# Stochastic partial differential equations driven by cylindrical Lévy processes

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# Stochastic Partial Differential Equations Driven by Cylindrical Lévy Processes

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**Abstract**

Cylindrical Lévy processes provide a unified framework for different kinds of infinite-dimensional Lévy noise considered in the literature. In the recent papers by Jakubowski and Riedle, the integral with respect to cylindrical Lévy processes was defined. The aim of the thesis is to prove the existence and uniqueness of solutions to the stochastic differential equations driven by such processes. Equations with specific kinds of noises have been considered for many years, motivating the study of this generalised equation.

To be precise, we are considering the equation $\mathrm{d}X(t) = F(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t)$. Here $L$ is a cylindrical Lévy process from $U$ to $L^0(\Omega)$. Typically, $F$ contains some differential operator. We assume that $F$ and $G$ satisfy monotonicity and coercivity assumptions and solve this equation in the so-called variational approach. We cover the case of non-square-integrable noise of diagonal structure. We derive conditions when the behaviour of jumps of the cylindrical Lévy process enables the use of the interlacing construction.

In another approach, we construct an integral with respect to cylindrical Lévy process integrable with some power $p$, $1 \leq p < 2$, in the Banach space setting. We show existence of solutions in the semigroup approach.

Thirdly, we consider a canonical stable cylindrical Lévy process. Assuming that the functions appearing in the equation map between domains of certain powers of a generator of a strongly continuous semigroup we prove existence and uniqueness of solutions using tightness arguments and the Yamada–Watanabe theorem. In the proofs we make use of tail and moment inequalities, which are new in the case of cylindrical noise.

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# Contents

# Chapter 1

# Introduction

### Stochastic PDEs

There are at least two ways to formulate stochastic partial differential equations (SPDEs) and various concepts of solutions. The first approach is through the random fields considered by Walsh, see e.g. [105]. He considered PDEs perturbed by a space-time Wiener process, for instance the heat equation on some domain $\mathcal{O} \subset \mathbb{R}^d$

$$\mathrm{d}u(t,x) = \Delta u(t,x) + \mathrm{d}W(t,x), \qquad t \geq 0, x \in \mathcal{O} \tag{1.1}$$

with a two-parameter real-valued Gaussian process $W$. A separate question that arises apart from the existence and uniqueness of solutions is about the regularity of solutions, that is: in what function space do the functions $u(t,\cdot)$ live and in what sense one can get the continuity of $t \mapsto u(t,\cdot)$? Results of Walsh were generalised in many directions. The case of Lévy stable noise was done by Balan [5] and Chong [22]. They proved the existence of solutions for various equations e.g.

$$\mathrm{d}u(t,x) = \Delta u(t,x) + \sigma(u(t,x))\mathrm{d}L(t,x), \qquad t \geq 0, x \in \mathcal{O} \tag{1.2}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$.

The second approach is to consider the equation (1.1) in function spaces from the very outset and to solve it using techniques from functional analysis. Instead of (1.1) one writes

$$\mathrm{d}X(t) = \Delta X(t) + \mathrm{d}W(t), \tag{1.3}$$

where $X$ and $W$ are Hilbert or Banach space-valued processes. More generally, one considers the following non-linear evolution equation driven by Lévy noise

$$\mathrm{d}X(t) = \big(AX(t) + F(X(t))\big)\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t), \tag{1.4}$$

where $X$ is an unknown process taking values in a Hilbert space $H$ satisfying the initial condition $X(0) = X_0$, $L$ is a Lévy process in a Hilbert space $U$, $A$ is a generator of a strongly continuous semigroup $(S(t) : t \geq 0)$ on the Hilbert space $H$, $F\colon H \to H$ and $G\colon H \to \mathcal{L}(U,H)$ maps into the space of linear operators from $U$ to $H$.

Inside the functional setting, we can identify a semigroup and variational formulations of the evolution equation which lead to various definitions of the concept of a solution. The first one, the semigroup approach, was developed by Da Prato and Zabczyk, see e.g. [26]. Equations with Lévy-type noise were considered, for instance by Peszat and Zabczyk [74] who proved the existence and uniqueness of solutions to (1.4) driven by a $U$-valued Lévy process.

There are many concepts of a solution to (1.4). When applied to a specific SPDE, it turns out that one cannot hope for the solution $X$ to be in the domain of the generator $A$. For this reason one either considers a weak solution satisfying

$$\langle X(t), h\rangle = \langle X_0, h\rangle + \int_0^t \langle F(X(s)), A^*h\rangle\,\mathrm{d}s + \int_0^t G^*(X(s))h\,\mathrm{d}L(s), \qquad h \in H,$$

or a mild solution satisfying

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s).$$

Moreover, the solutions can be weak or strong in the classical probabilistic sense: $X$ constructed on an arbitrary probability space or $X$ constructed together with a probability space and the noise $L$. The standard argument to prove the existence and uniqueness of solutions to such equations, assuming the operators $F$ and $G$ are Lipschitz as functions of $X$, uses a fixed point argument in a suitably chosen function space.

Solutions to the evolution equation in the functional formulation (1.4) can also be constructed without an explicit reference to the semigroup. Another approach is the so-called 'variational formulation'. Under some assumptions, equation (1.4) can be rewritten as follows. Assume that there is a Banach space $V$ densely and continuously embedded in $H$ and consider

$$\mathrm{d}X(t) = \tilde{F}(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t), \tag{1.5}$$

with $F\colon V \to V^*$ and $G\colon V \to \mathcal{L}(U,H)$. Typically, $\tilde{F}$ equals to some differential operator e.g. the Laplacian. The conditions imposed on $\tilde{F}$ and $G$ and techniques of constructing solutions are quite different than in the semigroup approach: it is assumed that $\tilde{F}$ and $G$ satisfy the so-called monotonicity and coercivity conditions. The solution is obtained as a weak limit of solutions to the equation (1.5) projected onto $n$-dimensional subspaces. It requires careful analysis of the norms of those solutions in each of the spaces on which $\tilde{F}$ and $G$ are defined and the use of a special version of the Itô formula.

In the thesis we pursue SPDEs both in the variational formulation (Chapter 4) and in the semigroup approach (Chapters 5 and 6).

### Models for the noise

The second modelling choice one needs to make apart from the precise mathematical formulation of the evolution equation, which we described above, is to choose some model for the noise. As already pointed out above the first two options are either Wiener or Lévy processes.

Perhaps the simplest process one can think of in the context of (1.3) is a Wiener process taking values in a Hilbert space. This leads also to the concept of a cylindrical Wiener process: given a separable Hilbert space $U$ it seems natural to define a process evolving like independent standard real-valued Brownian motions along the basis vectors and satisfying $\mathbb{E}\left[\langle W(t),u\rangle\langle W(s),v\rangle\right] = (t\wedge s)\langle u,v\rangle$. Such a process would need to be constructed as $\sum\limits_{n=1}^{\infty} w_n(t)e_n$, where $(e_n)$ is an orthonormal basis and $(w_n)$ is a sequence of independent standard Wiener processes. It turns out that such sum does not converge in $U$. However, one can define a cylindrical process with similar properties, the term 'cylindrical' referring to the fact that it does not take values in the Hilbert space $U$ under consideration. Instead, $W(t)$ maps from the Hilbert space into the space of real-valued random variables $W(t)\colon U \to L^2(\Omega;\mathbb{R})$ and is given by

$$W(t)u = \sum_{n=1}^{\infty} w_n(t)\langle u, e_n\rangle, \qquad u \in U. \tag{1.6}$$

In the thesis we consider the cylindrical counterpart of Hilbert-space valued Lévy processes. Cylindrical Lévy processes were studied in a systematic manner by Applebaum and Riedle in [2]. They are families of continuous linear mappings $L(t)\colon U \to L^0(\Omega,\mathbb{R})$ from a separable Hilbert space into the space of real-valued random variables satisfying the following: for all $n \in \mathbb{N}$ and $u_1,\dots,u_n \in U$ the finite-dimensional projections $\big(L(t)u_1,\dots,L(t)u_n : t \geq 0\big)$ are Lévy processes in $\mathbb{R}^n$. In some cases we we will more generally, take $L$ defined on a dual of a separable Banach space.

This definition builds on the well-established theory of cylindrical measures and cylindrical random variables. The research in the field was carried out in the 70s in France by Schwartz, Badrikian, Chevet, Maurey and others, see e.g. [3, 4, 64, 97]. In the following years some research was carried out on cylindrical local martingales and stochastic integration with respect to cylindrical processes, e.g. Métivier and Pellaumail [68], Kurtz and Protter [58], Brzeźniak and Zabczyk [18].

The abstract definition of cylindrical Lévy processes stated above includes various classes of processes. In particular it extends the notion of the cylindrical Brownian motion by allowing non-Gaussian infinitely divisible distributions possibly with infinite moments and having discontinuous paths. Moreover, one can show that cylindrical Wiener processes are necessarily of the diagonal form (1.6) with independent terms (the Karhunen–Loève Theorem). On the contrary, a general cylindrical Lévy processes cannot be written in the form (1.6), since that would mean that its Lévy measure is concentrated only on the set of multiples of the basis vectors $(e_n)$, that is on $\{\lambda e_n : \lambda \in \mathbb{R}, n \in \mathbb{N}\}$.

Nonetheless, processes of the form (1.6) with independent one-dimensional Lévy processes $\ell_n$ instead of $w_n$ are cylindrical Lévy processes and have attracted considerable attention. For instance as shown in Priola and Zabczyk [78], equation (1.4) with such diagonal noise, $F = 0$, diagonal operator $A$ and additive noise can be reduced to an infinite system of independent equations in one dimension. This model was investigated in the last 10 years by many authors. It was shown that as soon as the noise is a cylindrical process, the solution can be very irregular, for instance it may have no càdlàg modification, see Brzeźniak et al. [12] and Liu and Zhai [61].

Also as shown in Dalang and Quer-Sardanyons [27] for the Gaussian case and Griffiths and Riedle [40] for the Lévy case, the noises used in the random field approach (see (1.2)) can also be reformulated as a cylindrical Lévy process. In contrast to the Gaussian setting, the correspondence holds only for a subclass of cylindrical Lévy processes. In another approach, Balan [6] considered spatially correlated Lévy noise on nuclear spaces of test functions by means of the Fourier transform.

## Our objectives

In the thesis we show the existence and uniqueness of solutions to the evolution equations (1.4) and (1.5) in the case when $L$ is a cylindrical process without weak second moments i.e.

$\mathbb{E}\left[|L(t)u|^2\right] = \infty$ for some $u \in U$. Distributions with infinite moments are important from the point of view of applications due to their fat tails, see e.g. [80] for a discussion of data networks modelling.

The usual approach to the construction of solution to a stochastic differential equation driven by a Lévy process requires the use of the Lévy–Itô decomposition, that is separating the small and the large jumps. For a classical Lévy process $L$ one writes $L = L_1 + L_2$, with $L_1$ having jumps bounded from above and $L_2$ being of bounded variation and having only finitely many jumps on any compact interval. It follows that $L_1$ has all moments finite and therefore the equation driven by $L_1$ can usually be solved using a fixed point argument in some $L^2$-space. In this way, one solves the equation up to the stopping time $\tau$ of the first jump of $L_2$. The second step is to include the jumps of $L_2$ by re-defining the solution after time $\tau$. Cylindrical processes do not take values in a vector space, in contrast, they are families indexed by time and a vector space, so there is no natural way of defining the above decomposition nor the notion of the first 'large' jump. Therefore the usual technique does not work for cylindrical Lévy processes.

We propose two ways to address the problem of no semimartingale decomposition. For the processes of the diagonal form (1.6) one can in fact derive a decomposition using the decompositions of the one-dimensional components. However, as shown in Chapters 3 and 4, it turns out that one needs to choose a different cut-off level in each dimension to ensure convergence. Secondly, for general cylindrical Lévy processes one cannot avoid the problem and needs to deal with the process 'in one piece', that is, without separating the 'small' and the 'large' jumps. Often the use of non-standard techniques is required, see for instance in Jakubowski and Riedle [49], where the stochastic integral was constructed using tightness by decoupling, or Chapter 6 in this thesis, where we derive existence of solutions to the evolution equation driven by an $\alpha$-stable noise using moment estimates for $p < \alpha$.

In the thesis, we present existence and uniqueness results in three settings: a variational solution for (1.5), a mild solution for (1.4) formulated in Banach spaces and thirdly, a mild solution for (1.4) driven by a stable cylindrical Lévy process. We now describe the structure of the thesis. The literature and details about our contribution will be presented at the beginning of each chapter.

## Outline of the thesis

We start with the preliminary Chapter 2, where we give the precise definition of the cylindrical Lévy processes. We review the attempts to define a stochastic integral with respect to cylindrical Lévy processes and similar cylindrical processes. Some preliminary definitions and formulas are recalled also at the beginning of each chapter.

In the following Chapter 3 we extend some auxiliary results from the papers of Applebaum and Riedle. We improve the decomposition into a drift and a cylindrical martingale and, more importantly, improve the definition of the integral from Riedle [83]. We define certain stopping times describing the jump times of the cylindrical Lévy processes and show their irregular behaviour. We prove that the formula for the angle bracket process of a stochastic integral holds also in the cylindrical case. Those results, which in some cases are counterparts of the results from the classical stochastic analysis, are needed later in the construction of solutions to SPDEs.

The three main chapters which follow all lead to an existence and uniqueness results for equations of the form (1.4) or (1.5).

In Chapter 4 we consider the equation in the variational setting assuming the standard monotonicity and coercivity conditions. We start with the case of the noise with finite second moment. The existence of a solution in the non-square-integrable case is established for the subclass of diagonal cylindrical Lévy processes using a Lévy–Itô decomposition and stopping time arguments. We derive conditions on the Lévy characteristics, which guarantee that there is a Lévy–Itô decomposition. We show that in the case of the most common non-integrable processes, that is stable processes, one can construct stopping times suitable for the construction of a solution.

Chapter 5 is devoted to the construction of a Banach space-valued integral and existence of a mild solution to (1.4). Dealing with Banach space-valued processes poses some difficulties and requires geometric assumptions about the spaces. In earlier chapters a special role was played by the Hilbert-Schmidt operators acting between Hilbert spaces, because they map cylindrical random variables into classical vector-valued random variables. In the Banach space setting, we propose to consider integrands taking values in the space of $p$-summing operators, $p \in [1, 2]$, which generalise to Banach spaces the notion of Hilbert-Schmidt operators. By a famous result of Kwapień and Schwartz they have the same crucial property of mapping cylindrical random variables into classical vector-valued random variables. In order to prove the continuity of the integral operator, we need to assume certain geometric properties for the Banach spaces,

$p$-integrability of the cylindrical Lévy process and that it has paths of bounded $p$-variation. The crucial part of the proof is an application of an inequality by Schwartz on moments of the cylindrical measures. Existence and uniqueness follow in a standard manner by the Banach fixed point theorem assuming that the coefficient of the equation are Lipschitz.

Chapter 6 is devoted to the existence of a mild solution to the equation with an $\alpha$-stable cylindrical Lévy process, $\alpha \in (1, 2)$ i.e. a cylindrical process with the characteristic function $\mathbb{E}\left[e^{iL(t)u}\right] = e^{-t\|u\|^{\alpha}}$ for $u \in U$. We start by proving tail and moment estimates for the stochastic integrals with respect to a stable cylindrical Lévy process. We construct a càdlàg solution as a limit of the Picard iterations using tightness arguments in the Skorokhod space. Since we use the Skorokhod representation theorem, we obtain a weak solution defined on a different probability space. The most intricate part of the proof is the identification of limits, which requires estimates of $p$-th moments of approximations, $p < \alpha$. We get the existence of the strong solution from the pathwise uniqueness and the Yamada–Watanabe theorem.

# Chapter 2

# Notation and preliminaries

Throughout the thesis, $E$ and $F$ denote separable Banach spaces with duals $E^*$ and $F^*$, $U$ and $H$ denote separable Hilbert spaces, $(\Omega, \mathcal{F}, P)$ is a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and $L^0(\Omega, \mathcal{F}, P; E)$ denotes the space of equivalence classes of $E$-valued random variables, which will be always equipped with the topology of convergence in probability. Finally, $L^p(\Omega, \mathcal{F}, P; E)$ is the space of equivalence classes of random variables, which are Bochner integrable with power $p$, for $p \geq 1$. If $x^*$ is a functional on $E$, we often denote its value on $x$ by $\langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$. The closed unit ball of $E$ is denoted by $B_E$ and the ball of radius $r$ by $B_E(0, r)$. $S_E$ is the sphere of radius 1 in $E$. We write $\mathcal{P}$ for the predictable $\sigma$-algebra on $[0, T] \times \Omega$ and $\mathcal{B}(S)$ for the $\sigma$-algebra of Borel sets of a topological space $S$. The Lebesgue measure is denoted by Leb.

If $\psi$ is an operator from $E$ into $F$, we denote its operator norm by $\|\psi\|$, or if we want to make it clear what the domain and codomain of $\psi$ are, by $\|\psi\|_{\mathcal{L}(E,F)}$. The space of Hilbert-Schmidt operators from $U$ to $H$ is denoted by $L_{\mathrm{HS}}(U, H)$ and the norm in this space by $\|\cdot\|_{L_{\mathrm{HS}}(U,H)}$. The symbol $L_1(H)$ denotes the space of nuclear operators on $H$. $\ell^p(\mathbb{R})$ is the space of $p$-summable real-valued sequences, $\ell^p(\mathbb{R}_+)$ is the subset consisting of non-negative sequences.

Let $T > 0$. The space of $E$-valued càdlàg functions on $[0, T]$ is called the Skorokhod space and is denoted by $D([0, T]; E)$. We equip it with the Skorokhod topology as specified in [8, 48]. Similarly, $D_-([0, T]; E)$ is the space of càglàd functions.

We say that a sequence of $E$-valued processes $\Psi_n$ converges to $\Psi$ on $[0, T]$ uniformly in probability (ucp) if for any $\varepsilon > 0$

$$P\left(\sup_{t\in[0,T]} \|\Psi_n(t) - \Psi(t)\| \geq \varepsilon\right) \to 0 \quad \text{as} \quad n \to \infty.$$

## 2.1 Cylindrical Lévy processes

For $x_1^*, \dots, x_n^* \in E^*$ and a Borel set $B \in \mathcal{B}(\mathbb{R}^n)$ a *cylindrical set* is defined by

$$C(x_1^*, \dots x_n^*; B) = \{x \in E : (\langle x_1^*, x\rangle, \dots, \langle x_n^*, x\rangle) \in B\}.$$

The algebra of all cylindrical sets such that $x_1^*, \dots x_n^* \in \Gamma \subset E^*$ is denoted by $\mathcal{Z}(E, \Gamma)$. The algebra of all cylindrical sets is denoted by $\mathcal{Z}(E) = \mathcal{Z}(E, E^*)$. If $\Gamma$ is a finite set, then $\mathcal{Z}(E, \Gamma)$ is a $\sigma$-algebra. A function $\mu\colon \mathcal{Z}(E) \to [0, \infty]$ is a *cylindrical measure* if its restriction the $\sigma$-algebra $\mathcal{Z}(E, \Gamma)$ is a measure for every finite $\Gamma \subset E^*$. If $\mu(E) < \infty$ we call it finite and if $\mu(E) = 1$ we call it a cylindrical probability measure.

Let $p \geq 0$. A measure $\mu$ on $\mathcal{B}(E)$ is said to be of order $p$ if

$$\int_E \|x\|^p \,\mu(\mathrm{d}x) < \infty.$$

Similarly, a cylindrical measure $\mu$ is said to be of weak order $p$ if

$$\int_E |\langle x^*, x\rangle|^p \,\mu(\mathrm{d}x) < \infty, \qquad x^* \in E^*.$$

For any operator $\psi\colon E \to F$ the pushforward cylindrical measure is denoted with $\psi(\mu) = \mu \circ \psi^{-1}$, i.e. $(\psi(\mu))(C) = \mu(\psi^{-1}(C))$ for any $C \in \mathcal{Z}(F)$. An operator $\psi$ is called *p-Radonifying* if for every cylindrical probability $\mu$ on $E$ of weak order $p$, $\psi(\mu)$ extends to a measure on $F$ of order $p$. It is well known that for any $p > 0$ an operator acting between Hilbert spaces $\psi\colon U \to H$ is 0-Radonifying if and only if it is $p$-Radonifying and if and only if it is Hilbert-Schmidt, see [101, Th. VI.5.2].

A *cylindrical random variable* is a linear and continuous mapping

$$X\colon E^* \to L^0(\Omega, \mathcal{F}, P; \mathbb{R}).$$

Many classical notions from probability theory have their cylindrical counterparts. Cylindrical distribution of $X$ is defined by

$$\mu\colon \mathcal{Z}(E) \to [0,1], \qquad \mu(C(x_1^*,\dots,x_n^*;B)) = P\left((Xx_1^*,\dots,Xx_n^*) \in B\right).$$

The characteristic function of a cylindrical random variable $X$ (resp. cylindrical probability measure $\mu$) is defined as a function $\varphi\colon E^* \to \mathbb{C}$

$$\varphi_X(x^*) = \mathbb{E}\left[e^{iXx^*}\right], \qquad \left(\text{resp. } \varphi_\mu(x^*) = \int_E e^{i\langle x^*,x\rangle}\,\mu(\mathrm{d}x)\right).$$

A cylindrical random variable is said to be *induced* by a classical random variable if there exists a random variable $Y\colon \Omega \to E$ such that

$$Xx^* = \langle x^*, Y\rangle, \qquad x^* \in E^*.$$

An operator $\psi\colon E \to F$ is $p$-Radonifying if for every weakly $p$-integrable cylindrical random variable $X\colon E^* \to L^p(\Omega,\mathcal{F},P)$ the variable $X\psi^*\colon F^* \to L^p(\Omega,\mathcal{F},P)$ is induced by a classical $p$-integrable random variable.

The following definition was given in [2], but we make two changes: we allow an arbitrary filtration satisfying the usual conditions and we required above the cylindrical random variables to be continuous as the mappings to $L^0(\Omega,\mathcal{F},P;\mathbb{R})$ equipped with the convergence in probability.

**Definition 2.1.** A *cylindrical Lévy process* is a family of cylindrical random variables $(L(t): t \geq 0)$, where $L(t)\colon E^* \to L^0(\Omega,\mathcal{F},P;\mathbb{R})$ such that for any $n \in \mathbb{N}$ and $x_1^*,\dots,x_n^* \in E^*$ the $n$-dimensional process

$$((L(t)x_1^*,\dots L(t)x_n^*) : t \geq 0) \tag{2.1}$$

is a Lévy process in $\mathbb{R}^n$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

The characteristic function of a cylindrical Lévy process is characterised by a triplet $(p,q,\nu)$, where $b\colon E^* \to \mathbb{R}$ is a continuous function, $q\colon E^* \to \mathbb{R}$ is a quadratic form and $\nu$ is a cylindrical measure on $\mathcal{Z}(E)$ such that

$$\int_E \left(\langle x^*,x\rangle^2 \wedge 1\right)\nu(\mathrm{d}x) < \infty, \qquad x^* \in E^*$$

see [2, Th. 2.7] and [82, Th. 3.4]. We have

$$\begin{aligned}\varphi_{L(t)}(x^*) &= \exp\left(t\Psi(x^*)\right) \\ \Psi(x^*) &= ib(x^*) - \frac{1}{2}q(x^*) + \int_E \left(e^{i\langle x^*,x\rangle} - 1 - \mathbb{1}_{B_{\mathbb{R}}}(\langle x^*,x\rangle)\langle x^*,x\rangle\right)\nu(\mathrm{d}x).\end{aligned} \tag{2.2}$$

Cylindrical Lévy process $L$ is said to be weakly $p$-integrable ($p \geq 0$) if $\mathbb{E}\left[|L(t)x^*|^p\right] < \infty$ for all $x^* \in E^*$ and all (or equivalently some) $t > 0$. We will frequently make use of the fact that if $L$ is weakly $p$-integrable then by the closed graph theorem for each $t \geq 0$ the mapping $L(t)\colon E^* \to L^p(\Omega, \mathcal{F}, P)$ is continuous. We say that a weakly integrable (i.e. weakly 1-integrable) cylindrical Lévy process $L$ is weakly mean-zero if $\mathbb{E}\left[L(t)x^*\right] = 0$ for all $t > 0$ and $x^* \in E^*$. In that case we also say that it is a cylindrical Lévy martingale because then all projections (2.1) are martingales. If $L$ is weakly square-integrable (i.e. $\mathbb{E}\left[|L(t)x^*|^2\right] < \infty$ for all $x^* \in E^*$), the covariance operator $Q\colon E^* \to E^{**}$ of $L$ is defined by

$$\langle Qx^*, y^*\rangle = \mathbb{E}\left[\left(L(1)x^* - \mathbb{E}\left[L(1)x^*\right]\right)\left(L(1)y^* - \mathbb{E}\left[L(1)y^*\right]\right)\right], \qquad x^*, y^* \in E^*.$$

If $L$ is defined on a Hilbert space $U$, then the operator $Q\colon U \to U$ is non-negative and symmetric and therefore has a unique non-negative square root, which we denote by $Q^{1/2}$, see [88, Th. 12.33].

## 2.2 Examples of cylindrical Lévy processes

The following examples of cylindrical Lévy process were studied in [84, 85]

**Example 2.2.** Every classical $E$-valued Lévy process $L$ in $E$ induces a cylindrical process by the formula

$$\tilde{L}(t)\colon E^* \to L^0(\Omega, \mathcal{F}, P; \mathbb{R}), \qquad \tilde{L}(t)x^* = \langle x^*, L(t)\rangle.$$

**Example 2.3.** Let $(e_k)$ be an orthonormal basis of a Hilbert space $U$. A cylindrical Lévy process is called *diagonal* if

$$L(t)u = \sum_{k=1}^{\infty} \ell_k(t)\langle u, e_k\rangle, \qquad t \geq 0, u \in U, \tag{2.3}$$

where $(\ell_k)$ is a sequence of independent, real-valued Lévy processes. Conditions for the a.s. convergence of the series and continuity of $L(t)$ are given in [84, Lem. 4.2]. Denote the characteristics (with respect to the standard truncation function $\mathbb{1}_{B_{\mathbb{R}}}$) of $\ell_k$ by $(b_k, s_k, \rho_k)$ for each $k \in \mathbb{N}$. The sum converges and defines a cylindrical Lévy process if and only if the characteristic functions of $\ell_k$ are equicontinuous at 0 and the following three conditions are satisfied for every $(\alpha_k) \in \ell^2(\mathbb{R})$:

$$\text{(i)} \quad \sum_{k=1}^{\infty} \mathbb{1}_{B_{\mathbb{R}}}(\alpha_k)|\alpha_k| \left| b_k + \int_{1<|x|\le|\alpha_k|^{-1}} x\,\rho_k(\mathrm{d}x) \right| < \infty, \tag{2.4}$$

$$\text{(ii)} \quad (s_k) \in \ell^{\infty}(\mathbb{R}), \tag{2.5}$$

$$\text{(iii)} \quad \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left( |\alpha_k x|^2 \wedge 1 \right) \rho_k(\mathrm{d}x) < \infty. \tag{2.6}$$

SPDEs with additive noise of this type have been studied by various authors, see e.g. [12, 78, 77, 61, 75, 62].

**Example 2.4.** Let $N$ be a Poisson process and let $(X_k)$ be a sequence of independent identically distributed cylindrical random variables on $E^*$. The process

$$L(t) = \sum_{k=1}^{N(t)} X_k$$

is called a cylindrical compound Poisson process.

**Example 2.5.** A cylindrical Lévy process $W$ on a Hilbert space $U$ with characteristic function

$$\varphi_{W(t)}(u) = e^{-t\|u\|^2}, \qquad t \ge 0, u \in U,$$

is called a standard cylindrical Wiener process.

**Example 2.6.** If the characteristic function of a cylindrical Lévy process $L$ is $\varphi_{L(t)}(x^*) = e^{-t\|x^*\|^{\alpha}}$ for $x^* \in E^*$, then $L$ is called a canonical $\alpha$-stable cylindrical Lévy process. If $E = U$ is a Hilbert space with an orthonormal basis $(e_k)$, then the Lévy measure $\nu$ according to [85, Lem. 2.4] satisfies

$$\nu \circ \pi_{e_1,\dots,e_n}^{-1}(B) = \frac{\alpha}{c_\alpha} \int_{S_{\mathbb{R}^n}} \int_0^{\infty} \mathbb{1}_B(r\xi) \frac{1}{r^{1+\alpha}}\,\mathrm{d}r\,\lambda_n(\mathrm{d}\xi), \qquad B \in \mathcal{B}(\mathbb{R}^n), \tag{2.7}$$

where $\pi_{e_1,\dots,e_n}\colon U \to \mathbb{R}^n$ is given by $\pi_{e_1,\dots,e_n}(u) = (\langle u, e_1\rangle, \dots, \langle u, e_n\rangle)$, the constant $c_\alpha$ is defined as

$$c_\alpha := \begin{cases} -\alpha\cos(\frac{\alpha\pi}{2})\Gamma(-\alpha), & \text{for } \alpha \neq 1, \\ \frac{\pi}{2}, & \text{for } \alpha = 1, \end{cases} \tag{2.8}$$

and the measure $\lambda_n$ on $S_{\mathbb{R}^n}$ is uniform with the total mass

$$\lambda_n(S_{\mathbb{R}^n}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})}. \tag{2.9}$$

**Example 2.7.** Let $\mathcal{O} \subset \mathbb{R}^d$ and let $N$ be a Poisson random measure on $\mathcal{B}(\mathbb{R}_+ \times \mathcal{O}) \otimes \mathcal{B}(\mathbb{R})$ with intensity $\mathrm{Leb} \otimes \nu$ and

$$\nu(\mathrm{d}x) = \frac{1}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} \frac{1}{|x|^{1+\alpha}} \mathrm{d}x.$$

A canonical stable Lévy space-time white noise on $\mathbb{R}_+ \times \mathcal{O}$ is a function $Y\colon \mathcal{B}(\mathbb{R}_+ \times \mathcal{O}) \times \Omega \to \mathbb{R}$

$$Y(B) = \int_{B\times\mathbb{R}} y N(\mathrm{d}x, \mathrm{d}y).$$

A cylindrical Lévy process on $L(t)\colon L^\alpha(\mathcal{O}) \to L^0(\Omega, \mathcal{F}, P)$ is defined by extending the following

$$L(t)\mathbb{1}_A = Y([0,t] \times A)$$

for all bounded $A \in \mathcal{B}(\mathcal{O})$.

## 2.3 Stochastic integration with respect to cylindrical Lévy processes

There have been several attempts to define stochastic integration with respect to cylindrical Lévy processes. Métivier and Pellaumail [68], following an earlier article [37], present an integral with respect to weakly square-integrable cylindrical martingales in Banach spaces using the Doléans measures. Their integral is also a cylindrical process. They obtain a vector-valued integral in the setting of Hilbert spaces. In [67] the authors include also the case of cylindrical square-integrable local martingales. Since every Lévy local martingale is a martingale, this extension does not cover non-integrable cylindrical Lévy processes, see [79, Ex. 29, p. 49].

The integral constructed by Kurtz and Protter in [58] introduce the integration with respect to the cylindrical semimartingales (called $H^{\#}$-semimartingales in that publication). They impose a certain boundedness condition on the integrator (calling them special $H^{\#}$-semimartingales), which guarantee continuity of the integral mapping. The integrands are predictable, $H$-valued and attain values in a compact set with high probability. Since the integrands are vector-valued rather than operator-valued, the integral is a real-valued process.

In [18] Brzeźniak and Zabczyk considered processes of the form $Y(t) = W(\ell(t))$, where $\ell$ is a subordinator and $W$ is a cylindrical Brownian motion on a separable Hilbert space $H$, taking values in a Banach space $U$, such that $H$ is embedded in $U$ with a $\gamma$-Radonifying embedding. If $\ell$ is $\frac{\alpha}{2}$-stable, then $Y$ is a canonical stable cylindrical Lévy process on $H$, see [18, Rem. 2.5(4)]. In order to define the integral the authors use the fact that $Y$ is $U$-valued and rewrite the integral with respect to $Y$ as an integral with respect to a Poisson random measure.

In [84] Riedle defined the integral for general $L$ as in Definition 2.1 and deterministic integrands by mimicking Pettis' idea. The integrability criteria are expressed in terms of the characteristics of the cylindrical Lévy process, similarly to Chojnowska-Michalik [20]. This theory was sufficient to consider equations with additive noise, see [54, 55].

A novel approach to stochastic integration is presented in [49] by Jakubowski and Riedle. In that paper the integrator is a general cylindrical Lévy process as defined in Definition 2.1. The integrands are càglàd adapted processes. The proofs are based on tightness arguments and decoupled tangent sequences. The following is a direct consequence of [49, Th. 5.1]

**Theorem 2.8** (Jakubowski-Riedle)**.** *If $\Psi_n$ converge to $\Psi$ in probability in the Skorokhod space $D_-([0,T]; L_{\mathrm{HS}}(U,H))$, then for every $t \in [0,T]$*

$$\int_0^t \Psi_n(s)\,\mathrm{d}L(s) \to \int_0^t \Psi(s)\,\mathrm{d}L(s),$$

*in probability.*

Often it is not known a priori if the solution of an SPDE has a càglàd modification. For that reason we will present ways to define the integral for predictable (not necessarily càglàd) integrands and in this way extend [49]. However, our result will require some additional structural assumptions on the cylindrical Lévy process.

# Chapter 3

# Extensions of the existing theory of cylindrical Lévy processes

## 3.1 Lévy–Itô decomposition

In this section we improve the Lévy–Itô decomposition of [2, Cor. 3.12] by relaxing the moments assumption and showing that the continuity of $L(t)$ implies the continuity of the components in the decomposition.

**Proposition 3.1.** Suppose that $L$ is weakly integrable. There exists a linear functional $B\colon E^* \to \mathbb{R}$, a cylindrical Wiener process and a cylindrical Lévy martingale $M$ such that

$$L(t)x^* = B(x^*)t + W(t)x^* + M(t)x^*, \qquad t \geq 0, x^* \in E^*,$$

where $M$ is given by

$$M(t)x^* := \int_{\mathbb{R}\setminus\{0\}} \beta \, \tilde{N}_{x^*}(t, \mathrm{d}\beta) \tag{3.1}$$

and $\tilde{N}_{x^*}$ is the compensated Poisson random measure associated with $(L(t)x^* : t \geq 0)$.

*Proof.* The characteristics of the Lévy process $(L(t)x^* : t \geq 0)$ are $(b(x^*), q(x^*), x^*(\nu))$, cf. (2.2) and its Lévy–Itô decomposition is

$$L(t)x^* = b(x^*)t + q(x^*)W_{x^*}(t) + \int_{B_{\mathbb{R}}} \beta \, \tilde{N}_{x^*}(t, \mathrm{d}\beta) + \int_{B_{\mathbb{R}}^c} \beta \, N_{x^*}(t, \mathrm{d}\beta), \tag{3.2}$$

where $W_{x^*}$ is a one-dimensional standard Wiener process. Let $Bx^* := \mathbb{E}\left[L(1)x^*\right]$. Then

$$Bx^* = p(x^*) + \int_{B_{\mathbb{R}}^c} \beta\, \nu \circ (x^*)^{-1}(\mathrm{d}\beta)$$

and (3.2) can be rewritten as

$$L(t)x^* = tBx^* + q(x^*)W_{x^*}(t) + \int_{\mathbb{R}} \beta\, \tilde{N}_{x^*}(t, \mathrm{d}\beta).$$

Let $M$ be given by (3.1) and $W$ be defined as $W(t)x^* = q(x^*)W_{x^*}(t)$ for $t \geq 0$ and $x^* \in E^*$.

We show that each component of the decomposition is continuous. Continuity of $B$ follows from the continuity of $L(1)$ as a mapping into $L^1(\Omega, \mathcal{F}, P; \mathbb{R})$:

$$|Bx^*| \leq \mathbb{E}\left[|L(1)x^*|\right] \leq \|L(1)\|_{\mathcal{L}(E^*, L^1)} \|x^*\|.$$

Thus $\tilde{L}(t) := L(t) - tB = W(t) + M(t)$ is also continuous. According to [101, Prop. IV.3.4] the continuity of a cylindrical random variable is equivalent to the continuity of its characteristic function. A close inspection of the proof reveals that in fact the continuity of the characteristic function at 0 guarantees continuity of the cylindrical random variable. We show that if $x_n^* \to 0$, then $\varphi_{W(t)}(x_n^*) \to 1$ and $\varphi_{M(t)}(x_n^*) \to 1$. Suppose for contradiction that for some $\varepsilon > 0$ there is a subsequence $(n_k)$ such that $\varphi_{W(t)}(x_{n_k}^*) \leq 1 - \varepsilon$. We have

$$\varphi_{W(t)}(x^*) = \exp\left(-t\frac{1}{2}q(x^*)\right) > 0.$$

Hence $\varphi_{W(t)}(x^*) = \left|\varphi_{W(t)}(x^*)\right|$. Then

$$1 \leftarrow \left|\varphi_{\tilde{L}(t)}(x_{n_k}^*)\right| = \varphi_{W(t)}(x_{n_k}^*)\left|\varphi_{M(t)}(u_{n_k}^*)\right| \leq 1 - \varepsilon,$$

which is a contradiction. This finishes the proof of the fact that $\varphi_{W(t)}(x_n^*) \to 1$. Secondly,

$$\varphi_{M(t)}(x_n^*) = \frac{\varphi_{\tilde{L}(t)}(x_n^*)}{\varphi_{W(t)}(x_n^*)} \to 1$$

as $n \to \infty$, which finishes the proof of the continuity of $M(t)$. □

**Remark 3.2.** The decomposition (3.2) is obviously still valid for general (not necessarily integrable) cylindrical Lévy processes. As pointed out in [2] the components are not linear in $x^*$, so the only decomposition into cylindrical Lévy processes one can have in general is into

$$W(t) \qquad \text{and} \qquad b(x^*)t + q(x^*)W_{x^*}(t) + \int_{B_{\mathbb{R}}} \beta\, \tilde{N}_{x^*}(t\mathrm{d}\beta) + \int_{B_{\mathbb{R}}^c} \beta\, N_{x^*}(t, \mathrm{d}\beta).$$

By the same method as above one can show that both components are continuous.

**Remark 3.3.** The same reasoning gives the following more general result. Let $Z, X, Y \colon E^* \to L^0(\Omega, \mathcal{F}, P; \mathbb{R})$ be linear mappings. Assume that $\varphi_X \geq 0$ in some neighbourhood of 0. Moreover, assume that $X$ and $Y$ are independent and that $Z = X + Y$. If $Z$ is continuous, then $X$ and $Y$ are. Note that the assumption that $\varphi_X$ is real in a neighbourhood of 0 cannot be dropped as the following deterministic example from [101, Ex. 2(c), p. 403] shows: let $f$ be a discontinuous linear function on $E^*$ and

$$Xx^* := f(x^*), \qquad Yx^* := -f(x^*), \qquad \text{for } x^* \in E^*.$$

In this way $X + Y = 0$ is clearly a continuous cylindrical random variable but neither $X$ nor $Y$ are.

## 3.2 Stochastic integration with respect to weakly square-integrable processes

In this Section we improve the results of Riedle [83]. The construction of the integral in that paper is correct but far from optimal as the space of admissible integrands is small and unnatural. The construction consists of two steps and we only alter the second step. This integral will be used in Chapters 3 and 4.

Suppose that $L$ is a weakly square-integrable cylindrical Lévy process with the covariance operator denoted by $Q$. We now define a reproducing kernel Hilbert space associated to $L$; see e.g. [74, Def. 7.2] or [101, Sec. 111.1.2 and 111.1.3], where a more general case of variables in Banach spaces is considered. In our setting, we define the reproducing kernel Hilbert space associated to $L$ as $\mathcal{H} = Q^{1/2}U$. Let $Q^{-1/2}$ be the inverse of $Q^{1/2} \colon (\operatorname{Ker} Q^{1/2})^{\perp} \to \mathcal{H}$, see [76, App. C]. The space $\mathcal{H}$ is equipped with the scalar product

$$\langle u, v \rangle_{\mathcal{H}} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U, \qquad u, v \in \mathcal{H}.$$

Since for $u \in \mathcal{H}$

$$\|u\|_U = \left\|Q^{1/2}Q^{-1/2}u\right\|_U \leq \left\|Q^{1/2}\right\|\left\|Q^{-1/2}u\right\|_U = \left\|Q^{1/2}\right\|\|u\|_{\mathcal{H}},$$

the embedding $\mathcal{H} \subset U$ is continuous. Thus $L$ restricted to $\mathcal{H}$ is still a cylindrical Lévy process.

Let $(e_k)$ be an orthonormal basis of $(\operatorname{Ker} Q^{1/2})^{\perp}$. Then $(Q^{1/2}e_k)$ is an orthonormal basis of $\mathcal{H}$, see [76, Prop. C.0.3]. Thus for a linear operator $\psi\colon \mathcal{H} \to H$ we have

$$\|\psi\|^2_{L_{\mathrm{HS}}(\mathcal{H},H)} = \sum_{k=1}^{\infty}\left\|\psi Q^{1/2}e_k\right\|^2 = \left\|\psi Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)}.$$

In particular $\psi \in L_{\mathrm{HS}}(\mathcal{H}, H)$ if and only if $\psi Q^{1/2} \in L_{\mathrm{HS}}(U, H)$.

In the first step we define

$$\int_0^T \Psi(s)\,\mathrm{d}L(s)$$

for simple integrands $\Psi$ of the form

$$\Psi(t) = \Psi_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{N-1} \Psi_k \mathbb{1}_{(t_k,t_{k+1}]}(t), \qquad t \in [0,T], \tag{3.3}$$

where each $\Psi_k\colon \Omega \to L_{\mathrm{HS}}(\mathcal{H}, H)$ is $\mathcal{F}_{t_k}$-measurable and $0 = t_1 < t_2 < \cdots < t_N = T$ is a partition of $[0,T]$ and each $\Psi_k$ takes only finitely many values. The space of such processes is denoted by $\Lambda_0^S$. Let $\psi\colon \mathcal{H} \to H$ be a Hilbert-Schmidt operator. Then $(L(t) - L(s))\,\psi^*$ is induced by an $H$-valued random variable, which we denote $J_{s,t}(\psi)$ or simply $(L(t) - L(s))\,\psi^*$. In this way $\langle J_{s,t}(\psi), h\rangle_H = (L(t) - L(s))\,(\psi^* h)$ for all $h \in H$. Let

$$\Psi_k = \sum_{j=1}^{m_k} \psi_{k,j} \mathbb{1}_{A_{k,j}}, \tag{3.4}$$

where for each $k$ the sets $A_{k,1}, \ldots, A_{k,m_k} \in \mathcal{F}_{t_k}$ form a partition of $\Omega$. Then we define

$$J_{s,t}(\Psi_k) = \sum_{j=1}^{m_k} \mathbb{1}_{A_{k,j}} J_{s,t}(\psi_{k,j})$$

and finally

$$\int_0^T \Psi(t)\,\mathrm{d}L(t) := \sum_{k=0}^{N-1} J_{t_k,t_{k+1}}(\Psi_k).$$

**Lemma 3.4.** Suppose that $L$ is weakly mean-zero weakly square-integrable cylindrical Lévy process. Then for $\Psi \in \Lambda_0^S$ we have

$$\mathbb{E}\left[\left\|\int_0^T \Psi(s)\,\mathrm{d}L(s)\right\|^2\right] = \mathbb{E}\left[\int_0^T \left\|\Psi(s)Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \mathrm{d}s\right]. \tag{3.5}$$

*Proof.* The result follows from [83, Cor. 4.3] by observing that, if $Q_W$ denotes the covariance operator of the Gaussian part of $L$, then

$$\langle Qu, v\rangle = \langle Q_W u, v\rangle + \int_U \langle u, x\rangle\langle v, x\rangle\,\nu(\mathrm{d}x), \qquad u, v \in U. \qquad \square$$

Let $\Lambda^2(0,T; L_{\mathrm{HS}}(\mathcal{H}, H))$ be the space of equivalence classes of processes taking values in the space of the Hilbert-Schmidt operators from $\mathcal{H}$ to $H$, which are predictable and satisfy

$$\mathbb{E}\left[\int_0^T \left\|\Psi(s)Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \mathrm{d}s\right] < \infty.$$

It is naturally equipped with the norm

$$\|\Psi\|_\Lambda = \left(\int_0^T \mathbb{E}\left[\left\|\Psi(s)Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)}\right] \mathrm{d}s\right)^{1/2}.$$

By Proposition 4.22 in [26], $\Lambda_0^S$ is a dense subspace of $\Lambda^2(0,T; L_{\mathrm{HS}}(\mathcal{H}, H))$. The Itô isometry in Lemma 3.4 shows that the mapping

$$\Lambda_0^S \ni \Psi \mapsto \int_0^T \Psi(s)\,\mathrm{d}L(s) \in L^2(\Omega, \mathcal{F}, P; H) \tag{3.6}$$

is continuous, if $\Lambda_0^S$ is equipped with the norm $\|\cdot\|_\Lambda$. We extend the mapping (3.6) to $\Lambda^2(0,T; L_{\mathrm{HS}}(\mathcal{H}, H))$ by continuity. One obtains by repeating the proof of [83, Cor. 4.4] the following:

**Theorem 3.5.** *For any weakly square-integrable cylindrical Lévy process $L$ and for any $\Psi \in$*

$\Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$ *the process*

$$\left(I(t) := \int_0^t \Psi(s)\,\mathrm{d}L(s) : t\in[0,T]\right)$$

*has a càdlàg modification and if $L$ is a cylindrical martingale, then the process $I$ is càdlàg, mean-zero, square-integrable martingale and* (3.5) *holds.*

The above construction is straightforward and natural. It was however pointed to us by the referee of [52] that in the case when $Q$ is diagonal one can reduce the problem of defining the integral with respect to weakly square-integrable cylindrical Lévy process to the case of integration with respect to a classical process. Let $(e_n)$ be an orthonormal basis consisting of eigenvectors of $Q$ with the corresponding sequence of eigenvalues $(\gamma_n)$ i.e. $Qe_n = \gamma_n e_n$. Let

$$\tilde{L}(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2^n}}(L(t)e_n)e_n.$$

Note that the series converges in $L^2(\Omega;U)$ since

$$\mathbb{E}\left[\left\|\sum_{k=n}^{m}\frac{1}{\sqrt{2^k}}(L(t)e_k)e_k\right\|^2\right] = \mathbb{E}\left[\sum_{k=n}^{m}\frac{1}{2^k}(L(t)e_k)^2\right] = \sum_{k=n}^{m}\frac{1}{2^k}\left\|Q^{1/2}e_k\right\|^2 \le \left\|Q^{1/2}\right\|^2\sum_{k=n}^{m}\frac{1}{2^k},$$

which converges to 0 as $n,m\to\infty$.

Let $\tilde{Q}$ be the covariance operator of $\tilde{L}$ and let $\tilde{\mathcal{H}}$ be the reproducing kernel Hilbert space associated to $\tilde{L}$. We have

$$\langle \tilde{Q}e_k, e_n\rangle = \mathbb{E}\left[\langle \tilde{L}(1), e_k\rangle\langle \tilde{L}(1), e_n\rangle\right] = \frac{1}{\sqrt{2^k 2^n}}\mathbb{E}\left[L(t)e_k L(t)e_n\right] = \frac{1}{\sqrt{2^k 2^n}}\langle Qe_k, e_n\rangle.$$

It follows that $\tilde{Q}e_n = \frac{1}{2^n}\gamma_n e_n$. For $\psi\in L_{\mathrm{HS}}(\mathcal{H},H)$ define

$$\tilde{\psi}(u) = \sum_{n=1}^{\infty}\langle u, e_n\rangle\sqrt{2^n}\psi(e_n) \tag{3.7}$$

for $u\in\tilde{\mathcal{H}}$. We have

$$\left\|\tilde{\psi}\tilde{Q}^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} = \sum_{n=1}^{\infty}\left\|\tilde{\psi}\tilde{Q}^{1/2}e_n\right\|^2 = \sum_{n=1}^{\infty}\frac{1}{2^n}\gamma_n\left\|\tilde{\psi}(e_n)\right\|^2$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \gamma_n \left\| \sqrt{2^n} \psi(e_n) \right\|^2 = \sum_{n=1}^{\infty} \gamma_n \|\psi(e_n)\|^2 = \sum_{n=1}^{\infty} \left\| \psi Q^{1/2}(e_n) \right\|^2 = \left\| \psi Q^{1/2} \right\|^2_{L_{\mathrm{HS}}(U,H)},$$

which shows that the series in (3.7) indeed converges for $u \in \tilde{\mathcal{H}}$ and that $\tilde{\psi} \in L_{\mathrm{HS}}(\tilde{\mathcal{H}}, H)$ if and only if $\psi \in L_{\mathrm{HS}}(\mathcal{H}, H)$ with equal norms.

We have for $h \in H$

$$\langle \tilde{\psi}(\tilde{L}(t)), h \rangle = \sum_{n=1}^{\infty} \langle \tilde{L}(t), e_n \rangle \sqrt{2^n} \langle \psi(e_n), h \rangle = \sum_{n=1}^{\infty} L(t) e_n \langle \psi(e_n), h \rangle.$$

On the other hand

$$\langle J_t(\psi), h \rangle = L(t)(\psi^* h) = \sum_{n=1}^{\infty} L(t) e_n \langle \psi^* h, e_n \rangle.$$

Thus $\tilde{\psi}(\tilde{L}(t)) = J_t(\psi)$ and it follows that

$$\int_0^T \Psi(s) \, \mathrm{d}L(s) = \int_0^T \tilde{\Psi}(s) \, \mathrm{d}\tilde{L}(s).$$

for simple processes $\Psi \in \Lambda_0^S$. By continuity, both integrals coincide for any process $\Psi \in \Lambda^2(0, T; L_{\mathrm{HS}}(\mathcal{H}, H))$.

## 3.3 Jumps of cylindrical Lévy processes

Usually, constructing a solution to a stochastic differential equation with a non-integrable noise requires one to carry out an analysis of jumps of the noise, see e.g. [74, Sec. 9.7] or [47, Th. IV.9.1]. In what follows we show that the behaviour of the jumps of a cylindrical process is much more irregular than those of a classical process. This is due to the fact that their jumps accumulate at zero, or more precisely that the stopping times used in the interlacing technique are equal to 0 a.s. A similar phenomenon was observed by Balan [5, Rem. 6.7] in the random field setting, which arose due to the unboundedness of the domain.

For a bounded sequence of positive real numbers $c = (c_j)$ we define the sequence of stopping times by

$$\tau_{c,n}(k) := \inf \left\{ t \geq 0 : \sum_{j=1}^{n} (\Delta L(t) e_j)^2 \, c_j^2 > k^2 \right\} \qquad \text{for each } k > 0, n \in \mathbb{N}.$$

The stopping time $\tau_{c,n}(k)$ is the first time that the $n$-dimensional Lévy process

$$\big((L(t)(c_1e_1),\dots,L(t)(c_ne_n)) : \, t\ge 0\big)$$

has a jump of size larger than $k$. Since $\tau_{c,n}(k)$ is non-increasing in $n$, we can define another sequence of stopping times by

$$\tau_c(k) := \lim_{n\to\infty} \tau_{c,n}(k) \qquad \text{for } k>0. \tag{3.8}$$

In what follows we show that the distribution of the stopping time $\tau_c(k)$ depends on

$$m_c(k) := \sup_{n\in\mathbb{N}} \nu\Bigg(\bigg\{u\in U : \sum_{j=1}^n \langle u,e_j\rangle^2 c_j^2 > k^2\bigg\}\Bigg) \qquad \text{for } k>0, \tag{3.9}$$

where $\nu$ is the cylindrical Lévy measure of $L$.

**Proposition 3.6.** Fix a cylindrical Lévy process $L$ and $c\in\ell^\infty(\mathbb{R}_+)$.

(1) We have for each $k>0$:

  (i) $m_c(k)=0 \Leftrightarrow \tau_c(k)=\infty$ $P$-a.s;

  (ii) $m_c(k)\in(0,\infty) \Leftrightarrow \tau_c(k)$ is exponentially distributed with parameter $m_c(k)$;

  (iii) $m_c(k)=\infty \Leftrightarrow \tau_c(k)=0$ $P$-a.s.

(2) We have: $\lim\limits_{k\to\infty} m_c(k)=0 \Leftrightarrow \lim\limits_{k\to\infty}\tau_c(k)=\infty$ $P$-a.s.

*Proof.* (1) Define the mapping

$$\pi_{c,n}\colon U\to U, \qquad \pi_{c,n}(u)=\sum_{j=1}^n c_j\langle u,e_j\rangle e_j.$$

Then $\tau_{c,n}(k)$ is the time of the first jump of size larger than $k$ of the genuine Lévy process $L_{c,n}$ defined by

$$L_{c,n}(t)=\sum_{j=1}^n c_jL(t)(e_j)e_j, \qquad t\ge 0.$$

As the Lévy measure $\nu_{c,n}$ of $L_{c,n}$ is given by $\nu_{c,n} := \big(\nu\circ(\pi_{c,n})^{-1}\big)|_{U\setminus\{0\}}$, the stopping time

$\tau_{c,n}(k)$ is exponentially distributed with parameter

$$\lambda_{c,n}(k) := \nu_{c,n}\big(\{u \in U : \|u\| > k\}\big) = \nu\bigg(\bigg\{u \in U : \sum_{j=1}^{n} c_j^2 \langle u, e_j\rangle^2 > k^2\bigg\}\bigg).$$

We see that $\lambda_{c,n}(k) \nearrow m_c(k)$ as $n \to \infty$.

(i): Clearly, $m_c(k) = 0$ if and only if $\lambda_n(k) = 0$ for all $n \in \mathbb{N}$. Equivalently, $L_{c,n}$ has no jumps larger than $k$ for all $n \in \mathbb{N}$. This means precisely that $\tau_{c,n}(k) = \infty$ a.s. for all $n$ and consequently that $\tau_c(k) = \infty$ a.s.

(ii), (iii): the characteristic function $\varphi_{\tau_{c,n}(k)}$ of $\tau_{c,n}(k)$ is given by

$$\varphi_{\tau_{c,n}(k)} \colon \mathbb{R} \to \mathbb{C}, \qquad \varphi_{\tau_{c,n}(k)}(x) = \frac{\lambda_n(k)}{\lambda_n(k) - ix}.$$

The characteristic function $\varphi_{\tau_{c,n}(k)}$ converges to the characteristic function either of the exponential distribution with parameter $m_c(k)$ (in the case (ii)) or of the Dirac measure in 0 (in the case (iii)).

For establishing (2), note that monotonicity of $k \mapsto \tau_c(k)$ yields

$$P\Big(\lim_{k\to\infty} \tau_c(k) = \infty\Big) = P\bigg(\bigcap_{t\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcap_{l\geq k} \{\tau_c(l) > t\}\bigg) = \lim_{t\to\infty} \lim_{k\to\infty} P\left(\tau_c(k) > t\right).$$

Since $P(\tau_c(k) > t) = \exp(-t m_c(k))$, the proof of (2) is completed. □

Note that if $\nu$ is a Lévy measure of a Hilbert space-valued Lévy process, then

$$m_c(k) = \nu\bigg(\bigg\{u \in U : \sum_{j=1}^{\infty} \langle u, e_j\rangle^2 c_j^2 > k\bigg\}\bigg). \tag{3.10}$$

For $c_j = 1, j \in \mathbb{N}$, we get an even simpler representation: $m_c(k) = \nu(U \setminus B_U(0,k))$. In this case it is well known that the time of the first jump of size larger than $k$

$$\sigma(k) := \inf\{t \geq 0 : \|\Delta L(t)\| > k\}$$

has exponential distribution with parameter $m_c(k)$, see [92, Th. 21.3]. Moreover, since the Lévy measure is finite outside any neighbourhood of 0 one obtains that $m_c(k) \to 0$ as $k \to \infty$. This shows that $\sigma(k) \to \infty$ a.s.

On the other hand, if the noise is cylindrical the stopping times $\tau_{c,n}(k)$ may accumulate at zero, i.e. $\tau_c(k) = 0$ $P$-a.s. Also, when $\nu$ is a cylindrical measure we must use the formula (3.9) rather than (3.10) as the set $\left\{u \in U : \sum_{j=1}^{\infty} \langle u, e_j\rangle^2 c_j^2 > k\right\}$ is not cylindrical.

**Example 3.7.** The most natural, that is constant, truncation level is not suitable for most non-integrable processes. For instance let $L$ be the canonical stable cylindrical Lévy process on $U$ from Example 2.6. Then as stated in [85] (see the proof of Th. 5.1)

$$\lim_{n\to\infty} \nu\left(\left\{u \in U : \sum_{j=1}^{n} \langle u, e_j\rangle^2 > k^2\right\}\right) = \lim_{n\to\infty} \frac{1}{c_\alpha k} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)} = \infty.$$

An example where $c_j$ can be taken constant will be presented in the Example 4.17.

**Remark 3.8.** A similar technique was used in [22] in the random field formulation to deal with the unboundedness of the domain $\mathcal{O}$ on which an SPDE is considered. Let $N$ be a random measure on $\mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}$. The author adjusts the threshold for the size of the jump according to its location:

$$\tau(k) := \inf\left\{t \geq 0 : \int_0^t \int_{\mathcal{O}} \int_{\mathbb{R}} \mathbb{1}_{\{|z|>kh(x)\}} \tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}z) \neq 0\right\},$$

for $h(x) = 1 + |x|^\eta$, This enables the author to obtain a sequence $\tau(k)$, which converges to $+\infty$ as $k \to \infty$.

## 3.4 Diagonal cylindrical Lévy processes

A typical case of a diagonal Lévy process of Example 2.3 is when the components $\ell_k$ have same distribution up to constants i.e. when $\ell_k = \sigma_k h_k$ for $(\sigma_k) \subset \mathbb{R}$ and a sequence of independent identically distributed Lévy processes $(h_k)$.

**Lemma 3.9.** Let

$$L(t)u = \sum_{k=1}^{\infty} \langle u, e_k\rangle \sigma_k h_k(t), \qquad u \in U,$$

where $h_k$ are independent identically distributed symmetric square-integrable non-trivial Lévy martingales in $\mathbb{R}$ with characteristics $(0, s_k, \rho)$.

(i) $L$ is a weakly square-integrable cylindrical Lévy process if and only if the sequences $(\sigma_k)$ and $(s_k)$ are bounded.

(ii) $L$ is induced by a $U$-valued Lévy process if and only if $(\sigma_k)$ and $(s_k)$ belong to $\ell^2(\mathbb{R})$.

*Proof.* The Gaussian case has been considered in [84, Ex. 4.3]. Therefore we consider only the case when $h_k$ are purely discontinuous non-trivial Lévy processes.

(i): Suppose that $(\sigma_k)$ is bounded. In order to show that $L$ is a cylindrical Lévy process we verify conditions (2.4)–(2.6). Condition (2.4) is satisfied due to symmetry. We only need to verify condition (2.6). Note that the Lévy measure of $\sigma_k h_k$ is given by $\rho_j = \rho \circ m_{\sigma_j}^{-1}$, where $m_{\sigma_j} \colon \mathbb{R} \to \mathbb{R}$ is defined by $m_{\sigma_j}(x) = \sigma_j x$. Then

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \left( |\alpha_k x|^2 \wedge 1 \right) \rho_k(\mathrm{d}x) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left( |\alpha_k \sigma_k x|^2 \wedge 1 \right) \rho(\mathrm{d}x) \le \int_{\mathbb{R}} x^2 \, \rho(\mathrm{d}x) \sum_{k=1}^{\infty} \alpha_k^2 \sigma_k^2 < \infty.$$

The characteristic functions of $\sigma_k h_k(1)$ satisfy

$$\varphi_{\sigma_k h_k(1)}(x) = \mathbb{E}\left[ e^{ix\sigma_k h_k(1)} \right] = \varphi_{h_1(1)}(\sigma_k x), \qquad \text{for } i \in \mathbb{N} \text{ and } x \in \mathbb{R}.$$

Since $\varphi_{h_1(1)}$ is continuous at 0, for any $\varepsilon > 0$ we can find $\delta > 0$ such that $\left|1 - \varphi_{h_1(1)}(x)\right| < \varepsilon$ for $|x| < \delta$. Then for $|x| < \delta' := \delta / \|\sigma\|_{\ell^\infty}$ we have $\left|1 - \varphi_{\sigma_i h_i(1)}(x)\right| = \left|1 - \varphi_{h_1(1)}(\sigma_i x)\right| < \varepsilon$ for $|x| < \delta'$. This proves the equicontinuity of $\varphi_{\sigma_i h_i(1)}$ at 0. Thus $L(t)$ is well defined and continuous by [84, Lem. 4.2]

Secondly we check that it is weakly square-integrable. Fix $u \in U$ and let

$$S_n = \sum_{k=1}^{n} \ell_k(t) \langle u, e_k \rangle, \qquad S = \sum_{k=1}^{\infty} \ell_k(t) \langle u, e_k \rangle.$$

Then

$$\mathbb{E}\left[S_n^2\right] = \sum_{k=1}^{n} \mathbb{E}\left[\ell_k^2(t)\right] \langle u, e_k \rangle^2 = \mathbb{E}\left[h_1^2(t)\right] \sum_{k=1}^{n} \langle u, e_k \rangle^2 \sigma_k^2 \le \mathbb{E}\left[h_1^2(t)\right] \sum_{k=1}^{\infty} \langle u, e_k \rangle^2 \sigma_k^2 < \infty.$$

Therefore the sequence $(S_n)$ is uniformly integrable and by the Vitali theorem $S$ is square-integrable and $\mathbb{E}\left[|S_n - S|^2\right] \to 0$.

Conversely, if $(\sigma_k)$ is unbounded then there exists a subsequence $(\sigma_{n_k})$ such that $(1/\sigma_{n_k})$ belongs to $\ell^2(\mathbb{R})$. For a sequence defined by

$$\alpha_i := \begin{cases} \frac{1}{\sigma_{n_k}}, & \text{if } i = n_k \text{ for some } k, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{k=1}^{\infty}\int_{\mathbb{R}}\Big(|\alpha_k x|^2\wedge 1\Big)\,\rho_k(\mathrm{d}x)=\sum_{k=1}^{\infty}\int_{\mathbb{R}}\Big(|\alpha_{n_k}\sigma_{n_k}x|^2\wedge 1\Big)\,\rho(\mathrm{d}x)=\sum_{k=1}^{\infty}\int_{\mathbb{R}}\big(x^2\wedge 1\big)\,\rho(\mathrm{d}x)=\infty.$$

(ii): Applying Priola and Zabczyk [77, Lem. 2.3], $L$ is is induced by a $U$-valued process if and only if

$$\sum_{k=1}^{\infty}\int_{\mathbb{R}}\big(x^2\wedge 1\big)\,\rho_k(\mathrm{d}x)<+\infty.$$

Now, if $\sigma\in\ell^2(\mathbb{R})$, then we estimate

$$\sum_{k=1}^{\infty}\int_{\mathbb{R}}\big(x^2\wedge 1\big)\,\rho_k(\mathrm{d}x)=\sum_{k=1}^{\infty}\int_{\mathbb{R}}\big((\sigma_k x)^2\wedge 1\big)\,\rho(\mathrm{d}x)\le\sum_{k=1}^{\infty}\sigma_k^2\int_{\mathbb{R}}x^2\,\rho(\mathrm{d}x)<\infty,$$

which proves that $L$ is induced by a $U$-valued process. Conversely, suppose that $(\sigma_n)\notin\ell^2(\mathbb{R})$ and let $a>0$ be such that $\rho(B_{\mathbb{R}}(0,a)^c)>0$. We have

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}\big((\sigma_n x)^2\wedge 1\big)\,\rho(\mathrm{d}x)\ge\sum_{n=1}^{\infty}\big((\sigma_n a)^2\wedge 1\big)\,\rho(B_{\mathbb{R}}(0,a)^c).$$

Now observe that either there is a subsequence $(\sigma_{n_k})$ with $|\sigma_{n_k}|\ge\frac{1}{a}$, in which case

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}\big((\sigma_n x)^2\wedge 1\big)\,\rho(\mathrm{d}x)\ge\sum_{k=1}^{\infty}\big((\sigma_{n_k} a)^2\wedge 1\big)\,\rho(B_{\mathbb{R}}(0,a)^c)\ge\sum_{k=1}^{\infty}\rho(B_{\mathbb{R}}(0,a)^c)=\infty,$$

or there exists $N\in\mathbb{N}$ such that for all $n\ge N$ we have $|\sigma_n|<\frac{1}{a}$, in which case

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}\big((\sigma_n x)^2\wedge 1\big)\,\rho(\mathrm{d}x)\ge\sum_{n=N}^{\infty}(\sigma_n a)^2\rho(B_{\mathbb{R}}(0,a)^c)=\infty$$

because $\sigma\notin\ell^2(\mathbb{R})$. □

**Lemma 3.10.** If $L$ is a diagonal cylindrical Lévy process on a Hilbert space $U$, then its cylindrical Lévy measure $\nu$ uniquely extends to a $\sigma$-finite measure on $\mathcal{B}(U)$ and is of the form

$$\nu(A)=\sum_{k=1}^{\infty}\rho_k\circ m_{e_k}^{-1}(A)\qquad\text{for all }A\in\mathcal{B}(U),\tag{3.11}$$

where $m_{e_k}\colon \mathbb{R}\to U$ is given by $m_{e_k}(x)=xe_k$.

*Proof.* It is shown in [84, Lem. 4.2] that the cylindrical Lévy measure $\nu$ of $L$ is given by (3.11) but as a finitely additive set function restricted to the cylindrical algebra $\mathcal{Z}(E)$. In order to show that $\nu$ is $\sigma$-additive, let $A_1, A_2, \dots \in \mathcal{Z}(E)$ be pairwise disjoint sets with $A := \bigcup_{k=1}^{\infty} A_k \in \mathcal{Z}(U)$. The Tonelli theorem implies

$$\begin{aligned}\nu(A) = \sum_{k=1}^{\infty} \rho_k \circ m_{e_k}^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \rho_k \circ m_{e_k}^{-1}(A_i)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \rho_k \circ m_{e_k}^{-1}(A_i) = \sum_{i=1}^{\infty} \nu(A_i).\end{aligned}$$

By the Carathéodory extension theorem (see [94, Th. 6.1]) $\nu$ extends to a measure on the $\sigma$-algebra generated by $\mathcal{Z}(U)$, which is equal to $\mathcal{B}(U)$ by [101, Th. I.2.1]. □

## 3.5 Angle bracket process

Recall that for a square-integrable càdlàg martingale $M$ in $H$, the angle bracket process $(\langle M, M\rangle(t) : t \ge 0)$ is defined as the unique increasing, predictable process such that $(\|M(t)\|^2 - \langle M, M\rangle(t) : t \ge 0)$ is a martingale; see e.g. [66, Sec. 20]. The operator-valued angle bracket $\langle\langle M, M\rangle\rangle$ is defined as the unique increasing, predictable process taking values in the space of non-negative, nuclear operators such that $\big(M(t)\otimes M(t) - \langle\langle M, M\rangle\rangle(t) : t \ge 0\big)$ is a martingale. The formula for the angle brackets of an integral with respect to a genuine Lévy martingale is well known, see e.g. [74, Cor. 8.17] or [66]. Now we establish the same result for the integral with respect to the cylindrical Lévy martingales.

**Proposition 3.11.** Let $L$ be a weakly mean-zero, weakly square-integrable cylindrical Lévy process. Take $\Psi \in \Lambda^2(0,T; L_{\mathrm{HS}}(\mathcal{H}, H))$ and let

$$I(t) := \int_0^t \Psi(s)\,\mathrm{d}L(s), \qquad \text{for } t \in [0,T].$$

Then

(i) $\langle I, I\rangle(t) = \displaystyle\int_0^t \left\|\Psi(s)Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \mathrm{d}s$ for all $t \in [0,T]$ $P$-a.s.

(ii) $\langle\langle I(\Psi), I(\Psi)\rangle\rangle(t) = \int_0^t \Psi(s)Q\Psi(s)^* \, \mathrm{d}s$ for all $t \in [0,T]$ $P$-a.s.

*Proof.* Note that $I$ is a càdlàg, mean-zero, square-integrable martingale by Theorem 3.5 and thus the angle brackets exist. We now prove part (i).

Let $\Psi \in \Lambda_0^S$ be given by (3.3) and (3.4). Our aim is to verify that the process $X$ defined by

$$X(t) := \left\| \int_0^t \Psi(s) \, \mathrm{d}L(s) \right\|_H^2 - \int_0^t \left\| \Psi(s) Q^{1/2} \right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s, \qquad t \in [0,T],$$

is a martingale. We first show that for any $h \in H$,

$$\left( \left\langle \int_0^t \Psi(s) \, \mathrm{d}L(s), h \right\rangle_H^2 - \int_0^t \left\| Q^{1/2} \Psi(s)^* h \right\|_U^2 \mathrm{d}s : t \in [0,T] \right) \tag{3.12}$$

is a martingale. Fix $0 \le s < t \le T$ and assume without loss of generality that $s = t_{k_0}$ for some $k_0$ and $t = t_{N_0}$ for some $N_0$. We have

$$\begin{aligned}
&\left\langle \int_0^t \Psi(s) \, \mathrm{d}L(s), h \right\rangle_H^2 \\
&= \left( \sum_{k=1}^{N_0-1} \sum_{i=1}^{m_k} \mathbb{1}_{A_{k,i}} (L(t_{k+1}) - L(t_k))(\psi_{k,i}^* h) \right)^2 \\
&= \sum_{k,l=1}^{N_0-1} \sum_{i=1}^{m_k} \sum_{j=1}^{m_l} \mathbb{1}_{A_{k,i}} \mathbb{1}_{A_{l,j}} (L(t_{k+1}) - L(t_k))(\psi_{k,i}^* h)(L(t_{l+1}) - L(t_l))(\psi_{l,j}^* h).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\left[ \mathbb{1}_{A_{i,k}} \mathbb{1}_{A_{j,l}} (L(t_{k+1}) - L(t_k))(\psi_{k,i}^* h)(L(t_{l+1}) - L(t_l))(\psi_{l,j}^* h) | \mathcal{F}_s \right] \\
&\quad = \begin{cases} \mathbb{1}_{A_{k,i}} \mathbb{1}_{A_{l,j}} (L(t_{k+1}) - L(t_k)(\psi_{k,i}^* h)(L(t_{l+1}) - L(t_l))(\psi_{l,j}^* h), & k, l < k_0, \\ \mathbb{1}_{A_{k,i}} (t_{k+1} - t_k) \langle Q \psi_{k,i}^* h, \psi_{k,i}^* h \rangle_U, & k = l \ge k_0 \text{ and } i = j, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus

$$\mathbb{E}\left[ \left\langle \int_0^t \Psi(s) \, \mathrm{d}L(s), h \right\rangle_H^2 \middle| \mathcal{F}_s \right]$$

$$= \sum_{k,l<k_0} \sum_{i=1}^{m_k} \sum_{j=1}^{m_l} \mathbb{1}_{A_{k,i}} \mathbb{1}_{A_{l,j}} (L(t_{k+1}) - L(t_k)(\psi_{k,i}^* h)(L(t_{l+1}) - L(t_l))(\psi_{l,j}^* h)$$

$$+ \sum_{k=k_0}^{N_0-1} \sum_{i=1}^{m_k} \mathbb{1}_{A_{k,i}} (t_{k+1} - t_k) \langle Q\psi_{k,i}^* h, \psi_{k,i}^* h\rangle_U$$

$$= \left\langle \int_0^s \Psi(r) \, \mathrm{d}L(r), h \right\rangle_H^2 + \int_s^t \langle Q\Psi(r)^*, \Psi(r)^*\rangle_U \, \mathrm{d}r,$$

which can be rewritten as

$$\mathbb{E}\left[ \left\langle \int_0^t \Psi(r) \, \mathrm{d}L(r), h \right\rangle_H^2 - \int_0^t \left\| Q^{1/2}\Psi(r)^* h \right\|_U^2 \mathrm{d}r \middle| \mathcal{F}_s \right]$$
$$= \left\langle \int_0^s \Psi(u) \, \mathrm{d}L(r), h \right\rangle_H^2 - \int_0^s \left\| Q^{1/2}\Psi(r)^* h \right\|_U^2 \mathrm{d}r$$

i.e. (3.12) is a martingale.

Let $(f_j)$ be an orthonormal basis of $H$. Substituting $h = f_j$ in (3.12) we obtain that

$$X_n(t) := \sum_{j=1}^n \left( \left\langle \int_0^t \Psi(s) \, \mathrm{d}L(s), f_j \right\rangle_H^2 - \int_0^t \| Q^{1/2}\Psi(s)^* f_j \|_U^2 \, \mathrm{d}s \right)$$

defines a martingale. Moreover, by the Parseval identity for each $t \in [0, T]$, almost surely $X_n(t)$ converges to $X(t)$. Note that $X_n$ is dominated by

$$\left\| \int_0^t \Psi(s) \, \mathrm{d}L(s) \right\|_H^2 + \int_0^t \left\| \Psi(s) Q^{1/2} \right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s,$$

which is integrable. By the Lebesgue dominated convergence theorem $X$ is a martingale.

Take $\Psi \in \Lambda^2(0, T; L_{\mathrm{HS}}(\mathcal{H}, H))$. There exists a sequence $(\Psi_n) \subset \Lambda_0^S$ such that $\|\Psi - \Psi_n\|_\Lambda \to 0$ and $\int_0^T \Psi_n(s) \, \mathrm{d}L(s) \to \int_0^T \Psi(s) \, \mathrm{d}L(s)$ in $L^2(\Omega, \mathcal{F}, P; H)$. Fix $0 \leq s \leq t$. We know that for $\Psi_n$ the integral process is a martingale i.e.

$$\mathbb{E}\left[ \left\| \int_0^t \Psi_n(r) \, \mathrm{d}L(r) \right\|_H^2 - \int_0^t \left\| \Psi_n(r) Q^{1/2} \right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}r \middle| \mathcal{F}_s \right]$$
$$= \left\| \int_0^s \Psi_n(r) \, \mathrm{d}L(r) \right\|_H^2 - \int_0^s \left\| \Psi_n(r) Q^{1/2} \right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}r. \quad (3.13)$$

Taking $n \to \infty$ gives that (3.13) holds with $\Psi_n$ replaced by $\Psi$.

We now show (ii). Taking $h = f_i + f_j$ and $h = f_i - f_j$ in (3.12), gives that

$$\begin{aligned}
&R(f_i, f_j)(t) \\
&:= \frac{1}{4}\left(\left\langle \int_0^t \Psi(s)\,\mathrm{d}L(s), f_i + f_j\right\rangle^2 - \int_0^t \left\|Q^{1/2}\Psi^*(s)(f_i + f_j)\right\|_U^2 \mathrm{d}s\right) \\
&\qquad - \frac{1}{4}\left(\left\langle \int_0^t \Psi(s)\,\mathrm{d}L(s), f_i - f_j\right\rangle^2 - \int_0^t \left\|Q^{1/2}\Psi^*(s)(f_i - f_j)\right\|_U^2 \mathrm{d}s\right) \\
&= \left\langle \int_0^t \Psi(s)\,\mathrm{d}L(s), f_i\right\rangle_H \left\langle \int_0^t \Psi(s)\,\mathrm{d}L(s), f_j\right\rangle_H - \int_0^t \langle Q^{1/2}\Psi(s)^* f_i, Q^{1/2}\Psi^*(s) f_j\rangle_U \,\mathrm{d}s
\end{aligned}$$

defines a martingale. Define for each $n \in \mathbb{N}$ a martingale $M_n$ by

$$M_n(t) := \sum_{i,j=1}^{n} R(f_i, f_j)(t) f_i \otimes f_j \qquad \text{for all } t \in [0, T].$$

We show that $M_n(t)$ converges to $M(t)$ in $L^1(\Omega; L_1(H))$ for each $t \in [0, T]$, where

$$M(t) := I(\Psi)(t) \otimes I(\Psi)(t) - \int_0^t \Psi(s) Q \Psi^*(s)\,\mathrm{d}s.$$

Observe that the following operators are non-negative

$$\begin{aligned}
A &:= h \otimes h - \sum_{i,j=1}^{n} \langle h, f_i\rangle_H \langle h, f_j\rangle_H f_i \otimes f_j = \sum_{i,j=n+1}^{\infty} \langle h, f_i\rangle_H \langle h, f_j\rangle_H f_i \otimes f_j, \\
B &:= \Psi(s) Q \Psi^*(s) - \sum_{i,j=1}^{n} \langle Q^{1/2}\Psi^*(s) f_i, Q^{1/2}\Psi^*(s) f_j\rangle_U f_i \otimes f_j \\
&= \sum_{i,j=n+1}^{\infty} \langle Q^{1/2}\Psi^*(s) f_i, Q^{1/2}\Psi^*(s) f_j\rangle_U f_i \otimes f_j.
\end{aligned}$$

Indeed for $x \in H$

$$\langle Ax, x\rangle_H = \sum_{i,j=n+1}^{\infty} \langle h, f_i\rangle_H \langle h, f_j\rangle_H \langle f_i, x\rangle_H \langle f_j, x\rangle_H = \langle \tilde{\pi}_n^{\perp} h, \tilde{\pi}_n^{\perp} x\rangle^2 \geq 0,$$

where $\tilde{\pi}_n^\perp$ is the projection onto $\operatorname{Span}(e_1\dots,e_n)^\perp$. For the second operator we have

$$\begin{aligned}
\langle Bx,x\rangle_H &= \sum_{i,j=n+1}^{\infty} \langle \Psi(s)Q\Psi^*(s)f_i,f_j\rangle_H \langle f_i,x\rangle\langle f_j,x\rangle \\
&= \left\langle Q^{1/2}\Psi^*(s)\sum_{i=n+1}^{\infty}\langle f_i,x\rangle_H f_i, Q^{1/2}\Psi^*(s)\sum_{j=n+1}^{\infty}\langle f_j,x\rangle_H f_j\right\rangle \\
&= \left\| Q^{1/2}\Psi^*(s)\tilde{\pi}_n^\perp x\right\|_U^2 \\
&\ge 0.
\end{aligned}$$

Thus the operators in the definition of $M_n(t)$ and $M(t)$ are non-negative and we have by [25, Th. 18.11(d)]

$$\begin{aligned}
&\|M(t)-M_n(t)\|_{L_1(H)} \\
&\le \left\| I(\Psi)(t)\otimes I(\Psi)(t) - \sum_{i,j=1}^{n} \langle I(t),f_i\rangle_H \langle I(t),f_j\rangle_H f_i\otimes f_j\right\|_{L_1(H)} \\
&\quad + \left\| \int_0^t \Psi(s)Q\Psi^*(s)\,\mathrm{d}s - \int_0^t \langle Q^{1/2}\Psi(s)^* f_i, Q^{1/2}\Psi^*(s)f_j\rangle_U \,\mathrm{d}s f_i\otimes f_j\right\|_{L_1(H)} \\
&= \operatorname{Tr}\left( I(\Psi)(t)\otimes I(\Psi)(t) - \sum_{i,j=1}^{n}\langle I(\Psi)(t),f_i\rangle_H\langle I(\Psi)(t),f_j\rangle_H f_i\otimes f_j\right) \\
&\qquad + \operatorname{Tr}\left(\int_0^t \Psi(s)Q\Psi^*(s)\,\mathrm{d}s - \sum_{i,j=1}^{n}\int_0^t \langle Q^{1/2}\Psi^*(s)f_i, Q^{1/2}\Psi^*(s)f_j\rangle_U\,\mathrm{d}s f_i\otimes f_j\right) \\
&= \sum_{k=n+1}^{\infty}\left( \langle I(\Psi)(t),f_k\rangle_H^2 + \int_0^t \left\| Q^{1/2}\Psi^*(s)f_k\right\|_U^2 \mathrm{d}s\right).
\end{aligned}$$

The Lebesgue theorem shows that $\mathbb{E}\left[\|M(t)-M_n(t)\|_{L_1(H)}\right]$ converges to 0, which establishes that $(M(t):t\in[0,T])$ is a martingale. □

Recall for the following result that $I(\Psi)$ defines a square-integrable martingale in $H$ for each $\Psi\in\Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$. Stochastic integration with respect to such martingales is introduced for example in [66] or [74].

**Lemma 3.12.** Let $L$ be a weakly mean-zero, weakly square-integrable cylindrical Lévy process with covariance operator $Q$ and let $\Psi\in\Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$. If $V$ is another separable Hilbert space and $\Theta$ is an $\mathcal{L}(H,V)$-valued stochastic process for which the stochastic integral

$$N(t) := \int_0^t \Theta(s)\,\mathrm{d}I(\Psi)(s) \qquad \text{for } t\in[0,T],$$

exists in the sense of [74, Sec. 8.2], then we have

$$\langle N,N\rangle(t) = \int_0^t \big\|\Theta(s)\big(\Psi(s)Q\Psi^*(s)\big)^{1/2}\big\|^2_{L_{\rm HS}(H,V)}\,\mathrm{d}s.$$

*Proof.* Since $I(\Psi)$ is an $H$-valued martingale, Theorem 8.2 in [74] guarantees that there exists the so-called martingale covariance of $I(\Psi)$, that is a predictable stochastic process $(C(t): t\geq 0)$ in the space of symmetric, non-negative, nuclear operators on $U$, such that

$$\big\langle\big\langle I(\Psi),I(\Psi)\big\rangle\big\rangle(t) = \int_0^t C(s)\,\mathrm{d}\big\langle I(\Psi),I(\Psi)\big\rangle(s) \qquad \text{for all } t\in[0,T] \ P\text{-a.s.}$$

By Part (i) of Proposition 3.11 we conclude that

$$\big\langle\big\langle I(\Psi),I(\Psi)\big\rangle\big\rangle(t) = \int_0^t C(s)\big\|\Psi(s)Q^{1/2}\big\|^2_{L_{\rm HS}(U,H)}\,\mathrm{d}s.$$

By comparing with the formula in Part (ii) of Proposition 3.11 we obtain

$$C(s) = \frac{\Psi(s)Q\Psi^*(s)}{\big\|\Psi(s)Q^{1/2}\big\|^2_{L_{\rm HS}(U,H)}}, \qquad \text{if } \Psi(s)Q^{1/2}\neq 0$$

and $C(s)=0$ otherwise. Applying [74, Th. 8.7(iv)] and Part (i) of Proposition 3.11 results in

$$\begin{aligned}\langle N,N\rangle(t) &= \int_0^t \left\|\Theta(s)\left(\frac{\Psi(s)Q\Psi^*(s)}{\big\|\Psi(s)Q^{1/2}\big\|^2_{L_{\rm HS}(U,H)}}\right)^{1/2}\right\|^2_{L_{\rm HS}(H,V)} \mathrm{d}\big\langle I(\Psi),I(\Psi)\big\rangle(s)\\ &= \int_0^t \left\|\Theta(s)\left(\frac{\Psi(s)Q\Psi^*(s)}{\big\|\Psi(s)Q^{1/2}\big\|^2_{L_{\rm HS}(U,H)}}\right)^{1/2}\right\|^2_{L_{\rm HS}(H,V)} \big\|\Psi(s)Q^{1/2}\big\|^2_{L_{\rm HS}(U,H)}\,\mathrm{d}s\\ &= \int_0^t \big\|\Theta(s)\big(\Psi(s)Q\Psi^*(s)\big)^{1/2}\big\|^2_{L_{\rm HS}(H,V)}\,\mathrm{d}s,\end{aligned}$$

which completes the proof. ☐

## 3.6 Stopping times

In the proof of existence of solution we will need the following lemma concerning the stochastic integral with respect to cylindrical Lévy processes and stopping times.

**Lemma 3.13.** Let $L$ be a weakly square-integrable cylindrical Lévy process and let $\Psi \in \Psi \in \Lambda^2(0,T;L_{\text{HS}}(\mathcal{H},H))$. Let $\tau$ be a stopping time with $P(\tau \le T) = 1$. Then

$$\int_0^{t\wedge\tau} \Psi(s)\,\mathrm{d}L(s) = \int_0^t \Psi(s)\mathbb{1}_{\{s\le\tau\}}\,\mathrm{d}L(s) \qquad \text{for all } t\in[0,T] \ P\text{-a.s.}. \tag{3.14}$$

*Proof.* The proof follows closely [76, Lem. 2.3.9], where the case of Hilbert space-valued Wiener process is considered.

Step 1. Suppose that $\Psi$ is simple given by (3.3) and that the stopping time is simple i.e. takes only finitely many values:

$$\tau = \sum_{j=1}^m a_j \mathbb{1}_{A_j},$$

with $A_j = \{\tau = a_j\} \in \mathcal{F}_{a_j}$. Then

$$\int_0^{t\wedge\tau} \Psi(s)\,\mathrm{d}L(s) = \sum_{j=1}^m \mathbb{1}_{A_j}\int_0^{t\wedge a_j} \Psi(s)\,\mathrm{d}L(s) = \sum_{j=1}^m \mathbb{1}_{A_j}\sum_{k=1}^{N-1} J_{t_k\wedge t\wedge a_j,t_{k+1}\wedge t\wedge a_j}(\Psi_k). \tag{3.15}$$

Proving first for simple and then passing to the limit we get that for any process $\Psi \in \Lambda^2(0,T;L_{\text{HS}}(\mathcal{H},H))$ and any bounded, real-valued, $\mathcal{F}_s$-measurable random variable $\Phi$ we have almost surely

$$\int_s^t \Phi\Psi(r)\Phi\,\mathrm{d}L(r) = \Phi\int_s^t \Psi(r)\,\mathrm{d}L(r). \tag{3.16}$$

Using this we obtain

$$\begin{aligned}
\int_0^t \Psi(s)\mathbb{1}_{\{s\le\tau\}}\,\mathrm{d}L(s) &= \int_0^t \Psi(s)(1-\mathbb{1}_{\{\tau<s\}})\,\mathrm{d}L(s)\\
&= \int_0^t \Psi(s)\,\mathrm{d}L(s) - \int_0^t \Psi(s)\mathbb{1}_{\{\tau<s\}}\,\mathrm{d}L(s)\\
&= \sum_{k=1}^{N-1} J_{t_k\wedge t,t_{k+1}\wedge t}(\Psi_k) - \int_0^t \Psi(s)\sum_{j:a_j<t}\mathbb{1}_{A_j}\mathbb{1}_{\{a_j<s\}}\,\mathrm{d}L(s)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{N-1} J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - \sum_{j:a_j<t} \int_{a_j}^{t} \Psi(s)\mathbb{1}_{A_j}\,\mathrm{d}L(s) \\
&= \sum_{k=1}^{N-1} J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - \sum_{j:a_j<t} \mathbb{1}_{A_j} \int_{a_j}^{t} \Psi(s)\,\mathrm{d}L(s) \\
&= \sum_{k=1}^{N-1} J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - \sum_{j=1}^{m} \mathbb{1}_{A_j} \sum_{k=1}^{N-1} J_{(t_k\wedge t)\vee a_j, (t_{k+1}\wedge t)\vee a_j}(\Psi_k) \\
&= \sum_{k=1}^{N-1} \sum_{j=1}^{m} \mathbb{1}_{A_j} \left( J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - J_{(t_k\wedge t)\vee a_j, (t_{k+1}\wedge t)\vee a_j}(\Psi_k) \right).
\end{aligned}
$$

Comparing with (3.15), must prove that

$$
J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - J_{(t_k\wedge t)\vee a_j, (t_{k+1}\wedge t)\vee a_j}(\Psi_k) = J_{t_k\wedge t\wedge a_j, t_{k+1}\wedge t\wedge a_j}(\Psi_k).
$$

For $a_j \le t_k \wedge t$ it simplifies to

$$
J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) = J_{a_j, a_j}(\Psi_k).
$$

For $t_k \wedge t < a_j < t_{k+1} \wedge t$ it simplifies to

$$
J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - J_{a_j, t_{k+1}\wedge t}(\Psi_k) = J_{t_k\wedge t, a_j}(\Psi_k).
$$

For $a_j \ge t_{k+1} \wedge t$ it simplifies to

$$
J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k) - J_{a_j, a_j}(\Psi_k) = J_{t_k\wedge t, t_{k+1}\wedge t}(\Psi_k).
$$

Step 2. Take simple $\Psi$ and a general stopping time. Then there exists a sequence of simple stopping times $(\tau_n)$ decreasing to $\tau$; see [26, Sec. 3.5]. By Step 1

$$
\int_0^{\tau_n\wedge t} \Psi(s)\,\mathrm{d}L(s) = \int_0^{t} \Psi(s)\mathbb{1}_{\{s\le\tau_n\}}\,\mathrm{d}L(s). \tag{3.17}
$$

Since the stochastic integral is càdlàg, it follows that

$$
\int_0^{\tau_n\wedge t} \Psi(s)\,\mathrm{d}L(s) \to \int_0^{\tau\wedge t} \Psi(s)\,\mathrm{d}L(s) \quad P\text{-a.s.}
$$

i.e. the left-hand side of (3.17) converges. Since $\mathbb{1}_{[0,\tau_n]}\Psi$ converges to $\mathbb{1}_{[0,\tau]}\Psi$ in the space $\Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$, it follows that the right-hand side of (3.17) converges.

Step 3. Now suppose that $\Psi \in \Psi \in \Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$ is general.

Take a sequence of simple integrands $\Psi_n$ converging to $\Psi$. Application of Proposition 3.1 gives that the process $L$ can be decomposed as

$$L(t)u = tB(u) + M(t)u, \qquad t \geq 0, u \in U, t \geq 0.$$

Recall from Section 2.3 that for stochastic integration we consider $L$ restricted to $\mathcal{H}$. It follows from the continuity of the embedding $\mathcal{H} \subset U$ that the operator $B$ is continuous on $\mathcal{H}$. By the Riesz theorem, there exists $h_0 \in \mathcal{H}$ such that $B(u) = \langle u, h_0 \rangle_{\mathcal{H}}$ for all $u \in \mathcal{H}$. Then

$$\int_0^t \Psi(s)\,\mathrm{d}L(s) = \int_0^t \Psi(s)h_0\,\mathrm{d}s + \int_0^t \Psi(s)\,\mathrm{d}M(s)$$

and the stochastic integral on the right-hand side is a martingale by Theorem 3.5. By the Doob inequality

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\int_0^t \Psi_n(s)\,\mathrm{d}M(s) - \int_0^t \Psi(s)\,\mathrm{d}M(s)\right\|_H^2\right] \\
&\leq 4\mathbb{E}\left[\left\|\int_0^T \Psi_n(s) - \Psi(s)\,\mathrm{d}M(s)\right\|_H^2\right] \\
&= 4\mathbb{E}\left[\int_0^T \left\|(\Psi_n(s) - \Psi(s))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right],
\end{aligned}$$

which converges to 0 as $n \to \infty$. For the Bochner integral we estimate using that $\|\cdot\|_{\mathcal{L}(\mathcal{H},H)} \leq \|\cdot\|_{L_{\mathrm{HS}}(\mathcal{H},H)}$

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\left\|\int_0^t \Psi_n(s)h_0\,\mathrm{d}s - \int_0^t \Psi(s)h_0\,\mathrm{d}s\right\|_H^2\right] &\leq T\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t \left\|(\Psi_n(s) - \Psi(s))\,h_0\right\|_H^2 \mathrm{d}s\right] \\
&= T\mathbb{E}\left[\int_0^T \left\|(\Psi_n(s) - \Psi(s))\,h_0\right\|_H^2 \mathrm{d}s\right] \\
&\leq T\|h_0\|_{\mathcal{H}}^2\mathbb{E}\left[\int_0^T \left\|\Psi_n(s) - \Psi(s)\right\|_{L_{\mathrm{HS}}(\mathcal{H},H)}^2 \mathrm{d}s\right],
\end{aligned}$$

which converges to 0 as $n \to \infty$. Therefore there is a subsequence $(n_k)$ such that outside a null set:

$$\int_0^t \Psi_{n_k}(s)\,\mathrm{d}L(s) \to \int_0^t \Psi(s)\,\mathrm{d}L(s) \text{ for all } t \in [0,T]$$

and finally that almost surely

$$\int_0^{t\wedge\tau} \Psi_{n_k}(s)\,\mathrm{d}L(s) \to \int_0^{t\wedge\tau} \Psi(s)\,\mathrm{d}L(s)$$

Since $\mathbb{1}_{[0,\tau]}\Psi_n \to \mathbb{1}_{[0,\tau]}\Psi$ in $\Psi \in \Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$ it follows that

$$\int_0^t \Psi_n(s)\mathbb{1}_{\{s\le\tau\}}\,\mathrm{d}L(s) \to \int_0^t \Psi(s)\mathbb{1}_{\{s\le\tau\}}\,\mathrm{d}L(s),$$

which finishes the proof. □

# Chapter 4

# Variational solutions

In this chapter we deal with existence and uniqueness of solutions for an SPDE in the variational setting assuming the monotonicity and coercivity conditions. We start by discussing the literature related to our work and then describe the difficulties coming from the cylindrical noise and our techniques.

The variational approach to stochastic equations was developed by Krylov and Rozovskii [53] who considered an equation with a $U$-valued Wiener process. This was later generalised in various directions. Equations in Banach spaces driven by semimartingales were studied in a series of papers by Gyöngy and Krylov [41, 42, 43]. Equations with locally monotone coefficients were considered by Liu and Röckner [60] in the Gaussian setting. Brzeźniak, Liu and Zhu [16] proved existence and uniqueness of solution to an equation driven by a cylindrical Brownian motion and Poisson random measure also under the local monotonicity assumption. These more general conditions imposed on the operators in the equation allow the authors to extend the class of SPDEs covered by this approach and include for instance the 2D Navier–Stokes equation.

The precise formulation is as follows. Let $V$ be a Banach space embedded in $H$ and consider

$$\mathrm{d}X(t) = F(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}L(t), \tag{4.1}$$

where $F\colon V \to V^*$ and $G\colon V \to \mathcal{L}(U,H)$. In this thesis we assume the simplest version of monotonicity on $F$ and $G$ but consider cylindrical noise $L$. We start with presenting the equation driven by a cylindrical Lévy process with weak second moments. The solution is the limit of the standard Galerkin approximation

$$X_n(t) = \tilde{P}_n X_0 + \int_0^t \tilde{P}_n F(X_n(s))\,\mathrm{d}s + \int_0^t \tilde{P}_n G(X_n(s-))P_n\,\mathrm{d}L(s),$$

where $\tilde{P}_n$ and $P_n$ are projections onto $n$-dimensional subspaces. We obtain the a priori estimates of the $L^2$-norm of $X_n$. The Banach–Alaoglu theorem implies the weak convergence of a subsequence of $(X_n)$. Identifying the limit as the solution requires the formula for the angle bracket process of the stochastic integral with respect to a cylindrical process from Section 3.5 and a special version of the Itô formula for $\|X(t)\|_H^2$ from Gyöngy and Krylov [43], which takes into account that the drift $F$ in equation (4.1) is $V^*$-valued.

The situation is rather different without finite weak second moments. We obtain the result for the diagonal cylindrical Lévy processes of Example 2.3. For $L$ with this structure it is possible to derive a cylindrical decomposition from the one-dimensional decompositions. We identify the conditions on the Lévy characteristics, which make the processes resulting from this decomposition well-defined. However, the standard stopping times used in the construction of solutions driven by a genuine $U$-valued process $L$ defined by $\tau(k) = \inf\{t \geq 0 : \|\Delta L(t)\| > k\}$ make no sense for cylindrical process. Instead we use the stopping times already introduced in Section 3.6, which are a limit of the corresponding stopping times for the $n$-dimensional projections scaled down in higher dimensions:

$$\tau(k) := \lim_{n\to\infty} \inf\left\{t \geq 0 : \sum_{j=1}^{n} (\Delta\ell_j(t)e_j)^2\, c_j^2 > k^2\right\}.$$

Intuitively, the weights $(c_j)$ compensate for a too slow decay of the mass of the Lévy measures in higher dimensions. Choosing the right convergence rate for $(c_j)$ as the dimension $n$ increases to infinity is crucial and enables us to consider for example the $\alpha$-stable noise as well as processes with regularly varying tails.

## 4.1 Square-integrable case

Let $(V, \|\cdot\|_V)$ be a separable reflexive Banach space with the dual $V^*$. Assume that $V$ is densely and continuously embedded into $H$. That is we have a Gelfand triple

$$V \subseteq H = H^* \subseteq V^*.$$

Further, denote with ${}_{V^*}\langle\cdot,\cdot\rangle_V$ the duality pairing of $V^*$ and $V$. We have

$$ {}_{V^*}\langle h, v\rangle_V = \langle h, v\rangle_H, \qquad \text{for all } h\in H, v\in V $$

and without loss of generality we may assume that $\|v\|_H \le \|v\|_V$ for $v\in V$ and $\|h\|_{V^*} \le \|h\|_H$ for $h\in H$.

We consider the equation (4.1) with the initial condition $X(0) = X_0$ for a square-integrable, $\mathcal{F}_0$–measurable random variable $X_0$. The driving noise is a cylindrical Lévy process on a separable Hilbert space $U$. We assume that $L$ is a weakly mean-zero and weakly square-integrable. The coefficients in equation (4.1) are given by functions $F\colon V\to V^*$ and $G\colon V\to L_{\mathrm{HS}}(U,H)$. We assume (and do not repeat it later on in this section) that: there are constants $\alpha,\lambda,\beta,c>0$ such that:

(A1) (Coercivity) for all $v\in V$ we have

$$ 2_{V^*}\langle F(v), v\rangle_V + \left\|G(v)Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} + \alpha\|v\|_V^2 \le \lambda\|v\|_H^2 + \beta; $$

(A2) (Monotonicity) for all $v_1, v_2\in V$, we have

$$ 2_{V^*}\langle F(v_1) - F(v_2), v_1 - v_2\rangle_V + \left\|(G(v_1) - G(v_2))Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \le \lambda\|v_1 - v_2\|_H; $$

(A3) (Linear growth) $\|F(v)\|_{V^*} \le c(1+\|v\|_V)$ for all $v\in V$;

(A4) (Hemicontinuity) the mapping $\mathbb{R}\ni s\mapsto {}_{V^*}\langle F(v_1+sv_2), v_3\rangle_V$ is continuous for all $v_1, v_2, v_3\in V$.

(A5) The cylindrical Lévy process $L$ is weakly mean-zero and weakly square-integrable. Its covariance operator $Q$ has eigenvectors $(e_j)$ forming an orthonormal basis of $U$.

Conditions of this form appear in most of the papers mentioned in the introduction. We now give the definition of a solution to (4.1), similarly as in Prévôt and Röckner [76, Def. 4.2.1] or Brzeźniak, Liu and Zhu [16, Def. 1.1].

**Definition 4.1.** A *variational solution* of (4.1) is a pair $(X,\bar{X})$ of an $H$-valued, càdlàg adapted process $X$ and a $V$-valued, predicable process $\bar{X}$ such that

(i) $X$ equals $\bar{X}$ almost everywhere-Leb$\otimes P$;

(ii) $P$-a.s. $\int_0^T \|\bar{X}(t)\|_V \,\mathrm{d}t < \infty$;

(iii) $X(t) = X_0 + \int_0^t F(\bar{X}(s))\,\mathrm{d}s + \int_0^t G(\bar{X}(s))\,\mathrm{d}L(s)$ for all $t \in [0,T]$ $P$-a.s. (4.2)

We say that the solution is *pathwise unique* if any two variational solutions $(X, \bar{X})$ and $(Y, \bar{Y})$ satisfy

$$P\big(X(t) = Y(t) \text{ for all } t \in [0,T]\big) = 1.$$

Since we later consider the case of a driving noise without finite moments and thus the solution cannot be expected to have finite moments, we do not require finite expectation of the solution in Definition 4.1 in contrast to most literature.

The main result of this section is the following theorem.

**Theorem 4.2.** *Under Assumptions* (A1)–(A5)*, equation* (4.1)*, with an $\mathcal{F}_0$-measurable initial value $X_0$ such that $\mathbb{E}\left[\|X_0\|_H^2\right] < \infty$, has a unique variational solution $(X, \bar{X})$. Moreover, the solution satisfies*

$$\int_0^T \mathbb{E}\left[\|\bar{X}(s)\|_V^2\right] \mathrm{d}s < \infty.$$

The proof of 4.2 is given in a series of lemmas. Before proceeding to the proof we first give some preparatory results on the Itô formula for the square of the norm.

The Itô formula in infinite-dimensional spaces is discussed for example in Métivier [66, Th. 27.2]. The Itô formula for the square of the norm is of particular interest. It is discussed for instance in Peszat and Zabczyk [74, Lem. D.3]. We need however a more general result taking into account the Gelfand triple. The problem is that the integrand of the Lebesgue integral in (4.2) is $V^*$-valued. This version of the Itô formula was given in Krylov and Rozovskii [53, Th. I.3.1] and can be seen as a stochastic version of an earlier result by Lions, see e.g. [100, Lem. III.1.2]. See Prévôt and Röckner [76, Th. 4.2.5 and Rem. 4.2.8] for a more modern treatment. These formulas work for the Wiener integrals, because of the path continuity assumption. More general theorem can be found in Gyöngy and Krylov [43]. We present here without proof, Theorem 2 of [43].

**Theorem 4.3.** *Let $M$ be an $H$-valued, càdlàg, square-integrable martingale, $\Phi$ be a progressively measurable $V^*$-valued process, $X_0$ be an $\mathcal{F}_0$-measurable $H$-valued random variable and* *define*

$$X(t) := X_0 + \int_0^t \Phi(s)\,\mathrm{d}s + M(t) \qquad \textit{for all } t \in [0,T].$$

*If there exists a $V$-valued process $\bar{X}$ such that $X$ and $\bar{X}$ are equal* Leb $\otimes P$*-almost everywhere, then $X$ has $P$-a.s. $H$-valued càdlàg trajectories and satisfies*

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + 2\int_0^t {}_{V^*}\langle \Phi(s), \bar{X}(s)\rangle_V\,\mathrm{d}s + 2\int_0^t X(s-)\,\mathrm{d}M(s) + [M,M](t).$$

Note that in this result, the integral $\int_0^t X(s-)\,\mathrm{d}M(s)$ is real-value since we identify $H = L_{\mathrm{HS}}(H,\mathbb{R})$; see e.g. [66] or [74].

We now apply this important result in the case when $M$ is an integral with respect to a cylindrical Lévy process. We obtain a formula for the expectation of the square of the norm multiplied by an exponential function. This will allow us to complete certain estimates of the second moments using the monotonicity (A2) below.

**Corollary 4.4.** Let $\Phi$ be a $V^*$-valued predictable process and $\Psi \in \Psi \in \Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H},H))$

$$\mathbb{E}\left[\int_0^T \|\Phi(s)\|_{V^*}\,\mathrm{d}s\right] < \infty. \tag{4.3}$$

If the stochastic process $X$ defined by

$$X(t) = X_0 + \int_0^t \Phi(s)\,\mathrm{d}s + \int_0^t \Psi(s)\,\mathrm{d}L(s) \qquad \text{for } t \in [0,T]$$

has a Leb $\otimes P$-version $\bar{X}$, which belongs to $L^2([0,T]\times\Omega;V)$, then

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_H^2\right] < \infty, \tag{4.4}$$

and for each $\lambda \geq 0$ we have

$$\begin{aligned}&\mathbb{E}\left[e^{-\lambda t}\|X(t)\|_H^2\right] \qquad (4.5)\\ &= \mathbb{E}\left[\|X_0\|_H^2\right] + \mathbb{E}\left[\int_0^t e^{-\lambda s}\left(2_{V^*}\langle \Phi(s), \bar{X}(s)\rangle_V + \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X(s)\|_H^2\right)\mathrm{d}s\right].\end{aligned}$$

*Proof.* Define a martingale $I$ by $I(t) := \int_0^t \Psi(s)\,\mathrm{d}L(s)$ for $t \in [0,T]$. The Itô formula for real-valued processes together with Theorem 4.3 imply

$$\begin{aligned}\mathrm{d}\big(e^{-\lambda t}\|X(t)\|_H^2\big) &= e^{-\lambda t}\mathrm{d}\|X(t)\|_H^2 - \lambda e^{-\lambda t}\|X(t)\|_H^2\,\mathrm{d}t\\ &= e^{-\lambda t}\left(2{}_{V^*}\langle\Phi(t),\bar{X}(t)\rangle_V\,\mathrm{d}t + 2X(t-)\,\mathrm{d}I(t)\right)\\ &\quad + e^{-\lambda t}\mathrm{d}[I,I](t) - \lambda e^{-\lambda t}\|X(t)\|_H^2\,\mathrm{d}t. \end{aligned}\tag{4.6}$$

For establishing (4.4), define the stopping time $\tau^R := \inf\{t \geq 0 : \|X(t)\|_H > R\} \wedge T$ for some $R > 0$. Taking $\lambda = 0$ in (4.6) we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t<\tau^R}\|X(t)\|_H^2\right] &\leq \mathbb{E}\left[\|X_0\|_H^2\right] + 2\mathbb{E}\left[\sup_{t<\tau^R}\int_0^t {}_{V^*}\langle\Phi(s),\bar{X}(s)\rangle_V\,\mathrm{d}s\right]\\ &\quad + 2\mathbb{E}\left[\sup_{t<\tau^R}\int_0^t X(s-)\,\mathrm{d}I(s)\right] + \mathbb{E}\left[\sup_{t<\tau^R}[I,I](t)\right].\end{aligned}\tag{4.7}$$

We have

$$2\mathbb{E}\left[\sup_{t<\tau^R}\int_0^t {}_{V^*}\langle\Phi(s),\bar{X}(s)\rangle_V\,\mathrm{d}s\right] \leq \mathbb{E}\left[\int_0^T\left(\|\Phi(s)\|_{V^*}^2 + \big\|\bar{X}(s)\big\|_V^2\right)\mathrm{d}s\right]. \tag{4.8}$$

By inequality (14) in [46] for $p = 1$ we derive

$$\mathbb{E}\left[\sup_{t<\tau^R}\int_0^t X(s-)\,\mathrm{d}I(s)\right] \leq 3\mathbb{E}\left[\left(\left\langle\int X(s-)\,\mathrm{d}I(s), \int X(s-)\,\mathrm{d}I(s)\right\rangle(\tau^R)\right)^{1/2}\right].$$

Applying Lemma 3.12 and identifying $H = L_{\mathrm{HS}}(H,\mathbb{R})$ yields

$$\left\langle\int X(s-)\,\mathrm{d}I(s), \int X(s-)\,\mathrm{d}I(s)\right\rangle(\tau^R) = \int_0^{\tau^R}\big\|X(s-)(\Psi(s)Q\Psi^*(s))^{1/2}\big\|_H^2\,\mathrm{d}s.$$

Taking into account that pathwise $X(s-) = X(s)$ for almost all $s \in [0,T]$ we conclude

$$\mathbb{E}\left[\sup_{t<\tau^R}\int_0^t X(s-)\,\mathrm{d}I(s)\right] \leq 3\mathbb{E}\left[\left(\int_0^{\tau^R}\|X(s)\|_H^2\big\|(\Psi(s)Q\Psi^*(s))^{1/2}\big\|_{L_{\mathrm{HS}}(H,H)}^2\,\mathrm{d}s\right)^{1/2}\right].$$

Since we have

$$\big\|(\Psi(s)Q\Psi^*(s))^{1/2}\big\|_{L_{\mathrm{HS}}(H,H)}^2 = \big\|Q^{1/2}\Psi^*(s)\big\|_{L_{\mathrm{HS}}(H,U)}^2 = \big\|\Psi(s)Q^{1/2}\big\|_{L_{\mathrm{HS}}(U,H)}^2,$$

we obtain by the inequality $\sqrt{ab} \le \frac{1}{6}a + \frac{3}{2}b$ for $a, b \ge 0$, that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t<\tau^R} \int_0^t X(s-)\,\mathrm{d}I(s)\right] \\
&\quad \le 3\mathbb{E}\left[\left(\left(\sup_{s<\tau^R} \|X(s)\|_H^2\right) \int_0^{\tau^R} \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right)^{1/2}\right] \\
&\quad \le \frac{1}{2}\mathbb{E}\left[\sup_{s<\tau^R} \|X(s)\|_H^2\right] + \frac{9}{2}\mathbb{E}\left[\int_0^T \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right]. 
\end{aligned} \tag{4.9}$$

Proposition 3.11 yields

$$\mathbb{E}\Big[[I,I](\tau^R)\Big] \le \mathbb{E}\Big[[I,I](T)\Big] = \mathbb{E}\Big[\langle I,I\rangle(T)\Big] = \mathbb{E}\left[\int_0^T \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right]. \tag{4.10}$$

Applying (4.8), (4.9) and (4.10) to (4.7) and rearranging, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t<\tau^R} \|X(t)\|_H^2\right] &\le 2\mathbb{E}\left[\|X_0\|_H^2\right] + 2\mathbb{E}\left[\int_0^T \|\Phi(s)\|_{V^*}^2 + \left\|\bar{X}(s)\right\|_V^2 \mathrm{d}s\right] \\
&\qquad + 11\mathbb{E}\left[\int_0^T \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right].
\end{aligned}$$

Taking $R \to \infty$ gives (4.4).

For establishing (4.5) we use some ideas from [76, Rem. 4.2.8]. Let $(\tau_k)$ be a sequence of increasing stopping times such that the process $\left(\int_0^{t\wedge\tau_k} e^{-\lambda s} X(s-)\,\mathrm{d}I(s) : t \in [0,T]\right)$ is a martingale for each $k \in \mathbb{N}$. Taking expectation in (4.6) for the stopped process results in

$$\begin{aligned}
&\mathbb{E}\left[e^{-\lambda t\wedge\tau_k} \|X(t\wedge\tau_k)\|_H^2\right] \\
&= \mathbb{E}\left[\|X_0\|_H^2 + \int_0^{t\wedge\tau_k} e^{-\lambda s}\left(2_{V^*}\langle \Phi(s), \bar{X}(s)\rangle_V - \lambda\|X(s)\|_H^2\right) \mathrm{d}s + \int_0^{t\wedge\tau_k} e^{-\lambda s}\,\mathrm{d}[I,I](s)\right].
\end{aligned} \tag{4.11}$$

Note that we could have changed $\bar{X}$ into $X$, because they are equal Leb$\otimes P$-a.e.

We now consider the integral with respect to the quadratic variation of $I$. Firstly, we show that for all deterministic, Lebesgue–Stieltjes-integrable functions $\Phi\colon [0,T] \to \mathbb{R}$ and for all stopping times $\tau$ we have

$$\mathbb{E}\left[\int_0^{T\wedge\tau} \Phi(s)\,\mathrm{d}[I,I](s)\right] = \mathbb{E}\left[\int_0^{T\wedge\tau} \Phi(s)\,\mathrm{d}\langle I,I\rangle(s)\right]. \tag{4.12}$$

Note firstly that by [66, Th. 18.6(2)] we know that $\|I\|_H^2 - [I,I]$ is a martingale. Secondly, $\|I\|_H^2 - \langle I,I\rangle$ is a martingale by definition. Combining it with the Optional stopping theorem [81, Cor. II.3.6] we have also

$$(\|I(s\wedge\tau)\|_H^2 - [I,I](s\wedge\tau) : s\in[0,T]) \text{ and } (\|I(s\wedge\tau)\|_H^2 - \langle I,I\rangle(s\wedge\tau) : s\in[0,T])$$

are martingales. Thus for any $s\in[0,T]$

$$\mathbb{E}\big[[I,I](s\wedge\tau)\big] = \mathbb{E}\left[\|I(s\wedge\tau)\|_H^2\right] = \mathbb{E}\big[\langle I,I\rangle(s\wedge\tau)\big].$$

Equation (4.12) follows for simple $\Phi$ given by (3.3) for which we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_0^{T\wedge\tau} \Phi(s)\,\mathrm{d}[I,I](s)\right] &= \sum_{k=1}^{N-1} \mathbb{E}\big[\Phi_k\left([I,I](t_{k+1}\wedge\tau) - [I,I](t_k\wedge\tau)\right)\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}\big[\mathbb{E}\big[\Phi_k\left([I,I](t_{k+1}\wedge\tau) - [I,I](t_k\wedge\tau)\right)|\mathcal{F}_{t_k\wedge\tau}\big]\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}\big[\Phi_k\mathbb{E}\big[[I,I](t_{k+1}\wedge\tau) - [I,I](t_k\wedge\tau)|\mathcal{F}_{t_k\wedge\tau}\big]\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}[\Phi_k]\mathbb{E}\big[[I,I](t_{k+1}\wedge\tau) - [I,I](t_k\wedge\tau)\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}[\Phi_k]\mathbb{E}\big[\langle I,I\rangle(t_{k+1}\wedge\tau) - \langle I,I\rangle(t_k\wedge\tau)\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}[\Phi_k]\mathbb{E}\big[\langle I,I\rangle(t_{k+1}\wedge\tau) - \langle I,I\rangle(t_k\wedge\tau)|\mathcal{F}_{t_k\wedge\tau}\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}\big[\mathbb{E}\big[\Phi_k(\langle I,I\rangle(t_{k+1}\wedge\tau) - \langle I,I\rangle(t_k\wedge\tau))|\mathcal{F}_{t_k\wedge\tau}\big]\big] \\
&= \sum_{k=1}^{N-1} \mathbb{E}\big[\Phi_k(\langle I,I\rangle(t_{k+1}\wedge\tau) - \langle I,I\rangle(t_k\wedge\tau))\big]
\end{aligned}$$

$$= \mathbb{E}\left[\int_0^{T\wedge\tau} \Phi(s)\,\mathrm{d}\langle I, I\rangle(s)\right].$$

Note that both integrals in (4.12) are defined pathwise. Therefore (4.12) follows by approximation for integrable function.

From Proposition 3.11 we conclude for (4.11) that

$$\begin{aligned}
&\mathbb{E}\left[e^{-\lambda t\wedge\tau_k}\|X(t\wedge\tau_k)\|_H^2\right] \\
&= \mathbb{E}\left[\|X_0\|_H^2 + \int_0^{t\wedge\tau_k} e^{-\lambda s}\left(2_{V^*}\langle\Phi(s), \bar{X}(s)\rangle_V + \left\|\Psi(s)Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} - \lambda\|X(s)\|_H^2\right)\mathrm{d}s\right].
\end{aligned}$$

An application of the Lebesgue theorem completes the proof. □

Let $(e_j)$ be an orthonormal basis of $U$ consisting of eigenvectors of $Q$ and denote with $P_n$ the projection onto $\mathrm{Span}(e_1, \dots, e_n)$. Let also $(f_j) \subset V$ be an orthonormal basis of $H$ and denote with $\tilde{P}_n$, the projection onto $\mathrm{Span}(f_1, \dots, f_n)$. The operator $\tilde{P}_n$ extends to a mapping $\tilde{P}_n \colon V^* \to V$ by defining

$$\tilde{P}_n v^* = \sum_{j=1}^n {}_{V^*}\langle v^*, f_j\rangle_V f_j, \qquad v^* \in V^*.$$

**Lemma 4.5.** Let $L_n(t)\colon \Omega \to U$ be given by

$$L_n(t) := \sum_{j=1}^n L(t)(e_j)e_j.$$

Then

(i) $L_n$ is a classical Lévy process satisfying

$$\int_0^t \Psi(s)P_n\,\mathrm{d}L(s) = \int_0^t \Psi(s)\,\mathrm{d}L_n(s). \tag{4.13}$$

(ii) The covariance operator $Q_n$ of $L_n$ is given by $Q_n = P_nQP_n = P_nQ = QP_n$.

*Proof.* (i) The integral on the LHS is understood in the sense of Subsection 2.3 whereas the integral on the RHS is the standard stochastic integral (as defined for example in Peszat and

Zabczyk [74]). For $t \geq 0$ and $u \in U$

$$L(t)(P_n u) = \sum_{j=1}^{n} L(t)(e_j)\langle u, e_j\rangle_U = \langle L_n(t), u\rangle_U.$$

Thus for $h \in H$ and $\Psi$ of the form $\Psi(u) = \mathbb{1}_A \mathbb{1}_{(t_1,t_2]}(s)\psi$ with $0 \leq t_1 \leq t_2 \leq t$ and $A \in \mathcal{F}_{t_1}$

$$\begin{aligned}
\left\langle \int_0^t \Psi(r) P_n \, \mathrm{d}L(r), h \right\rangle_H &= (L(t_2) - L(t_1))(P_n \psi^* h)\mathbb{1}_A \\
&= \langle L_n(t_2) - L_n(t_1), \psi^* h\rangle_U \mathbb{1}_A \\
&= \langle \psi(L_n(t_2) - L_n(t_1)), h\rangle_U \mathbb{1}_A \\
&= \left\langle \int_0^t \Psi(s) \, \mathrm{d}L_n(s), h \right\rangle_H .
\end{aligned}$$

We then get (4.13) for simple functions of the form (3.3)–(3.4) by the linearity of the integrals and finally for general $\Psi \in \Psi \in \Lambda^2(0, T; L_{\mathrm{HS}}(\mathcal{H}, H))$ by the continuity of both integrals.

(ii) The result follows from the calculation

$$\begin{aligned}
\langle Q_n x, y\rangle &= \mathbb{E}[\langle L_n(1), x\rangle\langle L_n(1), y\rangle] = \mathbb{E}[L(1)(P_n x)L(1)(P_n y)] \\
&= \langle Q(P_n x), P_n y\rangle = \langle P_n Q P_n x, y\rangle
\end{aligned}$$

and the assumption that $Q$ is diagonal. □

**Lemma 4.6.** The solutions of

$$X_n(t) = \tilde{P}_n X_0 + \int_0^t \tilde{P}_n F(X_n(s)) \, \mathrm{d}s + \int_0^t \tilde{P}_n G(X_n(s-)) P_n \, \mathrm{d}L(s) \tag{4.14}$$

obey $\sup_n \mathbb{E}\left[\int_0^T \|X_n(t)\|_V^2 \, \mathrm{d}t\right] < \infty$.

*Proof.* It follows from standard results, see e.g. [42, Th. 1] that for each $n \in \mathbb{N}$ equation (4.14) has a unique càdlàg strong solution in $V$. For the stopping times $\tau_n^R := \inf\{t \geq 0 : \|X_n(t)\|_V \geq R\}$, for $n \in \mathbb{N}$ and $R > 0$ denote the stopped process by $X_n^R$. The coercivity assumption (A1) and growth assumption (A3) imply that

$$\left\|G(v)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \leq -2_{V^*}\langle F(v), v\rangle_V - \alpha\|v\|_V^2 + \lambda\|v\|_H^2 + \beta \leq (4c - \alpha + \lambda)\|v\|_V^2 + \beta + 2c, \tag{4.15}$$

for all $v \in V$. Therefore, condition (4.3) is satisfied and Corollary 4.4 implies

$$\begin{aligned}&\mathbb{E}\left[\left\|X_n^R(t)\right\|_H^2\right]\\&=\mathbb{E}\left[\left\|\tilde{P}_n X_0\right\|_H^2\right]+\mathbb{E}\left[\int_0^t\left(2_{V^*}\langle\tilde{P}_n F(X_n^R(s)),X_n^R(s)\rangle_V+\left\|\tilde{P}_n G(X_n^R(s))Q_n^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)}\right)\mathrm{d}s\right].\end{aligned}$$

Since $X_n(s)$ belongs to the space spanned by $f_1, \dots f_n$ we have

$$_{V^*}\langle\tilde{P}_n F(X_n^R(s)),X_n^R(s)\rangle_V={}_{V^*}\langle F(X_n^R(s)),\tilde{P}_n X_n^R(s)\rangle_V={}_{V^*}\langle F(X_n^R(s)),X_n^R(s)\rangle_V. \quad (4.16)$$

Moreover,

$$\begin{aligned}\left\|\tilde{P}_n G(X_n^R(s))Q_n^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}&=\left\|\tilde{P}_n G(X_n^R(s))Q^{1/2}P_n\right\|_{L_{\mathrm{HS}}(U,H)}\\&\le\left\|G(X_n^R(s))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}\end{aligned} \quad (4.17)$$

and $\left\|\tilde{P}_n X_0\right\|_H^2 \le \|X_0\|_H^2$. Thus

$$\begin{aligned}&\mathbb{E}\left[\left\|X_n^R(t)\right\|_H^2\right]\\&\quad\le\mathbb{E}\left[\|X_0\|_H^2\right]+\mathbb{E}\left[\int_0^t\left(2_{V^*}\langle F(X_n^R(s)),X_n^R(s)\rangle_V+\left\|G(X_n^R(s))Q^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)}\right)\mathrm{d}s\right].\end{aligned}$$

Adding the expression $\mathbb{E}\left[\int_0^t \alpha\left\|X_n^R(s)\right\|_V^2 \mathrm{d}s\right]$ to both sides and using the coercivity assumption (A1), we obtain

$$\mathbb{E}\left[\left\|X_n^R(t)\right\|_H^2\right]+\mathbb{E}\left[\int_0^t\alpha\left\|X_n^R(s)\right\|_V^2\mathrm{d}s\right]\le\mathbb{E}\left[\|X_0\|_H^2\right]+\beta t+\lambda\mathbb{E}\left[\int_0^t\left\|X_n^R(s)\right\|_H^2\mathrm{d}s\right] \quad (4.18)$$

and consequently skipping the second term on the left-hand side and taking supremum yields

$$\mathbb{E}\left[\left\|X_n^R(t)\right\|_H^2\right]\le\mathbb{E}\|X_0\|_H^2+\beta t+\lambda\int_0^t\sup_{r\le s}\mathbb{E}\left\|X_n^R(r)\right\|_H^2\mathrm{d}s.$$

It follows that

$$\sup_{r\le t}\mathbb{E}\left[\left\|X_n^R(r)\right\|_H^2\right]\le\mathbb{E}\left[\|X_0\|_H^2\right]+\beta t+\lambda\int_0^t\sup_{r\le s}\mathbb{E}\left[\left\|X_n^R(r)\right\|_H^2\right]\mathrm{d}s. \quad (4.19)$$

The Gronwall inequality implies

$$\sup_{r\le t}\mathbb{E}\left[\left\|X_n^R(r)\right\|_H^2\right]\le \mathbb{E}\left[\|X_0\|_H^2\right]+\beta t+\int_0^t\left(\mathbb{E}\left[\|X_0\|_H^2\right]+\beta s\right)e^{\lambda(t-s)}\lambda\,\mathrm{d}s.$$

As the RHS does not depend on $R$, we can take $R\to\infty$ to obtain the bound for processes which are not stopped at $\tau_n^R$

$$\sup_n\sup_{t\in[0,T]}\mathbb{E}\left[\|X_n(t)\|_H^2\right]<\infty.$$

We conclude that the right-hand side of (4.18) is bounded, from which the claim follows. $\square$

**Lemma 4.7.** For the solution $X_n$ of (4.14) define $X_n^-(t):=X_n(t-)$. Then there exists a subsequence $(n_k)\subseteq\mathbb{N}$ such that:

(i) $X_{n_k}^-$ converges weakly to a predictable process $\bar{X}$ in $L^2([0,T]\times\Omega;V)$;

(ii) $\tilde{P}_{n_k}F(X_{n_k}^-)$ converges weakly to some predictable process $\xi$ in $L^2([0,T]\times\Omega;V^*)$;

(iii) $\tilde{P}_{n_k}G(X_{n_k}^-)P_{n_k}Q^{1/2}$ converges weakly to $\eta Q^{1/2}$ in $L^2([0,T]\times\Omega;L_{\mathrm{HS}}(U,H))$ for some predictable process $\eta\in L^2([0,T]\times\Omega;L_{\mathrm{HS}}(\mathcal{H},H))$;

(iv) $\int_0^\cdot\tilde{P}_{n_k}F(X_{n_k}^-(s))\,\mathrm{d}s$ converges weakly to $\int_0^\cdot\xi(s)\,\mathrm{d}s$ in $L^2([0,T]\times\Omega;V^*)$;

(v) $\int_0^\cdot\tilde{P}_{n_k}G(Y_{n_k}(s))P_{n_k}\,\mathrm{d}L(s)$ converges weakly to $\int_0^\cdot\eta(s)\,\mathrm{d}L(s)$ in $L^2([0,T]\times\Omega;H)$.

*Proof.* By combining Corollary III.2.13 and Theorem IV.1.1 in [30] we conclude that all the spaces in the Lemma are reflexive.

(i) A càdlàg function has at most countable number of discontinuity points, hence $X_n$ and $X_n^-$ are equal for Lebesque-almost all $t\in[0,T]$. It follows from Lemma 4.6 that $(X_n^-)$ is bounded in $L^2([0,T]\times\Omega;V)$ and thus by the Banach–Alaoglu theorem has a weakly convergent subsequence.

(ii) Note that each $F(X_n^-)$ is predictable, because $X_n^-$ is and $F$ is measurable. By the linear growth assumption (A3), $\left\|\tilde{P}_nF(X_n^-(t))\right\|_{V^*}\le\|F(X_n^-(t))\|_{V^*}\le c(1+\|X_n^-(t)\|_V)$. Hence, we get (ii) from part (i).

(iii) We have since $P_n$ and $Q$ commute:

$$\left\|\tilde{P}_nG(X_n^-(t))P_nQ^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2=\left\|\tilde{P}_nG(X_n^-(t))Q^{1/2}P_n\right\|_{L_{\mathrm{HS}}(U,H)}^2$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \left\| \tilde{P}_n G(X_n^-(t)) Q^{1/2} P_n e_k \right\|_H^2 \\
&\le \sum_{k=1}^{\infty} \left\| G(X_n^-(t)) Q^{1/2} P_n e_k \right\|_H^2 \\
&= \left\| G(X_n^-(t)) Q^{1/2} P_n \right\|_{L_{\mathrm{HS}}(H,U)}^2 \qquad (4.20) \\
&= \left\| P_n Q^{1/2} G(X_n^-(t))^* \right\|_{L_{\mathrm{HS}}(H,U)}^2 \\
&\le \left\| Q^{1/2} G(X_n^-(t))^* \right\|_{L_{\mathrm{HS}}(H,U)}^2 \\
&= \left\| G(X_n^-(t)) Q^{1/2} \right\|_{L_{\mathrm{HS}}(H,U)}^2 .
\end{aligned}
$$

It follows from (4.20), (4.15) and part (i) that the sequence $\tilde{P}_n G(X_n^-) P_n$ is bounded in $L^2([0,T] \times \Omega; L_{\mathrm{HS}}(\mathcal{H}, H))$. Again by the Banach–Alaoglu theorem it converges weakly to some $\eta$, which is our claim.

(iv) We will use the fact that a bounded operator maps weakly convergent sequences to weakly convergent sequences. Thus it is enough to show that the mapping

$$
K \colon L^2([0,T] \times \Omega; V^*) \to L^2([0,T] \times \Omega; V^*), \qquad K(\Psi) = \left( \int_0^t \Psi(s)\, \mathrm{d}s : t \in [0,T] \right)
$$

is continuous. By the Cauchy-Schwarz inequality

$$
\begin{aligned}
\mathbb{E}\left[ \int_0^T \left\| \int_0^t \Psi(s)\, \mathrm{d}s \right\|_{V^*}^2 \mathrm{d}t \right] &\le \mathbb{E}\left[ \int_0^T t \int_0^t \|\Psi(s)\|_{V^*}^2 \,\mathrm{d}s\, \mathrm{d}t \right] \\
&\le T \mathbb{E}\left[ \int_0^T \int_0^T \|\Psi(s)\|_{V^*}^2 \,\mathrm{d}s\, \mathrm{d}t \right] \\
&= T^2 \mathbb{E}\left[ \int_0^T \|\Psi(s)\|_{V^*}^2 \,\mathrm{d}s \right].
\end{aligned}
$$

(v) Similarly, for part (v) define the mapping

$$
K \colon \Psi \in \Lambda^2(0,T; L_{\mathrm{HS}}(\mathcal{H}, H)) \to L^2([0,T] \times \Omega; H), \qquad K(\Psi) = \left( \int_0^t \Psi(s)\, \mathrm{d}L(s) : t \in [0,T] \right).
$$

By the Itô isometry in Theorem 3.5

$$\mathbb{E}\left[\int_0^T \left\| \int_0^t \Psi(s)\,\mathrm{d}L(s) \right\|_H^2 \mathrm{d}t\right] = \int_0^T \mathbb{E}\left[\int_0^t \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right] \mathrm{d}t$$

$$\leq T\mathbb{E}\left[\int_0^T \left\|\Psi(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 \mathrm{d}s\right]. \qquad \square$$

*Proof of Theorem 4.2 (existence).* Using the notation for the limits in Lemma 4.7 we define

$$X(t) := X_0 + \int_0^t \xi(s)\,\mathrm{d}s + \int_0^t \eta(s)\,\mathrm{d}L(s), \qquad \text{for } t \in [0,T]. \tag{4.21}$$

Since both $V$ and $H$ are embedded in $V^*$, the expressions in Lemma 4.7(i), (iv) and (v) also converge weakly in $L^2([0,T]\times\Omega; V^*)$. The process $X$ is $V^*$-valued as the limit of the right-hand side of (4.14). On the other hand, the left-hand side of (4.14) is Leb$\otimes P$-almost everywhere equal to $X_n^-$. By Lemma 4.7(i) it converges weakly to the $V$-valued process $\bar{X}$. Hence, we obtain that $X = \bar{X}$ almost everywhere-Leb$\otimes P$ and Theorem 4.3 guarantees that $P$-almost surely $X$ is $H$-valued and càdlàg.

It is left to show that $\xi = F(\bar{X})$ and $\eta Q^{1/2} = G(\bar{X})Q^{1/2}$, Leb$\otimes P$-almost surely, which will be accomplished in two steps.

Step 1. We conclude from Corollary 4.4 using estimates similar to (4.16) and (4.17) that

$$\begin{aligned}
&\mathbb{E}\left[e^{-\lambda t}\|X_n(t)\|_H^2 - \|X_0\|_H^2\right] \\
&\leq \mathbb{E}\left[\int_0^t e^{-\lambda s}\left(2_{V^*}\langle F(X_n(s)), X_n(s)\rangle_V + \left\|G(X_n(s))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X_n(s)\|_H^2\right)\mathrm{d}s\right].
\end{aligned}$$

By adding and subtracting an arbitrary process $\Phi \in L^2([0,T]\times\Omega; V)$ we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda t}\|X_n(t)\|_H^2\right] &\leq \mathbb{E}\left[\|X_0\|_H^2\right] \\
&+ \mathbb{E}\bigg[\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle F(X_n(s)) - F(\Phi(s)), X_n(s) - \Phi(s)\rangle_V \\
&\quad + \left\|(G(X_n(s)) - G(\Phi(s)))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X_n(s) - \Phi(s)\|_H^2\Big)\mathrm{d}s\bigg] \\
&+ \mathbb{E}\bigg[\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle F(X_n(s)), \Phi(s)\rangle_V + 2_{V^*}\langle F(\Phi(s)), X_n(s) - \Phi(s)\rangle_V + \lambda\|\Phi(s)\|_H^2 \\
&\quad + 2\langle G(X_n(s))Q^{1/2}, G(\Phi(s))Q^{1/2}\rangle_{L_{\mathrm{HS}}(U,H)} - \left\|G(\Phi(s))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2
\end{aligned}$$

$$- 2\lambda\langle X_n(s), \Phi(s)\rangle_H\Big)\,\mathrm{d}s\Big].$$

Using the monotonicity condition (A2) the first integral is non-positive and thus we obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{-\lambda t}\|X_n(t)\|_H^2 - \|X_0\|_H^2\right] \qquad (4.22)\\
&\leq \mathbb{E}\Big[\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle F(X_n(s)), \Phi(s)\rangle_V + 2_{V^*}\langle F(\Phi(s)), X_n(s) - \Phi(s)\rangle_V + \lambda\|\Phi(s)\|_H^2\\
&\quad + 2\langle G(X_n(s))Q^{1/2}, G(\Phi(s))Q^{1/2}\rangle_{L_{\mathrm{HS}}(U,H)} - \left\|G(\Phi(s))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2\\
&\quad - 2\lambda\langle X_n(s), \Phi(s)\rangle_H\Big)\mathrm{d}s\Big].
\end{aligned}$$

On the other hand, since $\xi \in L^2([0,T]\times\Omega; V^*)$ and $\eta Q^{1/2} \in L^2([0,T]\times\Omega; L_{\mathrm{HS}}(U,H))$, we obtain from Corollary 4.4 that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle \xi(s), \bar{X}(s)\rangle_V + \left\|\eta(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X(s)\|_H^2\Big)\mathrm{d}s\right]\\
&\qquad = \mathbb{E}\left[e^{-\lambda t}\|X(t)\|_H^2\right] - \mathbb{E}\left[\|X_0\|_H^2\right]. \qquad (4.23)
\end{aligned}$$

Now we multiply (4.23) by a non-negative $\psi \in L^\infty([0,T];\mathbb{R}_+)$ and integrate from 0 to $T$. We get

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \psi(t)\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle \xi(s), \bar{X}(s)\rangle_V + \left\|\eta(s)Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X(s)\|_H^2\Big)\mathrm{d}s\,\mathrm{d}t\Big]\\
&\qquad = \mathbb{E}\left[\int_0^T \psi(t)\left(e^{-\lambda t}\|X(t)\|_H^2 - \|X_0\|_H^2\right)\mathrm{d}t\right].
\end{aligned}$$

By the Fatou lemma and the fact that $X = \bar{X}$ and $X_n = X_n^-$ a.e.-Leb$\otimes P$

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\|X(t)\|_H^2\,\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\left\|\bar{X}(t)\right\|_H\mathrm{d}t\right]\\
&= \sum_{k=1}^\infty \mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\langle \bar{X}(t), e_k\rangle_H^2\,\mathrm{d}t\right]\\
&= \sum_{k=1}^\infty \lim_{n\to\infty}\mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\langle X_n^-(t), e_k\rangle_H^2\,\mathrm{d}t\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} \mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\langle X_n^-(t), e_k\rangle_H^2 \,\mathrm{d}t\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\big\|X_n^-(t)\big\|_H^2 \,\mathrm{d}t\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \psi(t)e^{-\lambda t}\|X_n(t)\|_H^2 \,\mathrm{d}t\right].
\end{aligned}
$$

Hence by (4.22)

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \psi(t)\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle \xi(s), \bar{X}(s)\rangle_V + \big\|\eta(s)Q^{1/2}\big\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X(s)\|_H^2\Big)\mathrm{d}s\,\mathrm{d}t\right] \\
&\leq \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T \psi(t)\left(e^{-\lambda t}\|X_n(t)\|_H^2 - \|X_0\|_H^2\right)\mathrm{d}t\right] \\
&\leq \liminf_{n\to\infty} \mathbb{E}\bigg[\int_0^T \psi(t)\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle F(X_n(s)), \Phi(s)\rangle_V + 2_{V^*}\langle F(\Phi(s)), X_n(s)-\Phi(s)\rangle_V \\
&\quad + 2\langle G(X_n(s))Q^{1/2}, G(\Phi(s))Q^{1/2}\rangle_{L_{\mathrm{HS}}(U,H)} - \big\|G(\Phi(s))Q^{1/2}\big\|_{L_{\mathrm{HS}}(U,H)}^2 \\
&\quad - 2\lambda\langle X_n(s), \Phi(s)\rangle_H + \lambda\|\Phi(s)\|_H^2\Big)\mathrm{d}s\,\mathrm{d}t\bigg].
\end{aligned}
$$

Using the weak convergence of $X_n, F(X_n)$ and $G(X_n)$ we get

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \psi(t)\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle \xi(s), \bar{X}(s)\rangle_V + \big\|\eta(s)Q^{1/2}\big\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\|X(s)\|_H^2\Big)\mathrm{d}s\,\mathrm{d}t\right] \\
&\leq \mathbb{E}\bigg[\int_0^T \psi(t)\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle \xi(s), \Phi(s)\rangle_V + 2_{V^*}\langle F(\Phi(s)), \bar{X}(s)-\Phi(s)\rangle_V + \lambda\|\Phi(s)\|_H^2 \\
&\qquad\qquad + 2\langle \eta(s)Q^{1/2}, G(\Phi(s))Q^{1/2}\rangle_{L_{\mathrm{HS}}(U,H)} - \big\|G(\Phi(s))Q^{1/2}\big\|_{L_{\mathrm{HS}}(U,H)}^2 \\
&\qquad\qquad - 2\lambda\langle \bar{X}(s), \Phi(s)\rangle_H\Big)\mathrm{d}s\,\mathrm{d}t\bigg].
\end{aligned}
$$

Moving the terms from the right-hand to the left-hand side we arrive at

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^T \psi(t)\int_0^t e^{-\lambda s}\Big(2_{V^*}\langle \xi(s)-F(\Phi(s)), \bar{X}(s)-\Phi(s)\rangle_V \\
&\qquad + \big\|(\eta(s)-G(\Phi(s)))Q^{1/2}\big\|_{L_{\mathrm{HS}}(U,H)}^2 - \lambda\big\|\bar{X}(s)-\Phi(s)\big\|_H^2\Big)\mathrm{d}s\,\mathrm{d}t\bigg] \leq 0. \qquad (4.24)
\end{aligned}
$$

Step 2. Taking $\Phi = \bar{X}$ in (4.24), we get

$$\mathbb{E}\left[\int_0^T \psi(t) \int_0^t e^{-\lambda s} \left\| (\eta(s) - G(\bar{X}(s)))Q^{1/2} \right\|^2_{L_{\text{HS}}(U,H)} \mathrm{d}s\,\mathrm{d}t\right] \leq 0,$$

which shows that $\eta Q^{1/2} = G(\bar{X})Q^{1/2}$ almost everywhere-Leb $\otimes P$. Moreover, taking $\Phi = \bar{X} - \varepsilon\tilde{\Phi}v$ for some $\tilde{\Phi} \in L^\infty([0,T] \times \Omega; \mathbb{R})$, $v \in V$ and $\varepsilon > 0$ in (4.24) and neglecting the only non-negative term, we obtain

$$\mathbb{E}\left[\int_0^T \psi(t) \int_0^t e^{-\lambda s}\left(2\varepsilon\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s) - \varepsilon\tilde{\Phi}(s)v), v\rangle_V - \lambda\varepsilon^2 \left|\tilde{\Phi}(s)\right|^2 \|v\|_H^2\right) \mathrm{d}s\,\mathrm{d}t\right] \leq 0.$$

Dividing by $\varepsilon$, taking the limit $\varepsilon \to 0$, applying the hemicontinuity assumption (A4) and the Lebesgue theorem we get that

$$\mathbb{E}\left[\int_0^T \psi(t) \int_0^t e^{-\lambda s} 2\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V \,\mathrm{d}s\,\mathrm{d}t\right] \leq 0. \tag{4.25}$$

Indeed, if we recall that ${}_{V^*}\langle F(u), v\rangle_V \leq \|F(u)\|_{V^*}\|v\|_V \leq c(1 + \|u\|_V)\|v\|_V$ we can estimate for $\varepsilon \in (0,1)$,

$$\begin{aligned}&\left| e^{-\lambda s}\left(2\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s) - \varepsilon\tilde{\Phi}(s)v), v\rangle_V - \lambda\varepsilon\left|\tilde{\Phi}(s)\right|^2 \|v\|_H^2\right)\right| \\ &\qquad \leq \|\xi(s)\|_{V^*}\|v\|_V + c\left(1 + \left\|\bar{X}(s) - \varepsilon\tilde{\Phi}(s)v\right\|_V\right)\|v\|_V + \lambda\left(\operatorname{ess\,sup}\left|\tilde{\Phi}\right|\right)^2 \|v\|_H^2,\end{aligned}$$

which is integrable.

Now we claim that $\xi = F(\bar{X})$ almost everywhere-Leb $\otimes P$. Changing the order of integrals in (4.25) we get

$$\int_0^T \psi(t)\mathbb{E}\left[\int_0^t e^{-\lambda s} 2\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V \,\mathrm{d}s\right] \mathrm{d}t \leq 0 \tag{4.26}$$

for all $\psi \in L^\infty([0,T]; \mathbb{R}_+)$ and $\tilde{\Phi} \in L^\infty([0,T] \times \Omega; \mathbb{R})$. Let $A$ denote the set of all $t \in [0,T]$ such that

$$\mathbb{E}\left[\int_0^t e^{-\lambda s} 2\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V \,\mathrm{d}s\right] = 0.$$

We prove that $A$ has full Lebesgue measure. Let

$$A^+ := \left\{ t \in [0,T] : \mathbb{E}\left[ \int_0^t e^{-\lambda s} 2\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V \,\mathrm{d}s \right] > 0 \right\},$$
$$A^- := \left\{ t \in [0,T] : \mathbb{E}\left[ \int_0^t e^{-\lambda s} 2\tilde{\Phi}(s)_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V \,\mathrm{d}s \right] < 0 \right\}.$$

Assume for contradiction that either $A^+$ or $A^-$ has positive measure. If $\mathrm{Leb}(A^+) > 0$ we can take $\psi = \mathbb{1}_{A^+}$ and get a contradiction with (4.26). If $\mathrm{Leb}(A^-) > 0$ we take $\psi = \mathbb{1}_{A^-}$ and $-\tilde{\Phi}$ instead of $\tilde{\Phi}$. This again gives a contradiction with (4.26). Thus $A$ has full Lebesgue measure.

Similarly taking $\tilde{\Phi}\colon [0,T]\times\Omega \to \mathbb{R}_+$ given by $\tilde{\Phi}(s) := \operatorname{sgn}\left({}_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V\right) \mathbb{1}_{[0,t]}(s)$ for $s \in [0,T]$ and $t \in A$, we obtain that ${}_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V = 0$ a.e. in $[0,t]\times\Omega$. If we consider $t_n \uparrow T, t_n \in A$ we get that ${}_{V^*}\langle \xi(s) - F(\bar{X}(s)), v\rangle_V = 0$ a.e. in $[0,T]\times\Omega$. This gives $\xi = F(\bar{X})$ almost everywhere-Leb $\otimes P$. □

Uniqueness of the variational solution can be derived exactly as in [16] and is given for the sake of completeness. We first show the following lemma.

**Lemma 4.8.** Let $X$ and $Y$ be variational solutions of (4.1) with initial conditions $X_0$ and $Y_0$, respectively. Suppose that $X_0$ and $Y_0$ are square-integrable. Then for $t \geq 0$,

$$\mathbb{E}\left[ \|X(t) - Y(t)\|_H^2 \right] \leq e^{\lambda t}\mathbb{E}\left[ \|X_0 - Y_0\|_H^2 \right].$$

*Proof.* We have

$$X(t) - Y(t) = X_0 - Y_0 + \int_0^t \left(F(\bar{X}(s)) - F(\bar{Y}(s))\right) \mathrm{d}s + \int_0^t \left(G(\bar{X}(s)) - G(\bar{Y}(s))\right) \mathrm{d}L(s).$$

Let

$$\tau^R := \inf\left\{ t \geq 0 : \int_0^t \left\|\bar{X}(s)\right\|_V^2 \mathrm{d}s \vee \int_0^t \left\|\bar{Y}(s)\right\|_V^2 \mathrm{d}s \geq R \right\}.$$

We have by Lemma 3.13

$$X(t\wedge\tau^R) - Y(t\wedge\tau^R) = X_0 - Y_0 + \int_0^t (F(\bar{X}(s)) - F(\bar{Y}(s)))\mathbb{1}_{\{s\leq\tau^R\}}\,\mathrm{d}s$$
$$+ \int_0^t (G(\bar{X}(s)) - G(\bar{Y}(s)))\mathbb{1}_{\{s\leq\tau^R\}}\,\mathrm{d}L(s).$$

The assumptions of Corollary 4.4 are satisfied by (4.15) and (A3). By Corollary 4.4 we get

$$\begin{aligned}
&\mathbb{E}\left[\left\|X(t\wedge\tau^R)-Y(t\wedge\tau^R)\right\|_H^2\right] \\
&= \mathbb{E}\left[\|X_0-Y_0\|_H^2\right]+\mathbb{E}\bigg[\int_0^t\Big(2_{V^*}\langle F(\bar{X}(s))-F(\bar{Y}(s)),\bar{X}(s)-\bar{Y}(s)\rangle_V \\
&\qquad\qquad + \left\|(G(\bar{X}(s))-G(\bar{Y}(s)))Q^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)}^2\Big)\mathbb{1}_{\{s\leq\tau^R\}}\,\mathrm{d}s\bigg].
\end{aligned}$$

By monotonicity (A2)

$$\mathbb{E}\left[\left\|X(t\wedge\tau^R)-Y(t\wedge\tau^R)\right\|_H^2\right]\leq\mathbb{E}\left[\|X_0-Y_0\|_H^2\right]+\lambda\mathbb{E}\left[\int_0^t\|X(s)-Y(s)\|_H^2\mathbb{1}_{\{s\leq\tau^R\}}\,\mathrm{d}s\right].$$

It follows that

$$\mathbb{E}\left[\left\|X(t\wedge\tau^R)-Y(t\wedge\tau^R)\right\|_H^2\right]\leq\mathbb{E}\left[\|X_0-Y_0\|_H^2\right]+\lambda\mathbb{E}\left[\int_0^t\left\|X(s\wedge\tau^R)-Y(s\wedge\tau^R)\right\|_H^2\mathrm{d}s\right].$$

Similarly to (4.19) we get

$$\begin{aligned}
&\sup_{r\leq t}\mathbb{E}\left[\left\|X(r\wedge\tau^R)-Y(r\wedge\tau^R)\right\|_H^2\right] \\
&\qquad\leq\mathbb{E}\left[\|X_0-Y_0\|_H^2\right]+\lambda\int_0^t\sup_{r\leq s}\mathbb{E}\left[\left\|X(r\wedge\tau^R)-Y(r\wedge\tau^R)\right\|_H^2\right]\mathrm{d}s.
\end{aligned}$$

By the Gronwall inequality

$$\sup_{r\leq t}\mathbb{E}\left[\left\|X(r\wedge\tau^R)-Y(r\wedge\tau^R)\right\|_H^2\right]\leq e^{\lambda t}\mathbb{E}\left[\|X_0-Y_0\|_H^2\right].$$

Taking $R\to\infty$ we get the assertion by the monotone convergence theorem. □

*Proof of Theorem 4.2 (uniqueness).* From Lemma 4.8 we get that if $X$ and $Y$ are two solutions to (4.1) with the same initial condition, then $X$ and $Y$ are modifications of each other. Since both of them are càdlàg it follows that they are indistinguishable. □

## 4.2 Non-square-integrable case

### 4.2.1 Assumptions and a special Lévy–Itô decomposition

For the remainder of this chapter we assume that the cylindrical Lévy process is diagonal i.e. it is given by (2.3). The function $m_c$, defined in (3.9), reduces to

$$m_c(k) = \sum_{j=1}^{\infty} \rho_j \left( \left\{ x \in \mathbb{R} : |x| > \tfrac{k}{c_j} \right\} \right) \qquad \text{for all } k > 0, \tag{4.27}$$

see Lemma 3.10.

We showed a Lévy–Itô decomposition for weakly integrable cylindrical Lévy processes in Section 3.1. For general, non-integrable cylindrical Lévy processes, one can decompose the one-dimensional processes $(L(t)u : t \geq 0)$ but this does not lead to a Lévy–Itô decomposition into a sum of cylindrical processes. However, the specific form of the diagonal cylindrical Lévy processes allows to derive a proper Lévy–Itô decomposition using the decomposition of $\ell_j$. For $c = (c_j) \in \ell^\infty(\mathbb{R}_+)$ and $k > 0$ we obtain $\ell_j(t) = p_{c,k}^{(j)}(t) + m_{c,k}^{(j)}(t) + r_{c,k}^{(j)}(t)$ for all $t \geq 0$ where

$$p_{c,k}^{(j)}(t) := \left( b_j + \int_{1<|x|\leq k/c_j} x \, \rho_j(\mathrm{d}x) \right) t, \tag{4.28}$$

$$m_{c,k}^{(j)}(t) := \sqrt{s_j} W_j(t) + \int_{|x|\leq k/c_j} x \, \tilde{N}_j(t, \mathrm{d}x), \tag{4.29}$$

$$r_{c,k}^{(j)}(t) := \int_{|x|>k/c_j} x \, N_j(t, \mathrm{d}x). \tag{4.30}$$

Here, the process $W_j$ is a real-valued standard Brownian motion, $N_j$ is a Poisson random measure on $[0, \infty) \times \mathbb{R}$ with intensity measure $\mathrm{Leb} \otimes \rho_j$ associated to $\ell_j$ and $\tilde{N}_j$ denotes the compensated Poisson random measure.

In the next Lemma we show that under the following assumption the stopping times $\tau_c(k)$ defined in (3.8) do not accumulate at zero and the decomposition of $\ell_j$ leads to a decomposition of the cylindrical Lévy process:

(A6) there exists a sequence $c = (c_j) \in \ell^\infty(\mathbb{R}_+)$ such that

(i) $\left( p_{c,k}^{(j)}(1) \right)_{j \in \mathbb{N}} \in \ell^2(\mathbb{R})$ for each $k > 0$; (4.31)

(ii) $\sup_{j \in \mathbb{N}} \int_{|x|\leq k/c_j} x^2 \, \rho_j(\mathrm{d}x) < \infty$ for each $k > 0$; (4.32)

(iii) $\lim_{k\to\infty} m_c(k) = 0.$ (4.33)

**Lemma 4.9.** Assume that $L$ is a diagonal cylindrical Lévy process satisfying (A6). Then $L$ can be decomposed into $L(t) = P_{c,k}(t) + M_{c,k}(t) + R_{c,k}(t)$ for each $t \geq 0$ and $k > 0$, where $P_{c,k}$, $M_{c,k}$ and $R_{c,k}$ are cylindrical Lévy processes defined by

$$P_{c,k}(t)u := \sum_{j=1}^{\infty} p_{c,k}^{(j)}(t)\langle u, e_j\rangle, \;\; M_{c,k}(t)u := \sum_{j=1}^{\infty} m_{c,k}^{(j)}(t)\langle u, e_j\rangle, \;\; R_{c,k}(t)u := \sum_{j=1}^{\infty} r_{c,k}^{(j)}(t)\langle u, e_j\rangle.$$

The process $M_{c,k}$ is a weakly square-integrable cylindrical Lévy martingale and the stopping times $\tau_c(k)$, defined in (3.8), satisfy $\tau_c(k) \to \infty$ $P$-a.s. as $k \to \infty$.

*Proof.* We write $M_{c,k}(t) = X(t) + Y_{c,k}(t)$ for each $k > 0$ with

$$X(t)u := \sum_{j=1}^{\infty} \sqrt{s_j} W_j(t)\langle u, e_j\rangle, \qquad Y_{c,k}(t)u := \sum_{j=1}^{\infty} \int_{|x|\leq k/c_j} x\, \tilde{N}_j(t, \mathrm{d}x)\langle u, e_j\rangle,$$

for all $u \in U$. Since condition (2.5) implies

$$\mathbb{E}\left[|X(t)u|^2\right] = \sum_{j=1}^{\infty} |s_j|\langle u, e_j\rangle^2 \leq \|s\|_{\ell^\infty}\|u\|^2,$$

we obtain that $X(t)\colon U \to L^0(\Omega;\mathbb{R})$ is well defined, continuous and weakly square-integrable. We have

$$\mathbb{E}\left[|Y_{c,k}(t)u|^2\right] = t\sum_{j=1}^{\infty}\langle u, e_j\rangle^2 \int_{|x|\leq k/c_j} x^2\,\rho_j(\mathrm{d}x) \leq t\|u\|^2 \sup_{j\in\mathbb{N}} \int_{|x|\leq k/c_j} x^2\,\rho_j(\mathrm{d}x) < \infty$$

by (4.32). Consequently, $Y_{c,k}(t)$ and thus $M_{c,k}(t)$ are well defined, continuous and weakly square-integrable. By (4.31), the deterministic process $P_{c,k}$ is well defined. Since $R_{c,k} = L - M_{c,k} - P_{c,k}$ it follows that the series in the definition of $R_{c,k}$ converges and that $R_{c,k}(t)\colon U \to L^0(\Omega;\mathbb{R})$ is continuous for all $t \geq 0$. □

**Remark 4.10.**

(i) All the terms in the series (4.27) converge monotonically to 0 as $k \to \infty$. Therefore assumption (4.33) holds by the Lebesgue dominated convergence theorem provided that the series in (4.27) converges for some $k > 0$.

(ii) For a square summable sequence $(c_j)$, condition (4.33) is automatically satisfied. Indeed, we get by (2.6) that for $k=1$ the series in (4.27) converges.

(iii) If $c_j$ is constantly equal to 1, then condition (4.32) holds. Indeed, we observe that (2.6) must hold with $\wedge k^2$ instead of $\wedge 1$. Suppose for contradiction that the sequence $\big(\int_{|x|\le k} x^2\,\rho_j(\mathrm{d}x) : j\in\mathbb{N}\big)$ is unbounded. Then there exists a sequence $(\alpha_j)\in\ell^2(\mathbb{R})$ such that

$$\sum_{j=1}^{\infty}\alpha_j^2\int_{|x|\le k} x^2\,\rho_j(\mathrm{d}x)=\infty,$$

which contradicts (2.6) because then for $(\alpha_j)\in\ell^2((0,1))$

$$\sum_{j=1}^{\infty}\int_{\mathbb{R}}(\alpha_j x)^2\wedge k^2\,\rho_j(\mathrm{d}x)\ge\sum_{j=1}^{\infty}\alpha_j^2\int_{|x|\le\frac{k}{|\alpha_j|}} x^2\,\rho_j(\mathrm{d}x)\ge\sum_{j=1}^{\infty}\alpha_j^2\int_{|x|\le k} x^2\,\rho_j(\mathrm{d}x)=\infty.$$

The integration theory developed in Section 3.2 relies on finite weak moments of the cylindrical Lévy process. In the following, we extend this stochastic integral to the class of diagonal cylindrical Lévy processes under Assumption (A6) without requiring finite weak moments. For this purpose, by fixing a sequence $c\in\ell^\infty(\mathbb{R}_+)$ such that Assumption (A6) is satisfied and by using the notation (4.28)–(4.30) we define $L_{c,k}:=P_{c,k}+M_{c,k}$ for each $k>0$. Lemma 4.9 yields that it is a square-integrable cylindrical Lévy process. Denote by $Q_k$ the covariance operator of $M_{c,k}$ (for simplicity we omit the dependence on $c$). Let $\Lambda_{\mathrm{loc}}$ denote the space of predictable processes $\Psi\colon[0,T]\times\Omega\to\mathcal{L}(U,H)$ such that $\int_0^T\big\|\Psi(s)Q_k^{1/2}\big\|^2_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s<\infty$ for all $k\in\mathbb{N}$.

**Theorem 4.11.** *Assume that $L$ is a diagonal cylindrical Lévy process satisfying* (A6) *and let $\Psi$ be in $\Lambda_{\mathrm{loc}}$. Then there exists an increasing sequence of stopping times $(\varrho(k))$ with $\varrho(k)\to\infty$ $P$-a.s. as $k\to\infty$ such that $\Psi(\cdot)\mathbb{1}_{[0,\varrho(k)]}(\cdot)\in\Psi\in\Lambda^2(0,T;L_{\mathrm{HS}}(\mathcal{H}_k,H))$ for each $k\in\mathbb{N}$ and*

$$\left(\int_0^t\Psi(s)\mathbb{1}_{\{s\le\varrho(k)\}}\,\mathrm{d}L_{c,k}(s):\,t\in[0,T]\right)_{k\in\mathbb{N}}$$

*is a Cauchy sequence in the topology of uniform convergence in probability and its limit is independent of the sequence $c$ satisfying Assumption (A6).*

This Theorem enables us to define for each $\Psi \in \Lambda_{\text{loc}}$ the stochastic integrals

$$\int_0^\cdot \Psi(s)\,\mathrm{d}L(s) := \lim_{k\to\infty} \int_0^\cdot \Psi(s)\mathbb{1}_{[0,\varrho(k)]}(s)\,\mathrm{d}L_{c,k}(s),$$

where the limit is taken in the topology of uniform convergence in probability.

*Proof.* Since $\Psi \in \Lambda_{\text{loc}}$, the stopping times

$$\tilde{\tau}(k,n) := \inf\left\{t \geq 0 : \int_0^t \left\|\Psi(s)Q_k^{1/2}\right\|^2_{L_{\text{HS}}(U,H)}\,\mathrm{d}s > n\right\},$$

increase to infinity as $n \to \infty$ (we take $\inf \varnothing = +\infty$). For every $k$ there is $n_k$ such that $P(\tilde{\tau}(k,n_k) < \infty) \leq \frac{1}{2^k}$. By the Borel–Cantelli Lemma

$$P\left(\limsup_{k\to\infty}\{\tilde{\tau}(k,n_k) < \infty\}\right) = 0.$$

Consequently, the stopping times $\varrho_c(k) := \tau_c(k) \wedge \tilde{\tau}(k,n_k)$ converge to $+\infty$ a.s. by Lemma 4.9. Note that if $T \leq \varrho_c(k)$, then $L_{c,k} = L_{c,n}$ on $[0,T]$ and

$$\int_0^t \Psi(s)\mathbb{1}_{\{s\leq\varrho_c(k)\}}\,\mathrm{d}L_{c,k}(s) = \int_0^t \Psi(s)\mathbb{1}_{\{s\leq\varrho_c(n)\}}\,\mathrm{d}L_{c,n}(s)$$

for all $t \in [0,T]$. Consequently, we obtain for each $k \leq n$ and $\varepsilon > 0$ that

$$\begin{aligned}
&P\left(\sup_{t\in[0,T]}\left\|\int_0^t \Psi(s)\mathbb{1}_{\{s\leq\varrho_c(k)\}}\,\mathrm{d}L_{c,k}(s) - \int_0^t \Psi(s)\mathbb{1}_{\{s\leq\varrho_c(n)\}}\,\mathrm{d}L_{c,n}(s)\right\|_H \geq \varepsilon\right)\\
&\quad\leq P\left(\int_0^t \Psi(s)\mathbb{1}_{\{s\leq\varrho_c(k)\}}\,\mathrm{d}L_{c,k}(s) \neq \int_0^t \Psi(s)\mathbb{1}_{\{s\leq\varrho_c(n)\}}\,\mathrm{d}L_{c,n}(s) \text{ for some } t\in[0,T]\right)\\
&\quad\leq P\big(T > \varrho_c(k)\big) \to 0 \qquad \text{as } n,k\to\infty,
\end{aligned}$$

which establishes the claimed convergence.

The limit of the Cauchy sequence does not depend on the choice of the sequence $c$ satisfying (A6) because if $d$ is another sequence satisfying (A6), then $L_{c,k} = L_{d,n}$ for all $t \in [0,T]$ on $\{T \leq \tau_c(k) \wedge \tau_d(n)\}$ and

$$\int_0^t \Psi(s)\mathbb{1}_{\{s\leq\tau_c(k)\}}\,\mathrm{d}L_{c,k}(s) = \int_0^t \Psi(s)\mathbb{1}_{\{s\leq\tau_d(n)\}}\,\mathrm{d}L_{d,n}(s),$$

which completes the proof. ☐

### 4.2.2 Existence of a solution for the diagonal noise

Convergence of the sum (2.3) in the definition of the diagonal cylindrical Lévy process depends on the interplay between the drift part $b_j$ and the Lévy measure $\rho_j$ of the real-valued Lévy process, see condition (2.4). For this reason, we consider the general case of a cylindrical Lévy process with a possibly non-zero drift part. This part can be moved to the drift part of the equation under consideration. Furthermore, instead of the standard coercivity and monotonicity requirements, we introduce assumptions for each truncation level $k \in \mathbb{N}$. They involve the operators $Q_k$, which are the covariance operators of $M_{c,k}$. Assumptions of this form were introduced in Peszat and Zabczyk [74, Sec. 9.7] in the semigroup approach. Assume that there are constants $\alpha_k, \lambda_k, \beta_k, c_k > 0$ such that

(A1′) (coercivity) For every $k \in \mathbb{N}$ and $v \in V$ we have

$$2_{V^*}\langle F(v) + P_{c,k}(1)G^*(v), v\rangle_V + \left\|G(v)Q_k^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} + \alpha_k\|v\|_V^2 \leq \lambda_k\|v\|_H^2 + \beta_k;$$

(A2′) (monotonicity) For every $k \in \mathbb{N}$ and $v_1, v_2 \in V$ we have

$$2_{V^*}\langle F(v_1) - F(v_2) + P_{c,k}(1)\big(G^*(v_1) - G^*(v_2)\big), v_1 - v_2\rangle_V \\ + \left\|(G(v_1) - G(v_2))Q_k^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \leq \lambda_k\|v_1 - v_2\|_H;$$

(A3′) (linear growth) $\|F(v) + P_{c,k}(1)G^*(v)\|_{V^*} \leq c_k(1 + \|v\|_V)$ for all $v \in V$;

(A4′) (hemicontinuity) the mapping $\mathbb{R} \ni s \mapsto {}_{V^*}\langle F(v_1 + sv_2) + P_{c,k}(1)G^*(v_1 + sv_2), v_3\rangle_V$ is continuous for all $v_1, v_2, v_3 \in V$.

**Theorem 4.12.** *Assume that $L$ is a diagonal cylindrical Lévy process satisfying* (A6)*. If the coefficients $F$ and $G$ satisfy* (A1′)–(A4′)*, then equation* (4.1) *with an $\mathcal{F}_0$–measurable initial condition $X(0) = X_0$ has a pathwise unique variational solution $(X, \bar{X})$.*

*Proof.* We reduce the case of the general initial condition to the square-integrable one as in [1, Th. 6.2.3]. For $k \in \mathbb{N}$ let $\Omega_k = \{\|X_0\| \leq k\}$. Using the decomposition $L(t) = P_{c,k}(t) + M_{c,k}(t) + R_{c,k}(t)$, Lemma 4.9 guarantees that $M_{c,k}$ is a weakly square-integrable cylindrical

Lévy martingale with a diagonal covariance operator $Q_k$, and thus condition (A5) holds for $M_{c,k}$. According to Theorem 4.2 there exists a unique variational solution $(X_{c,k}, \bar{X}_{c,k})$ of

$$\mathrm{d}X(t) = \big(F(X(t)) + P_{c,k}(1)G^*(X(t))\big)\,\mathrm{d}t + G(X(t))\,\mathrm{d}M_{c,k}(t),$$

with the initial condition $X(0) = X_0 \mathbb{1}_{\Omega_k}$.

Step 1. We first show that for each $k \le n$ we have $X_{c,k} = X_{c,n}$ $P$-a.s. on $\{T < \tau_c(k)\} \cap \Omega_k$.

For each $t \in [0,T]$ we have

$$\begin{aligned} X_{c,k}(t) - X_{c,n}(t) = -X_0 \mathbb{1}_{\Omega_n \setminus \Omega_k} &+ \int_0^t \left(F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right)\right)\mathrm{d}s \\ &+ \int_0^t \left(P_{c,k}(1)G^*\left(\bar{X}_{c,k}(s)\right) - P_{c,n}(1)G^*\left(\bar{X}_{c,n}(s)\right)\right)\mathrm{d}s \\ &+ \int_0^t G\left(\bar{X}_{c,k}(s)\right)\mathrm{d}M_{c,k}(s) - \int_0^t G\left(\bar{X}_{c,n}(s)\right)\mathrm{d}M_{c,n}(s). \end{aligned}$$

Define a cylindrical Lévy process $Y_{c,k,n}$ by

$$Y_{c,k,n}(t)u := R_{c,k}(t)u - R_{c,n}(t)u = \sum_{j=1}^{\infty} \left( \int_{k/c_j < |x| \le n/c_j} x\, N_j(t, \mathrm{d}x) \right) \langle u, e_j \rangle$$

for all $t \ge 0$ and $u \in U$. The cylindrical martingale $M_{c,n}$ can be rewritten as

$$\begin{aligned} M_{c,n}(t)u &= \sum_{j=1}^{\infty} \left( m_{c,k}^{(j)}(t) + \int_{k/c_j < |x| \le n/c_j} x\, \tilde{N}_j(t, \mathrm{d}x) \right) \langle u, e_j \rangle \\ &= \sum_{j=1}^{\infty} \left( m_{c,k}^{(j)}(t) + \int_{k/c_j < |x| \le n/c_j} x\, N_j(t, \mathrm{d}x) - \int_{k/c_j < |x| \le n/c_j} x\, \rho_j(t, \mathrm{d}x) \right) \langle u, e_j \rangle \\ &= M_{c,k}(t)u + Y_{c,k,n}(t)u - (P_{c,n}(1)u - P_{c,k}(1)u)t. \end{aligned} \tag{4.34}$$

Applying this we get

$$\begin{aligned} X_{c,k}(t) - X_{c,n}(t) = -X_0 \mathbb{1}_{\Omega_n \setminus \Omega_k} &+ \int_0^t F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right)\,\mathrm{d}s \\ &+ \int_0^t P_{c,k}(1)G^*\left(\bar{X}_{c,k}(s)\right) - P_{c,n}(1)G^*\left(\bar{X}_{c,n}(s)\right)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \left(G\left(\bar{X}_{c,k}(s)\right) - G\left(\bar{X}_{c,n}(s)\right)\right) \mathrm{d}M_{c,k}(s) \\
&-\int_0^t G\left(\bar{X}_{c,n}(s)\right) \mathrm{d}Y_{c,k,n}(s) + \int_0^t (P_{c,n}(1) - P_{c,k}(1)) G^*\left(\bar{X}_{c,n}(s)\right) \,\mathrm{d}s \\
= &-X_0 \mathbb{1}_{\Omega_n \setminus \Omega_k} + \int_0^t F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right) \,\mathrm{d}s \\
&+\int_0^t P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right) \mathrm{d}s \\
&+\int_0^t \left(G\left(\bar{X}_{c,k}(s)\right) - G\left(\bar{X}_{c,n}(s)\right)\right) \mathrm{d}M_{c,k}(s) - \int_0^t G\left(\bar{X}_{c,n}(s)\right) \mathrm{d}Y_{c,k,n}(s).
\end{aligned}
$$

We introduce new notation

$$
\begin{aligned}
A(t) &:= X_{c,k}(t) - X_{c,n}(t) + \int_0^t G\left(\bar{X}_{c,n}(s)\right) \mathrm{d}Y_{c,k,n}(s), \\
I(t) &:= \int_0^t \left(G\left(\bar{X}_{c,k}(s)\right) - G\left(\bar{X}_{c,n}(s)\right)\right) \mathrm{d}M_{c,k}(s), \\
J(t) &:= \int_0^t (X_{c,k}(s-) - X_{c,n}(s-)) \,\mathrm{d}I(s).
\end{aligned}
$$

We have

$$
\begin{aligned}
A(t) = -X_0 \mathbb{1}_{\Omega_n \setminus \Omega_k} &+ \int_0^t \left(F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right)\right) \mathrm{d}s \\
&+ \int_0^t P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right) \mathrm{d}s + I(t).
\end{aligned} \tag{4.35}
$$

On $\{t < \tau_c(k)\}$ we have $A(t) = X_{c,k}(t) - X_{c,n}(t)$. Theorem 1 in [43] applied with $v(t) = \bar{X}_{c,k}(t) - \bar{X}_{c,n}(t)$ and $h(t) = I(t)$ implies existence of an $H$-valued, càdlàg process $\tilde{h}$, which is equal to $X_{c,k} - X_{c,n}$ almost everywhere-Leb $\otimes P$ on $\{(t,\omega) \in [0,T] \times \Omega : t < \tau(\omega)\}$. We show that $X_{c,k} - X_{c,n}$ and $\tilde{h}$ are indistinguishable. We have that

$$
\int_\Omega \int_0^{\tau_c(k)(\omega)} \mathbb{1}_{\tilde{h}(t,\omega) \neq (X_{c,k} - X_{c,n})(t,\omega)} \,\mathrm{d}t \, P(\mathrm{d}\omega) = 0.
$$

This implies that there exists $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$

$$
\int_0^{\tau_c(k)(\omega)} \mathbb{1}_{\tilde{h}(t,\omega) \neq (X_{c,k} - X_{c,n})(t,\omega)} \,\mathrm{d}t = 0.
$$

We obtain that for every $\omega \in \Omega_1$ there is a subset $A_{\omega,t} \subset [0, \tau_c(k)(\omega))$ with $\mathrm{Leb}(A_{\omega,t}) = \tau_c(k)(\omega)$ and such that $\tilde{h}(t,\omega) = (X_{c,k} - X_{c,n})(t,\omega)$ for all $t \in A_{\omega,t}$. Note that $X_{c,k} - X_{c,n}$ is a càdlàg process in $V^*$. Fix $\omega \in \Omega_1$ and $t \in [0, \tau_c(k)(\omega))$. Let $(t_n) \subset A_{\omega,t}$ be a sequence decreasing to $t$. It follows from $\tilde{h}(t_n,\omega) = (X_{c,k} - X_{c,n})(t_n,\omega)$ that $\tilde{h}(t,\omega) = (X_{c,k} - X_{c,n})(t,\omega)$. Thus $\tilde{h}(t,\omega) = (X_{c,k} - X_{c,n})(t,\omega)$ for all $t \in [0, \tau(\omega))$ and for all $\omega \in \Omega_1$.

Thus, in what follows, we assume that for $t < \tau_c(k)$ the process $X_{c,k} - X_{c,n}$ is $H$-valued, càdlàg and, and by [43, Th. 1] the Itô formula for the square of the norm holds on $\{t < \tau_c(k)\}$:

$$\begin{aligned}
&\|X_{c,k}(t) - X_{c,n}(t)\|_H^2 \\
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2\int_0^t {}_{V^*}\langle F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\
&\quad + 2\int_0^t {}_{V^*}\langle P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\
&\quad + \int_0^t X_{c,k}(s-) - X_{c,n}(s-)\,\mathrm{d}I(s) + [I,I](t).
\end{aligned} \tag{4.36}$$

We show that

$$\begin{aligned}
&\|A(t \wedge \tau_c(k))\|_H^2 \\
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2\int_0^{t \wedge \tau_c(k)} {}_{V^*}\langle F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\
&\quad + 2\int_0^{t \wedge \tau_c(k)} {}_{V^*}\langle P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\
&\quad + J(t \wedge \tau_c(k)) + [I,I](t \wedge \tau_c(k)).
\end{aligned} \tag{4.37}$$

It follows from (4.36) by taking the left limit at $t \wedge \tau_c(k)$ that

$$\begin{aligned}
&\|X_{c,k}((t \wedge \tau_c(k))-) - X_{c,n}((t \wedge \tau_c(k))-)\|_H^2 \\
&= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2\int_0^{t \wedge \tau_c(k)} {}_{V^*}\langle F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\
&\quad + 2\int_0^{t \wedge \tau_c(k)} {}_{V^*}\langle P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\
&\quad + J((t \wedge \tau_c(k))-) + [I,I]((t \wedge \tau_c(k))-).
\end{aligned} \tag{4.38}$$

By definition of $A$, $A(s) = X_{c,k}(s) - X_{c,n}(s)$ for $s < \tau_c(k)$. Taking the limits as $s \nearrow t \wedge \tau_c(k)$ we get that $A((t \wedge \tau_c(k))-) = X_{c,k}((t \wedge \tau_c(k))-) - X_{c,n}((t \wedge \tau_c(k))-)$. Since the only discontinuous processes in (4.35) are $A$ and $I$, it follows that $\Delta A(t \wedge \tau_c(k)) = \Delta I(t \wedge \tau_c(k))$. Thus

$$\begin{aligned}\|A(t \wedge \tau_c(k))\|_H^2 &= \|A((t \wedge \tau_c(k))-) + \Delta A(t \wedge \tau_c(k))\|_H^2 \\ &= \|X_{c,k}((t \wedge \tau_c(k))-) - X_{c,n}((t \wedge \tau_c(k))-) + \Delta I(t \wedge \tau_c(k))\|_H^2 \\ &= \|X_{c,k}((t \wedge \tau_c(k))-) + X_{c,n}((t \wedge \tau_c(k))-)\|_H^2 \\ &\quad + \langle \Delta I(t \wedge \tau_c(k)), X_{c,k}((t \wedge \tau_c(k))-) - X_{c,n}((t \wedge \tau_c(k))-)\rangle_H \\ &\quad + \|\Delta I(t \wedge \tau_c(k)\|_H^2.\end{aligned}$$

Applying (4.38) we obtain

$$\begin{aligned}&\|A(t \wedge \tau_c(k))\|_H^2 \\ &= \|X_0\|_H^2 \mathbb{1}_{\Omega_n \setminus \Omega_k} + 2\int_0^{t \wedge \tau_c(k)} {}_{V^*}\langle F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\ &\quad + 2\int_0^{t \wedge \tau_c(k)} {}_{V^*}\langle P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \\ &\quad + J((t \wedge \tau_c(k))-) + [I, I]((t \wedge \tau_c(k))-) \\ &\quad + \langle \Delta I(t \wedge \tau_c(k)), X_{c,k}((t \wedge \tau_c(k))-) - X_{c,n}((t \wedge \tau_c(k))-)\rangle_H + \|\Delta I(t \wedge \tau_c(k))\|_H^2\end{aligned} \tag{4.39}$$

The jump of the stochastic integral $J$ at $t \wedge \tau_c(k)$ equals to

$$\Delta J((t \wedge \tau_c(k))) = \langle \Delta I(t \wedge \tau_c(k)), X_{c,k}((t \wedge \tau_c(k))-) - X_{c,n}((t \wedge \tau_c(k))-)\rangle_H, \tag{4.40}$$

see [66, Prop. 24.3 and Sec. 26.4]. Similarly, the jump of the quadratic variation of $I$ at $t \wedge \tau_c(k)$ equals

$$\Delta[I, I](t \wedge \tau_c(k)) = \|\Delta I(t \wedge \tau_c(k))\|_H^2, \tag{4.41}$$

see [66, Th. 20.5(4)]. Applying (4.40) and (4.41) in (4.39) finishes the proof of (4.37).

We multiply both sides of (4.37) by $\mathbb{1}_{\Omega_k}$ and take expectation. For the term involving the quadratic variation, we use the fact that $\mathbb{E}\left[[I, I](t \wedge \tau_c(k)\right] = \mathbb{E}\left[\langle I, I\rangle(t \wedge \tau_c(k)\right]$ and Proposition 3.11. Recall for the following that the martingale property is invariant under multiplication by $\mathbb{1}_{\Omega_k}$, since $\Omega_k$ is $\mathcal{F}_0$-measurable. Thus $\mathbb{E}\left[J(t \wedge \tau_c(k))\mathbb{1}_{\Omega_k}\right] = 0$. We obtain

$$\mathbb{E}\left[\|A(t \wedge \tau_c(k))\|_H^2 \mathbb{1}_{\Omega_k}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[2\int_0^{t\wedge\tau_c(k)} {}_{V^*}\langle F\left(\bar{X}_{c,k}(s)\right) - F\left(\bar{X}_{c,n}(s)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \mathbb{1}_{\Omega_k}\right] \\
&\quad + \mathbb{E}\left[2\int_0^{t\wedge\tau_c(k)} {}_{V^*}\langle P_{c,k}(1)\left(G^*\left(\bar{X}_{c,k}(s)\right) - G^*\left(\bar{X}_{c,n}(s)\right)\right), \bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\rangle_V \,\mathrm{d}s \mathbb{1}_{\Omega_k}\right] \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_c(k)} \left\|\left(G\left(\bar{X}_{c,k}(s)\right) - G\left(\bar{X}_{c,n}(s)\right)\right) Q_k^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \,\mathrm{d}s \mathbb{1}_{\Omega_k}\right].
\end{aligned}
$$

This implies by the monotonicity (A1$'$) and the fact that $X_{c,k}$ and $\bar{X}_{c,k}$ are equal $\mathrm{Leb}\otimes P$-almost everywhere

$$
\begin{aligned}
\mathbb{E}\left[\|A(t\wedge\tau_c(k))\|_H^2 \mathbb{1}_{\Omega_k}\right] &\leq \lambda_k \mathbb{E}\left[\int_0^{t\wedge\tau_c(k)} \left\|\bar{X}_{c,k}(s) - \bar{X}_{c,n}(s)\right\|_H^2 \,\mathrm{d}s \mathbb{1}_{\Omega_k}\right] \\
&= \lambda_k \mathbb{E}\left[\int_0^{t\wedge\tau_c(k)} \|X_{c,k}(s) - X_{c,n}(s)\|_H^2 \,\mathrm{d}s \mathbb{1}_{\Omega_k}\right] \\
&\leq \lambda_k \mathbb{E}\left[\int_0^{t} \|A(s\wedge\tau_c(k))\|_H^2 \,\mathrm{d}s \mathbb{1}_{\Omega_k}\right].
\end{aligned}
$$

It follows by the Gronwall inequality that

$$
\mathbb{E}\left[\left\|X_{c,k}(t\wedge\tau_c(k)) - X_{c,n}(t\wedge\tau_c(k)) + \int_0^{t\wedge\tau_c(k)} G\left(\bar{X}_{c,n}(s)\right) \mathrm{d}Y_{c,k,n}(s)\right\|_H^2 \mathbb{1}_{\Omega_k}\right] = 0.
$$

Thus

$$
\left(X_{c,k}(t\wedge\tau_c(k)) - X_{c,n}(t\wedge\tau_c(k)) + \int_0^{t\wedge\tau_c(k)} G\left(\bar{X}_{c,n}(s)\right) \mathrm{d}Y_{c,k,n}(s)\right) \mathbb{1}_{\Omega_k} = 0 \quad \text{a.s.}
$$

In particular we obtain that

$$
X_{c,k}(t) - X_{c,n}(t) = 0 \text{ a.s. on } \{t < \tau_c(k)\} \cap \Omega_k.
$$

Step 2. The first part enables us to define

$$
X := X_{c,k} \quad \text{and} \quad \bar{X} := \bar{X}_{c,k} \qquad \text{on } \{t < \tau_c(k)\}. \tag{4.42}
$$

This definition does not depend on the choice of the sequence $c$: for, if $d$ is another sequence satisfying (A6), then one can show similarly as in Step 1, that $X_{c,k} = X_{d,n}$ on $\{t < \tau_c(k) \wedge \tau_d(n)\}$.

Since for each $k \in \mathbb{N}$ we have Leb $\otimes P$-almost everywhere

$$X\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k} = X_{c,k}\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k} = \bar{X}_{c,k}\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k} = \bar{X}\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k},$$

we obtain $X = \bar{X}$ almost everywhere-Leb $\otimes P$ by taking $k \to \infty$.

Step 3. We show that $(X, \bar{X})$ defined in (4.42) satisfies (4.2). Note that

$$\begin{aligned} &X(t)\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k} = X_{c,k}(t)\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k} \\ &= X_0\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k} + \mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k}\int_0^t F(\bar{X}_{c,k}(s))\,\mathrm{d}s + \mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k}\int_0^t G(\bar{X}_{c,k}(s))\,\mathrm{d}L_{c,k}(s). \end{aligned} \tag{4.43}$$

From the very definition (4.42) it follows

$$\begin{aligned} \lim_{k\to\infty}\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k}\int_0^t F(\bar{X}_{c,k}(s))\,\mathrm{d}s &= \lim_{k\to\infty}\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k}\int_0^t F(\bar{X}(s))\,\mathrm{d}s \\ &= \int_0^t F(\bar{X}(s))\,\mathrm{d}s. \end{aligned} \tag{4.44}$$

The last term in (4.43) can be rewritten as

$$\mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k}\int_0^t G(\bar{X}_{c,k}(s))\,\mathrm{d}L_{c,k}(s) = \mathbb{1}_{\{t<\tau_c(k)\}\cap\Omega_k}\int_0^{t\wedge\tau_c(k)} G(\bar{X}_{c,k}(s))\,\mathrm{d}L_{c,k}(s). \tag{4.45}$$

From Lemma 3.13 and the definition of the stochastic integral with respect to $L$ after Theorem 4.11, it follows that

$$\begin{aligned} \lim_{k\to\infty}\int_0^{t\wedge\tau_c(k)} G(\bar{X}_{c,k}(s))\,\mathrm{d}L_{c,k}(s) &= \lim_{k\to\infty}\int_0^t G(\bar{X}_{c,k}(s))\mathbb{1}_{\{s\le\tau_c(k)\}}\,\mathrm{d}L_{c,k}(s) \\ &= \lim_{k\to\infty}\int_0^t G(\bar{X}(s))\mathbb{1}_{\{s\le\tau_c(k)\}}\,\mathrm{d}L_{c,k}(s) \\ &= \int_0^t G(\bar{X}(s))\,\mathrm{d}L(s). \end{aligned} \tag{4.46}$$

By taking the limit $k \to \infty$ in (4.43), equalities (4.44) and (4.46) show

$$X(t) = X_0 + \int_0^t F(\bar{X}(s))\,\mathrm{d}s + \int_0^t G(\bar{X}(s))\,\mathrm{d}L(s),$$

which finishes the proof of the theorem. □

## 4.3 Examples

**Example 4.13.** [Two-sided stable process] Suppose $\ell_j = \sigma_j h_j$, where $h_j$ are identically distributed, symmetric $\alpha$-stable Lévy processes with the characteristic function

$$\varphi_{h_j(1)}(x) = e^{-|x|^\alpha}, \qquad x \in \mathbb{R}, \tag{4.47}$$

and $\sigma_j \in \mathbb{R}$; see [77, 78]. In this case, $\ell_j$ has Lévy measure $\rho_j = \rho \circ m_{\sigma_j}^{-1}$, where $m_{\sigma_j} : \mathbb{R} \to \mathbb{R}$ is given by $m_{\sigma_j}(x) = \sigma_j x$, the measure $\rho$ is defined as $\rho(\mathrm{d}x) = \frac{\alpha}{2c_\alpha}|x|^{-1-\alpha}\,\mathrm{d}x$ and $c_\alpha$ is defined in (2.8). By [84, Ex. 4.5], formula (2.3) defines a cylindrical Lévy process on $U$ if and only if $\sigma = (\sigma_j) \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$. Moreover, $L$ is induced by a classical process if and only if $\sigma \in \ell^\alpha(\mathbb{R})$.

**Corollary 4.14.** Suppose that $L$ is a diagonal process of Example 2.3 with $\ell_j = \sigma_j h_j$, where $h_j$ are identically distributed, symmetric $\alpha$-stable Lévy processes with the characteristic function (4.47) and $(\sigma_j) \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$. If the coefficients $F$ and $G$ satisfy (A1′)–(A4′), then equation (4.1) with an $\mathcal{F}_0$–measurable initial condition $X(0) = X_0$ has a pathwise unique variational solution $(X, \bar{X})$.

*Proof.* The result follows by Theorem 4.12 once we show that Assumption (A6) is satisfied for the sequence $(c_j) \in \ell^2(\mathbb{R}_+)$ defined by $c_j := |\sigma_j|^{\frac{\alpha}{2-\alpha}}$. Condition (4.31) is trivially satisfied because each $h_j$ has no drift and the Lévy measure is symmetric. Since

$$\int_{|x| \le \frac{k}{c_j}} x^2\,\rho_j(\mathrm{d}x) = \sigma_j^2 \int_{|x| \le \frac{k}{|c_j\sigma_j|}} x^2\,\rho(\mathrm{d}x) = \sigma_j^2 \frac{\alpha k^{2-\alpha}}{c_\alpha(2-\alpha)}|c_j\sigma_j|^{\alpha-2} = \frac{\alpha k^{2-\alpha}}{c_\alpha(2-\alpha)},$$

condition (4.32) is satisfied. We have

$$\sum_{j=1}^\infty c_j^2 = \sum_{j=1}^\infty |\sigma_j|^{\frac{2\alpha}{2-\alpha}} < \infty$$

i.e. $(c_j) \in \ell^2(\mathbb{R}_+)$. Application of Remark 4.10(ii) establishes condition (4.33). □

Note that choosing $c_j = 1$ is not possible unless the process is $U$-valued. Indeed when

$\sigma \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R}) \setminus \ell^{\alpha}(\mathbb{R})$ we have

$$\rho_j(B_{\mathbb{R}}(0,k)^c) = \rho\left(B_{\mathbb{R}}\left(0, \tfrac{k}{|\sigma_j|}\right)^c\right) = \frac{\alpha}{c_\alpha}\int_{k/|\sigma_j|}^{\infty} x^{-1-\alpha}\,\mathrm{d}x = \frac{1}{c_\alpha k^\alpha}|\sigma_j|^\alpha$$

and we get

$$m_c(k) = \frac{1}{\alpha k^\alpha}\sum_{j=1}^{\infty}|\sigma_j|^\alpha = \infty.$$

In such case $\tau_c(k) = 0$ a.s. for all $k$ by Proposition 3.6. By introducing the weights $(c_j)$ we compensate the fact that the mass of the span of the higher nodes decays too slowly.

**Example 4.15.** [One-sided stable process] We choose $\ell_j = \sigma_j h_j$ with $\sigma_j \in \mathbb{R}$ and $h_j$ a strictly $\alpha$-stable Lévy process with $\alpha \in (0,1) \cup (1,2)$ and with no negative jumps. Note, that we exclude $\alpha = 1$, since a 1-stable Lévy process is strictly stable if and only if its Lévy measure is symmetric. The characteristic function of $h_j(1)$ is given by

$$\varphi_{h_j(1)}(x) = \exp\left(-|x|^\alpha\left(1 - i\tan\tfrac{\pi\alpha}{2}\operatorname{sgn} x\right)\right),$$

see [92, Th. 14.15]. According to [92, Th. 14.7(iv),(vi)] the drift (corresponding to the truncation function constantly equal to 0) of $h_j$ equals to 0 in the case $\alpha < 1$ and the center of $h_j$ equals to 0 in the case $\alpha > 1$. Thus the characteristic function of $h_j(1)$ equals

$$\varphi_{h_j(1)}(x) = \begin{cases} \exp\left(\int_{\mathbb{R}}\left(e^{ixy} - 1\right)\rho(\mathrm{d}y)\right), & \alpha \in (0,1), \\ \exp\left(\int_{\mathbb{R}}\left(e^{ixy} - 1 - ixy\right)\rho(\mathrm{d}y)\right), & \alpha \in (1,2), \end{cases}$$

where $\rho$ is given by

$$\rho(\mathrm{d}x) = \frac{1}{c_\alpha}\frac{1}{x^{1+\alpha}}\mathbb{1}_{(0,\infty)}(x)\,\mathrm{d}x.$$

Transferring back to our usual truncation function $\mathbb{1}_{B_{\mathbb{R}}}$ we get

$$\varphi_{h_j(1)}(x) = \begin{cases} \exp\left(\left(\int_{\mathbb{R}}\left(e^{ixy} - 1 - ixy\mathbb{1}_{B_{\mathbb{R}}}(y)\right)\rho(\mathrm{d}y) + ix\int_{\mathbb{R}} y\mathbb{1}_{B_{\mathbb{R}}}(y)\,\rho(\mathrm{d}y)\right)\right), & \alpha \in (0,1), \\ \exp\left(\left(\int_0^\infty\left(e^{ixy} - 1 - ixy\mathbb{1}_{B_{\mathbb{R}}}(y)\right)\rho(\mathrm{d}y) - ix\int_{\mathbb{R}} y\mathbb{1}_{B^c_{\mathbb{R}}}(y)\,\rho(\mathrm{d}y)\right)\right), & \alpha \in (1,2). \end{cases}$$

We calculate $b_j$, which is the drift of $\sigma_j h_j$ corresponding to the truncation function $\mathbb{1}_{B_{\mathbb{R}}}$. When $\alpha \in (0,1)$ it is equal to

$$b_j = \int_{\mathbb{R}} y\mathbb{1}_{B_{\mathbb{R}}}(y)\,\rho\circ m_{\sigma_j}^{-1}(\mathrm{d}y) = \frac{1}{c_\alpha}\sigma_j\int_0^{\frac{1}{|\sigma_j|}} y^{-\alpha}\,\mathrm{d}y = \frac{1}{c_\alpha}\frac{1}{1-\alpha}\sigma_j|\sigma_j|^{\alpha-1}$$

and when $\alpha \in (1,2)$

$$b_j = -\int_{\mathbb{R}} y \mathbb{1}_{B^c_{\mathbb{R}}}(y)\, \rho \circ m_{\sigma_j}^{-1}(\mathrm{d}y) = -\frac{1}{c_\alpha}\sigma_j \int_{\frac{1}{|\sigma_j|}}^{\infty} y^{-\alpha}\, \mathrm{d}x = \frac{1}{c_\alpha}\frac{1}{1-\alpha}\sigma_j |\sigma_j|^{\alpha-1}.$$

It follows that the Lévy process $\sigma_j h_j$ has characteristics $(b_j, 0, \rho_j)$ given by

$$b_j = \frac{1}{c_\alpha(1-\alpha)}\sigma_j|\sigma_j|^{\alpha-1}, \qquad \rho_j(\mathrm{d}x) = \big(\rho \circ m_{\sigma_j}^{-1}\big)(\mathrm{d}x).$$

We claim that $L$ is a cylindrical Lévy process if and only if $\sigma \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$. Indeed, condition (2.4) reduces to

$$\sum_{j=1}^{\infty} |\alpha_j| \left| b_j + \int_{1<|x|\le 1/|\alpha_j|} x\, \big(\rho \circ m_{\sigma_j}^{-1}\big)(\mathrm{d}x)\right| = \frac{1}{c_\alpha |1-\alpha|}\sum_{j=1}^{\infty} |\alpha_j \sigma_j|^{\alpha} < \infty,$$

whereas condition (2.6) reads as

$$\sum_{j=1}^{\infty}\int_{\mathbb{R}} \Big(|\alpha_j x|^2 \wedge 1\Big)\, \rho_j(\mathrm{d}x) = \frac{2}{c_\alpha(2-\alpha)\alpha}\sum_{j=1}^{\infty}|\alpha_j\sigma_j|^{\alpha} < \infty.$$

Both are equivalent to $\sigma \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$.

**Corollary 4.16.** Suppose that $L$ is a diagonal process of Example 2.3 with $\ell_j = \sigma_j h_j$, where $(\sigma_j) \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$ and $h_j$ is strictly $\alpha$-stable Lévy process with $\alpha \in (0,1)\cup(1,2)$ and with no negative jumps. If the coefficients $F$ and $G$ satisfy (A1′)–(A4′), then equation (4.1) with an $\mathcal{F}_0$–measurable initial condition $X(0) = X_0$ has a pathwise unique variational solution $(X, \bar{X})$.

*Proof.* As explained in Example 4.15 under the assumptions of the corollary the sum (2.3) converges and defines a cylindrical Lévy process. We show that Assumption (A6) is satisfied with $c_j = |\sigma_j|^{\frac{\alpha}{2-\alpha}}$, since condition (4.31) can be calculated as

$$\sum_{j=1}^{\infty}\left(b_j + \int_{1<|x|\le \frac{k}{c_j}} x\, \rho_j(\mathrm{d}x)\right)^2 = \left(\frac{k^{1-\alpha}}{c_\alpha(1-\alpha)}\right)^2 \sum_{j=1}^{\infty}|\sigma_j|^{\frac{2\alpha}{2-\alpha}}.$$

Conditions (4.32) and (4.33) follow by the same arguments as in Example 4.13. Thus the result follows by Theorem 4.12. ◻

**Example 4.17.** Choosing a constant truncation level i.e. $c_j = 1$ is possible only in some very special cases. It is easy to construct an example of a non-integrable diagonal cylindrical process whose jumps do not accumulate at 0 if we take the process $\ell_j$, which are not identically distributed. Assume that $L$ has finitely many non-square-integrable components, say $\ell_1, \dots, \ell_N$. For instance $\ell_j$ could be a standard symmetric $\alpha$-stable process for $j = 1, \dots, N$. Assume that $\ell_j = \sigma_j h_j$ for $j = N+1, \dots$. Take $h_j$ as in Lemma 3.9 i.e. symmetric square-integrable Lévy martingales and assume additionally that the Lévy measure $\rho$ of $h_j$ has bounded support.

We verify (A6) with $c_j = 1$ for $j \in \mathbb{N}$. Observe that (4.31) holds due to the symmetry. Condition (2.5) implies that for any $(\alpha_j) \in \ell^2(\mathbb{R})$ with $|\alpha_j| \le 1$ we have

$$\infty > \sum_{j=N+1}^{\infty} \int_{\mathbb{R}} (\alpha_j x)^2 \wedge k^2 \, \rho_j(\mathrm{d}x) \ge \sum_{j=N+1}^{\infty} \alpha_j^2 \int_{|x| < \frac{k}{|\alpha_j|}} x^2 \, \rho_j(\mathrm{d}x) \ge \sum_{j=N+1}^{\infty} \alpha_j^2 \int_{|x|<k} x^2 \, \rho_j(\mathrm{d}x).$$

Therefore (4.32) holds. Finally for (4.33), we have

$$m_c(k) = \sum_{j=1}^{N} \rho_j \left( B_{\mathbb{R}}(0,k)^c \right) + \sum_{j=N+1}^{\infty} \rho \left( B_{\mathbb{R}}(0, k/|\sigma_j|)^c \right) < \infty$$

because the second sum equals to 0 for $k$ sufficiently large so that $B_{\mathbb{R}}(0, k/|\sigma_j|)^c$ is outside the support of $\rho$.

### 4.3.1 Processes with regularly varying tails

Recall that a measure $\mu$ concentrated on $(0, \infty)$ is said to have a regularly varying tail with index $\alpha$ if

$$\lim_{x \to \infty} \frac{\mu((\lambda x, \infty))}{\mu((x, \infty))} = \lambda^{-\alpha} \qquad \text{for all } \lambda > 0;$$

see [9, 35].

**Proposition 4.18.** Let

$$L(t)u = \sum_{j=1}^{\infty} \sigma_j h_j(t) \langle u, e_j \rangle, \qquad t \ge 0, u \in U, \tag{4.48}$$

with a sequence of independent and identically distributed Lévy processes $h_j$ with no negative jumps and having tails of regular variation of index $\alpha \in (0,1) \cup (1,2)$. Suppose that

(i) if $\alpha \in (0,1)$ that the characteristic function of $h_j(1)$ is given by

$$\varphi_{h_j(1)}(x) = \exp\left(\int_0^\infty \left(e^{ixy} - 1 - ixy\mathbb{1}_{B_\mathbb{R}}(y)\right)\rho(\mathrm{d}y) + ixb\right) \tag{4.49}$$

(ii) if $\alpha \in (1,2)$ that the characteristic function of $h_j(1)$ is given by

$$\varphi_{h_j(1)}(x) = \exp\left(\int_0^\infty \left(e^{ixy} - 1 - ixy\right)\rho(\mathrm{d}y)\right). \tag{4.50}$$

If either (i) or (ii) holds and if $(\sigma_j) \in \ell^{\frac{2\delta}{2-\delta}}(\mathbb{R})$ for some $\delta < \alpha$, then (4.48) defines a cylindrical Lévy process.

Secondly, if the coefficients $F$ and $G$ satisfy (A1′)–(A4′), then equation (4.1) with an $\mathcal{F}_0$–measurable initial condition $X(0) = X_0$ has a pathwise unique variational solution $(X, \bar{X})$.

*Proof.* Step 1. Proof of convergence of (4.48).

Measure $\rho$ restricted to the complement of the unit ball is finite and we can write $\rho|_{B_\mathbb{R}^c} = \lambda\rho^{(1)}$ for a probability measure $\rho^{(1)}$ concentrated on $B_\mathbb{R}^c$. According to [32, Prop. 0] the Lévy measure of an infinitely divisible distribution with regularly varying tails has regularly varying tails as well. We obtain that $\rho^{(1)}$ has regularly varying tails of index $\alpha$. For $x \geq 0$ define

$$\begin{aligned} V_\delta(x) &:= \int_x^\infty y^\delta \mathbb{1}_{(1,\infty)}(y)\,\rho(\mathrm{d}y) = \lambda \int_x^\infty y^\delta \mathbb{1}_{(1,\infty)}(y)\,\rho^{(1)}(\mathrm{d}y), \\ U_\zeta(x) &:= \int_1^x y^\zeta\,\rho(\mathrm{d}y) = \lambda \int_1^x y^\zeta\,\rho^{(1)}(\mathrm{d}y). \end{aligned}$$

In order to show that the sum in (4.48) verifies the conditions (2.4) and (2.6), we prove that for any $(\alpha_j) \in \ell^2(\mathbb{R})$

$$S_1 := \sum_{j=1}^\infty |\alpha_j| \left| b_j + \sigma_j \int_{\frac{1}{|\sigma_j|} < |x| < \frac{1}{|\alpha_j\sigma_j|}} x\,\rho(\mathrm{d}x) \right| < \infty, \tag{4.51}$$

$$S_2 := \sum_{j=1}^\infty \alpha_j^2 \sigma_j^2 \int_{|x| < \frac{1}{|\alpha_j\sigma_j|}} x^2\,\rho(\mathrm{d}x) < \infty, \tag{4.52}$$

$$S_3 := \sum_{j=1}^\infty \rho\left(\left[\tfrac{1}{|\alpha_j\sigma_j|}, \infty\right)\right) < \infty. \tag{4.53}$$

We verify condition (4.51) in two cases separately

Case 1: $\alpha < 1$. Recall that in (4.51) $b_j$ is the drift corresponding to the truncation function $\mathbb{1}_{B_\mathbb{R}}$. The characteristic function of $\sigma_j h_j(1)$ can be written as

$$\begin{aligned}
&\varphi_{\sigma_j h_j(1)}(x) \\
&= \exp\left(\int_\mathbb{R} \left(e^{ix\sigma_j y} - 1 - ix\sigma_j y \mathbb{1}_{B_\mathbb{R}}(y)\right) \rho(\mathrm{d}y) + i\sigma_j bx\right) \\
&= \exp\left(\int_\mathbb{R} (e^{ixy} - 1 - ixy\mathbb{1}_{B_\mathbb{R}}(y))\, \rho_j \circ m_{\sigma_j}^{-1}(\mathrm{d}y) + ix\sigma_j \int_\mathbb{R} y(1_{B_\mathbb{R}}(y\sigma_j) - \mathbb{1}_{B_\mathbb{R}}(y))\, \rho(\mathrm{d}y) + i\sigma_j bx\right).
\end{aligned}$$

The drift is thus given by

$$b_j = \sigma_j \left(b + \int_\mathbb{R} y(1_{B_\mathbb{R}}(y\sigma_j) - \mathbb{1}_{B_\mathbb{R}}(y))\, \rho(\mathrm{d}y)\right).$$

Assume without loss of generality that $|\sigma_j| < 1$. Then

$$b_j = \sigma_j \left(b + \int_{1 < y \le \frac{1}{|\sigma_j|}} y\, \rho(\mathrm{d}y)\right). \tag{4.54}$$

Thus

$$S_1 = \sum_{j=1}^{\infty} |\alpha_j \sigma_j| \left| b + \int_{1 < x < \frac{1}{|\alpha_j \sigma_j|}} x\, \rho(\mathrm{d}x)\right|,$$

which implies that

$$S_1 \le |b| \sum_{j=1}^{\infty} |\alpha_j \sigma_j| + \sum_{j=1}^{\infty} |\alpha_j \sigma_j| U_1 \left(\frac{1}{|\alpha_j \sigma_j|}\right). \tag{4.55}$$

We show that $V_\delta$ is regularly varying with index $-(\delta - \alpha)$. Let $X$ be distributed according to $P_X = \rho^{(1)}$ and let $x > 1$. We obtain

$$\begin{aligned}
\delta \int_x^\infty y^{\delta-1} \rho^{(1)}(y, \infty)\, \mathrm{d}y &= \delta \int_x^\infty y^{\delta-1}\, P(X \ge y)\, \mathrm{d}y \\
&= \delta \int_x^\infty y^{\delta-1} \int_y^\infty P_X(\mathrm{d}z)\, \mathrm{d}y \\
&= \delta \int_x^\infty \int_x^z y^{\delta-1}\, \mathrm{d}y\, P_X(\mathrm{d}z) \\
&= \int_x^\infty z^\delta - x^\delta\, P_X(\mathrm{d}z)
\end{aligned}$$

$$= \mathbb{E}\left[X^\delta \mathbb{1}_{X \geq x}\right] - x^\delta P(X \geq x)$$
$$= V_\delta(x) - x^\delta \rho^{(1)}(x, \infty).$$

Thus
$$V_\delta(x) = \delta \int_x^\infty y^{\delta-1} \rho^{(1)}(y,\infty)\,\mathrm{d}y + x^\delta \rho^{(1)}(x,\infty).$$

Theorem 1.5.11 in [9] implies that the function $x \mapsto \int_x^\infty y^{\delta-1}\rho^{(1)}(y,\infty)\,\mathrm{d}y$ is regularly varying with index $-(\delta-\alpha)$. Combining [98, point $3^0$, p. 18] with [9, Th. 1.4.1(iii)] one sees that the sum of two regularly varying functions is regularly varying. It follows that for $\delta < \alpha$ the function $V_\delta$ is regularly varying with index $-(\delta-\alpha)$.

It is easy to see that $U_1(\infty) = \infty$. Applying [35, Th. II.9.2] with $\zeta = 1$, $\eta = \delta$ gives that

$$\lim_{x \to \infty} \frac{x^{1-\delta} V_\delta(x)}{U_1(x)} = c.$$

Moreover, from the proof it follows that if $V_\delta$ is regularly varying with index $-(\delta-\alpha)$, then $c = \frac{1-\alpha}{\alpha-\delta} \in (0,\infty)$. There exists $M > 0$ such that for all $x > M$

$$\frac{c}{2} \leq \frac{x^{1-\delta} V_\delta(x)}{U_1(x)}. \tag{4.56}$$

This enables us to estimate (assuming $\frac{1}{|\alpha_j \sigma_j|} > M$ for $j \in \mathbb{N}$)

$$\sum_{j=1}^\infty |\alpha_j\sigma_j| U_1\left(\frac{1}{|\alpha_j\sigma_j|}\right) \leq \frac{2}{c}\sum_{j=1}^\infty |\alpha_j\sigma_j| \frac{1}{|\alpha_j\sigma_j|^{1-\delta}} V_\delta\left(\frac{1}{|\alpha_j\sigma_j|}\right) = \frac{2}{c}\sum_{j=1}^\infty |\alpha_j\sigma_j|^\delta V_\delta\left(\frac{1}{|\alpha_j\sigma_j|}\right)$$

Applying this in (4.55) we get

$$S_1 \leq |b| \sum_{j=1}^\infty |\alpha_j\sigma_j| + \frac{2}{c}\sum_{j=1}^\infty |\alpha_j\sigma_j|^\delta V_\delta\left(\frac{1}{|\alpha_j\sigma_j|}\right). \tag{4.57}$$

The first sum is finite since for $\delta < \alpha < 1$ we have $(\sigma_j) \in \ell^{\frac{2\delta}{2-\delta}}(\mathbb{R}) \subset \ell^2(\mathbb{R})$. The second sum in (4.57) is finite since $V_\delta\left(\frac{1}{|\alpha_j\sigma_j|}\right) \to 0$ as $j \to \infty$ and

$$\sum_{j=1}^\infty |\alpha_j\sigma_j|^\delta \leq \left(\sum_{j=1}^\infty |\sigma_j|^{2\delta/(2-\delta)}\right)^{(2-\delta)/2} \left(\sum_{j=1}^\infty \alpha_j^2\right)^{\delta/2} < \infty. \tag{4.58}$$

This finishes the proof of (4.51) in the case $\alpha \in (0,1)$.

Case 2: $\alpha > 1$. The characteristic function of $\sigma_j h_j(1)$ can be written as

$$\varphi_{\sigma_j h_j(1)}(x) = \exp\left(\int_0^\infty \left(e^{ixy} - 1 - ixy\mathbb{1}_{B_\mathbb{R}}(y)\right)\rho_j \circ m_{\sigma_j}^{-1}(\mathrm{d}y) - ix\int_\mathbb{R} y\mathbb{1}_{B_\mathbb{R}^c}(y)\,\rho\circ m_{\sigma_j}^{-1}(\mathrm{d}x)\right).$$

Thus we have

$$b_j = -\int_\mathbb{R} y\mathbb{1}_{B_\mathbb{R}^c}(x)\,\rho\circ m_{\sigma_j}^{-1}(\mathrm{d}y) = -\sigma_j\int_{y>\frac{1}{|\sigma_j|}} y\,\rho(\mathrm{d}y).$$

Therefore

$$S_1 = \sum_{j=1}^\infty |\alpha_j\sigma_j|\left|\int_{y\geq\frac{1}{|\alpha_j\sigma_j|}} y\,\rho(\mathrm{d}y)\right| = \sum_{j=1}^\infty |\alpha_j\sigma_j| V_1\left(\frac{1}{|\alpha_j\sigma_j|}\right).$$

By [9, Th. 1.4.1] we can write $\rho(x) = x^{-\alpha}l(x)$ with a slowly varying function $l$. By [35, Lem. 2, p. 277] with $\varepsilon = \alpha - \delta$ there exists $M > 0$ such that

$$\rho(x,\infty) \leq x^{-\alpha+\varepsilon} = x^{-\delta} \tag{4.59}$$

for $x > M$. We have

$$V_1(x) = \int_x^\infty \rho^{(1)}(y,\infty)\,\mathrm{d}y + x\rho^{(1)}(x,\infty).$$

For $x > M$ we estimate

$$V_1(x) \leq \int_x^\infty y^{-\delta}\,\mathrm{d}y + xx^{-\delta} = \frac{\delta}{\delta-1}x^{1-\delta}.$$

Assuming $\frac{1}{|\alpha_j\sigma_j|} > M$ for $j \in \mathbb{N}$ we get $V_1\left(\frac{1}{|\alpha_j\sigma_j|}\right) \leq \frac{\delta}{\delta-1}|\alpha_j\sigma_j|^{\delta-1}$ and finally,

$$S_1 \leq \frac{2\delta}{\delta-1}\sum_{j=1}^\infty |\alpha_j\sigma_j|^\delta < \infty.$$

We prove (4.52). Suppose without loss of generality that $|\alpha_j\sigma_j| < 1$. We have

$$S_2 = \sum_{j=1}^\infty\left(\alpha_j^2\sigma_j^2\int_{|x|\leq 1} x^2\,\rho(\mathrm{d}x) + \alpha_j^2\sigma_j^2\int_{1<|x|<\frac{1}{|\alpha_j\sigma_j|}} x^2\,\rho(\mathrm{d}x)\right)$$

$$= \sum_{j=1}^{\infty} \alpha_j^2 \sigma_j^2 \int_{|x|\le 1} x^2 \, \rho(\mathrm{d}x) + \sum_{j=1}^{\infty} \alpha_j^2 \sigma_j^2 U_2\left(\frac{1}{|\alpha_j \sigma_j|}\right).$$

By the same arguments as above we get the counterpart of (4.56):

$$\frac{c}{2} \le \frac{x^{2-\delta} V_\delta(x)}{U_2(x)}, \qquad \text{for } x > M. \tag{4.60}$$

Since both $(\alpha_j)$ and $(\sigma_j)$ tend to 0 we can assume without loss of generality that $\frac{1}{|\alpha_j \sigma_j|} > M$ for all $j \in \mathbb{N}$. We obtain

$$S_2 \le \sum_{j=1}^{\infty} \alpha_j^2 \sigma_j^2 \int_{|x|\le 1} x^2 \, \rho(\mathrm{d}x) + \frac{2}{c} \sum_{j=1}^{\infty} |\alpha_j \sigma_j|^\delta V_\delta\left(\frac{1}{|\alpha_j \sigma_j|}\right).$$

The first sum is finite because $(\sigma_j)$ is bounded. Since $V_\delta\left(\frac{1}{|\alpha_j \sigma_j|}\right) \to 0$ as $j \to \infty$ we get using (4.58) that if $(\sigma_j) \in \ell^{\frac{2\delta}{2-\delta}}(\mathbb{R})$, then $S_2 < \infty$.

We now show (4.53). Applying (4.59)

$$S_3 \le \sum_{j=1}^{\infty} |\alpha_j \sigma_j|^\delta,$$

which is finite by (4.58).

Step 2. Equicontinuity of the characteristic functions follows exactly as in the proof of Lemma 3.9, where we only used the boundedness of $(\sigma_j)$.

Step 3. Verification of (A6) with the sequence $c$ defined by $c_j = |\sigma_j|^{\frac{\delta}{2-\delta}}$.

We prove (4.31). Suppose that $\alpha < 1$. Recall that $b_j$ is given in (4.54). Then

$$\sum_{j=1}^{\infty} \left(p_{c,k}^{(j)}(1)\right)^2 = \sum_{j=1}^{\infty} \left(\sigma_j \left(b + \int_{1 < x \le \frac{k}{|c_j \sigma_j|}} x \, \rho(\mathrm{d}x)\right)\right)^2 \le 2b^2 \sum_{j=1}^{\infty} \sigma_j^2 + 2 \sum_{j=1}^{\infty} \sigma_j^2 U_1\left(\frac{k}{|c_j \sigma_j|}\right)^2. \tag{4.61}$$

Again, we have that (4.56) holds for $x > M$. If $N$ is chosen so that $\frac{k}{|c_j \sigma_j|} > M$ for $j > N$, we have

$$\sum_{j=N+1}^{\infty} \sigma_j^2 U_1\left(\frac{k}{|c_j \sigma_j|}\right)^2 \le \frac{2k^{2-2\delta}}{c^2} \sum_{j=N+1}^{\infty} \sigma_j^{2\delta} c_j^{2\delta-2} V_\delta\left(\frac{k}{|c_j \sigma_j|}\right)^2$$

$$= \frac{2k^{2-2\delta}}{c^2} \sum_{j=N+1}^{\infty} |\sigma_j|^{\frac{2\delta}{2-\delta}} V_\delta \left( \frac{k}{|c_j \sigma_j|} \right)^2.$$

It follows that the sums in (4.61) converge. We get the result for $\alpha \in (1, 2)$ similarly. For (4.32) we estimate using (4.60)

$$\begin{aligned}
\sigma_j^2 \int_{0<x<\frac{k}{|\sigma_j c_j|}} x^2 \, \rho(\mathrm{d}x) &= \sigma_j^2 \int_{0<x\le 1} x^2 \, \rho(\mathrm{d}x) + \sigma_j^2 U_2 \left( \frac{k}{c_j \sigma_j} \right) \\
&\le \sigma_j^2 \int_{0<x\le 1} x^2 \, \rho(\mathrm{d}x) + \frac{2k^{2-\delta}}{c} |\sigma_j c_j|^{\delta-2} V_\delta \left( \frac{k}{|\sigma_j c_j|} \right) \\
&= \sigma_j^2 \int_{0<x\le 1} x^2 \, \rho(\mathrm{d}x) + \frac{2k^{2-\delta}}{c} V_\delta \left( k |\sigma_j|^{-\frac{2}{2-\delta}} \right).
\end{aligned}$$

Both terms are clearly bounded in $j$. Since $(c_j) \in \ell^2(\mathbb{R})$ condition (4.33) follows by Remark 4.10(ii). □

**Remark 4.19.** The assumptions (4.49) and (4.50) involving the characteristic function can be alternatively formulated as follows: the processes $h_j$ have no Gaussian part and in the case $\alpha \in (1, 2)$ the centre is 0. This is corresponds to the requirement of strict stability in the stable case. In the case $\alpha \in (0, 1)$ we allow the processes to have a common drift $b$. The fact that the truncation function needs to be chosen differently for the cases $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$ can be clearly seen in the case of the one sided stable noise. If $h_j$ has characteristics $(0, 0, \rho)$ corresponding to the truncation function $\mathbb{1}_{B_\mathbb{R}}$, then

- if $\alpha \in (0, 1)$, then $L$ is a cylindrical Lévy process if and only if $(\sigma_j) \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$.
- if $\alpha \in (1, 2)$, then $L$ is a cylindrical Lévy process if and only if $(\sigma_j) \in \ell^{2\alpha}(\mathbb{R})$.

This means that for $\alpha \in (1, 2)$ the summability condition on $\sigma$ changes if we do not require the center to be 0.

**Remark 4.20.** Note that the conclusion in Proposition 4.48 is not optimal if applied to $\alpha$-stable noise. For, in Example 4.15 we can choose $\sigma \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{R})$ whereas here we have to choose $\sigma \in \ell^{\frac{2\delta}{2-\delta}}(\mathbb{R})$ for $\delta < \alpha$.

# Chapter 5

# Stochastic integration in Banach spaces and Stochastic Evolution Equation

In this chapter we consider the stochastic evolution equation (1.4) with coefficients acting on Banach spaces; the most complex issue is however the construction of a stochastic integral. Stochastic integration in Banach spaces is only possible in some subclasses of Banach spaces.

Dettweiler [28] defined an integral in $p$-uniformly smoothable Banach spaces using estimates of moments of the Lévy process, the $p$-variation and moments of the Lévy measure. Van Neerven, Veraar and Weis developed a stochastic integral in a different class of Banach spaces, that is UMD spaces, see [103]. They integrate with respect to a cylindrical Brownian motion and the integrands consist of $\gamma$-Radonifying operators, that is operators which Radonify the standard Gaussian cylindrical measure into a genuine measure. Outside the Gaussian setting, in Veraar and Yaroslavtsev [104] the integral is defined for cylindrical continuous local martingales. The integrands take values in the space of linear operators from a Hilbert space into a UMD Banach space. The authors use the notion of $\gamma$-Radonifying norm, which is possible because according to the Dambis–Dubins–Schwarz theorem every continuous local martingale is a time changed Brownian motion. Another approach to integration with jumps is through Poisson Random measures. An integral in another class of spaces called martingale type $p$ spaces, was considered by various authors see e.g. Rüdiger [87] or a survey article by Van Neerven, Veraar and Weis [102].

As shown in the aforementioned works, already the definition of the integral requires assuming some geometric properties of the Banach spaces involved. Let $E$ and $F$ be separable Banach spaces and suppose that $F$ is of martingale type $p$ for some $p \in [1,2]$. Our space of admissible integrands consists of predictable $p$-integrable processes taking values in the space $\Pi_p(E,F)$ of $p$-summing operators equipped with the $p$-summing norm $\pi_p$; see page 85 below for the precise definition. These operators are a generalisation of the Hilbert-Schmidt operators to Banach spaces (recall from the preliminaries that for Hilbert spaces $U$ and $H$ one has $\Pi_p(U,H) = L_{\mathrm{HS}}(U,H)$ for all $p$ with equivalent norms). They seem to be a natural choice also because according to the results of Kwapień and Schwartz the $p$-summing operators are $p$-Radonifying, which means that $\psi \in \Pi_p(E,F)$ map weakly $p$-integrable cylindrical random variable $(L(t)-L(s))$ into $(L(t)-L(s))\psi^*$ and the latter is induced by a genuine vector-valued random variable with finite $p$-th moment.

We show that if the cylindrical Lévy measure $\nu$ of $L$ satisfies

$$\int_{\mathbb{R}} |x|^p \, (\nu \circ (x^*)^{-1})(\mathrm{d}x) < \infty \tag{5.1}$$

for all $x^* \in E^*$, then the stochastic integral is continuous as a mapping between the spaces $L^p(\Omega \times [0,T]; \Pi_p(E,F))$ and $L^p(\Omega; F)$ or in other words that

$$\mathbb{E}\left[\left\|\int_0^T \Psi(s)\,\mathrm{d}L(s)\right\|^p\right] \le c\mathbb{E}\left[\int_0^T \pi_p(\Psi(s))^p\,\mathrm{d}s\right] \tag{5.2}$$

for any predictable $\Pi_p(E,F)$-valued integrand $\Psi$. The idea of utilising the continuity between certain $L^p$-spaces to construct the integral and solve SPDEs goes back to Saint Loubert Bié [90] who considered equations driven by the Poisson random measures. This construction was further generalised to integrals taking values in martingale type $p$ spaces by Brzeźniak and Hausenblas [13, App. C]. We present a similar result for integrals with respect to cylindrical Lévy processes, where the difficulty is to obtain the integral as a genuine Banach space-valued random variable.

The main ingredient in the proof of the continuity is a result by Schwartz [97], which gives a bound of the $p$-th moment of a Radonified measure by the weak $p$-th moments of the cylindrical measure: if $\mu$ is a cylindrical measure on $E$ and $u\colon E \to F$, then

$$\left(\int_F \|x\|^p \, \left(\mu \circ u^{-1}\right)(\mathrm{d}x)\right)^{1/p} \le \pi_p(u) \sup_{x^* \in B_{E^*}} \left(\int_{\mathbb{R}} |x|^p \, \left(\mu \circ (x^*)^{-1}\right)(\mathrm{d}x)\right)^{1/p}. \tag{5.3}$$

This Schwartz inequality can be viewed as a generalisation of the Pietsch factorisation theorem, which asserts that for any $p$-summing operator $u$ there exists a measure $\rho$ on $\mathcal{B}(B_{E^*})$ such that

$$\|ux\| \leq \pi_p(u) \left( \int_{B_{E^*}} |x^*(x)|^p \, \rho(\mathrm{d}x^*) \right)^{1/p}, \qquad x \in E.$$

Note that the right-hand side of (5.2) does not depend on the integrator, which does not match the weakly square-integrable case in Hilbert spaces, where the optimal bound is by

$$\mathbb{E}\left[ \int_0^T \left\| \Psi(s) Q^{1/2} \right\|^2_{L_{\mathrm{HS}}(U,H)} \, \mathrm{d}s \right],$$

with $Q$ the covariance operator of $L$. In the future work we plan to generalise the construction so that it depends on the cylindrical characteristics of the cylindrical Lévy process.

We prove existence and uniqueness of mild solutions for the equation (1.4) with $F\colon E \to F$ and $G\colon F \to \Pi_p(E,F)$ under the standard Lipschitz and linear growth assumptions on $F$ and $G$, this time requiring that $G$ is Lipschitz as a mapping to $\Pi_p(E,F)$. The result follows in the same manner as in Peszat and Zabczyk [74, Th. 9.29], however the proof of the stochastic continuity of the stochastic convolution requires certain auxiliary result about the convergence of operators in the $p$-summing norm.

## Preliminaries

We recall some notions on the Banach spaces theory from [89, 103], which we need in this chapter. Let $p \geq 1$. A Banach space is of martingale type $p$ if for some constant $C$ and for any discrete $E$-valued martingale $(M_k)_{k=1}^n$ one has

$$\sup_{k=1,\dots,n} \mathbb{E}\left[\|M_k\|^p\right] \leq C_p \sum_{k=1}^n \mathbb{E}\left[\|M_k - M_{k-1}\|^p\right],$$

with the convention that $M_0 = 0$.

A Banach space $E$ has the approximation property if for every $\varepsilon > 0$ and for every compact set $K \subset E$ there exists a finite rank operator $\psi\colon E \to E$ such that $\|\psi x - x\| \leq \varepsilon$ for all $x \in K$. The space $E$ has a metric approximation property if one can find operators $\psi$ as above with $\|\psi\| \leq 1$. $E$ has the Radon–Nikodym property if for every probability space $(\Omega, \mathcal{F}, P)$ and an absolutely continuous $E$-valued measure $\mu$ on $\mathcal{F}$, there exists a measurable function $f\colon \Omega \to E$

such that

$$\mu(A) = \int_A f(\omega)\, \mu(\mathrm{d}\omega), \qquad A \in \mathcal{F}.$$

It is well known that every reflexive Banach space has the Radon–Nikodym property; see [101, Cor. 2, p. 219].

An operator $\psi\colon E \to F$ is $p$-summing if there exists a constant $C$ such that for all $n \in \mathbb{N}$ and all finite sequences $x_1, \ldots, x_n \in E$

$$\sum_{k=1}^{n} \|\psi x_k\|^p \le C^p \sup_{x^* \in B_{E^*}} \sum_{k=1}^{n} |\langle x^*, x_k \rangle|^p,$$

see [29]. The $p$-summing norm of $\psi$ denoted as $\pi_p(\psi)$ is the smallest constant $C$ such that the above condition holds. If $E$ and $F$ are Hilbert spaces, the space $\Pi_p(E, F)$ of $p$-summing operators coincides with the space of Hilbert-Schmidt operators; see [29, Th. 4.10 and Cor. 4.13]. Moreover, the $p$-summing norms and the Hilbert-Schmidt norm in $L_{\mathrm{HS}}(E, F)$ are equivalent.

The $p$-Radonifying operators were defined in Section 2.1. We recall the following well-known characterisation from [101, Th. VI.5.4 and Th. VI.5.5].

**Theorem 5.1.** *Assume that either*

*(i) $p > 1$ or*

*(ii) $p = 1$ and $F$ has the Radon–Nikodym property.*

*Then the classes of $p$-Radonifying and $p$-summing operators from $E$ to $F$ coincide.*

## 5.1 Some results on $p$-summing operators

Our approach to stochastic integration with respect to a cylindrical Lévy process is based on a generalisation of Pietsch's factorisation theorem, which is due to Schwartz; see [97, p. 23-28] and [95]. For a measure $\mu$ on $\mathcal{B}(E)$ and $p \in [1, 2]$ we define

$$\|\mu\|_p := \left( \int_E \|x\|^p \mu(dx) \right)^{1/p},$$

and say that $\mu$ is of order $p$ if $\|\mu\|_p < \infty$. For a cylindrical measure $\mu$ on $\mathcal{Z}(E)$ we define

$$\|\mu\|_p^* = \sup_{x^* \in B_{E^*}} \|x^*(\mu)\|_p,$$

and we say that $\mu$ is of weak order $p$ if $\|\mu\|_p^* < \infty$.

**Theorem 5.2.** *For $p \in [1, 2]$, assume either that $p > 1$ or that $F$ has the Radon–Nikodym property, and let $\mu$ be a cylindrical probability measure on $\mathcal{Z}(E)$. If $u \colon E \to F$ is $p$-summing then*

$$\|u(\mu)\|_p \leq \pi_p(u)\|\mu\|_p^*. \tag{5.4}$$

*Proof.* See [97] or [95, 96] or Appendix A. □

For establishing continuity of the integral operator in the next section, we need a result on the convergence of $p$-summing operators between Banach spaces. In the case of Hilbert spaces, this convergence result can easily be seen:

**Lemma 5.3.** Suppose that $U$ and $H$ are separable Hilbert spaces and let $\psi \colon U \to H$ be a Hilbert-Schmidt operator. If $(\varphi_n)$ is a sequence of operators $\varphi_n \colon H \to H$ converging strongly to 0 as $n \to \infty$, then the composition $\varphi_n\psi$ converges to 0 in the Hilbert-Schmidt norm.

*Proof.* Let $(e_n)$ be an orthonormal basis of $U$ and write

$$\|\varphi_n\psi\|_{L_{\mathrm{HS}}(U,H)}^2 = \sum_{k=1}^{\infty}\|\varphi_n\psi e_k\|^2.$$

Every term in the above sum converges to 0 as $n \to \infty$ due to the strong convergence of $\varphi_n$. Let $M := \sup_{n\in\mathbb{N}}\|\varphi_n\|$, which is finite because every strongly convergent sequence is bounded. Since

$$\sum_{k=1}^{\infty}\|\varphi_n\psi e_k\|^2 \leq M\sum_{k=1}^{\infty}\|\psi e_k\|^2 = M\|\psi\|_{L_{\mathrm{HS}}(U,H)}^2,$$

it follows by the Lebesgue dominated convergence theorem that $\|\varphi_n\psi\|_{L_{\mathrm{HS}}(U,H)}^2 \to 0$. □

The following result extends this conclusion in Hilbert spaces to the Banach space setting by approximating $p$-summing operators with finite rank operators.

**Theorem 5.4.** *Suppose that $E$ is a reflexive Banach space or a Banach space with separable dual and that $E^{**}$ has the approximation property. If $\psi \colon E \to F$ is a $p$-summing operator and $(\varphi_n)$ is a sequence of operators $\varphi_n \colon F \to F$ converging strongly to 0 then we have $\pi_p(\varphi_n\psi) \to 0$ as $n \to \infty$.*

*Proof.* We first prove the assertion for finite rank operators $\psi\colon E \to F$, in which case we can assume that $\psi = \sum_{k=1}^{N} x_k^* \otimes y_k$ for some $x_k^* \in E^*$ and $y_k \in F$. Then $\varphi_n\psi = \sum_{k=1}^{N} x_k^* \otimes (\varphi_n y_k)$ and since $\pi_p(x^* \otimes y) = \|x^*\|\|y\|$ (see [29, p. 37]), we estimate

$$\pi_p(\varphi_n\psi) \leq \sum_{k=1}^{N} \pi_p(x_k^* \otimes (\varphi_n y_k)) = \sum_{k=1}^{N} \|x_k^*\|\|\varphi_n y_k\| \to 0,$$

because $\|\varphi_n y_k\| \to 0$ for every $k \in \{1, \ldots, N\}$.

Consider now the case of a general $p$-summing operator $\psi$. Under the assumptions on $E$ and $F$, by Corollary 1 in [91], the finite rank operators are dense in the space of $p$-summing operators. That is, there exists a sequence of finite rank operators $(\psi_k)$ such that $\pi_p(\psi_k - \psi) \to 0$ as $k \to \infty$. It follows that

$$\pi_p(\varphi_n\psi) \leq \pi_p(\varphi_n\psi_k) + \pi_p(\varphi_n(\psi - \psi_k)) \qquad \text{for all } k,\, n \in \mathbb{N}. \tag{5.5}$$

Fix $\varepsilon > 0$ and let $c := \sup\{\|\varphi_n\| : n \in \mathbb{N}\}$. Choose $k \in \mathbb{N}$ such that $\pi_p(\psi - \psi_k) \leq \frac{\varepsilon}{2c}$. Since $\psi_k$ is a finite rank operator, the argument above guarantees that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\pi_p(\varphi_n\psi_k) \leq \frac{\varepsilon}{2}$. Inequality (5.5) implies for all $n \geq n_0$ that

$$\pi_p(\varphi_n\psi) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

**Remark 5.5.** The proof of Theorem 5.4 relies on the density of finite rank operators in the space of $p$-summing operators. This holds under more general assumptions than assumed in Theorem 5.4; see [91, p. 384 and 388].

However, the result of Theorem 5.4 does not hold in the case of arbitrary Banach spaces as the following example adapted from [29, p. 38-39] shows. Choose $E = \ell^1(\mathbb{R})$ and $F = \ell^2(\mathbb{R})$ and equip both spaces with the canonical basis $(e_n)$, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$. We take $\psi = \mathrm{Id}\colon E \to F$, which is 1-Radonifying by the Grothendieck theorem; see [29, p. 38-39]. Furthermore, we define $\varphi_n = e_n \otimes e_n$, i.e. $\varphi_n(x) = x(n)e_n = (0, \ldots, 0, x(n), 0, \ldots)$ for a sequence $x = (x(n)) \in \ell^1(\mathbb{R})$. Then $\varphi_n$ converges to 0 strongly as $n \to \infty$, but since $\varphi_n\psi$ is finite rank we have $\pi_1(\varphi_n\psi) = \|e_n\|\|e_n\| = 1$ for all $n \in \mathbb{N}$. This counterexample shows that the assumptions on the space $E$ in Theorem 5.4 cannot be dropped.

## 5.2 Radonification of increments and stochastic integral

In this section we fix $p \in [1,2]$. Fix $0 \le s < t \le T$. An $\mathcal{F}_s$-measurable random variable $\Psi\colon \Omega \to \Pi_p(E,F)$ is called *simple* if it is of the form

$$\Psi = \sum_{k=1}^{m} \mathbb{1}_{A_k} \psi_k, \tag{5.6}$$

for some disjoint sets $A_1, \dots, A_m \in \mathcal{F}_s$ and $\psi_1, \dots, \psi_m \in \Pi_p(E,F)$. Assuming that $p > 1$ or for $p = 1$ with $F$ having the Radon–Nikodym property, each $p$-summing operator $\psi_k\colon E \to F$ is $p$-Radonifying, it follows that the cylindrical random variable $(L(t) - L(s))\psi_k^*$ is induced by a classical, $F$-valued random variable, which we denote by $J_{s,t}(\psi_k)\colon \Omega \to F$, that is

$$\big(L(t) - L(s)\big)(\psi_k^* x^*) = x^*(J_{s,t}(\psi_k)) \qquad \text{for all } x^* \in F^*.$$

This enables us to define the $F$-valued random variables

$$J_{s,t}(\Psi) := \sum_{k=1}^{m} \mathbb{1}_{A_k} J_{s,t}(\psi_k). \tag{5.7}$$

**Lemma 5.6.** (Radonification of the increments)
Assume that the cylindrical Lévy process $L$ has finite $p$-th weak moments. We also assume that if $p = 1$, then $F$ has the Radon–Nikodym property. For fixed $0 \le s < t \le T$, the random variable $J_{s,t}(\Psi)$ defined in (5.7) is satisfies

$$\left(\mathbb{E}\left[\|J_{s,t}(\Psi)\|^p\right]\right)^{1/p} \le \|L(t-s)\|_{\mathcal{L}(E^*, L^p(\Omega;\mathbb{R}))} \left(\mathbb{E}\left[\pi_p(\Psi)^p\right]\right)^{1/p}. \tag{5.8}$$

*Proof.* Let $\Psi$ be of the form (5.6). Since the sets $A_k$ are disjoint it follows that

$$\mathbb{E}\left[\|J_{s,t}(\Psi)\|^p\right] = \mathbb{E}\left[\left\|\sum_{k=1}^{m} \mathbb{1}_{A_k} J_{s,t}(\psi_k)\right\|^p\right] = \mathbb{E}\left[\sum_{k=1}^{m} \mathbb{1}_{A_k} \|J_{s,t}(\psi_k)\|^p\right].$$

Using the fact that each $A_k$ is $\mathcal{F}_s$-measurable and that $J_{s,t}(\psi_k)$ is independent of $\mathcal{F}_s$ we can calculate further

$$\mathbb{E}\big[\|J_{s,t}(\Psi)\|^p\big] = \sum_{k=1}^{m} \mathbb{E}\big[\mathbb{E}\big[\mathbb{1}_{A_k} \|J_{s,t}(\psi_k)\|^p | \mathcal{F}_s\big]\big] = \sum_{k=1}^{m} P(A_k)\mathbb{E}\big[\|J_{s,t}(\psi_k)\|^p\big]. \tag{5.9}$$

In order to estimate $\mathbb{E}\left[\|J_{s,t}(\psi_k)\|^p\right]$ we apply Theorem 5.2 to obtain that

$$\left(\mathbb{E}\left[\|J_{s,t}(\psi_k)\|^p\right]\right)^{1/p} \le \pi_p(\psi_k)\|L(t)-L(s)\|_p^*. \tag{5.10}$$

Since stationary increments of the real-value Lévy processes yield

$$\left(\mathbb{E}\left[|(L(t)-L(s))x^*|^p\right]\right)^{1/p} = \left(\mathbb{E}\left[|L(t-s)x^*|^p\right]\right)^{1/p} \qquad \text{for all } x^* \in E^*,$$

it follows that

$$\|L(t)-L(s)\|_p^* = \sup_{x^*\in B_{E^*}} \left(\mathbb{E}\left[|L(t-s)x^*|^p\right]\right)^{1/p} = \|L(t-s)\|_{\mathcal{L}(E^*,L^p(\Omega;\mathbb{R}))}. \tag{5.11}$$

Note, that by the closed graph theorem and the continuity of $L(t-s)\colon E^* \to L^0(\Omega,\mathcal{F},P;\mathbb{R})$, the mapping $L(t-s)\colon E^* \to L^p(\Omega,\mathcal{F},P;\mathbb{R})$ is continuous. This shows that the last expression in (5.11) is finite. Applying estimates (5.10) and (5.11) to (5.9) results in

$$\begin{aligned}\left(\mathbb{E}\left[\|J_{s,t}(\Psi)\|^p\right]\right)^{1/p} &\le \left(\sum_{k=1}^m P(A_k)\pi_p(\psi_k)^p\|L(t-s)\|^p_{\mathcal{L}(E^*,L^p(\Omega;\mathbb{R}))}\right)^{1/p}\\ &= \|L(t-s)\|_{\mathcal{L}(E^*,L^p(\Omega;\mathbb{R}))}\left(\mathbb{E}\big[\pi_p(\Psi)^p\big]\right)^{1/p},\end{aligned}$$

which proves (5.8). □

For defining the stochastic integral, let $\Lambda^p(0,T;\Pi_p(E,F))$ denote the space of equivalence classes of predictable processes $\Psi\colon [0,T]\times\Omega \to \Pi_p(E,F)$ such that

$$\|\Psi\|_\Lambda := \left(\mathbb{E}\left[\int_0^T \pi_p(\Psi(s))^p\,\mathrm{d}s\right]\right)^{1/p} < \infty,$$

that is $\Lambda^p(0,T;\Pi_p(E,F)) = L^p\big([0,T]\times\Omega,\mathcal{P},\mathrm{Leb}\otimes P;\Pi_p(E,F)\big)$. A simple stochastic process is of the form

$$\Psi(t) = \Psi_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{N-1} \Psi_k \mathbb{1}_{(t_k,t_{k+1}]}(t), \tag{5.12}$$

where $0 = t_0 < t_1 < \cdots < t_N = T$, and each $\Psi_k$ is an $\mathcal{F}_{t_k}$-measurable, $\Pi_p(E,F)$-valued random variable with $\mathbb{E}[\pi_p(\Psi_k)^p] < \infty$. We denote with $\Lambda_0^S(0,T;\Pi_p(E,F))$ the space of simple processes of the form (5.12) where each $\Psi_k$ is a simple random variable of the form

(5.6), i.e. taking only a finite number of values.

Since for stochastic processes in $\Lambda_0^S(0,T;\Pi_p(E,F))$ the Radonification of the increments is defined by the operator $J_{s,t}$ according to Lemma 5.6 , we can define the integral operator by

$$I\colon \Lambda_0^S(0,T;\Pi_p(E,F)) \to L^p(\Omega,\mathcal{F}_T,P;F), \qquad I(\Psi) := \sum_{k=0}^{N-1} J_{t_k,t_{k+1}}(\Psi_k). \tag{5.13}$$

**Lemma 5.7.** The space $\Lambda_0^S(0,T;\Pi_p(E,F))$ is dense in $\Lambda^p(0,T;\Pi_p(E,F))$ with respect to $\|\cdot\|_\Lambda$.

*Proof.* The result follows from the construction in the proof of [26, Prop. 4.22(ii)]. □

**Theorem 5.8.** *(stochastic integration)*
*Assume that the cylindrical Lévy process $L$ has the characteristics $(b,0,\nu)$ and satisfies*

$$\int_E |x^*(x)|^p \,\nu(\mathrm{d}x) < \infty, \qquad \text{for all } x^* \in E^* \tag{5.14}$$

*Suppose also that $F$ is of martingale type $p$ and that if $p=1$, then $F$ has the Radon–Nikodym property. Then the integral operator $I$ defined in (5.13) is continuous and extends to the operator*

$$I\colon \Lambda^p(0,T;\Pi_p(E,F)) \to L^p(\Omega,\mathcal{F}_T,P;F).$$

*Proof.* Let $\Psi$ in $\Lambda_0^S(0,T;\Pi_p(E,F))$ be given by (5.12) where $\Psi_k$ is of the form

$$\Psi_k = \sum_{i=1}^{m_k} 1_{A_{k,i}} \psi_{k,i},$$

for some disjoint sets $A_{k,1},\dots,A_{k,m_k} \in \mathcal{F}_{t_k}$ and $\psi_{k,1},\dots,\psi_{k,m_k} \in \Pi_p(E,F)$ for all $k \in \{0,\dots,N\}$.

By Proposition 3.1, the cylindrical Lévy process $L$ can be decomposed into a sum of a deterministic drift and cylindrical Lévy martingale $L = B + M$. Both $B$ and $M$ are cylindrical Lévy processes, and we can integrate separately with respect to $B$ and $M$:

$$I(\Psi) = I_B(\Psi) + I_M(\Psi). \tag{5.15}$$

For the first integral in (5.15) we calculate

$$\|I_B(\Psi)\|^p = \sup_{y^*\in B_{F^*}} \left| y^*\left(\int_0^T \Psi(s)\,\mathrm{d}B(s)\right)\right|^p = \sup_{y^*\in B_{F^*}} \left|\int_0^T B(1)(\Psi^*(s)y^*)\,\mathrm{d}s\right|^p.$$

By Hölder's inequality with $q = \frac{p}{p-1}$ and $q = \infty$ if $p = 1$ we obtain

$$\begin{aligned}\|I_B(\Psi)\|^p &\le \sup_{y^*\in B_{F^*}} T^{p/q} \int_0^T |B(1)(\Psi^*(s)y^*)|^p\,\mathrm{d}s \\ &\le T^{p/q}\|B(1)\|^p_{\mathcal{L}(E^*,\mathbb{R})} \int_0^T \|\Psi^*(s)\|^p_{\mathcal{L}(F^*,E^*)}\,\mathrm{d}s.\end{aligned}$$

Since $\|\Psi^*(s)\|_{\mathcal{L}(F^*,E^*)} = \|\Psi(s)\|_{\mathcal{L}(E,F)} \le \pi_p(\Psi(s))$ according to [29, p. 31], it follows that

$$\|I_B(\Psi)\|^p \le T^{p/q}\|B(1)\|^p_{\mathcal{L}(E^*,\mathbb{R})} \int_0^T \pi_p(\Psi(s))^p\,\mathrm{d}s. \tag{5.16}$$

For estimating the second term in (5.15), define the Banach space

$$R_p = \left\{X\colon (0,T]\times\Omega \to H:\ \text{measurable and } \sup_{t\in(0,T]} \frac{1}{t^{1/p}}\big(\mathbb{E}\left[\|X(t)\|^p\right]\big)^{1/p} < \infty\right\}$$

with the norm $\|X\|_{R_p} = \sup_{t\in(0,T]} \frac{1}{t^{1/p}} \left(\mathbb{E}\left[\|X(t)\|^p\right]\right)^{1/p}$. Note that the the Lévy measure of $M(1)x^*$ is given by $\nu\circ(x^*)^{-1}$. By standard properties of real-valued Lévy martingales, e.g. [74, Th. 8.23(i)], it follows that there exists a constant $c > 0$ such that

$$\mathbb{E}\big[|M(t)x^*|^p\big] \le ct\int_{\mathbb{R}} |\beta|^p\,(\nu\circ(x^*)^{-1})(\mathrm{d}\beta) \qquad \text{for all } x^*\in E^*.$$

It follows that we can consider the map $M\colon E^*\to R_p$ defined by $Mx^* = (M(t)x^* : t\in(0,T])$. To show that $M$ is continuous we use the closed graph theorem: let $x_n^*$ converge to $x^*$ in $E^*$ and $Mx_n^*$ to some $Y$ in $R_p$. It follows that $M(t)x_n^* \to Y(t)$ in $L^p(\Omega;H)$ for every $t\in(0,T]$. On the other hand, continuity of $M(t)\colon E^*\to L^1(\Omega,\mathcal{F},P;\mathbb{R})$ implies $M(t)x_n^*\to M(t)x^*$ in $L^0(\Omega;H)$. Thus, $Y(t) = M(t)x^*$ for all $t\in(0,T]$ a.s., and the closed graph theorem gives that $M\colon E^*\to R_p$ is continuous. It follows that

$$\|M(t_{k+1}-t_k)\|^p_{\mathcal{L}(E^*;L^p(\Omega;\mathbb{R}))} \le (t_{k+1}-t_k)\|M\|_{\mathcal{L}(E^*,R_p)}. \tag{5.17}$$

Let $J_{t_k,t_{k+1}}$ denote the operators defined in (5.7) with $L$ replaced by $M$. Since $F$ is of martingale type $p$ here exists a constant $C_p > 0$ such that Lemma 5.6 and inequality (5.17) imply

$$\begin{aligned}
\mathbb{E}\left[\|I_M(\Psi)\|^p\right] &= \mathbb{E}\left[\left\|\sum_{k=0}^{N-1} J_{t_k,t_{k+1}}(\Psi_k)\right\|^p\right] \\
&\le C_p \mathbb{E}\left[\sum_{k=0}^{N-1} \left\|J_{t_k,t_{k+1}}(\Psi_k)\right\|^p\right] \\
&\le C_p \sum_{k=0}^{N-1} \|M(t_{k+1}-t_k)\|^p_{\mathcal{L}(E^*;L^p(\Omega;\mathbb{R}))} \mathbb{E}\left[\pi_p(\Psi_k)^p\right] \\
&\le C_p \|M\|_{\mathcal{L}(E^*,R_p)} \mathbb{E}\left[\int_0^T \pi_p(\Psi(s))^p \,\mathrm{d}s\right].
\end{aligned}$$

Together with (5.16), this completes the proof. □

By rewriting condition (5.14) as

$$\int_{B_{\mathbb{R}}^c} |\beta|^p \,(\nu \circ (x^*)^{-1})(\mathrm{d}\beta) < \infty \quad \text{and} \quad \int_{B_{\mathbb{R}}} |\beta|^p \,(\nu \circ (x^*)^{-1})(\mathrm{d}\beta) < \infty \quad \text{for all } x^* \in E^*,$$

it follows that condition (5.14) is equivalent to

$$(L(t)x^* : t \ge 0) \text{ is } p\text{-integrable and has finite } p\text{-variation for each } x^* \in E^*,$$

see [10]. This is a natural requirement if we want to control the moments, see [90, 63] and Remark 5.10 below. The interplay between the integrability of the Lévy process and its Blumenthal–Getoor index was observed also in [21, 22].

**Example 5.9** (Gaussian case)**.** Note that if $p < 2$, then $L$ cannot have the Gaussian part for the assertion to hold. Indeed, let $W$ be a one-dimensional Wiener process and suppose for contradiction that

$$\mathbb{E}\left[\left|\int_0^T \Psi(t)\,\mathrm{d}W(t)\right|^p\right] \le C\mathbb{E}\left[\int_0^T |\Psi(t)|^p \,\mathrm{d}t\right] \tag{5.18}$$

for some constant $C$ and every real-valued predictable process $\Psi$ with $\mathbb{E}\left[\int_0^T |\Psi(t)|^2 \,\mathrm{d}t\right] < \infty$. Choose for each $n \in \mathbb{N}$ the stochastic process $\Psi_n(t) = \mathbb{1}_{[0,1/n]}(t)$ for $t \in [0,T]$. By [39, Sec.

3.478] we calculate

$$\mathbb{E}\left[\left|\int_0^T \Psi_n(t)\,\mathrm{d}W(t)\right|^p\right] = \mathbb{E}\left[\left|W\left(\frac{1}{n}\right)\right|^p\right] = \left(\frac{1}{n}\right)^{\frac{p}{2}} \frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}.$$

But on the other side, since $\mathbb{E}\left[\int_0^T |\Psi_n(t)|^p\,\mathrm{d}t\right] = \frac{1}{n}$, solving (5.18) for $n$ yields

$$n^{1-\frac{p}{2}} \leq \frac{C\sqrt{\pi}}{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)},$$

which results in a contradiction by taking the limit as $n \to \infty$.

**Example 5.10** (Stable case)**.** The canonical $\alpha$-stable cylindrical Lévy process has the characteristic function $\varphi_{L(1)}(x^*) = \exp(-\|x^*\|^\alpha)$ for each $x^* \in E^*$; see [85]. It follows that the real-valued Lévy process $(L(t)x^* : t \geq 0)$ is symmetric $\alpha$-stable with Lévy measure $\left(\nu \circ (x^*)^{-1}\right)(\mathrm{d}\beta) = c\frac{1}{|\beta|^{1+\alpha}}\mathrm{d}\beta$ for a constant $c > 0$. Condition (5.14) fails to hold since

$$\int_{B_{\mathbb{R}}} |\beta|^p \left(\nu \circ (x^*)^{-1}\right)(\mathrm{d}\beta) = \infty \text{ for } p \leq \alpha, \qquad \int_{B_{\mathbb{R}}^c} |\beta|^p \left(\nu \circ (x^*)^{-1}\right)(\mathrm{d}\beta) = \infty \text{ for } p \geq \alpha.$$

One can observe in a similar way as in the Gaussian case that the stochastic integral operator with respect to the $\alpha$-stable cylindrical Lévy process $L$ is not continuous. For simplicity, let $L$ be a one-dimensional stable Lévy process. If $\Psi_n(t) = \mathbb{1}_{[0,1/n]}(t)$, then in the inequality

$$\mathbb{E}\left[\left|\int_0^T \Psi_n(t)\ \mathrm{d}L(t)\right|^p\right] \leq C\mathbb{E}\left[\int_0^T |\Psi_n(t)|^p\,\mathrm{d}t\right], \tag{5.19}$$

the left-hand side is infinite for $p \geq \alpha$ so we assume that $p < \alpha$. Using the self-similarity of stable processes we calculate

$$\mathbb{E}\left[\left|\int_0^T \Psi_n(t)\ \mathrm{d}L(t)\right|^p\right] = \mathbb{E}\left[\left|L\left(\frac{1}{n}\right)\right|^p\right] = \mathbb{E}\left[\frac{1}{n^{p/\alpha}}|L(1)|^p\right].$$

Solving (5.19) for $n$ yields

$$n^{(\alpha-p)/\alpha} \leq \frac{C}{\mathbb{E}\left[|L(1)|^p\right]},$$

which results in a contradiction by taking the limit as $n \to \infty$. Therefore, the stochastic integral mapping with respect to the $\alpha$-stable process cannot be continuous as a mapping from $L^p([0,T]\times\Omega,\mathcal{P},\mathrm{Leb}\otimes P;\mathbb{R})$ to $L^p(\Omega,\mathcal{F}_T,P;\mathbb{R})$ for any $p>0$. A moment inequality with different powers on the left and right-hand sides was proven in the case of real-valued integrands and vector-valued integrators in [86]. They prove for any $\alpha$-stable Lévy process $L$ and $p<\alpha$ that

$$\mathbb{E}\left[\left(\sup_{t\le T}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\|\right)^p\right]\le C\mathbb{E}\left[\left(\int_0^T |\Psi(s)|^\alpha\,\mathrm{d}t\right)^{p/\alpha}\right]. \tag{5.20}$$

We devolop this topic further in Chapter 6.

**Example 5.11.** In the case of a diagonal cylindrical Lévy process, we claim that condition (5.14) is satisfied if and only if

$$\sum_{k=1}^\infty\left(\int_{\mathbb{R}}|\beta|^p\,\rho_k(\mathrm{d}\beta)\right)^{\frac{2}{2-p}}<\infty,$$

where $\rho_k$ is the Lévy measure of $\ell_k$. Indeed, by Lemma 3.10 condition (5.14) simplifies to

$$\int_E|\langle y,x\rangle|^p\,\nu(\mathrm{d}x)=\sum_{k=1}^\infty\int_{\mathbb{R}}|\langle y,\beta e_k\rangle|^p\,(\rho_k\circ m_{e_k}^{-1})(\mathrm{d}\beta)=\sum_{k=1}^\infty|\langle y,e_k\rangle|^p\int_{\mathbb{R}}|\beta|^p\,\rho_k(\mathrm{d}\beta)<\infty$$

for any $y\in E$. This is equivalent to

$$\sum_{k=1}^\infty\alpha_k\int_{\mathbb{R}}|\beta|^p\,\rho_k(\mathrm{d}\beta)<\infty\qquad\text{for any } (\alpha_k)\in\ell^{2/p}(\mathbb{R}_+),$$

which results in $\left(\int_{\mathbb{R}}|\beta|^p\,\rho_k(\mathrm{d}\beta)\right)_{k\in\mathbb{N}}\in\ell^{2/p}(\mathbb{R})^*=\ell^{2/(2-p)}(\mathbb{R})$.

**Example 5.12.** Another example are cylindrical compound Poisson processes

$$L(t)x^*=\sum_{k=1}^{N(t)}X_kx^*\qquad x^*\in E^*, t\ge 0,$$

see Example 2.4. Let $\lambda$ be the intensity of the Poisson process $N$ and let $\rho$ denote the cylindrical distribution of $X_k$. Since the Lévy measure of $(L(t)x^*:t\ge 0)$ is given by $\lambda\left(\rho\circ(x^*)^{-1}\right)$, it follows that condition (5.14) is satisfied if and only if

$$\int_E |x^*(x)|^p \,\rho(\mathrm{d}x) < \infty \qquad x^* \in E^*$$

i.e. if and only if $\rho$ is of weak order $p$.

## 5.3 Existence and uniqueness of solution

In this section we apply the developed integration theory to derive the existence of an evolution equation in a Banach space under standard assumptions. For this purpose, we consider

$$\begin{aligned} \mathrm{d}X(t) &= \big(AX(t) + B(X(t))\big)\,\mathrm{d}t + G\big(X(t)\big)\,\mathrm{d}L(t), \\ X(0) &= X_0, \end{aligned} \tag{5.21}$$

where $X_0$ is an $\mathcal{F}_0$-measurable random variable in a Banach space $F$ and the driving noise $L$ is a cylindrical Lévy process, $L(t)\colon E^* \to L^0(\Omega, \mathcal{F}, P; R)$, $t \geq 0$, with finite weak $p$-th moments. The operator $A$ is the generator of a $C_0$-semigroup $(S(t) : t \geq 0)$ on $F$ and $B\colon F \to F$ and $G\colon F \to \Pi_p(E, F)$ are some functions.

**Definition 5.13.** A mild solution of (5.21) is a predictable process $X$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|^p\right] < \infty$$

for some $p \geq 1$, and such that, for all $t \in [0, T]$, we have $P$-a.s.

$$X(t) = S(t)X_0 + \int_0^t S(t-s)B(X(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s).$$

We assume Lipschitz and linear growth condition on the coefficients $F$ and $G$ and an integrability assumption on the initial condition:

**Assumption 5.14.** For fixed $p \in [1, 2]$ we assume:

(A1) there exists a function $b \in L^1([0, T]; \mathbb{R})$ such that for any $x, x_1, x_2 \in F$

$$\begin{aligned} \|S(t)B(x)\| &\leq b(t)(1 + \|x\|), \\ \|S(t)(B(x_1) - B(x_2))\| &\leq b(t)\|x_1 - x_2\|. \end{aligned}$$

(A2) there exists a function $g \in L^p([0,T];\mathbb{R})$ such that for any $x, x_1, x_2 \in F$

$$\begin{aligned} \pi_p(S(t)G(x)) &\le g(t)(1+\|x\|), \\ \pi_p(S(t)(G(x_1)-G(x_2))) &\le g(t)\|x_1-x_2\|. \end{aligned}$$

(A3) $X_0 \in L^p(\Omega, \mathcal{F}_0, P; F)$.

**Theorem 5.15.** *Let $p \in [1,2]$. Suppose that the Banach spaces $E$ and $F$ satisfy that*

(a) *$E$ is reflexive or has separable dual,*

(b) *$F$ is of martingale type $p$ and $E^{**}$ has the approximation property,*

(c) *if $p = 1$, then $F$ has the Radon–Nikodym property.*

*If $L$ is a cylindrical Lévy process such that* (5.14) *holds, then conditions* (A1)–(A3) *imply that there exists a unique mild solution of* (5.21).

*Proof.* The proof follows closely the proof of [74, Th. 9.29]. We define the space

$$\mathcal{H}_T := \left\{ X\colon [0,T]\times\Omega \to F \text{ is predictable and } \sup_{t\in[0,T]} \mathbb{E}\left[\|X(t)\|^p\right] < \infty \right\},$$

and a family of norms for $\beta \ge 0$:

$$\|X\|_{T,\beta} := \left( \sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}\left[\|X(t)\|^p\right] \right)^{1/p}.$$

Define an operator $K\colon \mathcal{H}_T \to \mathcal{H}_T$ by $K(X) := K_0(X) + K_1(X) + K_2(X)$, where

$$\begin{aligned} K_0(X)(t) &:= S(t)X_0, \\ K_1(X)(t) &:= \int_0^t S(t-s)B(X(s))\,\mathrm{d}s, \\ K_2(X)(t) &:= \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s). \end{aligned}$$

For applying the Banach fixed point theorem, we first show that $K$ indeed maps to $\mathcal{H}_T$. The Bochner integral and the stochastic integral above are well defined because $X$ is predictable and for every $t \in [0,T]$ the mappings

$$[0,t] \times F \ni (s,x) \mapsto S(t-s)B(x), \qquad [0,t] \times F \ni (s,h) \mapsto S(t-s)G(x)$$

are continuous. The appropriate integrability condition follows from (5.22) and (5.23) below.

There exist constants $m \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq me^{\omega t}$ for each $t \geq 0$; see [74, Th. 9.2]. It follows that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|S(t)X_0\|^p\right] \leq m^p e^{p|\omega|T}\mathbb{E}\left[\|X_0\|^p\right] < \infty.$$

By Assumption (A1) and Hölder inequality, we obtain with $q = \frac{p}{p-1}$ that

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}&\left[\left\|\int_0^t S(t-s)B(X(s))\,\mathrm{d}s\right\|^p\right] \\
&\leq \sup_{t\in[0,T]} \mathbb{E}\left[\left(\int_0^t b(t-s)(1+\|X(s)\|)\,\mathrm{d}s\right)^p\right] \\
&\leq \sup_{t\in[0,T]} \mathbb{E}\left[\left(\int_0^t b(t-s)\,\mathrm{d}s\right)^{p/q}\int_0^t b(t-s)(1+\|X(s)\|)^p\,\mathrm{d}s\right] \\
&\leq \left(\int_0^T b(s)\,\mathrm{d}s\right)^{p/q} 2^{p-1}(1+\|X\|_{T,0}^p)\sup_{t\in[0,T]}\int_0^t b(t-s)\,\mathrm{d}s \\
&= \left(\int_0^T b(s)\,\mathrm{d}s\right)^{1+p/q} 2^{p-1}(1+\|X\|_{T,0}^p) \\
&< \infty.
\end{aligned} \tag{5.22}$$

Similarly, we conclude from Assumption (A2) and Theorem 5.8 that there exists a constant $c > 0$ such that

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}\left[\left\|\int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|^p\right] &\leq c\sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t \pi_p(S(t-s)G(X(s)))^p\,\mathrm{d}s\right] \\
&\leq c\sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t g(t-s)^p(1+\|X(s)\|)^p\,\mathrm{d}s\right] \\
&\leq c2^{p-1}(1+\|X\|_{T,0}^p)\int_0^T g(s)^p\,\mathrm{d}s \\
&< \infty.
\end{aligned} \tag{5.23}$$

Next, we establish that the process $K(X)$ is stochastically continuous. For this purpose, let $\varepsilon > 0$. For each $t \geq 0$ we obtain

$$\begin{aligned}
&\mathbb{E}\left[\|K_1(X)(t+\varepsilon) - K_1(X)(t)\|\right] \\
&= \mathbb{E}\left[\left\|\int_0^{t+\varepsilon} S(t+\varepsilon-s)B(X(s))\,\mathrm{d}s - \int_0^t S(t-s)B(X(s))\,\mathrm{d}s\right\|\right] \\
&= \mathbb{E}\left[\left\|\int_t^{t+\varepsilon} S(t+\varepsilon-s)B(X(s))\,\mathrm{d}s + \int_0^t (S(\varepsilon)-\mathrm{Id})S(t-s)B(X(s))\,\mathrm{d}s\right\|\right] \\
&\leq \mathbb{E}\left[\int_t^{t+\varepsilon} \|S(t+\varepsilon-s)B(X(s))\|\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \|(S(\varepsilon)-\mathrm{Id})S(t-s)B(X(s))\|\,\mathrm{d}s\right] \\
&=: I_1 + I_2.
\end{aligned}$$

Since $\|X(s)\| \leq 1 + \|X(s)\|^p$ for all $s \geq 0$, it follows, for $\varepsilon \to 0$, that

$$I_1 \leq \mathbb{E}\left[\int_t^{t+\varepsilon} b(t+\varepsilon-s)(1+\|X(s)\|)\,\mathrm{d}s\right] \leq \left(2 + \|X\|_{T,0}^p\right)\int_0^\varepsilon b(s)\,\mathrm{d}s \to 0.$$

Similarly we obtain

$$\begin{aligned}
\|(S(\varepsilon)-\mathrm{Id})S(t-s)B(X(s))\| &\leq \left(1 + me^{|\omega|}\right) b(t-s)(1+\|X(s)\|) \\
&\leq (2 + \|X(s)\|^p)\left(1 + me^{|\omega|}\right) b(t-s),
\end{aligned} \tag{5.24}$$

which is Leb$\otimes P$-integrable on $[0,t] \times \Omega$ because

$$\mathbb{E}\left[\int_0^t (2 + \|X(s)\|^p)\, b(t-s)\,\mathrm{d}s\right] \leq \left(2 + \|X\|_{T,0}^p\right)\int_0^t b(s)\,\mathrm{d}s < \infty. \tag{5.25}$$

Since the integrand in $I_2$ tends to 0 as $\varepsilon \to 0$ by the strong continuity of the semigroup, the Lebesgue dominated convergence theorem shows that $I_2$ tends to 0 as $\varepsilon \to 0$. For $K_2$ we obtain by Theorem 5.8 that there exists a constant $c > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\|K_2(X)(t+\varepsilon) - K_2(X)(t)\|^p\right] \\
&= \mathbb{E}\left[\left\|\int_0^{t+\varepsilon} S(t+\varepsilon-s)G(X(s))\,\mathrm{d}L(s) - \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|^p\right] \\
&= \mathbb{E}\left[\left\|\int_t^{t+\varepsilon} S(t+\varepsilon-s)G(X(s))\,\mathrm{d}L(s) + \int_0^t (S(\varepsilon)-\mathrm{Id})S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|^p\right]
\end{aligned}$$

$$
\begin{aligned}
&\le 2^{p-1}\mathbb{E}\left[\left\|\int_t^{t+\varepsilon} S(t+\varepsilon-s)G(X(s))\,\mathrm{d}L(s)\right\|^p + \left\|\int_0^t (S(\varepsilon)-\mathrm{Id})S(t-s)G(X(s))\,\mathrm{d}L(s)\right\|^p\right] \\
&\le c2^{p-1}\mathbb{E}\left[\int_t^{t+\varepsilon} \pi_p(S(t+\varepsilon-s)G(X(s)))^p\,\mathrm{d}s + \int_0^t \pi_p((S(\varepsilon)-\mathrm{Id})S(t-s)G(X(s)))^p\,\mathrm{d}s\right] \\
&\le c2^{p-1}\mathbb{E}\left[\int_t^{t+\varepsilon} 2^{p-1}g(t+\varepsilon-s)^p(1+\|X(s)\|^p)\,\mathrm{d}s + \int_0^t \pi_p((S(\varepsilon)-\mathrm{Id})S(t-s)G(X(s)))^p\,\mathrm{d}s\right] \\
&=: c2^{p-1}(2^{p-1}J_1+J_2),
\end{aligned}
$$

where

$$
J_1 = \mathbb{E}\left[\int_t^{t+\varepsilon} g(t+\varepsilon-s)^p(1+\|X(s)\|^p)\,\mathrm{d}s\right] \le \left(1+\|X\|_{T,0}^p\right)\int_t^{t+\varepsilon} g(t+\varepsilon-s)^p\,\mathrm{d}s \to 0
$$

as $\varepsilon \to 0$, and

$$
J_2 = \mathbb{E}\left[\int_0^t \pi_p((\mathrm{Id}-S(\varepsilon))S(t-s)G(X(s)))^p\,\mathrm{d}s\right].
$$

By Theorem 5.4 the integrand $\pi_p((\mathrm{Id}-S(\varepsilon))S(t-s)G(X(s)))^p$ converges to 0 pointwise on $[0,t]\times\Omega$. Moreover, it is bounded by $(1+me^{|\omega|})^p g(t-s)^p(2+\|X(s)\|^p)$, which is $\mathrm{Leb}\otimes P$-integrable. Thus, the Lebesgue theorem on dominated convergence implies that $J_2 \to 0$ as $\varepsilon \to 0$. This completes the proof of stochastic continuity of $K(X)$ from the right. Similarly, we have for $0 < \varepsilon \le t$

$$
\begin{aligned}
&\mathbb{E}\left[\|K_1(X)(t-\varepsilon)-K_1(X)(t)\|\right] \\
&= \mathbb{E}\left[\left\|\int_0^{t-\varepsilon} S(t-\varepsilon-s)B(X(s))\,\mathrm{d}s - \int_0^t S(t-s)B(X(s))\,\mathrm{d}s\right\|\right] \\
&= \mathbb{E}\left[\left\|\int_0^{t-\varepsilon} S(t-\varepsilon-s)\left(\mathrm{Id}-S(\varepsilon)\right)B(X(s))\,\mathrm{d}s + \int_{t-\varepsilon}^t S(t-s)B(X(s))\,\mathrm{d}s\right\|\right] \\
&\le \mathbb{E}\left[\int_{t-\varepsilon}^t \|S(t-s)B(X(s))\|\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^{t-\varepsilon} \|S(t-\varepsilon-s)\left(\mathrm{Id}-S(\varepsilon)\right)B(X(s))\|\,\mathrm{d}s\right] \\
&=: I_1 + I_2.
\end{aligned}
$$

The first integral converges to 0 by the same argument as above. For the second one we estimate

$$
\|S(t-\varepsilon-s)\left(\mathrm{Id}-S(\varepsilon)\right)B(X(s))\| \le me^{|\omega|T}\|\left(\mathrm{Id}-S(\varepsilon)\right)B(X(s))\| \to 0
$$

as $\varepsilon \to 0$ almost surely thanks to the strong continuity of $S$. Arguing as in (5.24) and (5.25) we see that we can apply the Lebesgue dominated convergence theorem to get that $I_2$ converges to 0 as $\varepsilon \to 0$. One similarly obtains stochastic continuity of $K_2(X)$. Thus the process $K(X)$ is stochastically continuous.

In particular, stochastic continuity guarantees the existence of a predicable modification of $K(X)$ by [74, Prop. 3.12]. In summary, we obtain that $K$ maps from $\mathcal{H}_T$ to $\mathcal{H}_T$. For applying Banach's fixed point theorem it is enough to show that $K$ is a contraction for some $\beta$. We have

$$\|K(X_1) - K(X_2)\|_{T,\beta}^p \leq 2^{p-1} \left( \|K_1(X_1) - K_1(X_2)\|_{T,\beta}^p + \|K_2(X_1) - K_2(X_2)\|_{T,\beta}^p \right).$$

We estimate each term on the right-hand side separately.

For the part corresponding to the drift we calculate

$$\begin{aligned}
&\|K_1(X_1) - K_1(X_2)\|_{T,\beta}^p \\
&\leq \sup_{t\in[0,T]} e^{-\beta t} \mathbb{E} \left[ \left( \int_0^t b(t-s) \|X_1(s) - X_2(s)\| \,\mathrm{d}s \right)^p \right] \\
&= \sup_{t\in[0,T]} e^{-\beta t} \mathbb{E} \left[ \left( \int_0^t b(t-s)^{1/q} b(t-s)^{1/p} \|X_1(s)) - X_2(s)\| \,\mathrm{d}s \right)^p \right] \\
&\leq \left( \int_0^T b(t-s) \,\mathrm{d}s \right)^{p/q} \sup_{t\in[0,T]} e^{-\beta t} \int_0^t b(t-s) \mathbb{E} \left[ \|X_1(s)) - X_2(s)\|^p \right] \mathrm{d}s \\
&= \left( \int_0^T b(s) \,\mathrm{d}s \right)^{p/q} \sup_{t\in[0,T]} e^{-\beta t} \int_0^t b(t-s) e^{\beta s} e^{-\beta s} \mathbb{E} \left[ \|X_1(s) - X_2(s)\|^p \right] \mathrm{d}s \\
&\leq \left( \int_0^T b(s) \,\mathrm{d}s \right)^{p/q} \|X_1 - X_2\|_{T,\beta}^p \sup_{t\in[0,T]} \int_0^t b(t-s) e^{-\beta(t-s)} \,\mathrm{d}s \\
&= C(\beta) \|X_1 - X_2\|_{T,\beta}^p.
\end{aligned}$$

with $C(\beta) = \left( \int_0^T b(s) \,\mathrm{d}s \right)^{p/q} \int_0^T b(s) e^{-\beta s} \,\mathrm{d}s \to 0$ as $\beta \to \infty$. In the following calculation for the part corresponding to the diffusion we use in the first inequality the continuity of the stochastic integral formulated in Theorem 5.8

$$\|K_2(X_1) - K_2(X_2)\|_{T,\beta}^p \leq c \sup_{t\in[0,T]} e^{-\beta t} \mathbb{E} \left[ \int_0^t \pi_p(S(t-s)(G(X_1(s)) - G(X_2(s))))^p \,\mathrm{d}s \right]$$

$$
\begin{aligned}
&\leq c \sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}\left[\int_0^t g(t-s)^p \|X_1(s)-X_2(s)\|^p \,\mathrm{d}s\right] \\
&= c \sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}\left[\int_0^t g(t-s)^p e^{\beta s} e^{-\beta s} \|X_1(s)-X_2(s)\|^p \,\mathrm{d}s\right] \\
&\leq c\|X_1 - X_2\|_{T,\beta}^p \sup_{t\in[0,T]} \int_0^t e^{-\beta(t-s)} g(t-s)^p \,\mathrm{d}s \\
&= C'(\beta)\|X_1 - X_2\|_{T,\beta}^p,
\end{aligned}
$$

where $C'(\beta) = c\int_0^T e^{-\beta s} g(s)^p \,\mathrm{d}s \to 0$ as $\beta \to \infty$. Consequently, the Banach fixed point theorem implies that there exists a unique $X \in \mathcal{H}_T$ such that $K(X) = X$ which completes the proof. □

**Remark 5.16.** Note that if $E$ and $F$ are Hilbert spaces, then they satisfy assumption (ii) in Theorem 5.8, see e.g. [93, Cor. 1, p. 109]. Thus if $p = 2$ and if $L$ has covariance equal to the identity we recover [74, Th. 9.29].

The following example shows that the $p$-summing norm of certain operators coming from specific SPDEs can be explicitly estimated.

**Example 5.17.** Let $(\mathcal{O}_i, \mathcal{A}_i, \mu_i)$, $i = 1, 2$, be measure spaces and take $E = L^r(\mathcal{O}_1, \mathcal{A}_1\mu_1)$ and $F = L^p(\mathcal{O}_2, \mathcal{A}_2, \mu_2)$. Define the operator $K\colon E \to F$

$$
K(\psi)(y) = \int_{\mathcal{O}_1} k(x,y)\psi(x)\,\mu_1(\mathrm{d}x), \qquad y \in \mathcal{O}_2,
$$

where $k\colon \mathcal{O}_1 \times \mathcal{O}_2 \to \mathbb{R}$ is measurable. It follows from the proofs of Proposition 4.4 and Corollary 4.5 in [17] that the $p$-summing norm of $K$ satisfies

$$
\pi_p(K) \leq \left(\int_{\mathcal{O}_2}\left(\int_{\mathcal{O}_1} |k(x,y)|^{\frac{r}{r-1}}\,\mu_1(\mathrm{d}x)\right)^{\frac{p(r-1)}{r}} \mu_2(\mathrm{d}y)\right)^{\frac{1}{p}}.
$$

Even more specifically it is shown in [17, Prop. 4.8] that, under some assumptions on the weights, the heat semigroup $S(t)$ on $\mathbb{R}^d$ is $p$-summing as a mapping between the weighted spaces $L^p(\mathbb{R}^d, \hat{\varrho}(x)\mathrm{d}x)$ and $L^p(\mathbb{E}^d, \hat{\varrho}(x)\mathrm{d}x)$ provided

$$
\int_{\mathbb{R}^d} \frac{\varrho(x)}{\hat{\varrho}(x)}\,\mathrm{d}x < \infty.
$$

## 5.4 The case of diagonal noise

For diagonal processes (see Example 2.3) the integrability assumption (5.14) can be relaxed to include for instance stable processes in the same way as in Subsection 4.2.2, where the existence of variational solutions was demonstrated. We show the details of the proof for completeness. In fact, here we assume that the coefficients are Lipschitz, which allows us to follow [74, Th. 9.34]. Recall that $P_{c,k}$, $M_{c,k}$ and $L_{c,k}$ were defined in Lemma 4.9 and that $Q_k$ denotes the covariance operator of $M_{c,k}$. We have

**Theorem 5.18.** *Suppose that for a diagonal cylindrical Lévy process $L$ there exists a sequence $(c_j)$ satisfying (4.31)–(4.33). Assume that for every $k \in \mathbb{N}$ there exist functions $a, a_k, b_k \colon [0,T] \to \mathbb{R}_+$ such that for every $h, g \in H$ and $0 \le s \le t \le T$ we have*

$$\begin{aligned} \|S(t-s)(B(h)-B(g))\| &\le a(s)\|h-g\|, \\ \|S(t-s)B(h)\| &\le a(s)(1+\|h\|), \\ \int_0^T a(t)\,\mathrm{d}t &< \infty, \\ \|P_{c,k}(1)(G^*(h)-G^*(g))S^*(t-s)\| &\le a_k(s)\|h-g\|, \\ \|P_{c,k}(1)G^*(h)S^*(t-s)\| &\le a_k(s)(1+\|h\|), \\ \int_0^T a_k(t)\,\mathrm{d}t &< \infty, \\ \left\|S(t-s)(G(h)-G(g))Q_k^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)} &\le b_k(s)\|h-g\|, \\ \left\|G(h)Q_k^{1/2}\right\|_{L_{\mathrm{HS}}(U,H)} &\le b_k(s)(1+\|h\|), \\ \int_0^T b_k^2(t)\,\mathrm{d}t &< \infty. \end{aligned}$$

*Then* (5.21) *has a mild solution, which is unique up to modification.*

*Proof.* The equation driven by $L_{c,k}(t) = P_{c,k}(t) + M_{c,k}(t)$ has unique mild solution by a slight generalisation of [83, Th. 5.1] (one needs to replace the constants in the Lipschitz conditions by functions $a$ and $b$ depending on $t$ and also use the Itô isometry from Theorem 3.5 in the estimates). We denote this solution with $X_{c,k}$,

$$
\begin{aligned}
X_{c,k}(t) = S(t)X_0 + \int_0^t S(t-s)B(X_{c,k}(s)) + P_{c,k}(1)G^*(X_{c,k}(s))S^*(t-s)\,\mathrm{d}s \\
+ \int_0^t S(t-s)G(X_{c,k}(s))\,\mathrm{d}M_{c,k}(s).
\end{aligned}
\tag{5.26}
$$

Similarly, for another sequence $d = (d_j)$ satisfying (4.31)–(4.32) and $n \in \mathbb{N}$ let $X_{d,n}$ be the solution of an equation driven by $L_{d,n}$. We claim that

$$
X_{c,k}(t) = X_{d,n}(t) \quad P\text{-a.s. on } \{t < \tau_c(k) \wedge \tau_d(n)\} \tag{5.27}
$$

for $k, n \in \mathbb{N}$. We have

$$
\begin{aligned}
X_{c,k}(t) - X_{d,n}(t) = &\int_0^t S(t-s)(B(X_{c,k}(s)) - B(X_{d,n}(s)))\,\mathrm{d}s \\
&+ \int_0^t P_{c,k}(1)G^*(X_{c,k}(s))S^*(t-s) - P_{d,n}(1)G^*(X_{d,n}(s))S^*(t-s)\,\mathrm{d}s \\
&+ \int_0^t S(t-s)G(X_{c,k}(s))\,\mathrm{d}M_{c,k}(s) - \int_0^t S(t-s)G(X_{d,n}(s))\,\mathrm{d}M_{d,n}(s).
\end{aligned}
$$

Note that on $\{t < \tau_c(k) \wedge \tau_d(n)\}$ we have calculating as in (4.34) $M_{d,n}(t)u = M_{c,k}(t)u + (P_{c,k}(t) - P_{d,n}(t))u$. We have

$$
\begin{aligned}
&(X_{c,k}(t) - X_{d,n}(t))\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}} \\
&= \int_0^t S(t-s)(B(X_{c,k}(s)) - B(X_{d,n}(s)))\,\mathrm{d}s\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}} \\
&\quad + \int_0^t P_{c,k}(1)(G^*(X_{c,k}(s))S^*(t-s) - G^*(X_{d,n}(s))S^*(t-s))\,\mathrm{d}s\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}} \\
&\quad + \int_0^t S(t-s)(G(X_{c,k}(s)) - G(X_{d,n}(s)))\,\mathrm{d}M_{c,k}(s)\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathbb{E}\left[\|X_{c,k}(t) - X_{d,n}(t)\|^2\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right] \\
&\le 3\mathbb{E}\left[\left\|\int_0^t S(t-s)(B(X_{c,k}(s)) - B(X_{d,n}(s)))\,\mathrm{d}s\right\|^2\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right] \\
&\quad + 3\mathbb{E}\left[\left\|\int_0^t P_{c,k}(1)(G^*(X_{c,k}(s))S^*(t-s) - G^*(X_{d,n}(s))S^*(t-s))\,\mathrm{d}s\right\|^2\mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ 3\mathbb{E}\left[\left\|\int_0^t S(t-s)(G(X_{c,k}(s)) - G(X_{d,n}(s)))\,\mathrm{d}M_{c,k}(s)\right\|^2 \mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right] \\
&= 3(I_1 + I_2 + I_3). \qquad (5.28)
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1 &= \mathbb{E}\left[\left\|\int_0^t S(t-s)(B(X_{c,k}(s)) - B(X_{d,n}(s)))\,\mathrm{d}s\right\|^2 \mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right] \\
&\le \mathbb{E}\left[\left(\int_0^t \|S(t-s)(B(X_{c,k}(s)) - B(X_{d,n}(s)))\|\mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right)^2\right] \\
&\le \mathbb{E}\left[\left(\int_0^t a_k(s)\|X_{c,k}(s) - X_{d,n}(s)\|\mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right)^2\right] \qquad (5.29) \\
&\le \int_0^t a(s)\,\mathrm{d}s\,\mathbb{E}\left[\int_0^t a(s)\|X_{c,k}(s) - X_{d,n}(s)\|^2\mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right],
\end{aligned}
$$

where we have used Hölder's inequality for $\sqrt{a}$ and $\sqrt{a}\|X_{c,k} - X_{d,n}\|\mathbb{1}_{[0,\tau_c(k)\wedge\tau_d(n))}$. Similarly,

$$
I_2 \le \int_0^t a_k(s)\,\mathrm{d}s\,\mathbb{E}\left[\int_0^t a_k(s)\|X_{c,k}(s)) - X_{d,n}(s)\|^2\mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right]. \qquad (5.30)
$$

Note that

$$
\begin{aligned}
&\left\|\int_0^t S(t-s)(G(X_{c,k}(s)) - G(X_{d,n}(s)))\,\mathrm{d}M_{c,k}(s)\right\|^2 \mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}} \\
&\qquad\qquad \le \left\|\int_0^t S(t-s)(G(X_{c,k}(s)) - G(X_{d,n}(s)))\mathbb{1}_{\{s\le\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}M_{c,k}(s)\right\|^2,
\end{aligned}
$$

since the left-hand side is 0 on $\{t \ge \tau_c(k)\}$ and both sides are equal on $\{t < \tau_c(k)\}$. Thus by the Itô isometry in Theorem 3.5

$$
\begin{aligned}
I_3 &\le \mathbb{E}\left[\left\|\int_0^t S(t-s)(G(X_{c,k}(s)) - G(X_{d,n}(s)))\mathbb{1}_{\{s\le\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}M_{c,k}(s)\right\|^2\right] \\
&= \mathbb{E}\left[\int_0^t \left\|S(t-s)(G(X_{c,k}(s)) - G(X_{d,n}(s)))Q_k^{1/2}\right\|^2_{L_{\mathrm{HS}}(U,H)} \mathbb{1}_{\{s\le\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right] \qquad (5.31) \\
&\le \int_0^t b_k^2(s)\mathbb{E}\left[\|X_{d,n}(s) - X_{c,k}(s)\|^2\mathbb{1}_{\{s\le\tau_c(k)\wedge\tau_d(n)\}}\right]\mathrm{d}s.
\end{aligned}
$$

Combining (5.28), (5.29), (5.30), (5.31) we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\|X_{c,k}(t) - X_{d,n}(t)\|^2 \mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right] \\
&= 3\int_0^t a(s)\,\mathrm{d}s\,\mathbb{E}\left[\int_0^t a(s)\|X_{c,k}(s) - X_{d,n}(s)\|^2 \mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right] \\
&\quad + 3\int_0^t a_k(s)\,\mathrm{d}s\,\mathbb{E}\left[\int_0^t a_k(s)\|X_{c,k}(s) - X_{d,n}(s)\|^2 \mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\,\mathrm{d}s\right] \\
&\quad + 3\int_0^t b_k^2(s)\mathbb{E}\left[\|X_{c,k}(s) - X_{d,n}(s)\|^2 \mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\right]\,\mathrm{d}s,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\mathbb{E}\Big[\big\|X_{c,k}(t) - X_{d,n}(t)\big\|^2 \mathbb{1}_{\{t\le\tau_c(k)\wedge\tau_d(n)\}}\Big] \\
&\qquad \le 3\left(\int_0^T a(s) + a_k(s)\,\mathrm{d}s + 1\right) \\
&\qquad\quad \times \int_0^t (a(s) + a_k(s) + b_k^2(s))\mathbb{E}\left[\|X_{c,k}(s) - X_{d,n}(s)\|^2 \mathbb{1}_{\{s<\tau_c(k)\wedge\tau_d(n)\}}\right]\,\mathrm{d}s
\end{aligned}
$$

It follows from the Gronwall inequality that $\mathbb{E}\left[\|X_{c,k}(t) - X_{d,n}(t)\|^2 \mathbb{1}_{\{t<\tau_c(k)\wedge\tau_d(n)\}}\right] = 0$, which proves the claim. Property (5.27) enables us to define $X(t) := X_{c,k}(t)$ on $\{t < \tau_c(k)\}$. This definition makes sense because for fixed $t$ and $\omega$, if we used another sequence $d$ and another constant $n$ to define $X(t)(\omega)$ we would obtain the same value for almost every $\omega$.

In order to show that $X$ is a solution we prove that for every $k \in \mathbb{N}$

$$
\begin{aligned}
X(t)\mathbb{1}_{\{t<\tau_c(k)\}} = \mathbb{1}_{\{t<\tau_c(k)\}}S(t)X_0 &+ \mathbb{1}_{\{t<\tau_c(k)\}}\int_0^t S(t-s)B(X(s))\,\mathrm{d}s \\
&+ \mathbb{1}_{\{t<\tau_c(k)\}}\int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s).
\end{aligned}
\tag{5.32}
$$

It is clear from (5.26) that

$$
X_{c,k}(t) = S(t)X_0 + \int_0^t S(t-s)B(X_{c,k}(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X_{c,k}(s))\,\mathrm{d}L_{c,k}(s).
$$

The result follows by repeating the calculations (4.44)–(4.46). □

# Chapter 6

# Equation with the canonical stable cylindrical Lévy process

The approach taken in the previous chapter is restricted by the requirement that the integral map is continuous as a mapping between $L^p$ spaces. This does not hold if the noise is the canonical stable cylindrical Lévy process whose characteristic function is given by $\varphi_{L(t)}(u) = e^{-t\|u\|^\alpha}$ for $u \in U$, see Example 2.6. Note in particular that the integral in (5.1) is infinite for every $p > 0$.

SPDEs with stable noise have attracted a lot of attention in recent years. Having tails fatter than the normal distribution, stable processes are a good candidate for modelling various phenomena. Infinite variance of those distributions makes many techniques that apply to the Brownian motion or square-integrable martingales unavailable. Notwithstanding, due to explicit formulas e.g. for the Lévy measure, they are quite tractable and allow for direct calculations.

So far equations with stable noise have been mostly considered in the random field approach, see Balan [5]. Her results were improved by Chong [22]. In a subsequent work Chong, Dalang and Humeau [23] characterised negative Sobolev spaces in which the solutions live. Equations with non-Lipschitz coefficients were considered by Mueller [71] for $\alpha \leq 1$, Mytnik [72] for $1 < \alpha < 2$ and Xiong, Yang and Zhou [107, 108]. Brzeźniak and Zabczyk [18] proved existence and regularity of solutions with a cylindrical noise as ours but of additive type by using the construction of stable cylindrical Lévy process as a subordinated cylindrical Brownian motion i.e. $L(t) = W(\ell(t))$, where $W$ is a cylindrical Brownian motion and $\ell$ is an $\frac{\alpha}{2}$-stable subordinator.

In the preparatory Sections 6.1 and 6.2 we generalise the tail inequality for stochastic integrals from Giné and Marcus [38] to the case of the integral with respect to the canonical stable cylindrical Lévy process on Hilbert spaces, that is we prove that

$$\sup_{r>0} r^\alpha P\left(\sup_{t\in[0,T]}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\|_H > r\right) \le C_\alpha \mathbb{E}\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s. \tag{6.1}$$

This immediately implies an inequality for the $p$-th moment of the integral. For $p<\alpha$ one has

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\|^p\right] \le C_{\alpha,p}\left(\mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\right)^{p/\alpha}. \tag{6.2}$$

We note that Rosiński and Woyczyński [86] use time change technique to get a sharper bound in the case of real-valued integrands and non-cylindrical noise whereas Brzeźniak and Hausenblas [13, Cor. C.2] obtain a similar inequality for integrals with respect to Poisson random measures.

We construct a càdlàg solution to the SPDE (1.4) with a canonical $\alpha$-stable cylindrical Lévy process $L$, $\alpha\in(1,2)$ as a limit of the Picard iterations

$$X_n(t) = S(t)X_0 + \int_0^t S(t-s)F(X_{n-1}(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X_{n-1}(s-))\,\mathrm{d}L(s).$$

Firstly, we show tightness of the approximating sequence by verifying the Aldous condition and a version of the compact containment condition. For this to work we must assume that the functions $F$ and $G$ in (1.4) map between the domains of the fractional powers of $(-A)$. We employ similar estimates of the norms in the domains of fractional powers of $(-A)$ as Hausenblas [44], where she considered equations driven by Poisson random measures. Secondly, the almost sure convergence of the sequence (on a different probability space) follows by the Skorokhod theorem. In order to apply this result we must rewrite the cylindrical Lévy process as a metric space-valued random variable. Thirdly, we identify the limit as the solution of the SPDE. Since we do not have the decomposition into large and small jumps at hand, we need to use careful estimates of the $p$-moments for $p<\alpha$ using the moment estimate (6.2). Pathwise uniqueness follows from a version of the Gronwall inequality due to Willet and Wong [106]. Finally, by Kurtz's generalisation of the Yamada–Watanabe theorem, see [56], this gives also existence of a solution on any given probability space i.e. existence of the strong solution.

In future research we are planning to investigate if some assumptions taken in this chapter can be relaxed, in particular the boundedness of $F$ and $G$ and the assumption that they map into the domains of the fractional powers of $(-A)$. Also our Hölder condition for $S(t)G$, see (A3), holds also for $t = 0$. This is less general than the condition in Peszat and Zabczyk [74, Ch. 9], where the authors assume a Lipschitz condition for $S(t)G$ and allow the Lipschitz constant to depend on $t$ possibly exploding at $t = 0$.

### Preliminaries

In this chapter we work solely in the setting of Hilbert spaces (denoted by $U$ and $H$). We end this short introduction by recalling some facts about stable measures from Linde [59]. A probability measure $\mu$ on a Hilbert space $H$ is called stable if for every $n \in \mathbb{N}$ there exists $\gamma_n > 0$ and $x_n \in H$ such that

$$\varphi_\mu(h)^n = \varphi_\mu(\gamma_n h) e^{i\langle x_n, h\rangle}, \qquad h \in H. \tag{6.3}$$

For every stable measure there exists a unique number $\alpha \in (0, 2]$ such that $\gamma_n$ in the above formula can be chosen as $\gamma_n = n^{1/\alpha}$. For $\alpha = 2$ the measure $\mu$ is Gaussian. In what follows we consider $\alpha \in (0, 2)$.

Each stable measure is infinitely divisible and its Lévy measure $\nu$ can be written as

$$\nu(B) = c_\alpha^{-1} \int_0^\infty \int_{S_H} \mathbb{1}_B(tx)\, \sigma(\mathrm{d}x) t^{-1-p}\, \mathrm{d}t, \qquad B \in \mathcal{B}(H),$$

where $\sigma$ is a finite measure on $S_H$ and $c_\alpha$ is defined in (2.8). The measure $\sigma$ is called the spectral measure of $\mu$ and is given by the formula

$$\sigma(A) = \alpha c_\alpha \nu\left(\left\{x \in H : \|x\| > 1, \frac{x}{\|x\|} \in A\right\}\right), \qquad A \in \mathcal{B}(S_H). \tag{6.4}$$

By [59, Prop. 7.5.4(iv)] for every Hilbert space there exists a constant $c > 0$ such that for every stable measure $\mu$ on $H$

$$\sup_{r>0} r^\alpha \mu(\|x\| > r) \le c \lim_{r\to\infty} r^\alpha \mu(\|x\| > r). \tag{6.5}$$

## 6.1 Tail estimate

We start with the following lemma:

**Lemma 6.1.** For any real-valued non-negative random variable $X$ and $r > 0$ we have

$$\mathbb{E}\left[X^2 \mathbb{1}_{\{X \le r\}}\right] = 2\int_0^r tP(t < X \le r)\,\mathrm{d}t.$$

*Proof.* Similarly to [26, Lem. 4.38] we obtain

$$\begin{aligned}\mathbb{E}\left[X^2 \mathbb{1}_{\{X\le r\}}\right] &= \mathbb{E}\left[\int_0^{X\mathbb{1}_{\{X\le r\}}} 2t\,\mathrm{d}t\right] = 2\mathbb{E}\left[\int_0^\infty t\mathbb{1}_{t<X\mathbb{1}_{\{X\le r\}}}\,\mathrm{d}t\right] \\ &= 2\int_0^\infty t\mathbb{E}\left[\mathbb{1}_{t<X\mathbb{1}_{\{X\le r\}}}\right]\mathrm{d}t = 2\int_0^r tP(t<X\le r)\,\mathrm{d}t. \qquad \square\end{aligned}$$

Let $\psi \in L_{\mathrm{HS}}(U,H)$. Then the cylindrical Lévy process $(L(t)\psi^* : t \ge 0)$ on $H$ is induced by a classical process denoted by $(J_t(\psi) : t \ge 0)$. In that case the cylindrical Lévy measure $\nu \circ \psi^{-1}$ of $J_t(\psi)$ extends to a classical Lévy measure, see [101, Th. VI.5.2.].

**Lemma 6.2.** For any canonical $\alpha$-stable cylindrical Lévy process $L$ on $U$ and $\psi \in L_{\mathrm{HS}}(U,H)$ we have

$$\sup_{r>0} r^\alpha P(\|J_t(\psi)\| > r) \le ct\left(\nu \circ \psi^{-1}\right)(B_H^c) \le c_{1,\alpha} t \|\psi\|^\alpha_{L_{\mathrm{HS}}(U,H)}, \tag{6.6}$$

where

$$c_{1,\alpha} = \frac{c\Gamma(\frac{1}{2})}{c_\alpha \Gamma(\frac{1+\alpha}{2})}$$

and $c$ is the constant (depending on $\alpha$) appearing in (6.5).

*Proof.* In the proof we use ideas from [85, Sec. 4]. Note that the characteristic function of $J_t(\psi)$ is given by $\varphi_{J_t(\psi)}(h) = e^{-t\|\psi^* h\|^\alpha}$ for $h \in H$ and thus it is a stable random variable, cf. (6.3). It follows by (6.5) that

$$\sup_{r>0} r^\alpha P(\|J_t(\psi)\| > r) \le c \lim_{r\to\infty} r^\alpha P(\|J_t(\psi)\| > r). \tag{6.7}$$

The Lévy measure of the infinitely divisible random variable $J_t(\psi^*)$ is $t\left(\nu \circ \psi^{-1}\right)$. Let $\sigma$ denote the spectral measure of $\nu \circ \psi^{-1}$. Combining [59, Cor. 6.7.3] and formula (6.4) we get

$$\lim_{r\to\infty} r^\alpha P(\|J_t(\psi)\| > r) = t\frac{\sigma(S_H)}{\alpha c_\alpha} = t\left(\nu \circ \psi^{-1}\right)(B_H^c). \tag{6.8}$$

Now the first inequality in (6.6) follows from (6.7) and (6.8).

We now prove the second inequality in (6.6). The operator $\psi$ has the decomposition

$$\psi = \sum_{n=1}^{\infty} \gamma_n e_n \otimes f_n,$$

where $(e_n)$ is an orthonormal system in $U$, $(f_n)$ is an orthonormal system in $H$ and $(\gamma_n) \subset \mathbb{R}$, see [29, Th. 4.1]. Note that

$$\|\psi\|^2_{L_{\mathrm{HS}}(U,H)} = \sum_{n=1}^{\infty} \|\psi e_n\|^2 = \sum_{n=1}^{\infty} \gamma_n^2. \tag{6.9}$$

Let $P_n \colon H \to H$ be a projection onto $\mathrm{Span}(f_1, \dots, f_n)$.

$$\|J_t(P_n\psi) - J_t(\psi)\|^2 = \left\| \sum_{j=n+1}^{\infty} L(t)(\psi^* f_j) f_j \right\|^2 \to 0$$

almost surely because the series $\|J_t(\psi)\|^2 = \sum_{j=1}^{\infty} L(t)(\psi^* f_j) f_j$ converges a.s. It follows from [59, Prop. 6.6.5] that the spectral measure of $J_1(P_n\psi)$ (denoted by $\sigma_n$) converges weakly to the spectral measure of $J_1(\psi)$ (denoted by $\sigma$). By the Portmanteau theorem $\sigma_n(S_H) \to \sigma(S_H)$ as $n \to \infty$. It follows from (6.4) that $\sigma(S_H) = \alpha c_\alpha \left(\nu \circ \psi^{-1}\right)(B_H^c)$ and $\sigma_n(S_H) = \alpha c_\alpha \left(\nu \circ \psi^{-1} \circ P_n^{-1}\right)(B_H^c)$ and thus

$$\left(\nu \circ \psi^{-1}\right)(B_H^c) = \lim_{n\to\infty} \left(\nu \circ \psi^{-1} \circ P_n^{-1}\right)(B_H^c). \tag{6.10}$$

We calculate

$$\begin{aligned}
\left(\nu \circ \psi^{-1} \circ P_n^{-1}\right)(B_H^c) &= \nu\left(\{u \in U : \psi P_n u \in B_H^c\}\right) \\
&= \nu\left(\left\{u \in U : \sum_{j=1}^{n} \gamma_j^2 \langle u, e_j\rangle^2 > 1\right\}\right) \\
&= \nu \circ \pi^{-1}_{e_1,\dots,e_n}\left(\left\{x \in \mathbb{R}^n : \sum_{j=1}^{n} \gamma_j^2 x_j^2 > 1\right\}\right).
\end{aligned}$$

By (2.7)

$$\left(\nu \circ \psi^{-1} \circ P_n^{-1}\right)(B_H^c) = \frac{\alpha}{c_\alpha} \int_{S_{\mathbb{R}^n}} \int_0^\infty \mathbb{1}\left(r^2 \sum_{j=1}^n x_j^2 \gamma_j^2 > 1\right) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r\, \lambda_n(\mathrm{d}x)$$
$$= \frac{\alpha}{c_\alpha} \int_{S_{\mathbb{R}^n}} \int_{r > \left(\sum_{j=1}^n x_j^2 \gamma_j^2\right)^{-1/2}} \frac{1}{r^{1+\alpha}} \,\mathrm{d}r\, \lambda_n(\mathrm{d}x)$$
$$= \frac{1}{c_\alpha} \int_{S_{\mathbb{R}^n}} \left(\sum_{j=1}^n x_j^2 \gamma_j^2\right)^{\alpha/2} \lambda_n(\mathrm{d}x).$$

Denote $\lambda_n^{(1)} := \frac{1}{\lambda_n(S_{\mathbb{R}^n})} \lambda_n$. By the Jensen inequality

$$\begin{aligned} \left(\nu \circ \psi^{-1} \circ P_n^{-1}\right)(B_H^c) &\le \frac{\lambda_n(S_{\mathbb{R}^n})}{c_\alpha} \left(\int_{S_{\mathbb{R}^n}} \sum_{j=1}^n x_j^2 \gamma_j^2 \,\lambda_n^{(1)}(\mathrm{d}x)\right)^{\alpha/2} \\ &= \frac{\lambda_n(S_{\mathbb{R}^n})}{c_\alpha} \left(\sum_{j=1}^n \gamma_j^2 \int_{S_{\mathbb{R}^n}} x_j^2 \,\lambda_n^{(1)}(\mathrm{d}x)\right)^{\alpha/2}. \end{aligned} \tag{6.11}$$

Note that

$$\sum_{j=1}^n \int_{S_{\mathbb{R}^n}} x_j^2 \,\lambda_n^{(1)}(\mathrm{d}x) = 1 \tag{6.12}$$

because $\sum_{j=1}^n x_j^2 = 1$ on $S_{\mathbb{R}^n}$. Since all the terms in the sum in (6.12) are equal it follows that

$$\int_{S_{\mathbb{R}^n}} x_j^2 \,\lambda_n^{(1)}(\mathrm{d}x) = \frac{1}{n}$$

for all $j = 1, \dots, n$. Applying this in (6.11) we obtain

$$\left(\nu \circ \psi^{-1} \circ P_n^{-1}\right)(B_H^c) \le \frac{\lambda_n(S_{\mathbb{R}^n})}{c_\alpha n^{\alpha/2}} \left(\sum_{j=1}^n \gamma_j^2\right)^{\alpha/2}. \tag{6.13}$$

Recall that $\frac{\Gamma(x+\beta)}{\Gamma(x)x^\beta} \to 1$ as $x \to \infty$. It follows from (2.9) that as $n \to \infty$

$$\frac{\lambda_n(S_{\mathbb{R}^n})}{n^{\alpha/2}} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2}) n^{\alpha/2}} \to \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})},$$

and finally taking the limit in (6.13) and using (6.9) and (6.10) we obtain the second inequality in (6.6). ☐

**Lemma 6.3.** Suppose that $\Psi$ is a simple $L_{\mathrm{HS}}(U,H)$-valued process taking only finitely many values. Then

$$\sup_{r>0} r^{\alpha} P\left(\sup_{t\in[0,T]}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\| > r\right) \le c_{2,\alpha}\mathbb{E}\left[\int_0^T \|\Psi(s)\|^{\alpha}_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right] \tag{6.14}$$

with $c_{2,\alpha} = c_{1,\alpha}\left(1+\frac{8}{2-\alpha}\right)$.

*Proof.* In the proof we follow [38, Lem. 3.3] and [5, Th. 4.3]. Denote for $t \in [0,T]$

$$I(t) := \int_0^t \Psi(s)\,\mathrm{d}L(s).$$

Suppose that the simple process $\Psi$ is based on the partition $\{s_1,\dots,s_K\}$. Let $\{t_1,\dots,t_N\}$ be a partition of $[0,T]$ containing $\{s_1,\dots,s_K\}$. We first show that

$$P\left(\max_{i=1,\dots,N}\|I(t_i)\| > r\right) \le c_{2,\alpha} r^{-\alpha}\mathbb{E}\left[\int_0^T \|\Psi(s)\|^{\alpha}_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]. \tag{6.15}$$

Fix $n \in \mathbb{N}$ and write $\Psi$ as

$$\Psi = \Psi_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{N-1} \Psi_i \mathbb{1}_{(t_i,t_{i+1}]}, \qquad \Psi_i = \sum_{j=1}^{m_i} \mathbb{1}_{A_{i,j}} \psi_{i,j},$$

where for each $i = 1,\dots,N$ the sets $A_{i,1},\dots,A_{i,m_i}$ form a partition of $\Omega$. We have

$$\begin{aligned} P\left(\max_{i=1,\dots,N}\|I(t_i)\| > r\right) &\le \sum_{i=1}^{N-1} P\left(\left\|J_{t_i,t_{i+1}}(\Psi_i)\right\| > r\right) \\ &\quad + P\left(\max_{k=1,\dots,N-1}\left\|\sum_{i=1}^{k} J_{t_i,t_{i+1}}(\Psi_i)\mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\Psi_i)\|\le r\}}\right\| > r\right) \\ &=: e_1 + e_2. \end{aligned} \tag{6.16}$$

We estimate each term on the right-hand side separately. We have

$$\begin{aligned}
P\left(\left\|J_{t_i,t_{i+1}}(\Psi_i)\right\| > r\right) &= P\left(\left\|\sum_{j=1}^{m_i} \mathbb{1}_{A_{i,j}} J_{t_i,t_{i+1}}(\psi_{i,j})\right\| > r\right) \\
&= \sum_{j=1}^{m_i} P(A_{i,j} \cap \{\left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\| > r\}) \\
&= \sum_{j=1}^{m_i} P(A_{i,j}) P\left(\left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\| > r\right),
\end{aligned}$$

where the last step follows from the fact that each $A_{i,j}$ is $\mathcal{F}_{t_i}$-measurable and $J_{t_i,t_{i+1}}(\psi_{i,j})$ is independent of $\mathcal{F}_{t_i}$. From Lemma 6.2 we obtain

$$\begin{aligned}
P\left(\left\|J_{t_i,t_{i+1}}(\Psi_i)\right\| > r\right) &\leq c_{1,\alpha} r^{-\alpha} (t_{i+1} - t_i) \sum_{j=1}^{m_i} P(A_{i,j}) \|\psi_{i,j}\|^{\alpha}_{L_{\mathrm{HS}}(U,H)} \\
&= c_{1,\alpha} r^{-\alpha} (t_{i+1} - t_i) \mathbb{E}\left[\|\Psi_i\|^{\alpha}_{L_{\mathrm{HS}}(U,H)}\right].
\end{aligned}$$

This proves that

$$e_1 \leq c_{1,\alpha} r^{-\alpha} \mathbb{E}\left[\int_0^T \|\Psi(s)\|^{\alpha}_{L_{\mathrm{HS}}(U,H)}\, \mathrm{d}s\right].$$

We estimate the second term on the right-hand side of (6.16). By the Doob inequality

$$e_2 \leq 4r^{-2} \max_{k=1,\dots,N-1} \mathbb{E}\left[\left\|\sum_{i=1}^{k} J_{t_i,t_{i+1}}(\Psi_i)\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\Psi_i)\| \leq r\}}\right].$$

Note that for any $s < t$ and $\psi \in L_{\mathrm{HS}}(U,H)$ we have

$$\varphi_{J_{s,t}(\psi)}(h) = e^{-(t-s)\|\psi^* h\|^{\alpha}} = e^{-(t-s)\|-\psi^*(h)\|^{\alpha}} = \varphi_{J_{s,t}(-\psi)}(h), \qquad h \in H.$$

Thus $\mathcal{L}(J_{s,t}(\psi)) = \mathcal{L}(J_{s,t}(-\psi)) = \mathcal{L}(-J_{s,t}(\psi))$ and consequently $\mathcal{L}\left(J_{s,t}(\psi)\mathbb{1}_{\{\|J_{s,t}(\psi)\| \leq r\}}\right) = \mathcal{L}\left(-J_{s,t}(\psi)\mathbb{1}_{\{\|J_{s,t}(\psi)\| \leq r\}}\right)$. It follows that $\mathbb{E}\left[J_{s,t}(\psi)\mathbb{1}_{\{\|J_{s,t}(\psi)\| \leq r\}}\right] = 0$. It follows by conditioning on $\mathcal{F}_{t_l}$ that for $i < l$

$$\mathbb{E}\left[\left\langle J_{t_i,t_{i+1}}(\Psi_i)\mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\Psi_i)\| \leq r\}}, J_{t_l,t_{l+1}}(\Psi_l)\mathbb{1}_{\{\|J_{t_l,t_{l+1}}(\Psi_l)\| \leq r\}}\right\rangle\right]$$

$$= \sum_{j=1}^{m_i} \sum_{k=1}^{m_l} \mathbb{E}\left[\left\langle J_{t_l,t_{l+1}}(\psi_{l,j}) \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\psi_{i,j})\| \le r\}}, \mathbb{E}\left[J_{t_l,t_{l+1}}(\psi_{l,k}) \mathbb{1}_{\{\|J_{t_l,t_{l+1}}(\Psi_{l,k})\| \le r\}}\right]\right\rangle \mathbb{1}_{A_{l,j}} \mathbb{1}_{A_{l,k}}\right]$$

$$= 0.$$

Thus

$$e_2 \le 4r^{-2} \sum_{i=1}^{N-1} \mathbb{E}\left[\left\|J_{t_i,t_{i+1}}(\Psi_i)\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\Psi_i)\| \le r\}}\right]. \tag{6.17}$$

We write using the independence of $J_{t_i,t_{i+1}}(\psi_{i,j})$ from $\mathcal{F}_{t_i}$

$$\begin{aligned}
\mathbb{E}&\left[\left\|J_{t_i,t_{i+1}}(\Psi_i)\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\Psi_i)\| \le r\}}\right] \\
&= \sum_{j=1}^{m_i} \mathbb{E}\left[\left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\psi_{i,j})\| \le r\}} \mathbb{1}_{A_{i,j}}\right] \\
&= \sum_{j=1}^{m_i} P(A_{i,j}) \mathbb{E}\left[\left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\psi_{i,j})\| \le r\}}\right].
\end{aligned} \tag{6.18}$$

By Lemma 6.1

$$\begin{aligned}
E\left[\left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\psi_{i,j})\| \le r\}}\right] &= 2\int_0^r tP\left(t < \left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\| \le r\right) \mathrm{d}t \\
&\le 2\int_0^r tP\left(t < \left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\|\right) \mathrm{d}t.
\end{aligned}$$

Thus by Lemma 6.2 we estimate further

$$\begin{aligned}
\mathbb{E}\left[\left\|J_{t_i,t_{i+1}}(\psi_{i,j})\right\|^2 \mathbb{1}_{\{\|J_{t_i,t_{i+1}}(\psi_{i,j})\| \le u\}}\right] &\le 2c_{1,\alpha}(t_{i+1} - t_i) \|\psi_{i,j}\|^\alpha_{L_{\mathrm{HS}}(U,H)} \int_0^r t^{1-\alpha} \,\mathrm{d}t \\
&= \frac{2c_{1,\alpha}}{2-\alpha}(t_{i+1} - t_i)\|\psi_{i,j}\|^\alpha_{L_{\mathrm{HS}}(U,H)} r^{2-\alpha}.
\end{aligned} \tag{6.19}$$

Combining (6.17), (6.18) and (6.19) we get

$$e_2 \le \frac{8c_{1,\alpha}}{2-\alpha} \sum_{i=1}^{N-1} \sum_{j=1}^{m_i} (t_{i+1} - t_i) P(A_{i,j}) \|\psi_{i,j}\|_{L_{\mathrm{HS}}(U,H)} r^{-\alpha} = \frac{8c_{1,\alpha}}{2-\alpha} r^{-\alpha} \mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)} \,\mathrm{d}s\right].$$

This finishes the proof of (6.15).

The process $(I(t) : t \in [0,T])$ is càdlàg and thus separable. It follows that there exists a sequence of partitions $0 = t_1^n < \ldots < t_{k_n}^n = T$ such that

$$P\left(\sup_{t\in[0,T]} \|I(t)\| > r\right) = \lim_{n\to\infty} P\left(\max_{i=1,\ldots,k_n} \|I(t_i^n)\| > r\right).$$

Combining this with (6.15) gives the result. ☐

We now proceed to defining the integral. The class of integrands will be called $\Lambda(\alpha)$.

**Definition 6.4.** We define the space $\Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$ as

$$\left\{\Psi\colon [0,T]\times\Omega \to L_{\mathrm{HS}}(U,H) : \Psi \text{ is predictable}, \mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right] < \infty\right\}$$

with the metric defined by

$$d_\alpha(\Psi,\Phi) := \begin{cases} \mathbb{E}\left[\int_0^T \|\Psi(s)-\Phi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right], & \text{if } \alpha \le 1, \\ \left(\mathbb{E}\left[\int_0^T \|\Psi(s)-\Phi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\right)^{1/\alpha}, & \text{if } \alpha > 1. \end{cases}$$

Furthermore, let $\Lambda_0(0,T;L_{\mathrm{HS}}(U,H))$ denote the subspace of $\Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H)))$ consisting of simple processes of the form (3.3) and let $\Lambda_0^S(0,T;L_{\mathrm{HS}}(U,H)))$ be the subspace of simple integrands for which additionally each $\Psi_i$ takes only finitely many values like in (3.4).

Note that for $\alpha \ge 1$ the space $\Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$ is in fact a Banach space.

**Lemma 6.5.** The space $\Lambda_0^S(0,T;L_{\mathrm{HS}}(U,H))$ is dense in $\Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$.

*Proof.* For a given $\Psi \in \Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$ we construct an approximating sequence $(\Psi_n) \subset \Lambda_0(0,T;L_{\mathrm{HS}}(U,H))$ like in [26, Prop. 4.22(ii)]. By construction, the range of each $\Psi_n$ is finite and thus we conclude that $\Psi_n \in \Lambda_0^S(0,T;L_{\mathrm{HS}}(U,H))$. ☐

We are now ready to construct the integral. For $\Psi \in \Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$ take $\Psi_n \in \Lambda_0^S(0,T;L_{\mathrm{HS}}(U,H))$ converging to $\Psi$. By Lemma 6.3

$$\sup_{r>0} r^\alpha P\left(\sup_{t\in[0,T]}\left\|\int_0^t (\Psi_n(s)-\Psi_m(s))\,\mathrm{d}L(s)\right\| > r\right) \le C\mathbb{E}\left[\int_0^T \|\Psi_n(s)-\Psi_m(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right],$$

which converges to 0 as $n,m\to\infty$. Thus

$$\int_0^t \Psi_n(s)\,\mathrm{d}L(s), n\in\mathbb{N}$$

is a Cauchy sequence in the uniform convergence in probability. There exists a unique limit and it is denoted by $\int_0^t \Psi(s)\,\mathrm{d}L(s)$. The integral also satisfies (6.14):

**Theorem 6.6.** *The inequality* (6.14) *holds for any predictable process* $\Psi \in \Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$.

*Proof.* We have by the Portmanteau theorem

$$\begin{aligned}
P\left(\sup_{t\in[0,T]}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\| > r\right) &\le \liminf_{n\to\infty} P\left(\sup_{t\in[0,T]}\left\|\int_0^t \Psi_n(s)\,\mathrm{d}L(s)\right\| > r\right)\\
&\le \liminf_{n\to\infty} cr^{-\alpha}\mathbb{E}\left[\int_0^T \|\Psi_n(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\\
&= cr^{-\alpha}\mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right].
\end{aligned}$$

Moving $r^{-\alpha}$ to the left-hand side and taking supremum over $r>0$ we get the claim. □

## 6.2 Moments of the stochastic integral

A moment inequality with different powers on the left and right-hand sides was proven in the case of real-valued integrands and vector-valued integrators in [86], see (5.20) above. By the Jensen inequality (5.20) gives also

$$\mathbb{E}\left[\left(\sup_{t\le T}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\|\right)^p\right] \le C\left(\mathbb{E}\left[\int_0^T |\Psi(s)|^\alpha\,\mathrm{d}t\right]\right)^{p/\alpha}. \tag{6.20}$$

In the following proposition we provide (6.20) in the more general case when $L$ is cylindrical and $\Psi$ is operator-valued. Unlike [86] we are not using the time change technique. A counterpart with analogous proof for the integrals with respect to square-integrable martingales can be found in [74, Th. 3.41, Th. 9.24].

**Proposition 6.7.** For $p<\alpha$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \Psi(s)\,\mathrm{d}L(s)\right\|^p\right] \le C_{\alpha,p}\left(\mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\right)^{p/\alpha},$$

where

$$C_{\alpha,p} := \frac{c_{2,\alpha}^{p/\alpha}\alpha}{\alpha-p}. \tag{6.21}$$

*Proof.* Let $X := \sup_{t\in[0,T]} \left\| \int_0^t \Psi(s)\,\mathrm{d}L(s) \right\|$ and also let $\xi := c_{2,\alpha}\mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]$. Note that Theorem 6.6 implies that

$$P(X > r) \leq 1 \wedge (r^{-\alpha}\xi).$$

Therefore we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t \Psi(s)\,\mathrm{d}L(s) \right\|^p\right] &= \mathbb{E}\left[X^p\right] \\
&= p\int_0^\infty r^{p-1} P(X > r)\,\mathrm{d}r \\
&\leq p\int_0^\infty r^{p-1}\left(1 \wedge (r^{-\alpha}\xi)\right)\mathrm{d}r \\
&= p\int_0^{\xi^{1/\alpha}} r^{p-1}\,\mathrm{d}r + p\xi\int_{\xi^{1/\alpha}}^\infty r^{p-1-\alpha}\,\mathrm{d}r \\
&= \left(1 + \frac{p}{\alpha - p}\right)\xi^{p/\alpha} \\
&= \frac{c_{2,\alpha}^{p/\alpha}\alpha}{\alpha - p}\left(\mathbb{E}\left[\int_0^T \|\Psi(s)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\right)^{p/\alpha},
\end{aligned}$$

where in the second equality we use [51, Lem. 3.4]. □

## 6.3 Tigthness criteria in the Skorokhod space

Recall that a sequence of random variables $(X_n)$ with values in a Polish space $S$ is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset S$ such that for all $n \in \mathbb{N}$ one has $P(X_n \in K) \geq 1 - \varepsilon$. We say that a sequence of càdlàg processes $(X_n)$ satisfies the Aldous condition if for any $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that for all sequences of stopping times $(\tau_n)$ such that $\tau_n + \delta \leq T$ one has

$$\sup_{n\in\mathbb{N}} \sup_{0<\theta\leq\delta} P(\|X_n(\tau_n + \theta) - X_n(\tau_n)\| \geq \eta) \leq \varepsilon.$$

The classical Prokhorov theorem asserts that a sequence of random variables is tight if and only if their laws $\mathcal{L}(X_n)$ are relatively compact in the weak topology in the space of Radon measures on $S$, see [101, Th. I.3.6]. We formulate and prove yet another version of the well-known result, which says that if a sequence of càdlàg processes $(X_n)$ satisfies the Aldous condition and for every fixed $t$ it attains values in compact sets with high probability then it is tight in the Skorokhod space $D([0,T];H)$. Detailed exposition of this method is presented in [34, Sec. 3.7 and 3.8]. Criteria specifically useful in the context of SPDEs are given in [69, 70], which we build upon here. These references include also some more straightforward tightness criteria based solely on various moment estimates. The reason for formulating another version is that the following one is especially easy to verify in our setting due to the tail estimate in Theorem 6.6.

**Theorem 6.8.** *Let $X_n$ be a sequence of $H$-valued càdlàg adapted processes. Assume that there exists a subspace $\Gamma$ with a norm $\|\cdot\|_\Gamma$ compactly embedded in $H$ such that*

$$\forall \varepsilon > 0 \, \exists R > 0 \, \forall t \in [0,T] \cap \mathbb{Q}, \, n \in \mathbb{N} : P\left(X_n(t) \in \Gamma \text{ and } \|X_n(t)\|_\Gamma \leq R\right) \geq 1 - \varepsilon \tag{6.22}$$

*and such that the Aldous condition holds. Then $(X_n)$ is tight in $D([0,T];H)$.*

*Proof.* Fix $\varepsilon > 0$. We arrange $[0,T] \cap \mathbb{Q}$ in a sequence $(t_k)$. For every $k \in \mathbb{N}$ we choose $R_k > 0$ such that for all $n \in \mathbb{N}$

$$P\left(X_n(t_k) \in \Gamma \text{ and } \|X_n(t_k)\|_\Gamma \leq R_k\right) \geq 1 - \frac{\varepsilon}{2^{k+1}}.$$

Let

$$B = \{x \in D([0,T];H) : x(t_k) \in \Gamma \text{ and } \|x(t_k)\|_\Gamma \leq R_k \text{ for all } k \in \mathbb{N}\}.$$

Then

$$\begin{aligned} P(X_n \notin B) &= P\left(\bigcup_{k=1}^{\infty} \{X_n(t_k) \in \Gamma \text{ and } \|X_n(t_k)\|_\Gamma \leq R_k\}^c\right) \\ &\leq \sum_{k=1}^{\infty} P(\{X_n(t_k) \in \Gamma \text{ and } \|X_n(t_k)\|_\Gamma \leq R_k\}^c) \\ &\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{6.23}$$

By [69, Lem. 7 and 8] the Aldous condition implies that there exists a measurable set $A \subset$

$D([0,T];H)$ such that

$$P(X_n \in A) \geq 1 - \frac{\varepsilon}{2}$$

and

$$\lim_{\delta \to 0} \sup_{x \in A} \omega(x, \delta) = 0, \tag{6.24}$$

where $\omega$ is the usual modulus of continuity in $D([0,T];H)$, see e.g. [8, Ch. 3]. We show that the assumptions of Theorem 1 in [69] are satisfied for the set $A \cap B$:

(i) there exists a dense subset $J \subset [0,T]$ such that for all $t \in J$ the set $\{x(t) : x \in A \cap B\}$ is relatively compact,

(ii) $\lim_{\delta \to 0} \sup_{x \in A \cap B} \omega(x, \delta) = 0.$

Note that condition (i) holds because of the choice of $B$ and the fact that each closed ball $B_\Gamma(0, R_k) = \{h \in \Gamma : \|h\|_\Gamma \leq R_k\}$ is relatively compact in $H$. Condition (ii) is satisfied by (6.24). Thus we get that $A \cap B$ is relatively compact and thus $\overline{A \cap B}$ is compact. By (6.23) we have that

$$P(X_n \in \overline{A \cap B}) \geq P(X_n \in A \cap B) \geq P(X_n \in A) - P(X_n \in B^c) \geq 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon.$$

This proves that $(X_n)$ is tight. □

## 6.4 Stochastic evolution equation

We consider the problem of existence of a mild solution for

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t-))\,\mathrm{d}L(t) \tag{6.25}$$

with an $\mathcal{F}_0$-measurable initial condition $X(0) = X_0$. Suppose that $A$ is a generator of a strongly continuous semigroup $S$ on $H$, $F\colon H \to H$ and $G\colon H \to L_{\mathrm{HS}}(U,H)$. Assume that $L$ is a canonical $\alpha$-stable cylindrical process on $U$ with $\alpha \in (1,2)$. We prove existence of a mild solution.

**Definition 6.9.** A filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$, a cylindrical Lévy process $L$ and a càdlàg process $X$ is a *weak mild solution* to (6.25) if

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X(s-))\,\mathrm{d}L(s). \tag{6.26}$$

holds $P$-a.s. for all $t \in [0,T]$. The solution is said to be *strong* if it can be constructed for arbitrary probability basis and canonical stable cylindrical Lévy process. We say that pathwise uniqueness holds if for any two càdlàg processes $X_1$ and $X_2$ defined on the same probability space and satisfying (6.26) we have

$$P(X_1(t) = X_2(t) \text{ for all } t \in [0,T]) = 1.$$

We list all the assumptions that we need to prove existence of solutions:

(A1) The semigroup $S$ satisfies the following additional conditions:

- (i) $S$ is a semigroup of contractions i.e. $\|S(t)\| \leq 1$ for $t \geq 0$,
- (ii) 0 belongs to the resolvent set $\rho(A)$,
- (iii) $\{\lambda \in \mathbb{C} : 0 < \omega < |\arg \lambda| \leq \pi\} \subset \rho(-A)$ with $\omega < \frac{\pi}{2}$.
- (iv) The embedding $D(A) \subset H$ is compact.

(A2) There is a constant $M_0$ such that for all $t \in (0,T]$ and $x \in H$

$$\begin{aligned} \|S(t)F(x)\|_{D((-A)^\delta)} &\leq M_0, \\ \|S(t)G(x)\|_{L_{\mathrm{HS}}(U,D((-A)^\delta))} &\leq M_0, \end{aligned}$$

for some $\delta > 0$.

(A3) Assume the following Hölder condition holds for any $q \in (1-\delta, 1)$ for some $\delta > 0$

$$\|S(t)(F(x) - F(y))\|_{L_{\mathrm{HS}}(U,H)} \leq c_F \|x - y\|^q, \qquad \text{for all } x, y \in H, t \in [0,T].$$

and

$$\|S(t)(G(x) - G(y))\|_{L_{\mathrm{HS}}(U,H)} \leq c_G \|x - y\|^q, \qquad \text{for all } x, y \in H, [0,T].$$

(A4) The functions $G\colon H \to L_{\mathrm{HS}}(U,H)$ and $F\colon H \to H$ are continuous.

By [73, Th. 2.5.2] bounedness and $\{\lambda \in C : 0 < \omega < |\arg \lambda| \leq \pi\} \subset \rho(-A)$ with $\omega < \frac{\pi}{2}$ implies that the semigroup is analytic.

**Remark 6.10.** For any $\varphi \in L_{\mathrm{HS}}(U, D((-A)^\delta))$ we have

$$\|\varphi\|^2_{L_{\mathrm{HS}}(U,H)} = \sum_{n=1}^\infty \|\varphi e_n\|^2 \leq \big\|(-A)^{-\delta}\big\|^2 \sum_{n=1}^\infty \|\varphi e_n\|^2_{D((-A)^\delta)} = \big\|(-A)^{-\delta}\big\|^2 \|\varphi\|^2_{L_{\mathrm{HS}}(U,D((-A)^\delta))}.$$

Thus (A2) implies

(A2′) There is a constant $M_0$ such that for all $t \in (0,T]$ and $x \in H$

$$\begin{aligned} \|S(t)F(x)\| &\leq M_0, \\ \|S(t)G(x)\|_{L_{\mathrm{HS}}(U,H)} &\leq M_0. \end{aligned}$$

**Proposition 6.11.** The boundedness condition (A2′) implies that the Hölder condition is equivalent to the following Lipschitz condition:

(A3′) There exist constants $C_F$ and $C_G$ such that

$$\begin{aligned} \|S(t)(F(x) - F(y))\| &\leq C_F \|x - y\|, \\ \|S(t)(G(x) - G(y))\|_{L_{\mathrm{HS}}(U,H)} &\leq C_G \|x - y\|. \end{aligned}$$

*Proof.* See [36, p. 3]. We deal with $F$ only, the proof for $G$ being completely analogous. The implication (A3) $\Longrightarrow$ (A3′) follows simply by taking $q \nearrow 1$. We prove (A3′) $\Longrightarrow$ (A3). If $\|x - y\| \leq 1$, then $\|x - y\| \leq \|x - y\|^q$. If $\|x - y\| > 1$, then we estimate using (A2′)

$$\|S(t)(F(x) - F(y))\| \leq M_0 \leq M_0 \|x - y\|.$$

Thus we may take $c_F = C_F \vee M_0$. □

**Remark 6.12.** In [50] it is assumed that

(i) $G\colon D((-A)^\delta) \to L_{\mathrm{HS}}(U, D((-A)^\delta))$ satisfies the linear growth condition and

(ii) $G\colon H \to L_{\mathrm{HS}}(U, H)$ is Lipschitz.

Condition (i) is weaker than our assumption (A2) because we require that a bigger space ($H$ rather than $D((-A)^\delta)$) is mapped into $L_{\mathrm{HS}}(U, D((-A)^\delta))$. The verification of condition (i) is delicate and requires using a characterisation of $D((-A)^\delta)$ as a fractional Sobolev space.

### 6.4.1 Pathwise uniqueness

We prove pathwise uniqueness using a special version of the Gronwall inequality from Theorem 2 in [106], which we recall without proof. We also need a generalisation of Lemma 3.13 for the stopped stochastic integral with respect to the stable cylindrical Lévy process.

**Theorem 6.13** (Willet, Wong, 1964)**.** *Suppose that the functions $v$, $w$, $vu$ and $wu^p$ defined on $[0,T]$ are integrable and non-negative. If*

$$u(t) \le \int_0^t v(s)u(s)\,\mathrm{d}s + \int_0^t w(s)u^p(s)\,\mathrm{d}s, \qquad t\in[0,T],$$

*with some $p\in[0,1)\cup(1,\infty)$, then*

$$u(t)\exp\left(-\int_0^t v(s)\,\mathrm{d}s\right) \le \left(q\int_0^t w(s)\exp\left(-q\int_0^s v(r)\,\mathrm{d}r\right)\mathrm{d}s\right)^{1/q}, \qquad t\in[0,T]$$

*with $q=1-p$.*

**Lemma 6.14.** Let $\Psi\in\Lambda^\alpha(0,T;L_{\mathrm{HS}}(U,H))$. For any stopping time $\tau$

$$\int_0^{t\wedge\tau}\Psi(s)\,\mathrm{d}L(s) = \int_0^t \Psi(s)\mathbb{1}_{\{s\le\tau\}}\,\mathrm{d}L(s).$$

*Proof.* Note that the random variable on the left-hand side is understood as the càdlàg process $\left(\int_0^t\Psi(s)\,\mathrm{d}L(s) : t\in[0,T]\right)$ evaluated at $t\wedge\tau$. For the integral introduced in Section 6.1 the proof can be done exactly as in Lemma 3.13 changing the convergence of the integrands in $L^2$ into the convergence in $L^\alpha$ and convergence of the integrals in $L^2$ into the ucp convergence. □

**Proposition 6.15.** Suppose that

$$T < \frac{1}{\alpha c_{2,\alpha}2^\alpha c_G^\alpha}. \tag{6.27}$$

Condition (A3) implies that the pathwise uniqueness holds for equation (6.26). More generally, if $X$ and $Y$ are two solutions with initial conditions $X_0$ and $Y_0$, then $X$ and $Y$ are a.s. equal on $\{X_0=Y_0\}$.

*Proof.* Suppose that $X$ and $Y$ are two solutions of (6.26). Then

$$X(t)-Y(t) = S(t)(X_0-Y_0) + \int_0^t S(t-s)\left(F(X(s))-F(Y(s))\right)\mathrm{d}s$$

$$+\int_0^t S(t-s)\left(G(X(s-))-G(Y(s-))\right)\mathrm{d}L(s).$$

Let

$$\tau_n=\inf\left\{t\geq 0:\|X(t)\|\vee\|Y(t)\|\geq n\right\}\wedge T.$$

By Lemma 6.14

$$\begin{aligned}(X(t\wedge\tau_n)-Y(t\wedge\tau_n))\mathbb{1}_{\{X_0=Y_0\}}&\\ &=\int_0^t S(t-s)\left(F(X(s))-F(Y(s))\right)\mathbb{1}_{\{s\leq\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}s\\ &\quad+\int_0^t S(t-s)\left(G(X(s-))-G(Y(s-))\right)\mathbb{1}_{\{s\leq\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}L(s).\end{aligned}$$

By the Hölder and Jensen inequalities we have for a measurable function $f\colon\Omega\times[0,t]\to H$ and $\alpha>p\geq 1$

$$E\left[\left\|\int_0^t f(s)\,\mathrm{d}s\right\|^p\right]\leq t^{p-\frac{p}{\alpha}}\mathbb{E}\left[\left(\int_0^t\|f(s)\|^\alpha\,\mathrm{d}s\right)^{p/\alpha}\right]\leq t^{p-\frac{p}{\alpha}}\left(\mathbb{E}\left[\int_0^t\|f(s)\|^\alpha\,\mathrm{d}s\right]\right)^{p/\alpha}. \quad (6.28)$$

We estimate by (A3) with $q=\frac{p}{\alpha}$

$$\begin{aligned}&\mathbb{E}\left[\|X(t)-Y(t)\|^p\mathbb{1}_{\{t<\tau_n\}\cap\{X_0=Y_0\}}\right]\\ &\leq\mathbb{E}\left[\|X(t\wedge\tau_n)-Y(t\wedge\tau_n)\|^p\mathbb{1}_{\{X_0=Y_0\}}\right]\\ &\leq 2^{p-1}\mathbb{E}\left[\left\|\int_0^t S(t-s)\left(F(X(s))-F(Y(s))\right)\mathbb{1}_{\{s\leq\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}s\right\|^p\right]\\ &\quad+2^{p-1}\mathbb{E}\left[\left\|\int_0^t S(t-s)\left(G(X(s-))-G(Y(s-))\right)\mathbb{1}_{\{s\leq\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}L(s)\right\|^p\right]\\ &\leq 2^{p-1}T^{p-\frac{p}{\alpha}}\left(\mathbb{E}\left[\int_0^t\|S(t-s)\left(F(X(s))-F(Y(s))\right)\|^\alpha\mathbb{1}_{\{s\leq\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}s\right]\right)^{p/\alpha}\\ &\quad+2^{p-1}C_{\alpha,p}\left(\mathbb{E}\left[\int_0^t\|S(t-s)\left(G(X(s-))-G(Y(s-))\right)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\mathbb{1}_{\{s\leq\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}s\right]\right)^{p/\alpha}\\ &\leq 2^{p-1}\left(T^{p-\frac{p}{\alpha}}c_F^p+C_{\alpha,p}c_G^p\right)\left(\mathbb{E}\left[\int_0^t\|X(s)-Y(s)\|^p\mathbb{1}_{\{s<\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}s\right]\right)^{p/\alpha},\end{aligned}$$

where the last equality follows from the fact that both $X$ and $Y$ are càdlàg and therefore for each $\omega\in\Omega$ the integrands $X$ and $X(\cdot-)$ differ only for countably many $s$. Since the constant

$C_{\alpha,p}$ defined in (6.21) converges to $+\infty$ as $p \nearrow \alpha$, there exists $p_0$ such that if $p \in [p_0, \alpha)$, then $T^{p-\frac{p}{\alpha}} c_F^p \leq C_{\alpha,p} c_G^p$. For such $p$, the previous inequality implies that

$$\begin{aligned}\mathbb{E}\left[\|X(t)-Y(t)\|^p \mathbb{1}_{\{t<\tau_n\}\cap\{X_0=Y_0\}}\right] \\ \leq 2^p C_{\alpha,p} c_G^p \left(\mathbb{E}\left[\int_0^t \|X(s)-Y(s)\|^p \mathbb{1}_{\{s<\tau_n\}\cap\{X_0=Y_0\}}\,\mathrm{d}s\right]\right)^{p/\alpha}.\end{aligned}$$

Raising this inequality to the power $\frac{\alpha}{p}$ we obtain

$$\begin{aligned}\left(\mathbb{E}\left[\|X(t)-Y(t)\|^p \mathbb{1}_{\{t<\tau_n\}\cap\{X_0=Y_0\}}\right]\right)^{\alpha/p} \\ \leq 2^\alpha C_{\alpha,p}^{\alpha/p} c_G^\alpha \int_0^t \mathbb{E}\left[\|X(s-)-Y(s-)\|^p \mathbb{1}_{\{s<\tau_n\}\cap\{X_0=Y_0\}}\right] \mathrm{d}s.\end{aligned}$$

Let

$$u_p(t) := \left(\mathbb{E}\left[\|X(t)-Y(t)\|^p \mathbb{1}_{\{t<\tau_n\}\cap\{X_0=Y_0\}}\right]\right)^{\alpha/p}.$$

We apply Theorem 6.13 with $v \equiv 0$, $w \equiv 2^\alpha C_{\alpha,p}^{\alpha/p} c_G^\alpha$ and $\frac{p}{\alpha}$ instead of $p$. Then $q = 1-\frac{p}{\alpha} = \frac{\alpha-p}{\alpha}$. We obtain

$$u_p(t) \leq \left(\frac{\alpha-p}{\alpha}\int_0^t 2^\alpha C_{\alpha,p}^{\alpha/p} c_G^\alpha \,\mathrm{d}s\right)^{\frac{\alpha}{\alpha-p}}.$$

Inserting the formula for $C_{\alpha,p}$ from (6.21) we get

$$u_p(t) \leq \left(\frac{\alpha-p}{\alpha}\left(\frac{c_{2,\alpha}\alpha}{\alpha-p}\right)^{\alpha/p} 2^\alpha c_G^\alpha t\right)^{\frac{\alpha}{\alpha-p}} = (\alpha-p)^{-\frac{\alpha}{p}}\left(\alpha^{-1+\frac{\alpha}{p}} c_{2,\alpha} 2^\alpha c_G^\alpha t\right)^{\frac{\alpha}{\alpha-p}}.$$

Since $(\alpha-p)^{-\frac{\alpha}{p}} \leq (\alpha-p)^{-2}$ and $\alpha^{-1+\frac{\alpha}{p}} \leq \alpha$ we get

$$u_p(t) \leq (\alpha-p)^{-2}\left(\alpha c_{2,\alpha} 2^\alpha c_G^\alpha t\right)^{\frac{\alpha}{\alpha-p}}.$$

Take $p \nearrow \alpha$. We have substituting $x = \frac{\alpha}{\alpha-p}$, $c_3 = \alpha c_{2,\alpha} 2^\alpha c_G^\alpha T < 1$ and applying the L'Hôspital rule twice

$$\begin{aligned}\lim_{p\nearrow\alpha}(\alpha-p)^{-2}\left(\alpha c_{2,\alpha}2^\alpha c_G^\alpha t\right)^{\frac{\alpha}{\alpha-p}} &= \frac{1}{\alpha^2}\lim_{x\to\infty} x^2 c_3^x = \frac{1}{\alpha^2}\lim_{x\to\infty}\frac{2x}{-c_3^{-x}\log(c_3)} \\ &= \frac{2}{-\log(c_3)\alpha^2}\lim_{x\to+\infty}\frac{1}{-\log(c_3)c_3^{-x}} = 0. \quad (6.29)\end{aligned}$$

Thus $\lim_{p\nearrow\alpha} u_p(t) = 0$. Fix $\varepsilon \in (0,1)$. There exists $p < \alpha$ such that $u_p(t) \leq \varepsilon^{\alpha(1+\alpha)}$, which implies that

$$\mathbb{E}\left[\|X(t)-Y(t)\|^p \mathbb{1}_{\{t<\tau_n\}\cap\{X_0=Y_0\}}\right] \leq \varepsilon^{p(1+\alpha)} \leq \varepsilon^{1+\alpha}.$$

By the Markov inequality

$$P(\|X(t)-Y(t)\| \geq \varepsilon, t < \tau_n, X_0 = Y_0) \leq \frac{1}{\varepsilon^p}\mathbb{E}\left[\|X(t)-Y(t)\|^p \mathbb{1}_{\{t<\tau_n\}\cap\{X_0=Y_0\}}\right] \leq \varepsilon.$$

Since $\varepsilon$ was arbitrary it follows that $X(t)$ and $Y(t)$ coincide on $\{t < \tau_n\} \cap \{X_0 = Y_0\}$. Taking $n \to \infty$ proves that $X$ and $Y$ are modifications of each other on $\{X_0 = Y_0\}$. Since they are both càdlàg, it follows that $X$ and $Y$ are indistinguishable on $\{X_0 = Y_0\}$. □

### 6.4.2 Analytical lemmas

We prove some results concerning the composition of a strongly continuous semigroup and Hilbert-Schmidt operators. We consider continuity of the semigroup considered as a mapping on the Skorokhod space.

**Lemma 6.16.** If $K \subset L_{\mathrm{HS}}(U,H)$ is compact, then $\sup_{\psi\in K}\|(S(t)-I)\psi\|_{L_{\mathrm{HS}}(U,H)} \to 0$ as $t \to 0$.

*Proof.* In the proof we use a method similar to the proof of [73, Th. 3.2]. Let $M = \sup_{t\in[0,T]}\|S(t)\|$. Take $\varepsilon_1 = \frac{\varepsilon}{2(1+M)}$ and choose covering of $K$ consisting of the balls $B(\psi_i, \varepsilon_1)$ for $i = 1,\dots,N$. By Lemma 5.3 it follows that $\|(I-S(s))\psi_i\|_{L_{\mathrm{HS}}(U,H)} \to 0$ as $s \to 0$ for every $i$. There exists $\delta$ such that for $s \leq \delta$ and $i = 1,\dots,N$ we have $\|(I-S(s))\psi_i\|_{L_{\mathrm{HS}}(U,H)} \leq \frac{\varepsilon}{2}$. For any $s \leq \delta$ and $\psi \in K$ we find the closest center $\psi_i$ and estimate

$$\begin{aligned}
\|(I-S(s))\psi\|_{L_{\mathrm{HS}}(U,H)} &\leq \|(I-S(s))\psi_i\|_{L_{\mathrm{HS}}(U,H)} + \|(I-S(s))(\psi-\psi_i)\|_{L_{\mathrm{HS}}(U,H)} \\
&\leq \frac{\varepsilon}{2} + (1+M)\varepsilon_1 \\
&= \varepsilon.
\end{aligned}$$

□

**Lemma 6.17.** Assume that $G\colon H \to L_{\mathrm{HS}}(U,H)$ is Lipschitz, that is there exists a constant $c_L$ such that

$$\|S(t)\,(G(x)-G(y))\|_{L_{\mathrm{HS}}(U,H)} \leq c_L\|x-y\|, \qquad x,y \in H.$$

Then the mapping

$$\Theta\colon D([0,T];H) \to D([0,T];L_{\mathrm{HS}}(U,H)), \qquad \Theta(x)(s) = S(T-s)G(x(s)).$$

is continuous.

*Proof.* We use the notation $\|x\|_\infty = \sup\limits_{s\in[0,T]} \|x(s)\|$ when $x \in D([0,T];H)$ and similarly $\|j\|_\infty = \sup\limits_{s\in[0,T]} |j(s)|$ when $j\colon [0,T] \to [0,T]$. Recall that the Skorokhod topology on $D([0,T];H)$ is induced by the metric

$$d(x,y) = \inf_{j\in\Pi} \left(\|x - y\circ j\|_\infty \vee \|\mathrm{Id} - j\|_\infty\right), \qquad x,y \in D([0,T];H),$$

where $\Pi$ is the set of all increasing bijections of $[0,T]$ and $\mathrm{Id}\colon [0,T] \to [0,T]$ is the identity mapping, see [8, p. 124]. By the elementary inequality

$$a \vee b \le a + b \le 2(a \vee b), \qquad a, b \ge 0,$$

we see that the metric $d$ is equivalent to the following one:

$$d^+(x,y) := \inf_{j\in\Pi} \left(\|x - y\circ j\|_\infty + \|\mathrm{Id} - j\|_\infty\right), \qquad x,y \in D([0,T];H).$$

Let $M := \sup_{t\in[0,T]} \|S(t)\|$.

Fix $x_n, x \in D([0,T];H)$ such that $x_n$ converge to $x$ and fix $\varepsilon > 0$. By [31, Prob. 1, p. 146], the image of $[0,T]$ by $x$ is relatively compact in $H$. By the continuity of $G$ the set $K := \{G(x(s)) : s \in [0,T]\}$ is relatively compact in $L_{\mathrm{HS}}(U,H)$. By Lemma 6.16 we obtain that for some $\delta > 0$

$$\sup_{s\le\delta} \sup_{\psi\in K} \|(I - S(s))\psi\|_{L_{\mathrm{HS}}(U,H)} \le \frac{\varepsilon}{2M}. \tag{6.30}$$

Without loss of generality we assume that $\delta \le \frac{\varepsilon}{c_G+1}$.

There exists $n_0$ such that for all $n \ge n_0$ we have $d^+(x_n, x) \le \frac{\delta}{2}$. By definition of the metric, for each $n \ge n_0$ there exists $j_n \in \Pi$ such that

$$\|x_n - x\circ j_n\|_\infty + \|\mathrm{Id} - j_n\|_\infty \le d^+(x_n,x) + \frac{\delta}{2}.$$

Thus for each $n \geq n_0$ there exists $j_n \in \Pi$ such that

$$\|x_n - x \circ j_n\|_\infty + \|\mathrm{Id} - j_n\| \leq \delta. \tag{6.31}$$

Since $G$ composed with the semigroup is Lipschitz

$$\begin{aligned}
&d^+(\Theta(x_n), \Theta(x)) \\
&= \inf_{j\in\Pi} \left( \sup_{s\in[0,T]} \|S(T-s)G(x_n(s)) - S(T-j(s))G(x(j(s)))\| + \|\mathrm{Id} - j\|_\infty \right) \\
&\leq \sup_{s\in[0,T]} \|S(T-s)G(x_n(s)) - S(T-j_n(s))G(x(j_n(s)))\| + \|\mathrm{Id} - j_n\|_\infty \\
&\leq \sup_{s\in[0,T]} \|S(T-s)\left(G(x_n(s)) - G(x(j_n(s)))\right)\|_{L_{\mathrm{HS}}(U,H)} \\
&\quad + \sup_{s\in[0,T]} \|\left(S(T-s) - S(T-j_n(s))\right) G(x(j_n(s)))\|_{L_{\mathrm{HS}}(U,H)} + \|\mathrm{Id} - j_n\|_\infty \\
&\leq c_G \sup_{s\in[0,T]} \|x_n(s) - x(j_n(s))\| \\
&\quad + \sup_{s\in[0,T]} \big\|S(T-(j_n(s)\vee s))\big(S(j_n(s)\vee s - s) - S(j_n(s)\vee s - j_n(s))\big)G(x(j_n(s))\big\|_{L_{\mathrm{HS}}(U,H)} \\
&\quad + \|\mathrm{Id} - j_n\|_\infty.
\end{aligned} \tag{6.32}$$

By (6.31) we have $|s - (j_n(s) \vee s)| \leq \delta$ for all $n \geq n_0$ and $s \in [0,T]$. Note that

$$S((j_n(s)\vee s) - s) - S((j_n(s)\vee s) - j_n(s)) = \begin{cases} \mathrm{Id} - S(s - j_n(s)), & \text{if } j_n(s) \leq s, \\ S(j_n(s) - s) - \mathrm{Id}, & \text{if } j_n(s) > s. \end{cases}$$

Then by (6.30)

$$\begin{aligned}
&\sup_{s\in[0,T]} \big\|S(T-(j_n(s)\vee s))\big(S((j_n(s)\vee s) - s) - S((j_n(s)\vee s) - j_n(s))\big)G(x(j_n(s))\big\|_{L_{\mathrm{HS}}(U,H)} \\
&\leq M \sup_{s\in[0,T]} \big\|\big(S((j_n(s)\vee s) - s) - S((j_n(s)\vee s) - j_n(s))\big)G(x(j_n(s))\big\|_{L_{\mathrm{HS}}(U,H)} \\
&\leq M \sup_{s\leq\delta} \sup_{\psi\in K} \|(\mathrm{Id} - S(s))\psi\|_{L_{\mathrm{HS}}(U,H)} \\
&\leq \frac{\varepsilon}{2}.
\end{aligned}$$

Now (6.32) gives for $n \geq n_0$

$$d^+(\Theta(x_n), \Theta(x)) \leq c_G\delta + \frac{\varepsilon}{2} + \delta = (c_G + 1)\delta + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

### 6.4.3 Proof of tightness

**Lemma 6.18.** Consider the Picard approximating sequence defined by

$$\begin{aligned} X_0(t) &= S(t)X_0, \\ X_n(t) &= S(t)X_0 + \int_0^t S(t-s)F(X_{n-1}(s))\,\mathrm{d}s + \int_0^t S(t-s)G(X_{n-1}(s-))\,\mathrm{d}L(s) \end{aligned} \tag{6.33}$$

for $n \geq 1$, where the stochastic integral is understood as in Jakubowski and Riedle [49], see also Theorem 2.8. If the assumptions (A1), (A2), (A4) hold, then the sequence $(X_n)$ is tight in $D([0,T]; H)$.

*Proof.* We prove tightness in 3 steps. Firstly we show that the definition of $X_n$ makes sense and that they are càdlàg, secondly we verify the Aldous condition and finally we verify condition (6.22).

Step 1. The Dilation Theorem was for the first time used to show existence of a càdlàg modification of a mild solution by Hausenblas and Seidler [45]. By the Dilation theorem [99, Th. I.8.1], since we assumed (A1), there exist a Hilbert space $\hat{H} \supset H$, a projection $P\colon \hat{H} \to H$ and a group of unitary operators $\hat{S}$ such that $S(t) = P\hat{S}(t)$ for $t \geq 0$. We now prove by induction that $X_n$ is well defined and has a càdlàg modification. Suppose that $X_{n-1}$ is càdlàg. Write

$$X_n(t) = S(t)X_0 + P\hat{S}(t)\int_0^t \hat{S}(-s)F(X_{n-1}(s))\,\mathrm{d}s + P\hat{S}(t)\int_0^t \hat{S}(-s)G(X_{n-1}(s-))\,\mathrm{d}L(s).$$

The Lebesgue integral is clearly well defined by the continuity of $F$ assumed in (A4). Thus the second term on the right-hand side is a continuous process. We prove that that the integrand

$$s \mapsto \hat{S}(-s)G(X_{n-1}(s-)) \tag{6.34}$$

is càglàd. Fix $s_0 < s \leq t$. We show that the right limit exists. We have

$$\left\| \hat{S}(-s)G(X_{n-1}(s-)) - \hat{S}(-s_0)G(X_{n-1}(s_0)) \right\|_{L_{\mathrm{HS}}(U,\hat{H})}$$

$$
\begin{aligned}
&\leq \left\| \left(\hat{S}(-s) - \hat{S}(-s_0)\right) G(X_{n-1}(s-)) \right\|_{L_{\mathrm{HS}}(U,\hat{H})} \\
&\quad + \left\| \hat{S}(-s_0) \left(G(X_{n-1}(s-)) - G(X_{n-1}(s_0))\right) \right\|_{L_{\mathrm{HS}}(U,\hat{H})}.
\end{aligned} \tag{6.35}
$$

The second term on the right-hand side tends to 0 as $s \searrow s_0$ by (A4) and the right-continuity of $X_{n-1}$ and the continuity of the embedding $H \subset \hat{H}$. For fixed $\omega \in \Omega$ it follows from (A4) and [31, Prob. 1, p. 146] that the set $\{G(X_{n-1}(s-)(\omega)) : s \in [0,t]\}$ is relatively compact in $L_{\mathrm{HS}}(U, \hat{H})$. Therefore, by Lemma 6.16 applied for $\hat{H}$ instead of $H$ the first term on the right-hand side of (6.35) tends to 0 as well. Similarly, one shows that the limit of $\hat{S}(-s)G(X_{n-1}(s-))$ from the left equals $\hat{S}(-s_0)G(X_{n-1}(s_0-))$. This proves that the mapping (6.34) is càglàd. Thus, the integral in (6.33) is well defined according to [49] and $X_n$ has a càdlàg modification.

Step 2. Fix $\varepsilon, \eta > 0$. We have for any $\theta > 0$ and a sequence of stopping times $(\tau_n)$ such that $\tau_n + \theta \leq T$

$$
\begin{aligned}
&X_n(\tau_n + \theta) - X_n(\tau_n) \\
&= S(\tau_n + \theta)X_0 + \int_0^{\tau_n+\theta} S(\tau_n + \theta - s)F(X_{n-1}(s-))\,\mathrm{d}s \\
&\quad + \int_0^{\tau_n+\theta} S(\tau_n + \theta - s)G(X_{n-1}(s-))\,\mathrm{d}L(s) \\
&\quad - S(\tau_n)X_0 - \int_0^{\tau_n} S(\tau_n - s)F(X_{n-1}(s-))\,\mathrm{d}s - \int_0^{\tau_n} S(\tau_n - s)G(X_{n-1}(s-))\,\mathrm{d}L(s) \\
&= S(\tau_n)\left(S(\theta) - I\right)X_0 + \int_0^{\tau_n} S(\tau_n - s)S(\theta)F(X_{n-1}(s-))\,\mathrm{d}s \\
&\quad + \int_{\tau_n}^{\tau_n+\theta} S(\tau_n + \theta - s)F(X_{n-1}(s-))\,\mathrm{d}s \\
&\quad + \int_0^{\tau_n} S(\tau_n - s)S(\theta)G(X_{n-1}(s-))\,\mathrm{d}L(s) + \int_{\tau_n}^{\tau_n+\theta} S(\tau_n + \theta - s)G(X_{n-1}(s-))\,\mathrm{d}L(s) \\
&\quad - \int_0^{\tau_n} S(\tau_n - s)F(X_{n-1}(s-))\,\mathrm{d}s - \int_0^{\tau_n} S(\tau_n - s)G(X_{n-1}(s-))\,\mathrm{d}L(s) \\
&= S(\tau_n)\left(S(\theta) - I\right)X_0 + \int_0^{\tau_n} S(\tau_n - s)\left(S(\theta) - I\right)F(X_{n-1}(s-))\,\mathrm{d}s \\
&\quad + \int_0^{\tau_n} S(\tau_n - s)\left(S(\theta) - I\right)G(X_{n-1}(s-))\,\mathrm{d}L(s) + \int_{\tau_n}^{\tau_n+\theta} S(\tau_n + \theta - s)F(X_{n-1}(s-))\,\mathrm{d}s \\
&\quad + \int_{\tau_n}^{\tau_n+\theta} S(\tau_n + \theta - s)G(X_{n-1}(s-))\,\mathrm{d}L(s).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&P(\|X_n(\tau_n+\theta)-X_n(\tau_n)\|\geq\eta)\\
&\qquad\leq P\left(\|S(\tau_n)\,(S(\theta)-I)\,X_0\|\geq\frac{\eta}{5}\right)\\
&\qquad\quad+P\left(\left\|\int_0^{\tau_n}(S(\theta)-I)\,S(\tau_n-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|\geq\frac{\eta}{5}\right)\\
&\qquad\quad+P\left(\left\|\int_0^{\tau_n}(S(\theta)-I)S(\tau_n-s)G(X_{n-1}(s-))\,\mathrm{d}L(s)\right\|\geq\frac{\eta}{5}\right)\\
&\qquad\quad+P\left(\left\|\int_{\tau_n}^{\tau_n+\theta}S(\tau_n+\theta-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|\geq\frac{\eta}{5}\right)\\
&\qquad\quad+P\left(\left\|\int_{\tau_n}^{\tau_n+\theta}S(\tau_n+\theta-s)G(X_{n-1}(s-))\,\mathrm{d}L(s)\right\|\geq\frac{\eta}{5}\right)\\
&\qquad=: e_1+e_2+e_3+e_4+e_5.
\end{aligned}
$$

Let $M:=\sup_{t\in[0,T]}\|S(t)\|$. For $e_1$ we have

$$
e_1=P\left(\|S(\tau_n)\,(S(\theta)-I)\,X_0\|\geq\frac{\eta}{5}\right)\leq P\left(\|(S(\theta)-I)X_0\|\geq\frac{\eta}{5M}\right).
$$

By the strong continuity, $e_1$ tends to 0 as $\theta$ tends to 0. By the Markov inequality we have

$$
\begin{aligned}
e_2&=P\left(\left\|\int_0^{\tau_n}(S(\theta)-I)\,S(\tau_n-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|\geq\frac{\eta}{5}\right)\\
&\leq\frac{5}{\eta}\mathbb{E}\left[\left\|\int_0^{T}\mathbb{1}_{[0,\tau_n]}(s)\,(S(\theta)-I)\,S(\tau_n-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|\right]\\
&\leq\frac{5}{\eta}\mathbb{E}\left[\int_0^{T}\mathbb{1}_{[0,\tau_n]}(s)\|(S(\theta)-I)\,S(\tau_n-s)F(X_{n-1}(s))\|\,\mathrm{d}s\right].
\end{aligned}
$$

We estimate

$$
\begin{aligned}
&\|(S(\theta)-I)S(\tau_n-s)F(X_{n-1}(s))\|_H\\
&\qquad\leq\|S(\theta)-I\|_{\mathcal{L}(D((-A)^\delta),H)}\|S(\tau_n-s)F(X_{n-1}(s))\|_{D((-A)^\delta)}.
\end{aligned}\tag{6.36}
$$

Theorem 2.6.13(d) in [73] states that for all $\delta\in(0,1]$ one has for some $C>0$

$$
\|S(t)-I\|_{\mathcal{L}(D((-A)^\delta),H)}\leq Ct^\delta.\tag{6.37}
$$

By (6.36) and (A2) we get $e_2 \leq C\theta^\delta M_0$. For the third term we get

$$e_3 \leq \frac{5^\alpha c_{2,\alpha}}{\eta^\alpha}\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_n]}(s)\|(S(\theta)-I)S(\tau_n - s)G(X_{n-1}(s-))\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right].$$

We estimate as in (6.36)

$$\begin{aligned}&\|(S(\theta)-I)S(\tau_n-s)G(X_{n-1}(s-))\|_{L_{\mathrm{HS}}(U,H)}\\&\qquad\qquad \leq \|S(\theta)-I\|_{\mathcal{L}(D((-A)^\delta),H)}\|S(\tau_n-s)G(X_{n-1}(s-))\|_{L_{\mathrm{HS}}(U,D((-A)^\delta))}.\end{aligned}$$

Using (6.37) and the boundedness (A2) we get $e_3 \leq \frac{5^\alpha c_{2,\alpha}}{\eta^\alpha} C M_0^\alpha T\theta^\delta$. By (A2$'$) and the Markov inequality

$$\begin{aligned}e_4 &= P\left(\left\|\int_{\tau_n}^{\tau_n+\theta} S(\tau_n+\theta-s)F(X_{n-1}(s))\,\mathrm{d}s\right\| \geq \frac{\eta}{5}\right)\\&\leq \frac{5}{\eta}\mathbb{E}\left[\left\|\int_0^T \mathbb{1}_{[\tau_n,\tau_n+\theta]}(s)S(\tau_n+\theta-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|\right]\\&\leq \frac{5}{\eta}\mathbb{E}\left[\int_0^T \mathbb{1}_{[\tau_n,\tau_n+\theta]}(s)\|S(\tau_n+\theta-s)F(X_{n-1}(s))\|\,\mathrm{d}s\right]\\&\leq \frac{5}{\eta}\theta M_0.\end{aligned}$$

Note that by (A2$'$) and tail estimate from Theorem 6.6

$$\begin{aligned}e_5 &= P\left(\left\|\int_0^T \mathbb{1}_{(\tau_n,\tau_n+\theta]}(s)S(\tau_n+\theta-s)G(X_{n-1}(s-))\,\mathrm{d}L(s)\right\| \geq \frac{\eta}{5}\right)\\&\leq \frac{5^\alpha c_{2,\alpha}}{\eta^\alpha}\mathbb{E}\left[\int_{\tau_n}^{\tau_n+\theta}\|S(\tau_n+\theta-s)G(X_{n-1}(s-))\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\\&\leq M_0^\alpha c_{2,\alpha}\frac{5^\alpha}{\eta^\alpha}\theta.\end{aligned}$$

Thus the Aldous condition holds.

Step 3. According to [7, Cor. 3.8.2, Th. 6.7.3] it follows from (A1)(iv) that the embedding of $D((-A)^\delta) \subset H$ is compact as well. Thus the closed ball $B_{D((-A)^\delta)}(0,R)$ is compact in $H$. Note that

$$P\left(X_n(t) \notin B_{D((-A)^\delta)}(0,R)\right) \leq P\left(\|S(t)X_0\|_{D((-A)^\delta)} > \frac{R}{3}\right)$$

$$+ P\left(\left\|\int_0^t S(t-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|_{D((-A)^\delta)} > \frac{R}{3}\right)$$
$$+ P\left(\left\|\int_0^t S(t-s)G(X_{n-1}(s-))\,\mathrm{d}L(s)\right\|_{D((-A)^\delta)} > \frac{R}{3}\right)$$

We have $P\left(\|S(t)X_0\|_{D((-A)^\delta)} > \frac{R}{3}\right) \to 0$ as $R \to \infty$ because $S(t)X_0$ is $D((-A)^\delta)$-valued by [73, Th. 6.13(c)]. Secondly, by (A2)

$$\begin{aligned} P\left(\left\|\int_0^t S(t-s)F(X_{n-1}(s))\,\mathrm{d}s\right\|_{D((-A)^\delta)} > \frac{R}{3}\right) & \\ &\le \frac{3}{R}\mathbb{E}\left[\int_0^T \|S(t-s)F(X_{n-1}(s))\|_{D((-A)^\delta)}\,\mathrm{d}s\right] \\ &\le \frac{3}{R}M_0T \end{aligned}$$

and

$$\begin{aligned} P\left(\left\|\int_0^t S(t-s)G(X_{n-1}(s-))\,\mathrm{d}L(s)\right\|_{D((-A)^\delta)} > \frac{R}{3}\right) & \\ &\le \frac{c_{2,\alpha}3^\alpha}{R^\alpha}\mathbb{E}\left[\int_0^t \|S(t-s)G(X_{n-1}(s-))\|^\alpha_{L_{\mathrm{HS}}(U,D((-A)^\delta))}\,\mathrm{d}s\right] \\ &\le \frac{c_{2,\alpha}3^\alpha}{R^\alpha}M_0^\alpha T. \end{aligned}$$

It follows from Theorem 6.8 that the sequence $(X_n)$ is tight. □

In the following result the Hilbert space $H \times H$ is equipped with the norm

$$\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}, \qquad x, y \in H.$$

**Corollary 6.19.** The sequence $(X_n, X_{n-1})$ is tight in $D([0,T]; H \times H)$.

*Proof.* Firstly, we verify the Aldous condition for the joint sequence $(X_n, X_{n-1}) \in H \times H$. Fix $\varepsilon, \eta > 0$. Since we already know that the sequence $(X_n)$ satisfies the Alous condition, there exists $\delta$ such that for all stopping times $\tau_n$ with $\tau_n + \delta \le T$

$$\sup_{n\in\mathbb{N}}\sup_{0<\theta\le\delta} P\left(\|X_n(\tau_n+\theta) - X_n(\tau_n)\| \ge \frac{\eta}{\sqrt{2}}\right) \le \frac{\varepsilon}{2}.$$

We have

$$
\begin{aligned}
&P(\|(X_n, X_{n-1})(\tau_n + \theta) - (X_n, X_{n-1})(\tau_n)\| \geq \eta) \\
&\qquad \leq P\left(\|X_n(\tau_n + \theta) - X_n(\tau_n)\| \geq \frac{\eta}{\sqrt{2}}\right) + P\left(\|X_{n-1}(\tau_n + \theta) - X_{n-1}(\tau_n)\| \geq \frac{\eta}{\sqrt{2}}\right) \\
&\qquad \leq \varepsilon
\end{aligned}
$$

for $n \in \mathbb{N}$ and $\theta \leq \delta$.

Secondly, we prove that (6.22) holds with the subspace $\Gamma = D((-A)^\delta) \times D((-A)^\delta)$. Choose $R$ such that $P\left(\|X_n(t)\|_{D((-A)^\delta)} \leq \frac{R}{\sqrt{2}}\right) \geq 1 - \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. We have with

$$P(\|(X_n, X_{n-1})(t)\|_\Gamma > R) \leq P\left(\|X_n(t)\|_\Gamma > \tfrac{R}{\sqrt{2}}\right) + P\left(\|X_{n-1}(t)\|_\Gamma > \tfrac{R}{\sqrt{2}}\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

### 6.4.4 Estimates of the norm of the difference between $X_n$ and $X_{n-1}$

**Lemma 6.20.** Let $c_H$ be a constant such that for all $p \in (1, \alpha)$

$$2^{p-1}\left(\frac{\alpha - 1}{\left(c_{2,\alpha}^{1/\alpha} \wedge c_{2,\alpha}\right)\alpha} c_F^p + c_G^p\right) \leq c_H.$$

Assume (A1)–(A4) and also suppose that

$$T \vee \left((c_{2,\alpha} \vee c_{2,\alpha}^{1/\alpha}) c_H (T \vee T^{1/\alpha})\right) < 1. \tag{6.38}$$

and additionally that

$$\int_0^T \mathbb{E}\left[\|S(t)G(X_0)\|_{L_{\mathrm{HS}}(U,H)}^\alpha\right] \mathrm{d}t + \int_0^T \mathbb{E}\left[\|S(t)F(X_0)\|^\alpha\right] \mathrm{d}t < \infty. \tag{6.39}$$

For each $t \in [0, T]$ the sequence $(X_n(t) - X_{n-1}(t))_n$ converges to 0 in probability.

*Proof.* Step 1. We prove that for every $t \in [0, T]$

$$\lim_{p \nearrow \alpha} \lim_{n \to \infty} \mathbb{E}\left[\|X_n(t) - X_{n-1}(t)\|^p\right] = 0. \tag{6.40}$$

We start with the following estimate

$$\begin{aligned}
&\mathbb{E}\left[\|X_1(t)-X_0(t)\|^p\right]\\
&=\mathbb{E}\left[\left\|\int_0^t S(t-s)F(X_0)\,\mathrm{d}s+\int_0^t S(t-s)G(X_0)\,\mathrm{d}L(s)\right\|^p\right]\\
&\le 2^{p-1}\left(\mathbb{E}\left[\left\|\int_0^t S(t-s)F(X_0)\,\mathrm{d}s\right\|^p\right]+\mathbb{E}\left[\left\|\int_0^t S(t-s)G(X_0)\,\mathrm{d}L(s)\right\|^p\right]\right).
\end{aligned}$$

Applying (6.28) together with Proposition 6.7 results in

$$\begin{aligned}
\mathbb{E}\left[\|X_1(t)-X_0(t)\|^p\right]&=2^{p-1}T^{p-\frac{p}{\alpha}}\left(\mathbb{E}\left[\int_0^T\|S(t-s)F(X_0)\|^\alpha\,\mathrm{d}s\right]\right)^{p/\alpha}\\
&\quad+2^{p-1}C_{\alpha,p}\left(\mathbb{E}\left[\int_0^t\|S(s)G(X_0)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\right)^{p/\alpha}\\
&\le 2^{\alpha-1}C_{\alpha,p}\Bigg(\frac{\alpha-1}{(c_{2,\alpha}^{1/\alpha}\wedge c_{2,\alpha})\alpha}\left(\mathbb{E}\left[\int_0^T\|S(t-s)F(X_0)\|^\alpha\,\mathrm{d}s\right]\right)^{p/\alpha}\\
&\quad+\left(\mathbb{E}\left[\int_0^t\|S(s)G(X_0)\|^\alpha_{L_{\mathrm{HS}}(U,H)}\,\mathrm{d}s\right]\right)^{p/\alpha}\Bigg),
\end{aligned}$$

where the last step follows from the fact that $T<1$. The right-hand side is finite by (6.39). Thus for some constant $c_I$ for every $p\in(1,\alpha)$ we have

$$\sup_{t\in[0,T]}\mathbb{E}\left[\|X_1(t)-X_0(t)\|^p\right]\le C_{\alpha,p}c_I. \tag{6.41}$$

We estimate again by (6.28) and Proposition 6.7

$$\begin{aligned}
&\mathbb{E}\left[\|X_n(t)-X_{n-1}(t)\|^p\right]\\
&=\mathbb{E}\Bigg[\Bigg\|\int_0^t S(t-s)\left(F(X_{n-1}(s))-F(X_{n-2}(s))\right)\mathrm{d}s\\
&\qquad+\int_0^t S(t-s)\left(G(X_{n-1}(s-))-G(X_{n-2}(s-))\right)\mathrm{d}L(s)\Bigg\|^p\Bigg]\\
&\le 2^{p-1}\mathbb{E}\left[\left\|\int_0^t S(t-s)\left(F(X_{n-1}(s-))-F(X_{n-2}(s-))\right)\mathrm{d}s\right\|^p\right]\\
&\quad+2^{p-1}\mathbb{E}\left[\left\|\int_0^t S(t-s)\left(G(X_{n-1}(s-))-G(X_{n-2}(s-))\right)\mathrm{d}L(s)\right\|^p\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{p-1} t^{p-\frac{p}{\alpha}} \left( \mathbb{E}\left[ \int_0^t \| S(t-s) \left( F(X_{n-1}(s-)) - F(X_{n-2}(s-)) \right) \|^{\alpha} \, \mathrm{d}s \right] \right)^{p/\alpha} \\
&\quad + 2^{p-1} C_{\alpha,p} \left( \mathbb{E}\left[ \int_0^t \| S(t-s) \left( G(X_{n-1}(s-)) - G(X_{n-2}(s-)) \right) \|^{\alpha}_{L_{\mathrm{HS}}(U,H)} \, \mathrm{d}s \right] \right)^{p/\alpha}.
\end{aligned}
$$

From (A3) with $q = \frac{p}{\alpha} < 1$ we have

$$
\begin{aligned}
&\mathbb{E}\left[ \| X_n(t) - X_{n-1}(t) \|^p \right] \\
&\leq 2^{p-1} t^{p-\frac{p}{\alpha}} \left( \mathbb{E}\left[ \int_0^t c_F^{\alpha} \| X_{n-1}(s-) - X_{n-2}(s-) \|^p \, \mathrm{d}s \right] \right)^{p/\alpha} \\
&\quad + 2^{p-1} C_{\alpha,p} \left( \int_0^t c_G^{\alpha} \mathbb{E}\left[ \| X_{n-1}(s-) - X_{n-2}(s-) \|^p \right] \mathrm{d}s \right)^{p/\alpha} \\
&\leq 2^{p-1} C_{\alpha,p} \left( \frac{\alpha - 1}{\left( c_{2,\alpha}^{1/\alpha} \wedge c_{2,\alpha} \right) \alpha} c_F^p + c_G^p \right) \left( \mathbb{E}\left[ \int_0^t \| X_{n-1}(s-) - X_{n-2}(s-) \|^p \, \mathrm{d}s \right] \right)^{p/\alpha} \\
&\leq C_{\alpha,p} c_H \left( \int_0^t \mathbb{E}\left[ \| X_{n-1}(t_1) - X_{n-2}(t_1) \|^p \right] \mathrm{d}t_1 \right)^{p/\alpha}.
\end{aligned}
$$

Iterating:

$$
\begin{aligned}
&\mathbb{E}\left[ \| X_n(t) - X_{n-1}(t) \|^p \right] \\
&\leq C_{\alpha,p} c_H \left( \int_0^t C_{\alpha,p} c_H \left( \int_0^{t_1} \mathbb{E}\left[ \| X_{n-2}(t_2) - X_{n-3}(t_2) \|^p \right] \mathrm{d}t_2 \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&= C_{\alpha,p}^{1+\frac{p}{\alpha}} c_H^{1+\frac{p}{\alpha}} \left( \int_0^t \left( \int_0^{t_1} \mathbb{E}\left[ \| X_{n-2}(t_2) - X_{n-3}(t_2) \|^p \right] \mathrm{d}t_2 \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&\leq \dots \\
&\leq C_{\alpha,p}^{1+\frac{p}{\alpha}+\left(\frac{p}{\alpha}\right)^2+\dots+\left(\frac{p}{\alpha}\right)^{n-2}} c_H^{1+\frac{p}{\alpha}+\frac{p^2}{\alpha^2}+\dots+\left(\frac{p}{\alpha}\right)^{n-2}} \\
&\qquad \times \left( \int_0^t \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-2}} \mathbb{E}\left[ \| X_1(t_{n-1}) - X_0(t_{n-1}) \|^p \right] \mathrm{d}t_{n-1} \right)^{p/\alpha} \dots \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&\leq C_{\alpha,p}^{1+\frac{p}{\alpha}+\left(\frac{p}{\alpha}\right)^2+\dots+\left(\frac{p}{\alpha}\right)^{n-1}} c_H^{1+\frac{p}{\alpha}+\frac{p^2}{\alpha^2}+\dots+\left(\frac{p}{\alpha}\right)^{n-2}} c_I^{\left(\frac{p}{\alpha}\right)^{n-1}}
\end{aligned}
$$

$$\times \left( \int_0^t \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-2}} \mathrm{d}t_{n-1} \right)^{p/\alpha} \dots \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha},$$

where the last inequality follows from (6.41). Since

$$\begin{aligned}
I &= \left( \int_0^t \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-2}} \mathrm{d}t_{n-1} \right)^{p/\alpha} \dots \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&= \left( \int_0^t \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-3}} t_{n-2}^{\frac{p}{\alpha}} \,\mathrm{d}t_{n-2} \right)^{p/\alpha} \dots \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&= \left( \int_0^t \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-4}} \left( \frac{1}{1+\frac{p}{\alpha}} t_{n-3}^{1+\frac{p}{\alpha}} \right)^{\frac{p}{\alpha}} \mathrm{d}t_{n-3} \right)^{p/\alpha} \dots \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&= \left( \frac{1}{1+\frac{p}{\alpha}} \right)^{\left(\frac{p}{\alpha}\right)^{n-2}} \left( \int_0^t \left( \int_0^{t_1} \dots \left( \int_0^{t_{n-4}} t_{n-3}^{\frac{p}{\alpha}+\left(\frac{p}{\alpha}\right)^2} \mathrm{d}t_{n-3} \right)^{p/\alpha} \dots \right)^{p/\alpha} \mathrm{d}t_1 \right)^{p/\alpha} \\
&= \cdots \\
&= t^{\frac{p}{\alpha}+\dots+\left(\frac{p}{\alpha}\right)^{n-1}} \prod_{k=2}^{n-1} \left( \frac{1-\frac{p}{\alpha}}{1-\left(\frac{p}{\alpha}\right)^k} \right)^{\left(\frac{p}{\alpha}\right)^{n-k}} \\
&= t^{\frac{p}{\alpha}\frac{1-\left(\frac{p}{\alpha}\right)^{n-2}}{1-\frac{p}{\alpha}}} \frac{(\alpha-p)^{\sum_{k=2}^{n-1}\left(\frac{p}{\alpha}\right)^{n-k}}}{\alpha^{\sum_{k=2}^{n-1}\left(\frac{p}{\alpha}\right)^{n-k}}} \prod_{k=2}^{n-1} \left( \frac{1}{1-\left(\frac{p}{\alpha}\right)^k} \right)^{\left(\frac{p}{\alpha}\right)^{n-k}}
\end{aligned}$$

we have inserting also the formula for the constant $C_{\alpha,p}$

$$\begin{aligned}
&\mathbb{E}\left[\|X_n(t) - X_{n-1}(t)\|^p\right] \\
&\le (c_{2,\alpha}^{p/\alpha}\alpha)^{\frac{1-\left(\frac{p}{\alpha}\right)^n}{1-\frac{p}{\alpha}}} c_H^{\frac{1-\left(\frac{p}{\alpha}\right)^n}{1-\frac{p}{\alpha}}} c_I^{\left(\frac{p}{\alpha}\right)^{n-1}} \left( \frac{1}{\alpha-p} \right)^{\sum_{k=0}^{n-1}\left(\frac{p}{\alpha}\right)^k} t^{\frac{p}{\alpha}\frac{1-\left(\frac{p}{\alpha}\right)^{n-2}}{1-\frac{p}{\alpha}}} \frac{(\alpha-p)^{\sum_{k=2}^{n-1}\left(\frac{p}{\alpha}\right)^{n-k}}}{\alpha^{\sum_{k=2}^{n-1}\left(\frac{p}{\alpha}\right)^{n-k}}} \\
&\quad \times \prod_{k=2}^{n-1} \left( \frac{1}{1-\left(\frac{p}{\alpha}\right)^k} \right)^{\left(\frac{p}{\alpha}\right)^{n-k}} \\
&\le (c_{2,\alpha}^{p/\alpha}\alpha)^{\frac{1-\left(\frac{p}{\alpha}\right)^n}{1-\frac{p}{\alpha}}} c_H^{\frac{1-\left(\frac{p}{\alpha}\right)^n}{1-\frac{p}{\alpha}}} c_I^{\left(\frac{p}{\alpha}\right)^{n-1}} \left( \frac{1}{\alpha-p} \right)^{1+\left(\frac{p}{\alpha}\right)^{n-1}} t^{\frac{p}{\alpha}\frac{1-\left(\frac{p}{\alpha}\right)^{n-2}}{1-\frac{p}{\alpha}}} \frac{1}{\alpha^{\frac{p}{\alpha}\frac{1-\left(\frac{p}{\alpha}\right)^{n-2}}{1-\frac{p}{\alpha}}}}
\end{aligned}$$

$$\times \prod_{k=2}^{n-1} \left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^k} \right)^{\left(\frac{p}{\alpha}\right)^{n-k}}$$
$$=: \xi(n,p).$$

We now prove that

$$\lim_{n\to\infty} \prod_{k=2}^{n-1} \left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^k} \right)^{\left(\frac{p}{\alpha}\right)^{n-k}} = 1. \tag{6.42}$$

Observe that

$$\prod_{k=2}^{n-1} \left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^k} \right)^{\left(\frac{p}{\alpha}\right)^{n-k}} \geq 1.$$

Therefore, taking the logarithms in (6.42) it is enough to prove prove that

$$\limsup_{n\to\infty} \sum_{k=2}^{n-1} \left(\frac{p}{\alpha}\right)^{n-k} \log\left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^k} \right) \leq 0. \tag{6.43}$$

We split the sum according to $k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $k > \left\lfloor \frac{n}{2} \right\rfloor$. Since $\log\left( \frac{1}{1-\left(\frac{p}{\alpha}\right)^k} \right) \leq \log\left( \frac{1}{1-\left(\frac{p}{\alpha}\right)^2} \right)$ we have

$$\begin{aligned} \sum_{k=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{p}{\alpha}\right)^{n-k} \log\left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^k} \right) &\leq \log\left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^2} \right) \sum_{k=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{p}{\alpha}\right)^{n-k} \\ &= \log\left( \frac{1}{1 - \left(\frac{p}{\alpha}\right)^2} \right) \left(\frac{p}{\alpha}\right)^{n-\left\lfloor \frac{n}{2} \right\rfloor} \frac{1 - \left(\frac{p}{\alpha}\right)^{\left\lfloor \frac{n}{2} \right\rfloor - 1}}{1 - \frac{p}{\alpha}} \end{aligned} \tag{6.44}$$

which converges to 0 as $n \to \infty$. Secondly, we use the inequality

$$\log\left( \frac{1}{1-x} \right) = -\log(1-x) \leq \log(4)x, \qquad x \in \left(0, \frac{1}{2}\right),$$

to estimate for $n$ such that $\left(\frac{p}{\alpha}\right)^{\lfloor \frac{n}{2} \rfloor+1} < \frac{1}{2}$

$$\begin{aligned} \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \left(\frac{p}{\alpha}\right)^{n-k} \log\left(\frac{1}{1-\left(\frac{p}{\alpha}\right)^k}\right) &\leq \log(4) \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \left(\frac{p}{\alpha}\right)^{n-k} \left(\frac{p}{\alpha}\right)^{k} \\ &= \log(4) \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \left(\frac{p}{\alpha}\right)^{n} \\ &= \log(4) \left(n-1-\left\lfloor \frac{n}{2} \right\rfloor\right) \left(\frac{p}{\alpha}\right)^{n} \end{aligned} \tag{6.45}$$

whcich converges to 0 as $n \to \infty$. Combining (6.44) and (6.45) finishes the proof of (6.43) and (6.42).

Therefore

$$\begin{aligned} \lim_{n\to\infty} \xi(n,p) &= (c_{2,\alpha}^{p/\alpha} \alpha c_H t^{p/\alpha})^{\frac{1}{1-\frac{p}{\alpha}}} \frac{1}{\alpha-p} \frac{1}{\alpha^{\frac{p}{\alpha}\frac{1}{1-\frac{p}{\alpha}}}} \\ &= (c_{2,\alpha}^{p/\alpha} c_H \alpha^{1-\frac{p}{\alpha}} t^{p/\alpha})^{\frac{1}{1-\frac{p}{\alpha}}} \frac{1}{\alpha-p} \\ &\leq \left((c_{2,\alpha} \vee c_{2,\alpha}^{1/\alpha}) c_H (t \vee t^{1/\alpha})\right)^{\frac{\alpha}{\alpha-p}} \frac{1}{\alpha-p}. \end{aligned} \tag{6.46}$$

Let $c_3 = (c_{2,\alpha} \vee c_{2,\alpha}^{1/\alpha}) c_H (t \vee t^{1/\alpha})$. Note that $c_3 < 1$ according to condition (6.38). One can show similarly as in (6.29) that the right-hand side of (6.46) converges to 0 as $p \nearrow \alpha$. Thus,

$$\lim_{p\nearrow\alpha} \lim_{n\to\infty} \xi(n,p) = 0$$

This finishes the proof of (6.40).

Step 2. Fix $\varepsilon \in (0,1)$. Equation (6.40) implies that for any $\varepsilon_1 > 0$ we can find $p_0$ such that if $p \in [p_0, \alpha)$, then $\lim_{n\to\infty} \mathbb{E}\left[\|X_n(t) - X_{n-1}(t)\|^p\right] \leq \frac{\varepsilon_1}{2}$. Then there exists $n_0(p)$ such that for $n \geq n_0(p)$ we have $\mathbb{E}\left[\|X_n(t) - X_{n-1}(t)\|^p\right] \leq \varepsilon_1$.

Take $\varepsilon_1 = \varepsilon^{1+\alpha}$. There exist $p < \alpha$ and $n_0(p)$ such that for all $n \geq n_0(p)$

$$\mathbb{E}\left[\|X_n(t) - X_{n-1}(t)\|^p\right] \leq \varepsilon^{1+\alpha}.$$

we estimate by the Markov inequality

$$P(\|X_n(t) - X_{n-1}(t)\| \geq \varepsilon) \leq \frac{1}{\varepsilon^p} \mathbb{E}\left[\|X_n(t) - X_{n-1}(t)\|^p\right] \leq \frac{1}{\varepsilon^\alpha} \varepsilon^{1+\alpha} = \varepsilon,$$

for $n \geq n_0(p)$. □

### 6.4.5 Cylindrical Lévy processes as metric space-valued random variables

We now give some preparatory lemmas, which lead to a conclusion that every cylindrical Lévy process can be viewed as a random variable taking values in a Polish space.

We recall some facts about continuity of evaluations with respect to the Skorokhod topology, see [48, Lem. VI.3.12, Lem. VI.3.14]. For a a càdlàg process $X$ let $\mathcal{J}(X) := \{t \geq 0 : P(\Delta X(t) \neq 0) > 0\}$. The set $\mathcal{J}(X)$ is at most countable.

Let $(S, d)$ be a metric space. Recall that for each $t \in [0, T]$ the function $\pi_t \colon D([0, T]; S) \to S$ defined by $\pi_t(x) = x(t)$ is measurable, see [8, Th. 12.5(ii)]. Recall from [8, p. 133-134] that for $t \in (0, T)$ the function $\pi_t$ is continuous at $x$ if and only if $x$ is continuous at $t$, whereas $\pi_0$ and $\pi_T$ are continuous everywhere. Thus if $X$ and $Y$ are two $S$-valued processes with the same distribution, then for any $B \in \mathcal{B}(S)$ we have $P(X(t) \in B) = P(X \in \pi_t^{-1}(B)) = P(Y \in \pi_t^{-1}(B)) = P(Y(t) \in B)$, which proves that $X(t)$ and $Y(t)$ have the same distribution.

**Lemma 6.21.** Each cylindrical Lévy process $L$ is continuous as a mapping

$$L \colon U \to L^0(\Omega, \mathcal{F}, P; D([0, T]; \mathbb{R})).$$

*Proof.* The result follows from the closed graph theorem for F-spaces, see [88, Th. 2.15]. Take $x_n \to x$ such that $L(\cdot)x_n \to X$ in probability in $D([0, T]; \mathbb{R})$. By the continuous mapping theorem [51, Lem. 4.3] we have that $L(t)x_n \to X(t)$ for $t \in \mathcal{J}(X)$. Since for each $t \in [0, T]$ the mapping $L(t) \colon U \to L^0(\Omega; \mathbb{R})$ is continuous, it follows that $L(t)x_n \to L(t)x$. Thus $X(t) = L(t)x$ a.s. for $t \in \mathcal{J}(X)$. Since both processes are càdlàg, we obtain that $X(t) = L(t)x$ for all $t \in [0, T]$ a.s. □

Let $(u_n)$ be countable dense sequence in $U$. With each cylindrical Lévy process $L$ we associate a mapping

$$A_L \colon \Omega \to \prod_{n=1}^{\infty} D([0, T]; \mathbb{R}), \qquad A_L(\omega) := ((L(\cdot)u_n)(\omega))_{n \in \mathbb{N}}. \tag{6.47}$$

**Lemma 6.22.** $\prod_{n=1}^{\infty} D([0, T]; \mathbb{R})$ is a separable and complete metric space. The function $A_L$ is measurable.

*Proof.* For the first statement see [33, Cor. 2.3.16, Th. 4.22, Th. 4.3.12]. We prove that $A_L$ is measurable. Recall that the base of the product topology consists of the sets of the form $\prod_{n=1}^{\infty} G_n$, where all $G_n$ are open and all except for finitely many are equal to the whole space $D([0,T];\mathbb{R})$. Since the space is separable, every open set is a countable union of the sets of this form, see [33, Cor. 4.1.16]. Thus, to prove that $A$ is measurable it is enough to check that for any $N \in \mathbb{N}$ and any open sets $G_1, \ldots, G_N$ the set

$$A_L^{-1}\left(\prod_{n=1}^{N} G_n \times \prod_{n=N+1}^{\infty} D([0,T];\mathbb{R})\right) \tag{6.48}$$

belongs to $\mathcal{F}$. The set (6.48) is equal to $\bigcap_{n=1}^{N} \{\omega \in \Omega : L(\cdot)u \in G_n\}$ and each set in the intersection is measurable since every Lévy process is measurable as a mapping from $\Omega$ to $D([0,T];\mathbb{R})$, see [8, p. 135]. $\square$

Note that $A_L$ uniquely determines $L$ as for $u \in U \setminus \{u_n : n \in \mathbb{N}\}$ we find a subsequence $(u_{n_k})$ converging to $u$ and then by the continuity of $L(t)$ we have that $L(t)u = \lim_{k\to\infty} (A_L(t))_{n_k}$, where the limit is taken in probability.

### 6.4.6 Distribution of the stochastic integral

We show that the stochastic integral has unique distribution. The same result for integrals with respect to Poisson random measures was shown by Brzeźniak and Hausenblas [14]. Our presentation is much simpler, at the expense of being less detailed: we neglect without making further comments sets of measure 0. In some aspects our lemma is less general than the result in [14]: we consider integrals at fixed time $t \in [0,T]$ rather than integrals viewed as processes in the Skorokhod space.

**Lemma 6.23.** Let $\Psi_1$ and $\Psi_2$ be two càglàd, adapted, $L_{\mathrm{HS}}(U,H)$-valued processes defined on $(\Omega_1, \mathcal{F}_1, P_1, (\mathcal{F}_t^1))$ and $(\Omega_2, \mathcal{F}_2, P_2, (\mathcal{F}_t^2))$ respectively and such that $\mathcal{L}(\Psi_1) = \mathcal{L}(\Psi_2)$. Let

$$L_1 \colon U \to L^0\left(\Omega_1, \mathcal{F}_1, P_1, (\mathcal{F}_t^1)\right), \qquad L_2 \colon U \to L^0\left(\Omega_2, \mathcal{F}_2, P_2, (\mathcal{F}_t^2)\right)$$

be two cylindrical Lévy process on $U$ with the same distribution. Then for every $t \in [0,T]$

$$\mathscr{L}\left(\int_0^t \Psi_1(s)\,\mathrm{d}L_1(s)\right) = \mathscr{L}\left(\int_0^t \Psi_2(s)\,\mathrm{d}L_2(s)\right).$$

*Proof.* We first consider the case when $\Psi_1$ and $\Psi_2$ are simple. Without loss of generality we may assume that the partitions of $[0,T]$ for $\Psi_1$ and $\Psi_2$ are same. That is we have for $t \in [0,T]$

$$\Psi_1(t) = \Psi_0^1 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{N-1} \Psi_i^1 \mathbb{1}_{(t_i,t_{i+1}]}(t), \qquad \Psi_2(t) = \Psi_0^2 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{N-1} \Psi_i^2 \mathbb{1}_{(t_i,t_{i+1}]}(t) \tag{6.49}$$

and

$$\Psi_i^1 = \sum_{j=1}^{m_i^1} \mathbb{1}_{A_{i,j}^1} \psi_{i,j}^1, \qquad \Psi_i^2 = \sum_{j=1}^{m_i^2} \mathbb{1}_{A_{i,j}^2} \psi_{i,j}^2, \tag{6.50}$$

and $\psi_{i,j}^1, \psi_{i,j}^2 \in L_{\mathrm{HS}}(U,H)$. We may assume that for each $i$ there are no repetitions in the sequence $\psi_{i,1}^1, \dots, \psi_{i,m_i^1}^1$ (and similarly for $\psi_{i,1}^2, \dots, \psi_{i,m_i^1}^2$). Since $\Psi_1$ and $\Psi_2$ have the same distribution it follows that $m_i^1 = m_i^2 =: m_i$ and that

$$\{\psi_{i,1}^1, \dots, \psi_{i,m_i}^1\} = \{\psi_{i,1}^2, \dots, \psi_{i,m_i}^2\}.$$

We may assume that $\psi_{i,1}^1 = \psi_{i,1}^2, \dots, \psi_{i,m_i}^1 = \psi_{i,m_i}^2$ and $P_1(A_{i,j}^1) = P_2(A_{i,j}^2)$. Then

$$I_1(t) = \int_0^t \Psi_1(s)\,\mathrm{d}L_1(s) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} \mathbb{1}_{A_{i,j}^1} J_{t_i,t_{i+1}}^1(\psi_{i,j}^1),$$

$$I_2(t) = \int_0^t \Psi_2(s)\,\mathrm{d}L_2(s) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} \mathbb{1}_{A_{i,j}^2} J_{t_i,t_{i+1}}^2(\psi_{i,j}^2).$$

Using the independent increments of $L_1$ and $L_2$ and the adaptedness of $\Psi_1$ and $\Psi_2$ we calculate the characteristic function

$$\begin{aligned}
\varphi_{I_1(t)}(h) &= \mathbb{E}\left[e^{i\langle I_1(t),h\rangle}\right] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\prod_{i=0}^{N-1}\Bigg(\sum_{j=1}^{m_i} e^{i(L_1(t_{i+1})-L_1(t_i))((\psi_{i,j}^1)^*h)} \mathbb{1}_{A_{i,j}^1}\Bigg)\Bigg|\mathcal{F}_{t_{N-1}}^1\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{i=0}^{N-2}\Bigg(\sum_{j=1}^{m_i} e^{i(L_1(t_{i+1})-L_1(t_i))((\psi_{i,j}^1)^*h)} \mathbb{1}_{A_{i,j}^1}\Bigg)\Bigg] \\
&\quad \times \sum_{j=1}^{m_{N-1}} P(A_{N-1,j})\mathbb{E}\left[e^{i(L_1(t_N)-L_1(t_{N-1}))((\psi_{N-1,j}^1)^*h)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \dots \\
&= \prod_{i=0}^{N-1} \sum_{j=1}^{m_i} P_1(A^1_{i,j}) \mathbb{E}\left[ e^{i(L_1(t_{i+1} - L_1(t_i))((\psi^1_{i,j})^* h)} \right] \\
&= \prod_{i=0}^{N-1} \sum_{j=1}^{m_i} P_2(A^2_{i,j}) \mathbb{E}\left[ e^{i(L_2(t_{i+1}) - L_2(t_i))((\psi^2_{i,j})^* h)} \right] \\
&= \dots \\
&= \varphi_{I_2(t)}(h).
\end{aligned}
$$

Now take $\Psi_1$ and $\Psi_2$ arbitrary càglàd adapted $L_{\mathrm{HS}}(U, H)$-valued processes. Note that it is enough to prove that we can find a sequences simple processes $(\Phi^1_n)$ and $(\Phi^2_n)$ of the form (6.49)–(6.50) approximating $\Psi_1$ and $\Psi_2$ and such that $\mathscr{L}(\Phi^1_n) = \mathscr{L}(\Phi^2_n)$. Indeed, for such sequences it follows that

$$
\varphi_{I_1(t)}(h) = \lim_{n \to \infty} \varphi_{\int_0^t \Phi^1_n(s)\, \mathrm{d}L_1(s)}(h) = \lim_{n \to \infty} \varphi_{\int_0^t \Phi^2_n(s)\, \mathrm{d}L_2(s)}(h) = \varphi_{I_1(t)}(h).
$$

Let $\pi_n = (t^n_0, \dots, t^n_{k_n})$ be a sequence of partitions of $[0, T]$ with mesh converging to 0. Let $\Phi^1_n(t) = \Psi_1(t)$ and $\Phi^2_n(t) = \Psi_2(t)$ for $t \in [t^n_k, t^n_{k+1})$. By Proposition VI.6.37 of [48] the processes $\Phi^1_n$ and $\Phi^2_n$ converge a.s. to $\Psi_1$ and $\Psi_2$ in the Skorokhod topology. □

### 6.4.7 Main result and proof

**Theorem 6.24.** *Under assumptions* (A1)–(A4) *equation* (6.26) *has unique mild solution.*

*Proof.* We prove the theorem in several steps. We first prove existence of a weak solution with additional assumptions on the initial condition and the time horizon $T$. Then we obtain a strong solution and finally remove those extra conditions.

Step 1. Existence of weak mild solution assuming (6.27), (6.38) and (6.39).

By Corollary 6.19 we have the joint tightness of the sequence $(X_n, X_{n-1})_{n=1,2,\dots}$. Then also the sequence $(A_L, X_n, X_{n-1})$ is tight and we select a convergent subsequence $(A_L, X_{n_k}, X_{n_k-1})$. By a version of the Skorokhod theorem in [15, Th. C.11] applied in the metric space

$$
\left( \prod_{n=1}^{\infty} D([0, T]; \mathbb{R}) \right) \times D([0, T]; H \times H),
$$

there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ and a sequence $(\bar{B}, \bar{X}_k, \bar{Y}_k)$ of random variables defined on $\bar{\Omega}$ with the same law:

$$\mathscr{L}(A_L, X_{n_k}, X_{n_k-1}) = \mathscr{L}(\bar{B}, \bar{X}_k, \bar{Y}_k), \qquad k \in \mathbb{N},$$

and such that $\bar{X}_k \to \bar{X}$ and $\bar{Y}_k \to \bar{Y}$ a.s. when $k \to \infty$. Note that in particular $\mathcal{L}(\bar{X}_n(0)) = \mathcal{L}(X_0)$ and $\mathcal{L}(\bar{X}_k(t), \bar{Y}_k(t)) = \mathcal{L}(X_{n_k}(t), X_{n_k-1}(t))$ for every $t \in [0,T]$.

We construct a cylindrical Lévy process $\bar{L}$ on $\bar{\Omega}$. Take $u \in U$ and a subsequence $(u_{n_k})$ of the sequence $(u_n)$ used in the construction of $A_L$ in Subsection 6.4.5 such that $u_{n_k} \to u$. The distribution of the whole sequence $\left(\bar{B}(t)\right)_{n_k}$ is same as the distribution of $(A_L(t))_{n_k} = L(t)(u_{n_k})$. Thus, $\left(\bar{B}(t)\right)_{n_k}$ converges in probability and we denote its limit as $\bar{L}(t)u$.

We show that $\bar{X}$ and $\bar{Y}$ are indistinguishable. By Lemma 6.20 for any $t \in [0,T]$ and $\varepsilon > 0$

$$\bar{P}\left(\left\|\bar{X}_k(t) - \bar{Y}_k(t)\right\| \geq \varepsilon\right) = P\left(\|X_{n_k}(t) - X_{n_k-1}(t)\| \geq \varepsilon\right) \to 0.$$

It follows that $\bar{X}_k(t) - \bar{Y}_k(t)$ converges to 0 in probability. By the continuous mapping theorem $\bar{X}_k(t) \to \bar{X}(t)$ a.s. and $\bar{Y}_k(t) \to \bar{Y}(t)$ a.s. for $t \in [0,T] \setminus (\mathcal{J}(\bar{X}) \cup \mathcal{J}(\bar{Y}))$. We apply [80, Th. 3.4] and obtain that

$$\bar{X}(t) = \bar{Y}(t) \text{ a.s. for } t \in [0,T] \setminus (\mathcal{J}(\bar{X}) \cup \mathcal{J}(\bar{Y})).$$

For $t \in \mathcal{J}(\bar{X}) \cup \mathcal{J}(\bar{Y})$, since $[0,T] \setminus (\mathcal{J}(\bar{X}) \cup \mathcal{J}(\bar{Y}))$ is dense in $[0,T]$ we can find a sequence $(t_n) \subset [0,T] \setminus (\mathcal{J}(\bar{X}) \cup \mathcal{J}(\bar{Y}))$ which decreases to $t$. Since $\bar{X}(t_n) = \bar{Y}(t_n)$ a.s. for each $n$ we get that also $\bar{X}(t) = \bar{Y}(t)$ a.s. This proves that $\bar{X}$ and $\bar{Y}$ are modifications of each other and since both processes are càdlàg, they are indistinguishable.

We show that the new sequence satisfies the same equation:

$$\bar{X}_k(t) = S(t)\bar{X}_k(0) + \int_0^t S(t-s)F(\bar{Y}_k(s))\,\mathrm{d}s + \int_0^t S(t-s)G(\bar{Y}_k(s-))\,\mathrm{d}\bar{L}(s). \tag{6.51}$$

We know that

$$\mathbb{E}\left[\left\|X_{n_k}(t) - S(t)X_0 - \int_0^t S(t-s)F(X_{n_k-1}(s))\,\mathrm{d}s - \int_0^t S(t-s)G(X_{n_k-1}(s-))\,\mathrm{d}L(s)\right\| \wedge 1\right] = 0$$

and $\mathcal{L}(X_{n_k}, X_{n_k-1}, L) = \mathscr{L}(\bar{X}_k, \bar{Y}_k, \bar{L})$. It follows by Lemma 6.23 that

$$\mathbb{E}\left[\left\|\bar{X}_k(t) - S(t)\bar{X}_k(0) - \int_0^t S(t-s)F(\bar{Y}_k(s))\,\mathrm{d}s - \int_0^t S(t-s)G(\bar{Y}_k(s-))\,\mathrm{d}\bar{L}(s)\right\| \wedge 1\right] = 0$$

and thus (6.51) holds a.s.

By Theorem 2.8

$$\int_0^t S(t-s)G(\bar{Y}_k(s-))\,\mathrm{d}\bar{L}(s) \to \int_0^t S(t-s)G(\bar{X}(s-))\,\mathrm{d}\bar{L}(s), \tag{6.52}$$

in probability, for every $t \in [0,T]$. Since it is assumed in (A4) that $F$ is continuous, it follows that for $s \in [0,t] \setminus \mathcal{J}(\bar{X})$ almost surely

$$S(t-s)F(\bar{Y}_k(s)) \to S(t-s)F(\bar{X}(s)).$$

The boundedness of $F$ in Remark 6.10 implies that we can apply the Lebesgue dominated convergence theorem to get that

$$\int_0^t S(t-s)F(\bar{Y}_k(s-))\,\mathrm{d}s \to \int_0^t S(t-s)F(\bar{X}(s-))\,\mathrm{d}s, \tag{6.53}$$

in $L^1(\Omega; H)$. Passing to a subsequence if necessary, we get that the above convergence holds in probability.

We take $k \to \infty$ in (6.51). By (6.52) and (6.53) the right-hand side converges in probability for every $t \in [0,T]$. The left-hand side converges a.s. for $t \notin \mathcal{J}(\bar{X})$ and we get

$$\bar{X}(t) = S(t)\bar{X}(0) + \int_0^t S(t-s)F(\bar{X}(s))\,\mathrm{d}s + \int_0^t S(t-s)G(\bar{X}(s-))\,\mathrm{d}\bar{L}(s)$$

almost surely for $t \notin \mathcal{J}(\bar{X})$. The right-hand side is càdlàg by the Dilation theorem and the left-hand side is by definition. Thus the conclusion holds for any $t \in [0,T]$.

Step 2. Existence of a strong solution with the additional assumptions (6.27), (6.38) and (6.39).

A very general version of Yamada-Watanabe theorem is presented in [56], see also [57]. We take the Polish spaces

$$S_1 = D([0,T];H), \qquad\qquad S_2 = H \times \prod_{n=1}^{\infty} D([0,T];\mathbb{R}),$$

an $S_2$-valued random variable $Y$ given by $Y = (X_0, A_L)$, see (6.47), and filtrations defined by

$$\mathcal{F}_t^X = \sigma(X(s) : s \le t), \qquad\qquad \mathcal{F}_t^Y = \sigma(X_0) \otimes \sigma(L(s)u : u \in U, s \le t)$$

for $t \in [0,T]$. By [48, Prop. VI.6.37], if we take a sequence of partitions $\pi_n = (t_0^n, \dots, t_{k_n}^n)$, $t_k^n = \frac{kT}{n}$, $k = 0, \dots, n$ with mesh converging to 0, then the discretised processes defined by

$$X_n(t) := X(t_k^n), \qquad \text{for } t \in [t_k^n, t_{k+1}^n),$$

converge a.s. to $X$ in the Skorokhod topology. We discretise both integrals appearing in the equation:

$$I_1(t) := -\sum_{k=0}^{n-1} S\left(t - \left(\frac{k}{n} \wedge t\right)\right) F\left(X\left(t \wedge \frac{k}{n}\right)\right) \frac{T}{n},$$
$$I_2(t) := \sum_{k=0}^{n-1} \left(L\left(t \wedge \frac{(k+1)T}{n}\right) - L\left(t \wedge \frac{kT}{n}\right)\right) \left(S\left(t - \left(\frac{k}{n} \wedge t\right)\right) G\left(X\left(t \wedge \frac{k}{n}-\right)\right)\right)^*.$$

Equation (6.25) can be written in the form of a constraint $\Gamma$ appearing in [56] given by

$$\lim_{n\to\infty} \mathbb{E}\left[1 \wedge \|X(t) - S(t)X_0 - I_1(t) - I_2(t)\|\right] = 0, \qquad t \ge 0.$$

We show that uniqueness in the sense of [56] holds but first we recall some notions from that paper. Two solutions $X_1$ and $X_2$ are jointly compatible with $Y$ if for any $t \in [0,T]$ and for any bounded measurable function $h\colon S_2 \to \mathbb{R}$

$$\mathbb{E}\left[h(Y) | \mathcal{F}_t^{X_1} \vee \mathcal{F}_t^{X_2} \vee \mathcal{F}_t^Y\right] = \mathbb{E}\left[h(Y) | \mathcal{F}_t^Y\right] \text{ a.s.}$$

It follows by taking $h$ which depends only on $((x_0, a(t+s) - a(t)) : (x_0, a) \in S_2, s \ge 0)$ that for any two compatible solutions $X_1$ and $X_2$, $L$ is a cylindrical Lévy process with respect to the filtration $(\mathcal{F}_t^{X_1} \vee \mathcal{F}_t^{X_2} \vee \mathcal{F}_t^Y)_{t\in[0,T]}$. Therefore pathwise uniqueness in Proposition 6.15 implies uniqueness of jointly compatible solutions. Lemma 2.10 in [56] implies that uniqueness in the sense of [56] holds. By Theorem 1.5 from [56] there exists a strong compatible solution and by Proposition 2.13 therein it is adapted.

Step 3. The case of the general initial condition.

We proceed as in the proof of [1, Th. 6.2.3]. Let $\Omega_n = \{\|X_0\| \leq n\}$ and let $X_n$ be the solution of the equation with the initial condition $X_0 \mathbb{1}_{\Omega_n}$. By Proposition 6.15 for $n > k$ the solutions $X_n$ and $X_k$ coincide on $\Omega_k$. Let $X = X_k$ on $\Omega_k$. Since $\Omega_n \nearrow \Omega$, it follows that $X$ is a solution. If $Y$ is another solution, then $Y = X_n = X$ on $\Omega_n$ again by Proposition 6.15 and thus pathwise uniqueness holds. This finishes the proof of the Theorem in the case when (6.27) and (6.38) hold.

Step 4. Arbitrary $T > 0$.

We find time $T_0$ such that $[0, T_0] \cup [T_0, 2T_0] \cup \ldots \cup [(n-1)T_0, nT_0] = [0, T]$ and (6.27) and (6.38) hold with $T$ replaced by $T_0$. The previous considerations show that there is a unique mild solution on $[0, T_0]$, say $X_1$. Similarly there exists a unique mild solution, which we call $X_2$ on $[T_0, 2T_0]$ with the initial condition $X_1(T_0)$. We continue this procedure and define $X(t) := X_k(t)$ for $t \in [(k-1)T_0, kT_0]$. □

# Appendix: Proof of the Schwarz inequality

We present the proof of Theorem 5.2 since it is not easily available in the literature.

## A.1 Special case

We first prove the theorem in the special case when

$$E^* \text{ has the metric approximation property and } F \text{ is reflexive.} \tag{A.1}$$

*Proof of Theorem 5.2 under assumption* (A.1). In the space of cylindrical probability measures on $E$ we say that $(\mu_j)$ converges to $\mu$ in the *cylindrical sense* if for any finite-dimensional Banach space $G$ and any bounded operator $v\colon E \to G$ we have that $v(\mu_j) \to v(\mu)$ weakly. Denote by $\mathcal{M}(F)$ the set of Radon probability measures on $F$. We observe that the cylindrical convergence is weaker than the weak convergence. Indeed, take $\mu_j \to \mu$ weakly on $F$ equipped with the weak topology. Take a finite-dimensional Banach space $G$, a continuous linear operator $v\colon F \to G$ and a bounded continuous function $f\colon G \to \mathbb{R}$. Then $v$ is weak-weak-continuous. However, $G$ is finite-dimensional, so in fact $v$ is weak-strong continuous. It follows that $\int_F f \circ v \, \mathrm{d}\mu_j \to \int_F f \circ v \, \mathrm{d}\mu$. Thus $\mu_j \to \mu$ cylindrically. The proof of the theorem is divided into four steps.

Step 1. Inequality (5.4) holds for convex combinations of Dirac deltas.

We take

$$\mu = \sum_{k=1}^{n} c_k \delta_{y_k}$$

with $\sum_{k=1}^{n} c_k = 1$ and $(y_k) \subseteq E$. For simplicity we write $\mu$ as

$$\mu = \sum_{k=1}^{n} c_k \delta_{c_k^{-1/p} x_k}$$

with $x_k = c_k^{1/p} y_k$. Note that $u(\mu) = \sum_{k=1}^{n} c_k \delta_{c_k^{-1/p} u(x_k)}$. We calculate

$$\|u(\mu)\|_p = \left( \sum_{k=1}^{n} c_k \left\| c_k^{1/p} u(x_k) \right\|^p \right)^{1/p} = \left( \sum_{k=1}^{n} \|u(x_k)\|^p \right)^{1/p} \le \pi_p(u) \sup_{x^* \in B_{E^*}} \left( \sum_{k=1}^{n} |x^*(x_k)|^p \right)^{1/p}, \tag{A.2}$$

where the last inequality follows from the definition of the $p$-summing norm. Secondly,

$$\|\mu\|_p^* = \sup_{x^* \in B_{E^*}} \|x^*(\mu)\|_p = \sup_{x^* \in B_{E^*}} \left( \sum_{k=1}^{n} c_k \left| c_k^{-1/p} x^*(x_k) \right|^p \right)^{1/p} = \sup_{x^* \in B_{E^*}} \left( \sum_{k=1}^{n} |x^*(x_k)|^p \right)^{1/p}. \tag{A.3}$$

Now, (5.4) follows from (A.2) and (A.3).

Step 2. For any cylindrical probability measure $\mu$ there exists a sequence of convex combinations of Dirac deltas $(\mu_j)$ such that $\mu_j \to \mu$ cylindrically and $\|\mu_j\|_p^* \le \|\mu\|_p^*$ for all $j$.

In the proof we follow Maurey [64, Prop. 6]. Since $E^*$ has the metric approximation property, by [19, Cor. 3.4] there exists a sequence of finite rank operators $\pi_j \colon E^* \to E^*$ converging to the identity strongly and such that $\|\pi_j\| \le 1$.

For each $j$ let $x^*_{j,1}, \ldots, x^*_{j,n_j}$ be a basis of $\pi_j(E^*)$. By the Auerbach lemma [29, Lem. 6.26], there exists a corresponding basis $a_{j,1}, \ldots, a_{j,n_j}$ of $\pi_j(E^*)^*$ such that $a_{j,i}(x_{j,k}) = \delta_{i,k}$. We extend $a_{j,i}$ to functionals on $E^*$ and show that they are weak$^*$-continuous. By [24, Cor. V.12.8] it is enough to verify sequential continuity. Let $y_n^* \in E^*$ converge to 0 in the weak$^*$-topology. By the compactness of $\pi_j$, $\pi_j(y_n^*) \to 0$ in norm. The equality

$$\pi_j(y_n^*) = \sum_{k=1}^{n_j} a_{j,k}(y_n^*) x^*_{j,k}$$

together with the uniqueness of the representation as a linear combination of the basis vectors imply that $a_{j,k}(y_n^*) \to 0$ as $n \to \infty$.

Weak*-continuity of $a_{j,k}$ proves that $a_{j,k} \in E$, see [88, p. 95]. Thus

$$\pi_j(x^*) = \sum_{k=1}^{n_j} x^*(a_{j,k}) x^*_{j,k}, \qquad x^* \in E^*$$

and $\pi_j^*$ is given by

$$\pi_j^*(x^{**}) = \sum_{k=1}^{n_j} x^{**}(x^*_{j,k}) a_{j,k}, \qquad x^{**} \in E^{**}.$$

From this representation we see that $\pi_j^* \colon E^{**} \to E$. We can therefore define a measure on $E$ by $\mu_j := (\pi_j^*)|_E(\mu)$. We have

$$\begin{aligned}
\|\mu_j\|_p^* &= \sup_{y^* \in B_{E^*}} \left( \int_E |y^*(x)|^p \, \mu_j(\mathrm{d}x) \right)^{1/p} \\
&= \sup_{y^* \in B_{E^*}} \left( \int_E \left| y^*(\pi_j^*(x)) \right|^p \mu(\mathrm{d}x) \right)^{1/p} \\
&= \sup_{y^* \in B_{E^*}} \left( \int_E |\pi_j(y^*)(x)|^p \, \mu(\mathrm{d}x) \right)^{1/p} \\
&\leq \|\pi_j\| \left( \sup_{y^* \in B_{E^*}} \int_E |y^*(x)|^p \, \mu(\mathrm{d}x) \right)^{1/p} \\
&\leq \|\mu\|_p^*.
\end{aligned}$$

We show that $\mu_j$ converge to $\mu$ in the cylindrical sense. Let $G$ be a finite-dimensional Banach space and $v \colon E \to G$ be linear and continuous. Note that $\left\|(v \circ \pi_j^* - v)^*\right\| = \|\pi_j(v^*) - v^*\|$ converges to 0, because $\pi_j$ converges to Id strongly and $v^*$ maps from $G^*$, which is a finite-dimensional space. Then [64, Prop. 3] guarantees that $v(\mu_j) \to v(\mu)$ weakly. Finally, every Radon probability measure can be approximated by convex combinations of Dirac deltas, see [11, Ex. 8.1.6].

Step 3. Let $S \subset \mathcal{M}(F)$ be the set of Radon probability measures $\mu$ on $F$ with $\|\mu\|_p \leq M$. Then $S$ is compact in the topology of weak convergence and closed in the topology of cylindrical convergence.

We firstly show that $S$ is closed and secondly that it is relatively compact. By definition of the weak topology in the set of probability measures, the map from $\mathcal{M}(F)$ to $\mathbb{R}$ defined by

$$\mu \mapsto \mu(f) := \int_F f(x)\,\mu(\mathrm{d}x) \tag{A.4}$$

is continuous for $f \in C_b(F;\mathbb{R})$. If $f \in C(F;\mathbb{R}_+)$, then the map (A.4) is lower semi-continuous. This follows from the fact that it is a supremum of continuous mappings $\mu \to \mu(f\mathbb{1}_{|f|\leq n} + n\mathbb{1}_{|f|>n})$. Taking $f(x) = \|x\|^p$ we get that $\mu \mapsto \|\mu\|_p$ is lower semi-continuous. Now, $S$ is closed as the preimage of $(-\infty, M]$.

We show that the set $S$ is relatively compact. Let $K = B_F(0,R)$. Then by the Chebyshev inequality

$$\mu(F \setminus K) \leq \frac{\|\mu\|_p^p}{R^p} \leq \frac{M^p}{R^p}.$$

Since $F$ is reflexive, we get by the Banach–Alaoglu theorem that $K$ is compact in the weak topology. By the Prokhorov theorem, see [101, Th. I.3.6], we get that $S$ is relatively compact. Note that the use of the Prokhorov theorem is justified, because the weak*-topology is completely regular Hausdorff, see [65, p. 223].

It follows that $S$ is compact in the topology of weak convergence. Since the topology of cylindrical convergence is weaker, it follows that $S$ is also compact relative to the topology of cylindrical convergence. From here, $S$ is closed.

Step 4. Let $\mu_j$ be a sequence of discrete measures obtained in Step 2. By Step 1 we have

$$\|u(\mu_j)\|_p \leq \pi_p(u)\|\mu_j\|_p^* \leq \pi_p(u)\|\mu\|_p^*. \tag{A.5}$$

Since $\mu_j \to \mu$ cylindrically it follows that for any finite-dimensional Banach space $G$, any linear and continuous $v\colon F \to G$ and any $f \in C_b(G,\mathbb{R})$ we have by the very definition of cylindrical convergence applied to $v \circ u$ that

$$\int_G f(x)\,v(u(\mu_j))(\mathrm{d}x) = \int_E f(v(u(x)))\,\mu_j(\mathrm{d}x) \to \int_E f(v(u(x)))\mu(\mathrm{d}x) = \int_G f(x)\,v(u(\mu))(\mathrm{d}x)$$

as $j \to \infty$. This gives that $u(\mu_j) \to u(\mu)$ cylindrically. Each $u(\mu_j)$ lies in the set $S$ from the previous step with $M := \pi_p(u)\|u\|_p^*$. By Theorem 5.1 $u(\mu)$ is a Radon measure i.e. $u(\mu) \in \mathcal{M}(F)$. By the closedness of $S$ in the cylindrical convergence, the limit $u(\mu)$ must also lie in $S$, which means that it satisfies $\|u(\mu)\|_p \leq \pi_p(u)\|\mu\|_p^*$. □

## A.2 General case

*Proof of Theorem 5.2 in the general case.* We assume first that $p > 1$.

Let $K = B_{E^*}$, which is a weak*-compact. We define a mapping

$$i_E\colon E \to C(K), \qquad i_E(x)(x^*) = \langle x^*, x \rangle.$$

One can show that $i_E$ is an isometry. By the Pietsch Factorisation Theorem [29, Th. 2.13] there exists a Borel probability $\nu$ on $K$, a closed subspace $X_p$ of $L^p(\nu)$ and an operator $\hat{u}\colon X_p \to F$ such that $j_p i_E(E) \subseteq X_p$ and $\hat{u} j_p^E i_E(x) = x$ for $x \in E$, where $j_p$ the embedding of $C(K)$ into $L^p(K)$ and $j_p^E$ is the restriction of $j_p$ to $i_E(E)$. That is, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\ u\ } & F \\ \Big\downarrow{\scriptstyle i_E} & & \Big\uparrow{\scriptstyle \hat{u}} \\ i_E(E) & \xrightarrow{\ j_p^E\ } & X_p \\ \cap\!\!\!| & & \cap\!\!\!| \\ C(K) & \xrightarrow{\ j_p\ } & L^p(K,\nu) \end{array}$$

The measure $\nu$ and the operator $\hat{u}$ can be chosen so that $\|\hat{u}\| = \pi_p(u)$.

We have

$$\|u(\mu)\|_p = \left\|\hat{u} j_p^E i_E(\mu)\right\|_p \le \|\hat{u}\|_{\mathcal{L}(X_p,F)} \left\|j_p^E i_E(\mu)\right\|_p = \pi_p(u) \|j_p i_E(\mu)\|_p \tag{A.6}$$

where the last equality follows from the fact that $X_p$ is isometrically embedded into $L^p(K,\nu)$. Note that the dual of $C(K)$ has the metric approximation property, see [89, p. 80] and that $L^p(K,\nu)$ is reflexive i.e. assumption (A.1) holds. It is well known that the inclusion $j_p$ of $C(K)$ into $L^p(K,\nu)$ is $p$-summing and $\pi_p(j_p) = 1$, see [29, Ex. 2.9(b)]. Thus, applying Theorem 5.2 for the cylindrical measure $i_E(\mu)$ on $C(K)$ and the mapping $j_p\colon C(K) \to L^p(K,\nu)$, we get

$$\|j_p i_E(\mu)\|_p \le \pi_p(j_p) \|i_E(\mu)\|_p^*. \tag{A.7}$$

We observe that

$$\|i_E(\mu)\|_p^* = \sup_{x^* \in B_{C(K)^*}} \|x^* i_E(\mu)\|_p \le \sup_{x^* \in B_{E^*}} \|x^*(\mu)\|_p = \|\mu\|_p^*, \tag{A.8}$$

where the last inequality follows from the fact that for each $x^* \in B_{C(K)^*}$ the functional $x^* i_E \colon E \to \mathbb{R}$ belongs to $B_{E^*}$. Thus, (A.7) implies

$$\|j_p i_E(\mu)\|_p \leq \|\mu\|_p^*. \tag{A.9}$$

Combining this with (A.6) establishes claim in the case $p > 1$.

In the case when $p = 1$, we assume that $F$ has the Radon–Nikodym property. By Theorem Theorem 5.1 $u(\mu)$ is a Radon measure on $F$. Now we repeat estimate (A.6). We cannot directly apply the previous case for the cylindrical measure $i_E(\mu)$ and the mapping $j_1 \colon C(K) \to L^1(K, \nu)$ as the space $L^1(K, \mu)$ is not reflexive. However, we observe that Steps 1 and 2 of Section A.1 do not require reflexivity. In Step 3. we obtain that $j_1 i(\mu)$ is a measure on $L^1(K, \nu)^{**}$. We can apply Step 4 with the space $L^1(K, \nu)^{**}$ instead of $F$ and in this way obtain inequality (A.7). Inequalities (A.8) and (A.9) follow and thus the claim is proven also in the case $p = 1$. □

# Bibliography

[1] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[2] D. Applebaum and M. Riedle. Cylindrical Lévy processes in Banach spaces. *Proc. Lond. Math. Soc.*, 101(3):697–726, 2010.

[3] A. Badrikian. *Séminaire sur les fonctions aléatoires linéaires et les mesures cylindriques*, volume 139 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[4] A. Badrikian and S. Chevet. *Measures Cylindriques, Espaces de Wiener et Fonctiones Aléatoires Gaussiennes*. Springer-Verlag, Berlin, 1974.

[5] R. M. Balan. SPDEs with $\alpha$-stable Lévy noise: a random field approach. *Int. J. Stoch. Anal.*, 2014.

[6] R. M. Balan. Integration with respect to Lévy colored noise, with applications to SPDEs. *Stochastics*, 87(3):363–381, 2015.

[7] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[8] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999.

[9] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

[10] R. M. Blumenthal and R. K. Getoor. Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.*, 10:493–516, 1961.

[11] V. I. Bogachev. *Measure theory. Vol. II.* Springer-Verlag, Berlin, 2007.

[12] Z. Brzeźniak, B. Goldys, P. Imkeller, S. Peszat, E. Priola, and J. Zabczyk. Time irregularity of generalized Ornstein-Uhlenbeck processes. *C. R. Math. Acad. Sci. Paris*, 348(5-6):273–276, 2010.

[13] Z. Brzeźniak and E. Hausenblas. Maximal regularity for stochastic convolutions driven by Lévy processes. *Probab. Theory Related Fields*, 145(3-4):615–637, 2009.

[14] Z. Brzeźniak and E. Hausenblas. Uniqueness in law of the Itô integral with respect to Lévy noise. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 37–57. Birkhäuser/Springer Basel AG, Basel, 2011.

[15] Z. Brzeźniak, E. Hausenblas, and P. A. Razafimandimby. Stochastic Reaction-diffusion Equations Driven by Jump Processes. *Potential Anal.*, 49(1):131–201, 2018.

[16] Z. Brzeźniak, W. Liu, and J. Zhu. Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Anal. Real World Appl.*, 17:283–310, 2014.

[17] Z. Brzeźniak and H. Long. A note on $\gamma$-radonifying and summing operators. In *Stochastic analysis*, volume 105 of *Banach Center Publ.*, pages 43–57. Polish Acad. Sci. Inst. Math., Warsaw, 2015.

[18] Z. Brzeźniak and J. Zabczyk. Regularity of Ornstein-Uhlenbeck processes driven by a Lévy white noise. *Potential Anal.*, 32(2):153–188, 2010.

[19] P. G. Casazza. Approximation properties. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 271–316. North-Holland, Amsterdam, 2001.

[20] A. Chojnowska-Michalik. On processes of Ornstein-Uhlenbeck type in Hilbert space. *Stochastics*, 21(3):251–286, 1987.

[21] C. Chong. Lévy-driven Volterra equations in space and time. *J. Theoret. Probab.*, 30(3):1014–1058, 2017.

[22] C. Chong. Stochastic PDEs with heavy-tailed noise. *Stochastic Process. Appl.*, 127(7):2262–2280, 2017.

[23] C. Chong, R. C. Dalang, and T. Humeau. Path properties of the solution to the stochastic heat equation with lévy noise. *Stochastics and Partial Differential Equations: Analysis and Computations*, 2018.

[24] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[25] J. B. Conway. *A course in operator theory*, volume 21 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000.

[26] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[27] R. C. Dalang and L. Quer-Sardanyons. Stochastic integrals for spde's: A comparison. *Expo. Math.*, 29(1):67–109, 2011.

[28] E. Dettweiler. Banach space valued processes with independent increments and stochastic integration. In *Probability in Banach spaces, IV (Oberwolfach, 1982)*, volume 990 of *Lecture Notes in Math.*, pages 54–83. Springer, Berlin-New York, 1983.

[29] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[30] J. Diestel and J. J. Uhl, Jr. *Vector measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[31] J. Dieudonné. *Foundations of Modern Analysis*. Number Vol. 1 in Foundations of Modern Analysis. Academic Press, 1969.

[32] P. Embrechts and C. M. Goldie. Comparing the tail of an infinitely divisible distribution with integrals of its Lévy measure. *Ann. Probab.*, 9(3):468–481, 1981.

[33] R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[34] S. N. Ethier and T. G. Kurtz. *Markov processes Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.

[35] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[36] R. Fiorenza. *Hölder and locally Hölder continuous functions, and open sets of class $C^k, C^{k,\lambda}$.* Frontiers in Mathematics. Birkhäuser/Springer, Cham, 2016.

[37] B. Gaveau. Intégrale stochastique radonifiante. *C. R. Acad. Sci. Paris Sér. A*, 276:617–620, 1973.

[38] E. Giné and M. B. Marcus. The central limit theorem for stochastic integrals with respect to Lévy processes. *Ann. Probab.*, 11(1):58–77, 1983.

[39] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Academic Press, 2007.

[40] M. Griffiths and M. Riedle. Modelling Lévy space-time white noises. `https://arxiv.org/abs/1907.04193`.

[41] I. Gyöngy. On stochastic equations with respect to semimartingales. III. *Stochastics*, 7(4):231–254, 1982.

[42] I. Gyöngy and N. V. Krylov. On stochastic equations with respect to semimartingales. I. *Stochastics*, 4(1):1–21, 1980/81.

[43] I. Gyöngy and N. V. Krylov. On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces. *Stochastics*, 6(3-4):153–173, 1981/82.

[44] E. Hausenblas. Existence, uniqueness and regularity of parabolic SPDEs driven by Poisson random measure. *Electron. J. Probab.*, 10:1496–1546, 2005.

[45] E. Hausenblas and J. Seidler. Stochastic convolutions driven by martingales: maximal inequalities and exponential integrability. *Stoch. Anal. Appl.*, 26(1):98–119, 2008.

[46] A. Ichikawa. Some inequalities for martingales and stochastic convolutions. *Stochastic Anal. Appl.*, 4(3):329–339, 1986.

[47] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.

[48] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2003.

[49] A. Jakubowski and M. Riedle. Stochastic integration with respect to cylindrical lévy processes. *Ann. Probab.*, 45(6B):4273–4306, 2017.

[50] A. Jentzen and M. Röckner. Regularity analysis for stochastic partial differential equations with nonlinear multiplicative trace class noise. *J. Differential Equations*, 252(1):114–136, 2012.

[51] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[52] T. Kosmala and M. Riedle. Variational solutions of stochastic partial differential equations with cylindrical Lévy noise. 2018. `https://arxiv.org/abs/1807.11418`.

[53] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. *Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki*, (14):71–146, 1979. Translated from.

[54] U. Kumar and M. Riedle. Invariant measure for the stochastic Cauchy problem driven by a cylindrical Lévy process. `https://arxiv.org/abs/1904.03118`.

[55] U. Kumar and M. Riedle. The stochastic Cauchy problem driven by a cylindrical Lévy process. *Electron. J. Probab*, 25(10):26 pp., 2020.

[56] T. G. Kurtz. The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.*, 12:951–965, 2007.

[57] T. G. Kurtz. Weak and strong solutions of general stochastic models. *Electron. Commun. Probab.*, 19:no. 58, 16, 2014.

[58] T. G. Kurtz and P. E. Protter. Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 197–285. Springer, Berlin, 1996.

[59] W. Linde. *Probability in Banach spaces—stable and infinitely divisible distributions*. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, second edition, 1986.

[60] W. Liu and M. Röckner. *Stochastic partial differential equations: an introduction.* Universitext. Springer, Cham, 2015.

[61] Y. Liu and J. Zhai. A note on time regularity of generalized Ornstein-Uhlenbeck processes with cylindrical stable noise. *C. R. Math. Acad. Sci. Paris*, 350(1-2):97–100, 2012.

[62] Y. Liu and J. Zhai. Time regularity of generalized Ornstein-Uhlenbeck processes with Lévy noises in Hilbert spaces. *J. Theoret. Probab.*, 29(3):843–866, 2016.

[63] H. Luschgy and G. Pagès. Moment estimates for Lévy processes. *Electron. Commun. Probab.*, 13:422–434, 2008.

[64] B. Maurey. Probabilités cylindriques, type et ordre. Applications radonifiantes. In *Séminaire Maurey-Schwartz Année 1972–1973: Espaces $L^p$ et applications radonifiantes, Exp. No. 1.* Centre de Math., École Polytech., Paris, 1973.

[65] R. E. Megginson. *An introduction to Banach space theory*, volume 183 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[66] M. Métivier. *Semimartingales: A course on stochastic processes*, volume 2 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin-New York, 1982.

[67] M. Métivier and J. Pellaumail. Cylindrical stochastic integral. *Publications mathématiques et informatique de Rennes*, (3):1–14, 1976. talk:3.

[68] M. Métivier and J. Pellaumail. *Stochastic integration.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London-Toronto, Ont., 1980.

[69] E. Motyl. Stochastic Navier-Stokes equations driven by Lévy noise in unbounded 3D domains. *Potential Anal.*, 38(3):863–912, 2013.

[70] E. Motyl. Stochastic hydrodynamic-type evolution equations driven by Lévy noise in 3D unbounded domains – abstract framework and applications. *Stochastic Process. Appl.*, 124(6):2052–2097, 2014.

[71] C. Mueller. The heat equation with Lévy noise. *Stochastic Process. Appl.*, 74(1):67–82, 1998.

[72] L. Mytnik. Stochastic partial differential equation driven by stable noise. *Probab. Theory Related Fields*, 123(2):157–201, 2002.

[73] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[74] S. Peszat and J. Zabczyk. *Stochastic partial differential equations with Lévy noise. An evolution equation approach.* Cambridge: Cambridge University Press, 2007.

[75] S. Peszat and J. Zabczyk. Time regularity of solutions to linear equations with Lévy noise in infinite dimensions. *Stochastic Process. Appl.*, 123(3):719–751, 2013.

[76] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations.* Springer, Berlin, 2007.

[77] E. Priola and J. Zabczyk. On linear evolution equations for a class of cylindrical Lévy noises. In *Stochastic partial differential equations and applications*, volume 25 of *Quad. Mat.*, pages 223–242. Dept. Math., Seconda Univ. Napoli, Caserta, 2010.

[78] E. Priola and J. Zabczyk. Structural properties of semilinear SPDEs driven by cylindrical stable processes. *Probab. Theory Related Fields*, 149(1-2):97–137, 2011.

[79] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, Second edition. Version 2.1, Corrected third printing, 2005.

[80] S. I. Resnick. *Heavy-tail phenomena. Probabilistic and statistical modeling.* New York, NY: Springer, 2007.

[81] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, corrected printing of third edition, 2005.

[82] M. Riedle. Infinitely divisible cylindrical measures on Banach spaces. *Studia Math.*, 207(3):235–256, 2011.

[83] M. Riedle. Stochastic integration with respect to cylindrical Lévy processes in Hilbert spaces: an $L^2$ approach. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 17(1):19, 2014.

[84] M. Riedle. Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes. *Potential Anal.*, 42(4):809–838, 2015.

[85] M. Riedle. Stable cylindrical Lévy processes and the stochastic Cauchy problem. *Electron. Commun. Probab.*, 23, 2018.

[86] J. Rosiński and W. A. Woyczyński. Moment inequalities for real and vector $p$-stable stochastic integrals. In *Probability in Banach spaces, V (Medford, Mass., 1984)*, volume 1153 of *Lecture Notes in Math.*, pages 369–386. Springer, Berlin, 1985.

[87] B. Rüdiger. Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces. *Stoch. Stoch. Rep.*, 76(3):213–242, 2004.

[88] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[89] R. A. Ryan. *Introduction to tensor products of Banach spaces.* Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[90] E. Saint Loubert Bié. Étude d'une edps conduite par un bruit poissonnien. *Prob. Theory and Related Fields*, 111(2):287–321, 1998.

[91] P. Saphar. Hypothèse d'approximation à l'ordre $p$ dans les espaces de Banach et approximation d'applications $p$ absolument sommantes. *Israel J. Math.*, 13:379–399 (1973), 1972.

[92] K.-I. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2013.

[93] H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1999.

[94] R. L. Schilling. *Measures, integrals and martingales.* New York: Cambridge University Press, 2005.

[95] L. Schwartz. Applications $p$-sommantes et $p$-radonifiantes. *Séminaire d'analyse fonctionnelle (dit "Maurey-Schwartz")*, 1972-73. Talk no 3.

[96] L. Schwartz. *Seminar Schwartz.* Department of Pure Mathematics, Department of Mathematics, Australian National University, Canberra, 1973. Notes on Pure Mathematics, No. 7 (1973).

[97] L. Schwartz. *Geometry and probability in Banach spaces*, volume 852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981. Based on notes taken by Paul R. Chernoff.

[98] E. Seneta. *Regularly varying functions.* Lecture Notes in Mathematics, Vol. 508. Springer-Verlag, Berlin-New York, 1976.

[99] B. Sz.-Nagy and C. Foiaş. *Harmonic analysis of operators on Hilbert space.* Translated from the French and revised. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970.

[100] R. Temam. *Navier-Stokes equations. Theory and numerical analysis*, volume 2 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[101] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan. *Probability distributions on Banach spaces*, volume 14 of *Mathematics and its Applications.* D. Reidel Publishing Co., Dordrecht, 1987.

[102] J. van Neerven, M. Veraar, and L. Weis. Stochastic integration in Banach spaces—a survey. In *Stochastic analysis: a series of lectures*, volume 68 of *Progr. Probab.*, pages 297–332. Birkhäuser/Springer, Basel, 2015.

[103] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007.

[104] M. Veraar and I. Yaroslavtsev. Cylindrical continuous martingales and stochastic integration in infinite dimensions. *Electron. J. Probab.*, 21:Paper No. 59, 53, 2016.

[105] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.

[106] D. Willett and J. S. W. Wong. On the discrete analogues of some generalizations of Gronwall's inequality. *Monatsh. Math.*, 69:362–367, 1965.

[107] J. Xiong and X. Yang. Existence and pathwise uniqueness to an SPDE driven by $\alpha$-stable colored noise. *Stochastic Process. Appl.*, 129(8):2681–2722, 2019.

[108] X. Yang and X. Zhou. Pathwise uniqueness for an SPDE with Hölder continuous coefficient driven by $\alpha$-stable noise. *Electron. J. Probab.*, 22:Paper No. 4, 48, 2017.