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# Membership-Function-Dependent Control Design and Stability Analysis of Interval Type-2 Sampled-Data Fuzzy-Model-Based Control System

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***Abstract*—This paper investigates the design and stability analysis of the interval type-2 (IT2) sampled-data (SD) fuzzy-model-based (FMB) control system with the optimal guaranteed cost performance. An IT2 Takagi–Sugeno (T–S) fuzzy model is applied to describe the dynamics of the nonlinear systems where the parameter uncertainties are captured by the lower and upper membership functions. To conduct the stability analysis for the SD FMB control system, a looped-functional approach taking the advantage of the information about the sampling periods is employed. Because of the SD control strategy, the state will be sampled at each sampling instant and the control signal generated by the IT2SD fuzzy controller will be kept by the zero-order holder (ZOH) during the sampling period, which will result in mismatched membership grades between IT2 T-S fuzzy model and IT2SD fuzzy controller that leads to the complexity in carrying out stability analysis. Thanks to the imperfect premise matching (IPM) concept, which allows the difference on the number of rules and the premise membership functions between model and controller, the design of the IT2SD fuzzy controller can be more flexible. To further relax the stability conditions and minimize the upper bound of the guaranteed cost index, the membership-function-dependent (MFD) stability analysis approach which can make use of the features of the IT2 membership functions is adopted. The performance of the control system can also be adjusted through the choice of the weighting matrices in the cost function. The stability conditions building on the Lyapunov stability theory and the performance conditions building on the concept of the guaranteed cost control in the shape of linear matrix inequalities (LMIs) are established to assure the system stability and acquire the optimal guaranteed cost performance. The proposed IT2SDFMB control design is tested on the inverted pendulum system and the simulation results verify the effectiveness of the proposed approach.**

***Index Terms*—Sampled-data (SD) control, interval type-2 (IT2) fuzzy control, looped-functional approach, membership-function-dependent (MFD) stability analysis approach, linear matrix inequalities (LMIs).**

## I. Introduction

NONLINEARITIES are inevitable in various real-world systems, such as mass-spring-damping systems, teleoperation systems, active magnetic bearing systems [1]–[5]. To handle the control problem of nonlinear systems which are ill-defined, the fuzzy control approaches have been successfully applied during the past years [6]. Among fuzzy control approaches, the fuzzy-model-based (FMB) control approach attracts much attention [7]. In the FMB control approach, Takagi-Sugeno (T-S) fuzzy model is an important tool which supports the stability analysis and control of nonlinear systems due to the favourable modeling property [8] and [9]. The type-1 fuzzy model has been the mainstream model in the fuzzy control, but the lack of the ability to tackle uncertainties directly is a drawback [10] and [11]. Under the circumstance, more and more attention is paid to type-2 fuzzy model which can capture uncertainties directly by the type-2 fuzzy sets [12]. However, the general type-2 fuzzy sets will result in the complex design process and high computational expense. Therefore, the interval type-2 (IT2) fuzzy sets which are the generalization of type-1 fuzzy sets and interval-valued fuzzy sets [13] are applied. The IT2 fuzzy model not only retains the ability of capturing uncertainties but also decreases the computational expense compared with the general type-2 fuzzy sets [14]. The work in [15] was the first paper proposing the IT2 fuzzy model, stability analysis and control synthesis techniques in IT2 FMB control framework. Since then, it has drawn the attention from the fuzzy control community and led to many follow-up works regarding different techniques and control methodologies [7] and [16].

Thanks to the fast renovation of the computer science and digital circuit technologies, control strategies are able to be operated on low-cost digital computer or microcontrollers that turns the control system into a sampled-data (SD) control system. The control signal of the SD control system is held by the zero-order holder (ZOH) during the sampling period leading to discontinuities of the control signal which complicates the system dynamics and increases the difficulty in the analysis [17]. To implement the control synthesis and stability analysis for the SD control system, a variety of approaches have been put forward and applied, such as input delay approach [18], discretization approach [19], equivalent jump system [20] and looped-functional approach [21]. In recent years, increasing attention has been paid to looped-functional approach as it is able to take the information of the sampling period into the consideration [22]. Furthermore, the limitation on the positivity of the functional is alleviated by the looped-functional approach because the functional is only demanded to be positive definite at each sampling instant [23]–[26]. Due to its prominent properties, looped-functional

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approach has been broadly applied in the SD control of fuzzy systems. In [27], looped-functional approach is applied to the investigation on the stabilization problem of the dissipativity-based T-S fuzzy system with the memory SD control. [28] applies the looped-functional approach to the investigation on the stability problem of the T-S fuzzy system where there exists the time delay under the SD control. In [29], fuzzy control approach and looped-functional approach are employed for investigating the SD control problem of the semi-Markovin jump systems and the condition of actuator saturation is included. Nevertheless, the control gain of each rule in these researches are often same or very close, which indicates the effect of fuzzy control approach is little and control design results are conservative.

To alleviate the conservative system analysis and control design results, parallel distributed compensation (PDC) technique is often utilized in the traditional analysis [30] and [31]. However, only sampled state variables can be observed by the SD fuzzy controller, which will lead to mismatched membership grades between fuzzy model and fuzzy controller, resulting in the inapplicability of the PDC technique that demands the number of rules and premise membership functions of the fuzzy model and fuzzy controller to be the same [7]. Fortunately, the problem of mismatched membership functions in SD FMB control system has been first handled in [32] and then the later proposed imperfect premising matching (IPM) concept [33] and membership-function-dependent (MFD) analysis techniques [7], [17] and [34] offer further idea to deal with this class of FMB control systems.

In addition to system stability, it is essential to take the control performance into account as well [35]. In [36], guaranteed cost control (GCC) was first proposed for the problem. The proposed GCC can not only achieve the system stabilization but also offer an upper bound of the given performance index. GCC approach was employed in [37] for time-varying delayed nonlinear systems. In [38], GCC approach was applied to discrete-time nonlinear systems with polytopic uncertainties. In [39], MFD approach and GCC approach were considered in the stability analysis of FMB control systems. [40] proposed a robust guaranteed cost SD control method for time-delay T-S fuzzy systems with uncertainties. It is worth noting that GCC approach used in some works has to know the initial states in advance, but this paper will optimize the guaranteed cost performance without relying on initial states [41]–[43].

On the basis of aforementioned contents, this paper investigates problems on system stability and guaranteed cost performance of an interval type-2 sampled-data fuzzy-model-based (IT2SDFMB) control system. The minimal upper bound of the guaranteed cost index will be obtained independent of the initial states. The IT2 T-S fuzzy model is utilized to represent nonlinear systems subject to parameter uncertainties. A looped-functional approach which is able to make use of the information related to sampling periods is applied to the stability analysis. Although the PDC technique is not applicable due to the mismatch issue, the design flexibility can be improved with the introduction of IPM concept which permits to freely choose the number of rules and the premise membership functions [33]. In addition, the MFD stability analysis approach which includes the information of membership functions into the stability analysis will be employed to further achieve less conservative results and lower the upper bound of the guaranteed cost index. However, the membership functions of the SD fuzzy controller and the fuzzy model might not correspond one by one in the SD control system, which is often not perceived and might affect the results of the stability analysis. In [17], the relation between the sampled states and continuous states is presented to circumvent the problem. Also, the control performance can be adjusted by choosing the weighting matrices in the cost function. Stability conditions based on the Lyapunov stability theory and performance conditions based on the concept of GCC in the shape of LMIs are derived for the assurance on system stability and the optimal guaranteed cost performance.

The main contributions of the paper are shown as below:

1) An IT2SD fuzzy controller is proposed to solve the stabilization problem of the nonlinear systems subject to parameter uncertainties based on FMB control design.
2) The guaranteed cost performance is taken into account to optimize the control performance. The minimal upper bound of the guaranteed cost index is acquired without relying on the initial states.
3) A looped-functional approach which can make use of the information about the sampling periods is employed to conduct the stability analysis for the SD control system. The concept of IPM which allows the free selection of the number of rules and the premise membership functions is introduced for a flexible design of the IT2SD fuzzy controller. With the utilization of the MFD stability analysis approach, further relaxed results and lower upper bound of the guaranteed cost index can be acquired.

The rest of the paper is organized as follows. IT2 T-S fuzzy model, IT2SD fuzzy controller, cost function and the lemma are shown in Section II. Section III presents the stability analysis of the IT2SDFMB control system with the consideration of the guaranteed cost performance. Section IV verifies the performance of the IT2SDFMB control system by simulations. Section V draws a conclusion.

*Notations:* Superscript "$-1$" represents the inverse of a matrix and superscript "$T$" represents the matrix transposition, "$Sym\{\mathbf{X}\}$" stands for $\mathbf{X}+\mathbf{X}^T$, "$*$" indicates symmetric elements of a symmetric matrix, "$diag\{\cdots\}$" indicates the diagonal matrix of which terms in the bracket are diagonal, "$\mathbf{I}_n$" stands for the $n\times n$ identity matrix and "$\mathbf{0}_{m\times n}$" stands for the $m\times n$ zero matrix, "$|x|$" denotes the absolute value of $x$.

## II. Preliminaries

The IT2 T-S fuzzy model, IT2SD fuzzy controller, cost function and lemmas which will be used are shown as below.

### A. IT2 T-S Fuzzy Model

An IT2 T-S fuzzy model [10] consisting of $p$ rules is applied to represent the nonlinear system. The $i_{th}$ rule of the IT2 T-S fuzzy model where the antecedents include IT2 fuzzy sets and the consequent is a linear dynamical system is as below:

Rule $i$: IF $f_1(\mathbf{x}(t))$ is $\tilde{M}_1^i$ AND … AND $f_\Psi(\mathbf{x}(t))$ is $\tilde{M}_\Psi^i$,
THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)$

where $\tilde{M}_\alpha^i$ means the IT2 fuzzy set of the $i_{th}$ rule corresponding to the $f_\alpha(\mathbf{x}(t))$, $i = 1, 2, 3, \ldots, p$ and $\alpha = 1, 2, 3, \ldots, \Psi$, $i$ and $\alpha$ are always positive; $\mathbf{x}(t) = [x_1(t)\ x_2(t)\ x_3(t)\ \cdots\ x_n(t)]^T$ denotes the state vector; $\mathbf{u}(t) = [u_1(t)\ u_2(t)\ u_3(t)\ \cdots\ u_m(t)]^T$ denotes the input vector; $\mathbf{A}_i$ is the known system matrix and $\mathbf{B}_i$ is the known input matrix. The firing strength of the $i_{th}$ rule is in the following interval sets:

$$\tilde{w}_i(\mathbf{x}(t)) \in [\underline{w}_i(\mathbf{x}(t)),\ \overline{w}_i(\mathbf{x}(t))],\ i = 1,\ 2,\ \ldots,\ p \qquad (1)$$

where $\underline{w}_i(\mathbf{x}(t)) = \prod_{\alpha=1}^{\Psi} \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\overline{w}_i(\mathbf{x}(t)) = \prod_{\alpha=1}^{\Psi} \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ indicates the lower grade of membership and $\overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ indicates the upper grade of membership that satisfy $0 \leq \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \leq \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \leq 1$, and then $0 \leq \underline{w}_i(\mathbf{x}(t)) \leq \overline{w}_i(\mathbf{x}(t)) \leq 1$. Furthermore, $\tilde{w}_i(\mathbf{x}(t))$ is defined as below [10] and [44]:

$$\tilde{w}_i(\mathbf{x}(t)) = \underline{\lambda}_i(\mathbf{x}(t))\underline{w}_i(\mathbf{x}(t)) + \overline{\lambda}_i(\mathbf{x}(t))\overline{w}_i(\mathbf{x}(t)) \geq 0,\ \text{for } \forall i \qquad (2)$$

where $0 \leq \underline{\lambda}_i(\mathbf{x}(t)) \leq 1$, $0 \leq \overline{\lambda}_i(\mathbf{x}(t)) \leq 1$, $\underline{\lambda}_i(\mathbf{x}(t)) + \overline{\lambda}_i(\mathbf{x}(t)) = 1$, for $\forall i$, $\underline{\lambda}_i(\mathbf{x}(t))$ and $\overline{\lambda}_i(\mathbf{x}(t))$ are nonlinear functions which exist but not necessarily to be known.

The IT2 T-S fuzzy model is shown as below:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(t))\Big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\Big), \qquad (3)$$

where $\sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(t)) = 1$.

### B. IT2SD Fuzzy Controller

An IT2SD fuzzy controller [10] having $c$ rules is applied to handle the nonlinear system. The $j_{th}$ rule of the IT2SD fuzzy controller is described as below:

Rule $j$: IF $g_1(\mathbf{x}(t_k))$ is $\tilde{N}_1^j$ AND … AND $g_\Omega(\mathbf{x}(t_k))$ is $\tilde{N}_\Omega^j$,
THEN $\mathbf{u}(t) = \mathbf{G}_j\mathbf{x}(t_k)$

where $\tilde{N}_\beta^j$ means the IT2 fuzzy set of the $j_{th}$ rule corresponding to the $g_\beta(\mathbf{x}(t_k))$, $j = 1, 2, 3, \ldots, c$, $k = 0, 1, 2, \ldots, \infty$ and $\beta = 1, 2, 3, \ldots, \Omega$, $c$ and $\Omega$ are always positive. $\mathbf{G}_j$ denotes the constant feedback gains to be acquired. The firing strength of the $j_{th}$ rule is in the following interval sets:

$$\tilde{m}_j(\mathbf{x}(t_k)) \in [\underline{m}_j(\mathbf{x}(t_k)),\ \overline{m}_j(\mathbf{x}(t_k))],\ j = 1,\ 2,\ \ldots,\ c \qquad (4)$$

where $\underline{m}_j(\mathbf{x}(t_k)) = \prod_{\beta=1}^{\Omega} \underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t_k)))$, $\overline{m}_j(\mathbf{x}(t_k)) = \prod_{\beta=1}^{\Omega} \overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t_k)))$, $\underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t_k)))$ indicates the lower grade of membership and $\overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t_k)))$ indicates the upper grade of membership that satisfy $0 \leq \underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t_k))) \leq \overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t_k))) \leq 1$, and then $0 \leq \underline{m}_j(\mathbf{x}(t_k)) \leq \overline{m}_j(\mathbf{x}(t_k)) \leq 1$. Furthermore, $\tilde{m}_j(\mathbf{x}(t_k))$ is defined as below:

$$\tilde{m}_j(\mathbf{x}(t_k)) = \frac{\underline{\kappa}_j(\mathbf{x}(t_k))\underline{m}_j(\mathbf{x}(t_k)) + \overline{\kappa}_j(\mathbf{x}(t_k))\overline{m}_j(\mathbf{x}(t_k))}{\sum_{l=1}^{c}(\underline{\kappa}_l(\mathbf{x}(t_k))\underline{m}_l(\mathbf{x}(t_k)) + \overline{\kappa}_l(\mathbf{x}(t_k))\overline{m}_l(\mathbf{x}(t_k)))} \geq 0,\ \text{for } \forall j \qquad (5)$$

where $0 \leq \underline{\kappa}_j(\mathbf{x}(t_k)) \leq 1$, $0 \leq \overline{\kappa}_j(\mathbf{x}(t_k)) \leq 1$, $\underline{\kappa}_j(\mathbf{x}(t_k)) + \overline{\kappa}_j(\mathbf{x}(t_k)) = 1$, for $\forall j$, $\underline{\kappa}_j(\mathbf{x}(t_k))$ and $\overline{\kappa}_j(\mathbf{x}(t_k))$ are predefined functions.

The inferred IT2SD fuzzy controller is shown as below:

$$\mathbf{u}(t) = \sum_{j=1}^{c} \tilde{m}_j(\mathbf{x}(t_k))\mathbf{G}_j\mathbf{x}(t_k),\ t_k \leq t < t_{k+1} \qquad (6)$$

where $\sum_{j=1}^{c} \tilde{m}_j(\mathbf{x}(t_k)) = 1$, and $h_s = t_{k+1} - t_k$ stands for the sampling period.

### C. Cost Function

The following index is introduced to evaluate the control performance in a quantitative way [32] and [45]:

$$J = \int_{t_0}^{\infty} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt, \qquad (7)$$

where $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are weighting matrices. $\mathbf{J}_1 = \mathbf{J}_1^T > 0$, $\mathbf{J}_3 = \mathbf{J}_3^T > 0$ and $\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} > 0$, which are predefined.

*Lemma 1:* [46] Let $\mathbf{x}$: $[\alpha,\ \beta]$ be a differentiable signal. For the vector $\varsigma$, matrices $\mathbf{Z}$=$\mathbf{Z}^T > 0$, $\mathbf{N}_1$ and $\mathbf{N}_2$, the inequality holds as below:

$$-\int_{\alpha}^{\beta} \dot{\mathbf{x}}^T(s)\mathbf{Z}\dot{\mathbf{x}}(s)ds \leq (\beta-\alpha)\varsigma^T\Big(\mathbf{N}_1\mathbf{Z}^{-1}\mathbf{N}_1^T + \frac{(\beta-\alpha)^2}{3}\mathbf{N}_2\mathbf{Z}^{-1}\mathbf{N}_2^T\Big)\varsigma + 2\varsigma^T\Big(\mathbf{N}_1\Big(\mathbf{x}(\beta) - \mathbf{x}(\alpha)\Big) - 2\mathbf{N}_2\int_{\alpha}^{\beta}\mathbf{x}(s)ds\Big) + 2(\beta-\alpha)\varsigma^T\mathbf{N}_2\Big(\mathbf{x}(\beta) + \mathbf{x}(\alpha)\Big). \qquad (8)$$

*Schur Complement Lemma:* [8] and [47] For the matrices $\mathbf{Q}$=$\mathbf{Q}^T$, $\mathbf{R}$=$\mathbf{R}^T > 0$ and $\mathbf{S}$, the inequality $\mathbf{Q} - \mathbf{S}^T\mathbf{R}^{-1}\mathbf{S} > 0$ is equivalent to $\begin{bmatrix} \mathbf{Q} & \mathbf{S}^T \\ * & \mathbf{R} \end{bmatrix} > 0$.

## III. Stability Analysis

From (3) and (6), the closed-loop system can be acquired:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\Big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{G}_j\mathbf{x}(t_k)\Big). \qquad (9)$$

In this subsection, stability conditions and performance conditions in the shape of LMIs are derived to assure the stability. The following notations are used to simplify representations:

$$\varsigma(t) = \begin{bmatrix} \mathbf{x}^T(t) & \dot{\mathbf{x}}^T(t) & \mathbf{x}^T(t_k) & \int_{t_k}^{t}\mathbf{x}^T(s)ds \end{bmatrix}^T,$$
$$\mathbf{e}_i = \begin{bmatrix} \mathbf{0}_{n\times(i-1)n} & \mathbf{I}_n & \mathbf{0}_{n\times(4-i)n} \end{bmatrix},\ i = 1,\ 2,\ 3,\ 4.$$

*Theorem 1:* Suppose $\mathbf{G}_j$ in (6) has been known beforehand. For a given constant sampling period $h_s$ and scalars $\varkappa_r$ ($r = 1, \ldots, 3$), if there exist matrices $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{R} = \mathbf{R}^T$, $\mathbf{S}_1 = \mathbf{S}_1^T$,

$\mathbf{S}_2$, $\mathbf{Z}=\mathbf{Z}^T>0$, $\mathbf{N}_{w,ij}$, $\tilde{\mathbf{X}}$, $\underline{\mathbf{T}}_{ijl}=\underline{\mathbf{T}}_{ijl}^T>0$, $\overline{\mathbf{T}}_{ijl}=\overline{\mathbf{T}}_{ijl}^T>0$, ($w=1,\ 2;\ i=1,\ \ldots,\ p;\ j=1,\ \ldots,\ c,\ l=1,\ \ldots,\ L$), such that satisfying LMIs shown as below:

$$\begin{bmatrix} \Phi_{1,ijl}+h_s\Phi_{2,ij} & \Lambda_1 \\ * & \Lambda_2 \end{bmatrix}<0, \tag{10}$$

$$\begin{bmatrix} \Phi_{1,ijl}+h_s\Phi_{3,ij} & h_s\mathbb{N}_{ij} & \Lambda_1 \\ * & -h_s\mathbb{Z} & \mathbf{0}_{2n\times 3n} \\ * & * & \Lambda_2 \end{bmatrix}<0, \tag{11}$$

where $\Phi_{1,ijl}=\Xi_1+\Xi_{3,1}+\Xi_{4,4,ij}+\Xi_{6,ij}+\underline{\mathbf{T}}_{ijl}-\overline{\mathbf{T}}_{ijl}-\sum_{r=1}^{p}\sum_{s=1}^{c}\underline{\nu}_{rsl}\underline{\mathbf{T}}_{rsl}+\sum_{r=1}^{p}\sum_{s=1}^{c}\overline{\nu}_{rsl}\overline{\mathbf{T}}_{rsl}$, $\Phi_{2,ij}=\Xi_{2,1}+\Xi_{3,2}+\Xi_{4,1}$, $\Phi_{3,ij}=\Xi_{2,2}+\Xi_{4,3,ij}$, $\Xi_1=Sym\left\{\mathbf{e}_2^T\mathbf{P}\mathbf{e}_1\right\}$, $\Xi_{2,1}=\mathbf{e}_3^T\mathbf{R}\mathbf{e}_3$, $\Xi_{2,2}=-\mathbf{e}_3^T\mathbf{R}\mathbf{e}_3$, $\Xi_{3,1}=-\begin{bmatrix}\mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T\end{bmatrix}\mathbf{S}_1\begin{bmatrix}\mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T\end{bmatrix}^T-Sym\left\{\begin{bmatrix}\mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T\end{bmatrix}\mathbf{S}_2\mathbf{e}_3\right\}$, $\Xi_{3,2}=Sym\left\{\begin{bmatrix}\mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T\end{bmatrix}\mathbf{S}_1\begin{bmatrix}\mathbf{e}_2^T & \mathbf{e}_1^T\end{bmatrix}^T+\begin{bmatrix}\mathbf{e}_2^T & \mathbf{e}_1^T\end{bmatrix}\mathbf{S}_2\mathbf{e}_3\right\}$, $\Xi_{4,1}=\mathbf{e}_2^T\mathbf{Z}\mathbf{e}_2$, $\Xi_{4,3,ij}=Sym\left\{\mathbf{N}_{2,ij}(\mathbf{e}_1+\mathbf{e}_3)\right\}$, $\Xi_{4,4,ij}=Sym\left\{\mathbf{N}_{1,ij}(\mathbf{e}_1-\mathbf{e}_3)-2\mathbf{N}_{2,ij}\mathbf{e}_4\right\}$, $\Xi_{6,ij}=Sym\left\{\mathbf{U}(\mathbf{A}_i\mathbf{e}_1+\mathbf{B}_i\mathbf{G}_j\mathbf{e}_3-\mathbf{e}_2)\right\}$, $\mathbf{U}=\varkappa_1\mathbf{e}_1^T\tilde{\mathbf{X}}+\varkappa_2\mathbf{e}_2^T\tilde{\mathbf{X}}+\varkappa_3\mathbf{e}_3^T\tilde{\mathbf{X}}$, $\mathbb{N}_{ij}=\begin{bmatrix}\mathbf{N}_{1,ij} & h_s\mathbf{N}_{2,ij}\end{bmatrix}$, $\mathbb{Z}=diag\{\mathbf{Z},\ 3\mathbf{Z}\}$, $\Lambda_1=\begin{bmatrix}\mathbf{e}_1^T\mathbf{J}_1+\mathbf{e}_3^T\mathbf{G}_j^T\mathbf{J}_2^T & \mathbf{e}_1^T\mathbf{J}_2+\mathbf{e}_3^T\mathbf{G}_j^T\mathbf{J}_3\end{bmatrix}$, $\Lambda_2=-\begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ * & \mathbf{J}_3\end{bmatrix}$, the closed-loop system (9) is asymptotically stable. In addition, (7) satisfies the inequality shown as below:

$$J<\mathbf{x}^T(t_0)\mathbf{P}\mathbf{x}(t_0), \tag{12}$$

*Proof:* A quadratic Lyapunov function and looped-functionals are constructed as below:

$$V(t)=\sum_{i=1}^{4}V_i(t),\ t\in[t_k,t_{k+1}), \tag{13}$$

$$V_1(t)=\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t), \tag{14}$$

$$V_2(t)=(t-t_k)(t_{k+1}-t)\mathbf{x}^T(t_k)\mathbf{R}\mathbf{x}(t_k), \tag{15}$$

$$\begin{aligned} V_3(t)=(t_{k+1}-t)&\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(t_k)\\ \int_{t_k}^{t}\mathbf{x}(s)ds\end{bmatrix}^T \\ &\times\left(\mathbf{S}_1\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(t_k)\\ \int_{t_k}^{t}\mathbf{x}(s)ds\end{bmatrix}+2\mathbf{S}_2\mathbf{x}(t_k)\right),\end{aligned} \tag{16}$$

$$V_4(t)=(t_{k+1}-t)\int_{t_k}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Z}\dot{\mathbf{x}}(s)ds. \tag{17}$$

$\dot{V}_i(t)$ $(i=1,\ 2,\ 3,\ 4)$ can be obtained as below:

$$\begin{aligned}\dot{V}_1(t)&=\dot{\mathbf{x}}^T(t)\mathbf{P}\mathbf{x}(t)+\mathbf{x}^T(t)\mathbf{P}\dot{\mathbf{x}}(t)\\ &=Sym\left\{\dot{\mathbf{x}}^T(t)\mathbf{P}\mathbf{x}(t)\right\}\\ &=\varsigma^T(t)\Xi_1\varsigma(t).\end{aligned} \tag{18}$$

$$\begin{aligned}\dot{V}_2(t)&=(t_{k+1}-t)\mathbf{x}^T(t_k)\mathbf{R}\mathbf{x}(t_k)-(t-t_k)\mathbf{x}^T(t_k)\mathbf{R}\mathbf{x}(t_k)\\ &=\varsigma^T(t)\Big((t_{k+1}-t)\Xi_{2,1}+(t-t_k)\Xi_{2,2}\Big)\varsigma(t),\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{V}_3(t)=&-\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(t_k)\\ \int_{t_k}^{t}\mathbf{x}(s)ds\end{bmatrix}^T\left(\mathbf{S}_1\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(t_k)\\ \int_{t_k}^{t}\mathbf{x}(s)ds\end{bmatrix}+2\mathbf{S}_2\mathbf{x}(t_k)\right)\\ &+2(t_{k+1}-t)\left(\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(t_k)\\ \int_{t_k}^{t}\mathbf{x}(s)ds\end{bmatrix}^T\mathbf{S}_1\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{x}(t)\end{bmatrix}\right.\\ &\left.+\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{x}(t)\end{bmatrix}^T\mathbf{S}_2\mathbf{x}(t_k)\right)\\ &=\varsigma^T(t)\Big(\Xi_{3,1}+(t_{k+1}-t)\Xi_{3,2}\Big)\varsigma(t).\end{aligned} \tag{20}$$

From Lemma 1, the following can be acquired:

$$\begin{aligned}&\dot{V}_4(t)=(t_{k+1}-t)\dot{\mathbf{x}}^T(t)\mathbf{Z}\dot{\mathbf{x}}(t)-\int_{t_k}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Z}\dot{\mathbf{x}}(s)ds\\ &\leq\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{v=1}^{p}\sum_{h=1}^{c}\tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\tilde{w}_v(\mathbf{x}(t))\tilde{m}_h(\mathbf{x}(t_k))\varsigma^T(t)\Big(\\ &(t_{k+1}-t)\Xi_{4,1}+(t-t_k)\left(\Xi_{4,2,ijvh}+\Xi_{4,3,ij}\right)+\Xi_{4,4,ij}\Big)\varsigma(t),\end{aligned} \tag{21}$$

where $\Xi_{4,2,ijvh}=\mathbf{N}_{1,ij}\mathbf{Z}^{-1}\mathbf{N}_{1,vh}^T+\frac{h_s^2}{3}\mathbf{N}_{2,ij}\mathbf{Z}^{-1}\mathbf{N}_{2,vh}^T$.

The following can be obtained based on the cost function:

$$\begin{aligned}&\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}^T\begin{bmatrix}\mathbf{J}_1 & \mathbf{J}_2\\ * & \mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}\\ &=\mathbf{x}^T(t)\mathbf{J}_1\mathbf{x}(t)+\sum_{j=1}^{c}\tilde{m}_j(\mathbf{x}(t_k))\mathbf{x}^T(t_k)\mathbf{G}_j^T\mathbf{J}_2^T\mathbf{x}(t)\\ &+\sum_{j=1}^{c}\tilde{m}_j(\mathbf{x}(t_k))\mathbf{x}^T(t)\mathbf{J}_2\mathbf{G}_j\mathbf{x}(t_k)\\ &+\sum_{j=1}^{c}\tilde{m}_j(\mathbf{x}(t_k))\mathbf{x}^T(t_k)\mathbf{G}_j^T\mathbf{J}_3\sum_{q=1}^{c}\tilde{m}_q(\mathbf{x}(t_k))\mathbf{G}_q\mathbf{x}(t_k)\\ &=\sum_{j=1}^{c}\sum_{q=1}^{c}\tilde{m}_j(\mathbf{x}(t_k))\tilde{m}_q(\mathbf{x}(t_k))\varsigma^T(t)\Xi_{5,jq}\varsigma(t),\end{aligned} \tag{22}$$

where $\Xi_{5,jq}=\mathbf{e}_1^T\mathbf{J}_1\mathbf{e}_1+Sym\{\mathbf{e}_1^T\mathbf{J}_2\mathbf{G}_j\mathbf{e}_3\}+\mathbf{e}_3^T\mathbf{G}_j^T\mathbf{J}_3\mathbf{G}_q\mathbf{e}_3$.

Based on (9), the equality shown as below holds

$$\begin{aligned}\Gamma&=2\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\varsigma^T(t)\mathbf{U}(\mathbf{A}_i\mathbf{x}(t)+\mathbf{B}_i\mathbf{G}_j\mathbf{x}(t_k)-\dot{\mathbf{x}}(t))\\ &=\sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\varsigma^T(t)\Xi_{6,ij}\varsigma(t)=0,\end{aligned} \tag{23}$$

where $\mathbf{U}=\varkappa_1\mathbf{e}_1^T\tilde{\mathbf{X}}+\varkappa_2\mathbf{e}_2^T\tilde{\mathbf{X}}+\varkappa_3\mathbf{e}_3^T\tilde{\mathbf{X}}$.

In the paper, the MFD stability analysis approach is applied to further relax stability conditions and reduce the upper bound of the guaranteed cost index. Assume the modeling domain $\Upsilon$ is divided into $L$ connected sub-domains $\Upsilon_l$, i.e., $\Upsilon=\bigcup_{l=1}^{L}\Upsilon_l$. Define $\overline{\nu}_{ijl}=\overline{w}_{il}\overline{m}_{jl}$ and $\underline{\nu}_{ijl}=\underline{w}_{il}\underline{m}_{jl}$ satisfying $0\leq\underline{\nu}_{ijl}\leq\tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\leq\overline{\nu}_{ijl}\leq 1$, where $\underline{w}_{il}$, $\overline{w}_{il}$ denote the lower, upper bounds for $\tilde{w}_i(\mathbf{x}(t))$ in the $l$th sub-domain and $\underline{m}_{jl}$, $\overline{m}_{jl}$ denote the lower, upper bounds for $\tilde{m}_j(\mathbf{x}(t_k))$ in the $l$th sub-domain. Through the usage of the $\overline{\nu}_{ijl}$, $\underline{\nu}_{ijl}$ and slack matrices $\overline{\mathbf{T}}_{ijl} \geq 0$, $\underline{\mathbf{T}}_{ijl} \geq 0$, the inequalities shown as below hold

$$\sum_{l=1}^{L}\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\overline{\nu}_{ijl} - \tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\Big)\overline{\mathbf{T}}_{ijl} \geq 0, \tag{24}$$

$$\sum_{l=1}^{L}\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k)) - \underline{\nu}_{ijl}\Big)\underline{\mathbf{T}}_{ijl} \geq 0. \tag{25}$$

Using (18)-(25) for $t \in [t_k,\ t_{k+1})$, it follows that

$$\begin{aligned}
&\dot{V}(t) + \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\
&= \dot{V}(t) + \Gamma + \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \\
&\leq \sum_{l=1}^{L}\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{v=1}^{p}\sum_{h=1}^{c}\sum_{q=1}^{c} \xi_l \tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\tilde{w}_v(\mathbf{x}(t))\tilde{m}_h(\mathbf{x}(t_k)) \\
&\tilde{m}_q(\mathbf{x}(t_k))\varsigma^T(t)\Big(\Xi_1 + \Xi_{3,1} + \Xi_{4,4,ij} + \Xi_{5,jq} + \Xi_{6,ij} \\
&+ (t_{k+1} - t)\left(\Xi_{2,1} + \Xi_{3,2} + \Xi_{4,1}\right) \\
&+ (t - t_k)\left(\Xi_{2,2} + \Xi_{4,2,ijvh} + \Xi_{4,3,ij}\right)\Big)\varsigma(t) \\
&\leq \sum_{l=1}^{L}\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{v=1}^{p}\sum_{h=1}^{c}\sum_{q=1}^{c} \xi_l \tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\tilde{w}_v(\mathbf{x}(t))\tilde{m}_h(\mathbf{x}(t_k)) \\
&\tilde{m}_q(\mathbf{x}(t_k))\varsigma^T(t)\Big(\Phi_{1,ijl} + \Xi_{5,jq} + (t_{k+1} - t)\Phi_{2,ij} \\
&+ (t - t_k)\left(\Phi_{3,ij} + \Xi_{4,2,ijvh}\right)\Big)\varsigma(t) \\
&= \sum_{l=1}^{L}\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{v=1}^{p}\sum_{h=1}^{c}\sum_{q=1}^{c} \xi_l \tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t_k))\tilde{w}_v(\mathbf{x}(t))\tilde{m}_h(\mathbf{x}(t_k)) \\
&\tilde{m}_q(\mathbf{x}(t_k))\varsigma^T(t)\bigg(\frac{(t_{k+1} - t)}{h_s}\left(\Phi_{1,ijl} + \Xi_{5,jq} + h_s\Phi_{2,ij}\right) \\
&+ \frac{(t - t_k)}{h_s}\Big(\Phi_{1,ijl} + \Xi_{5,jq} + h_s(\Phi_{3,ij} + \Xi_{4,2,ijvh})\Big)\bigg)\varsigma(t),
\end{aligned} \tag{26}$$

where $\xi_l = 1$ when system is located at the sub-domain $\Upsilon_l$; otherwise $\xi_l = 0$; $\sum_{l=1}^{L}\xi_l = 1$.

Based on the convex combination technique [41], [42] and [48], the inequality (26) will be less than 0, if and only if

$$\Phi_{1,ijl} + \Xi_{5,jq} + h_s\Phi_{2,ij} < 0, \tag{27}$$

$$\Phi_{1,ijl} + \Xi_{5,jq} + h_s\left(\Phi_{3,ij} + \Xi_{4,2,ijvh}\right) < 0. \tag{28}$$

Through the application of the Schur complement, to require (27) to be valid is equivalent to require (10) to be valid, and (28) is equivalent to (11). In conclusion, if (26)$<0$ is satisfied, then the closed-loop system (9) will be asymptotically stable. In addition, the following can be obtained:

$$\begin{aligned}
&\int_{t_0}^{t_{k_l}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt < -\int_{t_0}^{t_{k_l}} \dot{V}(t)dt \\
&\int_{t_0}^{t_{k_l}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \\
&< -\int_{t_{k_0}}^{t_{k_l}} \dot{V}(t)dt - \sum_{i=0}^{k_0-1}\int_{t_i}^{t_{i+1}} \dot{V}(t)dt \\
&\int_{t_0}^{t_{k_l}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \\
&< -V(t_{k_l}) + V(t_{k_0}) - V(t_{k_0}) + V(t_{k_0-1}) - \cdots - V(t_1) + V(t_0) \\
&\int_{t_0}^{t_{k_l}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt < -V(t_{k_l}) + V(t_0).
\end{aligned} \tag{29}$$

When $t_{k_l} \to \infty$, (12) can be obtained. The proof is completed.

*Remark 1:* $\tilde{w}_i(\mathbf{x}(t))$ and $\tilde{m}_j(\mathbf{x}(t_k))$ are dependent on $\mathbf{x}(t)$ and $\mathbf{x}(t_k)$ respectively. The definition of regional boundaries will be a problem. If $\tilde{m}_j(\mathbf{x}(t_k))$ can be estimated by $\tilde{m}_j(\mathbf{x}(t))$, only $\mathbf{x}(t)$ will be considered [17]. Then the rational regional boundaries could be determined, which ensures the $\tilde{m}_j(\mathbf{x}(t_k))$ corresponding to $\tilde{w}_i(\mathbf{x}(t))$ is inside the defined regional boundary. The relationship of $\mathbf{x}(t)$ and $\mathbf{x}(t_k)$ during the sampling period is investigated to realize the estimation, which is shown as below:

$$\mathbf{x}(t) - \mathbf{x}(t_k) = \int_{t_k}^{t} \dot{\mathbf{x}}(s)ds. \tag{30}$$

Definie $|\dot{\mathbf{x}}| \leq \dot{\mathbf{x}}_{max}$ in the operating domain, then the following equation can be obtained through (30):

$$|\mathbf{x}(t) - \mathbf{x}(t_k)| \leq (t - t_k)\dot{\mathbf{x}}_{max} \leq h_s\dot{\mathbf{x}}_{max}. \tag{31}$$

From (30) and (31), we can obtain

$$\mathbf{x}(t_k) \in [\mathbf{x}(t) - h_s\dot{\mathbf{x}}_{max},\ \mathbf{x}(t) + h_s\dot{\mathbf{x}}_{max}] \tag{32}$$

which suggests that given any $\mathbf{x}(t)$, and suppose that $|\dot{\mathbf{x}}| \leq \dot{\mathbf{x}}_{max}$ is satisfied, then the range of $\mathbf{x}(t_k)$ is $[\mathbf{x}(t) - h_s\dot{\mathbf{x}}_{max},\ \mathbf{x}(t) + h_s\dot{\mathbf{x}}_{max}]$.

On the basis of Theorem 1, the following Theorem 2 is derived for the design of the IT2SD fuzzy controller.

*Theorem 2:* For a given constant sampling period $h_s$ and scalars $\varkappa_r$ $(r = 1,\ \ldots,\ 3)$, if there exist matrices $\mathbf{Y}_j$, $\mathbf{X}$, $\acute{\mathbf{P}} = \acute{\mathbf{P}}^T > 0$, $\acute{\mathbf{R}} = \acute{\mathbf{R}}^T$, $\acute{\mathbf{S}}_1 = \acute{\mathbf{S}}_1^T$, $\acute{\mathbf{S}}_2$, $\acute{\mathbf{Z}} = \acute{\mathbf{Z}}^T > 0$, $\acute{\mathbf{N}}_{w,ij}$, $\underline{\acute{\mathbf{T}}}_{ijl} = \underline{\acute{\mathbf{T}}}_{ijl}^T > 0$, $\acute{\overline{\mathbf{T}}}_{ijl} = \acute{\overline{\mathbf{T}}}_{ijl}^T > 0$, $(w = 1,\ 2;\ i = 1,\ \ldots,\ p;\ j = 1,\ \ldots,\ c,\ l = 1,\ \ldots,\ L)$, such that satisfying LMIs shown as below:

$$\begin{bmatrix} \acute{\Phi}_{1,ijl} + h_s\acute{\Phi}_{2,ij} & \acute{\Lambda}_1 \\ * & \Lambda_2 \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} \acute{\Phi}_{1,ijl} + h_s\acute{\Phi}_{3,ij} & h_s\acute{\mathbb{N}}_{ij} & \acute{\Lambda}_1 \\ * & -h_s\acute{\mathbb{Z}} & \mathbf{0}_{2n\times 3n} \\ * & * & \Lambda_2 \end{bmatrix} < 0, \tag{34}$$

where $\acute{\Phi}_{1,ijl} = \acute{\Xi}_1 + \acute{\Xi}_{3,1} + \acute{\Xi}_{4,4,ij} + \acute{\Xi}_{6,ij} + \underline{\acute{\mathbf{T}}}_{ijl} - \acute{\overline{\mathbf{T}}}_{ijl} - \sum_{r=1}^{p}\sum_{s=1}^{c}\underline{\nu}_{rsl}\underline{\acute{\mathbf{T}}}_{rsl} + \sum_{r=1}^{p}\sum_{s=1}^{c}\overline{\nu}_{rsl}\acute{\overline{\mathbf{T}}}_{rsl}$,

$\acute{\Phi}_{2,ij} = \acute{\Xi}_{2,1} + \acute{\Xi}_{3,2} + \acute{\Xi}_{4,1}$, $\acute{\Phi}_{3,ij} = \acute{\Xi}_{2,2} + \acute{\Xi}_{4,3,ij}$, $\acute{\Xi}_1 = Sym\{\mathbf{e}_2^T\acute{\mathbf{P}}\mathbf{e}_1\}$, $\acute{\Xi}_{2,1} = \mathbf{e}_3^T\acute{\mathbf{R}}\mathbf{e}_3$, $\acute{\Xi}_{2,2} = -\mathbf{e}_3^T\acute{\mathbf{R}}\mathbf{e}_3$, $\acute{\Xi}_{3,1} = -\begin{bmatrix} \mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T \end{bmatrix}\acute{\mathbf{S}}_1\begin{bmatrix} \mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T \end{bmatrix}^T - Sym\Big\{\begin{bmatrix} \mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T \end{bmatrix}\acute{\mathbf{S}}_2\mathbf{e}_3\Big\}$, $\acute{\Xi}_{3,2} = Sym\Big\{\begin{bmatrix} \mathbf{e}_1^T-\mathbf{e}_3^T & \mathbf{e}_4^T \end{bmatrix}\acute{\mathbf{S}}_1\begin{bmatrix} \mathbf{e}_2^T & \mathbf{e}_1^T \end{bmatrix}^T + \begin{bmatrix} \mathbf{e}_2^T & \mathbf{e}_1^T \end{bmatrix}\acute{\mathbf{S}}_2\mathbf{e}_3\Big\}$, $\acute{\Xi}_{4,1} = \mathbf{e}_2^T\acute{\mathbf{Z}}\mathbf{e}_2$, $\acute{\Xi}_{4,3,ij} = Sym\Big\{\acute{\mathbf{N}}_{2,ij}(\mathbf{e}_1 + \mathbf{e}_3)\Big\}$, $\acute{\Xi}_{4,4,ij} = Sym\Big\{\acute{\mathbf{N}}_{1,ij}(\mathbf{e}_1-\mathbf{e}_3) - 2\acute{\mathbf{N}}_{2,ij}\mathbf{e}_4\Big\}$, $\acute{\Xi}_{6,ij} = Sym\Big\{\acute{\mathbf{U}}(\mathbf{A}_i\mathbb{X}_1\mathbf{e}_1+\mathbf{B}_i\mathbf{Y}_j\mathbf{e}_3-\mathbb{X}_1\mathbf{e}_2)\Big\}$, $\acute{\mathbf{U}} = \varkappa_1\mathbf{e}_1^T+\varkappa_2\mathbf{e}_2^T+\varkappa_3\mathbf{e}_3^T$, $\acute{\mathbb{N}}_{ij} = \begin{bmatrix} \acute{\mathbf{N}}_{1,ij} & h_s\acute{\mathbf{N}}_{2,ij} \end{bmatrix}$, $\acute{\mathbb{Z}} = diag\{\acute{\mathbf{Z}},\ 3\acute{\mathbf{Z}}\}$, $\acute{\Lambda}_1 = \begin{bmatrix} \mathbf{e}_1^T\mathbb{X}_1^T\mathbf{J}_1+\mathbf{e}_3^T\mathbf{Y}_j^T\mathbf{J}_2^T & \mathbf{e}_1^T\mathbb{X}_1^T\mathbf{J}_2+\mathbf{e}_3^T\mathbf{Y}_j^T\mathbf{J}_3 \end{bmatrix}$, $\Lambda_2 = -\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ * & \mathbf{J}_3 \end{bmatrix}$, the closed-loop system (9) is asymptotically stable. In addition, (7) satisfies the inequality shown as below:

$$J < \mathbf{x}^T(t_0)\mathbb{X}_1^{-T}\acute{\mathbf{P}}\mathbb{X}_1^{-1}\mathbf{x}(t_0). \tag{35}$$

The gain matrices of the IT2SD fuzzy controller are retrieved by:

$$\mathbf{G}_j = \mathbf{Y}_j\mathbf{X}^{-1}. \tag{36}$$

*Proof:* The matrix shown as below is introduced:

$$\mathbb{X}_n = diag\{\mathbf{X},\ \ldots,\ \mathbf{X}\},$$

where $n$ denotes the number of $\mathbf{X}$ in $\mathbb{X}_n$.

Then, define $\acute{\mathbf{P}} = \mathbb{X}_1^T\mathbf{P}\mathbb{X}_1$, $\acute{\mathbf{R}} = \mathbb{X}_1^T\mathbf{R}\mathbb{X}_1$, $\acute{\mathbf{S}}_1 = \mathbb{X}_2^T\mathbf{S}_1\mathbb{X}_2$, $\acute{\mathbf{S}}_2 = \mathbb{X}_2^T\mathbf{S}_2\mathbb{X}_1$, $\acute{\mathbf{Z}} = \mathbb{X}_1^T\mathbf{Z}\mathbb{X}_1$, $\acute{\mathbf{N}}_{w,ij} = \mathbb{X}_4^T\mathbf{N}_{w,ij}\mathbb{X}_1$, $\tilde{\mathbf{X}} = \mathbf{X}^{-T}$ and $\acute{\mathbf{U}} = \varkappa_1\mathbf{e}_1^T + \varkappa_2\mathbf{e}_2^T + \varkappa_3\mathbf{e}_3^T$, $\acute{\underline{\mathbf{T}}}_{ijl} = \mathbb{X}_4^T\underline{\mathbf{T}}_{ijl}\mathbb{X}_4$, $\acute{\overline{\mathbf{T}}}_{ijl} = \mathbb{X}_4^T\overline{\mathbf{T}}_{ijl}\mathbb{X}_4$, $(w = 1,\ 2;\ i = 1,\ \ldots,\ p;\ j = 1,\ \ldots,\ c,\ l = 1,\ \ldots,\ L)$, $\Delta_1 = diag\{\mathbb{X}_4,\ \mathbf{I}_{3n}\}$, $\Delta_2 = diag\{\mathbb{X}_6,\ \mathbf{I}_{3n}\}$. Pre- and post-multiplying (10) by $\Delta_1^T$ and $\Delta_1$, and pre- and post-multiplying (11) by $\Delta_2^T$ and $\Delta_2$, then (33) and (34) are obtained.

The minimization process of the upper bound of the guaranteed cost index is shown as follows:

$$J < \mathbf{x}^T(t_0)\mathbb{X}_1^{-T}\acute{\mathbf{P}}\mathbb{X}_1^{-1}\mathbf{x}(t_0) < \Upsilon. \tag{37}$$

In this paper, there is no need knowing $\mathbf{x}(t_0)$ but all the vertex points $\mathbf{x}_z(t_0)$ of a polyhedron containing $\mathbf{x}(t_0)$ are known. Then $\mathbf{x}(t_0)$ can be described as below:

$$\mathbf{x}(t_0) = \sum_{z=1}^{n_z} m_z\mathbf{x}_z(t_0), \tag{38}$$

where $m_z \geq 0$, and $\sum_{z=1}^{n_z} m_z = 1$. Then the following equation can be obtained:

$$\sum_{z=1}^{n_z} m_z\mathbf{x}_z^T(t_0)\mathbb{X}_1^{-T}\acute{\mathbf{P}}\mathbb{X}_1^{-1}\sum_{l=1}^{n_l} m_l\mathbf{x}_l(t_0) < \Upsilon. \tag{39}$$

Through the application of the Schur complement, the inequality shown as below can be obtained:

$$\begin{bmatrix} -\Upsilon & \mathbf{x}_z^T(t_0) \\ * & -\mathbb{X}_1\acute{\mathbf{P}}^{-1}\mathbb{X}_1^T \end{bmatrix} < 0. \tag{40}$$

To deal with the term $-\mathbb{X}_1\acute{\mathbf{P}}^{-1}\mathbb{X}_1^T$, the following inequality is introduced [42]:

$$-\mathbb{X}_1\acute{\mathbf{P}}^{-1}\mathbb{X}_1^T \leq -\mathbb{X}_1 - \mathbb{X}_1^T + \acute{\mathbf{P}}. \tag{41}$$

From (40) and (41), we can obtain

$$\begin{bmatrix} -\Upsilon & \mathbf{x}_z^T(t_0) \\ * & -\mathbb{X}_1 - \mathbb{X}_1^T + \acute{\mathbf{P}} \end{bmatrix} < 0. \tag{42}$$

In conclusion, the minimum upper bound of the guaranteed cost index can be obtained by solving the generalized eigenvalue minimization problem (GEVP) shown as below:

$$\text{minimize } \Upsilon \tag{43}$$

subject to: (33), (34) and (42).

## IV. Simulation Example

In this section, the proposed IT2SD fuzzy controller is employed to stabilize the inverted pendulum system for showing the advantages.

The dynamic model of the inverted pendulum system is described as below [17] and [49]:

$$\dot{x}_1(t) = x_2(t), \tag{44}$$

$$\dot{x}_2(t) = \frac{\begin{pmatrix} g\sin(x_1(t)) - am_pl(x_2(t))^2\sin(2x_1(t))/2 \\ -a\cos(x_1(t))u(t) \end{pmatrix}}{4l/3 - am_pl(\cos(x_1(t)))^2}, \tag{45}$$

where $x_1(t)$ stands for the angle (rad), $x_2(t)$ stands for the angular velocity (rad/s), $g$ stands for the gravity acceleration which is 9.8 m/s$^2$, $m_p \in [m_{p_{min}},\ m_{p_{max}}] = [2,\ 3]$kg stands for the mass of the pendulum, $M_c \in [M_{c_{min}},\ M_{c_{max}}] = [8,\ 12]$kg stands for the mass of the cart, $a = \frac{1}{m_p+M_c}$, $l$ stands for the half length of the pendulum which is 0.5 m, $u(t)$ stands for the force which is applied to the cart (N).

A 4-rule IT2 T–S fuzzy model in the shape of (3) is applied to the description of the inverted pendulum system. Define $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ f_{1_{min}} & 0 \end{bmatrix}$, $\mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ f_{1_{max}} & 0 \end{bmatrix}$, $\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ f_{2_{min}} \end{bmatrix}$, $\mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ f_{2_{max}} \end{bmatrix}$, where $f_{1_{min}} = 10.0078$, $f_{1_{max}} = 18.4800$, $f_{2_{min}} = -0.1765$, and $f_{2_{max}} = -0.0261$. The operation domains of $x_1(t)$ and $x_2(t)$ considered in the control process are $[-\frac{5\pi}{12},\ \frac{5\pi}{12}]$ and $[-5,\ 5]$, respectively. Table I shows the lower and upper membership functions for the IT2 T-S fuzzy model of the inverted pendulum system where $f_1(\mathbf{x}(t) = \frac{g-am_pl(x_2(t))^2\cos(x_1(t))}{4l/3-am_pl(\cos(x_1(t)))^2}\left(\frac{\sin(x_1(t))}{x_1(t)}\right)$ and $f_2(\mathbf{x}(t) = -\frac{a\cos(x_1(t))}{4l/3-am_pl(\cos(x_1(t)))^2}$. A 2-rule IT2SD fuzzy controller is applied to fulfil the control objectives on the inverted pendulum system, and the membership functions of the IT2SD fuzzy controller are defined as below: $\overline{m}_1(x_1(t_k)) = \{0$ for $x_1(t_k) < -\frac{5\pi}{12}$; $\frac{x_1(t_k)+5\pi/12}{5\pi/12}$ for $-\frac{5\pi}{12} \leq x_1(t_k) \leq 0$; $\frac{-x_1(t_k)+5\pi/12}{5\pi/12}$ for $0 \leq x_1(t_k) \leq \frac{5\pi}{12}$; 0 for $x_1(t_k) > \frac{5\pi}{12}\}$, $\underline{m}_1(x_1(t_k)) = \{0$ for $x_1(t_k) < -\frac{5\pi}{12}$; $\frac{0.9(x_1(t_k)+5\pi/12)}{5\pi/12}$ for $-\frac{5\pi}{12} \leq x_1(t_k) \leq 0$; $\frac{0.9(-x_1(t_k)+5\pi/12)}{5\pi/12}$ for $0 \leq x_1(t_k) \leq \frac{5\pi}{12}$; 0 for $x_1(t_k) >$

TABLE I
LOWER AND UPPER MEMBERSHIP FUNCTIONS FOR THE IT2 T-S FUZZY MODEL OF THE INVERTED PENDULUM SYSTEM.

| | |
|---|---|
| $\underline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t))) = \frac{-f_1(\mathbf{x}(t))+f_{1_{max}}}{f_{1_{max}}-f_{1_{min}}}$;<br>$\overline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}(t))) = \frac{f_1(\mathbf{x}(t))-f_{1_{min}}}{f_{1_{max}}-f_{1_{min}}}$;<br>with $x_2(t)=0, m_p = m_{p_{max}}$ and $M_c = M_{c_{min}}$ | $\underline{\mu}_{\tilde{M}_2^1}(f_2(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_2^3}(f_2(\mathbf{x}(t))) = \frac{-f_2(\mathbf{x}(t))+f_{2_{max}}}{f_{2_{max}}-f_{2_{min}}}$;<br>$\overline{\mu}_{\tilde{M}_2^2}(f_2(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_2^4}(f_2(\mathbf{x}(t))) = \frac{f_2(\mathbf{x}(t))-f_{2_{min}}}{f_{2_{max}}-f_{2_{min}}}$;<br>with $m_p = m_{p_{max}}$ and $M_c = M_{c_{max}}$ |
| $\overline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t))) = \frac{-f_1(\mathbf{x}(t))+f_{1_{max}}}{f_{1_{max}}-f_{1_{min}}}$;<br>$\underline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}(t))) = \frac{f_1(\mathbf{x}(t))-f_{1_{min}}}{f_{1_{max}}-f_{1_{min}}}$;<br>with $x_2(t)=x_{2_{max}}, m_p = m_{p_{max}}$ and $M_c = M_{c_{min}}$ | $\overline{\mu}_{\tilde{M}_2^1}(f_2(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_2^3}(f_2(\mathbf{x}(t))) = \frac{-f_2(\mathbf{x}(t))+f_{2_{max}}}{f_{2_{max}}-f_{2_{min}}}$;<br>$\underline{\mu}_{\tilde{M}_2^2}(f_2(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_2^4}(f_2(\mathbf{x}(t))) = \frac{f_2(\mathbf{x}(t))-f_{2_{min}}}{f_{2_{max}}-f_{2_{min}}}$;<br>with $m_p = m_{p_{min}}$ and $M_c = M_{c_{min}}$ |

TABLE II
WEIGHTING MATRICES $\mathbf{J}_1$, $\mathbf{J}_2$ AND $\mathbf{J}_3$ FOR THE 3 CASES.

| Case | $\mathbf{J}_1$ | $\mathbf{J}_2$ | $\mathbf{J}_3$ |
|---|---|---|---|
| 1 | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | 0.01 |
| 2 | $\begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | 0.01 |
| 3 | $\begin{bmatrix} 2000 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | 0.01 |

TABLE III
RESULTS FOR CASE 1.

| Information of Membership Functions | Minimal $\Upsilon$ | Control Gains |
|---|---|---|
| No Information | 18117.2 | $\mathbf{G}_1 = \begin{bmatrix} 1133.5 & 277.1 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1133.5 & 277.1 \end{bmatrix}$ |
| Global Boundary Information | 16470.9 | $\mathbf{G}_1 = \begin{bmatrix} 1069.7 & 270.1 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1069.7 & 270.1 \end{bmatrix}$ |
| Regional Boundary Information with $L=3$ | 15546.9 | $\mathbf{G}_1 = \begin{bmatrix} 1025.8 & 266.4 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1026.7 & 266.6 \end{bmatrix}$ |
| Regional Boundary Information with $L=5$ | 13360.3 | $\mathbf{G}_1 = \begin{bmatrix} 977.4 & 265.2 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 984.1 & 266.4 \end{bmatrix}$ |
| Regional Boundary Information with $L=7$ | 15629.5 | $\mathbf{G}_1 = \begin{bmatrix} 887.0 & 247.3 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 912.4 & 254.3 \end{bmatrix}$ |
| Regional Boundary Information with $L=9$ | 15251.7 | $\mathbf{G}_1 = \begin{bmatrix} 851.9 & 240.1 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 891.3 & 251.5 \end{bmatrix}$ |

$\frac{5\pi}{12}\}$, $\overline{m}_2(x_1(t_k)) = 1-\underline{m}_1(x_1(t_k))$, $\underline{m}_2(x_1(t_k)) = 1-\overline{m}_1(x_1(t_k))$, and $\underline{\kappa}_j(\mathbf{x}(t_k)) = \overline{\kappa}_j(\mathbf{x}(t_k)) = 0.5$, for $j=1,\ 2$.

The sampling period of the control system is set as $0.02s$. The parameters $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ are chosen as 0.8, 0.1 and 1.0, respectively. The membership functions are dependent on $x_1(t)$. $\dot{x}_1(t) = x_2(t) \in [-5,\ 5]$, then $|x_1(t)-x_1(t_k)| \leq 0.1$ can be found based on Remark 1, which will also be verified by simulations. Through the above information, the range of $x_1(t_k)$ is obtained as $[x_1(t)-0.1,\ x_1(t)+0.1]$.

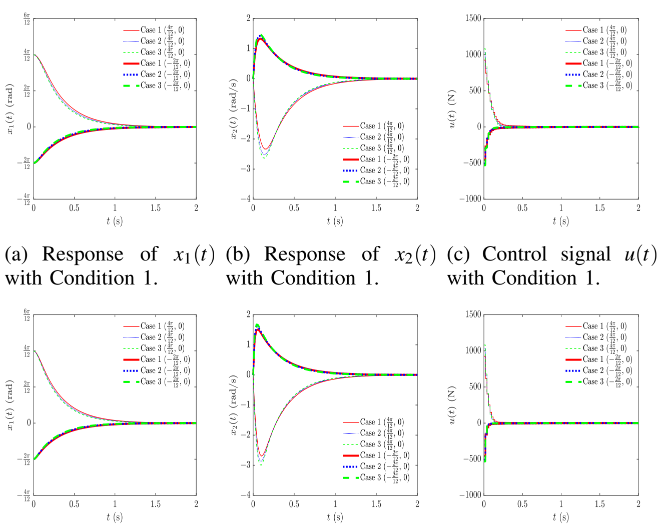
To demonstrate the effect on the guaranteed cost performance imposed by the MFD stability analysis approach, different information of membership functions is used in the controller design. In addition, the three cases shown in Table II are also considered to further illustrate how the guaranteed cost performance is affected by the MFD stability analysis approach with the different weighting matrices in the cost function. For Case 1, it can be seen from Table III that the biggest value of the minimal $\Upsilon$ which is 18117.2 occurs when no information is used; when the regional boundary information with $L=5$ is considered, the smallest value which is 13360.3 appears. As seen from Table IV and Table V, the values of the minimal $\Upsilon$ for Case 2 and Case 3 both keep declining from no information to regional boundary information with $L=9$. The results verify that the MFD stability analysis technique has a significant influence on further improving the control performance. When the appropriate information of membership functions is used, the upper bound of the guaranteed cost index can be reduced. In addition, when the regional boundary information with $L=7$ or $L=9$ is used, the minimal $\Upsilon$ for Case 1 is bigger than the minimal $\Upsilon$ for Case 2 and the minimal $\Upsilon$ for Case 3, while the minimal $\Upsilon$ for Case 2 is the smallest. Except the above two situations, with the same information of membership functions, the value of the

TABLE IV
RESULTS FOR CASE 2.

| Information of Membership Functions | Minimal $\Upsilon$ | Control Gains |
|---|---|---|
| No Information | 19317.1 | $\mathbf{G}_1 = \begin{bmatrix} 1166.8 & 278.4 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1166.8 & 278.4 \end{bmatrix}$ |
| Global Boundary Information | 17824.9 | $\mathbf{G}_1 = \begin{bmatrix} 1106.0 & 271.7 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1106.0 & 271.7 \end{bmatrix}$ |
| Regional Boundary Information with $L=3$ | 16674.8 | $\mathbf{G}_1 = \begin{bmatrix} 1071.7 & 269.4 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1072.4 & 269.5 \end{bmatrix}$ |
| Regional Boundary Information with $L=5$ | 14624.1 | $\mathbf{G}_1 = \begin{bmatrix} 1025.9 & 268.0 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1032.4 & 269.2 \end{bmatrix}$ |
| Regional Boundary Information with $L=7$ | 13704.8 | $\mathbf{G}_1 = \begin{bmatrix} 977.3 & 261.5 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 1002.7 & 266.9 \end{bmatrix}$ |
| Regional Boundary Information with $L=9$ | 12979.0 | $\mathbf{G}_1 = \begin{bmatrix} 946.5 & 257.2 \end{bmatrix}$<br>$\mathbf{G}_2 = \begin{bmatrix} 994.4 & 268.0 \end{bmatrix}$ |

TABLE V
RESULTS FOR CASE 3.

| Information of Membership Functions | Minimal $\Upsilon$ | Control Gains |
|---|---|---|
| No Information | 20402.1 | $\mathbf{G}_1 = \begin{bmatrix} 1198.3 & 279.9 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1198.3 & 279.9 \end{bmatrix}$ |
| Global Boundary Information | 18863.9 | $\mathbf{G}_1 = \begin{bmatrix} 1141.6 & 273.5 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1141.6 & 273.5 \end{bmatrix}$ |
| Regional Boundary Information with $L = 3$ | 18048.9 | $\mathbf{G}_1 = \begin{bmatrix} 1104.0 & 270.5 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1105.3 & 270.8 \end{bmatrix}$ |
| Regional Boundary Information with $L = 5$ | 15994.9 | $\mathbf{G}_1 = \begin{bmatrix} 1060.6 & 268.8 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1069.2 & 270.3 \end{bmatrix}$ |
| Regional Boundary Information with $L = 7$ | 14822.6 | $\mathbf{G}_1 = \begin{bmatrix} 1023.2 & 265.0 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1049.6 & 270.0 \end{bmatrix}$ |
| Regional Boundary Information with $L = 9$ | 14148.3 | $\mathbf{G}_1 = \begin{bmatrix} 992.4 & 260.7 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1041.4 & 270.8 \end{bmatrix}$ |



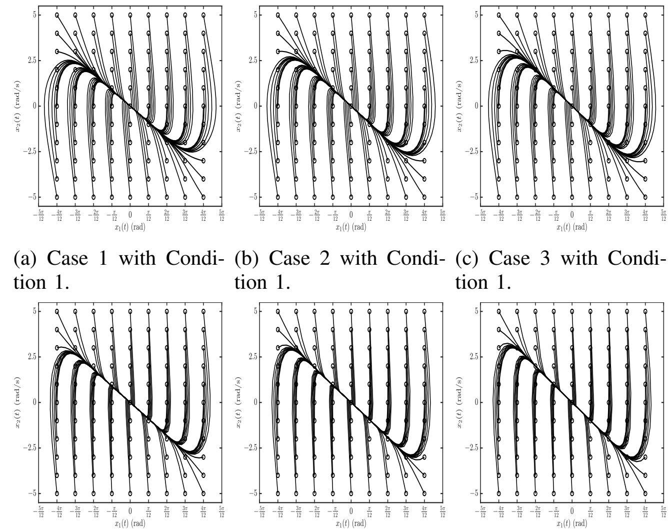
(a) Response of $x_1(t)$ with Condition 1. (b) Response of $x_2(t)$ with Condition 1. (c) Control signal $u(t)$ with Condition 1.

(d) Response of $x_1(t)$ with Condition 2. (e) Response of $x_2(t)$ with Condition 2. (f) Control signal $u(t)$ with Condition 2.

Fig. 1. Response curves.

minimal $\Upsilon$ is great when the weighting coefficient in the $x_1(t)$ position of the cost function is large. The results demonstrate the large weighting coefficient in the $x_1(t)$ position of the cost function will lead to the great upper bound of the guaranteed cost index in most instances.

In addition, Table III, Table IV and Table V show the feedback gains obtained by the different information of membership functions. It can be seen from three tables, the control gains of two different fuzzy rules are totally same when no information or global boundary information is considered. The same control gains demonstrate that the controller turns out to be a linear one. However, the difference between $\mathbf{G}_1$ and $\mathbf{G}_2$ will increase when more information of membership functions



(a) Case 1 with Condition 1. (b) Case 2 with Condition 1. (c) Case 3 with Condition 1.

(d) Case 1 with Condition 2. (e) Case 2 with Condition 2. (f) Case 3 with Condition 2.

Fig. 2. Phase portraits of $x_1(t)$ and $x_2(t)$.

is used. The outcome shows that the MFD stability analysis approach is helpful to achieve more relaxed results and has more potential to address the nonlinearity.

To further demonstrate the impact on the control performance caused by the weighting matrices in the cost function, the feedback gains for three cases with the initial states ($\frac{4\pi}{12}$, 0) and ($-\frac{2\pi}{12}$, 0) under the regional boundary information with $L = 9$ are applied to obtain the transient responses shown in Fig. 1. Furthermore, to demonstrate the uncertainties handling ability of the proposed IT2SD fuzzy controller, two sets of $m_p$ and $M_c$ which are Condition 1 ($m_p = 3$kg and $M_c = 12$kg) and Condition 2 ($m_p = 2$kg and $M_c = 8$kg) are utilized. Referring to Fig. 1, the settling time of $x_1(t)$ (the time required for $x_1(t)$ arriving at and staying within $\pm 2\%$ of the steady state as defined in [50]) is compared among the three cases. With respect to $m_p = 3$kg and $M_c = 12$kg, when the initial state is ($\frac{4\pi}{12}$, 0), the settling time of $x_1(t)$ for Case 3 is around 1.12s which is shorter than Case 1 requiring around 1.24s and Case 2 requiring around 1.18s; when the initial state is ($-\frac{2\pi}{12}$, 0), $x_1(t)$ for Case 3 is also the fastest which needs around 1.08s while $x_1(t)$ for Case 1 is the slowest which requires around 1.20s. With respect to $m_p = 2$kg and $M_c = 8$kg, when the initial state is ($\frac{4\pi}{12}$, 0), the settling time of $x_1(t)$ for Case 3 is around 1.08s which is shorter than Case 1 requiring around 1.18s and Case 2 requiring around 1.12s; when the initial state is ($-\frac{2\pi}{12}$, 0), $x_1(t)$ for Case 3 is also the fastest which needs around 1.06s while $x_1(t)$ for Case 1 is the slowest which requires around 1.16s. The results verify that the larger weighting coefficient in the $x_1(t)$ position of the cost function suppresses $x_1(t)$ harder. In addition, the results illustrate that the proposed IT2SD fuzzy controller can drive the inverted pendulum system subject to parameter uncertainties to the equilibrium, which is further verified by the phase portraits. As shown in Fig. 2, the inverted pendulum system with each initial state under two sets of $m_p$ and $M_c$ can be stabilized.

To sum up, the results of the minimal $\Upsilon$, feedback gains, transient responses and phase portraits verify the proposed approach can obtain the optimal guaranteed cost performance and the relaxed design results. Meanwhile, the proposed approach is capable of coping with the stabilization problem of the nonlinear systems subject to parameter uncertainties.

## V. Conclusion

This paper presents an IT2SDFMB control system with the consideration of the guaranteed cost performance. Nonlinear systems subject to uncertainty are modeled by the IT2 T–S fuzzy model that the parameter uncertainties can be captured by the lower and upper membership functions. Looped-functional approach making use of the information of the sampling periods is applied to facilitate the stability analysis of the SD FMB control system. The design flexibility is enhanced by employing the IPM concept which allows the free selection on the number of rules and the premise membership functions. The MFD stability analysis approach is applied to utilize the information of membership functions for achieving more relaxed results and minimizing the upper bound of the guaranteed cost index. Through the use of the weighting matrices in the cost function, the performance of the IT2SDFMB control system can be adjusted to satisfy different requirements as well. A simulation example of the inverted pendulum system is presented to demonstrate the effectiveness and merits of the proposed approach. In the future, improving the control performance by adaptively fine-tuning the IT2 parameters and adopting the polynomial fuzzy model will be potential directions. In addition, applications of IT2 FMB control design will be taken in to account, such as the tracking control system, the event-triggered control system and the time-varying system.

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