omega$  (which is equivalent to the Steklov problem) or on a smooth curve inside  $\Omega$ , the eigenvalue asymptotics has the same order as for the regular Dirichlet problem, namely,  $\lambda_k \sim Ck^{-1}$ , see also general results in [1], [11], [7].

Recently, the interest to the critical case has revived, due to some new applications, see, e.g., [18], [12]. So, a class of singular measures was considered in [9]. For a singular measure  $P = V\mu$  in  $\mathbb{R}^2$ , where  $\mu$  is Ahlfors  $\alpha$ -regular,  $\alpha \in (0, 2)$ , and  $V$  belongs to a certain Orlicz class, an estimate for the eigenvalues of the problem  $-\lambda\Delta u = Pu$  was obtained,  $|\lambda_k| \leq C(V, \mu)k^{-1}$ . Thus, unlike other cases, the order in the eigenvalue estimate does not depend on the (Hausdorff) dimension of the support of the measure. The question of sharpness of these estimates was not touched upon. In another field, the critical case was involved in studies related to noncommutative integration, see [12].

In this paper, we consider a class of singular measures in the critical case and prove order sharp estimates and asymptotic formulae for eigenvalues for the corresponding spectral problems. Namely, the measure  $P = P_{sing}$  is supposed to be supported on a compact Lipschitz surface  $\Sigma$  of codimension  $\mathfrak{d}$  in  $\mathbb{R}^{\mathbf{N}}$  and absolutely continuous with respect to the the surface measure  $\mu_\Sigma$  induced by the embedding of  $\Sigma$  into  $\mathbb{R}^{\mathbf{N}}$ ,  $P = V\mu_\Sigma$ . We find that the eigenvalues  $\lambda_k$  have asymptotics of order  $k^{-1}$ , so the order of asymptotics does not depend on the dimension or codimension of the surface. We consider the case of compactly supported measures only and therefore do not touch upon effects related with infinity, which, as it is known, may influence the eigenvalue behaviour drastically, even for a nice measure, see, e.g., [2]. If  $P$  is concentrated on several surfaces, of different dimensions, their contributions to the eigenvalue asymptotics add up. The same happens if the measure has both an absolutely continuous and a singular parts.

In the usual way, the estimates and asymptotics of eigenvalues of the weighted problem, by means of the Birman-Schwinger principle, produce similar results for the number of negative eigenvalues of the Schrödinger-like operator as the coupling constant grows. This relation was explored in [9].

Some of the results of the paper were presented in the short note [14] without proofs.

## 2 Setting and main results

Since the pathbreaking papers by M.Sh. Birman and M.Z. Solomyak, it is known that it is often useful to reduce a spectral problem for a differential equation to an eigenvalue problem for a compact operator. We consider a class of compact operators encompassing the above problems, as well as their pseudodifferential versions, as particular cases. Let  $\mathfrak{A}$  be an operator in a bounded domain  $\Omega \subset \mathbb{R}^{\mathbf{N}}$ , such that the localization of  $\mathfrak{A}$  to a proper subdomain  $\Omega' \subset \Omega$  is a pseudodifferential operator of order  $-l = -\mathbf{N}/2$ , up to a smoothing additive term. The operators  $(-\Delta)^{-l/2}$  and  $(-\Delta + 1)^{-l/2}$

in  $\mathbb{R}^d$ , restricted to  $\Omega$ , or  $(-\Delta_{\mathcal{D}})^{-1/2}$ , where  $-\Delta_{\mathcal{D}}$  is the Dirichlet Laplacian in  $\Omega$ , are typical examples of such  $\mathfrak{A}$ . By localisation we mean multiplication of  $\mathfrak{A}$  on both sides by a smooth cut-off function  $\varsigma \in C_0^\infty(\Omega)$ ,  $\varsigma|_{\Omega'} = 1$ . Our results do not depend on the particular localisation chosen, see Section 3; it is convenient to assume that the above multiplication has already been performed and the cut-offs are incorporated in  $\mathfrak{A}$ .

Let  $\Sigma$  be a compact Lipschitz surface in  $\Omega'$ , of dimension  $d$ ,  $1 \leq d \leq \mathbf{N} - 1$  and, correspondingly, of codimension  $\mathfrak{d} = \mathbf{N} - d$ . Thus, locally, in appropriate coordinates,  $X = (\mathbf{x}; \mathbf{y}) := (x_1, \dots, x_d; y_{d+1}, \dots, y_{\mathfrak{d}})$ , the corresponding piece of the surface  $\Sigma$  is described by an equation  $\mathbf{y} = \varphi(\mathbf{x})$ , where  $\varphi$  is a Lipschitz vector-function. For brevity, we describe our constructions in a single co-ordinate neighborhood; the resulting formulae are glued together in a standard manner. The embedding  $\Sigma \subset \mathbb{R}^{\mathbf{N}}$  generates the surface measure  $\mu_\Sigma$  on  $\Sigma$ ,  $d\mu_\Sigma = \sigma(\mathbf{x})d\mathbf{x}$ ,  $\sigma(\mathbf{x}) = [\det(\mathbf{1} + (\nabla\varphi)^*(\nabla\varphi))]^{\frac{1}{2}}d\mathbf{x}$ , which, in turn, generates a singular measure on  $\Omega$ , supported in  $\Sigma$ , which we also denote by  $\mu_\Sigma$ , as long as this does not cause confusion.

Let  $\mu$  be a Radon measure on  $\Omega$ . We denote by  $\mathbf{M}$  its support, the smallest closed set of full measure. The Orlicz space  $L^\Psi(\mu)$ ,  $\Psi(t) = (1+t)\log(1+t) - t$ , consists of functions  $V$  on  $\mathbf{M}$ , satisfying  $\int_{\mathbf{M}} \Psi(|V(X)|)d\mu(X) < \infty$ . For a subset  $E \subset \mathbb{R}^{\mathbf{N}}$ , we define the norm

$$\|V\|_E^{(av, \Psi; \mu)} := \sup \left\{ \left| \int_{E \cap \mathbf{M}} Vg d\mu \right| : \int \Phi(|g|)d\mu \leq \mu(E \cap \mathbf{M}) \right\}, \quad (2.1)$$

if  $\mu(E \cap \mathbf{M}) > 0$ , and  $\|V\|_E^{(av, \Psi)} = 0$  otherwise; here  $\Phi$  is the Orlicz complementary function to  $\Psi$ ,  $\Phi(t) = e^t - 1 - t$ . Such *averaged* norms, first introduced by M.Z. Solomyak in [19], have played an important rôle in the study of the eigenvalue distribution in the critical case. For a real-valued function  $V \in L_1(\mathbf{M})$ , we consider the quadratic form

$$\mathbf{F}_V[u] = \mathbf{F}_V[u; \mathfrak{A}; \mu] := \int_{\Omega} V(X)|(\mathfrak{A}u)(X)|^2 d\mu(X), \quad u \in L_2(\Omega). \quad (2.2)$$

We will see later that this is a bounded quadratic form in  $L_2(\Omega)$  and it defines a selfadjoint operator, as soon as  $V$  belongs to  $L^\Psi(\Sigma, \mu)$ . This operator is denoted by  $\mathbf{T}(V, \mu) = \mathbf{T}(V, \mathfrak{A}, \mu)$ .

The case of principal interest for us is  $\mu$  being the measure  $\mu_\Sigma$  and  $\mathbf{M}$  being the surface  $\Sigma$ . Here we will use the notation  $L^\Psi = L^\Psi(\Sigma)$  for the Orlicz space,  $\|V\|_E^{(av, \Psi)}$  for the averaged norm and  $\mathbf{T}(V) = \mathbf{T}(V, \mathfrak{A})$  for the operator defined by the form (2.2) with  $\mu = \mu_\Sigma$ .

For a compact self-adjoint operator  $\mathbf{T}$  in a Hilbert space, we denote by  $\lambda_k^\pm(\mathbf{T})$  the positive (negative) eigenvalues of  $\mathbf{T}$  in the non-increasing order of their absolute values, repeated according to their multiplicities. By  $n_\pm(\lambda, \mathbf{T})$  we denote the counting function of  $\lambda_k^\pm$ . The notation  $n(\lambda, \mathbf{T})$  is used for the counting function of singular numbers of the (not necessarily self-adjoint) operator  $\mathbf{T}$ . When the operator is associated with a quadratic form  $\mathbf{F}$ , the notation  $n_\pm(\lambda, \mathbf{F})$  etc. is sometimes used.

Our first main result is the following eigenvalue estimate.

**Theorem 2.1.** *Let  $\Sigma$  be a compact Lipschitz surface of dimension  $d < \mathbf{N}$  in  $\Omega' \subset \mathbb{R}^{\mathbf{N}}$  and  $V \in L^{\Psi}(\Sigma)$ . Then for the operator  $\mathbf{T} = \mathbf{T}(V, \Sigma, \mathfrak{A})$ , the estimate*

$$n(\lambda, \mathbf{T}) \leq C \|V\|_{\Sigma}^{(av, \Psi)} \lambda^{-1} \quad (2.3)$$

*holds with a constant  $C$  depending on the surface  $\Sigma$  and the operator  $\mathfrak{A}$  but independent of the function  $V$ .*

Theorem 2.1 extends to the case of singular measures supported on surfaces in  $\mathbb{R}^{\mathbf{N}}$  the estimates obtained by M.Z. Solomyak in [19] for domains (cubes) in  $\mathbb{R}^{\mathbf{N}}$  for an even  $\mathbf{N}$  and in [21] for an odd  $\mathbf{N}$ . In both cases, the operator  $\mathfrak{A}_0 = (1 - \Delta)^{-\mathbf{N}/4}$  played the role of  $\mathfrak{A}$ . The passage to a more general  $\mathfrak{A}$  is easy and is carried out at the end of Section 5.3.

Theorem 2.1 follows from a spectral estimate in a more general setting extending the considerations in [9].

**Definition 2.2.** Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^{\mathbf{N}}$ . We say that it is  $\alpha$ -Ahlfors regular (an  $\alpha$ -AR-measure),  $\alpha \in (0, \mathbf{N}]$ , if there exist positive constants  $c_0$  and  $c_1$  such that

$$c_0 r^{\alpha} \leq \mu(B(X, r)) \leq c_1 r^{\alpha} \quad (2.4)$$

for all  $0 < r \leq \text{diam}(\text{supp } \mu)$  and all  $X \in \text{supp } \mu$ , where  $B(X, r)$  is the ball of radius  $r$  centred at  $X$  and the constants  $c_0$  and  $c_1$  are independent of the balls.

If  $\mu$  is an  $\alpha$ -AR-measure, then it is equivalent to the  $\alpha$ -dimensional Hausdorff measure on its support (see, e.g., [5, Lemma 1.2]). The measure  $\mu_{\Sigma}$  for a compact Lipschitz surface of dimension  $d$  in  $\mathbb{R}^{\mathbf{N}}$  is, obviously,  $d$ -AR.

**Theorem 2.3.** *Let  $\mu$  be a compactly supported  $d$ -AR measure,  $0 < d \leq \mathbf{N}$ , and  $V \in L^{\Psi}(\mu)$ . Then for the operator  $\mathbf{T}(V, \mathfrak{A}_0, \mu)$ , the estimate*

$$n_{\pm}(\lambda, \mathbf{T}(V, \mu, \Omega)) \leq C(d, \mu) \|V\|_{\mathbf{M}}^{(av, \Psi)} \lambda^{-1} \quad (2.5)$$

*holds with a constant  $C(d, \mu)$  depending on the domain  $\Omega$  and the measure  $\mu$ , but not on the function  $V$ .*

To formulate the result on the eigenvalue asymptotics, we need more notation. According to the Rademacher theorem, the function  $\varphi$  is differentiable  $\mu_{\Sigma}$ -almost everywhere. At such ‘regular’ points  $X_0$ , the tangent  $d$ -dimensional plane  $T_{X_0}\Sigma$  and the normal  $\mathfrak{d}$ -dimensional plane  $N_{X_0}\Sigma$  exist. The principal symbol  $a_{-l}(X, \Xi)$  of the operator  $\mathfrak{A}$  can be expressed in a neighbourhood of such a point in the coordinates  $(\mathbf{x}, \mathbf{y}; \xi, \eta)$ , with  $\mathbf{x} \in T_{X_0}\Sigma$ ,  $\mathbf{y} \in N_{X_0}\Sigma$  and the corresponding co-variables  $\xi, \eta$ ; we denote it again  $a_{-l}(X_0; \xi, \eta)$ . In this notation, we set

$$r_{-d}(X_0, \xi) = (2\pi)^{-\mathfrak{d}} \int_{N_{X_0}\Sigma} |a_{-l}(X_0; \xi, \eta)|^2 d\eta, \quad \xi \in T_{X_0}^* \Sigma. \quad (2.6)$$

This function is defined almost everywhere on  $T^*\Sigma$  and is order  $-d$  homogeneous in  $\xi$ . Now we formulate our result on eigenvalue asymptotics.

**Theorem 2.4.** *Under the conditions of Theorem 2.1, the counting function for the eigenvalues of the operator  $\mathbf{T}$  has the following asymptotics*

$$n_{\pm}(\lambda, \mathbf{T}(V, \Sigma)) \sim \lambda^{-1} C_{\pm}(V, \Sigma), \quad C_{\pm} = (2\pi)^{-d} \int_{S^* \Sigma} V_{\pm}(X) r_{-d}(X, \xi) d\mu_{\Sigma}(X) d\xi, \quad (2.7)$$

where the integration is performed over the cosphere bundle of  $\Sigma$  and  $V_{\pm}(X)$  denotes the positive, respectively, negative part of the function  $V$ .

If several Lipschitz surfaces, of possibly different dimensions, are present and the measure  $P$  has a possibly nontrivial absolutely continuous part, the above eigenvalue asymptotics still holds, with the coefficient being the sum of the coefficients calculated for all components of the measure.

An important particular case of our general considerations concerns  $\mathfrak{A}$  being an appropriate negative power of the Laplace operator with some boundary conditions. More precisely,  $\mathfrak{A}_0 = (-\Delta)^{-l/4} + S$ , where  $S$  is an operator smoothing inside  $\Omega$  (typically, a singular Green operator.) In this case, the eigenvalue problem for the operator  $\mathbf{T}$  can be reduced, up to negligible terms (which we disregard throughout this section), to the weighted polyharmonic eigenvalue problem

$$\lambda(-\Delta)^l f = P f \quad (2.8)$$

understood in the distributional sense:

$$\lambda \int_{\Omega} f(-\Delta)^l h dX = \int_{\Omega} f h P, \quad h \in \mathcal{D}(\Omega), \quad (2.9)$$

or, for  $P = V\mu_{\Sigma}$ ,

$$\lambda \int_{\Omega} f(-\Delta)^l h dX = \int_{\Sigma} f h V d\mu_{\Sigma}, \quad h \in \mathcal{D}(\Omega), \quad (2.10)$$

with some boundary conditions understood, again, in the distributional sense. If the geometry of  $\Sigma$  is sufficiently ‘nice’, the spectral problem (2.9), (2.10) can be expressed more explicitly. For example, if  $\mathbf{N} = 2$ ,  $l = 1$ ,  $P = V\mu_{\Sigma}$ , and  $\Sigma$  is a Lipschitz curve inside  $\Omega$ , we arrive (see, e.g., [1]) at the *transmission (conjugation) problem*

$$-\Delta f = 0 \text{ outside } \Sigma, \quad \lambda[f_n](X) = V(X)f(X) \text{ on } \Sigma, \quad f \in H^1(\Omega), \quad (2.11)$$

where  $[f_n]$  is the jump of the normal derivative  $f_n$  at  $\Sigma$ . Note that if  $\Sigma$  is the boundary of  $\Omega$ , we obtain a Steklov type problem, associated with the Neumann-to-Dirichlet operator,

$$-\Delta f = 0 \text{ for } X \in \Omega, \quad \lambda f_n(X) = V(X)f(X) \text{ on } \Sigma. \quad (2.12)$$

This case is *not* covered by the reasoning in this paper as the surface  $\Sigma$  is *not* contained in  $\Omega$ . We stress here that the question on the eigenvalue asymptotics for the Steklov problem with Lipschitz boundary is still open, even in the two-dimensional case, while

for the transmission problem with a nice weight on a Lipschitz surface the eigenvalue asymptotics was established in [16].

Let now, still for  $\mathbf{N} = 2$ ,  $\Sigma \subset \Omega$ , the measure  $P$  have both absolutely continuous and singular parts,  $P = V_0 dX + V_1 \mu_\Sigma$  for a Lipschitz curve  $\Sigma \subset \Omega$ , with  $V_0 \in L^\Psi(\Omega)$  and  $V_1 \in L^\Psi(\Sigma)$ . The eigenvalue problem (2.9) takes the form

$$-\lambda \Delta f(X) = V_0(X)f(X) \text{ for } X \in \Omega \setminus \Sigma, \quad \lambda [f_n](X) = V_1(X)f(X) \text{ on } \Sigma. \quad (2.13)$$

So, here the spectral parameter is present both in the differential equation and the transmission condition. Our results show that they both contribute to the leading term in the eigenvalue asymptotics. *Boundary* problems of this type have been considered by A. Kozhevnikov, see [10].

Let us now pass to the case  $\mathbf{N} = 4$ ,  $l = 2$ . Here we have the following choice for the dimension  $d$  of  $\Sigma$ :  $d = 1, 2$ , or  $3$ . For  $d = 3$ ,  $\mathfrak{d} = 1$ , we arrive again at a transmission problem similar to (2.11):

$$\Delta^2 f(X) = 0 \text{ for } X \in \Omega \setminus \Sigma, \quad \lambda [(-\Delta f)_n](X) = V(X)f(X) \text{ on } \Sigma, \quad (2.14)$$

$$f \in H^2(\Omega).$$

Next, for  $d = 2$ ,  $\mathfrak{d} = 2$ , let  $X = (\mathbf{x}, \mathbf{y}) \in \Omega \subset \mathbb{R}^4$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ , let  $\Sigma = \{X : \mathbf{y} = 0\} \cap \Omega$  be (a piece of) the two-dimensional plane in  $\mathbb{R}^4$ , and let  $V = V(\mathbf{x})$ . Then, after some simple calculations, we arrive at the following problem

$$\Delta^2 f(X) = 0 \text{ for } X \in \Omega \setminus \Sigma, \quad \lambda \lim_{\delta \rightarrow 0} \int_{|\mathbf{y}|=\delta} (-\Delta f)_n(\mathbf{x}, \mathbf{y}) ds(\mathbf{y}) = V(\mathbf{x})f(\mathbf{x}, 0), \quad (2.15)$$

where, for each fixed  $\mathbf{x}$ , the integral of the normal derivative of the Laplacian of  $f$  is taken over the  $\delta$ -circle in the  $\mathbf{y}$ -plane. Finally, for  $d = 1$ , when the manifold  $\Sigma$  is the line  $\mathbf{y} = 0$  in the coordinates  $X = (\mathbf{x}, \mathbf{y})$ ,  $\mathbf{x} \in \mathbb{R}^1$ ,  $\mathbf{y} \in \mathbb{R}^3$  in  $\mathbb{R}^4$ , the resulting problem is

$$\Delta^2 f(X) = 0 \text{ for } X \in \Omega \setminus \Sigma, \quad \lambda \lim_{\delta \rightarrow 0} \int_{|\mathbf{y}|=\delta} (-\Delta f)_n(\mathbf{x}, \mathbf{y}) ds(\mathbf{y}) = V(\mathbf{x})f(\mathbf{x}, 0), \quad (2.16)$$

where, for each fixed  $\mathbf{x}$ , the integration is performed over the 2D sphere  $|\mathbf{y}| = \delta$ . One can interpret the conditions on the surface  $\Sigma$  in (2.15), (2.16) as multi-dimensional versions of the transmission conditions in (2.11), (2.14). Our results show that all these spectral problems have the same order of the eigenvalue asymptotics.

If the measure  $P$  has a nonzero absolutely continuous part with density  $V_0$ , one should replace in (2.14)-(2.16) the equation  $\Delta^2 f(X) = 0$  with  $\lambda \Delta^2 f(X) = V_0(X)f(X)$ , so, again, we arrive at spectral problems containing the spectral parameter both in the equation and in the transmission conditions.

As we already mentioned, if the support of the singular measure consists of several disjoint surfaces, of possibly different dimensions, their contributions to the eigenvalue estimates have the same order and the coefficients in the asymptotics add up.

**Corollary 2.5.** *Let a measure  $P$  with a compact support in  $\Omega \subset \mathbb{R}^N$  have the absolutely continuous part  $P_{ac} = V_0(X)dX$ ,  $V_0 \in L^\Psi(\Omega, dx)$  and the singular part  $P_{sing} = \sum P_j$ , where  $P_j = V_j\mu_{\Sigma_j}$ ,  $\Sigma_j$  are disjoint Lipschitz surfaces of dimension  $d_j < N$ ,  $\mu_{\Sigma_j}$  are the measures induced by the embeddings of  $\Sigma_j$  into  $\mathbb{R}^N$ , and  $V_j \in L^\Psi(\Sigma_j, \mu_{\Sigma_j})$ . Then for the operator in  $L_2(\Omega)$  defined by the quadratic form  $\mathbf{F}_P[u] = \int_\Omega |(\mathfrak{A}u)(X)|^2 P$  in  $L^2(\Omega)$ , the following asymptotic formula holds*

$$n_\pm(\lambda, \mathbf{F}) \sim \lambda^{-1} C(P, \mathfrak{A}), \lambda \rightarrow 0; \quad (2.17)$$

here  $C(P, \mathfrak{A})$  is the sum of the asymptotic coefficients in (2.7) corresponding to all surfaces  $\Sigma_j$ , plus the term coming from  $P_{ac}$ ,  $C(P, \mathfrak{A}) = C(P_{ac}, \mathfrak{A}) + \sum C(V_j, \Sigma_j)$ , where

$$C(P_{ac}, \mathfrak{A}) = (2\pi)^{-N} \int_{S^* \Omega} V_{0,\pm}(X) |a_{-l}(X, \Xi)|^2 dX d\Xi. \quad (2.18)$$

The disjointness conditions above can be considerably relaxed, see Sect. 7.

### 3 Some reductions

This section contains some technical observations that are used further on in the paper to reduce general eigenvalue estimates and asymptotics to more convenient setting. Similar arguments for spectral problems for differential operators are a well known part of mathematical folklore. They have been used systematically, starting with the papers by M.Sh Birman and M.Z. Solomyak, for the past 50 years. They are usually proved in a line or two. Our pseudodifferential versions require some additional, somewhat technical, reasoning, which nevertheless follows the classical pattern, and the results are quite natural. Readers familiar with the classical versions and not interested in these details can skip this section without detriment to understanding the rest of the paper. For brevity, we take a surface of dimension  $d = N$  to mean a domain in  $\mathbb{R}^N$ . For an operator  $\mathbf{T}$ , we denote by  $\mathbf{D}^\pm(\mathbf{T})$  the quantity

$$\mathbf{D}^\pm(\mathbf{T}) := \limsup_{\lambda \rightarrow 0} \lambda n_\pm(\lambda, \mathbf{T}) = \limsup_{k \rightarrow \infty} \pm k \lambda_k^\pm(\mathbf{T}).$$

**Observation 1:** Localization 1. It is sufficient to prove the eigenvalue estimate for a surface being described by only one coordinate neighbourhood. Indeed, if  $\Sigma$  is split into a finite number  $J$  of disjoint surfaces  $\Sigma_j$ ,  $j = 1, \dots, J$  of possibly different dimensions, and  $P_j = V_j\mu_{\Sigma_j}$  is the restriction of  $P$  to  $\Sigma_j$ , then  $\mathbf{D}^\pm(\Sigma, \mathfrak{A}) \leq J \sum_j \mathbf{D}^\pm(\Sigma_j, \mathfrak{A})$ . This property follows from the Ky Fan inequality for the sum of operators,  $\mathbf{T}(V, \Sigma, \mathfrak{A}) = \sum_j \mathbf{T}(V_j, \Sigma_j, \mathfrak{A})$ .

**Observation 2:** Lower order terms. Let  $\mathfrak{B}$  be a pseudodifferential operator of order  $\beta < -l$ . Then  $\mathbf{D}^\pm(V, \Sigma, \mathfrak{B}) = 0$ . This follows from the (already mentioned) result in [3] that a finite singular measure gives a lower order contribution to the eigenvalue distribution for higher order problems.



**Observation 3:** Perturbations by lower order terms. Let  $\mathfrak{B}$  be as above. Then

$$\mathbf{D}^\pm(V, \Sigma, \mathfrak{A} + \mathfrak{B}) = \mathbf{D}^\pm(V, \Sigma, \mathfrak{A}). \quad (3.1)$$

This follows from the inequality

$$(1 - \epsilon)|a|^2 + \left(1 - \frac{1}{\epsilon}\right)|b|^2 \leq |a + b|^2 \leq (1 + \epsilon)|a|^2 + \left(1 + \frac{1}{\epsilon}\right)|b|^2$$

with  $a = (\mathfrak{A}u)(X)$ ,  $b = (\mathfrak{B}u)(X)$  and the previous observation.

**Observation 4:** Localization 2. Let  $\Omega_1$  be an open subset in  $\Omega$  such that  $\Sigma \cap \overline{\Omega_1} = \emptyset$ , and let  $V \in L^\Psi(\Sigma)$ . Then for the eigenvalues of the operator  $\mathbf{T}(V, \Omega_1)$  on  $L_2(\Omega_1)$  defined by the form  $\mathbf{F}_{V, \Omega_1}[u] = \int_\Sigma V(X)|(\mathfrak{A}u)(X)|^2 d\mu$ ,  $u \in L^2(\Omega_1)$ , one has the following estimate  $n_\pm(\lambda, \mathbf{T}(V, \Omega_1)) = o(\lambda^{-1})$  as  $\lambda \rightarrow 0$ .

*Proof.* Let  $\chi \in C_0^\infty$  be a smooth function equal to 0 in a neighbourhood of  $\overline{\Omega_1}$  and to 1 in a neighbourhood of  $\Sigma$ . Then

$$\mathbf{F}_{V, \Omega_1}[u] = \int_\Sigma V(X)|(\chi(X)\mathfrak{A}u)(X)|^2 d\mu = \int_\Sigma V(X)|([\chi(X), \mathfrak{A}]u)(X)|^2 d\mu. \quad (3.2)$$

The commutator  $[\chi(X), \mathfrak{A}]$  is a pseudodifferential operator of order  $-l - 1$ , and therefore, by Observation 2, the eigenvalues of the corresponding operator  $\mathbf{T}(V, \Omega_1)$  decay faster than  $k^{-1}$ .

It follows, in particular, that the eigenvalue counting function gets a lower order perturbation if we perturb the operator  $\mathfrak{A}$  outside a neighborhood of the surface  $\Sigma$ . In particular, this gives us freedom in choosing cut-off functions away from  $\Sigma$  or adding operators smoothing away from the boundary – the possibility already mentioned.

The next, more complicated, statement is used in the study of eigenvalue asymptotics. It says that if two measures have supports separated by a positive distance, then, up to a lower order term, the counting functions behave additively with respect to the measures. This includes the important case when absolutely continuous measures are present. If both measures are absolutely continuous, this is a classical fact.

**Lemma 3.1.** (*Localization 3*) Let  $P = P_1 + P_2$ , where  $P_j = V_j \mu_{\Sigma_j}$  is a measure supported on a compact Lipschitz surface  $\Sigma_j$  of dimension  $d_j \in [1, \mathbf{N}]$ ,  $j = 1, 2$  (the cases  $d_1 = \mathbf{N}$  and  $d_2 = \mathbf{N}$  correspond to  $\Sigma_1$ , respectively  $\Sigma_2$ , being domains in  $\Omega \subset \mathbb{R}^{\mathbf{N}}$  and the measures being absolutely continuous with respect to the Lebesgue measure). Suppose that  $\text{dist}(\text{supp } P_1, \text{supp } P_2) > 0$ . Then

$$n_\pm(\lambda, \mathbf{T}(P_1 + P_2)) = n_\pm(\lambda, \mathbf{T}(P_1)) + n_\pm(\lambda, \mathbf{T}(P_2)) + o(\lambda^{-1}) \text{ as } \lambda \rightarrow 0. \quad (3.3)$$

*Proof.* Consider two disjoint open sets  $\Omega_1, \Omega_2 \subset \Omega$ , such that  $\overline{\Sigma_j} \subset \Omega_j$  and  $\overline{\Omega_1} \cup \overline{\Omega_2} \supset \Omega$ . Every function  $u \in L^2(\Omega)$  splits into the (orthogonal) sum  $u = u_1 \oplus u_2$ ,  $u_j \in L^2(\Omega_j)$ .

The quadratic form of the operator  $\mathbf{T}(P_1 + P_2)$  splits as follows

$$\begin{aligned}
\mathbf{F}(P_1, P_2)[u] &:= (\mathbf{T}(P_1 + P_2)u, u)_{L^2(\Omega)} \\
&= \int_{\Omega} V_1(X) |\mathfrak{A}(u_1)(X) + \mathfrak{A}(u_2)(X)|^2 d\mu_{\Sigma_1} + \int_{\Omega} V_2(X) |\mathfrak{A}(u_1)(X) + \mathfrak{A}(u_2)(X)|^2 d\mu_{\Sigma_2} \\
&= \mathbf{F}_1(P_1)[u_1] + \mathbf{F}_2(P_2)[u_2] + \mathbf{F}_R[u_1, u_2] \\
&:= \int_{\Omega} V_1(X) |\mathfrak{A}(u_1)(X)|^2 d\mu_{\Sigma_1} + \int_{\Omega} V_2(X) |\mathfrak{A}(u_2)(X)|^2 d\mu_{\Sigma_1} + \mathbf{F}_R[u_1, u_2],
\end{aligned} \tag{3.4}$$

where the remainder term  $\mathbf{F}_R[u_1, u_2]$  is a form in functions  $u_j \in L^2(\Omega_j)$  with the following property: if a term in  $\mathbf{F}_R$  contains  $V_j$ , then it necessarily contains  $u_{3-j}$ , so it always contains a measure and a function with disjoint supports. If such a term has the form  $\int V_1 |\mathfrak{A}u_2|^2 d\mu_{\Sigma_1}$ , the corresponding operator  $\mathbf{T}$  satisfies  $n_{\pm}(\lambda, \mathbf{T}) = o(\lambda^{-1})$  by Observation 4. If, on the other hand, such term has the form  $\int V_1 (\mathfrak{A}u_1) \overline{(\mathfrak{A}u_2)} d\mu_{\Sigma_1}$ , then by the Schwartz inequality,

$$\left| \int V_1 (\mathfrak{A}u_1) \overline{(\mathfrak{A}u_2)} d\mu_{\Sigma_1} \right| \leq \left( \int |V_1| |\mathfrak{A}u_1|^2 d\mu_{\Sigma_1} \right)^{1/2} \left( \int |V_1| |\mathfrak{A}u_2|^2 d\mu_{\Sigma_1} \right)^{1/2},$$

and the last factor, again, provides the required  $o$  estimate for the eigenvalues. Now we observe that the forms  $\mathbf{F}_1[u_1]$ ,  $\mathbf{F}_2[u_2]$  in (3.4) act on orthogonal subspaces. Then the spectrum of the sum of the corresponding operators  $\mathbf{T}_1, \mathbf{T}_2$  equals the union of the spectra of the summands, and hence  $n_{\pm}(\lambda, \mathbf{T}_1 + \mathbf{T}_2) = n_{\pm}(\lambda, \mathbf{T}_1) + n_{\pm}(\lambda, \mathbf{T}_2)$ . Since the term  $\mathbf{F}_R$  in (3.4) makes a weaker contribution,

$$n_{\pm}(\lambda, \mathbf{T}(P_1 + P_2)) = n_{\pm}(\lambda, \mathbf{T}_1) + n_{\pm}(\lambda, \mathbf{T}_2) + o(\lambda^{-1}). \tag{3.5}$$

Now consider the operator  $\mathbf{T}(P_1)$ . It has the quadratic form  $\mathbf{F}(P_1)[u] = \int V_1(X) |\mathfrak{A}(u_1 \oplus u_2)|^2 d\mu_{\Sigma_1}$ . Similarly to (3.4), we represent it as

$$\mathbf{F}(P_1)[u] = \int_{\Omega} V_1(X) |\mathfrak{A}(u_1)(X)|^2 d\mu_{\Sigma_1} + \mathbf{F}_{R_1}[u_1, u_2], \tag{3.6}$$

with  $\mathbf{F}_{R_1}$  having the same structure as  $\mathbf{F}_R$  in (3.4). Again, the form  $\mathbf{F}_{R_1}$  generates an operator with eigenvalues decaying faster than  $k^{-1}$ , and we obtain

$$n_{\pm}(\lambda, \mathbf{T}(P_1)) = n_{\pm}(\lambda, \mathbf{T}_1) + o(\lambda^{-1}). \tag{3.7}$$

In the same way,

$$n_{\pm}(\lambda, \mathbf{T}(P_2)) = n_{\pm}(\lambda, \mathbf{T}_2) + o(\lambda^{-1}). \tag{3.8}$$

Finally, we substitute (3.7), (3.8) into (3.5) to obtain (3.3).  $\square$

The following corollary of Lemma 3.1 allows one to separate the positive and the negative parts of the function  $V$  when studying the distribution of the positive and the negative eigenvalues of  $\mathbf{T}(V, \Sigma)$ .

**Corollary 3.2.** *Let  $\Sigma$  be a Lipschitz surface and  $V \in L^\Psi(\Sigma)$ . Let  $\Sigma_\pm$  be relatively open subsets of  $\Sigma$  such that  $V_\pm \geq 0$  in  $\Sigma_\pm$ ,  $V = 0$  in  $\Sigma \setminus (\Sigma_+ \cup \Sigma_-)$ , and  $\text{dist}(\overline{\Sigma_+}, \overline{\Sigma_-}) > 0$ . Then*

$$n_\pm(\lambda, \mathbf{T}(V, \Sigma)) = n_\pm(\lambda, \mathbf{T}(V_\pm, \Sigma)) + o(\lambda^{-1}) \text{ as } \lambda \rightarrow 0. \quad (3.9)$$

In other words, up to a lower order remainder, the behaviour of the positive, respectively negative, eigenvalues of the operator  $\mathbf{T}(V)$  is determined by the positive, respectively negative, part of the density  $V$ . To prove this property, we can use (3.5), taking as  $P_1$  the restriction of the measure  $P$  to the set  $\Sigma_+$ , and as  $P_2$  its restriction to  $\Sigma_-$ , and recall that  $n_-(\lambda, \mathbf{T}(V, \Sigma_+)) = n_+(\lambda, \mathbf{T}(V, \Sigma_-)) = 0$ .

## 4 Geometry considerations

Our proof of Theorems 2.1 and 2.3 relies upon certain geometric observations that might be of an independent interest.

Let  $\mathcal{A} \subset \mathbb{R}^{\mathbf{N}}$  be a  $k$ -dimensional affine subspace,  $0 \leq k < \mathbf{N}$ ,  $\mathcal{A} = a + \mathcal{L}$ , where  $a \in \mathbb{R}^{\mathbf{N}}$  and  $\mathcal{L}$  is a  $k$ -dimensional linear subspace in  $\mathbb{R}^{\mathbf{N}}$ . In the case  $k = 0$ ,  $\mathcal{L}$  is a 0-dimensional linear subspace, i.e  $\mathcal{L} = \{0\}$ , and  $\mathcal{A}$  is just the singleton  $\{a\}$ . The polar plane of  $\mathcal{A}$  is the  $\mathbf{N} - k$  dimensional linear subspace of  $\mathbb{R}^{\mathbf{N}}$ ,

$$\mathcal{A}^\perp := \mathcal{L}^\perp = \{Y \in \mathbb{R}^{\mathbf{N}} : (X, Y) = 0 \text{ for all } X \in \mathcal{L}\}, \quad (4.1)$$

where  $(\cdot, \cdot)$  denotes the standard inner product in  $\mathbb{R}^{\mathbf{N}}$ . We will say that  $\mathcal{A}$  is orthogonal to a vector  $b \in \mathbb{R}^{\mathbf{N}} \setminus \{0\}$  if  $b \in \mathcal{A}^\perp$ . For  $k = \mathbf{N} - 1$ ,  $\mathcal{A}^\perp$  is the one-dimensional linear subspace spanned by  $b$ . For a linear subspace  $\mathcal{M}$  in  $\mathbb{R}^{\mathbf{N}}$ , we denote by  $\mathcal{M}_\mathbb{S}$  the trace of  $\mathcal{M}$  on  $\mathbb{S}^{\mathbf{N}-1}$ ,  $\mathcal{M}_\mathbb{S} := \mathcal{M} \cap \mathbb{S}^{\mathbf{N}-1}$ , where  $\mathbb{S}^{\mathbf{N}-1}$  is the unit sphere in  $\mathbb{R}^{\mathbf{N}}$ ,  $\mathbb{S}^{\mathbf{N}-1} := \{X \in \mathbb{R}^{\mathbf{N}} : |X| = 1\}$ .

**Lemma 4.1.** *Let  $\mathfrak{B}$  be an at most countable family of proper linear subspaces of  $\mathbb{R}^{\mathbf{N}}$ . Then there exists an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_\mathbf{N}$  of  $\mathbb{R}^{\mathbf{N}}$  such that*

$$\mathbf{e}_j \notin \cup_{\mathcal{M} \in \mathfrak{B}} \mathcal{M}, \quad j = 1, \dots, \mathbf{N}.$$

*Proof.* The proof is by induction on  $\mathbf{N}$ . There is nothing to prove if  $\mathbf{N} = 1$  since the only proper linear subspace of  $\mathbb{R}$  is  $\mathcal{M} = \{0\}$ . Suppose that the statement is true for  $\mathbf{N} = \mathbf{N}_0$  and let us prove it for  $\mathbf{N} = \mathbf{N}_0 + 1$ . Let  $\mathfrak{B}'$  be the (possibly empty) subset of  $\mathfrak{B}$  consisting of all  $\mathbf{N}_0$ -dimensional  $\mathcal{M} \in \mathfrak{B}$ . For every such  $\mathcal{M}$ ,  $\mathcal{M}^\perp$  is 1-dimensional and  $\mathcal{M}_\mathbb{S}^\perp$  consists of two points. Since  $\cup_{\mathcal{M} \in \mathfrak{B}'} \mathcal{M}_\mathbb{S}^\perp$  is an at most countable union of spheres of dimension at most  $\mathbf{N}_0 - 1$ , the set

$$\Theta := (\cup_{\mathcal{M} \in \mathfrak{B}'} \mathcal{M}_\mathbb{S}^\perp) \cup (\cup_{\mathcal{M} \in \mathfrak{B}} \mathcal{M}_\mathbb{S})$$

has  $\mathbf{N}_0$ -dimensional Lebesgue measure equal to 0. So,  $\Theta \neq \mathbb{S}^{\mathbf{N}_0}$ . Take any vector  $\mathbf{e}_{\mathbf{N}_0+1} \in \mathbb{S}^{\mathbf{N}_0} \setminus \Theta$  and let  $\mathcal{M}_0$  be the  $\mathbf{N}_0$ -dimensional linear subspace of  $\mathbb{R}^{\mathbf{N}_0+1}$  orthogonal to  $\mathbf{e}_{\mathbf{N}_0+1}$ . Since

$$\mathbf{e}_{\mathbf{N}_0+1} \notin \cup_{\mathcal{M} \in \mathfrak{B}'} \mathcal{M}_\mathbb{S}^\perp,$$

$\mathcal{M}_0$  does not coincide with any element of  $\mathfrak{B}'$ , and hence the dimension of  $\mathcal{M}_0 \cap \mathcal{M}$  is at most  $\mathbf{N}_0 - 1$  for every  $\mathcal{M} \in \mathfrak{B}$ . Then, by the inductive assumption, there exists an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_{\mathbf{N}_0}$  of  $\mathcal{M}_0$  such that

$$\mathbf{e}_j \notin \cup_{\mathcal{M} \in \mathfrak{B}} (\mathcal{M}_0 \cap \mathcal{M}) = \mathcal{M}_0 \cap (\cup_{\mathcal{M} \in \mathfrak{B}} \mathcal{M}), \quad j = 1, \dots, \mathbf{N}_0.$$

It is clear that  $\mathbf{e}_1, \dots, \mathbf{e}_{\mathbf{N}_0}, \mathbf{e}_{\mathbf{N}_0+1}$  is an orthonormal basis of  $\mathbb{R}^{\mathbf{N}_0+1}$  and

$$\mathbf{e}_j \notin \cup_{\mathcal{M} \in \mathfrak{B}} \mathcal{M}, \quad j = 1, \dots, \mathbf{N}_0 + 1.$$

□

**Lemma 4.2.** *Let  $\mu$  be a  $\sigma$ -finite Radon measure on  $\mathbb{R}^{\mathbf{N}}$  such that  $\mu(\mathcal{A}) = 0$  for every  $(k-1)$ -dimensional affine subspace  $\mathcal{A} \subset \mathbb{R}^{\mathbf{N}}$  for some  $k \in [1, \mathbf{N}-1]$ . Then the set  $\mathfrak{X}_k$  of  $k$ -dimensional affine subspaces  $\mathcal{E}$  of  $\mathbb{R}^{\mathbf{N}}$  such that  $\mu(\mathcal{E}) > 0$  is at most countable.*

*Proof.* The proof is similar to that of [9, Lemma 2.13]. It is sufficient to prove the lemma for compactly supported measures as the general case then follows easily from the assumption that  $\mu$  is  $\sigma$ -finite. Suppose that  $\mathfrak{X}_k$  is uncountable. Then there exists a  $\delta > 0$  such that the set

$$\mathfrak{X}_{k,\delta} := \{\mathcal{E} \in \mathfrak{X}_k : \mu(\mathcal{E}) > \delta\}$$

is infinite. Otherwise,  $\mathfrak{X}_k = \bigcup_{m \in \mathbb{N}} \mathfrak{X}_{k, \frac{1}{m}}$  would have been at most countable. Now take distinct  $\mathcal{E}_1, \dots, \mathcal{E}_j, \dots \in \mathfrak{X}_{k,\delta}$ . We have  $\mu_k(\mathcal{E}_j) > \delta$  for all  $j \in \mathbb{N}$ .

Since  $\mathcal{E}_j \cap \mathcal{E}_{j'}, j \neq j'$ , is an affine subspace of dimension at most  $k-1$ ,

$$\mu \left( \bigcup_{j \neq j'} (\mathcal{E}_j \cap \mathcal{E}_{j'}) \right) = 0.$$

Let

$$\tilde{\mathcal{E}}_j := \mathcal{E}_j \setminus \bigcup_{j' \neq j} (\mathcal{E}_{j'} \cap \mathcal{E}_j).$$

Then  $\tilde{\mathcal{E}}_j \cap \tilde{\mathcal{E}}_{j'} = \emptyset, j \neq j'$  and  $\mu(\tilde{\mathcal{E}}_j) = \mu(\mathcal{E}_j)$ . So, since  $\mu$  has compact support,

$$\sum_{j \in \mathbb{N}} \mu(\tilde{\mathcal{E}}_j) = \mu \left( \bigcup_{j \in \mathbb{N}} (\tilde{\mathcal{E}}_j) \right) < \infty.$$

On the other hand,  $\mu(\tilde{\mathcal{E}}_j) = \mu(\mathcal{E}_j) > \delta$  implies that  $\sum_{j \in \mathbb{N}} \mu_k(\tilde{\mathcal{E}}_j) \geq \sum_{j \in \mathbb{N}} \delta = \infty$ . This contradiction means that  $\mathfrak{X}_k$  is at most countable. □

We arrive at our main geometric statement, considerably more general than what we actually need here.

**Theorem 4.3.** *Let  $\mu$  be a  $\sigma$ -finite Radon measure on  $\mathbb{R}^{\mathbf{N}}$  without point masses. Then there exists an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_{\mathbf{N}}$  of  $\mathbb{R}^{\mathbf{N}}$  such that  $\mu(\mathcal{E}) = 0$  for every  $(\mathbf{N}-1)$ -dimensional affine subspace  $\mathcal{E}$  of  $\mathbb{R}^{\mathbf{N}}$  orthogonal to an element of this basis.*

*Proof.* Let us first explain the idea of the proof. By Lemma 4.2 for  $k = 1$ , there is an at most countable set of one-dimensional affine subspaces whose  $\mu$ -measure is positive. Subtracting the portion of measure  $\mu$  living on all these subspaces and applying Lemma 4.2 to the resulting measure, this time with  $k = 2$ , we conclude that at most countably many 2-dimensional affine subspaces are charged positively. We subtract the part of our measure living on these subspaces, and so on. This procedure is repeated in all dimensions, after which it turns out that the remaining measure is zero on *all* affine subspaces in  $\mathbb{R}^{\mathbf{N}}$ , and the proof is completed by applying Lemma 4.1.

Now the formal proof. Denote  $\mu = \mu_1$ . Lemma 4.2 with  $k = 1$  implies that the set  $\mathfrak{X}_1$  of all 1-dimensional affine subspaces  $\mathcal{E}$  of  $\mathbb{R}^{\mathbf{N}}$  such that  $\mu_1(\mathcal{E}) > 0$  is at most countable. We introduce

$\mathcal{Z}_1 := \cup_{\mathcal{E} \in \mathfrak{X}_1} \mathcal{E}$ ,  $\mu_1^0(E) := \mu_1(E \cap \mathcal{Z}_1)$  for every Borel set  $E \subseteq \mathbb{R}^{\mathbf{N}}$ , and  $\mu_2 := \mu_1 - \mu_1^0$ .

Measure  $\mu_2$  annuls every 1-dimensional affine subspace  $\mathcal{E}$ . Indeed,

$$\begin{aligned} \mu_2(\mathcal{E}) &\leq \mu_1(\mathcal{E}) = 0 \quad \text{if } \mathcal{E} \notin \mathfrak{X}_1, \\ \mu_2(\mathcal{E}) &= \mu_1(\mathcal{E}) - \mu_1^0(\mathcal{E}) = \mu_1(\mathcal{E}) - \mu_1(\mathcal{E}) = 0 \quad \text{if } \mathcal{E} \in \mathfrak{X}_1. \end{aligned}$$

Therefore we can apply Lemma 4.2 again, this time with  $k = 2$  and  $\mu = \mu_2$ .

Repeating this procedure, we obtain at the  $k$ -th step a  $\sigma$ -finite Radon measure  $\mu_k$  on  $\mathbb{R}^{\mathbf{N}}$  such that  $\mu_k(\mathcal{A}) = 0$  for every  $(k - 1)$ -dimensional affine subspace  $\mathcal{A}$  of  $\mathbb{R}^{\mathbf{N}}$  and the set  $\mathfrak{X}_k$  of all  $k$ -dimensional affine subspaces  $\mathcal{E}$  of  $\mathbb{R}^{\mathbf{N}}$  with  $\mu_k(\mathcal{E}) > 0$  is at most countable. Similarly to the above, we set  $\mathcal{Z}_k := \cup_{\mathcal{E} \in \mathfrak{X}_k} \mathcal{E}$ ,  $\mu_k^0(E) := \mu_k(E \cap \mathcal{Z}_k)$  for every Borel set  $E \subseteq \mathbb{R}^{\mathbf{N}}$ , and  $\mu_{k+1} := \mu_k - \mu_k^0$ . Then, similarly to the above,  $\mu_{k+1}(\mathcal{E}) = 0$  for every  $k$ -dimensional affine subspace  $\mathcal{E}$ .

For every  $\mathcal{E} \in \mathfrak{X}_k$ ,  $\mathcal{E}^\perp$  is an  $(\mathbf{N} - k)$ -dimensional linear subspace of  $\mathbb{R}^{\mathbf{N}}$ ,  $1 \leq k \leq \mathbf{N} - 1$ . By Lemma 4.1, there exists an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_{\mathbf{N}}$  of  $\mathbb{R}^{\mathbf{N}}$  such that

$$\mathbf{e}_j \notin \mathfrak{E} := \cup_{k=1}^{\mathbf{N}-1} \cup_{\mathcal{E} \in \mathfrak{X}_k} \mathcal{E}^\perp, \quad j = 1, \dots, \mathbf{N}. \quad (4.2)$$

Take any  $j = 1, \dots, \mathbf{N}$ . Let  $\mathcal{E}_0$  be an  $(\mathbf{N} - 1)$ -dimensional affine subspace orthogonal to  $\mathbf{e}_j$ . Then for any  $k = 1, \dots, \mathbf{N} - 1$ ,

$$\mathcal{E} \in \mathfrak{X}_k \implies \mathcal{E} \not\subseteq \mathcal{E}_0. \quad (4.3)$$

Indeed, if  $\mathcal{E} \subseteq \mathcal{E}_0$ , then

$$\mathbf{e}_j \in \mathcal{E}_0^\perp \subseteq \mathcal{E}^\perp \subseteq \mathfrak{E},$$

which contradicts (4.2).

Since  $\mathcal{E}_0 \notin \mathfrak{X}_{\mathbf{N}-1}$ ,  $\mu_{\mathbf{N}-1}(\mathcal{E}_0) = 0$ . For any  $\mathcal{E} \in \mathfrak{X}_k$ , (4.3) implies that  $\mathcal{E}_0 \cap \mathcal{E}$  is an affine subspace of dimension at most  $k - 1$ . Then  $\mu_k(\mathcal{E}_0 \cap \mathcal{E}) = 0$  and

$$\mu_k^0(\mathcal{E}_0) = \mu_k(\mathcal{E}_0 \cap \mathcal{Z}_k) \leq \sum_{\mathcal{E} \in \mathfrak{X}_k} \mu_k(\mathcal{E}_0 \cap \mathcal{E}) = 0.$$

Hence

$$\begin{aligned} 0 &= \mu_{\mathbf{N}-1}(\mathcal{E}_0) = \mu_{\mathbf{N}-2}(\mathcal{E}_0) - \mu_{\mathbf{N}-2}^0(\mathcal{E}_0) = \mu_{\mathbf{N}-2}(\mathcal{E}_0) = \mu_{\mathbf{N}-3}(\mathcal{E}_0) \\ &= \cdots = \mu_1(\mathcal{E}_0) = \mu(\mathcal{E}_0). \end{aligned}$$

□

## 5 Estimates

The method of deriving eigenvalue estimates for non-smooth spectral problems developed by M.Sh. Birman and M.Z. Solomyak in the late 60's has remained since then the most efficient approach to such singular problems. The method is based on constructing piecewise polynomial approximations of functions in Sobolev spaces. We follow the presentation in [19], with necessary modifications caused by the presence of singular measures.

### 5.1 A homogeneous Hölder inequality

We recall that for a set  $G \subseteq \mathbb{R}^{\mathbf{N}}$  of positive Lebesgue measure, the Sobolev norm and the homogeneous Sobolev semi-norm are defined by

$$\|u\|_{H^s(G)}^2 := \sum_{|\nu| \leq s} \|\partial^\nu u\|_{L^2(G)}^2, \quad \|u\|_{H_{\text{hom}}^s(G)}^2 := \sum_{|\nu| = s} \|\partial^\nu u\|_{L^2(G)}^2$$

for an integer  $s$ , and by

$$\begin{aligned} \|u\|_{H^s(G)}^2 &:= \|u\|_{H^m(G)}^2 + \|u\|_{H_{\text{hom}}^s(G)}^2, \\ \|u\|_{H_{\text{hom}}^s(G)}^2 &:= \sum_{|\nu| = m} \int_G \int_G \frac{|(\partial^\nu u)(X) - (\partial^\nu u)(Y)|^2}{|X - Y|_2^{\mathbf{N}+1}} dx dy \end{aligned}$$

for a half-integer  $s = m + \frac{1}{2}$ . The semi-norm  $\|\cdot\|_{H_{\text{hom}}^s(G)}$  possesses the homogeneity property  $\|u(R\cdot)\|_{H_{\text{hom}}^s(G)} = R^{s-\mathbf{N}/2} \|u\|_{H_{\text{hom}}^s(RG)}$  for all  $R > 0$ . Our case of interest is  $s = l = \frac{\mathbf{N}}{2}$ ; here the homogeneous semi-norm is invariant under dilations. Although this semi-norm is not, in general, invariant under rotations, it is easy to see that under rotations, this semi-norm transforms to an equivalent one, and there exist positive constants  $m_{\mathbf{N}}, M_{\mathbf{N}}$  depending only on  $\mathbf{N}$  and such that

$$\begin{aligned} m_{\mathbf{N}} \|u(\mathbf{Z}(\cdot))\|_{H_{\text{hom}}^l(G)} &\leq \|u\|_{W^{\mathbf{N}/2,2}(\mathbf{Z}(G))} \leq M_{\mathbf{N}} \|u(\mathbf{Z}(\cdot))\|_{H_{\text{hom}}^l(G)}, \quad (5.1) \\ &\forall u \in H^l(G) \end{aligned}$$

for every affine transformation  $\mathbf{Z}$  of the form  $X = \mathbf{Z}(Y) = RUY + X_0$ , with  $X_0 \in \mathbb{R}^{\mathbf{N}}$ ,  $R > 0$ , and an orthogonal matrix  $U \in \mathbf{O}(\mathbf{N})$ .

Recall that  $\Psi(t) = (1+t) \ln(1+t) - t$  and  $\Phi(t) = e^t - 1 - t$  are complementary Orlicz functions.

**Lemma 5.1.** (cf. [13, Corollary 11.8/2]) *Let  $G \subset \mathbb{R}^N$  be a bounded set with Lipschitz boundary. If a positive Radon measure  $\mu$  on  $\overline{G}$  satisfies, for some  $\alpha > 0$ ,*

$$\mu(B(X, r)) \leq Kr^\alpha, \quad \forall X \in \overline{G} \quad \text{and} \quad \forall r \in \left(0, \frac{1}{2}\right), \quad (5.2)$$

*then the inequality*

$$\|w^2\|_{\Psi, \overline{G}, \mu} \leq A_1 \|w\|_{H^1(G)}^2, \quad \forall w \in H^1(G) \cap C(\overline{G})$$

*holds with a constant  $A_1 = A_1(G, \alpha, K)$ .*

*Proof.* The proof relies on [13, Theorem 11.8] and is almost identical to that of [9, Lemma 5.2].  $\square$

This statement implies the boundedness of the trace operator  $\Gamma : H^1(G) \rightarrow L^2(\Sigma)$ ,  $f \mapsto f_\Sigma$ , used at least twice here.

We will also need the following version of the Poincaré inequality that can be traced back to S.L. Sobolev (a very detailed proof can be found in [21, Theorem 2.4] for  $G$  being a cube). For every bounded set  $G \subset \mathbb{R}^N$  with Lipschitz boundary and every  $s > 0$ , there exists a constant  $C_s(G) > 0$  such that

$$\|u\|_{H^s(G)} \leq C_s(G) \|u\|_{H_{\text{hom}}^s(G)} \quad (5.3)$$

for all  $u \in H^s(G)$  orthogonal in  $L^2(G)$  to every polynomial of degree strictly less than  $s$ . We will denote the optimal constant in (5.3) with  $s = N/2$  by  $C(G) := C_{N/2}(G)$ .

Our eigenvalue estimates rely on the following inequality.

**Lemma 5.2.** *Let  $Q \subset \mathbb{R}^N$  be a cube and  $\mu = \mu_\Sigma$ .*

$$\int_Q |f(X)|^2 d\mu(X) \leq A_2 \|V\|_{\frac{av, \Psi}{Q, \mu}} \|w\|_{H_{\text{hom}}^l(Q)}^2 \quad (5.4)$$

*for all  $f \in H^l(Q)$  orthogonal in  $L^2(Q)$  to every polynomial of degree strictly less than  $l$ , with constant  $A_2$  depending only on  $N$  and  $\Sigma$ .*

*Proof.* If  $Q = Q_1$  is a unit cube, this is an immediate generalization of the proof of [9, Lemma 5.3]. For a cube of an arbitrary size, one should make a dilation (compression) in (5.4) of  $Q$  to  $Q_1$ . Under this transformation, the homogeneous norm in (5.4) is invariant. As for the norm  $\|V\|_{\frac{av, \Psi}{Q, \mu}}$ , it is not invariant under dilations, but is ‘almost’ invariant in the sense that the dilated (compressed) norm is majorized by the initial one, with constant depending on  $c_0, c_1$  in (2.4), independently of the dilation coefficient. Estimate (5.4) is proved in [9, Lemma 5.4], and the proof does not depend on dimension and uses nothing but the Ahlfors regularity of  $\mu$ , so it applies to our case without changes.  $\square$

## 5.2 Coverings and piecewise polynomial approximations

The construction to follow is an adaptation of the method developed by M.Z. Solomyak's in [19]. It stems from the approach initiated by M.Sh. Birman, M.Z. Solomyak, and G. Rozenblum in the late 60's–early 70's, see the nice exposition in [4]. Recently, it was used again for obtaining eigenvalue estimates for the weighted (poly-)harmonic operator in the critical case, see [9], [21]. We present the main idea and the structure of the proof first, and then fill in the required details.

**Theorem 5.3.** *Let  $\Sigma$  be a compact Lipschitz surface of dimension  $d$  in  $G \subset \mathbb{R}^{\mathbf{N}}$  and  $V \in L^{\Psi}(\Sigma)$ . Denote by  $\mathring{\mathbf{T}}(V, \Sigma)$  the operator defined by the sesquilinear form  $\int V(X)|f(X)|^2 d\mu_{\Sigma}$ ,  $f \in \mathring{H}^1(G)$ . Then*

$$n_{\pm}(\lambda, \mathring{\mathbf{T}}(V, \Sigma)) \leq C\lambda^{-1} \|V\|_{\Sigma}^{av, \Psi}, \quad (5.5)$$

with a constant  $C$  independent of  $V$ .

*Proof.* For  $d = \mathbf{N}$ , this result has been proved in [19] for even  $\mathbf{N}$  and in [21] for odd  $\mathbf{N}$ . So, let  $d < \mathbf{N}$ . It suffices to consider the case  $V \geq 0$ . We use one of the possible formulations of the variational principle for compact self-adjoint operators. If  $\mathbf{T} \geq 0$  is such an operator on a Hilbert space  $\mathcal{H}$ , with the quadratic form  $\mathbf{t}[f]$ , then for the eigenvalue counting function  $n(\lambda, \mathbf{T})$ ,

$$n(\lambda, \mathbf{T}) = \min \text{codim} \{ \mathcal{Y} \subset \mathcal{H} : \mathbf{t}[f] \leq \lambda \|f\|^2, f \in \mathcal{Y} \}. \quad (5.6)$$

Here the codimension  $\text{codim } \mathcal{Y}$  is the number of linearly independent functionals that have  $\mathcal{Y}$  as their common null space. Equality (5.6) hints at how to prove eigenvalue estimates. Let us, for some  $\lambda > 0$ , construct *some* subspace  $\mathcal{Y} = \mathcal{Y}(\lambda)$  on which the inequality in (5.6) holds. Then the quantity  $n(\lambda, \mathbf{T})$  is not greater than the codimension of the subspace we constructed,  $n(\lambda, \mathbf{T}) \leq \text{codim}(\mathcal{Y})$ . The more efficient we are in constructing  $\mathcal{Y}$ , the sharper the estimate is.

Let  $\mathcal{H}$  be the Sobolev space  $\mathring{H}^1(G)$ . In constructing the subspace  $\mathcal{Y}$ , we take some finite covering  $\Upsilon = \Upsilon(\lambda)$  of  $G$  by cubes. With each cube  $Q \in \Upsilon$ , we associate a set of functionals, the  $L^2(Q)$ -scalar products with polynomials of degree less than  $l$ . Thus, with each cube,  $\mathbf{d}(\mathbf{N}, l)$ , the dimension of the space of polynomials of interest, functionals are associated, altogether  $|\Upsilon|\mathbf{d}(\mathbf{N}, l)$  of them, so, the common null space  $\mathcal{Y} = \mathcal{Y}[\Upsilon]$  of these functionals has just this codimension (or less). Let  $\Lambda(\Upsilon)$  be defined as  $\sup_{f \in \mathcal{Y}[\Upsilon]} \mathbf{t}[f] / \|f\|^2$ . By the variational principal (5.6),

$$n(\Lambda(\Upsilon), \mathbf{T}) \leq |\Upsilon|\mathbf{d}(\mathbf{N}, l). \quad (5.7)$$

So, given  $\lambda > 0$ , we need to construct a covering  $\Upsilon = \Upsilon(\lambda)$  such that  $\Lambda(\Upsilon) \leq \lambda$ , and at the same time,  $|\Upsilon|$  should be under control, and its value will produce the required estimate via (5.7).



In our case, the quadratic form defining our operator is  $\mathbf{t}[f] = \int_G V(X)|f(X)|^2 d\mu$ , with  $V \geq 0$  and measure  $\mu = \mu_\Sigma$ . The same integral over a cube  $Q$  will be denoted by  $\mathbf{t}_Q[f]$ . If  $f \in \mathcal{Y}[\Upsilon]$  then on each cube  $Q$  of the covering  $\Upsilon$ , the restriction  $f_Q$  of  $f$  to  $Q$  is orthogonal to polynomials of degree less than  $l$ . Suppose that for *such* functions  $f_Q$  an estimate of the form

$$\mathbf{t}_Q[f] = \mathbf{t}[f_Q] \leq \mathbf{J}(Q) \|f\|_{H_{\text{hom}}^l(Q)}^2, \quad f \in \mathcal{Y}[\Upsilon] \quad (5.8)$$

holds, with some function of cubes  $\mathbf{J}(Q)$ . We need further the function  $\mathbf{J}$  to be *upper semiadditive*; this means that if  $Q_i$  is a family of disjoint cubes, all of them inside a cube  $Q^0$ , then

$$\sum \mathbf{J}(Q_i) \leq \mathbf{J}(Q^0). \quad (5.9)$$

We sum (5.8) over all cubes in the covering  $\Upsilon$  to obtain

$$\begin{aligned} \mathbf{t}[f] &\leq \sum_{Q \in \Upsilon} \mathbf{t}_Q[f] = \sum_{Q \in \Upsilon} \mathbf{t}[f_Q] \leq \sum_{Q \in \Upsilon} \mathbf{J}(Q) \|f\|_{H_{\text{hom}}^l(Q)}^2 \leq \\ &\leq \max_{Q \in \Upsilon} \mathbf{J}(Q) \sum_{Q \in \Upsilon} \|f\|_{H_{\text{hom}}^l(Q)}^2. \end{aligned} \quad (5.10)$$

Now, suppose that the covering  $\Upsilon$  has finite multiplicity, i.e. every point in  $G$  is covered by no more than  $\kappa = \kappa(\mathbf{N})$  different cubes in  $\Upsilon$ . Then the right-hand side in (5.10) can be majorized by  $\kappa \|f\|_{H_{\text{hom}}^l(G)}^2$ , which gives us

$$\mathbf{t}_Q[f] \leq \kappa \max_{Q \in \Upsilon} \mathbf{J}(Q) \|f\|_{H_{\text{hom}}^l(G)}^2. \quad (5.11)$$

We finally arrive at the estimate

$$\Lambda(\Upsilon) \leq \kappa \max_{Q \in \Upsilon} \mathbf{J}(Q). \quad (5.12)$$

So, we reach our aim, the estimate (5.5), as soon as we construct the covering  $\Upsilon$  such that

$$(1) \quad \kappa \max_{Q \in \Upsilon} \mathbf{J}(Q) < \lambda \quad \text{and} \quad (2) \quad |\Upsilon| \mathbf{d}(\mathbf{N}, l) \leq C \lambda^{-1} \|V\|_{\Sigma}^{av, \Psi}. \quad (5.13)$$

We set  $\mathbf{J}(Q) = A_2 \|V\|_{\Sigma \cap Q}^{av, \Psi}$  with  $A_2$  being the constant in (5.4). By Lemma 5.2, the required inequality (5.8) is satisfied for this particular choice of  $\mathbf{J}$ . Next, this function  $\mathbf{J}$  is upper semi-additive in the sense of (5.9). This property of the averaged Orlicz norm with respect to the Lebesgue measure was established in [19, Lemma 3] and then extended to AR-measures, see Lemma 2.8 in [9].

Now we construct the covering  $\Upsilon(\lambda)$ . First, by our Theorem 4.3, there exists a cube  $Q_0$  such that for every cube in  $\mathbb{R}^N$  with edges parallel to the ones of  $Q_0$  (we call such cubes parallel to  $Q_0$ ), its faces have zero  $\mu_\Sigma$ -measure. We fix such a cube  $Q_0$  and in the future *all* cubes under our consideration will be parallel to  $Q_0$ . Consider a ‘large’ cube

$\mathbf{Q}$  such that the cube concentric with  $\mathbf{Q}$  and with three times shorter edges still contains  $G$  inside. We replace in the eigenvalue problem the domain  $G$  by this larger cube  $\mathbf{Q}$ . By the usual variational principle, any eigenvalue estimate in  $\mathbf{Q}$  implies automatically the same estimate for the initial problem in  $G$ .

For any point  $X \in \Sigma$ , we consider the family of cubes  $Q_X(t)$ ,  $0 < t < \infty$ , of size  $t$  centred at  $X$ . By our choice of  $Q_0$ , the function  $\mu_\Sigma(Q_X(t))$  is continuous. Moreover, the function  $\rho_X(t) = \mathbf{J}(Q_X(t))$  is a continuous function of  $t$  and it tends to zero as  $t \rightarrow 0$ . The proof of this, rather elementary, fact is presented in [9] for the case  $\mathbf{N} = 2$ ,  $d = 1$ , and a very detailed proof for  $d = \mathbf{N}$  is included in [21]. Both proofs carry over to our case automatically, without any modifications, since they are dimension-independent and it is only the continuity of  $\mu_\Sigma(Q_X(t))$  that is used there.

The function  $\rho_X(t)$  stabilizes for large  $t$  to  $\rho_\infty$ , when  $Q_X(t) \supset \mathbf{Q}$ . With some constant  $\mathbf{k}$ , to be determined later, and  $\lambda < \mathbf{k}\rho_\infty$ , we find by continuity a value of  $t = t(X)$  such that  $\rho_X(t_X) = \lambda\mathbf{k}^{-1}$ . The set of all cubes  $\{Q_X(t(X)), X \in \Sigma\}$  forms a covering of  $\Sigma$ , and by the Besikovich covering lemma (see, e.g., [8, Ch. 1, Theorem 1.1]) one can find a finite sub-covering  $\Upsilon = \Upsilon(\lambda)$  of finite multiplicity  $\kappa$ ; moreover, this covering can be split into finitely many families,  $\Upsilon_{\mathbf{n}}$ ,  $\mathbf{n} \leq \mathbf{n}_0$ , with  $\mathbf{n}_0$  depending only on the dimension  $\mathbf{N}$ , so that the cubes in each family are disjoint. This will be the covering we are looking for.

In order to estimate the quantity of cubes in  $\Upsilon$ , we use the fact that the function  $\mathbf{J}$  is upper semi-additive. Therefore, for each of the families  $\Upsilon_{\mathbf{n}}$ ,

$$|\Upsilon_{\mathbf{n}}|\mathbf{k}^{-1}\lambda = \sum_{Q \in \Upsilon_{\mathbf{n}}} \mathbf{J}(Q) \leq \mathbf{J}(\mathbf{Q}) = A_2 \|V\|_{\Sigma}^{(av, \Psi)}, \quad \mathbf{n} = 1, \dots, \mathbf{n}_0,$$

and hence

$$|\Upsilon| = \sum |\Upsilon_{\mathbf{n}}| \leq \mathbf{n}_0 \lambda^{-1} \mathbf{k} A_2 \|V\|_{\Sigma}^{(av, \Psi)}. \quad (5.14)$$

Now we can choose the constant  $\mathbf{k}$  and arrive at the required estimate. Indeed, by (5.7), (5.12),

$$n(\kappa\lambda\mathbf{k}^{-1}, \mathbf{T}) \leq \lambda^{-1} A_2 \mathbf{n}_0 \mathbf{k} d(\mathbf{N}, l) \|V\|_{\Sigma}^{(av, \Psi)}. \quad (5.15)$$

Taking  $\mathbf{k} = \kappa$ , we obtain estimate (5.5).  $\square$

The approach, just presented, was called ‘piecewise polynomial approximation’ by its authors. It measures how fast a function in the Sobolev space can be approximated in the weighted  $L_2$  norm by functions that are polynomial on cubes in the covering  $\Upsilon$ .

### 5.3 Eigenvalue estimates

Now we can conclude the *Proof* of Theorem 2.1. Fix a bounded domain  $G = \Omega' \subset \Omega$  with smooth boundary. We consider first the case when  $\mathfrak{A} = \mathfrak{A}_0 = (-\Delta_{\mathcal{D}} + 1)^{-1/2}$ , where  $(-\Delta_{\mathcal{D}})$  is the Laplace operator in  $G$  with the Dirichlet boundary conditions. Having proved the eigenvalue estimate for this case, we will then use a simple argument to justify the required estimate for a general operator  $\mathfrak{A}$ .

Consider the quadratic form  $\mathbf{F}_V[u, \Sigma, \mathfrak{A}_0]$ ,  $u \in L_2(\Omega')$ . Denote  $f = \mathfrak{A}_0 u$ , so that  $u = (-\Delta + 1)^{1/2} f$ ,  $f \in H^l(\Omega)$ . The quadratic form  $\mathbf{F}_V[u, \mathfrak{A}_0]$  is thus transformed to  $\mathbf{F}_V^{(l)}[f] = \int_{\Sigma} V|f|^2 d\mu$ ,  $f \in \mathring{H}^l(G)$ . The operator  $\mathring{\mathbf{T}}(V, \Sigma)$  defined by the quadratic form  $\mathbf{F}_V^{(l)}$  on the Hilbert space  $\mathring{H}^l(\Omega)$  is exactly the operator considered in Theorem 5.3. Due to our substitution,  $u = (-\Delta + 1)^{1/2} f$ , the operator  $\mathring{\mathbf{T}}(V, \Sigma)$  on  $\mathring{H}^l(\Omega)$  is similar to the operator  $\mathbf{T}(V, \Sigma)$  on  $L^2(\Omega)$ , therefore they have the same spectrum, and the required estimate (2.3) for this case follows.

Now we pass to the general case, again for  $V \geq 0$ . As explained in Sections 2 and 3, we may assume that the pseudodifferential operator  $\mathfrak{A}$  contains cut-offs to  $\Omega'$ . We denote by  $\Gamma$  the operator of restriction of functions in  $H^l(\Omega')$  to  $\Sigma$ ; this is a bounded operator  $\Gamma : H^l \rightarrow L^2(\Sigma)$  by Lemma 5.1. We set  $|V| = W^2$ ,  $W \geq 0$ . Then the quadratic form  $\mathbf{F}_{|V|, \mathfrak{A}}[u]$  can be represented as follows

$$\mathbf{F}_{V, \mathfrak{A}}[u] = \langle W\Gamma\mathfrak{A}u, W\Gamma\mathfrak{A}u \rangle_{L_2(\Sigma, \mu)} = \langle (W\Gamma\mathfrak{A})^*(W\Gamma\mathfrak{A})u, u \rangle_{L^2(\Omega')}. \quad (5.16)$$

Therefore,  $\mathbf{T}(|V|, \Sigma, \mathfrak{A}) = \mathfrak{A}^*\Gamma^*|V|\Gamma\mathfrak{A}$ . Thus, compared with the operator  $\mathbf{T}(|V|, \Sigma, \mathfrak{A}_0)$ , we have

$$\mathbf{T}(|V|, \Sigma, \mathfrak{A}) = (\mathfrak{A}_0^{-1}\mathfrak{A})^*\mathbf{T}(|V|, \Sigma, \mathfrak{A}_0)(\mathfrak{A}_0^{-1}\mathfrak{A}). \quad (5.17)$$

Since  $\mathfrak{A}$  is a pseudodifferential operator of order  $-l$ ,  $\mathfrak{A}_0^{-1}\mathfrak{A}$  is a bounded operator on  $L_2(\Omega')$ , together with  $(\mathfrak{A}_0^{-1}\mathfrak{A})^*$ . Hence the eigenvalue estimate for  $\mathbf{T}(|V|, \Sigma, \mathfrak{A}_0)$ , already justified, is preserved after multiplication by bounded operators.

## 6 Approximation of the weight

To perform the last reduction, we use systematically the asymptotic perturbation lemma by M.Sh. Birman and M.Z. Solomyak (see, e.g., Lemma 1.5 in [3] or Lemma 6.1 in [15]). By this fundamental lemma, if for an operator  $\mathbf{T}$ , there exist a family of approximating operators  $\mathbf{T}_\varepsilon$  such that for their eigenvalues the asymptotic formula  $n_\pm(\lambda, \mathbf{T}_\varepsilon) \sim A_{\pm, \varepsilon}\lambda^{-q}$  is known and for the difference  $\mathbf{T}'_\varepsilon = \mathbf{T} - \mathbf{T}_\varepsilon$  the eigenvalue estimate  $\limsup \lambda^q \nu(\lambda, \mathbf{T}'_\varepsilon) \leq \varepsilon$  is proved, then the asymptotic formula  $n_\pm(\lambda, \mathbf{T}) \sim A_\pm \lambda^{-q}$  holds with coefficients  $A_\pm = \lim A_{\pm, \varepsilon}$ . In our case, the estimates will be provided by Theorem 2.1. So, let  $V \in L^\Psi(\Sigma)$ . We approximate  $V$  by sufficiently regular functions.

**Lemma 6.1.** *For any  $\varepsilon > 0$ , there exists  $V_\varepsilon \in C_0^\infty(\Omega)$  such that  $\|V - V_\varepsilon\|^{a\nu, \Psi}(\Sigma) < \varepsilon$ .*

*Proof.* For  $d = \mathbf{N}$  this is a well known statement about the density of smooth functions. So, let  $d < \mathbf{N}$ . As usual, we can assume that the surface  $\Sigma$  is covered by one local chart  $\Sigma : \mathbf{y} = \varphi(\mathbf{x})$ ,  $\mathbf{x} \in \mathcal{D} \subset \mathbb{R}^d$ . First, we truncate the function  $V$  at some level, i.e. we set

$$V'_\varepsilon(X) = V(X) \text{ if } |V(X)| \leq N, \quad V'_\varepsilon(X) = N \text{ sign } V(X) \text{ otherwise.}$$

The function  $V'_\varepsilon$  is bounded and, since  $V \in L^\Psi(\Sigma)$ , the cut-off level  $N$  can be chosen so that  $\|V - V'_\varepsilon\|_\Sigma^{av,\Psi} < \varepsilon/3$ . Next, we extend the function  $V'_\varepsilon$  defined on  $\Sigma$  to  $\mathcal{D} \times \mathbb{R}^d$  by setting  $V''_\varepsilon(\mathbf{x}, \mathbf{y}) = V'_\varepsilon(\mathbf{x}, \varphi(\mathbf{x}))$ ,  $\mathbf{x} \in \mathcal{D}$ . On  $\Sigma$ , we still have  $\|V - V''_\varepsilon\|_\Sigma^{av,\Psi} < \varepsilon/3$ . The resulting function  $V''_\varepsilon$  belongs to  $L^2(\mathcal{D} \times (-r, r))$  for any  $r > 0$ . Now consider the convolution of  $V''_\varepsilon$  with some mollifier  $\omega(\mathbf{x})$ . We obtain a smooth function  $V'''_\varepsilon$  depending, again, on  $\mathbf{x}$  only. The mollifier  $\omega$  can be chosen so that  $\|V''_\varepsilon - V'''_\varepsilon\|_\Sigma^{av,\Psi} \leq C\|V''_\varepsilon(\cdot, \mathbf{y}) - V'''_\varepsilon(\cdot, \mathbf{y})\|_{L^2(\mathcal{D})} < \varepsilon/3$ . Finally, multiplying  $V'''_\varepsilon$  by a smooth compactly supported cut-off function  $\chi$ , which equals 1 near  $\Sigma$ , we obtain the required approximation  $V_\varepsilon$ .  $\square$

We need approximating functions to satisfy an additional property (cf. Corollary 3.2).

**Lemma 6.2.** *The function  $V_\varepsilon$  can be approximated by a function  $\tilde{V}_\varepsilon \in C_0^\infty(\Omega)$  such that  $\|V_\varepsilon - \tilde{V}_\varepsilon\|_\Sigma^{av,\Psi} < \varepsilon$  and there exist open sets  $\Omega_\pm \subset \Omega$  such that  $\pm\tilde{V}_\varepsilon(X) \geq 0$  for  $X \in \Omega_\pm$ ,  $\tilde{V}_\varepsilon(X) = 0$  for  $X \in \Omega \setminus (\Omega_+ \cup \Omega_-)$ , and, finally,  $\text{dist}(\Omega_+, \Omega_-) > 0$ .*

*Proof.* Let the cylinder  $\mathcal{D} \times [-N, N]$  contain the support of  $V_\varepsilon$ . Consider the closed set  $\Omega_0 = \{X \in \overline{\mathcal{D}} \times [-N, N] : V_\varepsilon(X) = 0\}$ . Since  $V_\varepsilon$  is uniformly continuous on the compact set  $\overline{\mathcal{D}} \times [-N, N]$ , for any  $\delta_1 > 0$  there exists  $\delta_2 > 0$  such that  $|V_\varepsilon(X)| < \delta_1$  for  $X$  in the  $\delta_2$ -neighborhood of  $\Omega_0$ . Take a function  $\chi_\varepsilon \in C_0^\infty(\Omega)$  such that  $\chi_\varepsilon(X) \in [0, 1]$ ,  $\chi_\varepsilon(X) = 0$  in the  $\delta_2/2$  neighborhood of  $\Omega_0$  and  $\chi_\varepsilon(X) = 1$  outside the  $\delta_2$  neighborhood of  $\Omega_0$ . Set  $\tilde{V}_\varepsilon = \chi_\varepsilon V_\varepsilon$ . Then the sets  $\Omega_\pm = \{X : \pm\tilde{V}_\varepsilon(X) > 0\}$  satisfy the conditions of Lemma. Indeed, if  $|X_+ - X_-| < \delta_2/2$  and  $\pm\tilde{V}_\varepsilon(X_\pm) > 0$ , then, by continuity of  $V_\varepsilon$ , there must exist a point  $X_0 \in \Omega_0$  on the straight interval connecting  $X_+$  and  $X_-$ , such that at least one of the distances  $|X_\pm - X_0|$  is smaller than  $\delta_2/2$ , and this contradicts the construction of  $\Omega_\pm$ . Moreover,  $|V_\varepsilon - \tilde{V}_\varepsilon| \leq \delta_1$  everywhere in  $\Omega$ , and, by choosing  $\delta_1$  sufficiently small, we obtain the required approximation property.  $\square$

## 7 Asymptotic formula

We now present the proof of Theorem 2.4 for the case when there is just one surface  $\Sigma$ .

*Proof.* By the asymptotic perturbation lemma, together with our Lemmas 6.1, 6.2 and Theorem 2.1, it is sufficient to prove the eigenvalue asymptotics formula for the function  $\tilde{V}_\varepsilon$ ,  $\varepsilon > 0$  as above. We will denote it simply by  $V$  in what follows.

By Corollary 3.2, to study the asymptotics of positive (negative) eigenvalues of  $\mathbf{T}(V)$ , it is sufficient to consider the operator with a non-negative (non-positive) function  $V$ .

So, with a smooth  $V \geq 0$ , we perform a reduction to the integral operator. We have already made the first step. We set  $V = W^2$  with  $W \in C_0^\infty$ ,  $W \geq 0$ , and denote by  $\Gamma$  the operator of restriction of functions in  $H^1(\Omega)$  to the surface  $\Sigma$ ,  $\Gamma : H^1(\Omega) \rightarrow L^2(\Sigma, \mu_\Sigma)$ . The form (2.2) can now be written as

$\mathbf{F}_V[u] = \langle (W\Gamma\mathfrak{A})u, (W\Gamma\mathfrak{A})u \rangle_{L_2(\Sigma, \mu_\Sigma)} = \langle (W\Gamma\mathfrak{A})^*(W\Gamma\mathfrak{A})u, u \rangle_{L_2(\Omega)}$ , with  $(W\Gamma\mathfrak{A})$  considered as acting from  $L_2(\Omega)$  to  $L_2(\Sigma, \mu_\Sigma)$ .

It follows that  $\mathbf{T}(V) = (W\Gamma\mathfrak{A})^*(W\Gamma\mathfrak{A})$ . The nonzero eigenvalues of the operator  $\mathbf{T}$  in  $L_2(\Omega)$  coincide with the nonzero eigenvalues of the operator  $\mathbf{L} = (W\Gamma\mathfrak{A})(W\Gamma\mathfrak{A})^* = W[\Gamma(\mathfrak{A}\mathfrak{A}^*)\Gamma^*]W$  in  $L_2(\Sigma, \mu_\Sigma)$ . Here  $\mathfrak{A}\mathfrak{A}^*$  is, up to a smoothing term, a nonnegative pseudodifferential operator of order  $-2l = -\mathbf{N}$ , with the principal symbol  $|a_{-l}(X, \Xi)|^2$ , thus it is a weakly polar integral operator. The corresponding Schwartz kernel  $\mathcal{L}(x, y)$  has the leading singularity  $A(X) \log|x - y| + \vartheta(X, X - Y)$ , where  $\vartheta(X, X - Y)$  is a function positively homogeneous in  $X - Y$  of order zero and smooth in the first variable. Therefore, the operator  $\mathbf{L}$  is an integral operator on  $\Sigma$ ,

$$(\mathbf{L}v)(X) = \int_{\Sigma} W(X)\mathcal{L}(X, Y)W(Y)v(Y)d\mu_{\Sigma}(Y). \quad (7.1)$$

The function  $W$ , defined initially on  $\Sigma$ , is, in fact, the trace on  $\Sigma$  of a smooth function  $\tilde{W}(X) = (\tilde{V}(X))^{1/2}$  defined on  $\mathbb{R}^{\mathbf{N}}$ . Hence, the operator in (7.1) takes the form  $(\mathbf{L}v)(X) = \int_{\Sigma} \tilde{\mathcal{K}}(X, Y)v(Y)d\mu_{\Sigma}(Y)$ , where  $\tilde{\mathcal{K}}$  is the integral kernel of the pseudodifferential operator with the principal symbol  $\tilde{V}(X)|a_{-l}(X, \Xi)|^2$ . We are now in the setting of the paper [16] (see also [15]). By Theorem 6.4 there, the asymptotics of the eigenvalues of a weakly polar integral operator on a Lipschitz surface is given exactly by formula (2.7). We mention that the proof in those papers uses the asymptotic perturbation Lemma 1.5 in [3] in the analysis of operator convergence when a Lipschitz surface is approximated by smooth ones in a special way. Thus, this beautiful invention by M. Birman and M. Solomyak is used twice in our study, in quite different settings.  $\square$

Now we justify the eigenvalue asymptotic formula (2.17) for the case when there are several Lipschitz surfaces of possibly different dimensions, including, possibly, dimension  $d = \mathbf{N}$ , codimension  $\mathfrak{d} = 0$ , i.e. a domain in  $\Omega$  with an absolutely continuous measure. In order to avoid excessive complications in the proof, we impose a geometric restriction.

**Theorem 7.1.** *Let  $\Sigma_j$ ,  $j = 1, \dots, J$  be Lipschitz surfaces of dimension  $d_j$ ,  $1 \leq d_j \leq \mathbf{N}$  with measures  $P_j = V_j\mu_{\Sigma_j}$ ,  $V_j \in L_{\mu_{\Sigma_j}}^{\Psi}$ . We assume that the surfaces of the same dimension are disjoint. Then the eigenvalues of the operator  $\mathbf{T}(P, \mathfrak{A}) = \sum \mathbf{T}(P_j, \mathfrak{A})$  satisfy the asymptotic formula*

$$v_{\pm}(\lambda, \mathbf{T}(P, \mathfrak{A})) \sim \lambda^{-1} \sum_j C^{\pm}(V_j, \Sigma_j, \mathfrak{A}), \quad (7.2)$$

where  $C^{\pm}(V_j, \Sigma_j, \mathfrak{A})$  are given by (2.7) with  $V_j, \Sigma_j$  in place of  $V, \Sigma$ .

*Proof.* If the surfaces  $\Sigma_j$  are disjoint, so that all mutual distances are positive, the result follows by the inductive application of Lemma 3.1. Otherwise, we construct an approximation of the measure  $P = \sum P_j$  by measures with disjoint surfaces  $\Sigma_j$ .

Relabelling the surfaces if necessary, we can assume that their dimensions increase with  $j$ , i.e.  $d_1 \leq d_2 \leq \dots \leq d_J$ . We start with  $\Sigma_1$ . Consider the  $\delta$ -neighborhood  $U_1(\delta)$  of  $\Sigma_1$  in  $\mathbb{R}^N$ . If  $d_j > d_1$ , then the surface measure  $\mu_{\Sigma_j}(\Sigma_j \cap U_1(\delta))$ ,  $j > 1$ , decays at least as  $\delta^{d_j-d_1}$  for  $\delta \rightarrow 0$ . If  $d_j = d_1$ , then, according to our assumption,  $\Sigma_j \cap U_1(\delta) = \emptyset$  for all sufficiently small  $\delta > 0$ . Either way,  $\mu_{\Sigma_j}(\Sigma_j \cap U_1(\delta))$ ,  $j > 1$ , tends to zero as  $\delta \rightarrow 0$ . Therefore, for  $\delta$  small enough, the averaged  $\Psi$ -norm of each  $V_j$ ,  $j > 1$  over  $\Sigma_j \cap U_1(\delta)$  can be made arbitrarily small. We set  $V_j^{(1)}(X) = V_j(X)(1 - \chi(X))$ ,  $j > 1$ , where  $\chi$  is the characteristic function of  $U_1(\delta)$ . Then the averaged norm of  $V_j - V_j^{(1)}$  is small and  $\Sigma_1$  is separated from the support of  $V_j^{(1)}$ ,  $j > 1$ . Next we consider  $\Sigma_2^{(1)} = \Sigma_2 \cap \text{supp } V_2^{(1)}$ . This piece of the surface  $\Sigma_2$  is separated from  $\Sigma_1$ , but may cross  $\Sigma_j$ ,  $j > 2$ . We repeat with  $\Sigma_2^{(1)}$  the same procedure as above, considering its sufficiently small neighborhood in  $\mathbb{R}^N$  and then killing the remaining measures in this neighborhood. In this way, after a finite number of steps, we arrive at a system of separated measures, to which Lemma 3.1 can be applied, with further application of the already proved case of Theorem 2.4. In this construction, we introduce perturbations of  $V_j$  with small averaged  $\Psi$ -norms. By Theorem 2.1 and, again, Lemma 1.5 in [3], the asymptotic eigenvalue formula is thus justified in full.  $\square$

*Acknowledgments.* G.R. : The paper received a support from the RScF (Project 20-11-20032)

## References

- [1] M.S.Agranovich. *Potential type operators and transmission problems for strongly elliptic second-order systems in Lipschitz domains*. *Funct. Anal. and Appl.* **43**(2009) 165–183.
- [2] M.Sh. Birman; A. Laptev; M. Z. Solomyak. *The negative discrete spectrum of the operator  $(-\Delta)^l - \alpha V$  in  $\mathbb{R}^d$  for  $d$  even and  $2l \geq d$* . *Ark. Mat.* **35** (1997), 87–126.
- [3] M.Sh. Birman, M. Z. Solomyak. *Spectral asymptotics of nonsmooth elliptic operators*. I, *Trans. Moscow math. Soc.* **27** (1972); English: **27**, (1975) 1-52 .
- [4] M.Sh. Birman, M. Z. Solomyak. *Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory*. (Russian) Tenth Mathematical School (Russian), *Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1974*. pp. 5-189. English translation in: *American Mathematical Society Translations, Ser. 2* **114** (1980).
- [5] G. David and S. Semmes, *Fractured Fractals and Broken Dreams*. Clarendon Press, Oxford, 1997
- [6] J.Ercolano, M.Schechter. *Spectral theory for operators generated by elliptic boundary problems with eigenvalue parameter in boundary conditions*. I. *Comm. Pure Appl. Math.* **18** (1965), 83–105.
- [7] G.Grubb. *Spectral asymptotics for nonsmooth singular Green operators* *Comm. Part. Diff. Eq.* **39**, 3 (2014), 530-573
- [8] M. de Guzmán, *Differentiation of Integrals in  $\mathbb{R}^n$* . Springer, Berlin–Heidelberg–New York, 1975.

- [9] M. Karuhanga, E. Shargorodsky *On negative eigenvalues of two-dimensional Schrödinger operators with singular potentials.* J. Math. Phys. **61** 5 (2020), 051509(1–26)
- [10] A. Kozhevnikov. *The asymptotic behavior of the eigenvalues of an elliptic boundary value problem with  $\lambda$  in the equation and in the boundary condition.*(Russian) Uspehi Mat. Nauk **31** 4 (1976) 265–266.
- [11] A. Kozhevnikov, S.Yakubov. *On operators generated by elliptic boundary problems with a spectral parameter in boundary conditions.* Integral Equations and Operator Theory. **23** (1995) 205–231
- [12] S. Lord, F. Sukochev, D. Zanin. *A last theorem of Kalton and finiteness of Connes' integral.* J. Funct. Anal. **279** (2020) 108664(1-64).
- [13] V. Maz'ya. *Sobolev Spaces With Applications to Elliptic Partial Differential Equations.* Springer. Berlin, Heidelberg 2011
- [14] G.Rozenblum, E.Shargorodsky. *Eigenvalue asymptotics for weighted polyharmonic operator with a singular measure in the critical case* Submitted to Funct. Anal. Appl.
- [15] G.Rozenblum, G.Tashchiyan. *Eigenvalue asymptotics for potential type operators on Lipschitz surfaces.* Russian J. Math. Phys. **13** 3 (2006) 326–339
- [16] G.Rozenblum, G.Tashchiyan *Eigenvalue asymptotics for potential type operators on Lipschitz surfaces of codimension greater than 1.* Opuscula Math. **38** 5 (2018) 733–758.
- [17] E. Shargorodsky *On negative eigenvalues of two-dimensional Schrödinger operators.* Proc. Lond. Math. Soc. (3) **108** (2014), 2, 441–483.
- [18] E. Shargorodsky. *An estimate for the Morse index of a Stokes wave* Arch. Rat. Mech. Anal. **209** 1 (2013) 41–59.
- [19] M.Solomyak *Piecewise-polynomial approximation of functions from  $H^l((0, 1)^d)$ ,  $2l = d$ , and applications to the spectral theory of the Schrödinger operator,*Israel. J. Math. **86** (1994) 253–275.
- [20] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, New Jersey, 1970.
- [21] F.Sukochev, D. Zanin. *Cwikel-Solomyak estimates on tori and Euclidean spaces.* arXiv:2008.04494