New Supersymmetric Defects in Three and Six Dimensions

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Abstract

In this thesis we assemble recent results on BPS defect operators in $\mathcal{N} = (2, 0)$ and ABJM theory. Following a brief review of prerequisite material, we first construct a locally 1/2-BPS surface operator in the abelian $\mathcal{N} = (2, 0)$ theory, whose conformal anomaly we compute. The comparison of the anomaly coefficients at $N = 1$ to the holographic result is suggestive of a general linear relation between them, which we go on to prove using the framework of defect CFT. We then show how this approach can be used to find an expansion of bulk operators in terms of excitations of the defect. Along the way, we comment on surfaces with conical singularities and derive some useful technical results regarding the representation theory of certain superconformal algebras. Secondly, we revisit the known 1/6-BPS Wilson loops in ABJM theory, which we reinterpret as deformations around a bosonic loop operator. We proceed to adapt this construction to the $\mathcal{N} = 4$ case, and show that supersymmetric deformations of the 1/2-BPS Wilson loop yield a plethora of previously unknown operators which preserve various amounts of super- and conformal symmetry.

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Publications

This thesis is based on three collaborative publications. Chapters 3 and 4 closely follow, with slight alterations, the two articles

[i] N. Drukker, M. Probst, and M. Trépanier, "Surface operators in the 6d $\mathcal{N} = (2,0)$ theory," *J. Phys.*, vol. A53, no. 36, p. 365401, 2020.

[ii] N. Drukker, M. Probst, and M. Trépanier, "Defect CFT techniques in the 6d $\mathcal{N} = (2,0)$ theory," *JHEP*, vol. 03, p. 261, 2021.

Chapter 5 contains work first published in Chapter 2 of

[iii] N. Drukker *et al.*, "Roadmap on Wilson loops in 3d Chern-Simons-matter theories," *J. Phys.*, vol. A53, no. 173001, 2020,

coauthored with Gabriel Nagaoka, Marcia Tenser, and Maxime Trépanier, under the supervision of Dr Nadav Drukker and Dr Diego Trancanelli. The material presented in the second half of Chapter 5 is the subject of an upcoming publication in collaboration with Dr Nadav Drukker, Dr Diego Trancanelli, Ziwen Kong, and Marcia Tenser.

Contents

Chapter 1

Introduction

Almost a century after its inception, quantum field theory remains an intriguing subject of study and despite substantial advances, many features continue to elude satisfactory description in a coherent framework. While for most purposes of high energy particle physics, perturbative expansions in the coupling parameters, now understood in great detail, are perfectly sufficient, the study of strongly coupled systems continues to pose serious mathematical challenges. The most considerable progress towards understanding such systems has been made for theories that are highly constrained by either conformal or supersymmetry, such as naturally arise in the context of string and M-theory.

The most celebrated of these is maximally supersymmetric Yang-Mills theory in four dimensions, which has seen successful application of a wide range of techniques, including integrability [1], holography [2, 3], and the conformal bootstrap [4]. This theory has been shown to enjoy conformal symmetry for all values of the coupling constant [5, 6], making it a useful testing ground for methods relying on a large amount of symmetry.

On the other hand, the theory has often been cited as a reasonable toy model of quantum chromodynamics, and much work has been expended in attempts to reformulate and eventually answer open questions in nonabelian gauge theory in the language of $\mathcal{N} = 4$ super-Yang-Mills theory. A prominent example is the problem of quark confinement. In 1974, Wilson had argued that whether or not a theory confines is indicated by the expectation value of the phase factor W, now known as the Wilson loop, picked up by a heavy probe particle moving along a closed contour: In theories that confine, $\log \langle W \rangle$ scales like the area enclosed by the contour, while for theories that do not, it scales like the perimeter [7]. While a satisfactory description of the dynamics of confinement is still lacking, Wilson loops have since been a central tool in the study of gauge theories, including super-Yang-Mills theory. One of the most impressive results bridging the gap between weak and strong coupling is the exact calculation of the expectation value of the 1/2-BPS circular Wilson loop using supersymmetric localization, precisely matching the perturbative computation on one side and the holographic result at strong coupling on the other [8–10].

This milestone result suggests the systematic study of defect operators preserving some super- and conformal symmetry as a promising approach to superconformal theories in general. BPS Wilson loops have since been constructed in three, four, and five dimensional gauge theory [11–14]. In addition to Wilson loops, so called disorder line operators may be defined by allowing bulk fields to be singular along a prescribed contour, and supersymmetric constructions have been considered in [15].

While $\mathcal{N} = 4$ super-Yang-Mills may be understood as the worldvolume theory of a stack of D3-branes in type IIB string theory, analogous constructions in M-theory give rise to rather more exotic, but nevertheless very interesting theories that have received particular attention in recent years. M-theory contains fundamental extended objects, membranes and five-branes, whose respective worldvolume descriptions at low energies enjoy superconformal symmetry and are commonly referred to as ABJM and, somewhat obscurely, $\mathcal{N} = (2, 0)$ theory. However, in contrast to string theory, the explicit construction of these theories has proven rather involved, especially for the M5-brane, where a Lagrangian description has not been found. Instead of applying traditional methods with diminishing returns, one is therefore well-advised to explore new approaches.

The great success with which the Wilson line has been leveraged in $\mathcal{N} = 4$ super-Yang-Mills suggests the study of defect operators in M-brane worldvolumes as one avenue towards a better understanding of these theories. Since ABJM is a Chern-Simons gauge theory, it admits Wilson loops, supersymmetric versions of which have been known as early as 2008 [16]. While Yang-Mills theory fails to be conformal in dimensions other than four, three-dimensional Chern-Simons theory turns out to admit a surprising wealth of superconformal Wilson loops [17–21]. Since then, these operators have been studied extensively in ABJM and related theories, although a complete classification is still lacking. The correct way to proceed is less obvious in the $\mathcal{N} = (2, 0)$ theory. The fact that there exist supersymmetric configurations of M2- ending on M5-branes [22, 23] suggests the existence of surface rather than line operators. Framed in holographic terms, these were first studied in [24, 25]. The first attempts at an intrinsically field theoretic construction of these operators were made in [26, 27], but none of these preserve supersymmetry.

This thesis collects a number of results, obtained over the last four years, which grew out of attempts to address some of these glaring gaps in our understanding of BPS defect operators in M-brane worldvolume theories. It is structured as follows. Chapter 2 provides a brief review of superconformal symmetry, the M-theory origins of the theories in questions, as well as some standard constructions relating to defect operators and holography, both to establish notation and terminology, as well as to assemble classical results that will be used throughout the remainder of this work. The bulk of the thesis is concerned with surface defects in the $\mathcal{N} = (2, 0)$ theory. In Chapter 3, a locally BPS surface operator is defined for the abelian theory, and its conformal anomaly evaluated both in field theory and holographically at large N. In the following chapter, the coefficients governing the couplings of such defects to the stress tensor multiplet are extracted from the representation theory of the preserved superconformal algebra, providing linear relations between the anomaly coefficients at any N. These chapters are based on [28] and [29], respectively. Moving from six to three dimensions, in chapter 5, based on Section 2 of [30], we present a new look at the previously known supersymmetric Wilson loops in ABJM. We develop a formalism in which these loops can be written uniformly as continuous deformations of a particular bosonic loop, providing a natural construction of their moduli space. We then go on topresent the preliminary results of ongoing work [31], in which we adapt and generalise these techniques to $\mathcal{N} = 4$ sCSM theories with arbitrary quivers, making progress towards a full classification. We end with a short survey of possible extensions of this work and indulge in some rank speculation regarding future directions.

Chapter 2

Review

2.1 Superconformal Field Theory

2.1.1 Motivation

It is well known that, in the search for nontrivial extensions of Poincaré symmetry, one is presented with the two options of conformal and supersymmetry [32]. Algebras that contain either supersymmetries or conformal generators can be constructed in any spacetime dimension, and have been studied in great detail in the literature. Surprisingly, the requirement that an algebra contain both super- and conformal symmetry proves much more restrictive, and the space of theories with these symmetries is consequently rather more limited. However, it still includes many interesting theories that naturally arise in the study of string theory or are otherwise of interest. The considerable degree of symmetry renders such theories accessible to a wide array of methods and thus provides a great technical advantage, balancing the potential drawback of limited applicability.

2.1.2 Superconformal algebra

In anticipation of the technical sections in Chapters 4 and 5, we briefly review the structure and classification of superconformal algebras, following [33–35]. Since local field theories only admit up to 16 supercharges [34], and algebras with more supersymmetry consequently cannot arise as the symmetry algebras of interesting theories, we ignore them. Furthermore, for simplicity, we consider only Minkowski spacetimes of dimension $d \geq 3$. Since the structure of the minimal spinor representation depends intimately on the number of space and time dimensions, this overview is by necessity somewhat schematic.

The usual Poincaré algebra is spanned by $d(d-1)/2$ Lorentz transformations $\mathsf{M}_{\mu\nu}$, which form an $\mathfrak{so}(1,d-1)$, and d translations P_μ transforming in its vector representation. The simplest nontrivial (non-central) extension of this algebra is by a dilatation D, which commutes with rotations and satisfies

$$[\mathsf{D}, \mathsf{P}_\mu] = +\mathsf{P}_\mu. \tag{2.1}$$

In other words, P_μ has scaling dimension $+1$. To extend to full conformal symmetry, we adjoin another copy of the vector representation comprising the special conformal generators

K_μ, which are assigned scaling dimension -1 and satisfy

$$[\mathsf{P}_\mu, \mathsf{K}_\nu] = 2\left(\mathsf{M}_{\mu\nu} - \eta_{\mu\nu}\mathsf{D}\right). \tag{2.2}$$

We are left with a total $(d+1)(d+2)/2$ generators, which span the conformal algebra $\mathfrak{so}(2,d)$.

A supersymmetric extension of the Poincaré algebra, on the other hand, is constructed by adjoining $\mathcal{N}$ sets of fermionic operators Q^I_α, which each transform in a minimal spinor representation of the Lorentz group and commute with translations. Their anticommutator is taken to be of the form[1]

$$\{\mathsf{Q}^I_\alpha, \mathsf{Q}^J_\beta\} = \delta^{IJ}\left(\gamma^\mu C^{-1}\right)_{\alpha\beta}\mathsf{P}_\mu. \tag{2.3}$$

In addition, this algebra admits outer automorphisms R^{IJ}, which rotate the $\mathcal{N}$ families of supercharges.

In order to combine the super-Poincaré and conformal algebras, we must specify commutators of D and K_μ with Q^I_α. From (2.1) it is easily seen that Q^I_α has scaling dimension $+\frac{1}{2}$. Similarly, using the Jacobi identity, we find that

$$[\mathsf{D}, [\mathsf{K}_\mu, \mathsf{Q}^I_\alpha]] = -\frac{1}{2}[K_\mu, \mathsf{Q}^I_\alpha]. \tag{2.4}$$

Since neither the conformal nor the super-Poincaré algebra include a generator of that scaling dimension, we are required to add a generator S^I_α of scaling dimension $-\frac{1}{2}$ as a fermionic counterpart to K_μ . The structure of the (anti-)commutators $[\mathsf{P},\mathsf{S}], [\mathsf{K},\mathsf{Q}], \{\mathsf{S},\mathsf{S}\}$ is then largely fixed by the requirement that only terms of the correct scaling dimension appear:

$$[\mathsf{P}_\mu, \mathsf{S}^I_\alpha] \sim (\gamma_\mu)_\alpha{}^\beta \mathsf{Q}^I_\beta, \tag{2.5}$$
$$[\mathsf{K}_\mu, \mathsf{Q}^I_\alpha] \sim (\gamma_\mu)_\alpha{}^\beta \mathsf{S}^I_\beta, \tag{2.6}$$
$$\{\mathsf{S}^I_\alpha, \mathsf{S}^J_\beta\} \sim \delta^{IJ}\left(\gamma^\mu C^{-1}\right)_{\alpha\beta}\mathsf{K}_\mu. \tag{2.7}$$

The sole remaining anticommutator is that of Q and S. By inspection of the scaling dimension, it can only contain M and D. However, it can be shown that this is insufficient to define a closed Lie superalgebra. Indeed, it is necessary to include the R-symmetry generators R^{IJ} in the algebra as well such that the R-symmetry now becomes an *inner* automorphism.

The numerical coefficients in the (anti-)commutators can now be fixed by imposing the (super) Jacobi identities. The structure of a superconformal algebra turns out to be very restrictive, such that there are only 14 such algebras with at most 16 supercharges and $d \geq 3$. In particular, theories in spacetime dimension $d > 6$ do not admit superconformal symmetry. We will be interested only in the cases $d = 3, 6$, where these algebras and their respective bosonic subalgebras are given by:

$$\begin{aligned} d=3: &\quad \mathfrak{osp}(\mathcal{N}|4) \supset \mathfrak{so}(2,3)_{\text{conf}} \oplus \mathfrak{so}(\mathcal{N})_R, \qquad \mathcal{N} = 1,2,3,4,5,6,8 \\ d=6: &\quad \mathfrak{osp}(8|\mathcal{N}) \supset \mathfrak{so}(2,6)_{\text{conf}} \oplus \mathfrak{so}(2\mathcal{N}+1)_R, \qquad \mathcal{N} = 1,2. \end{aligned} \tag{2.8}$$

[1]Here, as in later sections, we largely follow the notation of [36], where C denotes the charge conjugation matrix.

While the space of available algebras itself is very rigid, their representation theory is richer than that of the ordinary conformal algebra. In particular, the generic representation will include more then one primary state. Highest weight representations are obtained by acting with super-Poincaré charges Q on a superprimary, i.e. a state $|J,R\rangle_\Delta$ annihilated by every S (and, consequently, all K_μ), where J and R indicate the quantum numbers with respect to Lorentz and R-symmetry. As in ordinary CFTs, enforcing nonnegativity of the norm of the superdescendant $\mathsf{Q}^n|J,R\rangle_\Delta$ at every level n places unitarity bounds $\Delta \geq \Delta_{\text{min}}(J,R)$ on the scaling dimension. Multiplets whose superprimaries saturate this bound contain states of zero norm, which may therefore be consistently deleted, and are therefore called short multiplets. However, for multiplets of superconformal symmetry, in addition to the continuum there is a finite series $\Delta_i < \Delta_{\text{min}}$ of allowed values for Δ. The scaling dimensions of primaries sitting at these values therefore cannot receive quantum corrections, and are thus protected.

An indispensable operator in a local field theory, which for the $\mathcal{N}=(2,0)$ theory will be studied in great detail in 4, is the stress tensor $T^{\mu\nu}$, which is the conserved current associated with translations:

$$\mathsf{P}_\mu = \int_{X^{d-1}} T_{\mu\nu}\, n^\nu. \tag{2.9}$$

Although it is itself not a superprimary, we can infer some of the structure of the corresponding multiplet from the algebra itself. The commutator $[\mathsf{Q},\mathsf{P}]=0$ implies that acting with Q on $T^{\mu\nu}$ gives a total derivative, i.e. a conformal descendant. The stress tensor is therefore the highest level primary in the multiplet. Similarly, $\{\mathsf{Q},\mathsf{Q}\} \sim \mathsf{P}$ implies that the supersymmetry current J^μ_α is, up to descendants, mapped to $T^{\mu\nu}$ under Q. Finally, the fact that $[\mathsf{R},\mathsf{Q}] \sim \mathsf{Q}$ implies that the R-symmetry current j^μ occupies the same multiplet as J^μ_α and $T^{\mu\nu}$.

2.2 M-brane worldvolume theories

The existence of superconformal algebras with stress tensor multiplets motivates the search for field theories with exactly those symmetries. Such theories can be constructed by hand, but their existence and many of their properties can also be inferred from brane constructions in string and M-theory. In this thesis we will be concerned with the $\mathcal{N}=(2,0)$ and ABJM theory, which provide the settings for Chapters 3, 4, and 5, respectively, and whose motivation as the low-energy description of M-branes we briefly review here. At low energies, M-theory is well known to be described by 11d supergravity. This theory has 32 real supercharges and includes a four-form field strength $F_4 = \mathrm{d}C_3$ whose equation of motion reads

$$0 = \mathrm{d} * F_4 + F_4 \wedge F_4. \tag{2.10}$$

The second term is due to a Chern-Simons type self-interaction $C_3 \wedge F_4 \wedge F_4$. It is instructive to introduce a dual field strength $F_7 = *F_4 + C_3 \wedge F_4$. Equation (2.10) can then be recast as $\mathrm{d}F_7 = 0$, implying that, at least locally, $F_7 = \mathrm{d}C_6$. Clearly, F_4 and F_7 are sourced by extended objects of dimension 3 and 6, respectively. In other words, F_4 couples electrically to M2- and magnetically to M5-branes. In the simplest case, we expect these objects to preserve some amount of supersymmetry. This suspicion is confirmed by the existence of 1/2-BPS soliton solutions to the supergravity equations of motion corresponding to such sources [37, 38].

The charges of these objects can be computed by integrating the respective fluxes:

$$Q_2 = \int_{X^7} *F_4 + C_3 \wedge F_4, \tag{2.11}$$

$$Q_5 = \int_{X^4} F_4, \tag{2.12}$$

where $X^{4,7}$ are the boundaries of transverse volumes intersecting the branes in a single point. These expressions allow us to deduce some simple results about their allowed interactions [22, 23]. Firstly, conservation of Q_5 implies that M5-branes must be closed, since otherwise the enclosing domain X^4 may be continuously contracted to a point. However, the additional term in the definition of Q_2, which comes from the Chern-Simons interaction of F_4 and is needed to ensure homotopy invariance, allows for M2- ending on M5-branes. Concretely, for an M2-brane with a boundary ∂M, X^7 may be deformed to the product of an S^3 surrounding ∂M and an S^4 in orthogonal directions. The integral then picks up only the second term, and we find

$$Q_2 = \int_{S^4} F_4 \int_{S^3} C_3 = Q_5 \int_{S^3} C_3. \tag{2.13}$$

Clearly, charge conservation can be maintained as long as the boundary of the M2-brane is entirely contained within an M5-brane. Such a configuration can be shown to preserve up to 1/4 of the supersymmetry [22]. From the perspective of the M5-brane, the boundary of the M2-brane forms a 1/2-BPS surface defect. The corresponding soliton solutions to the M5-brane equations of motion have been studied in the literature under the moniker of self-dual strings [39].

The low-energy degrees of freedom propagating on a brane are just the Goldstone modes associated with the broken continuous symmetries. Specifically, the 16 broken supercharges correspond to (on-shell) 8 fermionic zero modes, and since the excitations of the brane must fall into multiplets of the preserved supersymmetry, we expect 8 bosonic zero modes as well. These can be partially accounted for by the broken translational symmetry in the $10 - p$ directions transverse to the Mp-brane. For $p = 2$, the number of degrees of freedom corresponding to small transverse displacements matches that of the fermionic zero modes. By contrast, the M5-brane only admits five transverse excitations, and the remaining three bosonic zero modes must be represented another way. The unique supermultiplet containing the correct number of fields and five scalars is the tensor multiplet, where the the three extra bosons make up a selfdual three-form field strength. At the IR fixed point, we expect all higher modes to decouple, and the resulting theory describing the zero modes to be superconformal.

2.2.1 M2-brane: ABJM

According to the arguments in the previous section, the low-energy degrees of freedom of a single M2-brane should be described by a three-dimensional $\mathcal{N} = 8$ SCFT. The theory in question turns out to be a Chern-Simons theory with gauge group $U(1) \times U(1)$ at respective Chern-Simons level ± 1 and coupled to bifundamental matter [40]. Note that the Chern-Simons coupling is classically conformal, and that the presence of two gauge potentials in addition to the Goldstone fields does not contradict our earlier counting, since Chern-Simons fields do not represent propagating degrees of freedom.

While it can still be adapted to the description of two coincident M2-branes, the $\mathcal{N} = 8$ theory does not generalise beyond that. In order to describe a stack of N M2-branes, the supersymmetry has to be relaxed to $\mathcal{N} = 6$, leading to ABJM theory, a super-Chern-Simons matter theory with gauge group $U(N)_{-k} \times U(N)_k$ which describes N parallel M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity in the transverse directions [41]. From there, the theory can be generalised further, and indeed many of its interesting features are preserved in less restrictive settings. In particular, in Chapter 5 we relax supersymmetry to $\mathcal{N} = 4$ and consider more general quiver gauge theories.

2.2.2 M5-brane: $\mathcal{N} = (2, 0)$

An equally intriguing but much more elusive theory arises when one considers the world-volume of an M5-brane instead. This theory enjoys $\mathcal{N} = (2, 0)$ superconformal symmetry, which, as mentioned, is the largest possible amount of superconformal symmetry in the largest spacetime dimension that still admits any superconformal algebras. It is therefore tempting to view it as a "mother theory" which upon dimensional reduction gives rise to lower dimensional theories preserving some portion of the superconformal symmetry. Indeed, compactifying this theory on Riemann surfaces gives rise an intricate web of dualities between 2d and 4d theories, termed the AGT correspondence [42].

An explicit definition of the theory even for a single M5-brane is, however, hard to come by. The main difficulty is posed by the self-dual three-form field strength, for which a satisfactory manifestly Lorentz invariant action without auxiliary fields, even in the abelian case, has not yet been found (see [43, 44] for recent approaches). The situation is even less clear for multiple M5-branes, owing to the intrinsic difficulty of describing nonabelian gauge 2-forms. Furthermore, it is known that the theory does not admit exactly marginal deformations, rendering the tool of perturbation theory, which has been deployed with great success in the study of $\mathcal{N} = 4$ SYM, all but useless [45]. One is therefore forced to adopt a less direct approach. In Chapters 3 and 4, we probe the theory using 1/2-BPS surface defects, computing, among other things, the conformal anomaly associated with such objects.

2.3 Defect Operators

Usually the most immediately accessible objects of study in quantum field theory are local operators and their correlation functions. However, it has long been understood that the spectrum of a given QFT may include nonlocal excitations supported on a submanifold of spacetime as well. Perhaps the most obvious of these extended objects arise when considering a spacetime with a boundary, which may be understood as a defect of codimension $q = 1$ [46, 47], but a wide variety of defect operators of various codimension have since been described in the literature (see [48] and references therein for an overview). While these operators are interesting in their own right, they are particularly useful as a tool in the study of theories which do not readily yield to a description in terms of local operators, particularly if they preserve some amount of the bulk symmetry. Once one has good control over a defect operator, varying the underlying submanifold on which it is supported supplies a huge arsenal of operators one can use to probe the theory. Far and away the most famous of these operators is the Wilson line of gauge theory, which we will use to illustrate some of the salient features of defect operators. The pure Wilson line measures the holonomy of the gauge connection along a prescribed contour C:

$$W_C = \mathcal{P} \exp \mathrm{i} \int_C A_\mu \mathrm{d}x^\mu. \tag{2.14}$$

For generic C, the Wilson line breaks all geometric bulk symmetries, but choosing C to be a straight line (circle), W_C preserves a translation (rotation). Similarly, W_C can often be modified to preserve additional symmetries enjoyed by the bulk theory. The most celebrated example is the 1/2-BPS circular Wilson loop in $\mathcal{N} = 4$ super-Yang-Mills theory, which preserves rigid conformal symmetry along the circle as well as 8 supercharges and some R-symmetry. These symmetries have been leveraged to compute its expectation value for arbitrary 't Hooft coupling λ and any rank of the gauge group N, providing a spectacular interpolation between strong coupling results obtained holographically on one side and the perturbative expansion on the other. This remarkable result suggests that superconformal defects might be of some utility in less well understood settings as well. This is the point of view we adopt throughout this thesis.

2.3.1 Defect CFT

Conformal symmetry in particular places powerful constraints on the excitations of a defect and its interactions with the bulk, which we exploit in Chapter 4 for the case of surface defects in the $\mathcal{N} = (2,0)$ theory, and briefly introduce here. One of the key features of any *conformal* defect is that its excitations are described by a structure very similar to a conformal field theory. Indeed, the degrees of freedom supported by such a defect obey the axioms of a CFT, with the exception of locality, as is to be expected, since the defect exchanges energy with the bulk it is embedded in [46, 47]. In particular, these defect fields form multiplets of the preserved conformal symmetry, and their correlation functions are subject to the usual constraints. The most important example of these is the displacement operator $\mathbb{D}_m$, which encodes the defects response to an infinitesimal deformation of its supporting manifold. For the Wilson loop, this can be expressed explicitly as an insertion of fields into the line:

$$\frac{\delta W}{\delta x^\mu(\tau)} = \mathrm{i}\mathcal{P}\left(F_{\mu\nu}\dot{x}^\nu(\tau) \exp \mathrm{i} \int A_\mu \mathrm{d}x^\mu \right) = W[\mathrm{i}F_{\mu\nu}\dot{x}^\nu]. \tag{2.15}$$

We conclude that, for a line in the x^0 direction, $\mathbb{D}_m = \mathrm{i}F_{m0}$. The conformal symmetry preserved by the line now implies that

$$\langle W[\mathbb{D}_m(\tau)\mathbb{D}_n(0)] \rangle = \frac{C_{\mathbb{D}}\delta_{mn}}{\tau^4}. \tag{2.16}$$

While in ordinary CFTs, the coefficient of the two-point function can be absorbed by a field redefinition, this is impossible for a dCFT: The defect fields are defined in terms of the bulk fields, and their normalisation is therefore fixed. Consequently, the coefficient $C_{\mathbb{D}}$ is a meaningful quantity characterising the defect.

Put another way, the breaking of translation invariance in the directions orthogonal to the defect manifests as a correction to the conservation of the transverse components of the energy-momentum tensor $T^{\mu\nu}$ localised on the defect. Splitting coordinates $x = x_\perp + x_\parallel$ along directions orthogonal and perpendicular to the defect, we may write

$$\partial_\mu T^{\mu m} = \delta(x_\perp) W[\mathbb{D}^m(x_\parallel)]. \tag{2.17}$$

This equation suggests a relation between the energy radiated by the defect, as given by the normalisation of the one-point function of T^{0m} in the presence of W, and the dCFT quantity $C_{\mathbb{D}}$, which is therefore sometimes called the Bremsstrahlung function [49]. However, such relations have been proven only in settings with some supersymmetry [49, 50]. For defects breaking other continuous bulk symmetries, (2.17) is easily generalised, and each broken current gives rise to a defect field whose quantum numbers under the preserved symmetries may be easily read off. In the superconformal case, the bulk R-symmetry and supersymmetry currrents share a multiplet of the bulk superconformal algebra with the energy momentum tensor, and this structure is mirrored in the displacement multiplet on the defect side, whose members encode the breaking of the corresponding bulk symmetries on the defect. In Chapter 4, we will use supersymmetry to derive linear relations between the normalisation constants of their two-point functions, which, in turn, are related to the conformal anomaly associated with a surface.

2.4 Conformal Anomaly

Like any other symmetry of a QFT, the classical conformal invariance of any given CFT may be anomalous, i.e. spoiled at the quantum level. This manifests itself in the failure of the theory to be Weyl invariant on arbitrary curved backgrounds. Concretely, consider a theory defined on a (closed) d-dimensional spacetime with metric $g_{\mu\nu}$. The partition function is a nonlocal functional of the metric, $Z \equiv Z[g_{\mu\nu}]$. Under an infinitesimal Weyl transformation $g_{\mu\nu} \to (1+\omega) g_{\mu\nu}$, the free energy varies as

$$\frac{\delta \log Z}{\delta \omega(x)} = \frac{1}{Z} \frac{\delta Z}{\delta \omega(x)} = \frac{1}{Z} \frac{\delta Z}{\delta g_{\mu\nu}(x)} g_{\mu\nu(x)} = \sqrt{g} \left\langle T^\mu_\mu(x) \right\rangle. \tag{2.18}$$

While classically the stress tensor in a CFT must be traceless, this is generally no longer true in an anomalous theory.

The expectation value of T^μ_μ, sometimes referred to as the *anomaly density* $\mathcal{A}$, is a local functional of the metric $g_{\mu\nu}$. Crucially, the total conformal anomaly

$$A = \int \mathrm{d}^d x \sqrt{g}\, \mathcal{A} \tag{2.19}$$

is itself a conformal invariant. To see this, note first that A measures the change of $\log Z$ under a *constant* Weyl transformation:

$$\delta_{\omega_0} \log Z = \omega_0 A. \tag{2.20}$$

Then, the variation of A under a *general* infinitesimal Weyl transformation satisfies

$$\begin{aligned} \omega_0\, \delta_\omega A &= \delta_\omega \delta_{\omega_0} \log Z = \delta_{\omega_0} \delta_\omega \log Z \\ &= \delta_{\omega_0} \int \mathrm{d}^d x \sqrt{g}\, \mathcal{A}\, \omega \\ &= \int \mathrm{d}^d x \left(\mathcal{A}\, \delta_{\omega_0} \sqrt{g} + \sqrt{g}\, \delta_{\omega_0} \mathcal{A}\right) \omega, \end{aligned} \tag{2.21}$$

where in the second step we used that any two Weyl transformations commute: $[\delta_\omega, \delta_{\omega_0}] = 0$. This is a special case of the Wess-Zumino consistency condition, which restricts the form of general anomalies [51,52]. Each term contributing to $\mathcal{A}$ is a scalar constructed from m factors of the metric $g_{\mu\nu}$ and n factors of its inverse $g^{\mu\nu}$, both of which are dimensionless, as well as derivatives ∂_μ. Since the free energy, and therefore its variation under a Weyl transformation, must also be dimensionless, there are precisely d derivatives in each summand. Furthermore, in order for all indices to be fully contracted, $2n = 2m + d$. It follows immediately that $\mathcal{A}$ vanishes identically if d is odd. Although we have not derived the explicit form of $\mathcal{A}$, we can now compute its variation under constant Weyl transformations:

$$\delta_{\omega_0} \mathcal{A} = \omega_0\, (m - n)\, \mathcal{A} = -\frac{d}{2} \omega_0\, \mathcal{A}. \tag{2.22}$$

Noting that $\delta_{\omega_0} \sqrt{g} = +\frac{d}{2} \omega_0\, \mathcal{A}$, we find that the two terms in the last line of (2.21) cancel and thus, for an arbitrary Weyl transformation,

$$\delta_\omega A = 0, \tag{2.23}$$

as claimed. This places strong constraints on the terms that can possibly appear in $\mathcal{A}$. Two types of anomaly terms may be distinguished, based on their behaviour under a Weyl transformation: Type A anomalies are local functionals of the metric which change by a total derivative, while type B anomalies are themselves Weyl invariants. In $d = 2$, $\mathcal{A}$ is proportional to the Ricci scalar R, which is a type A anomaly; type B anomalies are absent. In higher dimensions $d = 2n$, there is one type A anomaly, namely the Euler density, as well as a number of type B anomalies which may be constructed by contracting n Weyl tensors.

These terms have been completely classified [52,53]. The anomaly density of a given CFT is then specified by the respective *anomaly coefficients* multiplying the available anomaly densities, which carry highly nontrivial information about the theory.

Beyond the bulk anomaly terms discussed above, defect operators may contribute to the conformal anomaly as well. By the same arguments as above, they must integrate to conformal invariants, and can therefore be enumerated explicitly. In particular, nonsingular odd-dimensional defects do not exhibit a conformal anomaly. In addition to the induced metric, the anomaly may now depend on additional geometric data associated with the embedding of the defect, leading to a somewhat richer structure of anomaly terms than in the bulk. For a defect defined over a surface Σ, the available terms, regardless of codimension, are the 2d Ricci scalar R^{Σ} (type A) as well as the trace of the pulled back bulk Weyl tensor $\mathrm{Tr}W$ and the Willmore functional $H^2 + 4\mathrm{Tr}P$, constructed from the mean curvature H^{μ} of the embedding and the pullback of the bulk Schouten tensor $P_{\mu\nu}$ (type B). Like the bulk theory, the defect itself can then be characterised by its anomaly coefficients. It is these numbers that we compute for surface operators in the two regimes at $N = 1$ and $N \gg 1$ in Chapter 3 and constrain at any N in Chapter 4.

2.5 Holography

Over the last 20 years, the idea that d-dimensional CFTs are dual to quantum gravity on a background geometry involving an AdS_{d+1} factor has gained widespread acceptance [2, 3]. In Maldacena's seminal paper, it was noted that the near-horizon geometry of the solution to type IIB supergravity sourced by a stack of N D3-branes is $AdS_5 \times S^5$, where the branes sit at the boundary of AdS_5 and the curvature radii of both AdS_5 and S^5 are given in terms of the string coupling g_s and length ℓ_s as $R = \ell_s(4\pi g_s N)^{\frac{1}{4}}$. As long as R is much larger than the string and Planck lengths (or, equivalently, $g_s N \gg 1$, $N \gg 1$), one can reliably expand around this solution. The same degrees of freedom should be described by the worldvolume theory of the D3-branes, which is known to be $\mathcal{N} = 4$ super-Yang-Mills theory with gauge group $SU(N)$ and coupling constant $g_{YM}^2 = 4\pi g_s$, leading to the conjecture that, at large 't Hooft coupling $\lambda = g_{YM}^2 N$ and large N, the gauge theory is well-described by type IIB supergravity.

In this thesis, we will be particularly interested in the holographic duals of defect operators. Again, the analogy with Wilson loops is instructive. Under the AdS/CFT correspondence, fundamental Wilson lines in $\mathcal{N} = 4$ have been all but proven to correspond to open strings in $AdS_5 \times S^5$ whose worldsheets end on the contour at the conformal boundary [2, 54, 55]. Since the string tension, measured in units of the curvature radius R^{-1}, becomes infinite in the limit we consider, the holographic dual of the Wilson loop expectation value is simply the exponential of the classical Nambu-Goto action, i.e. the area of a minimal surface with suitable boundary conditions.

In the case of stacks of branes in M-theory which we are interested in, the 11d supergravity solutions are very similar, and analogous arguments apply. Concretely, the near-horizon geometry of a stack of M2-branes is $AdS_4 \times S^7$ [56, 57], with their respective radii of curvature given by

$$2L_{AdS_4} = L_{S^7} = (32\pi^2 N)^{\frac{1}{6}} \ell_P, \tag{2.24}$$

while for M5-branes, we find $AdS_7 \times S^4$ with

$$L_{AdS_7} = 2L_{S^4} = 2(\pi N)^{\frac{1}{3}} \ell_P. \tag{2.25}$$

Note that the isometry algebras of these spaces are $\mathfrak{so}(2, 3) \oplus \mathfrak{so}(8)$ and $\mathfrak{so}(2, 6) \oplus \mathfrak{so}(5)$, respectively, which precisely match the bosonic parts of maximal superconformal algebras in 3d and 6d listed in (2.8). Clearly, in the large N limit all of the curvature radii become very large compared to the Planck length, and the supergravity approximation is valid. While in

10d the string tension is a free parameter, the tension of M-branes is determined entirely by the Planck length:

$$T_{M2} = \frac{1}{(2\pi)^2 \ell_P^3}, \qquad T_{M5} = \frac{1}{(2\pi)^5 \ell_P^6}, \tag{2.26}$$

which, in turn, may be recast in terms of the curvature radii and N. We see that in either of the two backgrounds described above, the tensions of both M2- and M5-branes become infinite in units of the curvature radius L as $N \to \infty$. This means that the appropriate holographic dual of the expectation value of a defect operators in ABJM or $\mathcal{N} = (2,0)$ is the classical action of a brane extending into the AdS bulk and attached to the defect at the conformal boundary. This is precisely the setup we will use in Chapter 3 to derive the conformal anomaly of a surface operator at large N.

Chapter 3

Locally BPS Surfaces in $6d$ $\mathcal{N} = (2,0)$

3.1 Introduction

In this chapter, closely based on our publication [28], we consider surface operators, the most natural observables in the $\mathcal{N} = (2,0)$ theory [39]. In some ways the surface operators in six dimensions are analogous to Wilson loops in lower dimensional gauge theories: Wilson loops are the boundaries of fundamental strings, which arise as the dimensional reduction of M2-branes, and indeed one obtains Wilson loops in compactifications of the 6d theory with surface operators. Wilson loops are not only interesting due to their physical importance, they are also accessible to many perturbative and non-perturbative calculational tools in supersymmetric field theories: Feynman diagrams, holographic descriptions [54, 55, 58], localization [10], the defect CFT framework and associated OPE techniques [59, 60], integrability [61, 62], duality to scattering amplitudes [63] and more. See for instance a recent survey of these techniques, as applied to supersymmetric Wilson loops in ABJM theory [30].

We do not expect all these techniques to apply equally to surface operators in six dimensions, but it is worthwhile to examine which of them might. Here we take the first step in such an examination, defining the notion of a "locally BPS surface operator" and studying basic properties of their anomalies. This is mainly based on previous work [27, 64–67], which we modify and refine in several ways.

As reviewed in the next section, the expectation value of generic surface operators contains logarithmic divergences, due to the conformal anomaly. The anomaly depends on the geometry of the surface, as well as intrinsic properties of the operator which are captured by three numbers, known as *anomaly coefficients* [53].

The "locally BPS" operator couples to the scalar fields via a unit 5-vector n^i. This can be viewed as a coupling to an R-symmetry background, and for non-constant n^i we find a new anomaly, proportional to $(\partial n)^2$, with its own anomaly coefficient.

We perform explicit calculations of the three geometrical and one R-symmetry background coefficients in both the free theory at $N = 1$ and the holographic description valid at large N. An examination of our results reveals that the new anomaly coefficient matches (up to a sign) one of the geometric ones in both regimes. We present here a simple argument, relying on supersymmetry, why we expect this relation to hold for all N. A rigorous proof of this relation based on the application of defect CFT techniques to surface operators is the given in Chapter 4.

Beyond the study of $\mathcal{N} = (2, 0)$ superconformal symmetry, surface operators in conformal field theories have drawn interest within a number of different contexts. Recent work on entangling surfaces in 4d [68–71] and theories with boundaries [72–74] uses some techniques which apply in our case as well. In particular, the classification of local conformal invariants of surfaces is independent of the codimension and translates to the 6d case [75].

Surface operators in the $\mathcal{N} = (2, 0)$ theory have been studied both from a field theory perspective [27, 65–67] and using holography [24, 64]. Corresponding soliton solutions of the M5-brane equations of motion have been discussed in the literature under the moniker of self-dual strings [39].

The resemblance to Wilson loops is evident in both the field theoretic and the holographic approach. In the abelian theory, as is studied in Section 3.3, we define the surface operator in analogy to the Maldacena-Wilson loops [58] as

$$V_\Sigma = \exp \int_\Sigma \left(\mathrm{i}B^+ - n^i \Phi_i \mathrm{vol}_\Sigma \right) , \tag{3.1}$$

where B^+ is the pullback of the chiral 2-form to the surface Σ and Φ^i are the scalar fields.

Since for $N > 1$ there is no realisation of the theory in terms of fundamental fields, we cannot give an analogous definition of the surface operator. However, by analogy with Wilson loops [54, 55, 58], in the large N limit, these operators in the fundamental representation have a nice holographic dual as M2-branes ending on the surface and extending into the $AdS_7 \times S^4$ bulk, as discussed in Section 3.4. In the absence of a scalar coupling breaking the $\mathfrak{so}(5)$ R-symmetry, these would be delocalised on the S^4 [76, 77]. At leading order, we need only consider minimal 3-volumes [58, 64] (similar to the minimal surfaces of interest in the Wilson loop case [54, 55, 58]), and to find the anomaly, which is a local quantity, it is enough to understand the volume close to the boundary of AdS. High-rank (anti-)symmetric representations are dual to configurations involving M5 branes shrinking to the surface on the boundary of AdS_7 and have been considered in [78–81].

The definition in (3.1) includes BPS operators. Simple examples are the plane or sphere with constant unit length n^i. Other examples are briefly discussed in Section 3.3 and have since been analysed in great detail [82]. We call operators with generic Σ and unit length n^i "locally BPS", and show that they possess some nice properties, in particular that all power law divergences cancel.

In the next section we recall the relation of surface operator anomalies to logarithmic divergences and introduce the anomaly coefficients. We evaluate these anomaly coefficients for the two known realisations of the $\mathcal{N} = (2, 0)$ theory; first as the theory of a single M5-brane ($N = 1$) [83], for which the equations of motion are known [84], and second, using holography (for the large N limit) from M-theory on the $AdS_7 \times S^4$ background [2] found in [56]. The resulting anomaly coefficients are presented in equations (3.37) and (3.64). After performing the free field and holographic calculations, we address in Section 3.5 surfaces with singularities. We discuss our results in Section 3.6 and offer a simple argument for the relation between two of the anomaly coefficients. We collect some technical tools in appendices. Our conventions can be found in Appendix A.1. Details of the geometry of submanifolds are compiled in Appendix A.2. Appendix A.3 contains an alternative, more geometric derivation of the field theory results in Section 3.3.

3.2 Surface anomalies

The most natural quantities associated to surface operators in conformal field theories are their anomaly coefficients. To understand their origin, note that, unlike line operators, the expectation values of surface operators typically suffer from ultraviolet divergences, which cannot be removed by the addition of local counterterms. The regularised expectation value satisfies

$$\log \langle V_\Sigma \rangle \sim \log \epsilon \int_\Sigma \mathrm{vol}_\Sigma \mathcal{A}_\Sigma + \text{finite}, \tag{3.2}$$

where ϵ is a regulator, $\mathcal{A}_\Sigma$ is known as the *anomaly density*, and we suppressed possible power-law divergences. $\mathcal{A}_\Sigma$ is scheme independent and indicates an anomalous Weyl symmetry, since for a constant rescaling $g \to e^{2\omega} g$, the expectation value varies as

$$\log \langle V_\Sigma \rangle_{e^{2\omega} g} - \log \langle V_\Sigma \rangle_g = \omega \int_\Sigma \mathrm{vol}_\Sigma \mathcal{A}_\Sigma \,, \tag{3.3}$$

where the subscript $\langle \bullet \rangle_g$ denotes the background metric.

As seen in Section 2.4, the anomaly is constrained by the Wess-Zumino consistency condition [52, 53] to be conformally invariant. In dimensions $d \geq 3$, the local geometric conformal invariants for a 2d submanifold, which have been classified in [75], are

R^Σ : The Ricci scalar of the induced metric h_{ab} on Σ.

$H^2 + 4\,\mathrm{Tr}\,P$: H^μ is the mean curvature, P_{ab} the pullback of the bulk Schouten tensor (A.16).

$\mathrm{Tr}\,W$: W_{abcd} is the pullback of the bulk Weyl tensor.

As we allow for variable couplings to the scalars, parametrised by a unit 5-vector n^i, we find an extra potential type B Weyl anomaly associated with it:

$$(\partial n)^2 \equiv \partial^a n^i \partial_a n_i.$$

This is (up to total derivatives) the only quantity of the correct dimension that can be constructed using only n.

The anomaly density of a surface operator in any 6d $\mathcal{N} = (2,0)$ theory then takes the form

$$\mathcal{A}_\Sigma = \frac{1}{4\pi} \left[a_1 R^\Sigma + a_2 \left(H^2 + 4\mathrm{Tr}P \right) + b\mathrm{Tr}W + c\,(\partial n)^2 \right]. \tag{3.4}$$

The anomaly coefficients a_1, a_2, b and c depend on the theory (on N, that is) and the type of surface operator (which, at least at large N, is specified by the representation of the A_{N-1} algebra [85, 86]), but not on its geometry or n. Computing these coefficients is the goal of this chapter.

Let us mention that there exists another commonly used basis where

$$\mathcal{A}_\Sigma = \frac{1}{4\pi} \left[a R^\Sigma + b_1 \mathrm{Tr}\tilde{\mathrm{I\!I}}^2 + b_2 \mathrm{Tr}W + c(\partial n)^2 \right], \tag{3.5}$$

where $\tilde{\mathrm{I\!I}}^{\mu}_{ab}$ is the traceless part of the second fundamental form (see (A.22)). These bases are related through the Gauss-Codazzi equation (A.21). The relation between the coefficients is then

$$a_1 = -b_1 + a\,, \qquad 2a_2 = b_1\,, \qquad b = b_2 + b_1\,, \tag{3.6}$$

$$a = a_1 + 2a_2\,, \qquad b_1 = 2a_2\,, \qquad b_2 = b - 2a_2\,. \tag{3.7}$$

Some results about these anomaly coefficients are known for surface defects in generic CFTs. The bound $b_1 < 0$ was derived in [70] by showing that b_1 captures the 2-point function of the displacement operator, which is positive by unitarity. Similarly, it was shown in [70, 87] that b_2 is calculated by the one-point function of the stress tensor in the presence of the surface defect (this was also conjectured in [88]). Assuming that the average null energy condition holds in the presence of defects leads to a bound $b_2 > 0$ [71].

For the surface operators at hand, some of these anomaly coefficients were also calculated previously. At large N, the first such result was obtained for the 1/2-BPS sphere [24], which has total anomaly $-4N$, implying $a_1^{(N)} + 2a_2^{(N)} = -2N$ to leading order at large N. This was soon followed by the more detailed result $a_2^{(N)} = -N$ and $a_1^{(N)} = b^{(N)} = 0$ [64].

More recently, it was conjectured that $\mathcal{N} = (2, 0)$ supersymmetry imposes $b = 0$ (or $b_1 = -b_2$) for any N [89], which we will prove in the next chapter. a and b_2 were calculated at any $N > 1$ (and for any representation) by studying the holographic entanglement entropy in the presence of surface operators [71, 81, 90, 91]. This result is also supported by a recent calculation based on the superconformal index [92], which suggests that it is exact.

To our knowledge, the anomaly coefficient c has not previously been discussed.

3.3 Abelian theory with $N = 1$

In this section we study the anomaly coefficients of the surface operator in the abelian (2, 0) theory. This is the theory of a single M5-brane and the degrees of freedom form the tensor supermultiplet of the $\mathfrak{osp}(8^*|4)$ symmetry algebra. It consists of three fields [93, 94] (see [95, 96] for an overview of superconformal multiplets in various dimensions):

- A real closed self-dual 3-form $H = \mathrm{d}B^+$.
- A chiral spinor $\psi_{\alpha\check{\alpha}}$ subject to the symplectic Majorana condition $\bar{\psi} = -c\Omega\psi$ (A.14) where c and Ω are charge conjugation matrices, see (A.12).
- Five real scalar fields Φ^i.

These fields transform into each other under superconformal transformations as [83]

$$\delta_\varepsilon B^+_{\mu\nu} = \varepsilon(x)\gamma_{\mu\nu}\psi\,, \tag{3.8}$$

$$\delta_\varepsilon\psi = -\gamma^\mu\partial_\mu\Phi^i\check{\gamma}_i\bar{\varepsilon}(x) + \frac{1}{12}\gamma^{\mu\nu\rho}H_{\mu\nu\rho}\bar{\varepsilon}(x) + 4\Phi^i\check{\gamma}_i\varepsilon^1\,, \tag{3.9}$$

$$\delta_\varepsilon\Phi^i = -\varepsilon(x)\check{\gamma}^i\psi\,. \tag{3.10}$$

The parameter $\bar{\varepsilon}(x)$ is an antichiral spinor of the form $\bar{\varepsilon}_{\dot{\alpha}\check{\alpha}}(x) = \bar{\varepsilon}^0_{\dot{\alpha}\check{\alpha}} + (\bar{\gamma}_\mu)_{\dot{\alpha}}{}^{\beta}x^\mu\varepsilon^1_{\beta\check{\alpha}}$, where $\bar{\varepsilon}^0$ and ε^1 are constant spinors parametrising, respectively, the supersymmetry and special supersymmetry transformations. Our spinor conventions are summarised in Appendix A.1.

3.3.1 Surface operators and BPS condition

We define the surface operators V_Σ of the abelian theory as in (3.1). We restrict to space-like surfaces in flat 6d Minkowski space (with mostly positive signature). Null surfaces could be interesting by analogy with null polygonal Wilson loops, which are dual to scattering amplitudes in $\mathcal{N}=4$ SYM [63] (see for instance [97]), but lie beyond the scope of this work.

A surface operator is BPS provided that its variation under the supersymmetry transformations (3.8) vanishes:

$$\delta_\varepsilon V_\Sigma = -\int \varepsilon(x)\left[\frac{\mathrm{i}}{2}\gamma_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \epsilon^{ab} - n^i \check{\gamma}_i\right]\psi\,\mathrm{vol}_\Sigma\, V_\Sigma = 0\,. \tag{3.11}$$

Since this is an integral over the insertion of an operator ψ along the surface, this is satisfied only when the integrand vanishes at every point, leading to the projector equation

$$\varepsilon\Pi_- = 0\,, \qquad \Pi_- = \frac{1}{2} - \frac{\mathrm{i}}{4}\partial_a x^\mu \partial_b x^\nu \epsilon^{ab}\frac{n^i}{n^2}\gamma_{\mu\nu}\check{\gamma}_i\,. \tag{3.12}$$

If we impose that $n^2 \equiv n^i n^i = 1$, then Π_- is a half rank projector, otherwise it is a full rank matrix. In the case of a planar surface with constant unit n^i, this is a single condition, so the surface preserves 16 supercharges, i.e. is 1/2-BPS.[1]

In analogy to Wilson loops in 4d theories, it is natural to discuss "locally BPS operators" [55], where the equations (3.12) admit a solution at every point along the surface, but a global solution does not necessarily exist. This amounts to the requirement $n^2 = 1$, and as shown below, leads to the cancellation of all power-like divergences in the evaluation of the surface operator.

One can also look for surfaces, other than planes, that preserve some smaller fraction of the supersymmetry by relating $n^i(\sigma)$ to $x^\mu(\sigma)$ and its derivatives. One simple way to realise this is for surfaces with the geometry $\mathbb{R}\times S$, for some curve $S \subset \mathbb{R}^{1,4}$. Upon dimensional reduction this becomes a Wilson loop in 5d maximally supersymmetric Yang-Mills (or 4d upon further dimension reduction). Then one can choose n^i to follow the construction of globally BPS Wilson loops of [11] or [12] to find globally BPS surface operators. Indeed this was realised recently in [98] (see also [99]).

There are further examples of globally BPS surface operators, which do not follow this construction. The simplest is the spherical surface, but there are several other classes of such operators [82].

3.3.2 Propagators

Since the abelian theory is non-interacting, the expectation value of V_Σ reduces to

$$\log\langle V_\Sigma\rangle = \frac{1}{2}\int\left[-\left\langle B^+(\sigma)B^+(\tau)\right\rangle + \langle\Phi_i(\sigma)\Phi_j(\tau)\rangle n^i(\sigma)n^j(\tau)\sqrt{h(\sigma)h(\tau)}d^2\sigma\, d^2\tau\right], \tag{3.13}$$

[1] The BPS condition for a surface operator extended in the time-like direction can be obtained by Wick-rotation to

$$V_\Sigma^{\text{timelike}} = \exp\left[\mathrm{i}\int_\Sigma B^+ - \Phi\mathrm{vol}_\Sigma\right].$$

where h is the determinant of the induced metric on Σ. Evaluating this requires expressions for the propagators of the tensor and scalar fields.

While one would preferably derive the propagators from an action, none is readily available. Many actions for the abelian $\mathcal{N} = (2,0)$ theory have been proposed over the years, but they all suffer from some pathologies regarding the self-dual 2-form (see [40, 83, 100–102] for examples of available actions, and [43, 44, 103] and references therein for recent accounts of the various approaches in the abelian theory). In any case, gauge fixing and inverting the kinetic operator is not straightforward.

Tensor structure

We sidestep these obstacles by determining the propagators in other ways. The scalar propagator in flat 6d is fixed by conformal symmetry to be

$$\langle \Phi_i(x)\Phi_j(y)\rangle = \frac{C_\Phi \delta_{ij}}{|x-y|^4}\,. \tag{3.14}$$

The proportionality constant depends on the normalisation of the fields. It could be determined from an action, but in its absence it is fixed by supersymmetry below.

The more complicated question is the self-dual 2-form propagator. Let us start by considering an unconstrained 2-form field B with a free Maxwell type action

$$S_{\text{tot}} \propto \int \mathrm{d}^6x\, B^{\mu\nu}\left(-(\delta_\mu^\rho\delta_\nu^\sigma - \delta_\nu^\rho\delta_\mu^\sigma)\partial^2 + 4(1-\alpha)\partial_\mu\partial^\rho\delta_\nu^\sigma\right) B_{\rho\sigma}\,, \tag{3.15}$$

were α is a gauge fixing parameter. In Feynman gauge $\alpha = 1$, this gives the propagator

$$\langle B^{\mu\nu}(x)B_{\rho\sigma}(y)\rangle = \frac{C_B(\delta_\mu^\rho\delta_\nu^\sigma - \delta_\nu^\rho\delta_\mu^\sigma)}{|x-y|^4}\,. \tag{3.16}$$

Now we decompose the field into its self-dual and anti-self-dual parts $B_{\mu\nu} = B^+_{\mu\nu} + B^-_{\mu\nu}$ and try to deduce the propagators for each component.

Since there is no covariant 4-tensor satisfying the self-duality properties of a mixed correlator $\langle B^+B^-\rangle$, we can decompose

$$\langle BB\rangle = \langle B^+B^+\rangle + \langle B^-B^-\rangle\,. \tag{3.17}$$

The two terms on the right hand side need not be identical, but the difference between them should be parity-odd.[2] The only such term of the right scaling dimension which we can write

[2] The two-dimensional analogue is instructive. The propagator of a free boson in complex coordinate z is given by

$$\langle \phi(z)\phi(0)\rangle = \log|z|^2\,,$$

while for a (anti-)chiral boson one finds

$$\langle \phi_+(z)\phi_+(0)\rangle = \log z\,, \qquad \langle \phi_-(z)\phi_-(0)\rangle = \log \bar{z}\,.$$

Indeed the sum reproduces the free boson propagator, but the two differ by a parity-violating imaginary part.

down is

$$\langle B^+B^+\rangle - \langle B^-B^-\rangle \propto \epsilon^{\mu\nu}{}_{\rho\sigma\kappa\lambda}\frac{x^\kappa y^\lambda}{|x-y|^6}. \tag{3.18}$$

However, terms of this type do not contribute to (3.13), since the integration is symmetric in x and y. Therefore, for the purpose of our calculation we can take $\langle B^+B^+\rangle = \langle BB\rangle/2$. Note that in curved space we can add to the right hand side a term proportional to the Weyl tensor with all the required symmetries.

Normalisation

The normalisation of the tensor field propagator is fixed by the assumption that the surface operator defined in (3.1) corresponds to a single unit of quantised charge. First, for any closed surface Σ, we can rewrite the surface operator (without scalars) in terms of the field strength as

$$\exp\int_\Sigma \mathrm{i}B^+ = \exp\int_V \mathrm{i}H, \tag{3.19}$$

where $\partial V = \Sigma$. In order for this to be well-defined, any two such V with the same boundary must yield the same result. Equivalently, for every closed 3-manifold V

$$\int_V H \in 2\pi\mathbb{Z}\,, \tag{3.20}$$

and similarly for $*H$.

Now consider a flat surface operator in the (x^1, x^2) plane, which we view as a source for the self-dual B field. The solution to the equations of motion would be given by convoluting the propagator with this source. Using the expression in (3.16) and adding the factor 1/2 to account for restricting to the self-dual sector, we get

$$B_{\mu\nu}(x) = \int_{\mathbb{R}^2}\frac{C_B(\delta^1_\mu\delta^2_\nu - \delta^1_\nu\delta^2_\mu)}{2|x-y|^4}\,\mathrm{d}^2y\,. \tag{3.21}$$

Again, because we don't know the self-dual propagator, the field strength we obtain is not self-dual, but the quantisation condition should still be satisfied. Imposing that the charge enclosed in a transverse sphere is quantised leads to

$$\int_{S^3} *H = 2\pi^3 C_B = 2\pi \qquad \Rightarrow \qquad C_B = \frac{1}{\pi^2}\,. \tag{3.22}$$

The normalisation of the scalar propagator is then fixed by supersymmetry. A simple way to implement that is to compare with the classical BPS solution of the self-dual string [39] which gives[3] $2C_\Phi = C_B$. Overall, we are left with

$$\langle\Phi_i(x)\Phi_j(y)\rangle = \frac{\delta_{ij}}{2\pi^2\,|x-y|^4}\,, \tag{3.23a}$$

$$\left\langle B^+_{\mu\nu}(x)B^+_{\rho\sigma}(y)\right\rangle = \frac{\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}}{2\pi^2\,|x-y|^4}\,. \tag{3.23b}$$

[3] The absence of power-law divergences in the calculation in the next section is also a hint that this is indeed the correct proportionality.

We emphasise that this normalisation is obtained by imposing a quantisation condition on the self-dual sector of an unconstrained B-field. This follows the discussion in [100], however some caution is warranted. A proper treatment of the quantisation of a self-dual two-form could remove the factor $1/2$ on the right hand side of (3.21), halving the resulting anomaly coefficients.

The flat space propagators are sufficient to determine the anomaly coefficients a_1, a_2, and c. The calculation of b, however, requires the curved space propagator, where the right-hand side of (3.18) could pick up contributions whose integral does not vanish. Since we do not know how to fix these terms, we cannot determine b.

Note though that we can calculate the contribution of the scalars to the anomaly coefficient b. The propagator of a conformal scalar in a curved background can be expanded in powers of the geodesic distance [27], and the contribution to the anomaly coefficient b is read off as $-1/3$.

If we give up the requirement of self-duality, we can use the short-distance expansion of an unconstrained 2-form propagator on curved space, which has been computed in [65, 67], and again, the Weyl tensor of the background explicitly contributes to the curvature corrections. Halving that in order to account for self-duality and adding to it the contribution from the scalars, one obtains $b = -4/3$ [27]. This is in disagreement with the conjecture $b = 0$ [89] and should therefore be taken with a grain of salt.

3.3.3 Evaluation of the anomaly

With the propagators at hand, we can compute the expectation value of the surface operator by evaluating the integrals in (3.13). Generically, these integrals are divergent and must be regularised.

In this section we take a rather naive approach of placing a hard UV cutoff on the double integral (3.13), so as to restrict $|\sigma - \tau| > \epsilon$ (where the distance is measured with respect to the induced metric), the same regularisation that is used in [65]. A different regularisation is employed in [27], where the surface is assumed to be contained within a 5d linear subspace of $\mathbb{R}^6$ and the two copies of the surface are displaced by a distance ϵ in the 6th direction. This restriction to $\mathbb{R}^5$ must still yield the correct answer, since even for surfaces in 4d the geometric invariants in the anomaly (3.4) are independent of each other. Still, in Appendix A.3 we redo the calculation removing this assumption by displacing the two copies of the surface along geodesics in the direction of an arbitrary normal vector field. That approach could be important for the calculation of surface operators in four dimensions, where the restriction to a 3d linear subspace does not allow to resolve all the anomaly coefficients.

To find the anomalies we only need the short-distance behaviour of the propagators, so we use normal coordinates η^a about a point σ on Σ. The notations and required geometry are presented in Appendix A.2.

Starting from the scalar contribution to (3.13), the integrand is

$$\frac{1}{4\pi^2} \frac{n^i(\sigma) n^i(\tau)}{|x(\sigma) - x(\tau)|^4} \sqrt{h(\sigma)} \sqrt{h(\tau)} \,. \tag{3.24}$$

Using $n^i n^i = 1$ and (A.28), (A.26) we have

$$n^i(\sigma)n^i(\tau) = 1 - \frac{1}{2}\left(\partial_a n^i \partial_b n^i\right)\eta^a\eta^b + \mathcal{O}(\eta^3)\,, \tag{3.25}$$

$$\sqrt{h(\tau)} = 1 - \frac{1}{6}R^{\Sigma}_{ab}\eta^a\eta^b + \mathcal{O}(\eta^3)\,, \tag{3.26}$$

$$|x(\sigma) - x(\tau)|^2 = \eta^a\eta_a - \frac{1}{12}\mathrm{II}_{ab}\cdot\mathrm{II}_{cd}\eta^a\eta^b\eta^c\eta^d + \mathcal{O}(\eta^5)\,. \tag{3.27}$$

The integral computing the density of the scalar contribution to $\log\langle V_\Sigma\rangle$ is then

$$\frac{1}{4\pi^2}\int\frac{\mathrm{d}^2\eta}{|\eta|^4}\left[1 - \left(\frac{1}{6}R^{\Sigma}_{ab} + \frac{1}{2}\partial_a n^i\partial_b n^i\right)\eta^a\eta^b + \frac{1}{6|\eta|^2}\mathrm{II}_{ab}\cdot\mathrm{II}_{cd}\eta^a\eta^b\eta^c\eta^d + \mathcal{O}(\eta^3)\right]. \tag{3.28}$$

Using polar coordinates $\eta^a = \eta\, e^a(\varphi)$, where e is a 2d unit vector, and the identities

$$\int_0^{2\pi}\mathrm{d}\varphi\, e^a e^b = \pi\delta^{ab}, \qquad \int_0^{2\pi}\mathrm{d}\varphi\, e^a e^b e^c e^d = \frac{\pi}{4}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right), \tag{3.29}$$

we are left with the radial integral, for which we introduce the cutoff ϵ

$$\frac{1}{2\pi}\int_\epsilon\frac{\mathrm{d}\eta}{\eta^3}\left(1 - \frac{\eta^2}{48}\left(4R^\Sigma + 12\,(\partial n)^2 - H^2 - 2\mathrm{II}^{ab}\cdot\mathrm{II}_{ab}\right) + \mathcal{O}(\eta^3)\right) \tag{3.30}$$

$$= \frac{1}{4\pi\epsilon^2} + \frac{1}{32\pi}\left(2R^\Sigma - H^2 + 4\,(\partial n)^2\right)\log\epsilon + \text{finite}. \tag{3.31}$$

To get the expression in the second line we also used the Gauss-Codazzi equation (A.21).

The calculation of the contribution of the 2-form field is very similar. Expanding the tensor structure, we have

$$\frac{1}{2}\big\langle B^+(\sigma)B^+(\tau)\big\rangle = \frac{1}{8\pi^2}\frac{\delta_{\mu\rho}\delta_{\nu\sigma}}{|x(\sigma)-x(\tau)|^4}\,\mathrm{d}x^\mu(\sigma)\wedge\mathrm{d}x^\nu(\sigma)\otimes\mathrm{d}x^\rho(\tau)\wedge\mathrm{d}x^\sigma(\tau)\,. \tag{3.32}$$

In terms of η^a, the differential forms read (see (A.23))

$$\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\big|_\sigma = \varepsilon^{ab}v^\mu_a v^\nu_b\mathrm{d}^2\eta\,, \tag{3.33}$$

$$\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\big|_\tau = \varepsilon^{cd}\left(v^{[\rho}_c v^{\sigma]}_d + 2v^{[\rho}_c v^{\sigma]}_{de}\eta^e + \left(v^{[\rho}_{ce}v^{\sigma]}_{df} + v^{[\rho}_c v^{\sigma]}_{def}\right)\eta^e\eta^f + \mathcal{O}(\eta^3)\right)\mathrm{d}^2\eta\,. \tag{3.34}$$

Collecting terms and introducing a radial cutoff as above, we find the contribution

$$-\frac{1}{4\pi\epsilon^2} - \frac{1}{32\pi}\left(-2R^\Sigma + 3H^2\right)\log\epsilon + \text{finite}. \tag{3.35}$$

Finally, combining (3.31) and (3.35) we find that the quadratic divergences cancel and we are left with

$$\log\langle V_\Sigma\rangle = \frac{1}{8\pi}\log\epsilon\int_\Sigma\mathrm{vol}_\Sigma\left[R^\Sigma - H^2 + (\partial n)^2\right] + \text{finite}. \tag{3.36}$$

Comparing to (3.4), we can read off the anomaly coefficients

$$a_1^{(1)} = +\frac{1}{2}\,, \qquad a_2^{(1)} = -\frac{1}{2}\,, \qquad c^{(1)} = +\frac{1}{2}\,. \tag{3.37}$$

As discussed above, since we do not know the contribution of the Weyl tensor to the B-field propagator, we cannot determine $b^{(1)}$. According to the conjecture of [89] however, it should vanish.

Equation (3.36) differs from (3.4) by the absence of the $\mathrm{Tr}P$ term, which vanishes in flat space. Since H^2 doesn't vanish in flat space, it determines a_2 unambiguously. In curved space H^2 is necessarily accompanied by $4\mathrm{Tr}P$, as seen in Appendix A.3.

Finally, we reiterate that, depending on the form of the quantisation condition, the result for the anomaly coefficients may be divided by 2, see the discussion following (3.23). In any case, the abelian theory should have surface operators with an integer multiple of $iB^+ - n^i\Phi_i$ in (3.1), and for all of them it is still true that $a_1^{(1)} = -a_2^{(1)} = c^{(1)}$.

3.3.4 Generalising the scalar coupling

Note that the preceding calculation is applicable regardless of whether the operator is locally BPS or not, so we may relax the condition $n^2 = 1$. In that case the result for the anomaly coefficients is

$$a_1^{(1)} = \frac{n^2+1}{4}\,. \qquad a_2^{(1)} = -\frac{n^2+3}{8}\,, \qquad c^{(1)} = \frac{1}{2}\,. \tag{3.38}$$

If we replace $n^i \to in^i$, we recover the surface operator studied in [27]. An operator with $n^2 = 0$ was also studied in [65], but assuming a non-self-dual 2-form. The anomaly coefficients computed in [27,65] are respectively twice and four times the ones we obtain by substituting the values of n in (3.38), due to a difference in the overall normalisation of the propagator.

It would be interesting to study this system in the large n^2 limit. This is similar to the "ladder" limit of the cusped Wilson loop in $\mathcal{N} = 4$ SYM in 4d first suggested in [104] which is related to a special scaling limit of that theory, dubbed the "fishnet" model, a 6d version of which was discussed in [105].

3.4 Holographic description at large N

The holographic calculation of the Weyl anomaly for surface operators was pioneered by Graham and Witten in [64]. Here we present a rewriting of their argument, which we also generalise slightly to include operators extended on the S^4.

3.4.1 Surface operators

The $\mathcal{N} = (2,0)$ theory is described at large N by 11d supergravity on an asymptotically $AdS_7 \times S^4$ geometry [2]

$$\mathrm{d}s^2 = \frac{L^2}{y^2}\left(\mathrm{d}y^2 + g^{(0)} + g^{(1)}y^2\right) + \frac{L^2}{4}g^{(0)}_{S^4} + \mathcal{O}(y^2)\,, \qquad L = (8\pi N)^{1/3}\,l_P\,, \tag{3.39}$$

such that $g^{(0)}$ is the metric of the dual field theory[4] and $g^{(0)}_{S^4}$ is the metric of S^4. The background includes N units of F_4 flux

$$\frac{1}{(2\pi)^2 l_P^3}\int_{S^4} F_4 = 2\pi N\,. \tag{3.40}$$

The full form of the metric is determined by the supergravity equations of motion in the presence of these fluxes and by requiring the geometry to close smoothly in the interior. While the latter requires nonlocal information, the near-boundary expansion is fixed to the required order by local information about the boundary. Following [106, 107], the first term in this expansion was found in [64] as

$$g^{(1)}_{\mu\nu} = -P^{(0)}_{\mu\nu} \equiv -P_{\mu\nu}\big|_{g=g^{(0)}}\,. \tag{3.41}$$

At this order the S^4 is round, so to leading order the solution to (3.40) is simply

$$F_4 = \frac{3}{8}L^3 \mathrm{vol}_{S^4}\,. \tag{3.42}$$

The holographic description of the surface operators (3.1) is by M2-branes anchored along Σ on the boundary of AdS [58]. Using $\hat{\Sigma}$ for the world-volume of the M2-brane, it has a boundary at $y = 0$ with $\partial\hat{\Sigma} = \Sigma$. The expectation value of the surface operators is then given by the minimum of the M2-brane action, reading (in Euclidean signature and with all fermionic terms suppressed) [108]

$$\log\langle V_\Sigma\rangle \simeq -S_{\mathrm{M2}} = -T_{\mathrm{M2}}\int_{\hat{\Sigma}}(\mathrm{vol}_{\hat{\Sigma}} + \mathrm{i}A_3)\,, \qquad T_{\mathrm{M2}} = \frac{1}{4\pi^2 l_P^3} = \frac{2N}{\pi L^3}\,, \tag{3.43}$$

where T_{M2} is the tension of the brane, proportional to N. $\mathrm{vol}_{\hat{\Sigma}}$ is the volume form calculated from the induced metric and A_3 is the pullback of the 3-form potential.

3.4.2 Local supersymmetry

Before studying the M2-brane embeddings, let us note that the M2-brane minimizing (3.43) is also locally supersymmetric. The supergravity fields appearing there sit in multiplet with supersymmetry transformations

$$\delta A_{MNP} = -3\bar{\varepsilon}\Gamma_{[MN}\Psi_{P]}\,, \tag{3.44}$$

$$\delta\Psi_M = D_M\varepsilon + \frac{1}{288}\left(\Gamma^{PQRS}{}_M - 8\Gamma^{QRS}\delta^P_M\right)F_{PQRS}\varepsilon\,, \tag{3.45}$$

$$\delta E^{\bar{M}}_M = \bar{\varepsilon}\Gamma^{\bar{M}}\Psi_M\,, \tag{3.46}$$

where $E^{\bar{M}}_M$, Ψ_M and A_3 are respectively the vielbein, gravitino and 3-form potential of F_4 ($\bar{M} = 1, \ldots, 11$ is the frame index). Using these transformations, the variation of (3.43) is

$$\delta_\varepsilon S = T_{\mathrm{M2}}\int_{\hat{\Sigma}}\bar{\varepsilon}\left(\Gamma^{\hat{a}} - \frac{\mathrm{i}}{2}\varepsilon^{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{b}\hat{c}}\right)\Psi_{\hat{a}}\mathrm{vol}_{\hat{\Sigma}} = 0\,. \tag{3.47}$$

[4] Or in the same conformal class.

We here denote the coordinates on the world-volume by $\hat{\sigma}^{\hat{a}}$. The projector equation is then

$$\bar{\varepsilon}\Pi_- = 0, \quad \Pi_- = \frac{1}{2}\left[1 - \frac{\mathrm{i}}{6}\varepsilon^{\hat{a}\hat{b}\hat{c}}\Gamma_{\hat{a}\hat{b}\hat{c}}\right]. \tag{3.48}$$

The projector is again half-rank, so that the M2-brane locally preserves half of the supersymmetries (16 supercharges). These supercharges can be shown to agree with the field theory BPS condition (3.12) on Σ once we decompose x^M into coordinates on the boundary of AdS, x^μ, and the S^4 coordinates n^i.

3.4.3 Holographic calculation

To find the saddle points of the action (3.43), we parametrise the M2-brane by y, σ^a where σ^a are coordinates for Σ. We then use the static gauge to describe the embedding by $\{u^{a'}(y,\sigma), n^i(y,\sigma)\}$, where $u^{a'}$ are the normal directions to the surface Σ at $y = 0$. In this setup, the boundary conditions are $u^{a'}(y = 0, \sigma) = 0$ and $n^i(y = 0, \sigma) = n^i(\sigma)$ (where the right hand side has the n^i from (3.1)).

Because the metric (3.39) diverges at the boundary of AdS, the volume element on the M2-brane diverges as y^{-3}, which leads to divergences in the action. Finding the shape of the embedding requires knowledge of the full surface and is generally a hard problem. But since we are only interested in the logarithmically divergent part of the action, it is sufficient to solve the equations of motion for small y. We do this perturbatively following [64], mirroring the solution of the background supergravity equations above.

Using (3.41), the lowest order terms in the metric for our coordinates normal and tangent to the surface, are

$$g_{ab}(y,\sigma,u) = h_{ab} - P^{(0)}_{ab}y^2 + \partial_{a'}g^{(0)}_{ab}\Big|_{u=0}u^{a'} + \mathcal{O}(y^4, u^2)\,, \tag{3.49}$$

$$g_{aa'}(y,\sigma,u) = \mathcal{O}(y^2, u)\,, \tag{3.50}$$

$$g_{a'b'}(y,\sigma,u) = g^{(0)}_{a'b'}\Big|_{u=0} + \mathcal{O}(y^2, u)\,. \tag{3.51}$$

Here $h_{ab} = g^{(0)}_{ab}\Big|_{u=0}$ is the metric on Σ. Note that away from $y = 0$, this metric depends on $u^{a'}$ (for $y \neq 0$, generically $u^{a'} \neq 0$), as in the first line.

To write down the M2-brane action we need the induced metric $\hat{h}_{ab} = \partial_a X^M \partial_b X^N g_{MN}$ (including also the S^4 directions). We expand the embedding coordinates as

$$u^{a'}(y,\sigma) = \mathcal{O}(y^2)\,, \tag{3.52}$$

$$n^i(y,\sigma) = n^i(\sigma) + \mathcal{O}(y^2)\,. \tag{3.53}$$

It is easy to check that higher order terms are not required. Then the S^4 metric can be replaced with $g^{(0)}_{S^4} = \delta_{ij}\mathrm{d}n^i\mathrm{d}n^j$ and the second fundamental form is $\mathrm{I\!I}^{a'}_{ab} = -\frac{1}{2}g^{a'b'}\partial_{b'}g_{ab}$.

Dropping the explicit $\mathcal{O}(y^\star)$ as well as the subscript $|_{u=0}$ along with the superscript $^{(0)}$, since all the quantities are evaluated on the surface, we find

$$\hat{h}_{yy} \simeq \frac{L^2}{y^2}\left[1+\partial_y u^{a'}\partial_y u^{b'} g_{a'b'}\right], \tag{3.54}$$

$$\hat{h}_{ay} \simeq 0\,, \tag{3.55}$$

$$\hat{h}_{ab} \simeq \frac{L^2}{y^2}\left[h_{ab}+\left(-P_{ab}+\frac{1}{4}\partial_a n^i \partial_b n^j \delta_{ij}\right)y^2 - 2\mathrm{II}^{a'}_{ab}u^{b'}g_{a'b'}\right]. \tag{3.56}$$

The determinant of the metric is then

$$\det\hat{h} \simeq \frac{L^6}{y^6}\left(1+\partial_y u^{a'}\partial_y u^{b'}g_{a'b'} - 2H^{a'}u^{b'}g_{a'b'} + \left(-\mathrm{Tr}P+\frac{1}{4}\left(\partial n\right)^2\right)y^2\right)\det h\,, \tag{3.57}$$

while the pullback of the 3-form

$$A_3 = \frac{1}{3!}A_{ijk}\,\mathrm{d}n^i\wedge \mathrm{d}n^j \wedge \mathrm{d}n^k \sim \mathcal{O}(y)\,, \tag{3.58}$$

does not contribute to the divergences. We thus find the action

$$S_{\mathrm{M2}} \simeq \frac{L^3}{\left(2\pi\right)^2 l_P^3}\int_\Sigma \mathrm{vol}_\Sigma \int\limits_{y\geq\epsilon} \frac{\mathrm{d}y}{y^3}\left[1+\frac{1}{2}\left(\partial_y u^{a'}\right)^2 - H\cdot u + \left(-4\mathrm{Tr}P + (\partial n)^2\right)\frac{y^2}{8}\right]. \tag{3.59}$$

At order $\mathcal{O}(y^2)$, we need only solve for $u^{a'}(y)$, which has the equation of motion

$$y^3\partial_y\left(y^{-3}\partial_y u^{a'}\right)+H_{a'}\simeq 0 \qquad\Rightarrow\qquad u^{a'}\simeq\frac{1}{4}H^{a'}y^2\,. \tag{3.60}$$

The action evaluated at the classical solution is then

$$S_{\mathrm{M2}} \simeq \frac{L^3}{\left(2\pi\right)^2 l_P^3}\int_\Sigma \mathrm{vol}_\Sigma \int\limits_{y\geq\epsilon} \frac{\mathrm{d}y}{y^3}\left[1-\frac{y^2}{8}\left(H^2+4\mathrm{Tr}P\right)+\frac{y^2}{8}\left(\partial n\right)^2\right], \tag{3.61}$$

and we see that the anomaly indeed takes the form (3.4). The result is

$$\log\langle V_\Sigma\rangle = \frac{N}{4\pi}\log\epsilon\int_\Sigma \mathrm{vol}_\Sigma\left[-\left(H^2+4\mathrm{Tr}P\right)+(\partial n)^2\right]\log\epsilon + \text{finite}, \tag{3.62}$$

where we discarded an irrelevant term proportional to ϵ^{-2} (see the discussion below).

This result agrees with the original calculation of [64] and adds to it the coupling to $(\partial n)^2$. It is also consistent with the explicit calculation of the 1/2-BPS sphere [24], for which the anomaly is $-4N$. The anomaly coefficients at leading order in N are then

$$a_1^{(N)} = \mathcal{O}(N^0)\,, \qquad\qquad b^{(N)} = \mathcal{O}(N^0)\,, \tag{3.63}$$

$$a_2^{(N)} = -N+\mathcal{O}(N^0)\,, \qquad\qquad c^{(N)} = +N+\mathcal{O}(N^0)\,. \tag{3.64}$$

As in the case of Wilson loops in $\mathcal{N}=4$ SYM in 4d, we expect this holographic description to be correct in the locally BPS case when the scalar couplings satisfy $n^2=1$. Following [76, 77], the case of $n^2=0$ should be described by the same surface inside AdS_7, but completely smeared over the S^4. In this case we find the same result for the geometric anomaly coefficients as above, and, since the corresponding anomaly term vanishes identically, $c^{(N)}$ does not apply.

Power-law divergence

Note that in addition to the log divergence in (3.62), (3.61) produces also a power-law divergence

$$\frac{L^3}{(2\pi)^2\, l_P^3}\frac{\text{Area}(\Sigma)}{2\epsilon^2}\,. \tag{3.65}$$

While such divergences can be removed by the addition of a local counter-terms, in the field theory result (3.36), they cancelled without extra counter-terms (for the locally BPS operator).

A more elegant way of eliminating the power law divergences also in this holographic calculation follows the example of the locally BPS Wilson loops [55]. A careful treatment of the boundary conditions suggests that the natural action is a Legendre transform of (3.43), which differs from the action we used by a total derivative. This modification does not change the equations of motion, but gives a contribution on the boundary, where it precisely cancels the divergence above.

By looking at the M5-brane metric before the decoupling limit, we can identify the coordinate to use in the transform as $r^i = L^3 n^i/2y^2$. Defining its conjugate momentum by differentiating with respect to the boundary value of the coordinate (where $y = \epsilon$)

$$p_i(\sigma) = \frac{\delta S[x^\mu, r^i]}{\delta r^i} = -\frac{\epsilon^3 n^i}{L^3}\frac{\delta S[x^\mu, n^i, \epsilon]}{\delta \epsilon} = \frac{\epsilon^3 n^i}{L^3}\frac{L^3}{(2\pi)^2 l_P^3}\left(\frac{1}{\epsilon^3} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\right). \tag{3.66}$$

In the last equality we used the value of the classical action (3.61), undoing the integration, so the classical Lagrangian density.

The Legendre transformed action is then

$$\tilde{S}\left[x^\mu, p^i\right] = S\left[x^\mu, r^i\right] - \int_\Sigma p_i r^i \text{vol}_\Sigma = S\left[x^\mu, n^i, \epsilon\right] - \frac{L^3}{2(2\pi)^2 l_P^3 \epsilon^2}\int_\Sigma \text{vol}_\Sigma\,. \tag{3.67}$$

The last term exactly cancels the power law divergence in (3.65).

3.5 Surfaces with singularities

An interesting class of surface operators that has received some attention recently are surfaces with conical singularities. For these surfaces, it was found that the regularised expectation value typically diverges as [69, 109–111]

$$\log\langle V_{\Sigma_c}\rangle \sim A\log^2\epsilon + \mathcal{O}(\log\epsilon)\,. \tag{3.68}$$

Let us consider a conical defect (in flat space) of the form

$$x^\mu(r,s) = r\gamma^\mu(s)\,, \qquad \gamma^2 = 1\,, \qquad n^i(r,s) = \nu^i(s)\,. \tag{3.69}$$

We allow here also a "conical singularity" in the scalar couplings, which has s dependence even as $r \to 0$. It is possible to also allow x^μ and n^i to have higher order terms in r, but since those lead to subleading divergences, they are unimportant.

We can try to use the usual formula for the anomaly (3.4) by plugging in the geometric invariants

$$R^{\Sigma} = \Omega\delta(r)\,, \qquad H^2 = \frac{\kappa^2-1}{r^2}\,, \qquad (\partial n)^2 = \frac{(\partial_s\nu)^2}{r^2}\,, \tag{3.70}$$

where Ω is the deficit angle, and $\kappa = \ddot{\gamma}^2/|\dot{\gamma}|^2$ is the curvature of γ. Plugging into (3.4), the Ricci scalar gives a finite contribution, but H^2 and $(\partial n)^2$ diverge as $r \to 0$. Introducing a cutoff $\hat{\epsilon}$ on the r integration, this gives

$$\frac{1}{4\pi}\log\epsilon\log\hat{\epsilon}\int_\gamma a_2\left(1-\kappa^2(s)\right) - c(\partial_s\nu)^2\mathrm{d}s + \mathcal{O}(\log\epsilon)\,. \tag{3.71}$$

This expression is somewhat naive, as we should treat all divergences on the same footing and identify $\hat{\epsilon} = \epsilon$. But then we should not use (3.4) in the first place. Rather, we should go back one step and regularise the divergences that gave rise to the original $\log\epsilon$ divergence while also applying it to the r integration. As we show below, this leads to the expression in (3.71) with $\log\epsilon\log\hat{\epsilon} \to \frac{1}{2}\log^2\epsilon$. In both the free field case and the holographic realisation this factor of $1/2$ is a simple consequence of the usual coefficient of the quadratic term in the Taylor expansion, or, in other words, of an integral of the form $\int \log r\,\mathrm{d}\log r$.

This factor of $1/2$ was noticed already in the calculations of [69,109] and justified in [111] by a careful treatment of the holographic calculation, which is repeated below. It was also studied in the context of defect CFT in [70]. We think that the comparison of this to the free-field calculation and the universal nature of our result further elucidates this mismatch from the naive expectation. Our calculation is also more generic, for allowing arbitrary conical singularities and incorporating singularities of the scalar coupling.

Beside this factor $1/2$, it is interesting to compare the $\log^2\epsilon$ divergence of surface operators to the $\log\epsilon$ divergence of Wilson loops, the cusp anomalous dimension. In $\mathcal{N}=4$ SYM, the cusp anomalous dimension is a complicated function of the opening angle ϕ. At small angles, it is related to the Bremsstrahlung function, which encodes the radiation emitted by heavy probe particles and depends on the 't Hooft coupling λ and the rank of the gauge group N. It is therefore an interesting quantity to compute, and the exact Bremsstrahlung function has been obtained using supersymmetric localization in [49].

In constrast, here the functions a_2 and c seem to be simple functions of N, and the expression (3.74) is not an approximation for small angles, but the exact result. The relation to physical quantities is unclear as well. It would be interesting to interpret it as a Bremsstrahlung function, but computing the radiation emitted by a probe string in 6d would require a more careful treatment of the self-dual field strength.

We should also note, as already noticed in [69], that surfaces with "creases", i.e. codimension one singularities, do not lead to additional $\log^2\epsilon$ divergences and the expression (3.4) can be immediately applied to them.

3.5.1 Field theory

Here we do not rely on (3.36), but go further back to where the $\log\epsilon$ arises from an integral of the form (3.31)

$$\int_\epsilon^\rho \frac{\mathrm{d}\eta}{\eta} = -\log\epsilon + \text{finite}, \tag{3.72}$$

where η is a radial coordinate around the point x, and ρ is an IR cutoff related to the overall size of the surface, or at least a large smooth patch where we defined our local coordinate. Near the cone the smooth patch is bounded by the distance from x to the apex, which we denote by r. The integral instead gives

$$\int_\epsilon^r \frac{\mathrm{d}\eta}{\eta} = -\log\frac{\epsilon}{r}\,. \tag{3.73}$$

With this careful treatment of the log, we can go back to (3.4), plug in the expressions from (3.70) and integrate over r and with the same UV cutoff to find

$$\begin{aligned}\log\langle V_\Sigma\rangle &= -\frac{1}{4\pi}\int_\gamma \mathrm{d}s \int_\epsilon \frac{\mathrm{d}r}{r}\left[a_2\left(1-\kappa^2\right) - c(\partial_s n)^2\right]\log\frac{\epsilon}{r} + \text{finite} & (3.74)\\ &= \frac{1}{8\pi}\log^2\epsilon\int_\gamma \left[a_2(1-\kappa^2(s)) - c(\partial_s\nu)^2\right]\mathrm{d}s + \mathcal{O}(\log\epsilon)\,. & (3.75)\end{aligned}$$

3.5.2 Holography

The derivation in holography is similar. We first note that conformal symmetry fixes the form of the solution as

$$y(r,s) = ru(s) \tag{3.76}$$

To get to (3.62), we integrate over y, but the conformal ansatz suggests to impose the range $\epsilon \le y \le ru_{\max}$. Plugging the curvatures from (3.70) into equation (3.62) we arrive at

$$\log\langle V_\Sigma\rangle = -\frac{1}{4\pi}\int_\gamma \mathrm{d}s\int_\epsilon \frac{\mathrm{d}r}{r}\left[a_2(1-\kappa^2) - c(\partial_s n)^2\right]\log\frac{\epsilon}{ru_{\max}(s)} + \text{finite}. \tag{3.77}$$

which again gives the $\log^2\epsilon$ divergence with the same $1/2$ prefactor as in field theory (3.74).

3.5.3 Example: circular cone

As a simple example of a singular surface we compute explicitly the anomaly of a circular cone. Denoting the deficit angle by ϕ (see figure 3.1) and including an internal angle θ for the scalar coupling n^i, we parametrise the cone as follows

$$\gamma^\mu(s) = \begin{pmatrix}\cos\phi\sin s\\ \cos\phi\cos s\\ \sin\phi\end{pmatrix}, \qquad n^i(s) = \begin{pmatrix}\sin\theta\sin s\\ \sin\theta\cos s\\ \cos\theta\end{pmatrix}, \qquad 0\le r\,, \quad 0\le s<2\pi\,. \tag{3.78}$$

The conformal invariants are explicitly

$$\kappa^2 = \frac{1}{\cos^2\phi}\,, \qquad (\partial_s n)^2 = \frac{\sin^2\theta}{\cos^2\phi}\,. \tag{3.79}$$

The divergence is then

$$\log\langle V_\Sigma\rangle = -\frac{a_2\sin^2\phi + c\sin^2\theta}{4\cos\phi}\log^2\epsilon + \mathcal{O}(\log\epsilon)\,. \tag{3.80}$$

Notice that as long as the anomaly coefficients satisfy the relation $a_2 = -c$, which we have shown to hold in the abelian and large N case, the anomaly vanishes for configurations $\theta = \pm\phi$, which correspond generically to 1/8-BPS configurations.

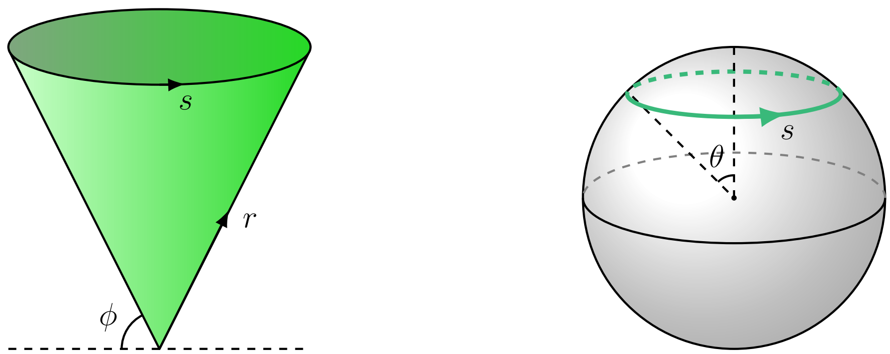


Figure 3.1: On the left, the surface wraps a (circular) cone with a deficit angle ϕ. On the right, the scalar coupling follows a circle at angle θ on S^2. For a fixed r, we have a curve that simultaneously traces the circles $\gamma(s)$ and $n^i(s)$.

3.6 Discussion

Making all N conjectures based on the asymptotics is a fools errand. This is especially true given that the abelian theory is not the same as the A_{N-1} theory at $N = 1$, since the latter is the empty theory. Nevertheless, both in field theory and holography we see that $a_2 = -c$, and there is some previous evidence for this to hold generally. The argument is based on the BPS Wilson loops of [11], where n^i is parallel to $\dot{x}^\mu$ and which have trivial expectation values. If we uplift them to the 6d theory we expect to find surface operators with no anomaly (and vanishing finite part as well). These operators satisfy $H^2 = (\partial n)^2$ and indeed they do not contribute to the anomaly[5] for $a_2 = -c$. The rigorous proof of this relation, based on defect CFT techniques, is the subject of the following chapter.

All our calculations are for a surface operator in the fundamental representation. It is expected that 1/2-BPS surface operators are classified by representations of the A_{N-1} algebra of the theory. At large N this is proven, since the asymptotically $AdS_7 \times S^4$ solutions of 11d supergravity preserving the symmetry algebra of 1/2-BPS surface operators can be classified in terms of Young diagrams [85, 86, 112].

[5] In the uplift we find only surfaces with trivial topology, so the anomaly vanishes regardless of a_1.

A calculation of anomalies of surface operators in arbitrary representations, based on bubbling geometries and holographic entanglement entropy, was undertaken in [91]. If we assume $b = 0$, then for the fundamental representation, their result reads

$$a_1^{(N)} = \frac{1}{2} - \frac{1}{2N}\,, \qquad a_2^{(N)} = -N + \frac{1}{2} + \frac{1}{2N}\,. \tag{3.81}$$

This is supported by an independent calculation using the superconformal index [92]. In the large N limit, this is indeed in agreement with our result.

The anomalies studied here are the most basic properties of surface operators, but their calculation is only a first step in understanding these observables and the mysterious theory they belong to. Planar and spherical surface operators preserve a part of the conformal group (and, when endowed with appropriate scalar coupling, also half the supersymmetries), such that their deformations behave like operators in a defect CFT. In the next chapter, we study this dCFT in some detail.

Chapter 4

The surface dCFT

4.1 Introduction

Having established a definition of locally BPS surface operators in the $\mathcal{N} = (2,0)$ theory in the previous chapter, we now study them in more detail, using the framework of defect CFT. This chapter is a slightly edited version of our publication [29].

We adopt the approach of the conformal bootstrap program [113–118] and use the symmetries preserved by the surface operators to constrain their correlators with other bulk operators, as well as local operator insertions on the surface. One of the virtues of this description is that it does not rely on a field realisation and therefore is applicable at any N.

We focus on 1/2-BPS defects because they preserve the largest amount of symmetry. These are surface operators defined over a plane and expected to be labeled by a representation of the ADE group of the $\mathcal{N} = (2,0)$ theory [85, 86, 119]. We consider local operator insertions into the defect, the simplest example encoding an infinitesimal geometric deformation of the plane itself. Because the plane preserves superconformal symmetry, the correlators of local operator insertions are constrained and obey the axioms of a dCFT—the 2- and 3-point functions are fixed up to a small set of numbers defining the dCFT.

Explicitly, consider a correlator involving such a surface operator V. While translating the plane along parallel directions leaves the correlator invariant, translations in directions transverse to the plane do not. Instead, the stress tensor receives a contribution from a contact term localised on the defect (at $x = 0$):

$$\partial_\mu T^{\mu m}(\sigma, x)\, V = V[\mathbb{D}^m(\sigma)]\delta^{(4)}(x). \tag{4.1}$$

The index $\mu = 1, \ldots, 6$ runs over all spacetime coordinates, while $m = 1, \ldots, 4$ are the coordinates transverse to the plane. We use the notation $V[\hat{\mathcal{O}}(\sigma)]$ to denote the planar surface operator with a defect operator $\hat{\mathcal{O}}$ inserted at a point σ on the plane.

Equation (4.1) is an operator equation, so it holds inside correlation functions. It defines $\mathbb{D}$, known as the displacement operator. In addition, because V preserves some supersymmetries, the displacement operator sits in a multiplet containing also contact terms for the divergence of the broken super- and R-current, which we label $\mathbb{Q}$ and $\mathbb{O}$, respectively.

It turns out that these defect operators play a pivotal role: not only are they highly constrained by the residual symmetry (which includes the 2d rigid superconformal symmetry[1]), but they also correspond to interesting physical quantities [70, 87]. Indeed it is easy to show that, as a consequence of (4.1), the insertion of a displacement operator $\mathbb{D}$ corresponds to small deformations of the plane, and thus captures the shape dependence of surface operators.

This chapter revolves around two correlators that capture physical properties of the defect. The first one is the 2-point function of displacement operators. Using the residual conformal symmetry of the plane and reading off the conformal dimension $\Delta_{\mathbb{D}} = 3$ from (4.1), the 2-point function is constrained up to a single coefficient $C_{\mathbb{D}}$ to be (the factor π^2 is for convenience):

$$\langle V[\mathbb{D}^m(\sigma)\mathbb{D}^n(0)]\rangle = \frac{C_{\mathbb{D}}\delta^{mn}}{\pi^2|\sigma|^6}\,. \tag{4.2}$$

The second operator we consider is the bulk stress tensor, which in the presence of the defect acquires an expectation value. Both the components of the tensor along the defect T^{ab} and orthogonal to it T^{mn} can have a nonzero 1-point function, and they are fixed by conformal invariance up to an arbitrary coefficient h_T to be

$$\left\langle T^{ab}(\sigma,x)V\right\rangle = \frac{h_T\eta^{ab}}{\pi^3x^6}\,, \qquad \langle T^{mn}(\sigma,x)V\rangle = -\frac{h_T(\delta^{mn}-2x^mx^n/x^2)}{\pi^3x^6}\,. \tag{4.3}$$

$T(\sigma, x)$ is inserted at a distance x from the defect, and obviously the correlators do not depend on the coordinate σ by translation invariance along the plane. $\eta^{ab} = \mathrm{diag}(-1, 1)$ is the Minkowski metric.

In theories with only conformal invariance the coefficients h_T and $C_{\mathbb{D}}$ are independent quantities [120], but in theories with enough supersymmetries one can use superconformal Ward identities to relate them [50]. For our surface operators we show in Section 4.3 that

$$h_T = \frac{3C_{\mathbb{D}}}{80}\,. \tag{4.4}$$

To derive this result, we obtain the transformations of the stress tensor multiplet under supersymmetry (4.19), which is also an important result of Section 4.3.

Analogous relations between h_T and $C_{\mathbb{D}}$ were first derived using the same techniques for the 1/2-BPS Wilson loops of 4d $\mathcal{N} = 2$ theories [50] and the 1/6-BPS bosonic loops of ABJM [121], proving the conjecture of [122, 123]. A similar analysis was also applied recently to surface operators in 4d $\mathcal{N} = 1$ theories [89]. All these different examples show how the language of dCFT is a powerful and universal tool to study superconformal defects.

More than simply equating different constants, the relation (4.4) has an important physical consequence: In Section 4.4 we relate the conformal anomaly coefficients b_1, b_2, c associated with the defect to $C_{\mathbb{D}}$, h_T and an additional constant $C_{\mathbb{O}}$ to be introduced in (4.9). In the language of anomaly coefficients, the result (4.4) along with the relative normalisations (4.12) of the operators in the displacement multiplet can then be rephrased as

$$c = -b_1/2\,, \qquad b_1 = -b_2\,, \tag{4.5}$$

[1]Note that the dCFT is not expected to contain a conserved stress tensor [74] and the rigid conformal symmetry is not necessarily enhanced to Virasoro symmetry.

or equivalently, with respect to the basis (3.5),

$$c = -a_2\,, \qquad b = 0. \tag{4.6}$$

We emphasize that these identities are a consequence of supersymmetry and hold for any 1/2-BPS operator of the $\mathcal{N} = (2, 0)$ theory and for any ADE group. In particular, the second identity agrees with the explicit holographic calculations of [28, 64, 124] and was conjectured to come from supersymmetry in [89]. The two remaining anomaly coefficients a and b_1 were calculated at $N = 1$ in [28] and for $N > 1$ using holographic entanglement entropy in the presence of surface operators [71, 81, 90, 91], and the superconformal index [92].

Finally, in Section 4.5 we expand our scope and consider the analog of the operator product expansion but for bulk operators in the presence of a defect—the defect operator expansion (dOE) [46, 125]. This expansion allows us to represent bulk operators near the defect in terms of insertions of defect operators. To understand what these defect operators are more generally, we classify unitary multiplets of the algebra preserved by the defect[2]. We then look at operators in the stress tensor multiplet and determine the short multiplets arising in their dOE. We find a new marginal defect operator, which we associate with the RG flow between the nonsupersymmetric and 1/2-BPS surface operator discussed in [124].[3]

In addition to this result, we find that the defect operator expansion provides a useful framework and makes the constraints imposed by the preserved symmetries manifest. In fact, in Section 4.5.4 we use the dOE and representation theory to give a different perspective on the relation (4.4). Unlike in Section 4.3, where (4.4) follows from a technical calculation, we are able to conclude directly that h_T and $C_{\mathbb{D}}$ must be related. This suggests a strategy for determining the minimal amount of supersymmetry required in order for the conjecture of [122], which relates these coefficients in the case of supersymmetric Wilson loops, to hold (see also [70] and references therein for a similar conjecture in the context of entanglement entropy).

Some auxiliary results are collected in appendices. Appendix A.1 summarises our conventions and the gamma matrices used throughout. In Appendix A.4 we show how to constrain correlators containing both bulk and defect operators using conformal symmetry. Appendix A.5 reviews the 2 algebras appearing in this chapter: the $\mathfrak{osp}(8^*|4)$ symmetry of the bulk theory and the $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$ symmetry preserved by the defect.

4.2 Displacement multiplet

As far as defect operators go, the displacement operator is pretty universal. As (4.1) suggests, any defect breaking translation symmetry contains that defect operator. For this reason, it has appeared in many contexts: the prototypical example is the 1/2-BPS Wilson line in $\mathcal{N} = 4$ SYM, where the study of deformations and operator insertions was initiated in [127], but many other examples have been studied over the years and follow the general analysis of [120].

[2]In the last stages of preparation of the paper which this chapter is based on, theclassification of unitary multiplets of $\mathfrak{osp}(4^*|2)$ presented in Section 4.5.2 also appeared in [126].

[3]This is analogous to the flow of Wilson line operators introduced in [76, 77].

In the case of $\mathcal{N} = (2,0)$, we are mostly interested in the multiplet which contains the displacement operator. Of the full superconformal algebra $\mathfrak{osp}(8^*|4)$, the 1/2-BPS plane preserves a 2d conformal algebra $\mathfrak{so}(2,2)_\parallel$ in the directions parallel to the plane, along with rotations of the transverse directions $\mathfrak{so}(4)_\perp$ and an $\mathfrak{so}(4)_R$ R-symmetry. In addition, it also preserves half the supersymmetries Q_+ (and $\bar{\mathsf{S}}_+$) such that $\mathsf{Q}_+ V = 0$. These are obtained by a half-rank projector $\mathsf{Q}_+ = \Pi_+ \mathsf{Q}$ whose explicit definition can be found in (A.71). The preserved generators form an $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$ subalgebra [128], detailed in Appendix A.5.2.

Importantly, in direct analogy to (4.1), the Ward identities associated to the remaining broken super- and R-symmetries also receive contributions localised on the defect, which give rise to defect operators $\mathbb{Q}$ and $\mathbb{O}^i$, encoding the nontrivial response of the defect to the broken generators. Explicitly, the conservation laws associated with the R-current j and the supercurrent J are broken as follows:

$$\begin{aligned} \partial_\mu T^{\mu m} V &= V[\mathbb{D}^m]\delta^{(4)}(x)\,, \\ \partial_\mu(\Pi_- J^\mu) V &= V[\mathbb{Q}]\delta^{(4)}(x)\,, \\ \partial_\mu j^{\mu i 5} V &= V[\mathbb{O}^i]\delta^{(4)}(x)\,. \end{aligned} \tag{4.7}$$

In this equation, $i = 1, \ldots, 4$ is the R-symmetry index of $\mathfrak{so}(4)_R$. The spinor indices of $J^\mu_{\alpha\check{\alpha}}$ and $\mathbb{Q}_{\alpha\check{\alpha}}$ are suppressed and follow the conventions outlined in appendix A.1 (see however footnote 5). For the definition of Π_-, see (A.71).

As mentioned previously, the (nonabelian) theory does not have a known field realisation, so we cannot write these operators in terms of fundamental fields. We can however derive some of their properties purely from representation theory. The full multiplet as derived in Appendix A.5.2 reads

$$\begin{aligned} \delta_+ \mathbb{D}_m &= \frac{1}{2}\varepsilon_+ \gamma_{am} \partial^a \mathbb{Q}\,, \\ \delta_+ \mathbb{Q} &= 2\varepsilon_+ \gamma_m \mathbb{D}^m - 2\varepsilon_+ \gamma_a \check{\gamma}_{i5} \partial^a \mathbb{O}^i\,, \\ \delta_+ \mathbb{O}_i &= -\frac{1}{2}\varepsilon_+ \check{\gamma}_{i5} \mathbb{Q}\,, \end{aligned} \tag{4.8}$$

where $\delta_+ = \varepsilon_+ \mathsf{Q}_+$ is a variation with respect to the preserved supercharges and $\varepsilon_+ = \varepsilon_+ \Pi_+$.

4.2.1 Superconformal Ward identity

The 2-point functions of these operators are easily found. Both $\mathbb{D}$ and $\mathbb{O}$ transform as scalars with respect to the 2d conformal symmetry, while $\mathbb{Q}$ is a spinor. Their conformal dimensions can also be read off from (4.7) and are given by $\Delta_\mathbb{D} = 3$, $\Delta_\mathbb{Q} = 5/2$ and $\Delta_\mathbb{O} = 2$. Consequently, using the preserved bosonic symmetries, their 2-point functions are (up to overall coefficients $C_\mathbb{D}$, $C_\mathbb{Q}$, $C_\mathbb{O}$):

$$\begin{aligned} \langle V[\mathbb{D}^m(\sigma)\mathbb{D}^n(0)]\rangle &= \frac{C_\mathbb{D}\delta^{mn}}{\pi^2 |\sigma|^6}\,, \\ \langle V[\mathbb{Q}(\sigma)\mathbb{Q}(0)]\rangle &= \frac{C_\mathbb{Q}\,(\gamma_a \sigma^a \Pi_-)}{\pi^2 |\sigma|^6}\,, \\ \langle V[\mathbb{O}_i(\sigma)\mathbb{O}_j(0)]\rangle &= \frac{C_\mathbb{O}\delta_{ij}}{\pi^2 |\sigma|^4}\,. \end{aligned} \tag{4.9}$$

As $\mathbb{Q}$ is a 2d spinor, its 2-point function should be written in terms of the corresponding 2d gamma matrices. In order to emphasize the relation between the respective symmetry algebras in 6d and 2d, we write these matrices as blocks of their 6d counterparts obtained by the projector Π_-.

We can now relate $C_{\mathbb{O}}$ and $C_{\mathbb{Q}}$ to $C_{\mathbb{D}}$ using superconformal Ward identities associated to the preserved supersymmetries. Apply the supersymmetry transformations (4.8) to the vanishing correlator $\langle V[\mathbb{Q}_{\beta\check{\beta}}\mathbb{O}_i]\rangle$ to find

$$-\frac{1}{2}\,(\check{\gamma}_{i5})_{\check{\alpha}}{}^{\check{\gamma}}\,\big\langle V[\mathbb{Q}_{\beta\check{\beta}}\mathbb{Q}_{\alpha\check{\gamma}}]\big\rangle = 2\,(\gamma_a\check{\gamma}_{j5}\Pi_- c\Omega)_{\alpha\beta\check{\alpha}\check{\beta}}\,\partial^a\big\langle V[\mathbb{O}^j\mathbb{O}_i]\big\rangle. \tag{4.10}$$

Substituting the explicit 2-point functions (4.9), we obtain the linear relation $C_{\mathbb{Q}} = -16C_{\mathbb{O}}$. In the same fashion, the Ward identity associated to $\langle V[\mathbb{Q}_{\beta\check{\beta}}\mathbb{D}_m]\rangle$ leads to

$$2\,(\gamma_n\Pi_- c\Omega)_{\alpha\check{\alpha}\beta\check{\beta}}\,\langle V[\mathbb{D}^n\mathbb{D}_m]\rangle = -\frac{1}{2}\,(\gamma_{am})_{\alpha}{}^{\gamma}\,\partial^a\big\langle V[\mathbb{Q}_{\beta\check{\beta}}\mathbb{Q}_{\gamma\check{\alpha}}]\big\rangle, \tag{4.11}$$

which serves to relate $C_{\mathbb{D}}$ to $C_{\mathbb{Q}}$. Altogether, we find that the normalisations of the 2-point functions obey

$$C_{\mathbb{D}} = -C_{\mathbb{Q}} = 16C_{\mathbb{O}}. \tag{4.12}$$

4.3 Stress tensor correlators

Some of the most important operators in any theory are the stress tensor and its multiplet. In the presence of the 1/2-BPS defect, their expectation values are highly constrained by the residual symmetry: typically the $\mathfrak{so}(2,2)_{\|}\oplus\mathfrak{so}(4)_{\perp}\oplus\mathfrak{so}(4)_R$ bosonic subalgebra of preserved symmetries is powerful enough to fix them up a to a single constant (see e.g. (4.3)).

In addition to the constraints imposed by conformal symmetry, supersymmetry relates correlators of different operators in the same multiplet. Adapting the strategy of [50,89,121], the key to deriving (4.4) is to focus on the correlator $\langle T^{\mu\nu}(x)V[\mathbb{D}^m(\sigma)]\rangle$, which is entirely fixed in terms of the constants $C_{\mathbb{D}}$ and h_T [120]. The kinematics of that correlator admit two independent tensor structures with their own coefficients. They are related to $C_{\mathbb{D}}$ by taking the divergence

$$\partial_\mu\langle T^{\mu m}V[\mathbb{D}^n]\rangle = \langle V[\mathbb{D}^m\mathbb{D}^n]\rangle \propto C_{\mathbb{D}}\,, \tag{4.13}$$

and to h_T by integrating the displacement operator over the surface, which simply translates the defect

$$\int_{\mathbb{R}^2} \mathrm{d}^2\sigma\langle T^{\mu\nu}(0,x)V[\mathbb{D}^m(\sigma)]\rangle = \partial^m\langle T^{\mu\nu}(0,x)V\rangle \propto h_T\,. \tag{4.14}$$

We stress that this does not provide in itself a relation between $C_{\mathbb{D}}$ and h_T, as can be checked using the explicit form of the correlators (see equation (6.2) of [70]).

Instead, we should use superconformal Ward identites to relate this correlator to $\langle O^{i5}V[\mathbb{O}^j]\rangle$, where O is the superconformal primary of the stress tensor multiplet. Because the latter admits only a single tensor structure, this would imply that $C_{\mathbb{D}}$ and h_T are related.

In order to derive this result, we need the explicit supersymmetry transformations of the stress tensor multiplet, which are summarised in (4.19). We also need the 1-point functions of the stress tensor appearing on the right-hand side of (4.14), which are derived in Section 4.3.2 (the 2-point functions of the displacement multiplet are given in (4.9)). Then, we use the supersymmetric Ward identities associated with correlators of the form $\langle \mathcal{O} V[\hat{\mathcal{O}}] \rangle$ to derive (4.4).

4.3.1 Stress tensor multiplet

We begin by obtaining explicit supersymmetry transformations for the stress tensor multiplet, whose content is derived from representation theory and can be found in [93], where it is presented as a massless graviton multiplet (see also [95,96] for an overview of superconformal multiplets in various dimensions).

The primaries of any multiplet are labelled by their transformation under Lorentz symmetry $[j_1, j_2, j_3]_{\mathfrak{su}(4)}$, R-symmetry $(R_1, R_2)_{\mathfrak{sp}(2)}$ as well as their conformal dimension Δ.[4] In the notation of [96], the stress tensor multiplet is the $D_1[0,0,0]_4^{(0,2)}$ multiplet (with representations written as $[j_1, j_2, j_3]_\Delta^{(R_1,R_2)}$). Its primaries are

- $T^{\mu\nu}$, the stress tensor ($[0,2,0]_6^{(0,0)} = \mathbf{20}$). It contains a null state, since $\partial_\mu T^{\mu\nu} = 0$, and has $20 - 6$ degrees of freedom.
- $J^\mu_{\alpha\check{\alpha}}$, the supercurrent ($[1,1,0]_{11/2}^{(1,0)} = \mathbf{20} \cdot \mathbf{4}$). It also has a null state $\partial_\mu J^\mu_{\alpha\check{\alpha}} = 0$, satisfies $(\bar{\gamma}_\mu)_{\dot{\alpha}}{}^{\beta} J^\mu_{\beta\check{\beta}} = 0$, and contains $80 - 16$ degrees of freedom.[5]
- $j^{\mu[IJ]}$, the R-current ($[0,1,0]_5^{(2,0)} = \mathbf{6} \cdot \mathbf{10}$). It has a null state $\partial_\mu j^{\mu IJ} = 0$, and contains $60 - 10$ degrees of freedom.
- $H^I_{\mu\nu\rho}$, a self-dual 3-form ($[2,0,0]_5^{(0,1)} = \mathbf{10} \cdot \mathbf{5}$) containing 50 degrees of freedom.
- $\chi^I_{\alpha\check{\alpha}}$, a fermion ($[1,0,0]_{9/2}^{(1,1)} = \mathbf{4} \cdot \mathbf{16}$) satisfying $(\check{\gamma}_I)_{\check{\alpha}}{}^{\check{\beta}} \chi^I_{\beta\check{\beta}} = 0$ and containing 64 degrees of freedom.
- $O^{(IJ)}$, a scalar ($[0,0,0]_4^{(0,2)} = \mathbf{14}$) with 14 degrees of freedom. It is the superprimary of the multiplet.

Together with their descendants, these form an on-shell multiplet with 128 bosonic operators (and a matching number of fermionic operators).

In addition to the operator content, we need below the explicit supersymmetry transformations, which have not been calculated before to the best of our knowledge. These can be

[4]These Dynkin labels are related to the usual $\mathfrak{so}(1,5)$ and $\mathfrak{so}(5)$ labels by

$$[j_1, j_2, j_3]_{\mathfrak{su}(4)} = [j_2, j_1, j_3]_{\mathfrak{so}(1,5)}\,, \qquad (R_1, R_2)_{\mathfrak{sp}(2)} = (R_2, R_1)_{\mathfrak{so}(5)}\,.$$

[5]Note that J transforms in the $[1,1,0]$ irrep. Since the tensor product of a vector and a chiral spinor decomposes into $[1,1,0] \oplus [0,0,1]$, we can write J with indices μ and α, provided we project out the antichiral spinor by requiring $(\bar{\gamma}_\mu)_{\dot{\alpha}}{}^{\beta} J^\mu_{\beta\check{\beta}} = 0$.

obtained in a variety of ways (e.g. oscillator constructions [93] and superspace transformations [34, 129]), but here we simply list the terms allowed by Lorentz and R-symmetry and fix the coefficients by requiring closure of the algebra, i.e. imposing that on every operator $\{\mathsf{Q},\mathsf{Q}\}\,\Phi = 2\mathsf{P}\Phi$. Importantly, imposing this condition is made easy because we already know the operator content.

We start from the superprimary O^{IJ}. Since Q transforms as $[1,0,0]^{(1,0)}_{1/2}$, we know from representation theory that the product $\mathsf{Q}O$ can contain

$$[1,0,0]^{(1,2)}_{9/2} \oplus [1,0,0]^{(1,1)}_{9/2}\,, \tag{4.15}$$

but as $[1,0,0]^{(1,2)}_{9/2}$ does not appear in the multiplet, we remove it. The remaining term $[1,0,0]^{(1,1)}_{9/2}$ can be constructed explicitly and is fixed up to a constant c_1

$$\mathsf{Q}_{\alpha\check{\alpha}} O^{IJ} = c_1 (\check{\gamma}^{(I}\chi^{J)})_{\alpha\check{\alpha}}\,. \tag{4.16}$$

The transformation of χ is more complicated but the same analysis leads to

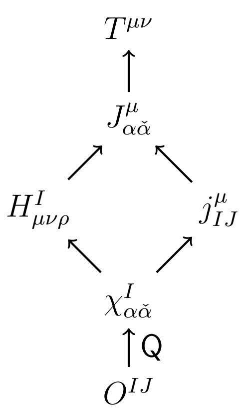
$$\begin{aligned}\mathsf{Q}_{\alpha\check{\alpha}}\chi^I_{\beta\check{\beta}} = {} & c_2(\gamma^{\mu\nu\rho})_{\alpha\beta}\left(\check{\gamma}^{IJ}+4\delta^{IJ}\right)_{\check{\alpha}\check{\beta}} H^J_{\mu\nu\rho} + c_3(\gamma_\mu)_{\alpha\beta}\left(\check{\gamma}^{IJK}+3\delta^{IJ}\check{\gamma}^K\right)_{\check{\alpha}\check{\beta}} j^\mu_{JK} \\ & + d_1(\gamma^\mu)_{\alpha\beta}\left(\check{\gamma}^J\right)_{\check{\alpha}\check{\beta}}\partial_\mu O^{IJ}\,.\end{aligned} \tag{4.17}$$

It is easy to check that

$$\left\{\mathsf{Q}_{\alpha\check{\alpha}},\mathsf{Q}_{\beta\check{\beta}}\right\} O^{IJ} = 2c_1 d_1 (\gamma^\mu)_{\alpha\beta}\Omega_{\check{\alpha}\check{\beta}}\partial_\mu O^{IJ}\,, \tag{4.18}$$

so the algebra closes provided $c_1 d_1 = 1$ (we identify $\mathsf{P}_\mu = \partial_\mu$, see (A.41)).

We can proceed this way for the full multiplet and build the supersymmetry transformations. Checking for closure of the algebra becomes a tedious (if straightforward) task and is not very illuminating, so we omit the details. The end result is (with $\delta = \varepsilon^{\alpha\check{\alpha}}\mathsf{Q}_{\alpha\check{\alpha}}$)



$$\begin{aligned}
\delta T^{\mu\nu} &= \frac{1}{2}\varepsilon\gamma^{\rho(\mu}\partial_\rho J^{\nu)}\,,\\
\delta J^\mu &= 2\varepsilon\gamma_\nu T^{\mu\nu} + \frac{2c_2}{5c_3}\left(6\eta^{\rho\mu}\left(\gamma^{\nu\sigma\lambda}+3\eta^{\sigma\nu}\gamma^\lambda\right) - \eta^{\mu\nu}\gamma^{\rho\sigma\lambda}\right)\check{\gamma}_I\partial_\nu H^I_{\rho\sigma\lambda}\\
&\quad + \frac{1}{10}\varepsilon\left(\gamma^{\mu\nu\rho}-4\eta^{\mu\rho}\gamma^\nu\right)\check{\gamma}^{IJ}\partial_\nu j_{\rho IJ}\,,\\
\delta j^\mu_{IJ} &= -\frac{1}{2}\varepsilon\check{\gamma}_{IJ}J^\mu + \frac{1}{5c_3}\varepsilon\gamma^{\mu\nu}\partial_\nu\check{\gamma}_{[I}\chi_{J]}\,,\\
\delta H^I_{\mu\nu\rho} &= \frac{c_3}{8c_2}\varepsilon\check{\gamma}^I\gamma_{[\mu\nu}J_{\rho]} + \frac{1}{120c_2}\varepsilon\gamma_\sigma\bar{\gamma}_{\mu\nu\rho}\partial^\sigma\chi^I\,,\\
\delta\chi^I &= c_2\varepsilon\gamma^{\mu\nu\rho}\left(\check{\gamma}^{IJ}+4\delta^{IJ}\right)H^J_{\mu\nu\rho} + c_3\varepsilon\gamma_\mu\left(\check{\gamma}^{IJK}+3\delta^{IJ}\check{\gamma}^K\right)j^\mu_{JK}\\
&\quad + \frac{1}{c_1}\varepsilon\gamma^\mu\check{\gamma}^J\partial_\mu O^{IJ}\,,\\
\delta O^{IJ} &= c_1\varepsilon\check{\gamma}^{(I}\chi^{J)}\,.
\end{aligned} \tag{4.19}$$

There are still some arbitrary constants c_i that remain unfixed and can be absorbed into the normalisations of O, χ and H. On the other hand, the normalisation of the conserved currents must match that of the algebra, so these operators cannot be rescaled. This can be seen by checking that the variation of the currents reproduces the corresponding commutator in (A.62). For example, the variation of j^μ computed using (4.19) is

$$\int \mathsf{Q}_{\alpha\check{\alpha}} j^0_{IJ} \mathrm{d}^5x = -\frac{1}{2}\int \left(\check{\gamma}_{IJ} J^0\right)_{\alpha\check{\alpha}} \mathrm{d}^5x = -\frac{1}{2}\left(\check{\gamma}_{IJ}\mathsf{Q}\right)_{\alpha\check{\alpha}} , \tag{4.20}$$

which is indeed the correct normalisation for the commutator $[\mathsf{Q}_{\alpha\check{\alpha}}, \mathsf{R}_{IJ}]$ of (A.62).

4.3.2 Defect without insertions

Among the operators of the stress tensor multiplet, some can acquire an expectation value in the presence of V. For the stress tensor, this happens when $h_T \neq 0$ in (4.3), and we can similarly constrain the 1-point functions of the other operators. This computation is done explicitly in Appendix A.4 and the only nonvanishing correlators are

$$\left\langle T^{ab} V \right\rangle = \frac{h_T \eta^{ab}}{\pi^3 x^6} , \qquad \left\langle T^{mn} V \right\rangle = -\frac{h_T}{\pi^3 x^6}\left(\delta^{mn} - 2\frac{x^m x^n}{x^2}\right) , \tag{4.21}$$

$$\left\langle H^5_{01m} V \right\rangle = \frac{h_H x_m}{\pi^3 x^6} , \qquad \left\langle H^5_{lmn} V \right\rangle = -\frac{h_H \varepsilon_{lmnp} x^p}{\pi^3 x^6} , \tag{4.22}$$

$$\left\langle O^{55} V \right\rangle = \frac{h_O}{\pi^3 x^4} , \qquad \left\langle O^{ij} V \right\rangle = -\frac{h_O \delta^{ij}}{4\pi^3 x^4} , \tag{4.23}$$

where h_O, h_H, and h_T are as yet undetermined constants. They are however related by the supersymmetry transformations (4.19) derived above. Specifically, consider the Ward identities associated with the preserved supersymmetries $\mathsf{Q}^+ = \Pi_+ \mathsf{Q}$ (with the projector Π_+ defined in (A.71))

$$\begin{aligned} 0 &= \left\langle \mathsf{Q}^+_{\alpha\check{\alpha}}(\chi^5_{\beta\check{\beta}} V) \right\rangle = -4\left(12c_2 h_H + \frac{h_O}{c_1}\right)\frac{[\Pi_+\gamma_m x^m \check{\gamma}^5]_{\alpha\check{\alpha}\beta\check{\beta}}}{\pi^3 x^6} , \\ 0 &= \left\langle \mathsf{Q}^+_{\alpha\check{\alpha}}(J^a_{\beta\check{\beta}} V) \right\rangle = 2\left(h_T + \frac{36c_2}{5c_3}h_H\right)\frac{[\Pi_+\gamma^a]_{\alpha\check{\alpha}\beta\check{\beta}}}{\pi^3 x^6} . \end{aligned} \tag{4.24}$$

These equations fix

$$h_O = -12c_1c_2h_H = \frac{5}{3}c_1c_3h_T, \tag{4.25}$$

and the correlators in (4.21) are fixed up to a single constant h_T.

4.3.3 Defect with an insertion

We are now in a position to derive the result (4.4) by relating $\langle O^{i5} V[\mathbb{O}^j]\rangle$ to $\langle T^{am} V[\mathbb{D}^n]\rangle$ using superconformal Ward identities. There are two Ward identities one could consider, $\langle \mathsf{Q}^+ \chi V[\mathbb{O}]\rangle = 0$ and $\langle \mathsf{Q}^+ J V[\mathbb{D}]\rangle = 0$, but one can check that they yield the same constraint, so we present only the first one.

The correlators we need are derived in Appendix A.4 by using the constraints of conformal symmetry. Importantly, the correlators $\langle OV[\mathbb{O}]\rangle$, $\langle \chi V[\mathbb{Q}]\rangle$ and $\langle HV[\mathbb{O}]\rangle$ are related to h_T by integrated relations like (4.14), while $\langle jV[\mathbb{O}]\rangle$ is related to $C_{\mathbb{D}}$ by (4.13), as we show below. They are

$$\begin{aligned}
\left\langle O^{i5}V[\mathbb{O}^j]\right\rangle &= \frac{C_{O\mathbb{O}}\delta^{ij}}{x^2(\sigma^2+x^2)^2}\,, & \left\langle \chi^5_{\alpha\check{\alpha}}V[\mathbb{Q}_{\beta\check{\beta}}]\right\rangle &= \frac{C_{\chi\mathbb{Q}}\left[\check{\gamma}^5\left(\gamma_a\sigma^a+\gamma_m x^m\right)\Pi_- c\Omega\right]_{\alpha\beta\check{\alpha}\check{\beta}}}{x^2\left(x^2+\sigma^2\right)^3}\,,\\
\left\langle j^{i5}_a V[\mathbb{O}^j]\right\rangle &= \frac{C_{j\mathbb{O}}\delta^{ij}\sigma_a}{x^2(\sigma^2+x^2)^3}\,, & \left\langle j^{i5}_m V[\mathbb{O}^j]\right\rangle &= \frac{C_{j\mathbb{O}}\delta^{ij}(x^2-\sigma^2)x_m}{2x^4(\sigma^2+x^2)^3}\,,\\
\left\langle H^i_{01m}V[\mathbb{O}^j]\right\rangle &= \frac{C_{H\mathbb{O}}\delta^{ij}x_m}{x^2(\sigma^2+x^2)^3}\,, & \left\langle H^i_{lmn}V[\mathbb{O}^j]\right\rangle &= \frac{C_{H\mathbb{O}}\delta^{ij}\varepsilon_{lmnp}x^p}{x^2(\sigma^2+x^2)^3}\,.
\end{aligned} \tag{4.26}$$

Explicitly, the Ward identity is

$$\begin{aligned}
0 &= \left\langle \mathsf{Q}^+_{\alpha\check{\alpha}}\left(\chi^5_{\beta\check{\beta}}V[\mathbb{O}^i]\right)\right\rangle\\
&= 6c_2\left[\Pi_+\gamma^{01m}(\check{\gamma}^5{}_J+4\delta^5_J)\right]_{\alpha\check{\alpha}\beta\check{\beta}}\left\langle H^J_{01m}V[\mathbb{O}^i]\right\rangle + 6c_2\left[\Pi_+\gamma^{lmn}(\check{\gamma}^5{}_J+4\delta^5_J)\right]_{\alpha\check{\alpha}\beta\check{\beta}}\left\langle H^J_{lmn}V[\mathbb{O}^i]\right\rangle\\
&\quad + 3c_3\left[\Pi_+\gamma^\mu\check{\gamma}_j\right]_{\alpha\check{\alpha}\beta\check{\beta}}\left\langle j^{5j}_\mu V[\mathbb{O}^i]\right\rangle + \frac{1}{c_1}\left[\Pi_+\gamma^\mu\check{\gamma}_J\right]_{\alpha\check{\alpha}\beta\check{\beta}}\partial_\mu\left\langle O^{5J}V[\mathbb{O}^i]\right\rangle\\
&\quad + \frac{1}{2}(\check{\gamma}^{i5})_{\check{\alpha}}{}^{\check{\gamma}}\left\langle \chi^5_{\beta\check{\beta}}V[\mathbb{Q}_{\alpha\check{\gamma}}]\right\rangle\,.
\end{aligned} \tag{4.27}$$

Plugging in the explicit forms of these correlators (4.26), and demanding that the terms proportional to $\gamma_a\sigma^a$ vanish, we obtain a linear relation

$$0 = 3c_3C_{j\mathbb{O}} + \frac{4}{c_1}C_{O\mathbb{O}} - C_{\chi\mathbb{Q}}\,. \tag{4.28}$$

The terms proportional $\gamma_m x^m$ give the same constraint.

Next, recall that $\mathbb{O}$ and $\mathbb{Q}$ respectively encode the action of a broken infinitesimal R-symmetry or supersymmetry variation. Therefore we can relate

$$0 = \left\langle \mathsf{R}_{j5}(O^{i5}(x)V)\right\rangle = \delta^{ij}\left\langle O^{55}(x)V\right\rangle - \left\langle O^{ij}(x)V\right\rangle + \int \mathrm{d}^2\sigma\left\langle O^{i5}(0,x)V[\mathbb{O}^j](\sigma)\right\rangle\,. \tag{4.29}$$

Using (4.21) and (4.26), we obtain

$$C_{O\mathbb{O}} = -\frac{5}{4\pi^4}h_O = -\frac{25c_1c_3}{12\pi^4}h_T\,. \tag{4.30}$$

A slightly more involved but entirely analogous calculation yields

$$C_{\chi\mathbb{Q}} = -\frac{5\cdot 8}{3\pi^4}h_T\,,\qquad C_{H\mathbb{O}} = \frac{5c_3}{36c_2\pi^4}h_T\,. \tag{4.31}$$

Finally, $C_{j\mathbb{O}}$ is related to the normalisation of the displacement operator multiplet by (4.7)

$$\partial_\mu\left\langle j^{\mu i5}(\sigma,x)V[\mathbb{O}^j(0)]\right\rangle = \left\langle V[\mathbb{O}^i(0)\mathbb{O}^j(\sigma)]\right\rangle\delta^{(4)}(x)\,. \tag{4.32}$$

Plugging the correlator of $j^{\mu i5}$ and $\mathbb{O}^j$ into the right hand side and integrating against a test function allows us to fix

$$C_{j\mathbb{O}} = -\frac{1}{\pi^4} C_{\mathbb{O}} = -\frac{1}{16\pi^4} C_{\mathbb{D}} \,. \tag{4.33}$$

Combining the above results into (4.28), we obtain

$$\frac{c_3}{\pi^4} \left(3C_{\mathbb{O}} - 5h_T\right) = 0 \qquad \Longrightarrow \qquad h_T = \frac{3C_{\mathbb{O}}}{5} = \frac{3C_{\mathbb{D}}}{80} \,, \tag{4.34}$$

which proves (4.4).

4.4 Relation to anomaly coefficients

In this section we explore the consequences of the relation between the coefficients $C_{\mathbb{D}}$ and h_T (4.4) for physical observables. These pieces of dCFT data appear in the Weyl anomaly of surface operators as defined in (3.5) and (3.5), and as we show below, the relations (4.12) and (4.34) relate the anomaly coefficients as (4.5).

The relation between correlators and anomaly coefficients is not specific to 2d defects in the $\mathcal{N} = (2,0)$ theory, but applies for any surface operator in a CFT. The anomaly coefficient b_1 was first shown to be related to $C_{\mathbb{D}}$ in [70], while the relation between b_2 and h_T was obtained in [70, 87]. Here we review their derivation and apply it to surface operators in the (2,0) theory to prove $c = -b_1/2$, $b_1 = -b_2$.

In a slightly different direction, the anomaly coefficients have also been discussed in the entanglement entropy literature, see [68, 70] and references therein.

4.4.1 Displacement operator

In order to isolate the contribution of $C_{\mathbb{D}}$ to the anomaly coefficients, we separately switch on each of the terms in (3.5). Since the displacement operator generates geometric deformations, one expects that inserting sufficiently many $\mathbb{D}^m$ into the planar surface operator V leads to a logarithmic divergence in the expectation value, signalling a conformal anomaly associated to the curvature of the surface. Similarly, inserting $\mathbb{O}^i$ to sufficient order will allow us to access the anomaly coefficient c associated with deformations in R-symmetry space.

To make this relation precise, we formally write deformations of the 1/2-BPS plane in terms of operator insertions

$$V_{\xi,\omega} = \exp\left[\int \mathrm{d}^2\sigma \xi_m(\sigma) P^m + \omega_i(\sigma) R^{i5}\right] V. \tag{4.35}$$

Here $P^m = \int \mathrm{d}^4x \partial_\mu T^{\mu m}$ generates translations transverse to the defect, while R-symmetry rotations are generated by $R^{i5} = \int \mathrm{d}^4x \partial_\mu j^{\mu i5}$. For constant parameters ξ, ω, the currents can be freely integrated and we recover the standard action of the charges P^m and R^{i5}.

Equation (4.35) is generally a complicated expression involving contact terms like (4.1), but also contact terms from P^m acting on defect operators and possibly other operators from the OPE. We can calculate its expectation value to quadratic order by expanding the exponential and noting that the 1-point functions of defect operators vanish:

$$\log \langle V_{\xi,\omega}\rangle - \log\langle V\rangle = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \Big(\langle V[\mathbb{D}_m\mathbb{D}_n]\rangle \xi^m\xi^n + \langle V[\mathbb{O}_i\mathbb{O}_j]\rangle\omega^i\omega^j\Big)\mathrm{d}^2\sigma\,\mathrm{d}^2\sigma' + \text{cubic}. \tag{4.36}$$

We can discard $\log\langle V\rangle$ since for the 1/2-BPS plane in a flat background, all anomaly terms vanish separately. Since the anomaly is quadratic in ξ and ω, it is related to the two point functions written here and we can safely ignore the higher order terms in the expansion.

To extract the anomaly coefficients, we study the UV divergence of the integrals in (4.36). The relevant correlators are found in (4.2) and (4.9). Fixing σ, the σ' integral can be evaluated explicitly by Taylor expanding $\xi^m(\sigma')$ and $\omega^i(\sigma')$ around σ. Starting with the second integrand and substituting $\tau = \sigma' - \sigma$,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^2}\langle V[\mathbb{O}_i(\sigma)\mathbb{O}_j(\sigma')]\rangle\omega^i(\sigma)\omega^j(\sigma')\,\mathrm{d}^2\sigma' \\ &\qquad = \frac{C_{\mathbb{O}}}{2\pi^2}\int_{\mathbb{R}^2}\frac{\delta_{ij}}{|\tau|^4}\omega^i(\sigma)\left[\omega^j(\sigma)+\tau^a\partial_a\omega^j(\sigma)+\frac{1}{2}\tau^a\tau^b\partial_a\partial_b\omega^j(\sigma)+\mathcal{O}(\tau^3)\right]\mathrm{d}^2\tau.\end{aligned} \tag{4.37}$$

While this integral leads to power law singularities as well, a logarithmic divergence arises only from the term quadratic in τ. We adopt polar coordinates $\tau^a = \tau e^a$ where e^a are orthonormal vectors parametrised by an angle φ. Borrowing the identities (3.29) from the previous chapter and dropping all but the logarithmic divergence, we obtain

$$\frac{C_{\mathbb{O}}}{4\pi^2}\pi\eta^{ab}\int_{\epsilon}\frac{\tau^3\mathrm{d}\tau}{\tau^4}\omega^i(\sigma)\partial_a\partial_b\omega^i(\sigma) = \frac{C_{\mathbb{O}}}{4\pi}(\partial\omega)^2\log\epsilon. \tag{4.38}$$

To leading order, the R-symmetry transformation in (4.35) takes the 1/2-BPS plane to a surface operator with $\partial_a n^i(\sigma) = \partial_a\omega^i$, so we can read the anomaly coefficient as

$$c = C_{\mathbb{O}}. \tag{4.39}$$

The logarithmic divergence of the first integrand in (4.36) can be evaluated in a similar way, and arises only from the fourth order in the Taylor expansion of ξ^n

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^2}\langle V[\mathbb{D}_m\mathbb{D}_n]\rangle\xi^m\xi^n\mathrm{d}^2\sigma' \\ &\qquad = \frac{C_{\mathbb{D}}}{2\pi^2}\int_{\mathbb{R}^2}\frac{\delta_{mn}}{|\tau|^6}\xi^m(\sigma)\left[\cdots+\frac{1}{24}\tau^a\tau^b\tau^c\tau^d\partial_a\partial_b\partial_c\partial_d\xi^n(\sigma)+\mathcal{O}(\tau^5)\right]\mathrm{d}^2\tau\,.\end{aligned} \tag{4.40}$$

Performing the angular integral with (3.29) leads to

$$\frac{C_{\mathbb{D}}}{48\pi^2}\frac{3\pi^2}{4}\int_{\epsilon}\frac{\tau^5\mathrm{d}\tau}{\tau^6}\xi_m(\sigma)(\partial^2)^2\xi^m(\sigma) = -\frac{C_{\mathbb{D}}}{64\pi}\partial^a\partial^b\xi_m(\sigma)\partial_a\partial_b\xi^m(\sigma)\log\epsilon\,. \tag{4.41}$$

This is the trace of the squared second fundamental form of the deformed surface (see (A.18)), which can be rewritten using the Gauss-Codazzi equation (A.21) as

$$\partial^a\partial^b\xi_m\partial_a\partial_b\xi^m = \mathrm{I\!I}^2 = 2\mathrm{Tr}\tilde{\mathrm{I\!I}}^2 + R^{\Sigma} - \mathrm{Tr}W\,. \tag{4.42}$$

Since we are on flat space, the Weyl tensor vanishes. The volume form for the deformed surface gets corrected, but to leading order in ξ does not affect the calculation. Therefore the contribution of this term to the anomaly density is

$$-\frac{C_{\mathbb{D}}}{64\pi}\int_{\Sigma}\left(2\mathrm{Tr}\tilde{\mathrm{II}}^2 + R^{\Sigma}\right)\mathrm{vol}_{\Sigma}\log\epsilon\,. \tag{4.43}$$

Note that the integral of R^{Σ} vanishes for small deformations of the plane. It therefore does not contribute to the anomaly, and we find

$$b_1 = -C_{\mathbb{D}}/8\,. \tag{4.44}$$

Using (4.12) along with (4.39) and (4.44) we find a relation for the anomaly coefficients

$$c = -b_1/2\,. \tag{4.45}$$

4.4.2 Stress tensor

The relation between b_2 and h_T is derived in a similar fashion, but instead of deforming the surface itself, we can relate the insertion of a stress tensor to a change in the background geometry.[6] The expectation value of the planar surface operator now receives a contribution from the metric variation:

$$\langle V\rangle_{\eta+\delta g} = \langle V\rangle_{\eta} - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^4}\delta g_{\mu\nu}(\sigma,x)\langle T^{\mu\nu}V\rangle_{\eta}\mathrm{d}^2\sigma\mathrm{d}^4x + \mathcal{O}(\delta g^2)\,. \tag{4.46}$$

In this equation, the subscript $\langle\bullet\rangle_g$ means the expectation value is calculated on a curved background metric g.

Since the insertion of a stress tensor sources a metric perturbation of linear order δg, we can only reproduce the anomaly to that order, which, expanding (3.5), is

$$\mathcal{A}|_{\delta g} = \frac{1}{4\pi}\left[-\frac{b_2}{10}\left(\partial_p^2\delta^{mn} - \partial^m\partial^n\right)\delta g_{mn} + \frac{3b_2}{20}\eta^{ab}\partial_p^2\delta g_{ab} + \partial_a(\dots)\right]. \tag{4.47}$$

These two terms are respectively associated to $\langle T^{mn}V\rangle$ and $\left\langle T^{ab}V\right\rangle$ in (4.46), and the total derivative drops out of the integral over the plane.

Using (4.21), we can evaluate the first term of (4.46). The logarithmic divergence arises as

$$\begin{aligned}\int_{\mathbb{R}^4}\delta g_{mn}\langle T^{mn}V\rangle\mathrm{d}^4x &= -\frac{h_T}{\pi^3}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\delta g_{mn}(\sigma,x)\frac{\delta^{mn}-2x^mx^n/x^2}{x^6} \\ &= -\frac{h_T}{\pi^3}\int_{\mathbb{R}^4}\frac{\mathrm{d}^4x}{x^6}\left(\dots+\frac{1}{2}\,\partial_{pq}\delta g_{mn}|_{x=0}\,x^px^q+\dots\right)\left(\delta^{mn}-2\frac{x^mx^n}{x^2}\right).\end{aligned} \tag{4.48}$$

[6] In the same way one can show that the bulk anomaly coefficients are related to the 2- and 3-point functions of the stress tensor [130].

In the second step we expanded $\delta g(x)$ in a Taylor series and dropped powers of x not contributing to the anomaly. We again switch to spherical coordinates $x^m = re^m$ and take note of the 4d analogue of (3.29)

$$\int \mathrm{vol}_{S^3} e^m e^n = \frac{\pi^2}{2}\delta^{mn}\,, \qquad \int \mathrm{vol}_{S^3} e^m e^n e^p e^q = \frac{\pi^2}{12}\left(\delta^{mn}\delta^{pq} + \delta^{mp}\delta^{nq} + \delta^{mq}\delta^{np}\right) . \tag{4.49}$$

The integral then becomes

$$-\frac{2\pi^2 h_T}{2\pi^3}\frac{2}{3}\int_\epsilon \frac{\mathrm{d}r}{r}\,\partial_{pq}\delta g_{mn}\big|_{x=0}\left(\delta^{mn}\delta^{pq} - \delta^{mp}\delta^{nq}\right) = \frac{1}{4\pi}\log\epsilon\left[\frac{2h_T}{3}\left(\partial_p^2\delta^{mn} - \partial^m\partial^n\right)\delta g_{mn}\right]_{x=0} . \tag{4.50}$$

Comparing against (4.47), we identify

$$h_T = \frac{3b_2}{10} . \tag{4.51}$$

The calculation for $\left\langle T^{ab}V\right\rangle$ is similar and gives the same result.

With expressions for b_1, b_2, c in terms of $C_{\mathbb{D}}$ and h_T in hand, we can finally translate the result of the previous section (4.34) into a constraint on the anomaly coefficients, and find

$$b_2 = -b_1\,, \tag{4.52}$$

as claimed.

A direct consequence of this relation (together with (4.45)) is that one only needs to calculate two nontrivial surface operators to calculate all the independent anomaly coefficients, for instance the sphere and cylinder.

4.5 Defect operator expansion

A useful tool in dCFT is the defect operator expansion (dOE), also known as the bulk-defect operator product expansion [46,125] (see [117] for a recent review of some dCFT techniques, including the dOE, in the context of the CFT bootstrap program). This is a convergent expansion representing bulk operators in terms of insertions of defect operators

$$\mathcal{O}_i(\sigma, x)V = \sum_k \frac{C^V_{ik}(x,\partial_\sigma)}{x^{\Delta_i - \hat{\Delta}_k}} V[\hat{\mathcal{O}}_k(\sigma)]\,, \tag{4.53}$$

where the sum is over defect primaries. The differential operators $C^V_{ik}(x,\partial_\sigma)$ are fixed by conformal symmetry. Their exact form can be obtained from the corresponding bulk-defect 2-point function of $\mathcal{O}_i$ and $\hat{\mathcal{O}}_k$ by equating

$$\left\langle \mathcal{O}_i(\sigma,x)V[\hat{\mathcal{O}}_k(0)]\right\rangle = \sum_j \frac{C^V_{ij}(x,\partial_\sigma)}{x^{\Delta_i-\hat{\Delta}_j}}\left\langle V[\hat{\mathcal{O}}_j(\sigma)\hat{\mathcal{O}}_k(0)]\right\rangle = \frac{1}{x^{\Delta_i-\hat{\Delta}_k}} C^V_{ik}(x,\partial_\sigma)\frac{C_{\hat{\mathcal{O}}_k}}{\sigma^{2\hat{\Delta}_k}}\,, \tag{4.54}$$

where we denote by $C_{\hat{\mathcal{O}}_k}$ the numerator of the 2-point function of $\hat{\mathcal{O}}_k$. Explicit expressions for C^V_{ik} can be found in [47, 120], but are not needed here.

The list of defect primaries appearing on the right-hand side of (4.53) can include the defect operators of Section 4.3 (namely the defect identity and the displacement operator multiplet), but it certainly includes more defect operators. This can be viewed as a consequence of the associativity of the OPE: since (4.53) maps bulk operators to defect operators and is valid in any correlator, all the CFT data of the bulk operators must be encoded, in some way, in the OPE of defect operators. Hence there must be at least as many defect degrees of freedom as bulk degrees of freedom.

Here we initiate the study of these other defect operators. We first classify the unitary multiplets of defect operators in Sections 4.5.1 and 4.5.2. This allows us to find the decomposition of the stress tensor multiplet in multiplets of the preserved algebra, see Figures 4.1 and 4.2.

After this detour into representation theory, we write the leading terms in the dOE for some operators and discuss the appearance of a new marginal operator. We finally comment on constraints imposed by supersymmetry and show how the dOE sheds light on the derivation of Section 4.3.

4.5.1 Representations of $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$

Defect operators sit in multiplets of the algebra preserved by the defect. For the 1/2-BPS plane V, the preserved algebra consists of two copies of $\mathfrak{osp}(4^*|2)$, so we are interested in constructing representations of $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$. The formulation of the algebra as a 2d superconformal algebra is reviewed in the appendix A.5.2, along with its embedding inside the bulk algebra $\mathfrak{osp}(8^*|4)$.

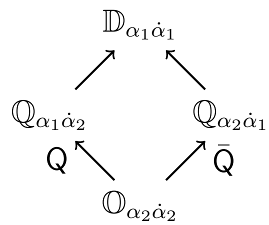
As usual, we can label primaries by their representation under the bosonic subalgebra, which here is

$$[\mathfrak{sl}(2) \oplus \mathfrak{su}(2)_\perp \oplus \mathfrak{su}(2)_R] \oplus [\mathfrak{sl}(2) \oplus \mathfrak{su}(2)_\perp \oplus \mathfrak{su}(2)_R] \,. \tag{4.55}$$

The corresponding labels are $[r_1, r_2]_h [\bar{r}_1, \bar{r}_2]_{\bar{h}}$, where r_1 and r_2 are the Dynkin labels for $\mathfrak{su}(2)_\perp$ and $\mathfrak{su}(2)_R$, and h is the conformal twist and labels representations of $\mathfrak{sl}(2)$. The labels $\bar{r}_1$, $\bar{r}_2$ and $\bar{h}$ are similar, but for the second subalgebra. We note that while (4.55) is equivalent to $\mathfrak{so}(2,2)_\parallel \oplus \mathfrak{so}(4)_\perp \oplus \mathfrak{so}(4)_R$, the factorisation above in terms of 2 algebras is dictated by supersymmetry, see A.5.2 for more details. The joint representation has conformal dimension $\hat{\Delta} = h + \bar{h}$ and spin $s = h - \bar{h}$.

The simplest nontrivial example of a multiplet of $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$ is the familiar displacement multiplet of section 4.2. Unlike our previous treatment however, here we label operators according to (4.55). In order to match that decomposition, we can express the superprimary $\mathbb{O}^i \sim (\check{\gamma}^i)^{\alpha_2 \dot{\alpha}_2} \mathbb{O}_{\alpha_2 \dot{\alpha}_2}$ in spinor indices. In this notation, the indices $\alpha = 1, 2$ are all $\mathfrak{su}(2)$ indices. We use $\alpha_1, \beta_1, \ldots$ for $\mathfrak{su}(2)_\perp$ and $\alpha_2, \beta_2, \ldots$ for $\mathfrak{su}(2)_R$; similarly for the second set of $\mathfrak{su}(2)$'s, but with dotted indices.

The values of h and $\bar{h}$ can also be read from (4.7), they are $h = \bar{h} = 1$ ($\mathbb{O}$ is a scalar of dimension 2). The representation of $\mathbb{O}$ is therefore $[0,1]_1[0,1]_1$. Acting with Q and $\bar{\mathsf{Q}}$ (which transform respectively as $[1,1]_{1/2}[0,0]_0$ and $[0,0]_0[1,1]_{1/2}$), one can build the full multiplet:



- $\mathbb{D}_{\alpha_1\dot\alpha_1}$, which transforms in the representation $[1,0]_{3/2}[1,0]_{3/2}$.
- $\mathbb{Q}_{\alpha_1\dot\alpha_2}$ and $\mathbb{Q}_{\alpha_2\dot\alpha_1}$ are respectively in $[1,0]_{3/2}[0,1]_1$ and $[0,1]_1[1,0]_{3/2}$. Together they form $\mathbb{Q}_{\alpha\breve\alpha}$ in (4.7).
- $\mathbb{O}_{\alpha_2\dot\alpha_2}$ is in the representation $[0,1]_1[0,1]_1$.

The structure of the multiplet as a product of two representations of $\mathfrak{osp}(4^*|2)$ is apparent in the diagram above. Under the action of Q, the operators transform as two multiplets of $\mathfrak{osp}(4^*|2)$, for instance the lower diagonal is

$$\mathsf{Q}_{\alpha_1\alpha_2}\mathbb{O}_{\beta_2\dot\beta_2} = c\epsilon_{\alpha_2\beta_2}\mathbb{Q}_{\alpha_1\dot\beta_2}\,, \qquad \mathsf{Q}_{\alpha_1\alpha_2}\mathbb{Q}_{\beta_1\dot\beta_2} = ic^{-1}\epsilon_{\alpha_1\beta_1}\partial\mathbb{O}_{\alpha_2\dot\beta_2}\,, \tag{4.56}$$

which is easily obtained from an ansatz as in Section 4.3.1 (the constant c is arbitrary). This is the simplest representation of $\mathfrak{osp}(4^*|2)$ and it contains the weights $[0,1]_1$ and $[1,0]_{3/2}$. Because it is ubiquitous, it is convenient to introduce some notation here and denote it $B[0,1]$, in anticipation of the results of Section 4.5.2.

4.5.2 Unitary multiplets of $\mathfrak{osp}(4^*|2)$

Since the algebra preserved by the defect factorises, we now turn our focus to general multiplets of a single copy of $\mathfrak{osp}(4^*|2)$. Importantly, we can classify allowed multiplets by working out the constraints imposed by unitarity.[7] This follows the method described in [33] used to classifiy multiplets in superconformal theories for $d \geq 3$.

The idea is the following. In radial quantisation, any operator $\mathcal{O}$ defines a corresponding state $|\mathcal{O}\rangle$. While $|\mathcal{O}\rangle$ has positive norm (by assumption), there is no guarantee that the norm of all the other states of the multiplet is also positive, as required by unitarity. Demanding that negative norm states are absent from the multiplet leads to a lower bound on the conformal dimension of the superprimary $h \geq h_A$. In particular, as we show below, at $h = h_A$ (4.59) some states become null, and the corresponding multiplets are the short multiplets A. In addition, we find yet shorter multiplets B with superprimary of conformal dimension h_B (4.60).

Consider the state $|\mathcal{O}\rangle$ of a superprimary operator in the representation $[r_1, r_2]_h$. Unitarity constrains the states $\mathsf{Q}|\mathcal{O}\rangle$ to satisfy

$$\|\mathsf{Q}|\mathcal{O}\rangle\|^2 = \langle\mathcal{O}|\{\mathsf{S},\mathsf{Q}\}|\mathcal{O}\rangle = \langle\mathcal{O}|D_+ + \sigma^i\mathsf{T}^i_{(1)} - 2\sigma^j\mathsf{T}^j_{(2)}|\mathcal{O}\rangle \geq 0\,, \tag{4.57}$$

where we use $\mathsf{Q}^\dagger_{\alpha_1\alpha_2} = \mathsf{S}^{\alpha_1\alpha_2}$ and the anticommutator (A.70), written in terms of $\mathfrak{su}(2)_\perp$ and $\mathfrak{su}(2)_R$ generators $\mathsf{T}^i_{(1,2)}$. We suppress the indices of Q and $|\mathcal{O}\rangle$, but the constraint should hold for any choice of Q, $|\mathcal{O}\rangle$, and linear combinations thereof.

The matrix elements $\langle s|\sigma^i\mathsf{T}^i|s\rangle$ are bounded by the eigenvalues of $\sigma^i\mathsf{T}^i$. Since σ^i is the fundamental representation, the product $\sigma^i\mathsf{T}^i$ can be decomposed as $[1]\otimes[r] = [r-1]\oplus[r+1]$, for both r_1 and r_2. The eigenvalues are expressed in terms of the quadratic Casimirs $C_2(j) =$

[7] The same analysis was also done in [126], which appeared as this paper was finalised.

$j(j+2)/4$ (using e.g. equation (2.38) of [33]), so that (4.57) takes the form

$$h \geq -\left(C_2(j_1) - C_2(1) - C_2(r_1)\right) + 2\left(C_2(j_2) - C_2(1) - C_2(r_2)\right), \qquad (4.58)$$

with j_1 and j_2 taking any values in $r_1 \pm 1$ and $r_2 \pm 1$. This assumes that both $r_1 > 0$ and $r_2 > 0$, otherwise the tensor product decomposition is simply $[1] \otimes [0] = [1]$ and $j = 1$.

For $r_1 > 0$, we then find that the strongest bound on the scaling dimension implied by (4.58) is

$$h \geq h_A = 1 + \frac{r_1}{2} + r_2\,. \qquad (4.59)$$

For $r_1 = 0$, we should instead take $j_1 = 1$ and we obtain

$$h \geq h_B = r_2, \qquad \text{if} \quad r_1 = 0\,. \qquad (4.60)$$

If these bounds are saturated, a subset of states become null and may be consistently removed from the multiplet.

While (4.59) and (4.60) are necessary conditions for unitarity, there could be, in principle, additional states whose norm becomes null (or negative), imposing further restrictions on h. It would be tedious to perform the above calculation for all states, but fortunately the conditions under which a representation is reducible (but not necessarily unitary) are listed by Kac in [131] (see also [132]). These match precisely the values obtained for the 4 choices of j_1 and j_2 in (4.58), which indicates that there are no further constraints.

We therefore conclude that for multiplets satisfying $h \geq h_A$, with h_A given in (4.59), there are no stronger constraints from requiring unitarity at higher levels. Generically, these are long multiplets, and they thus contain $2^4(r_1+1)(r_2+1)$ operators. Multiplets saturating the bound $h = h_A$ have a null state at level one, $\left|[r_1 - 1, r_2 + 1]_{h+1/2}\right\rangle$, and their dimension is reduced. The special case $r_1 = 0$ still leads to a unitary multiplet, but in this case the first null state is at level 2.

In the case $h_A > h \geq h_B$ (4.60) however, since h is below the unitarity bound h_A, some states in the multiplet would have a negative norm unless $h = h_B$ exactly: this is an isolated multiplet. It has a null state at level one, $\left|[1, r_2 + 1]_{h+1/2}\right\rangle$.

These short multiplets A and B are important to our discussion. For example, the $B[0,1]$ multiplet of Section 4.5.1 contains only $2+2$ operators, so it is indeed a short multiplet. From the argumentation above, the conformal dimension of its superprimary is thus fixed by unitarity to $h = h_B = 1$, in accordance with (4.7).

The broader question of determining the content of all short multiplets is interesting but lies beyond the scope of this work. However, specific short multiplets play a role in Section 4.5.3, and it is useful to know their content explicitly. It is sufficient for our present purposes to construct some representations heuristically by taking the tensor product decomposition of known multiplets. For instance, taking the product of two $B[0,1]$ multiplets, the superprimary decomposes into 2 representations $[0,1] \otimes [0,1] = [0,0] \oplus [0,2]$, so the tensor product gives 2 multiplets, which we identify as

$$B[0,1] \otimes B[0,1] = A[0,0] \oplus B[0,2]\,. \qquad (4.61)$$

The multiplet $A[0,0]$ contains the weights $[0,0]_1, [1,1]_{3/2}$ and $[2,0]_2$, while the multiplet $B[0,2]$ contains $[0,2]_2, [1,1]_{5/2}$ and $[0,0]_3$. Both of these representations appear as defect operators, see Figures 4.1 and 4.2 below.

4.5.3 The stress tensor dOE

Having gained some understanding of representations of the preserved algebra, we turn now to the main goal of this section: constructing the dOE (4.53) for the bulk operators of our theory. We focus on operators of the stress tensor multiplet (which should exist in any local quantum field theory), but the same analysis could be applied to other multiplets.

A naive way of thinking about (4.53) is as branching rules for the breaking of symmetry due to the presence of the defect. Indeed, it is natural to decompose, for example, the bulk superprimary O^{IJ} into representations of the preserved R-symmetry O^{55}, O^{i5} and O^{ij}, respectively the representations

$$[0,0][0,0]\,, \qquad [0,1][0,1]\,, \qquad [0,2][0,2]\,. \tag{4.62}$$

The dOE (4.53) is particularly simple for a trivial surface defect, where it is just the Taylor expansion of the bulk insertion:

$$O^{55}(x)I = I[O^{55}(0) + x^m\partial_m O^{55}(0) + \dots]\,, \tag{4.63}$$

While this expression merely amounts to a rewriting of the bulk degrees of freedom, the dOE becomes much more interesting if we consider a defect V which interacts with the bulk nontrivially.

A first sign that the dOE for general V contains additional terms is that the bulk operators couple to the defect identity $\mathbf{1_V}$ and the displacement multiplet (cf. for instance (4.21) and (4.26)). It is clear that these operators do not appear in the branching rules and encode additional interactions between bulk and defect degrees of freedom.

The second way in which the dOE is interesting is more subtle. The decomposition of operators in terms of the preserved algebra can be performed, as above, for all the operators in the stress tensor multiplet. The resulting representations can be organised in the multiplets of Figures 4.1 and 4.2 and the displacement multiplet, leading to the branching rules under the breaking of symmetry $\mathfrak{osp}(8^*|4) \to \mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$. The superprimaries of the multiplets in Figure 4.1 are easily identified as the defect counterparts of the operators O^{55} and O^{ij} by their representation, and with a bit of work this correspondence between bulk and defect operators can be also established for all the other operators.

Observe that the conformal dimension of these defect operators is, in some cases, lower than that of the corresponding bulk operators, leading to singular terms in the dOE. For instance, the dimension of $\hat{O}^{55}$ is 2, whereas the dimension of O^{55} is 4. A similar behavior occurs in the context of Wilson loops in 4d $\mathcal{N}=4$ SYM, where the 1/2-BPS line operator takes the form

$$W \sim \mathrm{Tr}\mathcal{P}\exp\mathrm{i}\int \left(A_\tau + \Phi^6\right)\mathrm{d}\tau. \tag{4.64}$$

In that case, the dOE of the stress tensor superprimary includes a defect operator of dimension 1, which can be understood as the insertion of Φ^6 in the line. Here, we do not have a field realisation of the $\mathcal{N}=(2,0)$ theory but $\hat{O}^{55}$ plays an analogous role.

Consider then the dOE for O^{55}. From Figures 4.1 and 4.2 we know some of the defect operators that can appear on the right-hand side of (4.53). This leads to

$$O^{55}(x)V = \frac{1}{x^4}C^V_{O\mathbf{1}}V[\mathbf{1}_V] + \frac{1}{x^2}C^V_{O\hat{O}}(x,\partial_\sigma)V[\hat{O}^{55}] + \frac{x_m}{x^2}C^V_{O\mathbb{D}}(x,\partial_\sigma)V[\mathbb{D}^m] + \dots \tag{4.65}$$

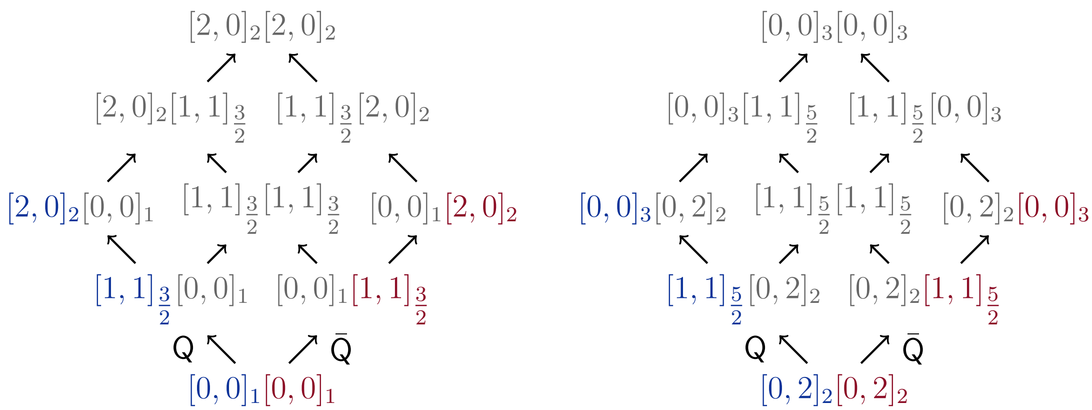


Figure 4.1: On the left, the $A[0,0]A[0,0]$ multiplet containing $32+32$ degrees of freedom. Its superprimary is $\hat{O}^{55}$. On the right, the $B[0,2]B[0,2]$ multiplet also containing 32+32 degrees of freedom. Its superprimary is $\hat{O}^{ij}$.

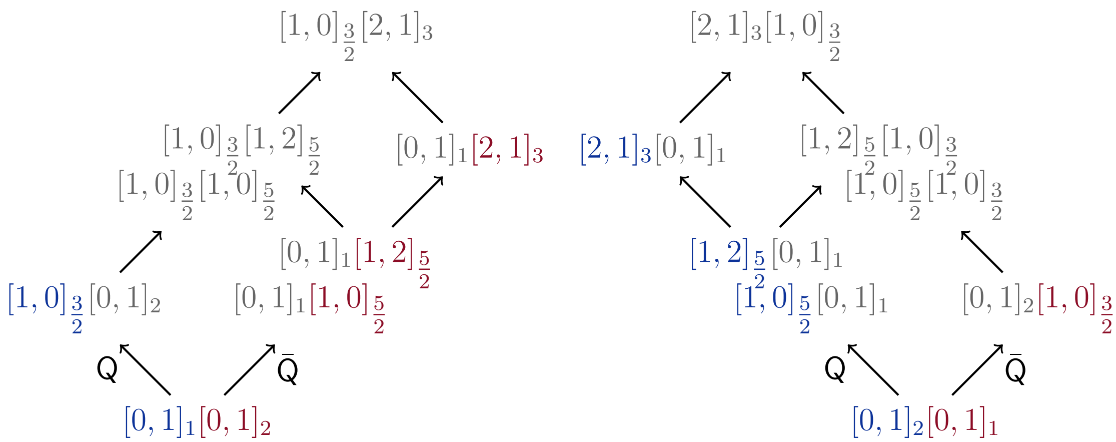


Figure 4.2: Multiplets $B[0,1]A[0,1]$ and $A[0,1]B[0,1]$. They both contain $32+32$ degrees of freedom.

The list of defect operators that may appear in this expansion is constrained by supersymmetry and can be treated systematically, but we do not pursue this direction further.

Equation (4.65) can be made more precise. The coefficients of the defect primaries encode the normalisation of bulk-defect correlators as in (4.54): 1-point functions such as (4.21) compute the coefficient of $\mathbf{1}_V$, 2-point functions such as (4.26) capture the coefficients of other defect primaries. Explicitly, $\langle O^{55}(x)V\rangle$ calculates the defect identity component of the dOE, such that

$$C^V_{O\mathbf{1}} = \frac{h_O}{\pi^3}. \tag{4.66}$$

The coefficient of the displacement operator can be found without computing $\langle O^{55}V[\mathbb{D}^m]\rangle$ explicitly, using the fact that the displacement operator is related to the broken translation symmetry. Integrating over the position of $\mathbb{D}^m$, we can replace it by a derivative:

$$\int \mathrm{d}^2\sigma \big\langle O^{55}(x) V[\mathbb{D}^m(\sigma)] \big\rangle = -\partial^m \big\langle O^{55}(x) V \big\rangle \,. \tag{4.67}$$

The left hand side is easily computed from (4.65) and related to $C_{\mathbb{D}}$ and $C^V_{O\mathbb{D}}$, while the right hand side is given in terms of h_O. Matching coefficients, we find

$$C^V_{O\mathbb{D}}(x, \partial_\sigma) = \frac{8h_O}{\pi^4 C_{\mathbb{D}}} \left(1 + \ldots\right). \tag{4.68}$$

By contrast, the coefficient $C^V_{O\hat{O}}$ is not obviously related to the remaining coefficients, and thus an independent piece of dCFT data.

4.5.4 Constraints from supersymmetry

We conclude this section by sketching an alternative derivation of the results of Section 4.3. It turns out that the dOE provides a simple and elegant way to understand the origin of the linear relations (4.24) and (4.28) without doing explicit calculations, by reframing them in terms of coefficients of displacement primaries in the stress tensor dOE. Indeed, the method we use can in principle be applied far more generally to obtain analogous constraints for the remaining dOE coefficients.

To reproduce these results, consider the dOE of χ^5. Following the analysis of Section 4.5.3, we decompose χ^5 into representations of the preserved algebra

$$[1,1][0,0] \oplus [1,0][0,1] \oplus [0,1][1,0] \oplus [0,0][1,1] \,, \tag{4.69}$$

which we label $\chi^5_{\alpha_1\alpha_2}$, $\chi^5_{\alpha_1\dot{\alpha}_2}$, $\chi^5_{\dot{\alpha}_1\alpha_2}$, $\chi^5_{\dot{\alpha}_1\dot{\alpha}_2}$. We only need the dOE of $\chi^5_{\alpha_1\dot{\alpha}_2}$, which takes the form

$$\chi^5_{\alpha_1\dot{\alpha}_2} V = \frac{1}{x^2} C^V_{\chi\mathbb{Q}}(x, \partial_\sigma) V[\mathbb{Q}_{\alpha_1\dot{\alpha}_2}] + \ldots \tag{4.70}$$

Again, there are other terms that could be included in this expansion, but they don't play a role in what follows so we ignore them. We also emphasise that (4.70) is related to the dOE of the stress tensor superprimary by supersymmetry.

We can now proceed as in Section 4.3 and find the constraints imposed by the preserved supersymmetries. Consider first acting with Q on the bulk operator $\chi^5_{\alpha_1\dot{\alpha}_2}$ to find

$$\mathsf{Q}\chi = H + j + \partial O \,, \tag{4.71}$$

with some coefficients. (The exact expression can be obtained by restricting (4.19) to the relevant representations of the preserved algebra.) Using the dOE on the right-hand side and focusing on the defect identity component gives

$$\big(\mathsf{Q}\chi(x)\big)V \sim \big(H(x) + j(x) + \partial O(x)\big)V \sim \frac{1}{x^5}\left(C_{H\mathbf{1}} + C_{j\mathbf{1}} + C_{\partial O\mathbf{1}}\right) V[\mathbf{1}_V] + \ldots \tag{4.72}$$

Note that $C_{j\mathbf{1}} = 0$ and $C_{\partial O\mathbf{1}}$ can be obtained from (4.67). We call this the "bulk" channel, since we calculate the action of Q on χ before taking the dOE.

The expression (4.72) is to be contrasted with the "defect" channel, where we first use (4.70) and then apply Q. Clearly, since $\mathbf{1}_V$ is not the variation of anything $\mathbf{1}_V \neq \mathsf{Q}(\dots)$, the result does not have an identity component. Consequently, the identity component of (4.72) must vanish as well, giving a linear constraint equivalent to (4.24) relating the normalisations of the stress tensor 1-point functions.

Similarly, (4.28) can be reproduced by focusing on the scalar displacement component of the same equation. The bulk channel gives schematically

$$\mathsf{Q}\chi V \sim \frac{1}{x^3}\left(C^V_{H\mathbb{O}} + C^V_{j\mathbb{O}} + C^V_{\partial O\mathbb{O}}\right) V[\mathbb{O}] + \dots \tag{4.73}$$

For the defect channel, we act on (4.70) with Q. From (4.56), we see that the variation only leads to descendants like $\partial\mathbb{O}$, and no primary. Since equality between defect and bulk channel must hold at the level of each defect operator, we conclude that the contribution of the displacement superprimary $\mathbb{O}$ to the bulk channel must vanish, and we obtain a linear constraint on the dOE coefficients $C^V_{j\mathbb{O}}, C^V_{H\mathbb{O}}, C^V_{\partial O\mathbb{O}}$, which is equivalent to (4.28). These two relations are only the simplest examples of a much larger set of constraints obeyed by the dOE coefficients. Indeed, equating the bulk and defect channel of any supercharge acting on any primary dOE at the level of each defect operator, it is straightforward to derive further such linear relations. These conditions greatly reduce the number of independent coefficients of stress tensor dOE coefficients, until we are left with what we could call a super-dOE, i.e. a set of dOEs which is fully consistent under the preserved supersymmetry.

4.6 Discussion

In this chapter, we studied insertions of local operators, including the displacement operator (4.1), into the 1/2-BPS plane. Other defect operators include excitations corresponding to inserting bulk operators near the defect—they are captured by the dOE (4.53).

One of our results is the classification of unitary multiplets of $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$, the algebra preserved by a 1/2-BPS defect, in Section 4.5.2. These multiplets are the building blocks for discussing other aspects of the dCFT, like its spectrum, the OPE of defect operators and the dOE. While in this work we focus on the dOE, it would also be interesting to pursue these other directions, for instance using the tools of the conformal bootstrap [117].

There are two important applications of the dOE (4.53) in our analysis: in Section 4.5.3 we use it to find new defect operators and in Section 4.5.4 we sketch how it makes the preserved symmetries manifest.

First, we use it to give the example of how the bulk stress tensor multiplet decomposes into defect multiplets. There are of course the operators $\mathbb{D}$, $\mathbb{Q}$ and $\mathbb{O}$ of the displacement multiplet, but also other defect multiplets whose operator content is shown in Figure 4.1 and 4.2. Although we focus on the stress tensor multiplet, this analysis could also be applied to any other multiplet of the $\mathcal{N} = (2,0)$ theory. In addition to the multiplets presented above, the dOE can include additional terms, and it would be interesting to obtain the selection rules as was done for 4d $\mathcal{N} = 4$ SYM in [133], by treating systematically all the superconformal Ward identities.

The important aspect of this decomposition of bulk operators is that it is convergent and encodes all the information of the bulk OPE, which opens the possiblity of studying the

$\mathcal{N} = (2,0)$ theory from the point of view of a 2d defect CFT. This direction could lead to additional constraints on the bulk theory, since the defect operators are not a trivial rewriting of those in the bulk. This is manifested for instance by the appearance of divergences in the dOE of O^{55} (4.65).

Instead, the dOE captures some important reorganisation of degrees of freedom in the dCFT. For instance, in the expansion of the bulk operator O^{55} (4.65) we find a defect operator which is of dimension 2 and therefore marginal (we expect it to be marginally irrelevant). The analogous expansion of the superprimary of the stress tensor multiplet is well understood in the context of Wilson loops in 4d $\mathcal{N} = 4$ SYM: using the definition of the 1/2-BPS Wilson loop (4.64) the marginal operator there corresponds to inserting Φ^6 into the line defect [77]. Here the interpretation is similar: inserting the analog of $\hat{O}^{55}$ in the non-supersymmetric surface operator triggers an RG flow which comes to a stop when $\hat{O}^{55}$ becomes marginal at the conformal fixed point, which is the 1/2-BPS surface operator. This flow is verified at large N in holography [124] and should hold more generally for all $\mathcal{N} = (2,0)$ theories.

A second use of the dOE is to make the preserved symmetries manifest. As we sketch in Section 4.5.4, we can explain the origin of the relation between h_T and $C_{\mathbb{D}}$ (4.4) simply by looking at the structure of multiplets of defect operators. This is to be contrasted with the derivation of Section 4.3, where the relation is the result of a calculation and not obvious from the outset. We believe this approach could shed light on determining the minimal amount of supersymmetry required to prove (4.4), that is whether it also holds for defects of the $\mathcal{N} = (1,0)$ theory, and more generally what are the necessary conditions to prove the conjecture of [122].

Finally, there are other interesting directions which we haven't explored here. For the Wilson line, a point of confluence between different techniques is the cusp, whose anomalous dimension at small angles is related to the Bremsstrahlung function [49] and can be calculated using integrability [61, 62, 134] and supersymmetric localization [135]. Its analog here are the conical singularities, which exhibit a $\log^2 \epsilon$ divergence, as discussed in Section 3.5. The coefficient of the divergence is entirely fixed by the behavior of the surface near the singularity, so it is natural to consider an operator inserting a conical singularity and to try and find its interpretation in the dCFT.

Another possibility is to study further the OPE for BPS operators. The $\mathcal{N} = (2,0)$ theory contains a sector isomorphic to a chiral algebra [136] which can be used to calculate for instance the 3-point functions of 1/4-BPS local operators. For 4d $\mathcal{N} = 2$ SCFTs, it was shown in [137] that the supercharges defining the cohomology are compatible with $\mathcal{N} = (2,2)$ surface defects, and it would be interesting to extend their construction to the $\mathcal{N} = (2,0)$ theory with 1/2-BPS surface defects. This could lead to exact results for a sector of the dOE and defect OPE.

It would also be interesting to study BPS operators in the context of the AGT correspondence. At large N one can use holography to calculate the expectation values, in the presence of the defect, of operators in the traceless symmetric representation of $\mathfrak{so}(5)_R$ [25], which contains in particular O^{IJ} in the stress tensor multiplet. Since the AGT correpondence can be used to calculate the expectation value of the stress tensor [92], it might also calculate expectation values for this larger class of operators at finite N.

Chapter 5

BPS Wilson loops in sCSM theories

Following our discussion of surface operators in the $\mathcal{N} = (2,0)$ theory, we now move on to the study of line defects in ABJM and related theories. Sections 5.1 and 5.2 are a modified and slightly expanded version of Chapter 2 in [30]. The remainder of this chapter contains material which will appear in an upcoming publication [31].

5.1 Background

While Wilson loops in gauge theories are important observables in general, they are of particular interest in Chern-Simons theories. The holonomies of a pure Chern-Simons gauge connection are topological and depend only on the homotopy class of the contours over which they are defined, and to whose knot invariants they are closely related [138]. Once the gauge sector is coupled to additional fields, the theory is no longer purely topological, and the Wilson loops themselves acquire a richer structure.

In supersymmetric gauge theories, loops preserving some fraction of the bulk supercharges may be constructed by explicitly coupling it to the superpartners of the gauge field. The most famous example of such an operator is the 1/2-BPS Maldacena-Wilson loop of $\mathcal{N} = 4$ super-Yang-Mills theory [54, 58],

$$W = \mathcal{P} \exp \int \mathrm{d}\varphi \left(A_\mu \dot{x}^\mu + \Phi^6 |\dot{x}| \right), \tag{5.1}$$

which may be defined either over a line or a circle. BPS Wilson loops can be defined over more general contours, by allowing for couplings to multiple scalars which vary along the contour in a way that is prescribed by $\dot{x}^\mu$. The most important of these are the loops of [11], which can be defined for completely arbitrary contours in $\mathbb{R}^4$, and those constructed in [12], whose contours are confined to an S^3; both of these classes are generically 1/16-BPS, with SUSY enhancement if the contours are further restricted to suitable subspaces. The classification of supersymmetric Wilson loops in $\mathcal{N} = 4$ has since been completed [139].

The study of the analogous objects in three dimensions was initiated in [17], whose authors judiciously coupled the ordinary Wilson loop of the $\mathcal{N} = 2$ theory to the auxiliary scalar of the vector multiplet to obtain a 1/2-BPS circular line operator:

$$W = \mathcal{P} \exp \mathrm{i} \int \mathrm{d}\varphi \left(A_\mu \dot{x}^\mu - \mathrm{i} |\dot{x}| \sigma \right). \tag{5.2}$$

Analogues of this loop for $\mathcal{N} > 2$ were constructed soon afterwards. In ABJM theory,[1] integrating out σ recasts the Wilson loop in terms of a particular bilinear of the scalar fields [16, 140, 141]:

$$W = \mathcal{P} \exp \mathrm{i} \int \mathrm{d}\varphi \left(A_\mu^{(1)} + 2\pi k^{-1} M_J^I C_I \bar{C}^J \right) \tag{5.3}$$

where $M = \mathrm{diag}(-1, -1, +1, +1)$, and $A^{(1)}$ is the gauge field associated with one of the gauge factors. An analogous construction for the remaining gauge factor yields an independent loop where $A^{(1)}$ is replaced by $A^{(2)}$ and the order of the scalars is reversed. These Wilson loops preserve four supercharges parametrised by $\theta_{12}^{\pm}$ and $\bar{\theta}_{\pm}^{12}$, accompanied by the special supersymmetries fixed to[2]

$$\epsilon_{12} = -i\theta_{12} \frac{\gamma_3}{|x|}, \qquad \bar{\epsilon}^{12} = -i \frac{\gamma_3}{|x|} \bar{\theta}^{12}, \tag{5.4}$$

and are therefore 1/6-BPS. In addition, they preserve a 1d rigid conformal algebra $\mathfrak{sl}_2(\mathbb{R})$ along the contour, a $\mathfrak{u}(1)$ comprised of rotations of the transverse directions, as well as an $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ subalgebra of the full $\mathfrak{su}(4)$ R-symmetry.

In order to construct loops preserving more supersymmetry, it is necessary to include couplings to the fermions. In order to preserve some conformal symmetry, W cannot involve dimensionful couplings, and therefore the fermions, which have scaling dimension $\Delta_\psi = 1$, should enter linearly. On the other hand, they transform in the bifundamental of $U(N_1) \times U(N_2)$, while the respective gauge connections are valued in the adjoint representation of either gauge factor. This puzzle was elegantly solved in [18], by promoting the modified gauge connection to a $U(N_1|N_2)$ supermatrix combining the two gauge fields, the scalar bilinears, and the fermions, schematically of the form

$$\mathcal{L} \sim \begin{pmatrix} A_\mu^{(1)} \dot{x}^\mu + C\bar{C}\, |\dot{x}| & \bar{\psi}\, |\dot{x}| \\ \psi\, |\dot{x}| & A_\mu^{(2)} \dot{x}^\mu + \bar{C} C\, |\dot{x}| \end{pmatrix}. \tag{5.5}$$

The Wilson loop then takes the form

$$W = \mathrm{sTr} \mathcal{P} \exp \mathrm{i} \int \mathcal{L} \mathrm{d}\varphi, \tag{5.6}$$

and the requirement that W preserve a given set of supercharges translates into conditions on the coefficients, which we suppressed in (5.5). A 1/2-BPS solution was found in [18], while further 1/6-BPS fermionic loops were first constructed in [19, 142]. While the discovery of these operators opens many directions of research, key aspects of their construction remain

[1] Our conventions are summarised in Appendix B.2.

[2] These conditions can equivalently be stated in terms of $\theta_{34} = -\bar{\theta}^{12}$, see appendix B.2.

riddled with intricacies. In particular, the coefficients of the fermions in the 1/2-BPS loop are antiperiodic, and none of the fermionic loops have hitherto been written in a manifestly gauge and reparametrisation invariant way.

By way of introduction to this chapter, we clarify some of these issues by providing a new formulation of 1/6-BPS Wilson loops in ABJM theory, which first appeared in [30]. We find that, in this new language, the generic 1/6-BPS fermionic operator can be written naturally as a deformation of a bosonic loop, shedding new light on how these loops preserve supersymmetry and their moduli space. In the main part of this chapter, based on upcoming work [31], we generalise our construction to the $\mathcal{N} = 4$ case, and report some previously unknown Wilson loops.

5.2 Deformation loops in ABJM

In order to give a unified account of the fermionic loops interpolating between the bosonic and 1/2-BPS loops, we consider a superconnection

$$\mathcal{L} = \mathcal{L}_{\text{bos}} + \Delta\mathcal{L}, \qquad \mathcal{L}_{\text{bos}} = \begin{pmatrix} \mathcal{A}^{(1)}_{\text{bos}} & 0 \\ 0 & \mathcal{A}^{(2)}_{\text{bos}} \end{pmatrix} + \frac{|\dot{x}|}{4|x|}\sigma_3, \tag{5.7}$$

where $\mathcal{A}^{(1,2)}$ are the modified gauge connections leading to 1/6-BPS bosonic loops, as in (5.3), and $\Delta\mathcal{L}$ may be block off-diagonal. In the limit $\Delta\mathcal{L} \to 0$, the constant term $\sigma_3 = \text{diag}(+\mathbb{I}_{N_1}, -\mathbb{I}_{N_2})$ can be exponentiated, and we recover the sum of the usual trace of bosonic connections.

In order for the loop defined by (5.7) to preserve supersymmetry, we require the superconnection to transform under the preserved supercharges as $\delta\mathcal{L} = \mathfrak{D}_\varphi G \equiv \partial_\varphi G - \mathrm{i}[\mathcal{L}, G]$ for some G [18, 143] (see appendix B.1). This relaxed notion of supersymmetry ensures that the variation takes the form of a supergauge transformation, under which loops of the form (5.6) are invariant. Consider then a deformation

$$\Delta\mathcal{L} = i|\dot{x}|\sigma_3 \left(\delta_+ G + \mathrm{i}\{G, G\}\right), \tag{5.8}$$

where δ_+ is parametrised by $\theta^+_{12}, \bar{\theta}^{12}_+$ (and $\epsilon^+_{12}, \bar{\epsilon}^{12}_+$ are given by (5.4)). The variation of $\Delta\mathcal{L}$ with respect to δ_+ assembles into a total derivative as required if

$$\delta_+^2 G = -i\sigma_3 \left(\partial_\varphi G + [\mathcal{L}_{\text{bos}}, G]\right), \tag{5.9}$$

which is satisfied for G comprised of C_1, $\bar{C}^1$ C_2, $\bar{C}^2$, breaking one $SU(2)$ of the residual R-symmetry (here, $\alpha, \bar{\alpha} \in \mathbb{C}^2$ are taken to be Grassmann odd and $i, j = 1, 2$)

$$G = \sqrt{\frac{2\pi i}{k}} \begin{pmatrix} 0 & \bar{\alpha}^i C_i \\ -\alpha_i \bar{C}^i & 0 \end{pmatrix}. \tag{5.10}$$

Using (5.9) one can show easily that $\delta_+\mathcal{L} = \mathfrak{D}_\varphi G$. Invariance under δ_- (parametrised by the remaining parameters $\theta^-_{12}, \bar{\theta}^{12}_-$) is ensured because $\delta_- G$ is related to $\delta_+ G$ by a gauge transformation, so that $\delta_-\mathcal{L}$ also takes the form of a total derivative.

The resulting family of 1/6 BPS loops is then parametrised by $\alpha, \bar{\alpha}$ and can be written explicitly as

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}^{(1)} & \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\rho_i^\alpha \bar{\psi}_\alpha^i \\ \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\psi_i^\alpha \bar{\rho}_\alpha^i & \mathcal{A}^{(2)} \end{pmatrix}, \qquad \begin{aligned} \mathcal{A}^{(1)} &= \mathcal{A}^{(1)}_{\text{bos}} - \tfrac{2\pi i}{k}|\dot{x}|\Delta M^i_j C_i \bar{C}^j + \tfrac{|\dot{x}|}{4|x|}\,, \\ \mathcal{A}^{(2)} &= \mathcal{A}^{(2)}_{\text{bos}} - \tfrac{2\pi i}{k}|\dot{x}|\Delta M^i_j \bar{C}^j C_i - \tfrac{|\dot{x}|}{4|x|}\,, \end{aligned} \tag{5.11}$$

$$\rho_i = 2\sqrt{2}\bar{\alpha}^j \theta^+_{ij}\Pi_+, \qquad \bar{\rho}^i = 2\sqrt{2}\Pi_+ \bar{\theta}^{ij}_+ \alpha_j, \qquad \Delta M^i_j = 2\bar{\alpha}^i \alpha_j, \qquad \Pi_\pm \equiv \frac{1}{2}\left(1 \pm \frac{\dot{x}^\mu \gamma_\mu}{|\dot{x}|}\right). \tag{5.12}$$

We note that (5.11) is related to the operators of [18, 19] by a gauge transformation

$$\exp i\Lambda, \qquad \Lambda = \left(\frac{\pi}{8} - \frac{\phi}{4}\right)\sigma_3 \tag{5.13}$$

where $0 < \phi < 2\pi$ is the polar angle and $\pi/8$ accounts for different conventions for $\rho, \bar{\rho}$. The fields transform as

$$\mathcal{A}^{(1)} \to \mathcal{A}^{(1)} - \frac{|\dot{x}|}{4|x|}, \qquad \mathcal{A}^{(2)} \to \mathcal{A}^{(2)} + \frac{|\dot{x}|}{4|x|}, \qquad \psi \to \sqrt{-i}e^{-i\phi/2}\psi, \qquad \bar{\psi} \to \sqrt{i}e^{i\phi/2}\bar{\psi}\,, \tag{5.14}$$

where the right-hand side reproduces the original formulation. The discontinuity of Λ at 2π yields a delta function term which, upon integration, exchanges the supertrace for a trace.

We stress that in contrast to previous formulations, this loop is manifestly reparametrisation invariant. It is also gauge invariant without the need for an additional twist matrix (see for instance [144]), since the couplings $\rho, \bar{\rho}$ and $M + \Delta M$ are periodic by construction. This comes, of course, at the expense of introducing a constant piece in the connection $\mathcal{L}_{\text{bos}}$, whose physical interpretation remains unclear.

We obtain the moduli space of 1/6 BPS deformations (5.8) by noting that any rescaling α and $\bar{\alpha}$ such that their product ΔM is unmodified can be absorbed by a gauge transformation. The resulting space is the conifold. This construction matches Class I of [19], while Class II is obtained by breaking the other $SU(2)$, i.e. coupling to C_3, $\bar{C}^3$, C_4 and $\bar{C}^4$ in $\mathcal{G}$. These two branches intersect at the origin singularity, i.e. $\mathcal{L}_{\text{bos}}$.

At particular points where ΔM has eigenvalues 2 and 0, the full matrix $M + \Delta M$ has enhanced $SU(3)$ symmetry. It is easy to see that commuting the 4 preserved supercharges with this $SU(3)$ symmetry gives rise to 12 supercharges, so these operators are 1/2-BPS, and we recover the loops of [18], as may be checked explicitly by performing the gauge transformation used above.

5.3 Deformation loops in $\mathcal{N} = 4$

Various generalisations to the above construction have been proposed. In [20, 21], the deformation point of view was extended to theories with fewer supersymmetries and more general quivers. This setting allows for much more general superconnections: Instead of just 2×2

blocks, as is appropriate for the two-node quiver of ABJM theory, one may introduce a gauge connection, together with the appropriate scalar bilinears, for each node in the quiver, with fermions on the off-diagonal for every edge connecting two nodes, and scalar bilinears in each off-off-diagonal block corresponding to a pair of nodes separated by two edges.

A further generalisation, explored in [21], is achieved by deforming around a more general bosonic loop. Indeed, one can replace the sum of two ordinary bosonic loops with the sum of two bosonic loops incorporating φ-dependence. In the language of ABJM theory, such loops take the form

$$\mathcal{A}^{\theta}_{\text{bos}} = A_\mu \dot{x}^\mu + 2\pi k^{-1} + (M + \Delta M^\theta)^I_J C_I \bar{C}^J \,, \tag{5.15}$$

$$\Delta M^\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-\cos\theta & 0 & -e^{-\mathrm{i}\varphi}\sin\theta \\ 0 & 0 & -1 & 0 \\ 0 & -e^{+\mathrm{i}\varphi}\sin\theta & 0 & -1+\cos\theta \end{pmatrix} . \tag{5.16}$$

It is not hard to show that this loop is generically 1/12-BPS [145], and as $\theta \to 0$, we recover the ordinary 1/6-BPS bosonic loop used in (5.7). The moduli spaces of deformations of either type of bosonic loop are quotients of $\mathbb{C}^n$ by a suitable group action of $\mathbb{C}^\times$, of which the conifold we found in the previous section is only the simplest example.

In this section, we will modify the above construction of supersymmetric Wilson loops in yet another way. Working in $\mathcal{N} = 4$ theory, we take as our point of departure a 1/2-BPS loop $W_{1/2}$, which we modify with respect to an arbitrary linear combination of the preserved supercharges. This construction reproduces the "hyperloops" of [21], but gives many additional loops preserving different amounts of super- and conformal symmetry. We take our Wilson loops to sit at a great circle of S^3 parametrised by φ and, throughout, adopt the notation of [21], summarised in Appendix B.4.

5.3.1 Preliminaries

While the supersymmetry variation of the bosonic loops considered above under any preserved supercharge vanishes identically, this is no longer the case for fermionic loops. Indeed, for the 1/2-BPS loop,

$$\mathsf{Q}\mathcal{L}_{1/2} = \sigma_3 \mathcal{D}^{\mathcal{L}_{1/2}}_{\varphi} H, \tag{5.17}$$

with H a nonvanishing supermatrix. Deformations around $W_{1/2}$ will, in addition to a supermatrix G and a supercharge Q, also depend on H associated with Q. We therefore begin by computing H and deriving some of its properties.

The superconformal algebra on S^3 is $\mathfrak{osp}(4|4)$, with bosonic subalgebra $\mathfrak{so}(1,4)_{\text{conf}} \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ (see appendix B.3). Consider a 1/2-BPS loop coupling only to two adjacent nodes and hypermultiplets charged under at least one of them. Each 1/2-BPS loop must preserve either $\mathfrak{su}(2)_L$ or $\mathfrak{su}(2)_R$. We are of course free to choose either, and in the following it should be understood that we are looking only at one branch of the moduli space. We will ignore issues related to such finite degeneracies. We pick the loop preserving $\mathfrak{su}(2)_L$, whose superconnection is given by

$$\mathcal{L}_{1/2} = \begin{pmatrix} \mathcal{A}_I & -i\bar{\alpha}\psi_{I\mathrm{i}-} \\ i\alpha\bar{\psi}_{I+}^{\mathrm{i}} & \mathcal{A}_{I+1} - \frac{1}{2} \end{pmatrix} . \tag{5.18}$$

where[3]

$$\mathcal{A}_I = A_{\varphi,I} - \frac{\mathrm{i}}{k}\big(-\nu_I + 2\tilde{\mu}_I{}^{\mathrm{i}}{}_{\mathrm{i}}\big)\,, \qquad \mathcal{A}_{I+1} = A_{\varphi,I+1} - \frac{i}{k}\big(-\nu_{I+1} + 2\tilde{\mu}_{I+1\mathrm{i}}{}^{\mathrm{i}}\big)\,, \tag{5.19}$$

and the parameters α and $\bar{\alpha}$ satisfy $\alpha\bar{\alpha} = 2i/k$. Note that the Wilson loop does not depend on $\alpha, \bar{\alpha}$. Instead of fixing its value, we leave it as a constant gauge parameter. One can introduce a φ-dependence in $\alpha, \bar{\alpha}$ at the expense of adding a $U(1)$ gauge field at the bottom right entry: $\mathcal{A}_{I+1} - \frac{1}{2} \to \mathcal{A}_{I+1} - \frac{1}{2} - i\alpha^{-1}\partial_\varphi\alpha$.

The eight supercharges preserved by (5.18) are

$$\mathsf{Q}_{\alpha}^{\dot{2}a}\,, \qquad \mathsf{Q}_{\bar{\alpha}}^{\mathrm{i}a}\,, \tag{5.20}$$

where α takes values l, r and $\bar{\alpha}$ is $\bar{l}$, $\bar{r}$. An explicit computation using the anticommutation relations in B.3 shows that the bosonic part of the superalgebra spanned by these charges is given by $\mathfrak{sl}_2(\mathbb{R})_{\|} \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{\tilde{\mathsf{M}}_\perp}$. In other words, in addition to half of the R-symmetry, the loop defined by (5.18) preserves full conformal symmetry along the contour, generated by rotations $\mathsf{M}_{\|}$ along the φ-direction and two of the four special conformal generators, which we denote $\mathsf{T}_\pm$, as well as a linear combination of $\bar{\mathsf{R}}_3$ and transverse rotations $\mathsf{M}_\perp$, which we denote $\tilde{\mathsf{M}}_\perp$. The preserved superalgebra is $\mathfrak{sl}(2|2)$.

A general superposition of the supercharges in (5.20) can be covariantly written as

$$\mathsf{Q}_\eta = \eta_a^\alpha \mathsf{Q}_\alpha^{\dot{2}a} + \bar{\eta}_a^\alpha (\sigma_1)_\alpha{}^{\bar{\beta}} \mathsf{Q}_{\bar{\beta}}^{\mathrm{i}a}. \tag{5.21}$$

The anticommutator of two supercharges reads

$$\{\mathsf{Q}_\eta, \mathsf{Q}_\rho\} = \mathrm{i}\left[\bar{\eta}_a^\alpha \rho_b^\beta + \bar{\rho}_a^\alpha \eta_b^\beta\right] \cdot \left[\epsilon^{ab}\begin{pmatrix} \mathsf{T}_+ & \mathsf{M}_{\|} \\ \mathsf{M}_{\|} & \mathsf{T}_- \end{pmatrix}_{\alpha\beta} + \epsilon^{ab}\epsilon_{\alpha\beta}\tilde{\mathsf{M}}_\perp + \frac{\mathrm{i}}{2}\epsilon_{\alpha\beta}\mathsf{R}^{ab}\right] . \tag{5.22}$$

In order to reduce clutter in the following, we use shorthand

$$\Pi_{ab} = \bar{\eta}_a^\alpha \begin{pmatrix} e^{+\mathrm{i}\varphi} & 1 \\ 1 & e^{-\mathrm{i}\varphi} \end{pmatrix}_{\alpha\beta} \eta_b^\beta, \qquad \lambda_{ab} = \epsilon_{\alpha\beta}\bar{\eta}_a^\alpha \eta_b^\beta, \tag{5.23}$$

as well as $\Pi = \Pi_- e^{-\mathrm{i}\varphi} + \Pi_0 + \Pi_+ e^{+\mathrm{i}\varphi} \equiv \epsilon^{ab}\Pi_{ab}$ and $\lambda = \epsilon^{ab}\lambda_{ab}$. In these terms, the square of a single supercharge takes the natural form

$$\mathsf{Q}^2 = \mathrm{i}\Pi_-\,\mathsf{T}_- + \mathrm{i}\Pi_0\,\mathsf{M}_{\|} + \mathrm{i}\Pi_+\,\mathsf{T}_+ + \mathrm{i}\lambda\,\tilde{\mathsf{M}}_\perp - \frac{1}{2}\lambda_{ab}\,\mathsf{R}^{ab}. \tag{5.24}$$

It will be convenient to define rotated scalars

$$r^1 \equiv (\eta\bar{v})_a q^a\,, \quad r^2 \equiv (\bar{\eta}v)_a q^a\,, \quad \bar{r}_1 \equiv -\epsilon^{ab}(\bar{\eta}v)_a \bar{q}_b\,, \quad \bar{r}_2 \equiv \epsilon^{ab}(\eta\bar{v})_a \bar{q}_b\,, \tag{5.25}$$

[3] Note that $\tilde{\mu}_I{}^{\mathrm{i}}{}_{\mathrm{i}} = -\tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}$, as these are by definition traceless, see Appendix B.4.

where $v_\alpha = (e^{+i\varphi}, 1)_\alpha$, $\bar{v}_\alpha = (1, e^{-i\varphi})_\alpha$. Note that this rotation is invertible if and only if $\Pi \neq 0$. We will later see that supercharges with $\Pi = 0$, i.e. whose square does not contain any of the conformal generators, behave quite differently from those that do. The supersymmetry transformations of the rotated scalars take the rather simple form

$$\mathsf{Q}r^1 = \Pi\psi_{\dot{2}+}\,, \quad \mathsf{Q}r^2 = -\Pi\psi_{\dot{1}-}\,, \quad \mathsf{Q}\bar{r}_1 = \Pi\bar{\psi}^{\dot{2}}_{-}\,, \quad \mathsf{Q}\bar{r}_2 = -\Pi\bar{\psi}^{\dot{1}}_{+}\,. \tag{5.26}$$

The supersymmetry condition $\mathsf{Q}W_{1/2} = 0$ can now be rewritten as a system of equations for the (off-)diagonal parts of $\mathcal{L}_{1/2}$ (see app B.1):

$$\mathsf{Q}\mathcal{L}^{\text{diag}}_{1/2} = \mathrm{i}\{\mathcal{L}^{\text{off}}_{1/2}, H\}, \tag{5.27}$$

$$\mathsf{Q}\mathcal{L}^{\text{off}}_{1/2} = \mathcal{D}^{\mathcal{L}^{\text{diag}}_{1/2}}_{\varphi} H. \tag{5.28}$$

Here, H is a matrix whose only nonzero entries are offdiagonal and Grassmann even. In terms of the coefficients $\eta, \bar{\eta}$, it reads

$$H = \begin{pmatrix} 0 & -\bar{\alpha}r^2 \\ -\alpha\bar{r}_2 & 0 \end{pmatrix}, \tag{5.29}$$

and, consequently,

$$H^2 = \bar{\alpha}\alpha \begin{pmatrix} r^2\bar{r}_2 & 0 \\ 0 & \bar{r}_2 r^2 \end{pmatrix}. \tag{5.30}$$

Using (5.26), it is now easily verified that

$$\mathsf{Q}H = -\mathrm{i}\Pi\mathcal{L}^{\text{off}}_{1/2}. \tag{5.31}$$

Acting with Q one more time, and using equation (5.28), we find

$$\begin{aligned} \mathsf{Q}^2 H &= -\mathrm{i}\Pi\,\mathcal{D}^{\mathcal{L}^{\text{diag}}_{1/2}}_{\varphi} H \\ &= -\mathrm{i}\mathcal{D}^{\mathcal{L}^{\text{diag}}_{1/2}}_{\varphi}(\Pi H) + \mathrm{i}H\,\partial_\varphi\Pi. \end{aligned} \tag{5.32}$$

5.3.2 General deformation

We want to systematically study continuous deformations $\mathcal{L}$ around $\mathcal{L}_{1/2}$ which share some supersymmetry with the latter. We first note that the only fermions whose supersymmetry variation under any of the supercharges (5.20) gives derivatives in the φ-direction are those appearing on the RHS of (5.26). As long as $\Pi \neq 0$, all of these can be obtained by taking supersymmetry variations of the rotated scalars, and without loss of generality we may therefore write

$$\mathcal{L} = \mathcal{L}_{1/2} + \mathsf{Q}G + B + C, \tag{5.33}$$

using a supercharge Q and an off-diagonal matrix G incorporating the rotated scalars

$$G = \begin{pmatrix} 0 & \bar{\beta}_a r^a \\ \beta^a \bar{r}_a & 0 \end{pmatrix}, \tag{5.34}$$

as well as a block-diagonal matrix B comprised of scalar bilinears, and a diagonal numerical matrix C. Since contributions proportional to $\mathbf{1}_{(N|M)}$ commute with everything else, they may be immediately exponentiated. Without loss of generality, we ignore such terms and take the constant contribution to be $C = \mathrm{diag}(0, c)$. The requirement that the loop defined by the deformed connection preserve the same supercharge Q takes the form

$$\mathsf{Q}\mathcal{L} = \sigma_3 \, \mathcal{D}^{\mathcal{L}}_{\varphi} \, (H + \Delta H) \, . \tag{5.35}$$

Using (5.27), these translate into a system of equations

$$0 = \mathsf{Q}B - \mathrm{i}\{\mathcal{L}_0^{\mathrm{off}}, \Delta H\} - \mathrm{i}\{\mathsf{Q}G, H + \Delta H\}, \tag{5.36}$$

$$0 = \mathsf{Q}^2 G - \mathcal{D}_{\varphi}^{\mathcal{L}_0^{\mathrm{diag}}} \Delta H + \mathrm{i}[B + C, H + \Delta H]. \tag{5.37}$$

The second of these equations allows us to compute ΔH. To that end, note that $\mathsf{Q}^2 G$ can be decomposed into a total covariant derivative, a term linear in scalars, and a cubic term as (see Appendix B.5)

$$\mathsf{Q}^2 G = -\mathrm{i}\mathcal{D}_{\varphi}^{\mathcal{L}_0^{\mathrm{diag}}} (\Pi G) + \left(\mathsf{Q}^2 G\right)_{\mathrm{linear}} + \left(\mathsf{Q}^2 G\right)_{\mathrm{cubic}} . \tag{5.38}$$

Comparing the terms involving covariant derivatives, we can immediately read off $\Delta H = -\mathrm{i}\Pi G$. Plugging this back into (5.36) and eliminating $\Pi \mathcal{L}_{1/2}^{\mathrm{off}}$ using (5.31), we find

$$0 = \mathsf{Q}\Big(B - \mathrm{i}\{G, H\} - \Pi G^2\Big). \tag{5.39}$$

Unless $\Pi = 0$, Q does not annihilate any scalar bilinears, and we conclude that

$$B = \mathrm{i}\{G, H\} + \Pi G^2. \tag{5.40}$$

Equation (5.36) is now identically satisfied. Turning to the offdiagonal part of the supersymmetry condition, i.e. equation (5.37), and plugging in our above results as well as $(\mathsf{Q}^2 G)_{\mathrm{cubic}} = -[H^2, G]$ (see app B.5), we obtain

$$0 = \left(\mathsf{Q}^2 G\right)_{\mathrm{linear}} + [C, \mathrm{i}H + \Pi G]. \tag{5.41}$$

Using the expressions for $(\mathsf{Q}^2 G)_{\mathrm{linear}}$ derived in Appendix B.5, equation (5.41) may now be recast as four first order ODEs for the coefficient functions $b^{1,2}, \bar{b}_{1,2}$:

$$\begin{aligned} 0 &= \partial_\varphi(\Pi b^1) - \mathrm{i}(c-1)\Pi b^1 \, , \\ 0 &= \partial_\varphi(\Pi b^2) - \mathrm{i}c(\alpha + \Pi b^2) \, , \\ 0 &= \partial_\varphi(\Pi \bar{b}_1) + \mathrm{i}(c-1)\Pi \bar{b}_1 \, , \\ 0 &= \partial_\varphi(\Pi \bar{b}_2) + \mathrm{i}c(-\bar{\alpha} + \Pi \bar{b}_2) \, . \end{aligned} \tag{5.42}$$

While for the unrotated scalars $q^a, \bar{q}_b$ we would have obtained four coupled equations, our choice of basis (5.25) diagonalises this system, which is now readily integrated. Denoting by $\hat{c}(\varphi)$ the antiderivative of $c(\varphi)$ and integration constants by $\beta^{1,2}, \bar{\beta}_{1,2}$, the four independent solutions are:

$$\Pi b^1 = \beta^1 e^{-\mathrm{i}\varphi + \mathrm{i}\hat{c}} \, , \quad \Pi b^2 = \beta^2 e^{+\mathrm{i}\hat{c}} - \alpha \, , \quad \Pi \bar{b}_1 = \bar{\beta}_1 e^{+\mathrm{i}\varphi - \mathrm{i}\hat{c}} \, , \quad \Pi \bar{b}_2 = \bar{\beta}_2 e^{-\mathrm{i}\hat{c}} + \bar{\alpha} \, . \tag{5.43}$$

Note that these functions are periodic if and only if $\hat{c}(\varphi+2\pi)-\hat{c}(\varphi)\in 2\pi\mathbb{Z}$, or, equivalently, if the 0-th Fourier coefficient of $c(\varphi)$ is an integer.

To summarise: For $\Pi \neq 0$, the most general supersymmetric deformation of $\mathcal{L}_{1/2}$ preserving a common supercharge Q is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathsf{Q}G + \mathrm{i}\{G, H\} + \Pi G^2 + C, \tag{5.44}$$

where the coefficient functions in G are given by (5.43). It is instructive to compare this expression to our previous deformation (5.8). Equation (5.44) now contains a term linear in G, while the quadratic term comes with a prefactor Π.

This construction, as well as the moduli spaces swept out by these deformations is the subject of upcoming work [31]. Without presuming to give a full classification of the resulting loops here, we conclude this chapter by remarking on two cases in particular.

5.3.3 Bosonic loops

Note that (5.31) implies that, as long as $\Pi \neq 0$, we are free to make the choice $G = -\mathrm{i}\Pi^{-1}H$, whose supersymmetry variation will precisely cancel the fermionic part $\mathcal{L}^{\text{off}}_{1/2}$ of the undeformed connection. Using (5.32) it is not hard to verify that this choice indeed satisfies the supersymmetry condition (5.41). Indeed, it does so even for arbitrary C, as is to be expected: As, by construction, bosonic superconnections are block-diagonal, they will commute with any choice of C, which can then be exponentiated directly, and we will therefore ignore. Physically, this freedom corresponds to the decoupling of the two connections in the diagonal blocks: While previously the supersymmetry variation of the compound superconnection was a total derivative, for block-diagonal connections it must vanish outright for each block, and the corresponding Wilson loops preserve Q independently.

Consequently, the moduli space associated with each supercharge preserved by $W_{1/2}$ that satisfies $\Pi \neq 0$ contains a bosonic loop. The explicit form of the superconnection, up to C, is readily obtained from (5.44) and (5.30):

$$\mathcal{L}_{\text{bos}} = \mathcal{L}^{\text{diag}}_{1/2} + \Pi^{-1}H^2 = \begin{pmatrix} A_{I,\varphi} + \frac{2\mathrm{i}}{k}\left(M_a^b\, \mu_I{}^a{}_b - \tilde{\mu}_I{}^{\mathrm{i}}{}_{\mathrm{i}}\right) & 0 \\ 0 & A_{I+1,\varphi} + \frac{2\mathrm{i}}{k}\left(M_a^b\, \mu_{I+1b}{}^a - \tilde{\mu}_{I+1}{}^{\mathrm{i}}{}_{\mathrm{i}}\right) - \frac{1}{2} \end{pmatrix}, \tag{5.45}$$

where the couplings to the scalars of the untwisted hypermultiplet are governed by

$$M_a^b = \Pi^{-1}\Pi_{ac}\epsilon^{cb}. \tag{5.46}$$

Note that by tracelessness of the μ's, M_a^b may be freely shifted by a multiple of δ_a^b. For specific choices of Q, this reproduces familiar operators previously described in the literature. Consider for instance the supercharge $\mathsf{Q} = Q_l^{\dot{2}2} + Q_{\bar{l}}^{\dot{1}1}$. It is easy to check that the only nonzero component Π_{ab} is $\Pi_{12} = 1$, and the corresponding coupling matrix is easily seen to be $M_a^b = -\frac{1}{2}\delta_a^1\delta_1^b + \frac{1}{2}\delta_a^2\delta_2^b$, which exactly reproduces the $\mathcal{N}=4$ equivalent of the bosonic loops (5.3) arising from the construction of Gaiotto and Yin [17]. More generally, we can turn on explicit φ-dependence in a controlled fashion by considering a supercharge

$$\mathsf{Q} = \cos\frac{\theta}{2}\,\mathsf{Q}_l^{\dot{1}1} + \sin\frac{\theta}{2}\,\mathsf{Q}_{\bar{r}}^{\dot{1}2} + \cos\frac{\theta}{2}\,\mathsf{Q}_l^{\dot{2}2} - \sin\frac{\theta}{2}\,\mathsf{Q}_r^{\dot{2}1}. \tag{5.47}$$

After a brief calculation, we find

$$M = -\frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-\mathrm{i}\varphi} \\ \sin\theta e^{+\mathrm{i}\varphi} & -\cos\theta \end{pmatrix}, \tag{5.48}$$

and the resulting loops are just the bosonic latitudes (5.15).

However, while certainly the simplest application of the above technique, this construction of bosonic loops already yields some previously unknown operators: In general the couplings to the scalar bilinears will be rational functions of $e^{\pm \mathrm{i}\varphi}$, which have not previously appeared in the literature.

5.3.4 $\Pi = 0$

So far, we have carefully avoided the case of supercharges Q which satisfy $\mathsf{Q}^2 \in \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{\tilde{\mathsf{M}}_\perp}$, since in that case $\Pi = 0$. While (5.44) continues to yield supersymmetric loops for these Q's, it is no longer the most general deformation available to us. The reason for this is threefold. Firstly, there are now fermionic terms which satisfy $\mathsf{Q}\,\psi \sim \partial_\varphi\, q$ (as required), but which cannot be written as $\psi = \mathsf{Q}\, q$. Secondly, the rotation to the new frame $r^1, r^2, \bar{r}_1, \bar{r}_2$ for the scalars is no longer invertible. The third and perhaps most interesting reason is the fact that Q now annihilates some scalar bilinears, specifically $\mathsf{Q}H^2 = 0$, as implied by (5.31). Therefore (5.39) no longer uniquely specifies the term B in our deformation. Instead, we find that inserting completely arbitrary profiles of H^2 into any loop preserving Q does not break Q. These peculiar results warrant a closer look at the supercharges in question. From (5.22), it is easy to translate the requirement $\Pi = 0$ into the conditions on $\eta, \bar{\eta}$.

Indeed, the contributions of $\mathsf{T}_\pm$ to Q^2 vanish iff

$$0 = \epsilon^{ab}\eta^l_a\bar{\eta}^l_b, \qquad 0 = \epsilon^{ab}\eta^r_a\bar{\eta}^r_b, \tag{5.49}$$

or, in other words, if $\eta^l, \bar{\eta}^l$ and $\eta^r, \bar{\eta}^r$, respectively, are linearly dependent:

$$\eta^l_a = t^l w_a\,, \qquad\qquad \eta^r_a = t^r z_a\,, \tag{5.50}$$

$$\bar{\eta}^l_a = \bar{t}^l w_a\,, \qquad\qquad \bar{\eta}^r_a = \bar{t}^r z_a\,. \tag{5.51}$$

Without loss of generality we can take v, w to be normalised. We find that

$$\begin{aligned} \Pi &= (\epsilon^{ab} z_a w_b)\,(\epsilon_{\alpha\beta}\bar{t}^\alpha t^\beta), \\ \lambda_{ab} &= (\epsilon_{\alpha\beta}\bar{t}^\alpha t^\beta) z_{(a} w_{b)} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta} \bar{t}^\alpha t^\beta (\epsilon^{cd} z_c w_d)\epsilon_{ab}. \end{aligned} \tag{5.52}$$

If in addition we demand that Q^2 does not contain any $\mathsf{M}_\parallel$, we must impose $\Pi = 0$. By (5.52), there are two possibilities:

$$\Pi = 0 \iff \begin{cases} 0 = \epsilon^{ab} z_a w_b & \Longrightarrow \lambda_{ab} = \lambda_{ba}, \quad \mathsf{Q}^2 \in \mathfrak{su}(2)_L, \text{ or} \\ 0 = \epsilon_{\alpha\beta}\bar{t}^\alpha t^\beta & \Longrightarrow \lambda_{ab} = -\lambda_{ba}, \quad \mathsf{Q}^2 \in \mathfrak{u}(1)_{\tilde{\mathsf{M}}_\perp}. \end{cases} \tag{5.53}$$

a) $\mathsf{Q}^2 \in \mathfrak{su}(2)_L$**.** Without loss of generality we can let $z = w$. We then have

$$\mathsf{Q}^2 \sim w_a w_b\, \mathsf{R}^{ab}. \tag{5.54}$$

b) $\mathsf{Q}^2 \in \mathfrak{u}(1)_{\tilde{M}_\perp}$**.** In this case,

$$\mathsf{Q}^2 \sim \epsilon^{ab} z_a w_b \, \tilde{\mathsf{M}}_\perp. \tag{5.55}$$

We see that Π can vanish in two distinct ways. While we no longer have the most general supersymmetric deformations at our disposal, there are explicit examples of loops continuously connected to $W_{1/2}$, which are associated with either of these cases. Indeed, the loops defined by superconnections

$$\mathcal{L}_{1/8} = \mathcal{L}_{1/2} + t\Delta\mathcal{L}_{1/8}, \tag{5.56}$$

$$\Delta\mathcal{L}_{1/8} = \begin{pmatrix} -\frac{2\mathrm{i}}{k}\nu_I & \mathrm{i}\left(\bar{\alpha}\psi_{I\dot{1}-} - \bar{\alpha}'\psi_{I\dot{2}+}\right) \\ -\mathrm{i}\left(\alpha\bar{\psi}^{\dot{1}}_{I+} + \alpha'\bar{\psi}^{\dot{2}}_{I-}\right) & -\frac{2\mathrm{i}}{k}\nu_{I+1} + 1 \end{pmatrix}, \tag{5.57}$$

with parameters $\bar{\alpha}'\alpha' = 2\mathrm{i}/k$ are 1/8-BPS for all values of t. The preserved supercharges $\mathsf{Q}_1^{1/8} \equiv \alpha\mathsf{Q}_{\bar{l}}^{\dot{1}1} - \alpha'\mathsf{Q}_l^{\dot{2}1}$ and $\mathsf{Q}_2^{1/8} \equiv \alpha\mathsf{Q}_{\bar{l}}^{\dot{1}2} - \alpha'\mathsf{Q}_l^{\dot{2}2}$ generate $\mathfrak{su}(2)_L$,

$$\{\mathsf{Q}_1^{1/8}, \mathsf{Q}_1^{1/8}\} = -\alpha\alpha' R_+, \tag{5.58}$$

$$\{\mathsf{Q}_1^{1/8}, \mathsf{Q}_2^{1/8}\} = \alpha\alpha' R_3, \tag{5.59}$$

$$\{\mathsf{Q}_2^{1/8}, \mathsf{Q}_2^{1/8}\} = \alpha\alpha' R_+, \tag{5.60}$$

and therefore correspond to case a). Note that this deformation is linear, as is the one obtained from (5.44) when setting $\Pi = 0$. Furthermore, a close comparison with (5.18) shows that at $t = 1$, we obtain another 1/2-BPS loop preserving the same supercharges as $W_{1/2}$.

By a slight modification of this deformation, we find a 1/4-BPS linear deformation realising case b): The loops associated with

$$\mathcal{L}_{1/4} = \mathcal{L}_{1/2} + t\Delta\mathcal{L}_{1/4}, \tag{5.61}$$

$$\Delta\mathcal{L}_{1/4} = \begin{pmatrix} -\frac{2\mathrm{i}}{k}\nu_I & \mathrm{i}\left(\bar{\alpha}\psi_{I\dot{1}-} - \bar{\alpha}'e^{+\mathrm{i}\varphi}\psi_{I\dot{2}+}\right) \\ -\mathrm{i}\left(\alpha\bar{\psi}^{\dot{1}}_{I+} + \alpha' e^{-\mathrm{i}\varphi}\bar{\psi}^{\dot{2}}_{I-}\right) & -\frac{2\mathrm{i}}{k}\nu_{I+1} \end{pmatrix} \tag{5.62}$$

preserve the supercharges

$$\begin{aligned} \mathsf{Q}_1^{1/4} &\equiv \alpha\mathsf{Q}_{\bar{l}}^{\dot{1}1} - \alpha'\mathsf{Q}_r^{\dot{2}1}, \\ \mathsf{Q}_2^{1/4} &\equiv \alpha\mathsf{Q}_{\bar{l}}^{\dot{1}2} - \alpha'\mathsf{Q}_r^{\dot{2}2}, \\ \mathsf{Q}_3^{1/4} &\equiv \alpha\mathsf{Q}_{\bar{r}}^{\dot{1}1} - \alpha'\mathsf{Q}_l^{\dot{2}1}, \\ \mathsf{Q}_4^{1/4} &\equiv \alpha\mathsf{Q}_{\bar{r}}^{\dot{1}2} - \alpha'\mathsf{Q}_l^{\dot{2}2}, \end{aligned} \tag{5.63}$$

whose only nonvanishing anticommutators are

$$\{\mathsf{Q}_1^{1/4}, \mathsf{Q}_4^{1/4}\} = -\{\mathsf{Q}_2^{1/4}, \mathsf{Q}_3^{1/4}\} = \alpha\alpha'\tilde{\mathsf{M}}_\perp. \tag{5.64}$$

Like before, at $t = 1$, we obtain a 1/2-BPS loop, but in a gauge where phases are introduced in the fermionic components.

5.4 Discussion

The machinery presented in Section (5.3.2) yields a cornucopia of new supersymmetric Wilson loops. The first challenge that the further study of this space poses is to find some criteria along which to classify these operators. For supercharges with $\Pi \neq 0$, the moduli space is a finite dimensional cone which contains bosonic loops, not dissimilar to the spaces studied in [21, 30]. By contrast, for supercharges with $\Pi = 0$ the moduli space is infinite dimensional and the precise structure of the correct deformation is as of yet unclear. An interesting set of questions arises in the intermediate case where Π does not vanish identically, but has one or more zeros on the unit circle. At these points, the coefficients in the deformation become singular, and the notion of a continuous deformation itself breaks down. It is therefore tempting to guess that the supercharges preserved by $\mathcal{L}_{1/2}$, and, by extension, the associated Wilson loops, should be properly classified based on the analytic structure of Π, concretely whether the two zeros of $\Pi(z = e^{+i\varphi})$ lie within or outside of the unit circle.

An ostensibly unrelated problem is that of symmetry enhancement. The generic loop we have constructed is 1/16-BPS, but we expect large subspaces where supersymmetry is enhanced to 1/8 or even 1/4-BPS, as happens for instance for the bosonic latitude and the two examples given in Section 5.3.4. The simplest strategy for finding these loops is to carefully examine the bosonic symmetries preserved by a loop, and to explicitly compute the commutators with the preserved supercharge. While loops preserving some amount of R-symmetry are easy to spot, the situation is more involved for special conformal and mixed symmetries. A somewhat more abstract geometric approach is to consider intersections of the moduli spaces constructed above for different supercharges. Ultimately, it would be desirable to entirely classify the possible subalgebras of the 1/2-BPS algebra (2|2) and establish a correspondence with classes of Wilson loops preserving them.

These and related questions are the subject of ongoing work [31], which we hope to report progress on soon.

Chapter 6

Conclusion

A recent surge of interest in the world of defect operators in supersymmetric field theories notwithstanding, there still remain many blank spots on the map, ripe for exploration. The efforts at unraveling the space of BPS defects have been continuously progressing both in breadth and in depth: On one hand, many more such operators have been found in recent years, and an algebraic classification of superconformal defects, while at this stage still in its infancy, is now underway [126]. On the other hand, the study of individual defects has seen great advances, and various methods that were previously used to study specific defects have gradually been developed into a set of tools that may be applied to defects of any codimension. The work presented in this thesis is intended to foray, along these two paths, into two distinct areas which, while far from uncharted territory, have only recently become navigable terrain. Firstly, we established an important supply route for the study of defects in the $\mathcal{N} = (2, 0)$ theory by giving an explicit definition of a locally BPS surface operator in the abelian case, whose conformal anomaly we computed. In order to bridge the chasm between the abelian theory and the holographic result at large N, we then provided a more algebraic characterisation of these objects in terms of its dCFT, which relies only on the preserved symmetry algebra and applies at all N, and whose results agree with the explicit computations. Secondly, we elucidated the structure of known circular Wilson loops in ABJM theory and its less supersymmetric cousins by giving a unified description of their construction and identifying the moduli space formed by these operators. We then proceeded to generalise our deformation approach to the $\mathcal{N} = 4$ case, obtaining many new line operators preserving various amounts of superconformal symmetry, and paving the way for a full classification of these objects in the future.

Comparing these two approaches, there are many immediate extensions of this work, many of which are already being pursued. For one, globally BPS defects in the $\mathcal{N} = (2, 0)$ theory were constructed from our definition by choosing the underlying surface and scalar coupling judiciously, revealing a rich spectrum of operators preserving different amounts of superconformal symmetry [82]. In the same work, the M2-brane configurations dual to these operators in the fundamental representation were given. By analogy with Wilson loops in $\mathcal{N} = 4$ super-Yang-Mills theory, whose holographic duals in the antisymmetric representation are given by D5-branes rather than strings [146], it has been argued that the analogous holographic duals to surface operators in high-dimensional representations are given by M5-branes, which are stabilised in the interior of $AdS_7 \times S^4$ by the four-form flux associated with the supergravity solution and which locally resemble $AdS_3 \times S^3$, with the three-sphere shrinking towards the conformal boundary. Computing the conformal anomaly according to this prescription would provide a vital data point in understanding the behaviour of these operators for more general representations.

Similar questions arise for Wilson loops in ABJM and related theories. In particular, the existence of a continuous, and, indeed, in some cases infinite dimensional moduli space of Wilson loops suggests a continuum of dual brane configurations, which until now have not been properly understood, even for the fundamental representation (see [30] for an account of the state of the art). Furthermore, while dCFT techniques have been applied with some success to both the bosonic and the 1/2-BPS loop [121, 147], a systematic treatment is still lacking. It may be illuminating to reexamine the deformations of superconnections we have considered in this work through the lens of coherent insertions of defect operators. A more intrinsic geometric understanding of the moduli spaces themselves is also desirable: The form of the deformation $Q(\cdot) + \{H, \cdot\} + (\cdot)^2$ is suggestive of a "supercovariant derivative" structure associated with translations along the moduli space, which would be interesting to develop further.

Finally, throughout this thesis, we have adopted a point of view that treats defects as nondynamical external probes, which are at most subject to infinitesimal excitations. This paradigm for instance allows us to interpret a Wilson loop as encoding the forces acting on, and energy radiated by, a heavy quark dragged along a given path, and is therefore heuristically useful. The defect operators we have considered in this thesis are, however, worth considering in their own right, as integral constituents of the theories they live in. In particular, it has long been hypothesized that the worldvolume of a stack of M5-branes supports a theory of selfdual strings, corresponding to M2-branes stretching between the M5's [22, 39, 148], of which the surface defects considered here are merely the large tension limit. The nature of these strings is still more obscure than that of the $\mathcal{N} = (2, 0)$ theory itself, but gradually our improved understanding of surface defects may allow us to color in some parts of the map.

Appendix A

Surfaces

A.1 Conventions

In Chapters 3 and 4 we work in Minkowski space with mostly positive signature. We make use of the following indices:

Index	Usage
$M = 1, \ldots, 11$	11d spacetime vector X^M
$A = 1, \ldots, 32$	11d spinors
$\mu = 1, \ldots, 6$	6d spacetime vectors x^μ
α ($\dot{\alpha}$) $= 1, \ldots, 4$	6d chiral (antichiral) spinors
$i = 1, \ldots, 5$	R-symmetry vectors
$\check{\alpha} = 1, \ldots, 4$	R-symmetry spinors
$a' = 1, \ldots, 4$	spacetime vectors orthogonal to the surface
$a = 1, 2$	worldsheet coordinates σ^a
$\hat{a} = 1, 2, 3$	worldvolume coordinates $\hat{\sigma}^{\hat{a}}$

Our usage of spinors is restricted to the supersymmetry transformations (3.8) and (3.46) but we include our conventions for completeness. In general we follow the NW-SE convention for index contraction:

$$\bar{\Phi}\Psi \equiv \bar{\Phi}^A \Psi_A \,. \tag{A.1}$$

The conjugate and transpose act as

$$(\Psi_A)^* = (\Psi^*)^A \,, \qquad \left(\mathcal{C}^{AB}\right)^T = \mathcal{C}^{BA} \,. \tag{A.2}$$

Below we detail the properties of gamma matrices in $d = 11$ and $d = 6$, and we state the reality condition on spinors. More details can be found in [83] and references therein.

A.1.1 $d = 11$ Clifford algebra

The 11d Clifford algebra is generated by the set of matrices $(\Gamma_M)_A{}^B$ satisfying

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}. \tag{A.3}$$

Here for readability M is used for flat spacetime, unlike (3.46) where it denotes curved spacetime.

The matrices may be chosen such that $\Gamma_0^\dagger = -\Gamma_0$ is antihermitian while the others are hermitian $\Gamma_M^\dagger = \Gamma_M$ $(M \neq 0)$. In addition, there is an orthogonal, real anti-symmetric matrix $\mathcal{C}_{AB}$ such that $\Gamma_M \mathcal{C} = -\left(\Gamma_M \mathcal{C}\right)^T$. $\mathcal{C}$ naturally defines a real structure by relating Ψ and $\Psi^\dagger$ as

$$\bar{\Psi} \equiv -i\Gamma_0 \Psi^\dagger = \mathcal{C}^\dagger \Psi. \tag{A.4}$$

This is the Majorana condition.

A.1.2 $d = 6$ Clifford algebra

An easy way to construct the 6d Clifford algebra is to decompose $\Gamma_M = \{\Gamma_\mu, \Gamma_i\}$ by introducing a chirality matrix $\Gamma_* = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5$. The matrices are then (in the chiral basis)

$$\Gamma_\mu = \begin{pmatrix} 0 & \bar{\gamma}_\mu \\ \gamma_\mu & 0 \end{pmatrix} \otimes I_4 \,, \qquad \Gamma_i = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \otimes \check{\gamma}_i \,, \qquad \Gamma_* = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \otimes I_4 \,, \tag{A.5}$$

where the algebra is

$$\bar{\gamma}_\mu \gamma_\nu + \bar{\gamma}_\nu \gamma_\mu = 2\eta_{\mu\nu} \,, \qquad \gamma_\mu \bar{\gamma}_\nu + \gamma_\nu \bar{\gamma}_\mu = 2\eta_{\mu\nu} \,, \qquad \{\check{\gamma}_i, \check{\gamma}_j\} = 2\delta_{ij} \,. \tag{A.6}$$

Since γ_μ and $\check{\gamma}_i$ commute, they define independent spinor representations. Explicitly, we decompose $A = (\dot{\alpha} \oplus \alpha) \otimes \check{\alpha}$, so that the indices are $(\gamma_\mu)_\alpha{}^{\dot{\beta}}$, $(\bar{\gamma}_\mu)_{\dot{\alpha}}{}^\beta$ and $(\check{\gamma}_i)_{\check{\alpha}}{}^{\check{\beta}}$. The chiral and antichiral representations are related through

$$\bar{\gamma}_\mu^\dagger = \gamma_0 \bar{\gamma}_\mu \gamma_0 \Rightarrow \begin{cases} \bar{\gamma}_0^\dagger = -\gamma_0 \,, & \\ \bar{\gamma}_\mu^\dagger = \gamma_\mu \,, & \mu \neq 0 \,. \end{cases} \tag{A.7}$$

The chirality operator gives 2 additional constraints

$$\gamma_{012345} = I \,, \qquad \bar{\gamma}_{012345} = -I \,, \tag{A.8}$$

with $\gamma_{\mu\nu\ldots\rho} \equiv \gamma_{[\mu}\bar{\gamma}_\nu \ldots \gamma_{\rho]}$ the antisymmetrised product of γ-matrices.[1] The charge conjugation matrix takes the form

$$\mathcal{C}_{AB} = \begin{pmatrix} 0 & c_{\dot{\alpha}\beta} \\ c_{\alpha\dot{\beta}} & 0 \end{pmatrix} \otimes \Omega_{\check{\alpha}\check{\beta}} \,, \qquad c \equiv c_{\dot{\alpha}\beta} \,, \tag{A.9}$$

and is used to lower (or raise) spinor indices. The matrix $\Omega_{\check{\alpha}\check{\beta}}$ is the real, antisymmetric symplectic metric of $\mathfrak{sp}(4)$ and c is unitary:

$$c^\dagger c = c^{\alpha\dot{\alpha}} c_{\dot{\alpha}\beta} = \delta_\alpha^\beta \,, \qquad c^* c^T = c^{\dot{\alpha}\alpha} c_{\alpha\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \,, \qquad \Omega^\dagger \Omega = \Omega^{\check{\alpha}\check{\beta}} \Omega_{\check{\beta}\check{\gamma}} = \delta_{\check{\gamma}}^{\check{\alpha}} \,. \tag{A.10}$$

They satisfy

$$(\gamma_\mu c) = -\left(\gamma_\mu c\right)^T \,, \qquad \left(\bar{\gamma}_\mu c^T\right) = -\left(\bar{\gamma}_\mu c^T\right)^T \,, \qquad (\check{\gamma}_i \Omega) = -\left(\check{\gamma}_i \Omega\right)^T \,. \tag{A.11}$$

[1] (Anti-)symmetrisation is understood with the appropriate combinatorial factors, i.e. $A_{[ab]} = \frac{1}{2} A_{ab} - A_{ba}$.

A representation of this algebra is given by

$$\begin{aligned}
&\gamma_0 = \bar\gamma_0 = iI_2 \otimes I_2\,, && \gamma_1 = -\bar\gamma_1 = -i\sigma_1 \otimes I_2\,, && \gamma_2 = -\bar\gamma_2 = -i\sigma_2 \otimes I_2\,,\\
&\gamma_3 = -\bar\gamma_3 = i\sigma_3 \otimes \sigma_1\,, && \gamma_4 = -\bar\gamma_4 = i\sigma_3 \otimes \sigma_2\,, && \gamma_5 = -\bar\gamma_5 = -i\sigma_3 \otimes \sigma_3\,,\\
&\check\gamma_1 = \sigma_1 \otimes \sigma_2\,, \quad \check\gamma_2 = \sigma_2 \otimes \sigma_2\,, && \check\gamma_3 = \sigma_3 \otimes \sigma_2\,, \quad \check\gamma_4 = I_2 \otimes \sigma_1\,, && \check\gamma_5 = I_2 \otimes \sigma_3,\\
&c = -c^T = \sigma_1 \otimes i\sigma_2\,, && \Omega = i\sigma_2 \otimes I_2\,. &&
\end{aligned} \tag{A.12}$$

A.1.3 Symplectic Majorana condition

In 6d the spinor Ψ decomposes into a chiral and an antichiral 6d spinor as

$$\Psi_A = \begin{pmatrix} \bar\chi_{\dot\alpha\check\alpha} \\ \psi_{\alpha\check\alpha} \end{pmatrix}, \qquad \bar\Psi^A \equiv \begin{pmatrix} -i(\psi^\dagger)^{\alpha\check\alpha}(\gamma_0)_\alpha{}^{\dot\alpha} & -i(\bar\chi^\dagger)^{\dot\alpha\check\alpha}(\bar\gamma_0)_{\dot\alpha}{}^{\alpha} \end{pmatrix} \equiv \begin{pmatrix} \bar\psi^{\dot\alpha\check\alpha} & \chi^{\alpha\check\alpha} \end{pmatrix}. \tag{A.13}$$

The Majorana condition on Ψ then translates to

$$\chi^{\alpha\check\alpha} = (c^\dagger\Omega^\dagger\bar\chi)^{\alpha\check\alpha} = (c\Omega\bar\chi)^{\alpha\check\alpha}\,, \qquad \bar\psi^{\dot\alpha\check\alpha} = (c^*\Omega^\dagger\psi)^{\dot\alpha\check\alpha} = -(c\Omega\psi)^{\dot\alpha\check\alpha}\,, \tag{A.14}$$

where in the second equality we use the properties of our representation. The inclusion of the symplectic form Ω in (A.14) is the reason these equations are known as the *symplectic Majorana condition*. The spinors $\bar\varepsilon^0$, ε^1, and ψ in (3.8) are of this type.

A.2 Geometry of submanifolds

In this appendix we assemble the geometry results used throughout Chapters 3 and 4 as well as in Appendix A.3. Sections A.2.1 and A.2.2 contain our conventions for Riemann curvature and the definition of the second fundamental form of an embedded submanifold as well as some standard results relating the two. In Section A.2.3 the second fundamental form is related to the coefficients of the normal coordinate expansion of the embedding.

A.2.1 Riemann curvature

We adopt the convention where the Riemann tensor is defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho}\,. \tag{A.15}$$

It is convenient to split it into a conformally invariant Weyl tensor $W_{\mu\nu\rho\sigma}$ and the Schouten tensor $P_{\mu\nu}$,

$$P_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} - \frac{R}{2(d-1)}g_{\mu\nu}\right), \tag{A.16}$$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu\rho}P_{\nu\sigma} + g_{\mu\sigma}P_{\nu\rho} + g_{\nu\rho}P_{\mu\sigma} - g_{\nu\sigma}P_{\mu\rho}\,. \tag{A.17}$$

A.2.2 Extrinsic curvature

We define the second fundamental form to be

$$\mathrm{I\!I}^{\mu}_{ab} = \left(\partial_a \partial_b x^{\lambda} + \partial_a x^{\rho} \partial_b x^{\sigma} \Gamma^{\lambda}_{\rho\sigma}\right) \left(\delta^{\mu}_{\lambda} - g_{\kappa\lambda} \partial^c x^{\kappa} \partial_c x^{\mu}\right) . \tag{A.18}$$

The second part is the projector to the components orthogonal to the surface (defined by its embedding $x^{\mu}(\sigma)$), while the first part is the action of the covariant derivative on the (pullback) of $x^{\lambda}(\sigma)$. The mean curvature vector is then

$$H^{\mu} = h^{ab} \mathrm{I\!I}^{\mu}_{ab} . \tag{A.19}$$

These invariants are related to the intrinsic curvature of Σ and M by the Gauss-Codazzi equation

$$R^{\Sigma}_{abcd} = R^{M}_{abcd} + 2 \mathrm{I\!I}^{\mu}_{a[b} \mathrm{I\!I}^{\nu}_{c]d} g_{\mu\nu} . \tag{A.20}$$

Contracting twice with h^{-1} and expanding the Riemann tensor in terms of the Weyl and Schouten tensors, we obtain

$$\left(H^2 + 4\mathrm{Tr} P\right) = 2R^{\Sigma} + 2\mathrm{Tr}\tilde{\mathrm{I\!I}}^2 - 2\mathrm{Tr} W , \tag{A.21}$$

where $\tilde{\mathrm{I\!I}}^{\mu}_{ab}$ is the traceless part of the second fundamental form

$$\tilde{\mathrm{I\!I}}^{\mu}_{ab} = \mathrm{I\!I}^{\mu}_{ab} - \frac{H^{\mu}}{2} h_{ab} . \tag{A.22}$$

A.2.3 Embedding in normal coordinates

Using these standard geometry results, we now derive the expressions needed for (3.27) and (3.33). Unlike in Section 3.3, we state here the result for a generic curved spacetime M. This allows us to perform the calculation in Appendix A.3 on curved space.

Let x^{μ} and η^a be Riemann normal coordinates on M and Σ about the same point. In terms of these, the embedding $\Sigma \hookrightarrow M$ may be expanded as

$$x^{\mu}(\eta) = x^{\mu}(0) + \eta^a v^{\mu}_a + \frac{1}{2} \eta^a \eta^b v^{\mu}_{ab} + \frac{1}{6} \eta^a \eta^b \eta^c v^{\mu}_{abc} + \mathcal{O}(\eta^4) . \tag{A.23}$$

These coefficients are constrained by the condition that straight lines in normal coordinates correspond to geodesics. In particular, a curve on Σ given by a straight line in η has constant speed and its curvature in M is normal to Σ at every point, which gives the constraints

$$\begin{aligned} \delta_{ab} &= v_a \cdot v_b , \\ 0 &= v_{ab} \cdot v_c , \\ 0 &= 3\, v_d \cdot v_{abc} + v_{ab} \cdot v_{cd} + v_{ac} \cdot v_{bd} + v_{ad} \cdot v_{bc} . \end{aligned} \tag{A.24}$$

Using (A.18) one easily checks that the second order coefficient equals the second fundamental form

$$\mathrm{I\!I}^{\mu}_{ab}|_{\eta=0} = v^{\mu}_{ab} . \tag{A.25}$$

The geodesic distance between $\xi(\eta)$ and the origin of the normal frame is found from (A.23)

$$|x(\eta) - x(0)|^2 = \eta^a \eta_a - \frac{1}{12} \mathrm{II}_{ab} \cdot \mathrm{II}_{cd} \eta^a \eta^b \eta^c \eta^d + \mathcal{O}(\eta^5) \,. \tag{A.26}$$

Furthermore, in normal coordinates, the metrics take the form

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} - \frac{1}{3} R^M_{\mu\rho\nu\sigma} \xi^\rho \xi^\sigma + \mathcal{O}(\xi^3) \,, \\ h_{ab} &= \delta_{ab} - \frac{1}{3} R^\Sigma_{acbd} \eta^c \eta^d + \mathcal{O}(\eta^3) \,, \end{aligned} \tag{A.27}$$

which yields an expansion for the volume factor

$$\sqrt{h(\eta)} = 1 - \frac{1}{6} R^\Sigma_{ab} \eta^a \eta^b + \mathcal{O}(\eta^3) \,. \tag{A.28}$$

A.3 Geodesic point-splitting

In this appendix we present an alternative regularisation of (3.13), essentially point splitting, displacing one copy of the surface operator by a distance ϵ in an arbitrary normal direction ν. This regularisation is used in [27,67], but there the vector ν is taken to be a constant, and therefore the method is only applicable if the operators are restricted to a codimension-one subspace.

The technology used to define this regularisation scheme applies for generic smooth embedded surfaces in a Riemannian manifold, and we present here a curved space calculation, as opposed to Section 3.3.3, where for brevity we restricted ourselves to flat space. However, we still have to restrict to conformally flat backgrounds, since otherwise we do not have a short-distance expansion for the propagator and therefore still cannot infer the anomaly coefficient b.

As expected, we recover the result (3.36) exactly, and thus verify scheme-independence.

A.3.1 Displacement map

We can regularise the integral (3.13) by displacing a copy of the surface a distance ϵ along a unit normal vector field ν. Under that map, which we denote by $\mathcal{T}$, the geodesic distance admits an expansion of the form

$$|\mathcal{T}(x^\mu(\sigma)) - x^\mu(\sigma + \eta)|^2 = \epsilon^2 + \eta^2 + \sum_{k=3}^{\infty} \sum_{l=0}^{k} f_l^{(k)} \eta^l \epsilon^{k-l} \,. \tag{A.29}$$

We can combine the terms of fixed k in terms of degree k polynomials $f^{(k)}$

$$\sum_{l=0}^{k} f_l^{(k)} \eta^l \epsilon^{k-l} = \epsilon^k f^{(k)}(\eta/\epsilon) \,. \tag{A.30}$$

We calculate the higher order terms in (A.29) explicitly in (A.35), but first we note that the only terms contributing to the divergent part are $f^{(3)}$ and $f^{(4)}$. To see that, the integrals computing the expectation value take the form

$$\int_0^\rho \frac{\eta^{m+1}\mathrm{d}\eta}{|\mathcal{T}(x^\mu(\sigma)) - x^\mu(\sigma+\eta)|^4}\,, \tag{A.31}$$

where ρ is an arbitrary but fixed IR cutoff. We can evaluate (A.31) by expanding the integrand in ϵ. Writing $s \equiv \eta/\epsilon$, we obtain

$$\epsilon^{m-2}\int_0^{\rho/\epsilon} \frac{s^{m+1}}{(1+s^2)^2}\left[1 - \frac{2f^{(3)}(s)}{1+s^2}\epsilon + \left(\frac{3(f^{(3)}(s))^2}{(1+s^2)^2} - \frac{2f^{(4)}(s)}{1+s^2}\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\right]\mathrm{d}s\,. \tag{A.32}$$

By application of Faà di Bruno's formula one checks that the terms in brackets of order ϵ^n contribute to the divergence only if $m+n \leq 2$. We can therefore safely ignore higher orders in ϵ. Only a finite number of terms remains to be computed and we find that the only divergent integrals (A.31) are:

$$m=0: \quad \frac{1}{2\epsilon^2} - \frac{1}{8\epsilon}\left(4f_0^{(3)} + \pi f_1^{(3)} + 4f_2^{(3)} + 3\pi f_3^{(3)}\right) + \left(-3(f_3^{(3)})^2 + 2f_4^{(4)}\right)\log\epsilon\,, \tag{A.33a}$$

$$m=1: \quad \frac{\pi}{4\epsilon} + 2f_3^{(3)}\log\epsilon\,, \tag{A.33b}$$

$$m=2: \quad -\log\epsilon\,. \tag{A.33c}$$

The relevant coefficients can be read off of the expansion of the geodesic distance up to combined order of 4 in η and ϵ. The second term on the left hand side of (A.29) can be expanded simply using the embedding (A.23). For the first term, we solve the geodesic equation order by order in the displacement ϵ to obtain

$$\mathcal{T}(x^\mu) = x^\mu + \epsilon\nu^\mu - \frac{\epsilon^2}{2}\Gamma^\mu_{\kappa\lambda}\nu^\kappa\nu^\lambda + \frac{\epsilon^3}{6}\left(-\partial_\nu\Gamma^\mu_{\rho\sigma} + 2\Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\rho\sigma}\right)\nu^\nu\nu^\rho\nu^\sigma + \mathcal{O}(\epsilon^4)\,. \tag{A.34}$$

Combining these expressions, and writing $\eta^a = \eta e^a(\varphi)$ as in (3.29) and onwards, the only two non-vanishing relevant coefficients read

$$\begin{aligned} f_2^{(3)} &= -e^a e^b \mathrm{I\!I}_{ab}\cdot\nu\,,\\ f_4^{(4)} &= -\frac{1}{12}e^a e^b e^c e^d \mathrm{I\!I}_{ab}\cdot\mathrm{I\!I}_{cd}\,. \end{aligned} \tag{A.35}$$

The first contributes to a scheme-dependent divergence ϵ^{-1}, while the second contributes to the anomaly.

A.3.2 Evaluation of the anomaly

With the displacement map (A.34) in hand, we can evaluate (3.13). The propagators on a conformally flat background can be obtained by considering curved space actions for a conformal scalar and a Maxwell-type 2-form and inverting the kinetic operators order by order, following [27] and [65]. We find:

$$\langle \Phi_i(x)\Phi_j(x+\xi)\rangle = \frac{\delta_{ij}}{\pi^2 |\xi|^4}\left[1+\frac{1}{3}P_{\mu\nu}\xi^\mu\xi^\nu + \mathcal{O}(\xi^3)\right], \tag{A.36}$$

$$\begin{aligned}\left\langle B^{+\mu\nu}(x)B^+_{\rho\sigma}(x+\xi)\right\rangle = \frac{1}{4\pi^2|\xi|^4}\Big[&\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu \\ &-\frac{4}{3}\left(4P^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}\delta_{\lambda\tau} + P_{\lambda[\rho}\delta^{[\mu}_{\sigma]}\delta^{\nu]}_\tau + \delta_{\lambda[\rho}P^{[\mu}_{\sigma]}\delta^{\nu]}_\tau\right)\xi^\lambda\xi^\tau + \mathcal{O}(\xi^3)\Big].\end{aligned} \tag{A.37}$$

To apply our regularisation, we should replace ξ by (A.29) in the denominator of the propagators before performing the integral over η. A priori, we should also perform the displacement in the numerator, since a term of order $\mathcal{O}(\epsilon)$ can contribute to the ϵ^{-1} divergence by multiplying (A.33a). However, one easily checks that the only terms of that order are accompanied by nonzero powers of η, and therefore do not contribute to the divergence of (3.13). We therefore drop the ϵ in the numerators of the propagators.

The expansion of the numerators is then assembled, as before, from (3.27) and (3.33), but in addition, since we are working on curved space, we obtain an additional term at $\mathcal{O}(\eta^2)$ explicitly involving $\mathrm{Tr}P$ from the propagators (A.36). Collecting terms in analogy to Section 3.3.3, and integrating out the angular coordinate using (3.29), we obtain the scalar contribution

$$\frac{1}{2\pi\epsilon^2} + \frac{H\cdot\nu}{4\pi\epsilon} + \frac{1}{16\pi}\left(2R^\Sigma - \left(H^2 + 4\mathrm{Tr}P\right) + 4\left(\partial n\right)^2\right)\log\epsilon + \text{finite}, \tag{A.38}$$

while the tensor field yields

$$-\frac{1}{2\pi\epsilon^2} - \frac{H\cdot\nu}{4\pi\epsilon} - \frac{1}{16\pi}\left(-2R^\Sigma + 3\left(H^2+4\mathrm{Tr}P\right)\right)\log\epsilon + \text{finite}. \tag{A.39}$$

Combining these terms, we find

$$\log\langle V_\Sigma\rangle = \frac{1}{4\pi}\log\epsilon\int_\Sigma \mathrm{vol}_\Sigma\left[R^\Sigma - \left(H^2+4\mathrm{Tr}P\right) + (\partial n)^2\right] + \text{finite}, \tag{A.40}$$

which agrees exactly with (3.36). Note that the scheme dependence, which is present in the simple pole of both (A.38) and (A.39), cancels in the final result, and the terms H^2 and $\mathrm{Tr}P$ combine to an anomaly term as in (3.4), as required.

A.4 Conformal Ward identities for defect correlators

In this appendix, we derive explicit expressions for the structure of the expectation values of stress tensor primaries in the presence of a flat conformal surface defect. Up to overall normalisation constants, which we further constrain in Section 4.3 using supersymmetry, these correlators are completely fixed by the bosonic symmetries (conformal and R-symmetry) preserved by the defect. We consider both defects with an insertion of a single primary of the displacement operator multiplet, and defects without such insertions. For brevity, we do not give an exhaustive list of such correlators and instead focus on those we require in the main text. More specifically, we compute only the expectation values of the primaries in the stress tensor multiplet, and some 2-point functions involving low-level primaries, namely $O^{IJ}, \chi^I_{\alpha\check{\alpha}}, H^I_{\lambda\mu\nu}$ in the stress tensor, and $\mathbb{O}^i, \mathbb{Q}_{\alpha\check{\alpha}}$ in the displacement multiplet. The remaining correlators can of course be calculated using the same method.

We proceed in two steps. First, we fix the dependence on σ and x by implementing the Ward identities associated with the conformal symmetry preserved by the defect as well as transverse rotational symmetry. For clarity, in this calculation we suppress the R-symmetry indices of the operators and leave the scaling dimensions general. Indeed, as much of the kinematics is easily generalised to defects of dimension p in arbitrary spacetime dimension $d = p + q$, we state the more general result wherever we can do so without obscuring the results we presently need. Secondly, we fix the R-symmetry tensor structure of these correlators by demanding invariance under the residual $\mathfrak{so}(4)_R$ symmetry. Throughout, we denote generic operators in the bulk $\mathcal{O}$ and on the defect $\hat{\mathcal{O}}$.

Many of the kinematical results have been obtained by different methods in the past. In particular, the embedding space formalism allows for the efficient computation of bosonic correlators [120]. However, it is not straightforwardly applicable to correlators involving fermions.

A.4.1 Defect without insertions

We want to solve the constraints that the residual conformal symmetry places on expectation values of the form $\langle \mathcal{O} V \rangle$ with $\mathcal{O}$ a bulk operator of scaling dimension Δ. The representation of the conformal algebra (A.60) acting on $\mathcal{O}$ is given in terms of the representation of $\mathcal{O}$ under Lorentz transformations $S_{\mu\nu}$ and is

$$\begin{gathered}\mathsf{P}_\mu = \partial_\mu, \qquad \mathsf{M}_{\mu\nu} = 2x_{[\mu}\partial_{\nu]} + S_{\mu\nu}, \qquad \mathsf{D} = -x^\mu\partial_\mu - \Delta\,, \\ \mathsf{K}_\mu = x^2\partial_\mu - 2x_\mu(x^\nu\partial_\nu + \Delta) + 2x^\nu S_{\nu\mu}.\end{gathered} \tag{A.41}$$

Treating separately the coordinates along the plane σ^a and tranverse x^m, translation invariance on the plane implies that $\langle \mathcal{O}(\sigma, x) V \rangle$ is a function of x^m only. The other Ward identities can be cast into the form:

$$\begin{aligned}0 &= S_{ab}\langle \mathcal{O} V \rangle\,, \\ 0 &= (x^m\partial_m + \Delta)\,\langle \mathcal{O} V \rangle\,, \\ 0 &= x^m S_{am}\langle \mathcal{O} V \rangle\,, \\ 0 &= (x_m\partial_n - x_n\partial_m)\,\langle \mathcal{O} V \rangle + S_{mn}\langle \mathcal{O} V \rangle\,.\end{aligned} \tag{A.42}$$

These constraints are now straightforwardly solved. We focus on scalars O, vectors j_μ, selfdual 3-forms $H_{\lambda\mu\nu}$ and traceless symmetric 2-tensors $T_{\mu\nu}$, as operators of those types make up the bosonic degrees of freedom of the stress tensor multiplet, while the correlators of fermionic operators with a scalar defect vanish identically.

For a Lorentz scalar O, all $S_{\mu\nu}$ vanish and the conformal Ward identities (A.42) are immediately solved to give

$$\langle O(\sigma, x) V \rangle = \frac{h_O}{x^\Delta}\,, \tag{A.43}$$

with h_O an as yet undetermined constant.

The transformation law for a vector reads

$$\left(S_{\mu\nu}j\right)_\rho = \delta_{\mu\rho}j_\nu - \delta_{\nu\rho}j_\mu\,, \tag{A.44}$$

which, plugged into (A.42) eventually leads to[2]

$$\langle j_a V\rangle = \langle j_m V\rangle = 0. \tag{A.45}$$

For higher spin bosonic operators, each Lorentz index separately transforms as (A.44). For a 3-form $H_{\lambda\mu\nu}$, the Ward identities (A.42) imply that the only components with nonvanishing expectation value in the presence of V are H_{abm} and H_{lmn}, and furthermore restricts the available terms for their one-point functions to

$$\langle H_{abm}(x)V\rangle \sim \frac{\epsilon_{ab}x_m}{x^{\Delta+1}}\,, \qquad \langle H_{lmn}(x)V\rangle \sim \frac{\epsilon_{lmnp}x^p}{x^{\Delta+1}}\,. \tag{A.46}$$

In this work, we are concerned with 3-forms which come with a selfduality condition, which serves to relate the proportionality constants in (A.46). We are left with

$$\langle H_{abm}(x)V\rangle = h_H\frac{\epsilon_{ab}x_m}{x^{\Delta+1}}\,, \qquad \langle H_{lmn}(x)V\rangle = h_H\frac{\epsilon_{lmnp}x^p}{x^{\Delta+1}}\,. \tag{A.47}$$

Lastly, we repeat the same analysis for a symmetric traceless 2-tensor. Exactly the same line of argument as above yields

$$\begin{aligned} \langle T_{ab}(x)V\rangle &= \frac{h_T}{x^\Delta}\delta_{ab}\,, \qquad \langle T_{am}(x)V\rangle = 0\,, \\ \langle T_{mn}(x)V\rangle &= \frac{h_T}{x^{\Delta+2}}\left(2x_m x_n - x^2\delta_{mn}\right)\,. \end{aligned} \tag{A.48}$$

We are now in a position to construct the correlator of V with any bosonic primary in the stress tensor multiplet. To that end, recall that, under the unbroken $\mathfrak{so}(5)_R$, O^{IJ} and $H^I_{\lambda\mu\nu}$ transform as a symmetric traceless 2-tensor and a vector, respectively, while the stress tensor $T_{\mu\nu}$ is an R-symmetry singlet.[3] Without explicitly applying the Ward identities associated with the preserved $\mathfrak{so}(4)_R$, we can fix the R-symmetry structure of the 1-point functions by writing down the available terms and, for O^{IJ}, implementing tracelessness. Plugging in the correct scaling dimensions $\Delta_O = 4$, $\Delta_H = 5$, and $\Delta_T = 6$, we find the only nonvanishing 1-point functions of stress tensor primaries in the presence of V are (4.21).

[2]More generally, for a p-dimensional defect in a spacetime of dimension $d = p + q$, one obtains

$$\langle j_a(x)V\rangle = 0\,, \qquad (q-2)\langle j_m(x)V\rangle = 0\,.$$

Indeed, for $q = 2$, the transverse components of j can take the form

$$\langle j_m(x)V\rangle \sim \frac{\epsilon_{mn}x^n}{x^{\Delta+1}}\,,$$

which is compatible with conservation.

[3]The R-symmetry current j^{IJ}_μ transforms as an antisymmetric tensor, but as seen above, its 1-point function vanishes identically regardless of the R-symmetry structure.

A.4.2 Defect with an insertion

We now repeat the above discussion for correlators $\left\langle \mathcal{O}(\sigma,x)V[\hat{\mathcal{O}}(\sigma')]\right\rangle$ involving a defect with an insertion of a displacement multiplet primary. The kinematical analysis is more involved than, but technically very similar to, the previous subsection. We use translation invariance to center $\hat{\mathcal{O}}$ at $\sigma'=0$ and suppress the arguments of $\mathcal{O}(\sigma,x)$. The conformal Ward identities may be cast into the form:

$$\begin{aligned}
0 &= \Big((\sigma_a\partial_b-\sigma_b\partial_a)+\hat{S}_{ab}+S_{ab}\Big)\left\langle \mathcal{O}V[\hat{\mathcal{O}}]\right\rangle ,\\
0 &= \Big((x_m\partial_n-x_n\partial_m)+\hat{S}_{mn}+S_{mn}\Big)\left\langle \mathcal{O}V[\hat{\mathcal{O}}]\right\rangle ,\\
0 &= \Big(\sigma^a\partial_a+x^m\partial_m+\Delta+\hat{\Delta}\Big)\left\langle \mathcal{O}V[\hat{\mathcal{O}}]\right\rangle ,\\
0 &= \Big(2x^mS_{am}+2\sigma^bS_{ab}+2\hat{\Delta}\sigma_a+(\sigma^2+x^2)\partial_a\Big)\left\langle \mathcal{O}V[\hat{\mathcal{O}}]\right\rangle .
\end{aligned} \tag{A.49}$$

For the simplest case of a scalar $\mathbb{O}$ on the defect and a scalar O in the bulk, (A.49) become particularly simple, and imply[4]

$$\langle O(\sigma,x)V[\mathbb{O}]\rangle = \frac{C_{O\mathbb{O}}}{x^{\Delta-\hat{\Delta}}(\sigma^2+x^2)^{\hat{\Delta}}}, \tag{A.50}$$

with $C_{\mathbb{O}O}$ some normalisation constant.

For a defect scalar $\mathbb{O}$ and a bulk vector j_μ we obtain:

$$\begin{aligned}
\langle j_a(\sigma,x)V[\mathbb{O}]\rangle &= \frac{C_{j\mathbb{O}}\sigma_a}{x^{\Delta-\hat{\Delta}-1}(\sigma^2+x^2)^{\hat{\Delta}+1}} ,\\
\langle j_m(\sigma,x)V[\mathbb{O}]\rangle &= \frac{C_{j\mathbb{O}}(x^2-\sigma^2)x_m}{2x^{\Delta-\hat{\Delta}+1}(\sigma^2+x^2)^{\hat{\Delta}+1}} .
\end{aligned} \tag{A.51}$$

Indeed, these correlators are exactly the same for defects of generic dimension and codimension. It is easily checked that (A.51) is compatible with conservation of j in the bulk if and only if $\Delta=d-1$ and $\hat{\Delta}=p$, which is indeed satisfied by the displacement superprimary $\mathbb{O}^i$ and the bulk R-symmetry current j^{IJ}_μ. The conservation equation

$$\partial_\mu\langle j^\mu(\sigma,x)V[\mathbb{O}]\rangle = \langle V[\mathbb{O}(\sigma)\mathbb{O}(0)]\rangle, \tag{A.52}$$

then allows us to fix $C_{\mathbb{O}}$ in terms of $C_{\mathbb{O}j}$ in equation (4.33). For the remaining required bosonic correlator, consider a defect scalar $\mathbb{O}$ and a bulk 3-form $H_{\lambda\mu\nu}$. The conformal Ward identities (A.49) imply that the only components of the correlator that do not vanish identically are

$$\begin{aligned}
\langle H_{abm}(\sigma,x)V[\mathbb{O}]\rangle &= \frac{h_H\epsilon_{ab}x_m}{x^{\Delta-\hat{\Delta}+1}(\sigma^2+x^2)^{\hat{\Delta}}} ,\\
\langle H_{lmn}(\sigma,x)V[\mathbb{O}]\rangle &= \frac{h_H\epsilon_{lmnp}x^p}{x^{\Delta-\hat{\Delta}+1}(\sigma^2+x^2)^{\hat{\Delta}}} ,
\end{aligned} \tag{A.53}$$

[4]In particular, inserting for $\mathbb{O}$ the defect identity operator $\mathbf{1}_V$, we recover the form of (A.43), as expected.

where, as for the 1-point function, we have used the selfduality of $H_{\lambda\mu\nu}$ to relate the two normalisation constants. Lastly, we compute the only correlator of fermions that we require in our analysis. Consider a bulk chiral spinor χ_α and a defect chiral spinor $\mathbb{Q}_\alpha$.[5] Their transformation laws are familiar:

$$(S_{\mu\nu}\chi)_\alpha = \frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha{}^\beta \chi_\beta\,, \qquad (S_{ab}\mathbb{Q})_\alpha = \frac{1}{2}\left(\gamma_{ab}\right)_\alpha{}^\beta \mathbb{Q}_\beta\,, \qquad (S_{mn}\mathbb{Q})_\alpha = \frac{1}{2}\left(\gamma_{mn}\right)_\alpha{}^\beta \mathbb{Q}_\beta\,. \tag{A.54}$$

In order to apply the Ward identities (A.49), we expand $\langle\chi_\alpha V[\mathbb{Q}_\beta]\rangle$ in terms of antisymmetrised products of gamma matrices. The only such matrices with the appropriate chirality properties are γ^μ and $\gamma^{\mu\nu\rho}$ (we can omit $\gamma^{\mu\nu\rho\sigma\tau}$ since it is related to γ^μ by duality):

$$\langle\chi_\alpha V[\mathbb{Q}_\beta]\rangle = a_\mu\left(\gamma^\mu c\right)_{\alpha\beta} + \frac{1}{3!}b_{\lambda\mu\nu}\left(\gamma^{\lambda\mu\nu}c\right)_{\alpha\beta}\,. \tag{A.55}$$

Writing out and simplifying the conformal Ward identities explicitly then leads to

$$\langle\chi_\alpha(\sigma,x)V[\mathbb{Q}_\beta]\rangle = \frac{c_{\chi\mathbb{Q}}\left[(\sigma_a\gamma^a + x_m\gamma^m)c\right]_{\alpha\beta}}{x^{\Delta-\hat{\Delta}}\sqrt{\sigma^2+x^2}^{1+2\hat{\Delta}}}. \tag{A.56}$$

Having completed the kinematic analysis, we can now restore the R-symmetry structure in order to construct the full bulk-defect 2-point functions. The Ward identities associated with the generators of $\mathfrak{so}(4)_R$ decouple from the kinematics, and therefore take a purely algebraic form (with $R, \hat{R}$ the representations of $\mathcal{O}, \hat{\mathcal{O}}$)

$$0 = \left(R^{ij} + \hat{R}^{ij}\right)\left\langle\mathcal{O}V[\hat{\mathcal{O}}]\right\rangle. \tag{A.57}$$

Among the bosonic 2-point functions we consider, the only nonvanishing ones are (we again suppress coordinate dependence and Lorentz indices):

$$\left\langle O^{i5}V[\mathbb{O}^j]\right\rangle \sim \delta^{ij}\,, \qquad \left\langle j^{i5}V[\mathbb{O}^j]\right\rangle \sim \delta^{ij}\,, \qquad \left\langle H^iV[\mathbb{O}^j]\right\rangle \sim \delta^{ij}\,. \tag{A.58}$$

To restore the correct R-symmetry structure of the fermionic 2-point function, recall that $\chi^I_{\alpha\check{\alpha}}$ transforms in the tensor product of the vector and spinor representation of $\mathfrak{so}(5)_R$ and is subject to a constraint $\check{\gamma}_I\chi^I = 0$, while $\mathbb{Q}_{\alpha\check{\alpha}}$ transforms as an ordinary R-symmetry spinor but obeys a constraint $\Pi_+\mathbb{Q} = 0$ mixing Lorentz and R-symmetry. Since we only need the correlator involving $\chi^5_{\check{\alpha}}$, we make the ansatz

$$\left\langle\chi^5_{\check{\alpha}}\mathbb{Q}_{\check{\beta}}\right\rangle \sim \left(\check{\gamma}^5\right)_{\check{\alpha}\check{\beta}}\,, \tag{A.59}$$

which is indeed compatible with (A.57).

With the kinematical data and R-symmetry structure in hand, we can now assemble the full 2-point functions. Plugging in the correct defect operator scaling dimensions $\Delta_{\mathbb{O}} = 2$ and $\Delta_{\mathbb{Q}} = 5/2$, we obtain (4.26).

[5] Since ultimately we are interested in a defect operator defined in terms of a chiral fermionic bulk current, we take $\mathbb{Q}$ to transform as a spinor under both parallel and transverse rotations, and consider only chiral objects.

A.5 Algebras

In this appendix we collect some results on the algebras $\mathfrak{osp}(8^*|4)$ and $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$. For a general reference on Lie superalgebra, see [149, 150] and references therein.

A.5.1 The algebra $\mathfrak{osp}(8^*|4)$

The quaternionic orthosymplectic algebra $\mathfrak{osp}(8^*|4) = D(4,2)$ is a 6d superconformal algebra containing 38 bosonic and 32 fermionic generators.[6] Its bosonic part $\mathfrak{so}(2,6) \oplus \mathfrak{so}(5)$ contains a 6d conformal algebra

$$\begin{aligned} [\mathsf{M}_{\mu\nu}, \mathsf{M}_{\rho\sigma}] &= 2\eta_{\sigma[\mu}\mathsf{M}_{\nu]\rho} - 2\eta_{\rho[\mu}\mathsf{M}_{\nu]\sigma} \,, & [\mathsf{P}_\mu, \mathsf{K}_\nu] &= 2\left(\mathsf{M}_{\mu\nu} + \eta_{\mu\nu}\mathsf{D}\right) \,, \\ [\mathsf{M}_{\mu\nu}, \mathsf{P}_\rho] &= 2\mathsf{P}_{[\mu}\eta_{\nu]\rho} \,, & [\mathsf{M}_{\mu\nu}, \mathsf{K}_\rho] &= 2\mathsf{K}_{[\mu}\eta_{\nu]\rho} \,, \\ [\mathsf{D}, \mathsf{P}_\mu] &= \mathsf{P}_\mu \,, & [\mathsf{D}, \mathsf{K}_\mu] &= -\mathsf{K}_\mu \,, \end{aligned} \tag{A.60}$$

along with an $\mathfrak{so}(5)$ R-symmetry

$$[\mathsf{R}_{IJ}, \mathsf{R}_{KL}] = 2\delta_{K[I}\mathsf{R}_{J]L} - 2\delta_{L[I}\mathsf{R}_{J]K} \,. \tag{A.61}$$

The fermionic generators Q and $\bar{\mathsf{S}}$ form a representation under that bosonic algebra and obey

$$\begin{aligned} [\mathsf{M}_{\mu\nu}, \mathsf{Q}_{\alpha\check{\alpha}}] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\mathsf{Q}\right)_{\alpha\check{\alpha}} \,, & \left[\mathsf{M}_{\mu\nu}, \bar{\mathsf{S}}_{\dot{\alpha}\check{\alpha}}\right] &= -\frac{1}{2}\left(\bar{\gamma}_{\mu\nu}\bar{\mathsf{S}}\right)_{\dot{\alpha}\check{\alpha}} \,, \\ [\mathsf{K}_\mu, \mathsf{Q}_{\alpha\check{\alpha}}] &= \left(\gamma_\mu\bar{\mathsf{S}}\right)_{\alpha\check{\alpha}} \,, & \left[\mathsf{P}_\mu, \bar{\mathsf{S}}_{\dot{\alpha}\check{\alpha}}\right] &= \left(\bar{\gamma}_\mu\mathsf{Q}\right)_{\dot{\alpha}\check{\alpha}} \,, \\ [\mathsf{D}, \mathsf{Q}_{\alpha\check{\alpha}}] &= \frac{1}{2}\mathsf{Q}_{\alpha\check{\alpha}} \,, & \left[\mathsf{D}, \bar{\mathsf{S}}_{\dot{\alpha}\check{\alpha}}\right] &= -\frac{1}{2}\bar{\mathsf{S}}_{\dot{\alpha}\check{\alpha}} \,, \\ [\mathsf{R}_{IJ}, \mathsf{Q}_{\alpha\check{\alpha}}] &= \frac{1}{2}\left(\check{\gamma}_{IJ}\mathsf{Q}\right)_{\alpha\check{\alpha}} \,, & \left[\mathsf{R}_{IJ}, \bar{\mathsf{S}}_{\dot{\alpha}\check{\alpha}}\right] &= \frac{1}{2}\left(\check{\gamma}_{IJ}\bar{\mathsf{S}}\right)_{\dot{\alpha}\check{\alpha}} \,. \end{aligned} \tag{A.62}$$

Finally, the anticommutator of Q generates a translation P, while the anticommutator of $\bar{\mathsf{S}}$ generates a special conformal transformation K

$$\begin{aligned} \left\{\mathsf{Q}_{\alpha\check{\alpha}}, \mathsf{Q}_{\beta\check{\beta}}\right\} &= 2\left(\gamma_\mu c\right)_{\alpha\beta}\Omega_{\check{\alpha}\check{\beta}}\mathsf{P}^\mu \,, \qquad \left\{\bar{\mathsf{S}}_{\dot{\alpha}\check{\alpha}}, \bar{\mathsf{S}}_{\dot{\beta}\check{\beta}}\right\} = 2\left(\bar{\gamma}_\mu c^T\right)_{\dot{\alpha}\dot{\beta}}\Omega_{\check{\alpha}\check{\beta}}\mathsf{K}^\mu \,, \\ \left\{\mathsf{Q}_{\alpha\check{\alpha}}, \bar{\mathsf{S}}_{\dot{\beta}\check{\beta}}\right\} &= 2\left[\left(\mathsf{D} + \frac{1}{2}\gamma_{\mu\nu}\mathsf{M}^{\mu\nu} + \check{\gamma}_{IJ}\mathsf{R}^{IJ}\right)c^T\Omega\right]_{\alpha\dot{\beta}\check{\alpha}\check{\beta}} \,. \end{aligned} \tag{A.63}$$

All the other commutators vanish.

Note that this algebra has a natural structure in terms of supermatrices. This point of view, along with its relation to the 6d algebra presented above, is elaborated in [83]. We also note that the $\mathfrak{so}(5)$ generators can be expressed in terms of $\mathfrak{sp}(2)$ generators by the relation

$$\mathsf{U}_{\check{\alpha}\check{\beta}} = \frac{1}{2}\left(\check{\gamma}_{IJ}\Omega\right)_{\check{\alpha}\check{\beta}}\mathsf{R}^{IJ} \,, \qquad \mathsf{R}_{IJ} = -\frac{1}{4}\left(\Omega^\dagger\check{\gamma}_{IJ}\right)^{\check{\alpha}\check{\beta}}\mathsf{U}_{\check{\alpha}\check{\beta}} \,. \tag{A.64}$$

The appropriate commutators are then

$$\begin{aligned} \left[\mathsf{U}_{\check{\alpha}\check{\beta}}, \mathsf{U}_{\check{\gamma}\check{\delta}}\right] &= 2\Omega_{\check{\alpha}(\check{\gamma}}\mathsf{U}_{\check{\delta})\check{\beta}} + 2\Omega_{\check{\beta}(\check{\gamma}}\mathsf{U}_{\check{\delta})\check{\alpha}} \,, \\ \left[\mathsf{U}_{\check{\alpha}\check{\beta}}, \mathsf{Q}_{\alpha\check{\gamma}}\right] &= 2\mathsf{Q}_{\alpha(\check{\alpha}}\Omega_{\check{\beta})\check{\gamma}} \,, \qquad \left[\mathsf{U}_{\check{\alpha}\check{\beta}}, \bar{\mathsf{S}}_{\dot{\alpha}\check{\gamma}}\right] = 2\bar{\mathsf{S}}_{\dot{\alpha}(\check{\alpha}}\Omega_{\check{\beta})\check{\gamma}} \,. \end{aligned} \tag{A.65}$$

[6] More precisely, it is a real form of $D(4,2)$ given by $\mathsf{P}_\mu^\dagger = \mathsf{K}^\mu$ (which also implies $(\mathsf{Q}_{\alpha\check{\alpha}})^\dagger = \mathsf{S}^{\alpha\check{\alpha}}$) and compatible with radial quantisation in Euclidean space. Hermitean generators can be obtained by redefining all generators $\mathsf{P} \to i\mathsf{P}$.

A.5.2 The subalgebra $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$

In the presence of the plane, the original symmetry $\mathfrak{osp}(8^*|4)$ is reduced to the subalgebra $\mathfrak{osp}(4^*|2) \oplus \mathfrak{osp}(4^*|2)$ [128], a real form of $D(2,1,\alpha) \oplus D(2,1,\alpha)$ with $\alpha = -1/2$. Each copy of the $\mathfrak{osp}(4^*|2)$ is a (rigid) 1d superconformal algebra, whose bosonic part is

$$\begin{gathered}[\mathsf{P}_+, \mathsf{K}_+] = 2\mathsf{D}_+ , \qquad [\mathsf{D}_+, \mathsf{P}_+] = \mathsf{P}_+ , \qquad [\mathsf{D}_+, \mathsf{K}_+] = -\mathsf{K}_+ , \\ \left[\mathsf{T}^i_{(a)}, \mathsf{T}^j_{(b)}\right] = -i\delta_{(ab)}\varepsilon^{ijk}\mathsf{T}^k_{(b)}, \qquad (a) = 1, 2.\end{gathered} \tag{A.66}$$

In addition to the 1d conformal algebra, there are 2 additional $\mathfrak{su}(2)$. Together, they form the "chiral" part of the $\mathfrak{so}(2,2)_\parallel \oplus \mathfrak{so}(4)_\perp \oplus \mathfrak{so}(4)_R$ preserved by the plane, with the "antichiral" part (denoted by a "$-$" subscript) given by the other $\mathfrak{osp}(4^*|2)$. They are related to the bulk generators by

$$\mathsf{P}_\pm = \frac{1}{2}(\mathsf{P}_0 \pm \mathsf{P}_1) , \qquad \mathsf{D}_\pm = \frac{1}{2}(\mathsf{D} \pm \mathsf{M}_{01}) , \qquad \mathsf{K}_\pm = \frac{1}{2}(-\mathsf{K}_0 \pm \mathsf{K}_1) , \tag{A.67}$$

where for definiteness we assume that the plane spans the directions $x^{0,1}$. The decomposition of $\mathfrak{so}(4)_{\perp,R}$ is given by the 't Hooft symbols

$$\mathsf{T}^{i_1}_{(1)} = \frac{i}{4}\eta^{i_1}_{mn}\mathsf{M}^{mn} , \qquad \mathsf{T}^{i_2}_{(2)} = -\frac{i}{4}\eta^{i_2}_{ij}\mathsf{R}^{ij} , \tag{A.68}$$

and similarly for $\bar{\mathsf{T}}$ in terms of the antichiral 't Hooft symbols $\bar{\eta}$.

In addition to these generators, the algebra includes supersymmetries $\mathsf{Q}_{\alpha_1\alpha_2}$ and special supersymmetries $\mathsf{S}_{\alpha_1\alpha_2}$ charged under both $\mathfrak{su}(2)$. These satisfy

$$\begin{aligned}
[\mathsf{K}_+, \mathsf{Q}_{\alpha_1\alpha_2}] &= -i\mathsf{S}_{\alpha_1\alpha_2} , & [\mathsf{P}_+, \mathsf{S}_{\alpha_1\alpha_2}] &= i\mathsf{Q}_{\alpha_1\alpha_2} , \\
[\mathsf{D}_+, \mathsf{Q}_{\alpha_1\alpha_2}] &= \frac{1}{2}\mathsf{Q}_{\alpha_1\alpha_2} , & [\mathsf{D}_+, \mathsf{S}_{\alpha_1\alpha_2}] &= -\frac{1}{2}\mathsf{S}_{\alpha_1\alpha_2} , \\
\left[\mathsf{T}^{i_1}_{(1)}, \mathsf{Q}_{\alpha_1\alpha_2}\right] &= \frac{1}{2}(\sigma^{i_1})_{\alpha_1}{}^{\beta_1}\mathsf{Q}_{\beta_1\alpha_2} , & \left[\mathsf{T}^{i_1}_{(1)}, \mathsf{S}_{\alpha_1\alpha_2}\right] &= \frac{1}{2}(\sigma^{i_1})_{\alpha_1}{}^{\beta_1}\mathsf{S}_{\beta_1\alpha_2} , \\
\left[\mathsf{T}^{i_2}_{(2)}, \mathsf{Q}_{\alpha_1\alpha_2}\right] &= \frac{1}{2}(\sigma^{i_2})_{\alpha_2}{}^{\beta_2}\mathsf{Q}_{\alpha_1\beta_2} , & \left[\mathsf{T}^{i_2}_{(2)}, \mathsf{S}_{\alpha_1\alpha_2}\right] &= \frac{1}{2}(\sigma^{i_2})_{\alpha_2}{}^{\beta_2}\mathsf{S}_{\alpha_1\beta_2} ,
\end{aligned} \tag{A.69}$$

where σ^i are the Pauli matrices. They anticommute to

$$\begin{gathered}\{\mathsf{Q}_{\alpha_1\alpha_2}, \mathsf{Q}_{\beta_1\beta_2}\} = 2i\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\mathsf{P}_+ , \qquad \{\mathsf{S}_{\alpha_1\alpha_2}, \mathsf{S}_{\beta_1\beta_2}\} = 2i\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\mathsf{K}_+ , \\ \{\mathsf{Q}_{\alpha_1\alpha_2}, \mathsf{S}_{\beta_1\beta_2}\} = 2\left[\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\mathsf{D}_+ + (\sigma^{i_1}\epsilon)_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\mathsf{T}^{i_1}_{(1)} - 2\epsilon_{\alpha_1\beta_1}(\sigma^{i_2}\epsilon)_{\alpha_2\beta_2}\mathsf{T}^{i_2}_{(2)}\right] .\end{gathered} \tag{A.70}$$

The ratio $\alpha = -1/2$ between the coefficients of $\mathsf{T}_{(1)}$ and $\mathsf{T}_{(2)}$ is a specific case of the exceptional Lie algebra $D(2,1;\alpha)$ (see [151] for the algebra with general α and its Kac-Moody extension).

The precise embedding of these supercharges inside $\mathsf{Q}_{\alpha\check{\alpha}}$ is obtained by restricting to the preserved supercharges $\Pi_+\mathsf{Q} = \mathsf{Q}$, where the projector is [28]

$$(\Pi_\pm)_{\alpha\check{\alpha}}{}^{\beta\check{\beta}} = \frac{1}{2}\left[1 \pm \gamma_{01}\check{\gamma}_5\right]_{\alpha\check{\alpha}}{}^{\beta\check{\beta}} , \qquad (\Pi_\pm)_{\dot{\alpha}\check{\alpha}}{}^{\dot{\beta}\check{\beta}} = \frac{1}{2}\left[1 \mp \bar{\gamma}_{01}\check{\gamma}_5\right]_{\dot{\alpha}\check{\alpha}}{}^{\dot{\beta}\check{\beta}} , \tag{A.71}$$

which has a different expression acting respectively on chiral and antichiral representations. This projector decomposes as

$$\frac{1}{2}\left[1+\gamma_{01}\check{\gamma}_5\right]=\frac{1}{2}\left[1+\gamma_{01}\right]\frac{1}{2}\left[1+\check{\gamma}_5\right]+\frac{1}{2}\left[1-\gamma_{01}\right]\frac{1}{2}\left[1-\check{\gamma}_5\right], \tag{A.72}$$

which gives, respectively for the two terms, two anticommuting supercharges $\bar{\mathsf{Q}}_{\dot{\alpha}_1\check{\alpha}_2}$ and $\mathsf{Q}_{\alpha_1\alpha_2}$. Their chirality is derived from the projector: $(1+\gamma_{01})$ projects onto the positive chirality component, which is correlated with the positive chirality under $\mathfrak{so}(4)_\perp$ since $\gamma_{01}=\gamma_{2345}$.

Subalgebra as an embedding inside $\mathfrak{osp}(8^*|4)$

Lastly, in Section 4.2 and 4.3 it is convenient to discuss the subalgebra directly within the larger $\mathfrak{osp}(8^*|4)$. Here we decompose some of the commutators of $\mathfrak{osp}(8^*|4)$ into preserved and broken generators directly with the projector. We make use of the following identities

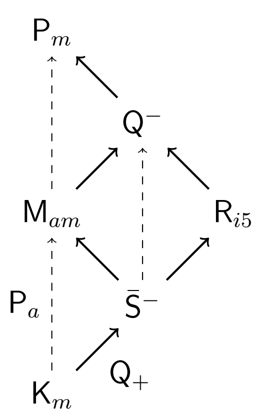
$$\begin{gathered}\Pi_\pm^\dagger=\Pi_\pm, \quad (\Pi_\pm\mathcal{C})^T=-\Pi_\pm\mathcal{C}^T\,,\\ [\Pi_\pm,\Gamma_a]=[\Pi_\pm,\check{\gamma}_5]=0\,,\\ \Pi_\pm\Gamma_m=\Gamma_m\Pi_\mp, \quad \Pi_\pm\Gamma_i=\Gamma_i\Pi_\mp\,.\end{gathered} \tag{A.73}$$

Note that here we don't differentiate between the action of Q and $\bar{\mathsf{Q}}$ for simplicity.

Using these properties, one can easily derive the induced subalgebra and its representation by acting with $\Pi_\pm$. The only nontrivial part of the preserved algebra is for the supercharges, which now obey

$$\begin{gathered}\left\{\mathsf{Q}^+_{\alpha\check{\alpha}},\mathsf{Q}^+_{\beta\check{\beta}}\right\}=2\left(\gamma_a\Pi_+c\Omega\right)_{\alpha\beta\check{\alpha}\check{\beta}}\mathsf{P}^a\,, \qquad \left\{\bar{\mathsf{S}}^+_{\dot{\alpha}\check{\alpha}},\bar{\mathsf{S}}^+_{\dot{\beta}\check{\beta}}\right\}=2\left(\bar{\gamma}_a\Pi_+c^T\Omega\right)_{\dot{\alpha}\dot{\beta}\check{\alpha}\check{\beta}}\mathsf{K}^a\,,\\ \left\{\mathsf{Q}^+_{\alpha\check{\alpha}},\bar{\mathsf{S}}^+_{\dot{\beta}\check{\beta}}\right\}=2\left[\left(\check{\gamma}_{ij}\mathsf{R}^{ij}+\mathsf{D}+\frac{1}{2}\gamma_{mn}\mathsf{M}^{mn}+\frac{1}{2}\gamma_{ab}\mathsf{M}^{ab}\right)\Pi_+c^T\Omega\right]_{\alpha\dot{\beta}\check{\alpha}\check{\beta}}\,.\end{gathered} \tag{A.74}$$

The broken generators satisfy



$$\begin{aligned}\left[\mathsf{Q}^+_{\alpha\check{\alpha}},\mathsf{P}_m\right]&=0,\\ \left\{\mathsf{Q}^+_{\alpha\check{\alpha}},\mathsf{Q}^-_{\beta\check{\beta}}\right\}&=2\left(\gamma_m\Pi_-c\Omega\right)_{\alpha\beta\check{\alpha}\check{\beta}}\mathsf{P}^m,\\ \left[\mathsf{Q}^+_{\alpha\check{\alpha}},\mathsf{R}_{i5}\right]&=-\frac{1}{2}\left(\check{\gamma}_{i5}\mathsf{Q}^-\right)_{\alpha\check{\alpha}}\,,\\ \left[\mathsf{Q}^+_{\alpha\check{\alpha}},\mathsf{M}_{am}\right]&=\frac{1}{2}\left(\gamma_{am}\mathsf{Q}^-\right)_{\alpha\check{\alpha}}\,,\\ \left\{\mathsf{Q}^+_{\alpha\check{\alpha}},\bar{\mathsf{S}}^-_{\dot{\beta}\check{\beta}}\right\}&=4\left[\left(\check{\gamma}_{i5}\mathsf{R}^{i5}+\frac{1}{2}\gamma_{am}\mathsf{M}^{am}\right)\Pi_-c^T\Omega\right]_{\alpha\dot{\beta}\check{\alpha}\check{\beta}}\,,\\ \left[\mathsf{Q}^+_{\alpha\check{\alpha}},\mathsf{K}_m\right]&=-\left(\gamma_m\bar{\mathsf{S}}^-\right)_{\alpha\check{\alpha}}\,.\end{aligned} \tag{A.75}$$

These transformations are related to (4.8) using (4.7) to write the displacement operator as contact terms in the presence of the defect:

$$\mathsf{R}^{i5}V=\int_{\mathbb{R}^2}\mathrm{d}^2\sigma V[\mathbb{O}^i(\sigma)]\,. \tag{A.76}$$

We can recover the full representation by acting with Q^+, e.g.,

$$\int_{\mathbb{R}^2} V[\mathsf{Q}^+\mathbb{O}^i(\sigma)]\mathrm{d}^2\sigma = \left[\mathsf{Q}^+, \mathsf{R}^{i5}\right] V = -\frac{1}{2}\check{\gamma}_{i5}\mathsf{Q}^- V = -\frac{1}{2}\int_{\mathbb{R}^2} \mathrm{d}^2\sigma V[\check{\gamma}_{i5}\mathbb{Q}^-(\sigma)]\,. \qquad \text{(A.77)}$$

The action of Q^+ on $\mathbb{Q}$ can similarly be read from (A.75), but it misses the descendant. These are fixed instead by requiring closure under the Jacobi identity as in (4.18) (see also for instance the discussion in Section 2 of [152]).

Appendix B

Loops

B.1 SUSY Condition

In this appendix, we derive the SUSY condition

$$\mathsf{Q}\mathcal{L} = \sigma_3 \mathcal{D}^{\mathcal{L}}_{\varphi} H \tag{B.1}$$

for a Wilson loop defined in terms of a superconnection $\mathcal{L}$.

Given a connection $\mathcal{L}$ (super or otherwise) and a closed contour parametrised by $\varphi \in [0, 2\pi]$, we write the generalised gauge holonomy

$$W_{\mathcal{L}} = \mathcal{P} \exp i \int_0^{2\pi} \mathrm{d}\varphi \mathcal{L}(\varphi), \tag{B.2}$$

where we take the path ordering to be right-to-left, i.e.

$$\mathcal{P}\mathcal{L}(\varphi_1)\mathcal{L}(\varphi_2) = \begin{cases} \mathcal{L}(\varphi_1)\mathcal{L}(\varphi_2), & \text{if } \varphi_1 > \varphi_2, \\ \mathcal{L}(\varphi_2)\mathcal{L}(\varphi_1), & \text{if } \varphi_1 < \varphi_2. \end{cases} \tag{B.3}$$

It is well established that $W_{\mathcal{L}}$ behaves naturally under an insertion of an integrated covariant derivative[1]

$$\mathcal{D}^{\mathcal{L}}_{\varphi} \tilde{H} = \partial_{\varphi} \tilde{H} - \mathrm{i}[\mathcal{L}, \tilde{H}]. \tag{B.4}$$

We have

$$W_{\mathcal{L}}[\int_0^{2\pi} \mathrm{d}\varphi \mathcal{D}^{\mathcal{L}}_{\varphi} \tilde{H}] = [W_{\mathcal{L}}, \tilde{H}_0], \tag{B.5}$$

where $\tilde{H}_0 \equiv \tilde{H}(0) = \tilde{H}(2\pi)$. If $\mathcal{L}$ and $\tilde{H}$ are purely bosonic, the trace of the commutator on the RHS vanishes, and $\operatorname{Tr} W_{\mathcal{L}}$ is unaffected by such insertions.

Now consider a superconnection $\mathcal{L} = \mathcal{L}_{\text{diag}} + \mathcal{L}_{\text{off}}$ which is an even supermatrix, such that $\mathcal{L}_{\text{diag}}$ is bosonic and $\mathcal{L}_{\text{off}}$ is fermionic. First, note that the ordinary trace is not cyclic when

[1] The opposite choice for path ordering would change the sign of the commutator term in the covariant derivative.

applied to products of supermatrices, such that the natural object to consider is instead $\mathrm{sTr} W_{\mathcal{L}}$. Further let $\overline{\mathcal{L}} = \mathcal{L}_{\mathrm{diag}} - \mathcal{L}_{\mathrm{off}}$. Then expand

$$W_{\mathcal{L}} = \sum_{n=0}^{\infty} \mathrm{i}^n \int_{\varphi_k > \varphi_{k+1}} \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_n \mathcal{L}_1 \dots \mathcal{L}_n, \tag{B.6}$$

where we wrote $\mathcal{L}_k = \mathcal{L}(\varphi_k)$. Now act with a supercharge Q on the loop. Note that since Q anticommutes with $\mathcal{L}_{\mathrm{off}}$, we have $\mathsf{Q}(\mathcal{L}_1 \mathcal{L}_2) = \mathsf{Q}\mathcal{L}_1 \cdot \mathcal{L}_2 + \overline{\mathcal{L}}_1 \cdot \mathsf{Q}\mathcal{L}_2$ and so on. We find (the $n = 0$ term is annihilated by Q)

$$\mathsf{Q} W_{\mathcal{L}} = \sum_{n=1}^{\infty} \mathrm{i}^n \int_{\varphi_k > \varphi_{k+1}} \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_n \sum_{m=1}^{n} \overline{\mathcal{L}}_1 \dots \overline{\mathcal{L}}_{m-1} \cdot \mathsf{Q}\mathcal{L}_m \cdot \mathcal{L}_{m+1} \dots \mathcal{L}_n. \tag{B.7}$$

For the next step, note that $\overline{\mathcal{L}}\sigma_3 = \sigma_3 \mathcal{L}$. Introducing a factor of $\mathbf{1} = \sigma_3^2$ in front of Q and commuting one of the σ_3's to the front, we find

$$\mathsf{Q} W_{\mathcal{L}} = \sigma_3 \sum_{n=1}^{\infty} \mathrm{i}^n \int_{\varphi_k > \varphi_{k+1}} \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_n \sum_{m=1}^{n} \mathcal{L}_1 \dots \mathcal{L}_{m-1} \cdot (\sigma_3 \mathsf{Q}\mathcal{L}_m) \cdot \mathcal{L}_{m+1} \dots \mathcal{L}_n. \tag{B.8}$$

This expression now naturally splits as

$$\mathsf{Q} W_{\mathcal{L}} = \mathrm{i}\sigma_3 \int_0^{2\pi} \mathrm{d}\varphi W_{2\pi,\varphi} \cdot (\sigma_3 Q \mathcal{L}(\varphi)) \cdot W_{\varphi,0} \tag{B.9}$$

$$= \mathrm{i}\sigma_3 W_{\mathcal{L}} [\int_0^{2\pi} \mathrm{d}\varphi \, \sigma_3 \mathsf{Q}\mathcal{L}(\varphi)]. \tag{B.10}$$

In light of (B.5), we require

$$\sigma_3 \mathsf{Q}\mathcal{L} = \mathcal{D}_{\varphi}^{\mathcal{L}} \tilde{H}. \tag{B.11}$$

Comparing Grassmann degrees in this equation, we see that $\tilde{H}$ must be an odd supermatrix, i.e. one with bosonic components on the offdiagonal and vice versa. Furthermore, by counting scaling dimensions, in our case the diagonal (i.e. fermionic) components of $\tilde{H}$ are zero. Finally, check that the supertrace of $\mathsf{Q} W_{\mathcal{L}}$ vanishes. Rewrite

$$\mathrm{sTr}\, \mathsf{Q} W_{\mathcal{L}} = \mathrm{i}\, \mathrm{sTr}\big(\sigma_3 [W_{\mathcal{L}}, \tilde{H}_0]\big) = \mathrm{i}\, \mathrm{Tr}\big([W_{\mathcal{L}}, \tilde{H}_0]\big), \tag{B.12}$$

which is readily shown to vanish. Expand

$$\begin{aligned} \mathrm{Tr}\left[W, \tilde{H}_0\right] &= \mathrm{Tr} \begin{pmatrix} w_{12}\tilde{h}_{21} - \tilde{h}_{12} w_{21} & * \\ * & w_{21}\tilde{h}_{12} - \tilde{h}_{21} w_{12} \end{pmatrix} \\ &= \mathrm{Tr}\left(w_{12}\tilde{h}_{21} - \tilde{h}_{12} w_{21} + w_{21}\tilde{h}_{12} - \tilde{h}_{21} w_{12}\right). \end{aligned} \tag{B.13}$$

Since $\tilde{h}_{12}, \tilde{h}_{21}$ are bosonic, there are no issues in using the cyclicity of the trace, and we find that, indeed,

$$\mathrm{sTr} \mathsf{Q} W_{\mathcal{L}} = 0. \tag{B.14}$$

Actually, we will work in the main text with $H = \sigma_3 \tilde{H}$. The supersymmetry condition written separately for diagonal and off-diagonal parts then reads

$$\mathsf{Q}\mathcal{L}_{\text{diag}} = \mathrm{i}\{\mathcal{L}_{\text{off}}, H\}, \tag{B.15}$$

$$\mathsf{Q}\mathcal{L}_{\text{off}} = \mathcal{D}_{\varphi}^{\mathcal{L}_{\text{diag}}} H. \tag{B.16}$$

If one prefers working instead with bosonic variations, introduce a Grassmann unit ξ and write $\delta = \xi \mathsf{Q}$. The analogous susy condition reads

$$\delta\mathcal{L} = \mathcal{D}_{\varphi}^{\mathcal{L}}(\xi H). \tag{B.17}$$

B.2 Conventions and SUSY Transformations in ABJM

We mostly adopt the conventions of [144] and denote the gauge group of ABJ(M) theory as $U(N_1) \times U(N_2)$. In addition to the gauge fields $A^{(1)}$ and $A^{(2)}$ transforming in the adjoint of their respective gauge group, the theory contains scalars C_I and $\bar{C}^I$ and fermions ψ_I^α and $\bar{\psi}_\alpha^I$ in the bifundamental, such that $C\bar{C}$ and $\bar{\psi}\psi$ ($\bar{C}C$ and $\psi\bar{\psi}$) transform in the adjoint of $U(N_1)$ ($U(N_2)$), with the R-symmetry index I transforming in the fundamental of $\mathfrak{su}(4)$. These fields assemble in a single supermultiplet satisfying

$$\begin{aligned}
\delta A_\mu^{(1)} &= -\frac{4\pi i}{k} C_I \psi_J^\alpha (\gamma_\mu)_\alpha{}^\beta \bar{\Theta}_\beta^{IJ} + \frac{4\pi i}{k} \Theta_{IJ}^\alpha (\gamma_\mu)_\alpha{}^\beta \bar{\psi}_\beta^I \bar{C}^J \,, \\
\delta A_\mu^{(2)} &= \frac{4\pi i}{k} \psi_I^\alpha C_J (\gamma_\mu)_\alpha{}^\beta \bar{\Theta}_\beta^{IJ} - \frac{4\pi i}{k} \Theta_{IJ}^\alpha (\gamma_\mu)_\alpha{}^\beta \bar{C}^I \bar{\psi}_\beta^J \,, \\
\delta \bar{\psi}_\beta^I &= 2i (\gamma^\mu)_\beta{}^\alpha \bar{\Theta}_\alpha^{IJ} D_\mu C_J + \frac{16\pi i}{k} \bar{\Theta}_\beta^{J[I} C_{[J} \bar{C}^{K]} C_{K]} - 2\bar{\epsilon}_\beta^{IJ} C_J \,, \\
\delta \psi_I^\beta &= -2i \Theta_{IJ}^\alpha (\gamma^\mu)_\alpha{}^\beta D_\mu \bar{C}^J - \frac{16\pi i}{k} \Theta_{J[I}^\beta \bar{C}^{[J} C_{K]} \bar{C}^{K]} - 2\epsilon_{IJ}^\beta \bar{C}^J \,, \\
\delta C_I &= 2\Theta_{IJ}^\alpha \bar{\psi}_\alpha^J \,, \\
\delta \bar{C}^I &= -2\psi_J^\alpha \bar{\Theta}_\alpha^{JI} \,,
\end{aligned} \tag{B.18}$$

for a (Euclidean) superconformal transformation parametrised by $\Theta_{IJ} = \theta_{IJ} + \epsilon_{IJ}(x \cdot \gamma)$ and $\bar{\Theta} = \bar{\theta}^{IJ} - (x \cdot \gamma)\bar{\epsilon}^{IJ}$. The parameters are related by $\bar{\theta}_\alpha^{IJ} = -\frac{1}{2}\varepsilon^{IJKL}\theta_{KL}^\beta \varepsilon_{\beta\alpha}$ (likewise $\bar{\epsilon}_\alpha^{IJ}$), but unlike in Minkowski space there is no reality condition (i.e. $\bar{\theta} \neq \theta^\dagger$). Omitted spinor indices follow the NW-SE summation convention. A review of the theory in these conventions along with an action can be found in [153].

B.3 $\mathcal{N} = 4$ Superconformal Algebra on S^3

The symmetries of an $\mathcal{N} = 4$ superconformal field theory in 3d form the algebra $D(2,2) = \mathfrak{osp}(4|4)$. Its bosonic subalgebra is (in the Euclidean case) $\mathfrak{so}(1,4)_{\text{conf}} \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. Although S^3 and $\mathbb{R}^3$ are conformal to each other, and therefore their conformal algebras agree, in the main text we specialise to the case of an S^3 of radius r embedded in $\mathbb{R}^4$

and parametrised by 4d coordinates x^i, $i = 1, 2, 3, 4$. Explicitly, the algebra of geometric symmetries is spanned by vector fields

$$\mathsf{M}_{ij} = x_i \partial_j - x_j \partial_i, \tag{B.19}$$
$$\mathsf{T}_i = r\partial_i - r^{-1} x_i x^j \partial_j \equiv r\mathsf{P}_i + r^{-1}\mathsf{K}_i, \tag{B.20}$$

where the rotations M_{ij} make up the S^3 isometry algebra $\mathfrak{so}(4)$ and the T_i generate conformal maps. Their commutators are easily computed:

$$[\mathsf{M}_{ij}, \mathsf{M}_{kl}] = -\delta_{ik}\mathsf{M}_{jl} + \delta_{il}\mathsf{M}_{jk} + \delta_{jk}\mathsf{M}_{il} - \delta_{jl}\mathsf{M}_{ik}, \tag{B.21}$$
$$[\mathsf{M}_{ij}, \mathsf{T}_k] = -\delta_{ik}\mathsf{T}_j + \delta_{jk}\mathsf{T}_i, \tag{B.22}$$
$$[\mathsf{T}_i, \mathsf{T}_j] = \mathsf{M}_{ij}. \tag{B.23}$$

The conformal algebra can be written more compactly by introducing indices $\mu, \nu = 0, 1, 2, 3, 4$ and defining $\mathsf{M}_{0i} = \mathsf{T}_i$. The commutators then become

$$[\mathsf{M}_{\mu\nu}, \mathsf{M}_{\rho\sigma}] = -\eta_{\mu\rho}\mathsf{M}_{\nu\sigma} + \eta_{\nu\rho}\mathsf{M}_{\mu\sigma} + \eta_{\mu\sigma}\mathsf{M}_{\nu\rho} - \eta_{\nu\sigma}\mathsf{M}_{\mu\rho}, \tag{B.24}$$

where $\eta = \mathrm{diag}(-, +, +, +, +)$. This makes the $\mathfrak{so}(1, 4)$ structure of the conformal algebra manifest. The R-symmetry algebra is spanned by two independent sets of $\mathfrak{su}(2)$ generators R_I, $\bar{\mathsf{R}}_I$ with the usual commutators

$$[\mathsf{R}_I, \mathsf{R}_J] = 2i\epsilon_{IJK}\mathsf{R}_K, \qquad [\bar{\mathsf{R}}_I, \bar{\mathsf{R}}_J] = 2i\epsilon_{IJK}\bar{\mathsf{R}}_K. \tag{B.25}$$

It is convenient to use instead the symmetric contractions $\mathsf{R}^{ab} = \left(\sigma^I\right)^{ab} \mathsf{R}_I$ and similarly for $\bar{\mathsf{R}}^{ab}$, where we raised one fundamental $\mathfrak{su}(2)$ index with ϵ^{ab}. The commutation relations then take the form

$$[\mathsf{R}^{ab}, \mathsf{R}^{cd}] = \epsilon^{ac}\mathsf{R}^{bd} + \epsilon^{ad}\mathsf{R}^{bc} + \epsilon^{bc}\mathsf{R}^{ad} + \epsilon^{bd}\mathsf{R}^{ac}, \tag{B.26}$$
$$[\bar{\mathsf{R}}^{\dot{a}\dot{b}}, \bar{\mathsf{R}}^{\dot{c}\dot{d}}] = \epsilon^{\dot{a}\dot{c}}\bar{\mathsf{R}}^{\dot{b}\dot{d}} + \epsilon^{\dot{a}\dot{d}}\bar{\mathsf{R}}^{\dot{b}\dot{c}} + \epsilon^{\dot{b}\dot{c}}\bar{\mathsf{R}}^{\dot{a}\dot{d}} + \epsilon^{\dot{b}\dot{d}}\bar{\mathsf{R}}^{\dot{a}\dot{c}}. \tag{B.27}$$

Furthermore, the algebra contains 16 supercharges which transform as spinors of $\mathfrak{so}(1, 4)_{\mathrm{conf}}$ and fundamental doublets of both $\mathfrak{su}(2)_{L,R}$. They therefore each carry a spinor index A and two R-symmetry indices $a, \dot{a}$. Using 5d gamma matrices $(\Gamma_\mu)_A{}^B$, the Lorentz transformation law of the supercharges reads

$$[\mathsf{M}_{\mu\nu}, \mathsf{Q}^{\dot{a}a}_A] = -\frac{1}{2}\left(\Gamma_{\mu\nu}\right)_A{}^B, \mathsf{Q}^{\dot{a}a}_B. \tag{B.28}$$

while under the R-charges we have

$$[\mathsf{R}_I, \mathsf{Q}^{\dot{a}a}_A] = \mathsf{Q}^{\dot{a}b}_A \left(\sigma_I\right)_b{}^a, \qquad [\bar{\mathsf{R}}_I, \mathsf{Q}^{\dot{a}a}_A] = \mathsf{Q}^{\dot{b}a}_A \left(\sigma_I\right)_{\dot{b}}{}^{\dot{a}}, \tag{B.29}$$

or, equivalently,

$$[\mathsf{R}^{bc}, \mathsf{Q}^{\dot{a}a}_A] = \epsilon^{ba}\mathsf{Q}^{\dot{a}c}_A + \epsilon^{ca}\mathsf{Q}^{\dot{a}b}_A, \qquad [\bar{\mathsf{R}}^{\dot{b}\dot{c}}, \mathsf{Q}^{\dot{a}a}_A] = \epsilon^{\dot{b}\dot{a}}\mathsf{Q}^{\dot{c}a}_A + \epsilon^{\dot{c}\dot{a}}\mathsf{Q}^{\dot{b}a}_A. \tag{B.30}$$

Finally, we must specify an anticommutator for the supercharges. Symmetry under the simultaneous exchange of the R-symmetry and spinor indices as well as the requirement that the super-Jacobi identity be satisfied fixes all coefficients up to an overall normalisation, which we may absorb into the definition of Q, and we find[2]

$$\{\mathsf{Q}^{\dot{a}a}_{A}, \mathsf{Q}^{\dot{b}b}_{B}\} = \epsilon^{\dot{a}\dot{b}}\epsilon^{ab}\left(\Gamma^{\mu\nu}C^{-1}\right)_{AB}\mathsf{M}_{\mu\nu} + \epsilon^{\dot{a}\dot{b}}C^{-1}_{AB}\mathsf{R}^{ab} + \epsilon^{ab}C^{-1}_{AB}\bar{\mathsf{R}}^{\dot{a}\dot{b}}. \tag{B.31}$$

B.4 3d $\mathcal{N} = 4$ sCSM theories

B.4.1 Quiver Structure

We consider an $\mathcal{N} = 4$ Chern-Simons-matter theory defined by a quiver which we take to be either linear, or circular with an even number of nodes. Locally, the quiver has the following structure:

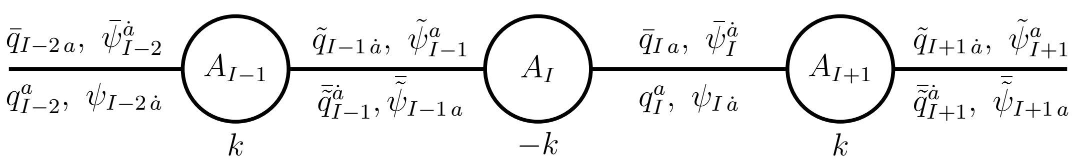


Figure B.1: The quiver and field content of the $\mathcal{N} = 4$ theory.

Each node corresponds to a gauge factor $U(N_I)$ and comes with an associated gauge field A_I. The edges carry hypermultiplets $(q^a, \psi_{\dot{a}})$ and twisted hypermultiplets $(\tilde{q}_{\dot{a}}, \tilde{\psi}^a)$, alternately. These matter multiplets can be decomposed into pairs of chiral multiplets. Figure B.2 shows the chiral scalar in this decomposition explicitly. As usual, the orientation of the arrows indicates the representation under the two gauge factors. For instance, the field q^2_I is in the $(\Box, \bar{\Box})$ of $U(N_I) \times U(N_{I+1})$ and $\bar{q}_{I\,1}$ is in the conjugate representation. Note that scalar bilinears comprising barred and unbarred q's transform in the adjoint. Concretely, $q^a_I \bar{q}_{I\,b}$ is in the adjoint of $U(N_I)$, $\bar{q}_{I,a} q^b_I$ is in the adjoint of $U(N_{I+1})$, and similarly for the twisted hypermultiplets.

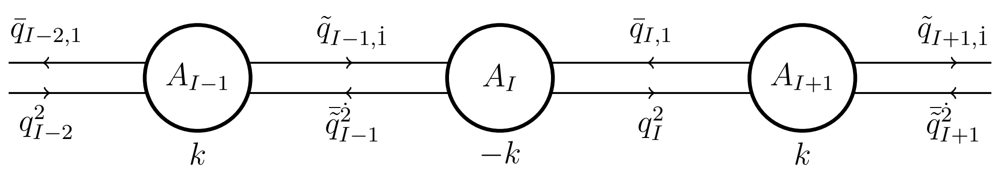


Figure B.2: The decomposition of the $\mathcal{N} = 4$ matter multiplets into pairs of chiral multiplets.

It is convenient to decompose the scalar bilinears as $\mathbf{2}_L \otimes \mathbf{2}_R = \mathbf{3} \oplus \mathbf{1}$. Concretely, we

[2]Here, C is the 5d charge conjugation matrix. Note that in 5d the charge conjugation matrix (and its inverse) are always antisymmetric, as is $\Gamma_\mu C^{-1}$, while $\Gamma_{\mu\nu} C^{-1}$ is always symmetric.

recast the bilinears in the adjoint of $U(N_I)$ in terms of traceless tensors μ_I and scalars ν_I as[3]

$$\mu_I{}^a{}_b = q_I^a \bar{q}_{I\,b} - \frac{1}{2}\delta^a_b q_I^c \bar{q}_{I\,c}\,, \qquad j_I^{a\dot{b}} = q_I^a \bar{\psi}_I^{\dot{b}} - \epsilon^{ac}\epsilon^{\dot{b}\dot{c}}\psi_{I\,\dot{c}}\bar{q}_{I\,c}\,, \tag{B.32}$$

$$\tilde{\mu}_I{}^{\dot{a}}{}_{\dot{b}} = \bar{\tilde{q}}^{\dot{a}}_{I-1}\tilde{q}_{I-1\,\dot{b}} - \frac{1}{2}\delta^{\dot{a}}_{\dot{b}}\bar{\tilde{q}}^{\dot{c}}_{I-1}\tilde{q}_{I-1\,\dot{c}}\,, \qquad \tilde{j}_I^{\dot{b}a} = \bar{\tilde{q}}^{\dot{b}}_{I-1}\tilde{\psi}^a_{I-1} - \epsilon^{\dot{b}\dot{c}}\epsilon^{ac}\bar{\tilde{\psi}}_{I-1\,c}\tilde{q}_{I-1\,\dot{c}}\,, \tag{B.33}$$

$$\nu_I = q_I^a \bar{q}_{I\,a}\,, \qquad \tilde{\nu}_I = \bar{\tilde{q}}^{\dot{a}}_{I-1}\tilde{q}_{I-1\,\dot{a}}\,. \tag{B.34}$$

Similar bilinears (with the appropriate replacement of hypermultiplets and twisted hypermultiplets) exist also for the other nodes. For example, for the $I+1$ node one can define $\nu_{I+1} = \bar{q}_{Ia} q_I^a$.

B.4.2 The SUSY transformations for $\mathcal{N} = 4$ on S^3

The supersymmetry transformations of this theory on S^3 were shown in [21] to be

$$\delta A_{\mu\,I} = \frac{i}{k}\xi_{a\dot{b}}\gamma_\mu (j_I^{a\dot{b}} - \tilde{j}_I^{\dot{b}a})\,, \qquad \begin{matrix} \delta q_I^a = \xi^{a\dot{b}}\psi_{I\,\dot{b}}\,, & \delta \bar{q}_{I\,a} = \xi_{a\dot{b}}\bar{\psi}_I^{\dot{b}}\,, \\ \delta \tilde{q}_{I-1\,\dot{b}} = -\xi_{a\dot{b}}\tilde{\psi}^a_{I-1}\,, & \delta\bar{\tilde{q}}^{\dot{b}}_{I-1} = -\xi^{a\dot{b}}\bar{\tilde{\psi}}_{I-1\,a}\,, \end{matrix} \tag{B.35}$$

$$\delta\psi_{I\,\dot{a}} = i\gamma^\mu \xi_{b\dot{a}} D_\mu q_I^b + i\zeta_{b\dot{a}} q_I^b - \frac{i}{k}\xi_{b\dot{a}}(\nu_I q_I^b - q_I^b \nu_{I+1}) + \frac{2i}{k}\xi_{b\dot{c}}\left(\tilde{\mu}_I{}^{\dot{c}}{}_{\dot{a}} q_I^b - q_I^b \tilde{\mu}_{I+1}{}_{\dot{a}}{}^{\dot{c}}\right), \tag{B.36}$$

$$\delta\bar{\psi}_I^{\dot{a}} = i\gamma^\mu \xi^{b\dot{a}} D_\mu \bar{q}_{I\,b} + i\zeta^{b\dot{a}}\bar{q}_{I\,b} - \frac{i}{k}\xi^{b\dot{a}}(\bar{q}_{I\,b}\nu_I - \nu_{I+1}\bar{q}_{I\,b}) + \frac{2i}{k}\xi^{b\dot{c}}\left(\bar{q}_{I\,b}\tilde{\mu}_I{}^{\dot{a}}{}_{\dot{c}} - \tilde{\mu}_{I+1}{}_{\dot{c}}{}^{\dot{a}}\bar{q}_{I\,b}\right), \tag{B.37}$$

$$\delta\tilde{\psi}^a_{I-1} = -i\gamma^\mu \xi^{a\dot{b}} D_\mu \tilde{q}_{I-1\,\dot{b}} - i\zeta^{a\dot{b}}\tilde{q}_{I-1\,\dot{b}} + \frac{i}{k}\xi^{a\dot{b}}(\tilde{q}_{I-1\,\dot{b}}\tilde{\nu}_I - \tilde{\nu}_{I-1}\tilde{q}_{I-1\,\dot{b}}) \tag{B.38}$$

$$- \frac{2i}{k}\xi^{b\dot{c}}\left(\tilde{q}_{I-1\,\dot{c}}\mu_I{}^a{}_b - \mu_{I-1}{}_b{}^a\tilde{q}_{I-1\,\dot{c}}\right), \tag{B.39}$$

$$\delta\bar{\tilde{\psi}}_{I-1\,a} = -i\gamma^\mu \xi_{a\dot{b}} D_\mu \bar{\tilde{q}}^{\dot{b}}_{I-1} - i\zeta_{a\dot{b}}\bar{\tilde{q}}^{\dot{b}}_{I-1} + \frac{i}{k}\xi_{a\dot{b}}(\tilde{\nu}_I\bar{\tilde{q}}^{\dot{b}}_{I-1} - \bar{\tilde{q}}^{\dot{b}}_{I-1}\tilde{\nu}_{I-1}) \tag{B.40}$$

$$- \frac{2i}{k}\xi_{b\dot{c}}\left(\mu_I{}^b{}_a\bar{\tilde{q}}^{\dot{c}}_{I-1} - \bar{\tilde{q}}^{\dot{c}}_{I-1}\mu_{I-1}{}_a{}^b\right). \tag{B.41}$$

Here, $\xi_{a\dot{b}}$ are the Killing spinors and $\zeta_{a\dot{b}} = \frac{1}{3}\gamma^\mu\nabla_\mu\xi_{a\dot{b}}$.

Specifically, each supersymmetry parameter $\xi_{a\dot{b}}$ is a linear combination of four Killing-spinors on S^3. We label them as ξ^l, $\xi^{\bar{l}}$, ξ^r, $\xi^{\bar{r}}$ and they obey

$$\nabla_\mu \xi^{l,\bar{l}} = \frac{i}{2}\gamma_\mu \xi^{l,\bar{l}}\,, \qquad \nabla_\mu \xi^{r,\bar{r}} = -\frac{i}{2}\gamma_\mu \xi^{r,\bar{r}}\,. \tag{B.42}$$

Along the circle we may take $\gamma_\varphi = \sigma_3$ and these reduce to [14]

$$\xi^l_\alpha = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \xi^{\bar{l}}_\alpha = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \xi^r_\alpha = \begin{pmatrix}e^{-i\varphi}\\0\end{pmatrix}, \qquad \xi^{\bar{r}}_\alpha = \begin{pmatrix}0\\e^{i\varphi}\end{pmatrix}. \tag{B.43}$$

From this one finds $\zeta^{l,\bar{l}}_{a\dot{b}} = \frac{i}{2}\xi^{l,\bar{l}}_{a\dot{b}}$ and $\zeta^{r,\bar{r}}_{a\dot{b}} = -\frac{i}{2}\xi^{r,\bar{r}}_{a\dot{b}}$. We take the gamma-matrices, $(\gamma^\mu)_\alpha{}^\beta$, to be the Pauli matrices. As usual, spinor indices are contracted according to the NW-SE convention.

[3] The $SU(2)_L \times SU(2)_R$ R-symmetry indices $a, b = 1, 2$ and $\dot{a}, \dot{b} = \dot{1}, \dot{2}$ are raised and lowered from the left using $\epsilon^{12} = \epsilon_{21} = 1$.

B.5 Covariant form of Q^2G [20/08]

Firstly, denote

$$N = \begin{pmatrix} \nu_I & 0 \\ 0 & \nu_{I+1} \end{pmatrix}. \tag{B.44}$$

Consider then the matrices spanning the space of G's:

$$g^a = \begin{pmatrix} 0 & q^a \\ 0 & 0 \end{pmatrix}, \qquad \bar{g}_a = \begin{pmatrix} 0 & 0 \\ \bar{q}_a & 0 \end{pmatrix}. \tag{B.45}$$

Using the supersymmetry transformations, it is tedious but straightforward to prove[4]

$$\begin{aligned} Q^2 g^a = &-\mathrm{i}\epsilon^{bc}(\bar{\eta}V\eta)_{bc}\mathcal{D}^{\mathcal{A}}_{\varphi} g^a \\ &- \epsilon^{ab}\lambda_{cb}g^c + \frac{1}{2}\epsilon^{bc}\lambda_{bc}g^a - \frac{\mathrm{i}}{2}\partial_\varphi(\epsilon^{bc}(\bar{\eta}V\eta)_{bc})g^a \\ &- \frac{2i}{k}\epsilon^{ab}(\bar{\eta}V\eta)_{bc}[N, g^c] - \epsilon^{bc}(\bar{\eta}V\eta)_{bc}[B_0, g^a], \end{aligned} \tag{B.46}$$

$$\begin{aligned} Q^2 \bar{g}_a = &-\mathrm{i}\epsilon^{bc}(\bar{\eta}V\eta)_{bc}\mathcal{D}^{\mathcal{A}}_{\varphi} \bar{g}_a \\ &- \epsilon^{bc}\lambda_{ba}\bar{g}_c + \frac{1}{2}\epsilon^{bc}\lambda_{bc}\bar{g}_a - \frac{\mathrm{i}}{2}\partial_\varphi(\epsilon^{bc}(\bar{\eta}V\eta)_{bc})\bar{g}_a \\ &+ \frac{2i}{k}(\bar{\eta}V\eta)_{ba}\epsilon^{bc}[N, \bar{g}_c] - \epsilon^{bc}(\bar{\eta}V\eta)_{bc}[B_0, \bar{g}_a]. \end{aligned} \tag{B.47}$$

Expanding $G = \bar{\beta}_a g^a + \beta^a \bar{g}_a$, it is then straightforward to derive that

$$Q^2 G = -\mathrm{i}\mathcal{D}_{\varphi}^{\mathcal{L}_0^{\mathrm{diag}}}(\Pi G) + \left(Q^2 G\right)_{\mathrm{linear}} + \left(Q^2 G\right)_{\mathrm{cubic}}, \tag{B.48}$$

where

$$\begin{aligned} \left(Q^2G\right)_{\mathrm{linear}} = &\left[\mathrm{i}\Pi\partial_\varphi\bar{\beta}_a + \frac{1}{2}\left(\lambda + \Pi + \mathrm{i}\partial_\varphi\Pi\right)\bar{\beta}_a + \lambda_{ab}\epsilon^{bc}\bar{\beta}_c\right] g^a \\ &+ \left[\mathrm{i}\Pi\partial_\varphi\beta^a + \frac{1}{2}\left(\lambda - \Pi + \mathrm{i}\partial_\varphi\Pi\right)\beta^a + \epsilon^{ab}\lambda_{bc}\beta^c\right]\bar{g}_a, \end{aligned} \tag{B.49}$$

$$\left(Q^2G\right)_{\mathrm{cubic}} = -[H^2, G]. \tag{B.50}$$

[4]Here, $V = \begin{pmatrix} e^{+\mathrm{i}\varphi} & 1 \\ 1 & e^{+\mathrm{i}\varphi} \end{pmatrix}$, such that $(\bar{\eta}v)_a(\eta\bar{v})_b = (\bar{\eta}V\eta)_{ab} = \Pi_{ab}$.

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