DOI:
10.1515/gmj-2022-2152

Document Version
Peer reviewed version

Citation for published version (APA):
Shargorodsky, E., Diening, L., & Karlovich, A. (2022). On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded. *Georgian Mathematical Journal*, *29*(3), 347-352. https://doi.org/10.1515/gmj-2022-2152

On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded

Lars Diening, Alexei Karlovich, and Eugene Shargorodsky

To Professor Stefan Samko on the occasion of his 80th birthday

ABSTRACT. We show that if the Hardy-Littewood maximal operator M is bounded on a reflexive variable exponent space $L^{p(\cdot)}(\mathbb{R}^d)$, then for every $q \in (1,\infty)$, the exponent $p(\cdot)$ admits, for all sufficiently small $\theta > 0$, the representation $1/p(x) = \theta/q + (1-\theta)/r(x)$, $x \in \mathbb{R}^d$ such that the operator M is bounded on the variable Lebesgue space $L^{r(\cdot)}(\mathbb{R}^d)$. This result can be applied for transferring properties like compactness of linear operators from standard Lebesgue spaces to variable Lebesgue spaces by using interpolation techniques.

1. Introduction

Let $L^0(\mathbb{R}^d)$ denote the space of all (equivalence classes of) Lebesgue measurable complex-valued functions on $\mathbb{R}^d$ with the topology of convergence in measure on sets of finite measure. Let $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R}^d)$ we denote the set of all functions $f \in L^0(\mathbb{R}^d)$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^d} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

It is easy to see that if $p(\cdot) = p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^d)$ is nothing but the standard Lebesgue space $L^p(\mathbb{R}^d)$. The space $L^{p(\cdot)}(\mathbb{R}^d)$ is referred to as a *variable Lebesgue space*.

Let $1 \le q < \infty$. Given $f \in L^q_{\rm loc}(\mathbb{R}^d)$, the q-th maximal operator is defined by

$$(M_q f)(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|}\int_Q |f(y)|^q dy\right)^{1/q},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ containing x (here, and throughout, cubes will be assumed to have their sides parallel to the coordinate axes). Note that $M := M_1$ is the usual Hardy-Littlewood maximal operator. By

1991 *Mathematics Subject Classification.* 46E30, 42B25.

$\mathcal{B}_M(\mathbb{R}^d)$ denote the set of all measurable a.e. finite functions $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$.

We will use the following standard notation:

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} p(x).$$

It is well known that the space $L^{p(\cdot)}(\mathbb{R}^d)$ is reflexive if and only if $1 < p_- \le p_+ < \infty$. In this case, its dual space is isomorphic to $L^{p'(\cdot)}(\mathbb{R}^d)$, where

$$1/p(x) + 1/p'(x) = 1, \quad x \in \mathbb{R}^d$$

(see, e.g., [**5**, Chap. 3]).

Suppose that $1 < p_- \le p_+ < \infty$ and there exist constants $c_0, c_\infty \in (0,\infty)$ and $p_\infty \in (1,\infty)$ such that

$$|p(x) - p(y)| \le \frac{c_0}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^d, \tag{1.1}$$

$$|p(x) - p_\infty| \le \frac{c_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^d. \tag{1.2}$$

Then $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ (see [**3**, Theorem 3.16] or [**5**, Theorem 4.3.8]). Following [**5**, Section 4.1] or [**3**, Section 2.1], we will say that $p(\cdot)$ is *globally log-Hölder continuous* if conditions (1.1)–(1.2) are satisfied. The class of all globally log-Hölder continuous exponents will be denoted by $\mathcal{P}^{\log}(\mathbb{R}^d)$.

Conditions (1.1) and (1.2) are optimal for the boundedness of M in the sense of modulus of continuity; the corresponding examples are contained in [**16**] and [**2**]. However, neither (1.1) nor (1.2) is necessary for $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$. Thus

$$\mathcal{P}^{\log}(\mathbb{R}^d) \subsetneqq \mathcal{B}_M(\mathbb{R}^d).$$

Here we mention results by Nekvinda [**14, 15**] and Lerner [**13**] and further discussion in the monographs [**3**, Chap. 4] and [**5**, Chaps. 4–5].

The following result was obtained in a somewhat more complete form by the first author (see [**4**, Theorem 8.1] or [**5**, Theorem 5.7.2]).

THEOREM 1.1. *Let $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ be a measurable function satisfying $1 < p_- \le p_+ < \infty$. The following statements are equivalent:*

(a) *M is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$;*
(b) *M is bounded on $L^{p'(\cdot)}(\mathbb{R}^d)$;*
(c) *there exists an $s \in (1/p_-, 1)$ such that M is bounded on $L^{sp(\cdot)}(\mathbb{R}^d)$;*
(d) *there exists a $q \in (1,\infty)$ such that M_q is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$.*

Rabinovich and Samko [**17**] (see also [**12**, Section 9.1.2]) observed that if a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$ satisfies $1 < p_- \le p_+ < \infty$, then it can be decomposed as

$$\frac{1}{p(x)} = \frac{\theta}{2} + \frac{1-\theta}{r(x)}, \quad x \in \mathbb{R}^d, \tag{1.3}$$

where $\theta \in (0,1)$ and $r(\cdot)$ satisfies $1 < r_- \le r_+ < \infty$ and belongs to $\mathcal{P}^{\log}(\mathbb{R}^d)$. This observation was important in the "transfer of the compactness techniques" from $L^2(\mathbb{R}^d)$ to $L^{p(\cdot)}(\mathbb{R}^d)$ by means of the one-sided interpolation of the compactness property between the spaces $L^{r(\cdot)}(\mathbb{R}^d)$ (where an operator is merely bounded) and $L^2(\mathbb{R}^d)$ (where an operator is compact).

The second author and Spitkovsky [**11**] exploited this idea for more general variable exponents $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ based on the following result obtained by the first author and published in [**11**, Theorem 4.1].

THEOREM 1.2. *Let* $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ *be a measurable function satisfying* $1 < p_- \le p_+ < \infty$. *If* $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$, *then there exist numbers* $q \in (1,\infty)$ *and* $\theta \in (0,1)$ *such that the variable exponent* $r(\cdot)$ *defined by*

$$\frac{1}{p(x)} = \frac{\theta}{q} + \frac{1-\theta}{r(x)}, \quad x \in \mathbb{R}^d, \tag{1.4}$$

belongs to $\mathcal{B}_M(\mathbb{R}^d)$.

The above theorem has the disadvantage that the constant q depends on the variable exponent $p(\cdot)$. It is desirable to avoid such a dependence and to find, for a given $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$, a $\theta \in (0,1)$ such that (1.3) holds and $r(\cdot)$ belongs to $\mathcal{B}_M(\mathbb{R}^d)$. This would allow one to simplify formulations of several results in the literature, where it was supposed that $p(\cdot)$ is of the form (1.3) with $r(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ and some sufficiently small $\theta \in (0,1)$ (see, e.g., [**7**, Corollary 2.1, Theorem 3.2], [**8**, Theorem 1.2], [**9**, Theorem 2.1]). The second author asked in [**10**, Section 4.4] whether for a given $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ satisfying $1 < p_- \le p_+ < \infty$, one can find a number $\tau_{p(\cdot)} \in (0,1]$ such that the variable exponent $r(\cdot)$ defined by (1.3) belongs to $\mathcal{B}_M(\mathbb{R}^d)$ *for every* $\theta \in (0,\tau_{p(\cdot)}]$.

Our main result is the following refinement of Theorem 1.2, which gives positive answers to the above questions.

THEOREM 1.3 (Main result). *Let* $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ *be a measurable function satisfying* $1 < p_- \le p_+ < \infty$. *Then* $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ *if and only if for every* $q \in (1,\infty)$, *there exists a number* $\Theta_{p(\cdot),q} \in (0,1)$ *such that for every* $\theta \in (0,\Theta_{p(\cdot),q}]$ *the variable exponent* $r(\cdot)$ *defined by* (1.4) *belongs to* $\mathcal{B}_M(\mathbb{R}^d)$.

Note that representation (1.4) implies that $0 < \theta/q \le 1/p(x) \le \theta/q + 1 - \theta < 1$ for $\theta > 0$, whence $1 < p_- \le p_+ < \infty$.

The paper is organized as follows. In Section 2, we formulate an interpolation lemma due to Cruz-Uribe [**1**], which immediately implies the proof of the sufficiency portion of Theorem 1.3. In Section 3, we show that if $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ satisfies $1 < p_- \le p_+ < \infty$, then the variable exponents $\left(\frac{1}{t}\left(\frac{p(\cdot)}{s}\right)'\right)'$ belong to $\mathcal{B}_M(\mathbb{R}^d)$ for all $s, t \ge 1$ sufficiently close to 1. Based on this result, we complete the proof of the necessity portion of Theorem 1.3 in Section 4.

2. Proof of the sufficiency portion of Theorem 1.3

The sufficiency portion is an immediate corollary of the following result obtained by Cruz-Uribe [**1**, Corollary 3] (see also [**6**, Corollary 2.5] for the case $1 < (p_j)_- \le (p_j)_+ < \infty$, $j = 0,1$) and the boundedness of the Hardy-Littlewood maximal operator M on the standard Lebesgue space $L^q(\mathbb{R}^d)$ with $q \in (1,\infty)$.

LEMMA 2.1. *If* $p_i(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ *for* $i = 0,1$, *then for every* $\theta \in (0,1)$, *the variable exponent* $p_\theta(\cdot)$ *defined by*

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_0(x)} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R}^d, \tag{2.1}$$

belongs to $\mathcal{B}_M(\mathbb{R}^d)$ *and*

$$\|M\|_{L^{p_\theta(\cdot)}\to L^{p_\theta(\cdot)}} \le 96\|M\|^{\theta}_{L^{p_0(\cdot)}\to L^{p_0(\cdot)}}\|M\|^{1-\theta}_{L^{p_1(\cdot)}\to L^{p_1(\cdot)}}. \tag{2.2}$$

Note that inequality (2.2) is stated in [**1**] with the constant 48, which seems to be a typo. This result was obtained as a consequence of the pointwise inequality $|T_f f| \le Mf \le 2T_f|f|$, where each T_f is a linear integral operator with a positive kernel. On the other hand, it was shown in [**1**, Theorem 1] that if T is a linear integral operator with a positive kernel that satisfies $\|Tf\|_{p_i(\cdot)} \le B_i\|f\|_{p_i(\cdot)}$ for $i = 0, 1$ and all $f \in L^{p_i(\cdot)}(\mathbb{R}^d)$ with B_i independent of f, then

$$\|Tf\|_{p_\theta(\cdot)} \le 48B_0^\theta B_1^{1-\theta}\|f\|_{p_\theta(\cdot)}.$$

3. Doubly iterated "left-openness and then duality" trick

Our construction is similar to that of the proof of [**11**, Theorem 4.1]. It is based on the consecutive application of the "left-openness" of the class $\mathcal{B}_M(\mathbb{R}^d)$ (see Theorem 1.1(c)) and then the "duality" of the class $\mathcal{B}_M(\mathbb{R}^d)$ (see Theorem 1.1(b)). In order to succeed, we repeat this procedure two times. The main novelty is that we can guarantee that the constructed exponents belong to $\mathcal{B}_M(\mathbb{R}^d)$ in certain ranges of parameters.

Lemma 3.1. *If* $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ *satisfies* $1 < p_- \le p_+ < \infty$, *then there exist* $s_0, t_0 \in (1,\infty)$ *such that*

$$\left(\frac{1}{t}\left(\frac{p(\cdot)}{s}\right)'\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad s \in [1, s_0], \quad t \in [1, t_0]. \tag{3.1}$$

Proof. By Theorem 1.1(c), there exists a number $s_0 \in (1,\infty)$ such that $p(\cdot)/s_0 \in \mathcal{B}_M(\mathbb{R}^d)$. Then it follows from Theorem 1.1(b) that $p'(\cdot)$ and $(p(\cdot)/s_0)'$ belong to $\mathcal{B}_M(\mathbb{R}^d)$. Applying Theorem 1.1(c) once again, we see that there exist $t_1, t_2 \in (1,\infty)$ such that

$$\frac{p'(\cdot)}{t_1} \in \mathcal{B}_M(\mathbb{R}^d), \quad \frac{1}{t_2}\left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d).$$

It follows from Lemma 2.1 (one can employ also a more elementary argument using Jensen's inequality as in [**6**, p. 43]) that

$$\frac{p'(\cdot)}{t} \in \mathcal{B}_M(\mathbb{R}^d) \text{ for all } t \in [1, t_1], \quad \frac{1}{t}\left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \text{ for all } t \in [1, t_2].$$

Put

$$t_0 := \min\{t_1, t_2\}.$$

Then it is clear that $t_0 \in (1,\infty)$ and

$$\frac{p'(\cdot)}{t} \in \mathcal{B}_M(\mathbb{R}^d), \quad \frac{1}{t}\left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad t \in [1, t_0]. \tag{3.2}$$

Take any $s \in [1, s_0]$ and set $\theta := \frac{s_0 - s}{s_0 - 1} \in [0,1]$. Then $s = \theta + (1-\theta)s_0$. Further, for $x \in \mathbb{R}^d$, we have

$$\left(\frac{1}{t}\left(\frac{p(x)}{s}\right)'\right)^{-1} = t\left(\frac{p(x)/s}{p(x)/s - 1}\right)^{-1} = t\frac{p(x) - s}{p(x)}$$

$$
\begin{aligned}
&= t\frac{p(x)(\theta+(1-\theta)) - (\theta+(1-\theta)s_0)}{p(x)} \\
&= t\theta\frac{p(x)-1}{p(x)} + t(1-\theta)\frac{p(x)-s_0}{p(x)} \\
&= \theta\left(\frac{p'(x)}{t}\right)^{-1} + (1-\theta)\left(\frac{1}{t}\left(\frac{p(x)}{s_0}\right)'\right)^{-1}.
\end{aligned} \tag{3.3}
$$

It follows from (3.2)–(3.3) and Lemma 2.1 that

$$
\frac{1}{t}\left(\frac{p(\cdot)}{s}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad s\in[1,s_0], \quad t\in[1,t_0].
$$

Applying Theorem 1.1(b) one more time, one arrives at (3.1). $\square$

4. Proof of the necessity portion of Theorem 1.3

Suppose that $q \in (1,\infty)$ and $p(\cdot)$ satisfies $1 < p_- \le p_+ < \infty$ and belongs to $\mathcal{B}_M(\mathbb{R}^d)$. We need to prove that $r(\cdot)$ defined by (1.4) belongs to $\mathcal{B}_M(\mathbb{R}^d)$ for all sufficiently small positive values of θ. We will show that one can choose s and t in such a way that $r(\cdot) = (\frac{1}{t}(\frac{p(\cdot)}{s})')'$, and use Lemma 3.1 to conclude that $r(\cdot)\in\mathcal{B}_M(\mathbb{R}^d)$.

Now, (1.4) is equivalent to

$$
\frac{1}{r(x)} = \frac{1}{1-\theta}\frac{1}{p(x)} - \frac{\theta}{1-\theta}\frac{1}{q}, \quad x\in\mathbb{R}^d,
$$

while

$$
\frac{1}{(\frac{1}{t}(\frac{p(x)}{s})')'} = 1 - t\left(1-\frac{s}{p(x)}\right) = st\frac{1}{p(x)} - (t-1), \quad x\in\mathbb{R}^d.
$$

So, we need to take s and t such that

$$
st = \frac{1}{1-\theta} \qquad \text{and} \qquad t-1 = \frac{\theta}{1-\theta}\frac{1}{q}.
$$

An easy calculation shows that these equations are equivalent to

$$
\theta = 1-\frac{1}{st} \qquad \text{and} \qquad t = \frac{q-1}{q-s}.
$$

Let $s_0\in(1,\infty)$ and $t_0\in(1,\infty)$ be such that (3.1) holds. Put

$$
t(s) := \frac{q-1}{q-s}, \quad 1<s<q. \tag{4.1}
$$

Since $1 < t(s) \to 1$ as $s\to 1$, there exists $s_1\in(1,s_0]$ such that

$$
1 < t(s) \le t_0 \quad \text{for all} \quad s\in(1,s_1],
$$

Let

$$
\theta(s) := 1-\frac{1}{st(s)} = 1-\frac{q-s}{s(q-1)} = \frac{q(s-1)}{s(q-1)} = \frac{q}{q-1}\left(1-\frac{1}{s}\right). \tag{4.2}
$$

Then $0<\theta(s)\to 0$ as $s\to 1$. So, there exists $s_2\in(1,s_1]$ such that

$$
\Theta_{p(\cdot),q} := \theta(s_2)\in(0,1).
$$

It is clear from (4.2) that $\theta(\cdot)$ is an increasing continuous function. Then

$$
\theta((1,s_2]) = (0,\Theta_{p(\cdot),q}].
$$

Take any $\theta \in (0, \Theta_{p(\cdot),q}]$. It follows from the above that there exists a unique $s \in (1, s_2]$ such that $\theta(s) = \theta$. For this s and $t := t(s)$,

$$r_\theta(\cdot) := \left(\frac{1}{t} \left(\frac{p(\cdot)}{s} \right)' \right)' \in \mathcal{B}_M(\mathbb{R}^d) \tag{4.3}$$

according to (3.1).

It follows from (4.1) that

$$q = \frac{st - 1}{t - 1}. \tag{4.4}$$

Combining (4.2), (4.3), (4.4), we get for $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\theta}{q} + \frac{1-\theta}{r_\theta(x)} &= \frac{\theta}{q} + (1-\theta) \left(1 - \frac{1}{\frac{1}{t}\left(\frac{p(x)}{s}\right)'} \right) \\ &= \left(1 - \frac{1}{st}\right) \frac{t-1}{st-1} + \frac{1}{st} \left(1 - t \left(1 - \frac{s}{p(x)} \right) \right) \\ &= \frac{t-1}{st} + \frac{1}{st}(1-t) + \frac{1}{st} \cdot \frac{ts}{p(x)} \\ &= \frac{1}{p(x)}. \end{aligned}$$

Thus $r_\theta(\cdot)$ satisfies (1.4) for every $\theta \in (0, \Theta_{p(\cdot),q}]$. □

Acknowledgments. This work was supported by national funds through the FCT Fundação para a Ciência e a Tecnologia, I.P. (Portuguese Foundation for Science and Technology) within the scope of the project UIDB/00297/2020 (Centro de Matemática e Aplicaes). Lars Diening was also funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) SFB 1283/2 2021 317210226.

References

[1] D. Cruz-Uribe. Interpolation of positive operators on variable Lebesgue spaces. *Math. Inequal. Appl.*, 15(3):639–644, 2012.

[2] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 28(1):223–238, 2003.

[3] D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. Foundations and harmonic analysis.

[4] L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.

[5] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[6] L. Diening, P. Hästö, and A. Nekvinda. Open problems in variable Lebesgue and Sobolev spaces. In *FSDONA04 Proceedings*, pages 38–58. Czech Acad. Sci., Milovy, Czech Republic, 2004.

[7] A. Fiorenza, A. Gogatishvili, and T. Kopaliani. Estimates for imaginary powers of the Laplace operator in variable Lebesgue spaces and applications. *Izv. Nats. Akad. Nauk Armenii Mat.*, 49(5):11–22, 2014.

[8] A. Gogatishvili and T. Kopaliani. On the Rubio de Francia's theorem in variable Lebesgue spaces. *Bull. TICMI*, 18(1):3–10, 2014.

[9] A. Gogatishvili and T. Kopaliani. Maximal multiplier operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces. *Bull. Sci. Math.*, 140(4):86–97, 2016.

[10] A. Y. Karlovich. Algebras of continuous Fourier multipliers on variable Lebesgue spaces. *Mediterr. J. Math.*, 17(4):Paper No. 102, 19, 2020.
[11] A. Y. Karlovich and I. M. Spitkovsky. Pseudodifferential operators on variable Lebesgue spaces. In *Operator theory, pseudo-differential equations, and mathematical physics*, volume 228 of *Oper. Theory Adv. Appl.*, pages 173–183. Birkhäuser/Springer Basel AG, Basel, 2013.
[12] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko. *Integral operators in non-standard function spaces. Vol. 1*, volume 248 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, [Cham], 2016. Variable exponent Lebesgue and amalgam spaces.
[13] A. K. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Z.*, 251(3):509–521, 2005.
[14] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.*, 7(2):255–265, 2004.
[15] A. Nekvinda. Maximal operator on variable Lebesgue spaces for almost monotone radial exponent. *J. Math. Anal. Appl.*, 337(2):1345–1365, 2008.
[16] L. Pick and M. Růžička. An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. *Expo. Math.*, 19(4):369–371, 2001.
[17] V. Rabinovich and S. Samko. Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces. *Integral Equations Operator Theory*, 60(4):507–537, 2008.

Universität Bielefeld, Fakultät für Mathematik, Postfach 10 01 31, D-33501 Bielefeld, Germany
E-mail address: lars.diening@uni-bielefeld.de

Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal
E-mail address: oyk@fct.unl.pt

Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom and Technische Universität Dresden, Fakultät Mathematik, 01062 Dresden, Germany
E-mail address: eugene.shargorodsky@kcl.ac.uk