DOI:
10.1016/j.mechmachtheory.2022.104722

Mobility of Overconstrained Parallel Mechanisms with Reconfigurable End-Effectors

Patrice Lambert[1], Lyndon da Cruz[2], Christos Bergeles[1]

[1] *School of Biomedical Engineering & Imaging Sciences, King's College London*

[2] *Wellcome/EPSRC Centre on Interventional and Surgical Sciences, University College London*

Corresponding author: Patrice Lambert, e-mail:patrice.lambert@kcl.ac.uk *This work was supported by an ERC Starting Grant [714562] and the Wellcome/EPSRC Centre for Medical Engineering [WT 203148/Z/16/Z].*

Abstract

Reconfigurable parallel mechanisms have legs or end-effectors with additional degrees of freedom or with the ability to change their mobility. This allows them to reconfigure into a variety of conventional parallel mechanisms. One type of reconfigurable parallel mechanism can be obtained by replacing the end-effector with a closed-loop linkage. The parts of this linkage that are connected to the legs may be considered as the end-effectors, having their own mobility relative to the base and a coupled mobility with each other. Classical methods for mobility analysis cannot be directly applied to this type of mechanism. This article presents a general method to calculate and represent the coupled mobility of the multiple end-effectors using nullspace operations on a matrix of mobility. The nullspace approach provides an efficient way to calculate the mobility in a closed-loop chain while the matrix of mobility allows for calculating the multiple end-effectors mobility interaction. The general method can be performed analytically to obtain insights on the mechanism in any configuration, or algorithmically to obtain numerical values of the mobility in a particular configuration. Two illustrative examples showcasing the method are presented.

Keywords: Parallel mechanisms, reconfigurable mechanisms, mobility analysis.

1. Introduction

Parallel mechanisms are made of a base link and an end-effector link connected in parallel by several kinematic chains. Their closed-loop architectures form the backbone structure of parallel robots and allow for a full control of the mechanism while only a subset of the mechanism joints is actuated. Due to their combination of low inertia and high stiffness, they are widely used in several applications, ranging from flight simulators to high-speed pick-and-place and haptic devices.

Parallel mechanisms with reconfigurable end-effectors are a different type of structure in which the end-effector is replaced by a closed-loop linkage. This linkage can be made of one or multiple closed-loops and some segments in this linkage are connected to the base by parallel kinematic chains that are called 'legs'. For a single closed-loop reconfigurable end-effector, the end-effector linkage is usually called a 'platform'. One advantage of parallel mechanisms with reconfigurable end-effectors is that the internal mobility of the end-effector allows for interaction with the environment from multiple contact points.

To the best of our knowledge, the first manipulator that was based on this structure is [23], which presented a planar version that used the reconfigurable end-effector as a gripper. Another early application was the modification of a Delta robot with a reconfigurable end-effector to provide rotational capability in [18] and later in [19]. The concept of parallel robots with reconfigurable end-effectors also has found applications in haptics [14] as a means to provide grasping capabilities. The concept has also been extended to more than two end-effectors in [5, 4]. A first attempt at the generalisation of the concept can be found in [17], where generic forms of parallel mechanisms with configurable platforms are presented. The article also included a method to calculate the Jacobian of those manipulators given that the mobility of the mechanism is already known and the mechanism is not overconstrained. Jacobian analysis was also described further in [13] along with statics, while some singularity analyses were presented in [15]. Recently, synthesis of some classes of parallel mechanisms with configurable platform has been investigated in [10, 21].

The kinematic methods presented above assume that the mobility of the mechanism is already known. Mobility analysis of parallel mechanisms usually refers to the instantaneous mobility of the end-effector at a given pose. When numerical values are used to describe the pose of the mechanism, a numerical matrix representing the screw system of mobility is obtained for this pose. If analytical vectors are used to describe the general pose of the mechanism, an analytical screw system of the mobility can be obtained for the set of all the poses of the mechanism. Various analytical methods have been proposed to calculate the mobility of conventional parallel mechanisms using constraint screw theory [9, 2, 6, 1, 22, 7]. Those different methods successfully take into account overconstraints in mechanisms, i.e constraints that create self-stresses when a kinematic loop is closed, to describe the mobility for a general pose. The methods generally use the sum of the constraint systems of all legs and then calculate its reciprocal system to obtain the mobility.

However, those methods cannot be directly applied to the mobility analysis of parallel mechanisms with reconfigurable end-effectors because of the coupled mobility between each end-effectors. This kinematic coupling has been recently addressed in [8], where a method was proposed to obtain the mobility of a particular link on a reconfigurable end-effector made of a single closed-loop linkage. The method uses the sum of the constraint systems of all legs through all paths on the reconfigurable platform that can be formed from the base to the platform effector. The method can therefore give a single screw system describing the mobility for each link on the platform relative to the base, but does not address how those mobilities are coupled nor the total number of Degrees of Freedom (DOF) of the mechanism.

In this article, we present a method to calculate the mobility of any of the end-effectors in a general reconfigurable end-effector relative to the base and its coupling to the mobility of the other end-effectors. The method is based on the representation of the coupled mobility via a matrix of mobility, which combines the screw systems of all legs into a single vector space, to which a nullspace approach to the internal mobility of closed-loops is applied successively for each independent closed-loop created by the reconfigurable end-effector until all DOF of the mechanism and their coupling are revealed. First, we will present in the next section an overview of the mobility analysis rules for various conventional mechanisms based on their topology. We will then explain why those rules are not suitable for parallel mechanisms with reconfigurable end-effectors and present the fundamentals of our method based on nullspace operations on a matrix of mobility. We will finally present two illustrative examples in which this method can be applied.

2. Instantaneous Mobility and Overconstraints in General Mechanisms

This section provides a unified framework for the calculation of the mobility and constraints in a general overconstrained mechanism based on Graph Theory and Screw Theory. The results presented in this section may be considered general knowledge but the framework in which they are presented is original and provides the necessary background for the rest of the article.

A general mechanism comprises a set of rigid links connected by a set of joints. The two main types of 1-DOF joints are revolute joints and prismatic joints. Multi-DOF joints, such as cylinder joints and spherical joints are also used, but since they can be represented as a series of 1-DOF joints, one can consider only 1-DOF joints without loss of generality. The topology of a mechanism describes which links are connected to each other, regardless of the type of joint or its geometrical position on the link. While each link may possess several joints, each joint in a mechanism strictly connects only two links. Since Graph Theory uses graphs to model pairwise relations between a set of objects, it is an appropriate tool to study the topology of mechanisms [16]. Using this representation, each vertex in the graph can be associated with a link while each edge is associated with a joint. 1 shows the topology graphs using the Graph Theory representation for a serial and a parallel mechanism, as well as the graphs for the well-known Delta mechanism and a hybrid robot made of a parallel mechanism connected to a serial chain.

Each joint imposes relative constraints between two links. The mobility and constraints imposed by a joint can be described by its twist space T, and its reciprocal wrench space W, respectively. Both are

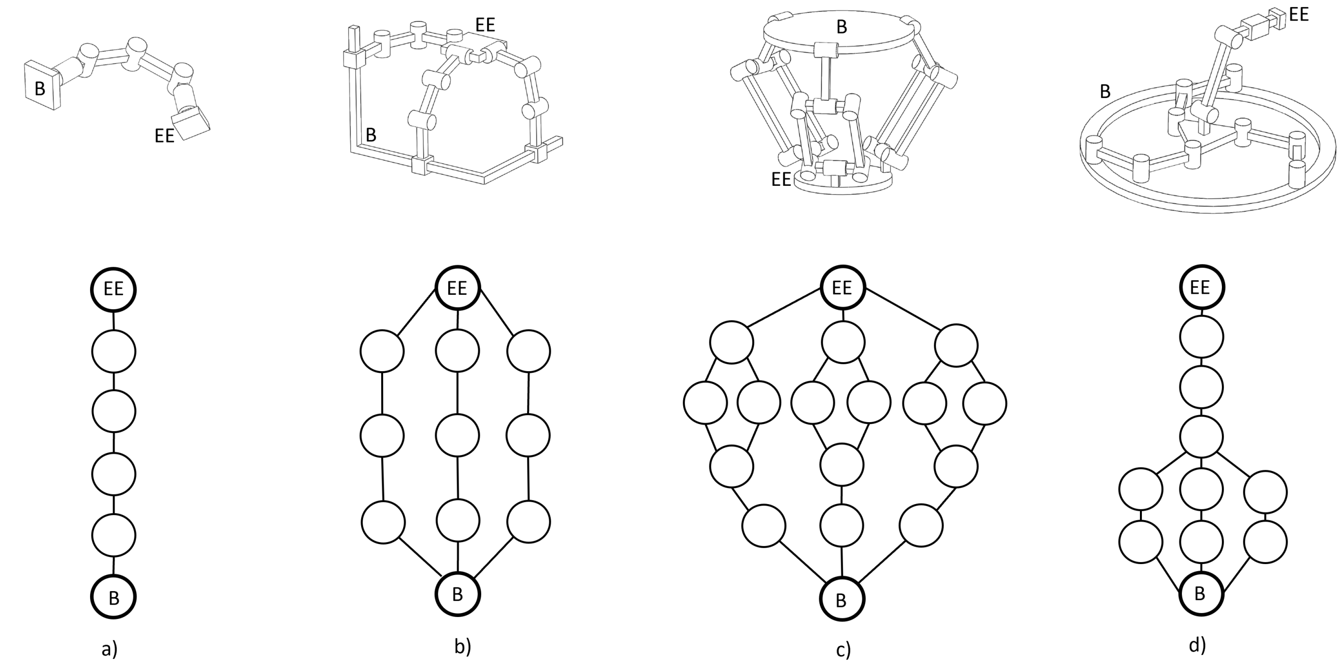

Figure 1: Examples of topological graphs using the Graph Theory representation for several types of mechanisms. The base (B) and the end-effector (EE) are connected via a set of links (vertices) and joints (edges). a) A 5 DOF serial arm. b) A 3 DOF three-legged parallel robot. c) A 3 DOF Delta robot. d) A 6 DOF hybrid robot made of a parallel mechanism connected to a serial chain.

expressed as screw systems. Starting from a topological graph, as those presented in 1, one can find the mobility and constraint systems between the end-effector (EE) and the base (B) using the rules of serial and parallel reductions. We will also explain later how those graph reduction rules can also be used to reveal the local mobilities and overconstraints in the mechanism.

In a serial reduction, a vertex (link) and its two connected edges (joints) are replaced by a single new edge. The mobility of the resulting edge corresponds to the sum of the mobility of the original two edges, $T = T_1 + T_2$, and its constraints correspond to the intersection of the constraints $W = W_1 \cap W_2 = T^r$. In a parallel reduction, two edges connecting the same vertices are replaced by a single new edge. The mobility of the resulting edge is obtained by a dual set of rules and corresponds to the intersection of the mobility of the original two edges, $T = T_1 \cap T_2$, and its constraints correspond to the sum of the constraints $W = W_1 + W_2 = T^r$. Serial and parallel reduction rules are summarised in 2. The twist space and the wrench space attached to an edge are reciprocal to each other such that $T = W^r$ and $W = T^r$. Since the sum of screw systems and the calculation of their reciprocals is easier to perform than the calculation of their intersection, it is generally advisable to calculate screw systems intersection as the reciprocal of the sum of the reciprocals of the original systems, such as

$$T_1 \cap T_2 = (T_1^r + T_2^r)^r \qquad\qquad W_1 \cap W_2 = (W_1^r + W_2^r)^r \,. \tag{1}$$

In each of the examples of 1, serial and parallel reductions can be successively used until only one edge is left connecting the base and the end-effector vertices. The twist space of this edge corresponds to the mobility of the end-effector relative to the base. For example, in a conventional parallel mechanism with p legs and q joints per leg, where T_{ij} is the twist space of joint j in leg i, the mobility T of the end-effector is calculated as

$$T = \left(\sum_{i=1}^{p} \left(\sum_{j=1}^{q} (T_{ij}) \right)^r \right)^r \tag{2}$$

Two other dual concepts, which do not influence the final end-effector mobility, can also be revealed by performing those graph reductions. The first one concerns the intersection of twists in a serial reduction.

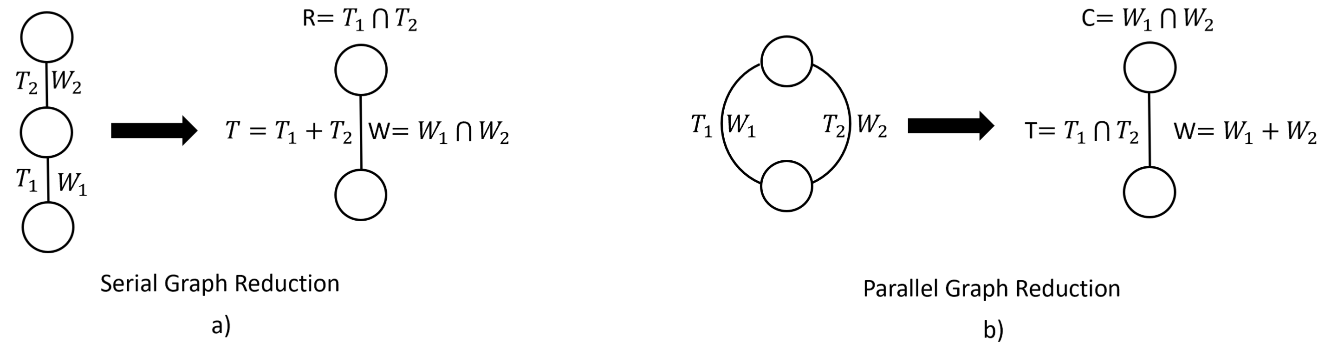

Figure 2: Graph reductions that can be used successively to determine the final twist T and its reciprocal wrench W between the base and the end-effector in a general mechanism. Each edge represents a joint for which a twist space T and a wrench space W are associated (a) a serial reduction (b) a parallel reduction.

If T_1 and T_2 in a serial reduction are not fully independent, i.e. $T_1 \cap T_2 \neq \{\mathbf{0}\}$, a local mobility system $R = T_1 \cap T_2$, i.e. a mobility that is independent of the motion of the two end vertices, exists. In this case, some mobilities in the serial chain are possible even when the two ends of the chain do not move relative to each other. This also means that the resulting DOF in this serial reduction will be smaller than the sum of the DOF of each edge such that $dim\,(T_1 + T_2) = dim\,(T_1) + dim\,(T_2) - dim\,(R)$. This local mobility in serial robots is usually called 'redundancy' and corresponds to the mobility left when the end-effector is motionless relative to the base. This also means that those mobility correspond to the mobility inside the single closed-loop formed when the effector is rigidly attached to the base. Each serial reduction may create its own distinct local mobility screw system R_i. The total number of redundant DOF for the local mobilities for a general serial open-loop chain with n joints, or equivalently the total mobility of a single closed-loop chain with n joints, is given by

$$\sum_{i=2}^{n} dim\,(R_i) = \sum_{i=2}^{n} dim \left(T_i \cap \sum_{j=1}^{i-1} T_j \right) , \tag{3}$$

with $R_i = T_i \cap \sum_{j=1}^{i-1} T_j$. Since serial reductions can be performed in any arbitrary order, the order in which the joints are labelled for $j = 1...n$ is unimportant and will lead to different but equivalent results. Equation 3 is particularly useful as it allows to calculate the number of DOF in a single closed-loop. A dual concept concerning overconstraints also occurs during parallel reductions. In this case, constraints that are applied by both parallel edges are revealed as $C = W_1 \cap W_2$. In a dual way to the local mobilities in a serial chain that does not create any movement on the end-effector, those constraints produce self-stresses in a closed-loop chain that do not create any resulting forces on the links. Thus, similarly, for n edges in parallel, the total number of overconstraints is given as

$$\sum_{i=2}^{n} dim\,(C_i) = \sum_{i=2}^{n} dim \left(W_i \cap \sum_{j=1}^{i-1} W_j \right) . \tag{4}$$

Using those notations, a modified Chebyshev–Grübler–Kutzbach mobility criterion for any general mechanism with m links and n f_i-DOF joints, including the mechanisms presented in 1, corresponds to

$$dim\,(T) + \sum dim\,(R) = 6\,(m - 1 - n) + \sum_{i=1}^{n} f_i + \sum dim\,(C) , \tag{5}$$

where the left part of the equation is the sum of the mobility of the end-effector and all local mobilities. On the right side of the equation, each of the $(n + 1 - m)$ mechanical closed-loop removes 6 DOF from the total DOF of the joints, and each overconstraint adds a DOF since those constraints are redundant. For a planar

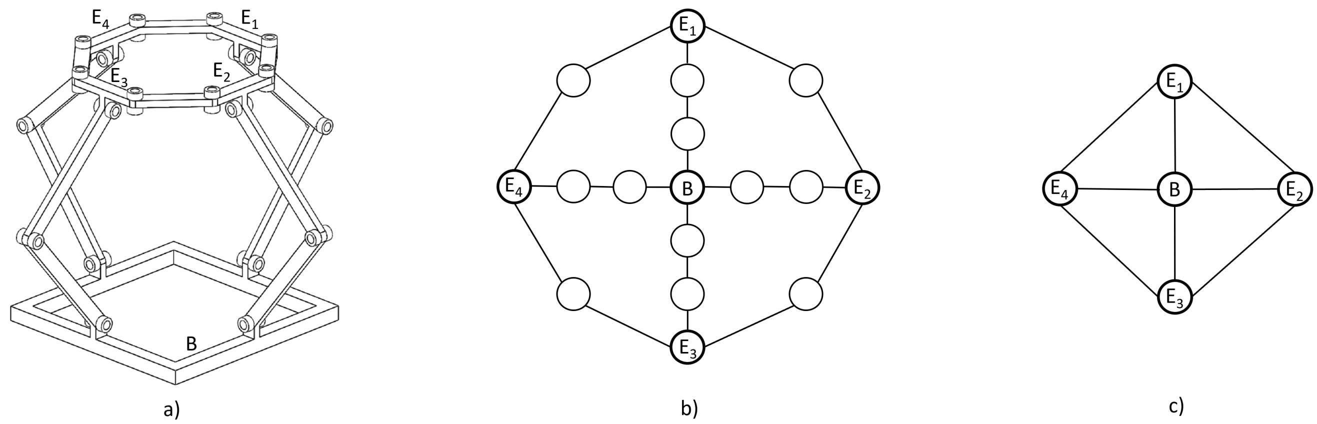

Figure 3: A parallel mechanism with a reconfigurable platform. Four legs are connected to the base (B) and to a rigid link (E_i) of the platform. The platform is made of eight revolute joints in a single closed-loop. (a) Schematic representation of the mechanism (b) The topology graph of the mechanism in which each vertex represents a link and each edge represents a joint. (c) The topology graph after graph reductions.

mechanism, either removing 3 DOF per closed-loop or removing 6 DOF per closed-loop but considering 3 planar overconstraints for each will lead to the same result. While 5 is valid for any mechanism, it does not provide the twist space of the motion, and the dimensions of $\sum dim\,(R)$ and $\sum dim\,(C)$ depend on the geometrical arrangement of the joints. It can be used however to validate the total number of DOF obtained from the graph reduction framework, for example when calculating the twist space of a parallel mechanism using 2.

In the next section, we will see how this general framework cannot be directly applied to parallel mechanisms with reconfigurable end-effectors.

3. Coupled Mobility in Parallel Mechanisms with Reconfigurable End-Effectors

In this section, we show the limitations of the general framework presented in the previous section when applied to parallel mechanisms with reconfigurable end-effectors. We then propose a solution to this problem based on an extension of the nullspace approach for single closed-loop mechanisms when applied to the concept of a matrix of mobility. The resulting method can be applied to any parallel mechanisms with reconfigurable end-effectors, including reconfigurable platforms and constitutes a direct method i.e it does not involve any numerical analysis and therefore symbolic manipulations can be used. This can lead to analytical results that are valid for any configuration of the mechanism when using a vector notation as input. Also, the method does not require any a priori insight on the mechanism and can therefore be automated directly into an algorithm to produce numerical results for a particular configuration when numerical values for the screw systems of each leg and end-effector segments are used as inputs.

3.1. Limitations of the General Mobility Analysis Method for Parallel Mechanisms with Reconfigurable End-Effectors

The graph reduction method described in the previous section covers all mechanisms that are represented by a series-parallel graph. However, the serial and parallel graph reductions, by definition, can not be used to reduce a non series-parallel graph to a single mobility system T and its reciprocal constraint system W between the base and the end-effector. Figure 3 shows an example of a parallel mechanism with a reconfigurable platform and its associated graph. The mechanism has a base link (B) and four identical legs L_1 to L_4 made of three revolute joints with parallel axes. The reconfigurable platform is made of a single planar closed-loop linkage comprising 8 revolute joints with parallel axes. Each leg is connected to a link of the platform separated by two platform joints to the adjacent legs. This mechanism will be explored further in Section 4 as the first example for illustrating the general method.

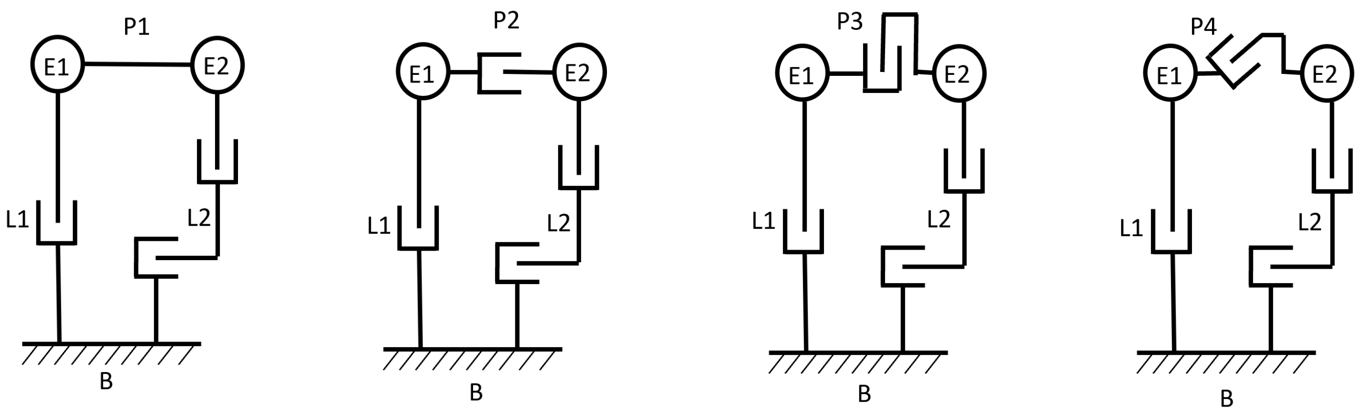

Figure 4: Four single closed-loops formed by two legs L_1 and L_2 connected in parallel with a rigid platform P_1 and reconfigurable platforms P_2, P_3, and P_4

We can assign a vertex to each link and an edge to each joint to create the topology graph of the mechanism, as shown in Figure 3 b). When using the serial and parallel reductions on which classical methods for the calculation of mobility are based, we can reduce any series-parallel mechanism to a single edge and its corresponding twist space. However, due to the presence of the reconfigurable platform, it is not possible in this case to reduce the graph to a single edge between the base and a single effector using the serial and parallel reductions presented in the previous section. The final graph of parallel mechanisms with reconfigurable platforms after graph reductions is called a wheel graph, as shown in Figure 3 c). The centre of the wheel represents the base, the spokes of the wheel are the legs of the mechanism, and the outer rim represents the reconfigurable platform. Beside the base vertex, the only remaining vertices are the links that are part of both the platform and the legs. Those links are considered the multiple end-effectors of the mechanisms because they cannot be simplified with serial and parallel reductions.

We now see that the mobilities of a mechanism with a reconfigurable platform cannot be described using a single screw system. Indeed, each end-effector has a specific mobility relative to the base, and this mobility may be coupled with the mobility of the remaining end-effectors. To address this problem, a method to calculate the mobility of each end-effector relative to the base was recently presented in [8]. This method is based on the sum of the constraints of all legs for all possible paths on the reconfigurable platform to reach one of the end-effectors. Although the reciprocal system of the resulting wrench gives the proper mobility for a specific end-effector, the coupled mobility between the multiple end-effectors, and therefore the total number of DOF, is not addressed by this method. In this article, we combine a modification of a recent method developed in [20] using the twist space intersection approach for the calculation of the instantaneous mobility of conventional parallel mechanisms with the concept of the matrix of mobility to solve the problem of the coupled mobility between the end-effectors.

3.2. Application of the Nullspace Approach to a Single-Loop Chain

To introduce the method of coupled mobility, let us first consider a simple case of two legs in 2 dimensions. 4 shows four examples of a system of two legs, L_1 and L_2, each connecting a base B to end-effectors E_1 and E_2. L_1 is composed of a single vertical translation while L_2 allows for both vertical and horizontal translations. The screw systems of L_1 and L_2 are denoted $\mathbb{S}_{L1}$ and $\mathbb{S}_{L2}$ respectively. Before the connection of the platform segment P, the total number of DOF of the mechanism is $dim\left(\mathbb{S}_{L1}\right) + dim\left(\mathbb{S}_{L2}\right) = 3$. When a platform P_i is connected between the legs, a single closed-loop is formed. If it is a rigid platform, as with P_1, the mobilities of E_1 and E_2 relative to the base become identical and are given by $T = \mathbb{S}_{L1} \cap \mathbb{S}_{L2}$, which is a normal parallel reduction and corresponds to the vertical translation in this case.

Let us now consider the platform P_2, which has a joint allowing for horizontal translation. Unlike P_1, this platform allows relative motion between E_1 and E_2. One can see that E_2 can move horizontally and vertically. However, the horizontal movement occurs without affecting E_1, while the vertical displacement is coupled. Even though all the joints are allowed to move, the total number of DOF for the system is 2 because of this coupled mobility. When considering platform P_3, we can notice that the platform constraints the horizontal displacement of E_2 but allow for E_1 and E_2 to move vertically independently and each end-effector counts for 1 DOF each, even if their DOF are in the same direction. Therefore, this mechanism also has 2 DOF. Lastly, if platform P_4 is connected, both joints in L_2 conserve their mobility, but their velocities become coupled by a ratio that depends on the angle of joint P_4. This last mechanism also has 2 DOF in total when we include the vertical translation.

For each of those examples, the total mobilities can be described using two distinct screw systems. The first one is the common mobility between the legs and is given by $R_1 = \mathbb{S}_{L1} \cap \mathbb{S}_{L2}$. The second one represents the intersection of the mobility of the platform with the relative mobility between the legs and is given by $R_2 = \mathbb{S}_P \cap (\mathbb{S}_{L1} + \mathbb{S}_{L2})$. This is in accordance with the mobility of a single closed-loop described in (3) by the screw systems R_i.

As mentioned in the previous section and illustrated by (1), the intersection of screw systems is generally performed by calculating the reciprocal of the sum of the reciprocals of the screw systems to be intersected. However, in this article, we will use the nullspace-associated approach recently presented in [20] to calculate the intersections. We will show later how an extension of this approach can be applied to the concept of the matrix of mobility.

So far we have discussed twist spaces in the general concept of vector spaces. In practice, notably for vector space operations in an algorithm, vector spaces must be expressed in the form of a vector basis for which the vector space comprises all vectors that can be obtained from all linear combinations of the vector basis. By definition, this basis is not unique and there is an infinity of valid screw bases that can be used to define a single screw system. For the rest of this article, we will make the distinction between a screw system $\mathbb{S}$, and its basis expressed in a matrix form S, where each column in S represents a basis vector of the system $\mathbb{S}$, noted $\mathbb{S} = \{S\}$. It is an important distinction as operators such as the sum and intersection have different meanings when applied to vector spaces or matrices. In addition, since a unique screw system $\mathbb{S}_1$ has an infinite number of valid bases S_i, the sign "$\equiv$" will be used to show the equivalence between two matrix bases with different elements representing the same screw system, such as in $\mathbb{S} = \{S_1\} = \{S_2\}$, with $S_1 \equiv S_2$.

In the original nullspace approach, the screw system $\mathbb{S}^{int}$ representing the intersection of two screw systems $\mathbb{S}_1 = \{S_1\}$ and $\mathbb{S}_2 = \{S_2\}$ is calculated as

$$\mathbb{S}^{int} = \mathbb{S}_1 \cap \mathbb{S}_2 = \{S_1 K_1\} = \{S_2 K_2\} \qquad \text{where} \qquad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} S_1 & -S_2 \end{bmatrix}\right), \tag{6}$$

with K_i having the same number of rows as the number of columns in S_i. This can be understood as the nullspace of $\begin{bmatrix} S_1 & -S_2 \end{bmatrix}$ represents the allowed velocity of each joint when $\mathbb{S}_1$ and $\mathbb{S}_2$ form a closed-loop, i.e. are connected in parallel. The allowed velocity of either chain, therefore, corresponds to their intersection, as is the case with a parallel reduction. The nullspace matrix K also represents a basis for its own vector space and any linear operation on its columns will produce a valid basis. The approach presented in [20] was also extended to the intersection of three or more kinematic chains for calculating the mobility of classical parallel mechanisms. In our case, we extend this approach to calculate the multiple screw systems created in a single closed-loop. Since K in Equation 6 represents the allowed velocity of each screw basis in a closed-loop, the allowed screw velocity in a loop formed by three chains with mobility $\mathbb{S}_{L1}$, $\mathbb{S}_{L2}$, and $\mathbb{S}_P$ as in Figure 4 is

$$K' = \begin{bmatrix} K_{L1} \\ K_{L2} \\ K_P \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} S_{L1} & -S_{L2} & S_P \end{bmatrix}\right), \tag{7}$$

assuming that the velocity for the platform segment P_i is for E_2 relative to E_1. The total number of DOF in a single loop corresponds to the number of columns in K'. While it was possible to express the mobility in (6) with a single screw system, we now need two distinct screw systems when connecting three chains. To obtain a basis for those screw systems, we need to separate the matrix K' into two sets of columns. The first set of columns, in which the elements of K_p are zero, represents the common mobility between legs, while the second set of columns represents the relative mobility. The general method for the common and relative mobilities of two end-effectors in a single closed-loop made of chains with mobilities $\mathbb{S}_{L1}$, $\mathbb{S}_{L2}$, and $\mathbb{S}_P$ is

$$\begin{aligned} \mathbb{S}_{L1} \cap \mathbb{S}_{L2} &= \{S_{L1}K_{L11}\} = \{S_{L2}K_{L21}\} \\ \mathbb{S}_{Pi} \cap (\mathbb{S}_{L1} + \mathbb{S}_{L2}) &= \{S_P K_{P2}\} \end{aligned} \qquad \text{where} \quad K' = \begin{bmatrix} K_{L11} & K_{L12} \\ K_{L21} & K_{L22} \\ \mathbf{0} & K_{P2} \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} S_{L1} & -S_{L2} & S_P \end{bmatrix}\right). \quad (8)$$

Applying Equations (8) to the four examples of Figure 4, we obtain the following:

$$S_{L1} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \quad S_{L2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{x}} & \hat{\mathbf{y}} \end{bmatrix} \qquad S_{P1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad S_{P2} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} \quad S_{P3} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \quad S_{P4} = \begin{bmatrix} \mathbf{0} \\ \lambda_1\hat{\mathbf{x}} + \lambda_2\hat{\mathbf{y}} \end{bmatrix}$$

$$K'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad K'_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad K'_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad K'_4 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_1 \\ 1 & \lambda_2 \\ 0 & 1 \end{bmatrix} \quad , \qquad (9)$$

$$\mathbb{S}_{L1} \cap \mathbb{S}_{L2} = \left\{\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix}\right\} \qquad \begin{aligned} \mathbb{S}_{P1} \cap (\mathbb{S}_{L1} + \mathbb{S}_{L2}) &= \left\{\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}\right\} & \mathbb{S}_{P2} \cap (\mathbb{S}_{L1} + \mathbb{S}_{L2}) &= \left\{\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix}\right\} \\ \mathbb{S}_{P3} \cap (\mathbb{S}_{L1} + \mathbb{S}_{L3}) &= \left\{\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix}\right\} & \mathbb{S}_{P4} \cap (\mathbb{S}_{L1} + \mathbb{S}_{L3}) &= \left\{\begin{bmatrix} \mathbf{0} \\ \lambda_1\hat{\mathbf{x}} + \lambda_2\hat{\mathbf{y}} \end{bmatrix}\right\} \end{aligned}$$

where $\hat{\mathbf{x}}$ corresponds to a unit vector in the horizontal direction, $\hat{\mathbf{y}}$ corresponds to a unit vector in the vertical direction, and the values of λ_1 and λ_2 depend on the angle of the prismatic joint in P_4. K'_i is the nullspace obtained with platform Pi. We have shown how the nullspace approach for the calculation of the intersection of screw systems can be extended to a single closed-loop. For example, Eq. 8 can be applied to any single closed-loop made of 3 serial chains that do not themself contain local mobilities to find all the mobilities inside the loop. Generalising this equation to a single loop formed by n serial chains, we obtain the $n-1$ screw systems describing all the DOF in a general single loop.

$$\begin{aligned} \mathbb{S}_2 \cap \mathbb{S}_1 &= \{S_2K_{21}\} \\ \mathbb{S}_3 \cap (\mathbb{S}_1 + \mathbb{S}_2) &= \{S_3K_{32}\} \\ &\vdots \\ \mathbb{S}_n \cap \textstyle\sum_{i=1}^{n-1} \mathbb{S}_i &= \left\{S_nK_{n,(n-1)}\right\} \end{aligned} \qquad \text{where} \quad K' = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1,(n-1)} \\ K_{21} & K_{22} & \dots & K_{2,(n-1)} \\ \mathbf{0} & K_{32} & \dots & K_{3,(n-1)} \\ \vdots & \mathbf{0} & \dots & K_{4,(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & K_{n,(n-1)} \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} S_1 & \dots & S_n \end{bmatrix}\right). \quad (10)$$

However, if we want to extend this procedure to a whole parallel mechanism with a reconfigurable end-effector, one needs to keep track of the influence of the connections of the previous end-effector segments on the mobility of the legs when connecting a new end-effector segment. This can be achieved with a matrix of mobility.

3.3. Nullspace Approach Applied on a Matrix of Mobility

While it is possible to describe the mobilities in a single loop with a set of independent screw systems as shown in (8), it is not the case anymore for the screw systems describing a mechanism with reconfigurable end-effectors as the effect on every closed-loop must be taken into account for each end-effector and its relative mobility with the rest of the end-effectors and the base. To keep track of the effect of each closed-loop on the global system, we will combine the screw systems of all n end-effectors, for which the row space dimension is 6, into a single vector space with a row dimension of $6n$. A similar type of matrix was introduced in [3], where it was used to describe the statics and overconstraints in a triple Koenigs joint. This matrix notation was also briefly introduced in [12, 11] to represent mobility, but without using the nullspace approach. This matrix notation allows for the representation of the coupling between multiple screw systems and is adapted here in this article to the mobility of parallel mechanisms with reconfigurable end-effectors while using the nullspace approach applied to multiple closed-loops. For a general parallel mechanism with n reconfigurable end-effectors and m DOF, the matrix of mobility T takes the general form

$$T_{(6n\times m)} = \begin{bmatrix} \$_{11} & \dots & \$_{1m} \\ \vdots & \ddots & \vdots \\ \$_{n1} & \dots & \$_{nm} \end{bmatrix} \tag{11}$$

where $\$_{ij}$ is a (6×1) single screw representing the motion of end-effector i associated with DOF j. The total mobility of end-effector i is obtain by the screw system spanned by the basis screws $\mathbb{S}_i = \{[\$_{i1} \dots \$_{im}]\}$. The screws $\$_{ij}$ may not be independent and therefore $dim\,(\mathbb{S}_i)$ might be smaller than m. Each column represents a single DOF for the whole mechanism and therefore the total number of DOF for the mechanism corresponds the number of columns m. Each DOF $j = 1 \dots m$ only creates motion in an end-effector i if $\$_{ij} \neq \mathbf{0}$. The columns of T represent a vector basis spanning a subspace of dimension m of a larger vector space of dimension $6n$ and any linear operation can be performed on the columns of T to obtain an equivalent vector basis. Finally, the relative mobility between two end-effectors can be obtained by the subtraction of the corresponding rows. For example, the relative mobility between end-effector i and k is given as $\mathbb{S}_{i-k} = \{[(\$_{i1} - \$_{k1}) \dots (\$_{im} - \$_{km})]\}$.

The first step in expressing the examples of Figure 4 in the framework of the matrix of mobility is to start with an initial matrix of mobility T_0, in which end-effector segments P_i are not already connected. Each column corresponds to a DOF of the total system, and each set of 6 rows corresponds to the mobility of a leg relative to the base. In our examples in Figure 4 , before the connection of the end-effector segment P_i, we have

$$T_0 = \begin{bmatrix} S_{L1} & \mathbf{0} \\ \mathbf{0} & S_{L2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0}_{(6\times1)} & \mathbf{0}_{(6\times1)} \\ \mathbf{0}_{(6\times1)} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix}. \tag{12}$$

Each matrix S_{L1} and S_{L2} represents a set of basis screws. In addition, each column in T now represents a basis for a larger system that comprises both legs, and in which the vectors are now of dimension 12 instead of the 6 dimensions of a regular screw system. As with the case of any vector space basis, this set is not unique and any linear operations on the columns of the matrix T_0 can be performed to obtain a new valid basis, including linear operations on columns affecting different legs.

The relative mobility between legs L_1 and L_2 can be obtained by the subtraction of the corresponding rows in matrix T. For example, in equation 12, the individual DOF of the relative mobility $T_{0,(1-2)}$ between end-effector 1 and 2 are given by

$$T_{0,(1-2)} = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & -\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} & -\begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix}, \tag{13}$$

which is a screw system of dimension 2 with 3 independent DOF. Now that we want to close the loop with an end-effector segment P_i, we extend this row with S_{Pi}, a basis of $\mathbb{S}_{Pi}$, as in Equation (7), to obtain $K_i' = \text{Null}\left(\begin{bmatrix} T_{0,(1-2)} & S_{Pi} \end{bmatrix}\right)$, which are numerically described in Equation (9) for each example. A new matrix ${}^iT$ is obtained by multiplying T_0 with a new matrix K_i, the part of the matrix K_i' that is affecting the legs. The whole procedure can be summarised as

$$K_i' = \begin{bmatrix} K_i \\ K_{Pi} \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} T_{0,(1-2)} & S_{Pi} \end{bmatrix}\right) \qquad {}^{i}T = T_0 K_i \,, \tag{14}$$

where the row dimension of K_i is the same as the column dimension of T_0. Equation (14) is the core step of the proposed method and can be used successively to connect all end-effector segments in a parallel mechanism with a reconfigurable end-effector. It is derived from our adaptation of the nullspace approach to single closed-loops presented in Equation (8) applied the whole matrix of mobility. Using the initial matrix T_0 in Equation (12) and the numerical values obtained in Equation (9) for each K_i', we obtain a matrix of mobility ${}^{i}T$ representing the coupled mobility for each of the examples i shown in Figure 4 by multiplying T_0 with the first three rows of K_i', i.e., the rows that correspond to the legs and not the platform.

$$\begin{gathered}
{}^{1}T = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix} \qquad\qquad {}^{2}T = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} \end{bmatrix} \\
{}^{3}T = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix} \qquad {}^{4}T = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \lambda_1 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} + \lambda_2 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & -\lambda_2 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \lambda_1 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} \end{bmatrix}
\end{gathered} \;. \tag{15}$$

Matrix equivalence "$\equiv$" is shown in ${}^{3}T$ and ${}^{4}T$ to illustrate that any linear column operations can still be performed on a matrix of mobility to obtain a valid representation. Those results are still equivalent to those described by Equations (8) and (9). However, now the whole single loop is represented in a single vector space of dimension 12. The procedure can then be extended for a parallel mechanism with a reconfigurable end-effector with n legs to obtain a total matrix of mobility with $6n$ rows. When we connect an end-effector segment P_i between leg j and k, Equation (14) can be used to modify the whole matrix $T_{(i-1)}$ to obtain T_i using $T_{i,(j-k)}$ and S_{Pi}. When connecting the following end-effector segment, the interaction of the previous segments will have already been taken into account on the whole mechanism since the matrix of mobility combines all legs in a single vector space.

We will now describe mathematically the procedure for a general parallel mechanism with a reconfigurable end-effector. Let us define S_{Li} as matrices with column vectors corresponding to a basis of the screw system $\mathbb{S}_{Li}$ of leg i. Each set of 6 rows in T_i corresponds to the mobility of an end-effector relative to the base. Similarly, S_{Pi} represents the screw system of the platform segment i connecting the end-effectors j and k. The relative mobility between end-effectors j and k is given by $T_{i,(j-k)}$, which is a matrix with 6 rows obtained when subtracting the rows of leg k from the rows of leg j in the matrix T_i. The whole procedure for calculating the coupled mobility of a n-legged parallel mechanism with a reconfigurable end-effector can then be summarised as

$$\begin{gathered}
T_0 = \begin{bmatrix} S_{L1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & S_{L2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & S_{Ln} \end{bmatrix} \\
K_i' = \begin{bmatrix} K_i \\ K_{Pi} \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} T_{(i-1),j-k} & S_{Pi} \end{bmatrix}\right) \\
T_i = T_{(i-1)} K_i \qquad T = \left(\cdots\left(\left(T_0 K_1\right) K_2\right) \cdots K_n\right)
\end{gathered} \;. \tag{16}$$

In this procedure, the nullspace approach proposed in Equation (8) provides a direct method to determine K', representing which DOF are allowed in a single closed-loop. We can then multiply the part of K' affecting only the end-effectors mobilities relatively to the base, i.e. K, with a matrix of mobility T_i that comprises

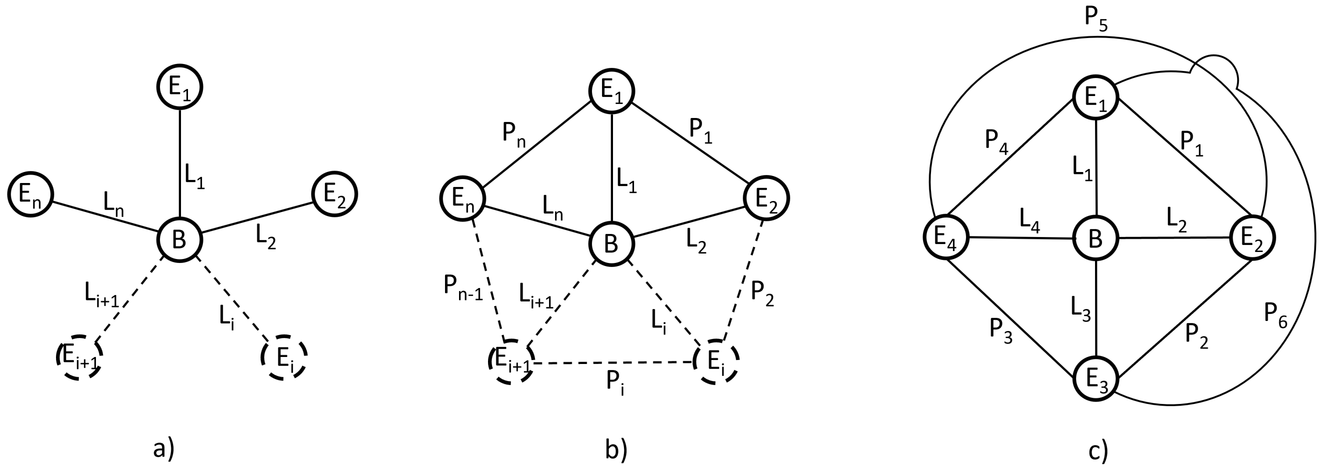

Figure 5: Graph representations of a) a system of n independent legs described by the matrix T_0, b) a parallel mechanism with a reconfigurable platform with n legs and n end-effector segments, c) a 4-legged parallel mechanism with a multi-loop reconfigurable end-effector using 6 end-effector segments

the mobility of all n end-effectors into a single vector space of dimension $6n$. This allows us to keep track of the global effects that the connection of each end-effector segment has on the mobility of the end-effectors. It is important to note that matrices K_i must be calculated *successively*, i.e the relative mobility in the nullspace calculation of K_i' is $T_{(i-1),j-k}$, thus not $T_{0,j-k}$, to takes into account the end-effector segments already connected.

The method can be applied to any general multi-loop linkage used as a reconfigurable end-effector, as long as each kinematic chain, i.e each end-effector segment, connects only two legs. The starting matrix of mobility T_0 always corresponds to a star graph where each edge of the star represents the mobility of a leg. For each end-effector segment i connecting leg j and k, equation 16 can be used to calculate the matrix K_i and the effects of this connection on all the legs of the mechanism. Figure 5 a) shows the graph for a general starting matrix T_0 in a parallel mechanism with n legs. Figure 5 b) shows a reconfigurable platform with n end-effector segments used as the reconfigurable end-effector. Figure 5 c) shows the graph representation of a 4-legged parallel mechanism with a multi-loop reconfigurable end-effector made of 6 end-effector segments.

3.4. Practicality of the Proposed Method

The proposed method can be directly implemented using either analytical vectors as inputs or numerical values as inputs. Each way presents some advantages and challenges.

On one hand, using analytical vectors which may include variables that depend on the configuration of the mechanism leads to an analytical solution that is valid in any configuration and can also provide insights on the nature of the mechanism, revealing conditions for which the mobility changes as it occurs in singular configurations for example. The main challenge in the analytical method is that the nullspace must be calculated by inspection. The method can be applied to fairly complex mechanisms. However, it can be that some mechanisms generate matrices for which the nullspace might be difficult to calculate analytically by inspection. In any case, this process using analytical notation would be difficult to automate as it requires some interpretation.

On the other hand, the process using numerical values as inputs can be automated in an algorithm. It does not depend on the selected reference frame nor the screw basis used. The main limitation is that it provides a solution only for a single configuration, and the robustness of its implementation as to be considered as it requires the calculation of nullspaces, which are sensitive to numerical round-off errors. However, several methods for the robust calculation for the rank of numerical matrices are readily available. For example, the use of singular value decomposition and the use of a threshold to include singular vectors related to very small singular values as also part of the nullspace.

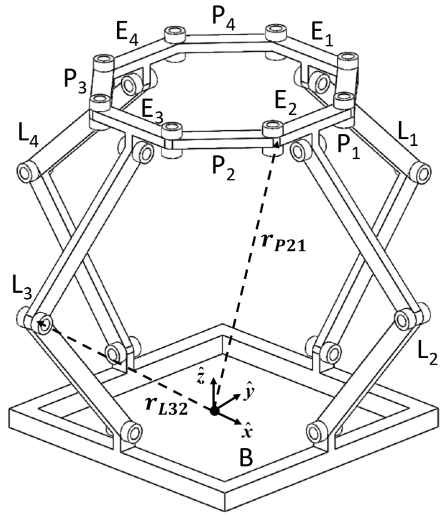

Figure 6: A 4-legged parallel mechanism with an 8R reconfigurable platform and its notation. Platform segment P_i has two parallel revolute joints and connects legs L_i and leg $L_{(i+1)}$. The links that are part of both the configurable platform and leg i are the end-effectors E_i. The position of each joint j of leg i is noted as $\mathbf{r_{Lij}}$ and the position of each joint j of platform segment i is noted as $\mathbf{r_{Pij}}$.

In both scenarios, one might want to minimise the number of operations and the size of the matrices involved in the calculation. To this effect, any column in $T_{i,(j-k)}$ for which the values are all 0 can be discarded from the calculation when connecting P_i, as those will produce their own vectors in the nullspace K_i. Those vectors will have all their elements as 0 except for the column in question which will be 1. This will leave those columns unchanged in T_i after the multiplication $T_{(i-1)}K_i$. In the next section, we will illustrate the method with two examples.

4. Examples

In this section, we apply the method described by Equation 16 to two examples. The first one has 4 legs and is made of planar chains. We will describe its mobility in any configuration using the analytical method and will also shown some singular configurations of the mechanism. We will then apply the algorithmic method to confirm the results obtained analytically. The second example has 3 non-planar legs and a spherical platform and we will calculate its mobility analytically.

4.1. Example 1

The first example used to illustrate the method is shown in Figure 6 with its notation. This is a 4-legged parallel mechanism with an 8R reconfigurable platform. Platform segment P_i has two parallel revolute joints and connects legs L_i and $L_{(i+1)}$. The links that are part of both the configurable platform and leg i are the end-effectors E_i. The position of each joint j of leg i is noted as $\mathbf{r_{Lij}}$ and the position of each joint j of platform segment i is noted as $\mathbf{r_{Pij}}$. This example will be used to show both the analytical method and its implementation in an algorithm.

4.1.1. Analytical Method Using Vector Notation

Since the mechanism has four legs, the initial matrix of mobility is given by

$$T_0 = \begin{bmatrix} S_{L1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_{L2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{L3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & S_{L4} \end{bmatrix}. \tag{17}$$

To find a valid basis for $\mathbb{S}_{L1} = \{S_{L1}\}$, we can directly assign a basis screw for each joint of leg L_1.

$$S_{L1} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{x}} \\ \mathbf{r_{L11}} \times \hat{\mathbf{x}} & \mathbf{r_{L12}} \times \hat{\mathbf{x}} & \mathbf{r_{L13}} \times \hat{\mathbf{x}} \end{bmatrix}, \tag{18}$$

where $\mathbf{r_{L1i}}$ is the vector position of joint i of leg L_1. It is well known that the mobility of a serial chain made of 3 parallel revolute joints corresponds to 2 translations in a plane and 1 rotation perpendicular to that plane. Using linear combinations on the three basis screws, this basis can be rearranged in the more convenient form

$$S_{L1} = \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{r_{L11}} \times \hat{\mathbf{x}} & (\mathbf{r_{L12}} - \mathbf{r_{L11}}) \times \hat{\mathbf{x}} & (\mathbf{r_{L13}} - \mathbf{r_{L11}}) \times \hat{\mathbf{x}} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix}. \tag{19}$$

Indeed, second and third basis vectors in Equation (18) can have their top part, the vector $\hat{\mathbf{x}}$, cancelled out by the first screw. Since their bottom part is perpendicular to $\hat{\mathbf{x}}$, it can be rearranged linearly in their $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components. Finally, the bottom part of the first screw can be cancelled out by the bottom part of the two remaining screws. Those linear operations on the basis screws show that leg L_1 can translate in the YZ plane and rotate around the $\hat{\mathbf{x}}$ axis at any point on that plane for any configuration. The screw system bases for the remaining three legs are obtained similarly as:

$$S_{L2} = \begin{bmatrix} \hat{\mathbf{y}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{x}} & \hat{\mathbf{z}} \end{bmatrix} \qquad S_{L3} = \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix} \qquad S_{L4} = \begin{bmatrix} \hat{\mathbf{y}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{x}} & \hat{\mathbf{z}} \end{bmatrix}. \tag{20}$$

We can assign those values in T_0 to obtain

$$T_0 = \begin{bmatrix} \begin{bmatrix}\hat{\mathbf{x}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{y}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{x}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{y}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} \end{bmatrix}. \tag{21}$$

Similarly, we can assign a screw basis to each joint of the platform segment P_i to obtain

$$S_{Pi} = \begin{bmatrix} \hat{\mathbf{z}} & \hat{\mathbf{z}} \\ \mathbf{r_{Pi1}} \times \hat{\mathbf{z}} & \mathbf{r_{Pi2}} \times \hat{\mathbf{z}} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{z}} & \mathbf{0} \\ \mathbf{r_{Pi2}} \times \hat{\mathbf{z}} & (\mathbf{r_{Pi2}} - \mathbf{r_{Pi1}}) \times \hat{\mathbf{z}} \end{bmatrix}, \tag{22}$$

where $\mathbf{r_{Pij}}$ is the position of the revolute joint j of platform segment P_i.

When connecting the first platform segment P_1, we first calculate the nullspace K_1'. As mentioned in 3.4, we can omit the last 6 columns of T_0 at this stage for simplicity as their relative mobility is zero and they would remain unchanged by the nullspace multiplication even if they were considered. We have

$$K_1' = \text{Null}\begin{bmatrix} T_{0,(1-2)} & S_{P1} \end{bmatrix} = \text{Null}\begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{y}} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}} & \hat{\mathbf{z}} & \mathbf{0} & -\hat{\mathbf{x}} & -\hat{\mathbf{z}} & \mathbf{r_{P11}} \times \hat{\mathbf{z}} & (\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}} \end{bmatrix}. \tag{23}$$

To perform the nullspace calculation analytically, one can refer again to Equation (3) for the allowed mobility in a single-loop. Following this equation, the set of independent vectors in K_1' is obtained by considering the intersection of each vector i in $\begin{bmatrix} T_{0,(1-2)} & S_{P1} \end{bmatrix}$ with the sum of the $i-1$ previous vectors. The first vector with a non-null intersection is the $\hat{\mathbf{z}}$ translation of the column 6 with column 3. The second one is the last column where the translation of the platform corresponds to a combination of columns 2 and 5. We therefore obtain

$$K_1' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & -1 & 0 & 0 & -\rho_1 \end{bmatrix}^T \qquad K_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & \lambda_1 & 0 & 0 & -1 & 0 \end{bmatrix}^T, \tag{24}$$

where the particular value of λ_1 depends on the orientation of the vector $(\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}}$ in the XY plane and is the ratio of displacement in the $\hat{\mathbf{y}}$ direction to the displacement in the $\hat{\mathbf{x}}$ direction as in

$$\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} + \lambda_1 \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} = \rho_1 \begin{bmatrix}\mathbf{0}\\ (\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}}\end{bmatrix} . \tag{25}$$

We can now multiply the first six columns of T_0 with K_1 to obtain

$$T_0 K_1 = T_1 = \begin{bmatrix} \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \lambda_1 \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{x}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{y}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} \end{bmatrix} . \tag{26}$$

This is the matrix of mobility when the platform segment P_1 is connected between leg L_1 and L_2. The first column means that the translation in the $\hat{\mathbf{z}}$ direction is now the same in both legs, and the second column means that a translation in $\hat{\mathbf{y}}$ in the leg L_1 is coupled to a translation in $\hat{\mathbf{x}}$ in leg L_2. Both effectors E_1 and E_2 have now lost their ability to provide rotation since the corresponding screw bases were not part of the nullspace.

We now repeat the same procedure for each platform segment. When connecting platform segment P_2, we can omit for now the last three columns of T_1, which only include S_{L4}, as those will produce columns of value $\begin{bmatrix}\mathbf{0}^T & \mathbf{0}^T\end{bmatrix}^T$ in $T_{1,(2-3)}$. We have

$$K_2' = \text{Null}\begin{bmatrix}T_{1,(2-3)} & S_{P2}\end{bmatrix} = \text{Null}\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \hat{\mathbf{z}} & -\hat{\mathbf{x}} & \mathbf{0} & -\hat{\mathbf{y}} & -\hat{\mathbf{z}} & \mathbf{r_{P21}} \times \hat{\mathbf{z}} & (\mathbf{r_{P22}} - \mathbf{r_{P21}}) \times \hat{\mathbf{z}} \end{bmatrix} , \tag{27}$$

which gives us

$$K_2' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \lambda_2 & 0 & 0 & \rho_2 \end{bmatrix}^T \qquad K_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \lambda_2 & 0 \end{bmatrix}^T \tag{28}$$

with

$$\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} + \lambda_2 \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} = \rho_2 \begin{bmatrix}\mathbf{0}\\ (\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}}\end{bmatrix} . \tag{29}$$

We can now multiply the first 5 columns of T_1 with K_2 to obtain

$$T_1 K_2 = T_2 = \begin{bmatrix} \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \lambda_1 \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \lambda_2 \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{y}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} \end{bmatrix} . \tag{30}$$

The same procedure is now repeated for platform segment P_3. This time, all columns of T_2 must be considered since no $\begin{bmatrix}\mathbf{0}^T & \mathbf{0}^T\end{bmatrix}^T$ column is formed in $T_{2,(3-4)}$. The whole procedure is shown below as

$$K_3' = \text{Null}\begin{bmatrix}T_{2,(3-4)} & S_{P3}\end{bmatrix} = \text{Null}\begin{bmatrix}\mathbf{0} & \mathbf{0} & -\hat{\mathbf{y}} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \hat{\mathbf{z}} & \lambda_2\hat{\mathbf{y}} & \mathbf{0} & -\hat{\mathbf{x}} & -\hat{\mathbf{z}} & \mathbf{r_{P31}} \times \hat{\mathbf{z}} & (\mathbf{r_{P32}} - \mathbf{r_{P31}}) \times \hat{\mathbf{z}}\end{bmatrix}$$

$$K_3' = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda_3/\lambda_2 & 0 & -1 & 0 & 0 & -\rho_3\end{bmatrix}^T \qquad K_3 = \begin{bmatrix}1 & 0 & 0 & 0 & 1 \\ 0 & \lambda_3/\lambda_2 & 0 & -1 & 0\end{bmatrix}^T$$

$$\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{x}}\end{bmatrix} + \lambda_3 \begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{y}}\end{bmatrix} = \rho_3 \begin{bmatrix}\mathbf{0} \\ (\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}}\end{bmatrix}$$

$$T_3 = T_2K_3 = \begin{bmatrix} \begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{z}}\end{bmatrix} & \frac{\lambda_3\lambda_1}{\lambda_2}\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{y}}\end{bmatrix} \\ \begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{z}}\end{bmatrix} & -\frac{\lambda_3}{\lambda_2}\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{x}}\end{bmatrix} \\ \begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{z}}\end{bmatrix} & \lambda_3\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{y}}\end{bmatrix} \\ \begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{x}}\end{bmatrix} \end{bmatrix} \quad . \tag{31}$$

Finally, the last segment P_4 is connected between leg L_4 and L_1. Since the first column for leg L_4 and L_1 are both the $\hat{\mathbf{z}}$ translation, this leads to a $\begin{bmatrix}\mathbf{0}^T & \mathbf{0}^T\end{bmatrix}^T$ column in in $T_{3,(4-1)}$. The nullspace approach gives us

$$K_4' = \text{Null}\begin{bmatrix}T_{3,(4-1)} & S_{P4}\end{bmatrix} = \text{Null}\begin{bmatrix}\mathbf{0} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{x}} - \frac{\lambda_3\lambda_1}{\lambda_2}\hat{\mathbf{y}} & \mathbf{r_{P41}} \times \hat{\mathbf{z}} & (\mathbf{r_{P42}} - \mathbf{r_{P41}}) \times \hat{\mathbf{z}}\end{bmatrix} . \tag{32}$$

The first vector of K_4' is indeed $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$, corresponding to the 0 elements of the first column of $T_{3,(4-1)}$. Columns 2 and 4 are in the same plane but they can only create a nullspace vector if they are aligned, i.e if the following condition is respected:

$$\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{x}}\end{bmatrix} + \frac{\lambda_3\lambda_1}{\lambda_2}\begin{bmatrix}\mathbf{0} \\ \hat{\mathbf{y}}\end{bmatrix} = \rho_4 \begin{bmatrix}\mathbf{0} \\ (\mathbf{r_{P42}} - \mathbf{r_{P41}}) \times \hat{\mathbf{z}}\end{bmatrix} . \tag{33}$$

As it was the case with the previous platform segments, we can define λ_4 as the ratio of $\hat{\mathbf{y}}$ displacement over $\hat{\mathbf{x}}$ displacement for P_4, which leads to the following condition

$$\lambda_4\lambda_2 = \lambda_3\lambda_1 . \tag{34}$$

If this condition is satisfied, the nullspace is given by

$$K_4' = \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1\end{bmatrix}^T \qquad K_4 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}^T , \tag{35}$$

which gives the unity matrix and leaves the matrix T_3 unchanged. If the condition in Equation (35) is not respected, the nullspace becomes $K_4' = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$ and the configurable platform completely loses its mobility, leaving only the common translation in $\hat{\mathbf{z}}$. When the condition in Equation 34 is respected, the final matrix of mobility is given by Equation 36. The individual mobility of end-effector i is described by the screw system spanned by the screws of the set of rows i. The two columns of T_4 mean that the mechanism has 2 DOF as a whole. One DOF produces identical translations in the $\hat{\mathbf{z}}$ direction in each leg. The second DOF produces either $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ translations in each leg for which the velocities are coupled by the various λ_i terms. Following the simple mobility criterion of Equation 5, this 2-DOF mechanism has 4 closed-loops, 20 1-DOF joints, and therefore possess $(6 \times 4) - 20 + 2 = 6$ overconstraints.

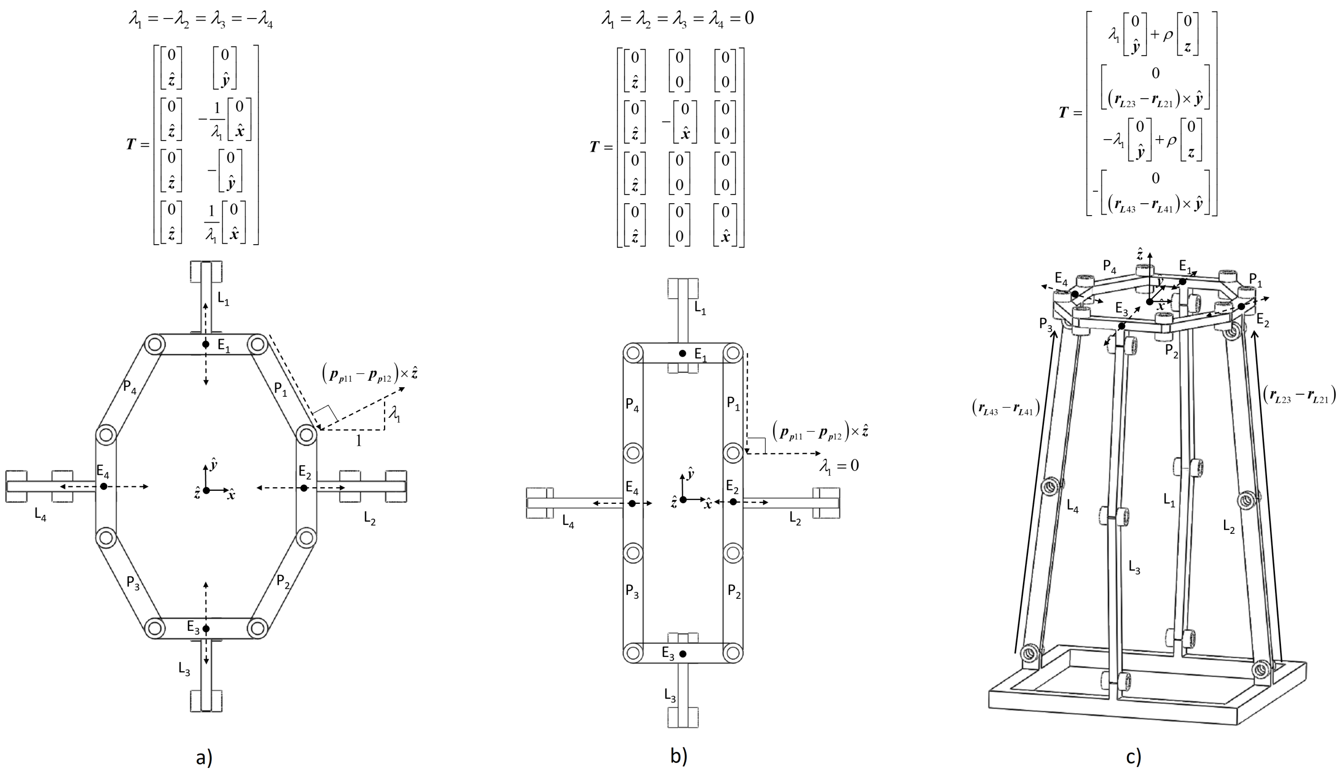

Figure 7: a) Top view of the mechanism. The reconfigurable platforms has 1 internal DOF and can also move as a whole in the $\hat{\mathbf{z}}$ direction. E_1 and E_3 move in opposite directions with same velocity while E_2 and E_4 also move in opposite directions with same velocity. b) The mechanism in a singular configuration of the platform. In this case, E_2 and E_4 becomes independent while E_1 and E_3 lose their mobility. The mechanism gain 1-DOF as a whole, shown by the 3 columns of the matrix of mobility. c) the mechanism in singular configuration of the legs. Effectors E_2 and E_4 can only translate in one direction, coupling the motion in the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions for leg E_1 and E_3

Let explore the conditions for which the mechanism conserves the platform DOF, represented by the second column in $T_4 = T_3 K_4 = T_3$. We can reorganise the basis of the matrix by multiplying this column with a scalar $\lambda_2/(\lambda_3\lambda_1)$ such that this DOF corresponds to a unit of velocity of E_1 in the positive $\hat{\mathbf{y}}$ direction.

$$T_4 = \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} & -\frac{1}{\lambda_1} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} & \frac{\lambda_2}{\lambda_1} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} & -\frac{\lambda_2}{\lambda_1\lambda_3} \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{x}} \end{bmatrix} \end{bmatrix} \equiv T_3 \, . \tag{36}$$

Figure 7 a) shows a case in which the mobility of the platform is conserved. The matrix of mobility not only provide the allowed DOF of each end-effector, but also the relative velocity of each end-effector in a given configuration. A general case in which none of the gradient values λ_i in $\lambda_4\lambda_2 = \lambda_3\lambda_1$ have the same value is also possible. We see in this example that one can use the method with analytical vector notation when possible to describe the full cycle mobility and obtain insights on the mechanism mobility for different platform segment length ratios. The method can also provide insights into the possible singular configurations of the mechanism.

4.2. Singular Configurations

In this section, we will consider two cases in which the mechanism is in a singular configuration. The first case corresponds to a platform singularity in which the mechanism gains 1 DOF. The second case is a leg singularity in which the mechanism loses 1 DOF.

First, Let consider that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, which means that all platform segments are oriented along the $\hat{\mathbf{y}}$ axis. In this configuration, represented in Figure 7 b), E_1 and E_3 reach their maximal distance from the centre of the platform. In this case, the connection of the first two platform segments are similar to the non-singular configuration and Equation (30) becomes

$$T_2 = \begin{bmatrix} \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix}\hat{\mathbf{y}}\\ \mathbf{0}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} \end{bmatrix}. \tag{37}$$

However, the results of connecting the third platform segment P_3 will be different due to the special conditions on the λ coefficients. Instead of Equation (31), we obtain

$$K_3' = \mathrm{Null}\begin{bmatrix} T_{2,(3-4)} & S_{P3} \end{bmatrix} = \mathrm{Null}\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\hat{\mathbf{y}} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \hat{\mathbf{z}} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{x}} & -\hat{\mathbf{z}} & \mathbf{r_{P31}} \times \hat{\mathbf{z}} & \hat{\mathbf{x}} \end{bmatrix}$$

$$K_3' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T \qquad K_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$T_3 = T_2 K_3 = \begin{bmatrix} \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \mathbf{0} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix} \end{bmatrix}. \tag{38}$$

Finally, connecting the fourth platform segment P_4, we obtain

$$K_4' = \mathrm{Null}\begin{bmatrix} T_{3,(4-1)} & S_{P4} \end{bmatrix} = \mathrm{Null}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{x}} & \mathbf{r_{P41}} \times \hat{\mathbf{z}} & \hat{\mathbf{x}} \end{bmatrix}$$

$$K_4' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}^T \qquad K_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \qquad T_4 = T_3 K_4 = T_3. \tag{39}$$

In this particular configuration, the distance between E_1 and E_3 in the platform reaches its maximum as shown in Figure 7 b), and their mobility become null. However, E_2 and E_4 can still perform infinitesimal motion in the $\hat{\mathbf{x}}$ direction. Moreover, the method proposed shows that the motion between E_2 and E_4 is now decoupled and the whole mechanism gains 1 DOF and has now a total of 3 DOF in this singular configuration, represented by the 3 columns of $T_4 = T_3$. Since the mechanism gained 1 DOF, this means it is a constraint singularity, i.e., a singularity in which some constraints become redundant, thus increasing the mobility of the mechanism.

On the other hand, we will now consider a leg singularity, in which legs L_2 and L_4 are extended to their maximum, as shown in 7 c). In this case the screw system of each leg becomes

$$
\begin{aligned}
S_{L1} &= \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix} \qquad S_{L2} = \begin{bmatrix} \hat{\mathbf{y}} & \hat{\mathbf{y}} & \mathbf{0} \\ \mathbf{r_{L21}} \times \hat{\mathbf{y}} & \mathbf{r_{L22}} \times \hat{\mathbf{y}} & (\mathbf{r_{L23}} - \mathbf{r_{L21}}) \times \hat{\mathbf{y}} \end{bmatrix} \\
S_{L3} &= \begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix} \qquad S_{L4} = \begin{bmatrix} \hat{\mathbf{y}} & \hat{\mathbf{y}} & \mathbf{0} \\ \mathbf{r_{L41}} \times \hat{\mathbf{y}} & \mathbf{r_{L42}} \times \hat{\mathbf{y}} & (\mathbf{r_{L43}} - \mathbf{r_{L41}}) \times \hat{\mathbf{y}} \end{bmatrix}
\end{aligned}, \tag{40}
$$

where the leg L_2 and L_4 lost their ability to translate in the $(\mathbf{r_{Li3}} - \mathbf{r_{Li1}})$ direction. When we close the first platform segment, equation 23 becomes

$$
\begin{gathered}
K_1' = \mathrm{Null}\begin{bmatrix} T_{0,(1-2)} & S_{P1} \end{bmatrix} = \\
\mathrm{Null}\begin{bmatrix} \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{y}} & -\hat{\mathbf{y}} & \mathbf{0} & \hat{\mathbf{z}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}} & \hat{\mathbf{z}} & -\mathbf{r_{L21}} \times \hat{\mathbf{y}} & -\mathbf{r_{L22}} \times \hat{\mathbf{y}} & -(\mathbf{r_{L23}} - \mathbf{r_{L21}}) \times \hat{\mathbf{y}} & \mathbf{r_{P11}} \times \hat{\mathbf{z}} & (\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}} \end{bmatrix}
\end{gathered} \tag{41}
$$

Unlike the non-singular configuration, which had a nullspace of dimension 2, the nullspace in this case is of dimension 1. We have

$$
K_1' = \begin{bmatrix} 0 & \lambda_1 & \rho & 0 & 0 & 1 & 0 & \lambda_2 \end{bmatrix}^T \qquad K_1 = \begin{bmatrix} 0 & \lambda_1 & \rho & 0 & 0 & 1 \end{bmatrix}^T \tag{42}
$$

where $(\mathbf{r_{L23}} - \mathbf{r_{L21}}) \times \hat{\mathbf{y}} = \lambda_1 \hat{\mathbf{y}} + \rho \hat{\mathbf{z}} + \lambda_2 (\mathbf{r_{P12}} - \mathbf{r_{P11}}) \times \hat{\mathbf{z}}$. Then the matrix of mobility T_1 of equation 26 becomes

$$
T_0 K_1 = T_1 = \begin{bmatrix}
\lambda_1 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} + \rho \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\begin{bmatrix} \mathbf{0} \\ (\mathbf{r_{L23}} - \mathbf{r_{L21}}) \times \hat{\mathbf{y}} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{bmatrix} \hat{\mathbf{y}} \\ \mathbf{0} \end{bmatrix} & \begin{bmatrix} \hat{\mathbf{y}} \\ \mathbf{r_{L42}} \times \hat{\mathbf{y}} \end{bmatrix} & \begin{bmatrix} \mathbf{0} \\ (\mathbf{r_{L43}} - \mathbf{r_{L41}}) \times \hat{\mathbf{y}} \end{bmatrix}
\end{bmatrix}. \tag{43}
$$

The same procedure using the method of Equation 16 is repeated for platform segment P_2, P_3, and P_4. Those steps will be omitted here since they are fairly similar to what is described in the non singular case, except that the $\hat{\mathbf{z}}$ translation is now coupled with the mobility of the platform. A important point is that leg L_2 and L_4 are symmetrical, thus producing the same translation in $\rho\hat{\mathbf{z}}$ for opposite but equal translations in $\hat{\mathbf{y}}$. The final matrix of mobility T_4 is given as

$$
T_4 = \begin{bmatrix}
\lambda_1 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} + \rho \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} \\
\begin{bmatrix} \mathbf{0} \\ (\mathbf{r_{L23}} - \mathbf{r_{L21}}) \times \hat{\mathbf{y}} \end{bmatrix} \\
-\lambda_1 \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{y}} \end{bmatrix} + \rho \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{z}} \end{bmatrix} \\
-\begin{bmatrix} \mathbf{0} \\ (\mathbf{r_{L43}} - \mathbf{r_{L41}}) \times \hat{\mathbf{y}} \end{bmatrix}
\end{bmatrix}. \tag{44}
$$

Since T_4 has only one column, the mechanism has only 1 DOF in this configuration.

We have shown with this example how the method can be applied analytically to provide the relative coupled velocity of all end-effectors in any configuration of the mechanism. The method also provides insights on the mobility conditions of the mechanism as well as a description of its singularities. Although the analytical method requires some inspection, the method does not require any a-priori intuition on the mobility of the mechanism and can therefore be fully automated in an algorithm.

4.3. Algorithm Method

The only inputs for the algorithm method are the numerical values describing the bases of the screw systems of each leg and each end-effector segment, as well as the information about which legs are connected by which end-effector segment. Let us consider the mechanism under the conditions shown in Figure 7 a) with $\lambda_1 = \lambda_3 = 1$, and $\lambda_2 = \lambda_4 = -1$, which means all end-effector segments are at 45°. Let also assume that the distance in the XY plane of the middle of the platform segment to the reference frame is $\sqrt{2}$. The numerical values for valid screw basis matrices for the platform segment in Equation (22) can therefore be given as:

$$S_{P1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \quad S_{P2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \quad S_{P3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \quad S_{P4} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{45}$$

Similarly, we can assign numerical values for the screw basis matrices S_{Li} for the legs from Equations (19) and (40).

$$S_{L1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad S_{L2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad S_{L3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad S_{L4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{46}$$

In appendix Appendix A, we provide a generic Matlab code based on the proposed method that can numerically calculate the instantaneous mobility of any parallel mechanism with a configurable platform. This algorithm can also be easily modified to calculate the mobility of any general reconfigurable end-effector such as in Figure 5 c) since Equation 16 can be applied successively on any pair of leg j and k. The inputs of the algorithm are the basis matrices of the legs and platforms and the output is the final matrix of mobility T. The algorithm is a direct method and does not use any numerical analysis procedures to evaluate the mobility of such mechanisms. It is assumed that the calculation of the rank of the matrices is robust and not subject to round-off numerical errors, as this was the case in this example using a double-precision floating point format. It should also be noted that the algorithm uses the whole matrix T_i in the multiplication $T_i K_{(i+1)}$ for each platform without discarding the $\begin{bmatrix}\mathbf{0}^T & \mathbf{0}^T\end{bmatrix}^T$ elements in $T_{i,(j-k)}$. Using the numerical inputs from Equations (45) and (46) for a $n = 4$-legged mechanism in the function of Appendix A, we obtain the following numerical results:

$$
T = \begin{bmatrix}
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0.1119 & -0.3354\\
-0.4743 & -0.1582\\
0 & 0\\
0 & 0\\
0 & 0\\
-0.1119 & 0.3354\\
0 & 0\\
-0.4743 & -0.1582\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
-0.1119 & 0.3354\\
-0.4743 & -0.1582\\
0 & 0\\
0.000 & 0.000\\
0 & 0\\
0.1119 & -0.3354\\
0 & 0\\
-0.4743 & -0.1582
\end{bmatrix}
\equiv
\begin{bmatrix}
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 1\\
1 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & -1\\
0 & 0\\
1 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & -1\\
1 & 0\\
0 & 0\\
0 & 0\\
0 & 0\\
0 & 1\\
0 & 0\\
1 & 0
\end{bmatrix}
=
\begin{bmatrix}
\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix}\\
\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix}\\
\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & -\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{y}}\end{bmatrix}\\
\begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{z}}\end{bmatrix} & \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{x}}\end{bmatrix}
\end{bmatrix}. \tag{47}
$$

The equivalence "$\equiv$" between the two vector bases in Equation (47) is obtained by linear operations on the columns of T. Row 20 in the numerical basis resulted in 0.000 instead of 0, showing that numerical round-off errors were introduced during the connection of the last platform segment, without affecting the calculation of the matrix rank in this case. The final result corresponds to the case of Figure 7 a) with $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = \lambda_4 = -1$, validating the analytical method proposed.

4.4. Example 2

In this second example, we will apply the method to the 3-legged parallel mechanism with multiple end-effectors shown in Figure 8 with its notation. This mechanism has a 6R spherical reconfigurable platform. Leg L_1 has three revolute joints while leg L_2 and L_3 each have four revolute joints. The first two joints of each leg are all oriented in the $\hat{\mathbf{x}}$ direction. The remaining joints of each leg are oriented towards the origin. Each platform segment has two joints that are also oriented towards the origin.

To describe the coupled mobility of each end-effector and the total number of DOF of this mechanism, we can apply directly the method presented in Equation 16. We will use the analytical vector representation to obtain a representation of the mobility of the mechanism in any configuration.

Let first define the matrices containing the basis screws of each leg and each platform segment. We have

$$
S_{L1} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L1}} \\ \mathbf{r_{L11}} \times \hat{\mathbf{x}} & \mathbf{r_{L12}} \times \hat{\mathbf{x}} & \mathbf{0} \end{bmatrix}
$$

$$
S_{L2} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L21}} & \hat{\mathbf{s}}_{\mathbf{L22}} \\ \mathbf{r_{L21}} \times \hat{\mathbf{x}} & \mathbf{r_{L22}} \times \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad S_{L3} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L31}} & \hat{\mathbf{s}}_{\mathbf{L32}} \\ \mathbf{r_{L31}} \times \hat{\mathbf{x}} & \mathbf{r_{L32}} \times \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{48}
$$

$$
S_{Pi} = \begin{bmatrix} \hat{\mathbf{s}}_{\mathbf{Pi1}} & \hat{\mathbf{s}}_{\mathbf{Pi2}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad i = 1 \ldots 3
$$

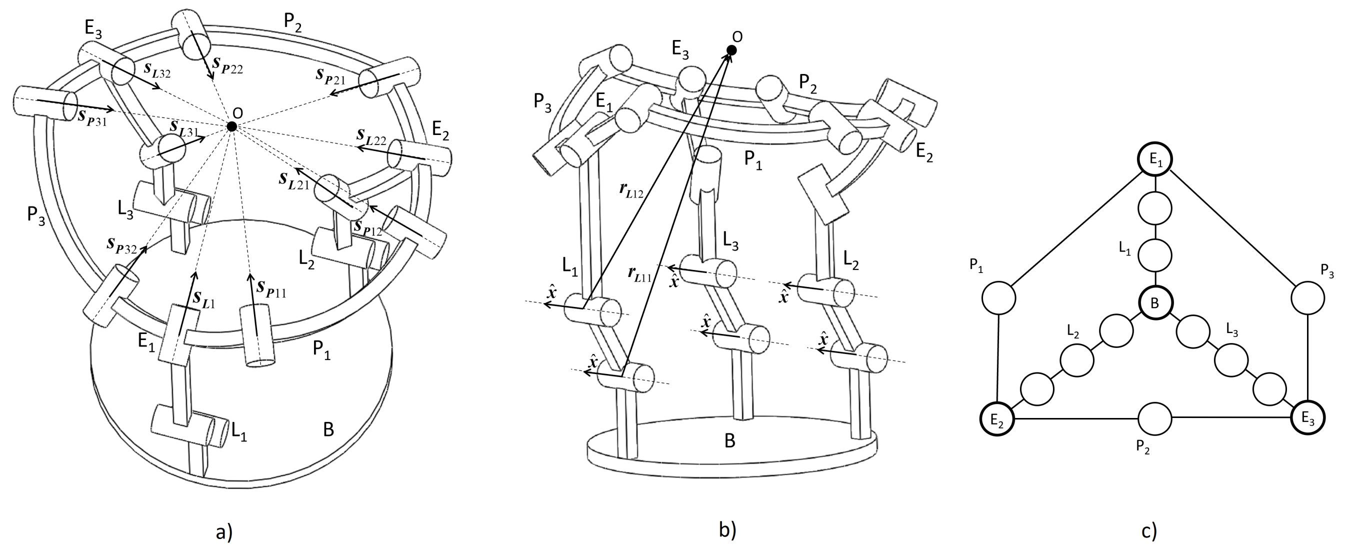

Figure 8: A 3-legged parallel mechanism with an 6R spherical reconfigurable platform. Leg L_1 has three joints while leg L_2 and L_3 have four joints each. The first two joints of each leg are all oriented in the $\hat{\mathbf{x}}$ direction. The remaining joints of each leg are oriented toward the origin. All joints in the platform are also oriented toward the origin. a) the notation for the platform. b) the notation for the parallel joints in the legs. c) the topology graph of the mechanism.

where the vectors $\hat{\mathbf{s}}$ are all oriented toward the origin. $\hat{\mathbf{s}}_{\mathbf{Pij}}$ is the orientation vector of the platform joint j of the platform segment i while $\hat{\mathbf{s}}_{\mathbf{Lij}}$ is the orientation vector of the joint j of the leg i.

We can now initiate the matrix of mobility T_0 representing the legs before the connection of the platform segments. Each set of two rows of 3D vectors represents a screw system for each leg.

$$T_0 = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{r_{L11}} \times \hat{\mathbf{x}} & \mathbf{r_{L12}} \times \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L21}} & \hat{\mathbf{s}}_{\mathbf{L22}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r_{L21}} \times \hat{\mathbf{x}} & \mathbf{r_{L22}} \times \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L31}} & \hat{\mathbf{s}}_{\mathbf{L32}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r_{L31}} \times \hat{\mathbf{x}} & \mathbf{r_{L32}} \times \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{49}$$

Connecting the platform segment P_1 between leg L_1 and L_2, we obtain

$$\begin{bmatrix} T_{0,(1-2)} & S_{P1} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{x}} & \hat{\mathbf{s}}_{\mathbf{L1}} & -\hat{\mathbf{x}} & -\hat{\mathbf{x}} & -\hat{\mathbf{s}}_{\mathbf{L21}} & -\hat{\mathbf{s}}_{\mathbf{L22}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathbf{s}}_{\mathbf{P11}} & \hat{\mathbf{s}}_{\mathbf{P12}} \\ \mathbf{r_{L11}} \times \hat{\mathbf{x}} & \mathbf{r_{L12}} \times \hat{\mathbf{x}} & \mathbf{0} & -\mathbf{r_{L21}} \times \hat{\mathbf{x}} & -\mathbf{r_{L22}} \times \hat{\mathbf{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{50}$$

Since the first link of both legs have the same length and the same orientation, this means that $\mathbf{r_{L11}} - \mathbf{r_{L12}} = \mathbf{r_{L21}} - \mathbf{r_{L22}}$. Using this relation, we can deduce by inspection that the four revolute joints of leg L_1 and L_2 oriented in the same $\hat{\mathbf{x}}$ direction are not independent. The same is true for the set of the 5 remaining joints since they are all oriented toward the same point. The nullspace of this matrix is therefore

$$K_1' = Null\left(\begin{bmatrix}T_{0,(1-2)} & S_{P1}\end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \lambda_{11} & 0 & 0 & 0 & 0 & -\lambda_{12} & -\lambda_{13} \\ 0 & 0 & 0 & 0 & 0 & -1 & \lambda_{21} & 0 & 0 & 0 & 0 & -\lambda_{22} & -\lambda_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T . \tag{51}$$

where $\quad \mathbf{\hat{s}_{L1}} = \lambda_{11}\mathbf{\hat{s}_{L22}} + \lambda_{12}\mathbf{\hat{s}_{P11}} + \lambda_{13}\mathbf{\hat{s}_{P12}} \quad$ and $\quad \mathbf{\hat{s}_{L21}} = \lambda_{21}\mathbf{\hat{s}_{L22}} + \lambda_{22}\mathbf{\hat{s}_{P11}} + \lambda_{23}\mathbf{\hat{s}_{P12}}$

The matrix K_1 corresponds to the 11 first rows of K_1', i.e, the rows corresponding to the legs. After connection of the first platform segment, we obtain the matrix of mobility T_1 as

$$T_0K_1 = T_1 = \begin{bmatrix} \mathbf{0} & \mathbf{\hat{s}_{L1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{r_{L11}} - \mathbf{r_{L12}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda_{11}\mathbf{\hat{s}_{L22}} & (\lambda_{21}\mathbf{\hat{s}_{L22}} - \mathbf{\hat{s}_{L21}}) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{r_{L21}} - \mathbf{r_{L22}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\hat{x}} & \mathbf{\hat{x}} & \mathbf{\hat{s}_{L31}} & \mathbf{\hat{s}_{L32}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r_{L31}} \times \mathbf{\hat{x}} & \mathbf{r_{L32}} \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} \end{bmatrix} . \tag{52}$$

The next step is to compute the second closed-loop by connecting the platform segment P_2 between Leg L_2 and L_3. We have

$$\begin{bmatrix}T_{1,(2-3)} & S_{P2}\end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{0} & \lambda_{11}\mathbf{\hat{s}_{L22}} & (\lambda_{21}\mathbf{\hat{s}_{L22}} - \mathbf{\hat{s}_{L21}}) & -\mathbf{\hat{x}} & -\mathbf{\hat{x}} & -\mathbf{\hat{s}_{L31}} & -\mathbf{\hat{s}_{L32}} & \mathbf{\hat{s}_{P21}} & \mathbf{\hat{s}_{P22}} \\ (\mathbf{r_{L21}} - \mathbf{r_{L22}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & -\mathbf{r_{L31}} \times \mathbf{\hat{x}} & -\mathbf{r_{L32}} \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{53}$$

Again, since the first link of Leg L_3 is the same length and orientation as in leg L_2, we can conclude that $\mathbf{r_{L21}} - \mathbf{r_{L22}} = \mathbf{r_{L31}} - \mathbf{r_{L32}}$. From the 6 screw vectors directly towards the origin, three of them must be dependent. The nullspace of this matrix is

$$K_2' = Null\left(\begin{bmatrix}T_{1,(2-3)} & S_{P2}\end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \lambda_{31} & -\lambda_{32} & -\lambda_{33} \\ 0 & 0 & 1 & 0 & 0 & 0 & \lambda_{41} & -\lambda_{42} & -\lambda_{43} \\ 0 & 0 & 0 & 0 & 0 & -1 & \lambda_{51} & -\lambda_{52} & -\lambda_{53} \end{bmatrix}^T . \tag{54}$$

$$\text{where} \quad \begin{aligned} \lambda_{11}\mathbf{\hat{s}_{L22}} &= \lambda_{31}\mathbf{\hat{s}_{L32}} + \lambda_{32}\mathbf{\hat{s}_{P21}} + \lambda_{33}\mathbf{\hat{s}_{P22}} \\ (\lambda_{21}\mathbf{\hat{s}_{L22}} - \mathbf{\hat{s}_{L21}}) &= \lambda_{41}\mathbf{\hat{s}_{L32}} + \lambda_{42}\mathbf{\hat{s}_{P21}} + \lambda_{43}\mathbf{\hat{s}_{P22}} \\ \mathbf{\hat{s}_{L31}} &= \lambda_{51}\mathbf{\hat{s}_{L32}} + \lambda_{52}\mathbf{\hat{s}_{P21}} + \lambda_{53}\mathbf{\hat{s}_{P22}} \end{aligned}$$

The matrix K_2 corresponds to the 7 first rows of K_2'. The new matrix of mobility T_2, obtained after the connection of the platform segment $P2$ is

$$T_1K_2 = T_2 = \begin{bmatrix} \mathbf{0} & \mathbf{\hat{s}_{L1}} & \mathbf{0} & \mathbf{0} \\ (\mathbf{r_{L11}} - \mathbf{r_{L12}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda_{11}\mathbf{\hat{s}_{L22}} & (\lambda_{21}\mathbf{\hat{s}_{L22}} - \mathbf{\hat{s}_{L21}}) & \mathbf{0} \\ (\mathbf{r_{L21}} - \mathbf{r_{L22}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda_{31}\mathbf{\hat{s}_{L32}} & \lambda_{41}\mathbf{\hat{s}_{L32}} & (\lambda_{51}\mathbf{\hat{s}_{L32}} - \mathbf{\hat{s}_{L31}}) \\ (\mathbf{r_{L31}} - \mathbf{r_{L32}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} . \tag{55}$$

We are now left with 4 DOF in total before the connection of the platform segment P_3. Keeping in mind that $\mathbf{r_{L11}} - \mathbf{r_{L12}} = \mathbf{r_{L31}} - \mathbf{r_{L32}}$, the close-loop created by the last platform segment is given as

$$\begin{bmatrix} T_{2,(3-1)} & S_{P3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & (\lambda_{31}\mathbf{\hat{s}_{L32}} - \mathbf{\hat{s}_{L1}}) & \lambda_{41}\mathbf{\hat{s}_{L32}} & (\lambda_{51}\mathbf{\hat{s}_{L32}} - \mathbf{\hat{s}_{L31}}) & \mathbf{\hat{s}_{P31}} & \mathbf{\hat{s}_{P32}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{56}$$

Since those screw vectors represent 5 rotation screws intersecting at the same point, at least two of them are dependent on the others. The nullspace is given as

$$K_3' = Null\left(\begin{bmatrix} T_{2,(3-1)} & S_{P3} \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\lambda_{61} & -\lambda_{62} & -\lambda_{63} \\ 0 & 0 & 1 & -\lambda_{71} & -\lambda_{72} & -\lambda_{73} \end{bmatrix}^T . \tag{57}$$

$$\text{where} \quad (\lambda_{31}\mathbf{\hat{s}_{L32}} - \mathbf{\hat{s}_{L1}}) = \lambda_{61}\left(\lambda_{51}\mathbf{\hat{s}_{L32}} - \mathbf{\hat{s}_{L31}}\right) + \lambda_{62}\mathbf{\hat{s}_{P31}} + \lambda_{63}\mathbf{\hat{s}_{P32}}$$
$$\lambda_{41}\mathbf{\hat{s}_{L32}} = \lambda_{71}\left(\lambda_{51}\mathbf{\hat{s}_{L32}} - \mathbf{\hat{s}_{L31}}\right) + \lambda_{72}\mathbf{\hat{s}_{P31}} + \lambda_{73}\mathbf{\hat{s}_{P32}}$$

Finally, using K_3 as the first four rows of K_3', we obtain the final matrix of mobility as

$$T_2 K_3 = T_3 = \begin{bmatrix} \mathbf{0} & \mathbf{\hat{s}_{L1}} & \mathbf{0} \\ (\mathbf{r_{L11}} - \mathbf{r_{L12}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda_{11}\mathbf{\hat{s}_{L22}} & (\lambda_{21}\mathbf{\hat{s}_{L22}} - \mathbf{\hat{s}_{L21}}) \\ (\mathbf{r_{L11}} - \mathbf{r_{L12}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\lambda_{61}\mathbf{\hat{s}_{L31}} + (\lambda_{31} - \lambda_{61}\lambda_{51})\,\mathbf{\hat{s}_{L32}}) & (\lambda_{71}\mathbf{\hat{s}_{L31}} + (\lambda_{41} - \lambda_{71}\lambda_{51})\,\mathbf{\hat{s}_{L32}}) \\ (\mathbf{r_{L11}} - \mathbf{r_{L12}}) \times \mathbf{\hat{x}} & \mathbf{0} & \mathbf{0} \end{bmatrix} . \tag{58}$$

The 3 columns of the matrix of mobility T_3 show that the mechanism of Figure 8 has 3 DOF in total, which was also confirmed by modelling the mechanism in CAD. The first DOF corresponds to a translation perpendicular to the first link of each leg. The direction of this translation will change with the configuration of the mechanism as $(\mathbf{r_{L11}} - \mathbf{r_{L12}})$ changes value. The second column is a rotation around the last joint of leg L_1 and the third column is a combination of rotations of the last two joints in leg L_2. The screw vectors for each of the 3 sets of 6 rows span the screw system for the mobility of an end-effector. While in the first DOF, the translations are identical for each legs, the second DOF, for example, shows that a rotation of $\mathbf{\hat{s}_{L1}}$ in leg L_1 will create rotation around $\mathbf{\hat{s}_{L22}}$ in Leg L_2, and around both $\mathbf{\hat{s}_{L31}}$ and $\mathbf{\hat{s}_{L32}}$ in leg L_3. The relative velocity between the joints are given by the various λ factors. The choice of a rotation around $\mathbf{\hat{s}_{L1}}$ as the reference for the second DOF is somewhat arbitrary, since it depends on which bases were used to describe the nullspaces. However, any linear operations on the columns of the matrix of mobility are still allowed to create a different but equivalent basis representing the 3 DOF of this mechanism.

5. Conclusion

This article presented a general method allowing for the analytical and numerical calculations of the mobility of overconstraint parallel mechanisms with reconfigurable end-effectors. By successively calculating the mobility for each closed-loop using the nullspace approach and by tracking its effect on the other loops using a matrix of mobility combining the mobility of all legs into a single vector space, the total number of DOF and the twist spaces for the coupled mobility of each leg are revealed. The method can also be used to analyse singular configurations. Two illustrative examples were presented to showcase how the method can be applied.

References

[1] Cammarata A (2017) A novel method to determine position and orientation errors in clearance-affected overconstrained mechanisms. Mechanism and Machine Theory 118:247–264

[2] Dai JS, Huang Z, Lipkin H (2006) Mobility of overconstrained parallel mechanisms. Journal of Mechanical Design

[3] Fayet M (2003) Distribution of wrench-twist duality in over-constrained mechanisms. J Mech Des 125(1):81–91

[4] Gallardo-Alvarado J (2020) A gough–stewart parallel manipulator with configurable platform and multiple end-effectors. Meccanica 55(3):597–613

[5] Gallardo-Alvarado J, Tinajero-Campos JH (2019) A parallel manipulator with planar configurable platform and three end-effectors. Mathematical Problems in Engineering 2019

[6] Huang Z, Liu J, Zeng D (2009) A general methodology for mobility analysis of mechanisms based on constraint screw theory. Science in China Series E: Technological Sciences 52(5):1337–1347

[7] Jia G, Huang H, Wang S, Li B (2021) Type synthesis of plane-symmetric deployable grasping parallel mechanisms using constraint force parallelogram law. Mechanism and Machine Theory 161:104,330

[8] Kang X, Dai JS (2019) Relevance and transferability for parallel mechanisms with reconfigurable platforms. Journal of Mechanisms and Robotics 11(3)

[9] Kong X, Gosselin CmM (2005) Mobility analysis of parallel mechanisms based on screw theory and the concept of equivalent serial kinematic chain. In: International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, vol 47446, pp 911–920

[10] Kong X, Jin Y (2016) Type synthesis of 3-dof multi-mode translational/spherical parallel mechanisms with lockable joints. Mechanism and Machine Theory 96:323–333

[11] Lambert P, Herder J (2013) Mobility analysis of non series-parallel mechanisms. In: New trends in mechanism and machine science, Springer, pp 63–71

[12] Lambert P, Herder JL (2014) Self dual topology of parallel mechanisms with configurable platforms. In: Computational Kinematics: Proceedings of the 6th International Workshop on Computational Kinematics (CK2013), Springer Science & Business Media, vol 15, p 291

[13] Lambert P, Herder JL (2016) Parallel robots with configurable platforms: fundamental aspects of a new class of robotic architectures. Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science 230(3):463–472

[14] Lambert P, Herder JL (2019) A 7-dof redundantly actuated parallel haptic device combining 6-dof manipulation and 1-dof grasping. Mechanism and Machine Theory 134:349–364

[15] Marchi T, Mottola G, Porta JM, Thomas F, Carricato M (2021) Position and singularity analysis of a class of planar parallel manipulators with a reconfigurable end-effector. Machines 9(1):7

[16] McCarthy JM, Soh GS (2010) Geometric design of linkages, vol 11. Springer Science & Business Media

[17] Mohamed MG, Gosselin CM (2005) Design and analysis of kinematically redundant parallel manipulators with configurable platforms. IEEE Transactions on Robotics 21(3):277–287

[18] Nabat V, de la O RODRIGUEZ M, Company O, Krut S, Pierrot F (2005) Par4: very high speed parallel robot for pick-and-place. In: 2005 IEEE/RSJ International Conference on intelligent robots and systems, IEEE, pp 553–558

[19] Pfurner M (2015) Analysis of a delta like parallel mechanism with an overconstrained serial chain as platform. In: Proc. 14th World Congress in Mechanism and Machine Science

[20] Song Y, Kang X, Dai JS (2020) Instantaneous mobility analysis using the twist space intersection approach for parallel mechanisms. Mechanism and Machine Theory 151:103,866

[21] Tian C, Zhang D, Tang H, Wu C (2021) Structure synthesis of reconfigurable generalized parallel mechanisms with configurable platforms. Mechanism and Machine Theory 160:104,281

[22] Xu Y, Liu W, Yao J, Zhao Y (2015) A method for force analysis of the overconstrained lower mobility parallel mechanism. Mechanism and Machine Theory 88:31–48

[23] Yi BJ, Na HY, Lee JH, Hong YS, Oh SR, Suh IH, Kim WK (2002) Design of a parallel-type gripper mechanism. The International Journal of Robotics Research 21(7):661–676

Appendix A. Matlab Script for Automated Mobility Calculation

In this appendix, we show the Matlab script used to numerically calculate the instantaneous mobility of the mechanism shown in the example. The script assumes that round-off errors do not affect the calculation of the rank of matrices. This script can be used for any parallel mechanism with reconfigurable platform with n legs and n platform segments.

Matlab script:

```
function T = mobility(n,SL,SP)

%Inputs:
    %n: number of legs = number of platform segments
    %SL: cell array containing the basis matrices SLi of leg i. SL = {[SL1], [SL2], ... , [SLn]}
    %SP: cell array containing the basis matrices SPi of platform i. SP = {[SP1], [SP2], ... , [SPn]}
%Output:
    %T: Final matrix of mobility

Tdim(1) = 1;% Initialisation of the column dimension of T0;

%Building empty T0
for i = 1:n
    SizeLegtemp = size(SL{i}); SizeLeg(i) = SizeLegtemp(2); %column dimension of SLi
    Tdim(i+1) = Tdim(i)+SizeLeg(i); %starting column of SLi
end
T0 = zeros(6*n,Tdim(n+1)-1);

%Building T0
for i = 1:n
    T0(6*(i-1)+1:6*i,Tdim(i):Tdim(i+1)-1) = SL{i};
end

T = T0; %initialisation of T

%closing (n-1) first platform segments
for i=1:(n-1)
    SizeTtemp = size(T); SizeT = SizeTtemp(2);
    Kp = null([T(6*(i-1)+1:6*i,:)-T(6*i+1:6*(i+1),:) SP{i}]);
    K = Kp(1:SizeT,:);
```

```
    T = T*K;
end

%closing last platform segment
SizeTtemp = size(T); SizeT = SizeTtemp(2);
Kp = null([T(6*(3)+1:6*4,:)-T(1:6,:) SP{n}]);
K = Kp(1:SizeT,:);
T = T*K;
```