Formulas for Brumer–Stark Units

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Submitted for the degree of Doctor of Philosophy in Mathematics

Abstract

Let F be a totally real number field. There have been three p-adic formulas conjectured by Dasgupta and Dasgupta–Spieß for the Brumer–Stark units of F. These formulas are conjectured to be equal by Dasgupta–Spieß. In this thesis we first show that two of these formulas are equal in the case that F is a cubic field. This proof uses only elementary methods involving calculations of Shintani sets. We then present joint work with Dasgupta which proves that all three of the conjectural formulas are equal for any totally real field F. Finally, work of Dasgupta–Kakde has shown that one of the conjectural formulas is equal to the Brumer–Stark unit up to a root of unity. Recent work of Bullach–Burns–Daoud–Seo proves the minus part of the eTNC away from 2, for finite abelian CM extensions of totally real fields. We show that this recent work implies that the formulas hold up to a 2-power root of unity.

Acknowledgments

I am extremely grateful for the generous support and mentorship of my supervisor Mahesh Kakde. Throughout my PhD studies his patient answers to my many questions has made the last few years so enjoyable. Mahesh has offered much excellent advice during this time, both in terms of directions for my research and with more general academic guidance. I would also like to thank Mahesh for his valuable comments on earlier versions of this thesis.

I should also like to express my gratitude to Samit Dasgupta for his collaboration which has improved some of the contents of this thesis. I would also like to thank Samit for generously giving his time and for many helpful conversations over the course of my PhD. Additionally, for supporting a visit to Duke University where much of the collaboration was conducted.

I thank David Burns for guiding my research throughout my PhD and for always making himself available for many useful meetings. I also thank David for his comments on an earlier version of this thesis.

I am grateful to my academic friends: Dominik Bullach, Alexandre Daoud, Tomasz Kosmala, Asuka Kumon, Thomas Sharpe, and Kwok-Wing "Ghaleo" Tsoi, for many tea breaks and much cake. In particular, I thank Dominik for carefully reading the introduction and background sections of this thesis.

I am thankful to Thanasis Bouganis and Luis Garcia for a very enjoyable and engaging viva. I am also grateful for the many helpful comments which have improved the final version of this thesis.

I would like to thank my father, Tim, and my siblings, Jacob and Isabelle, for their constant care throughout what has been a particularly difficult time for our family.

I am so thankful for the incredible support Kaman Lyu has shown me throughout our time together. Many times her willingness to listen to my frequent mathematical problems and difficulties has led me to a greater understanding. I am also grateful for Kaman's reading of this thesis and many helpful comments which have improved the grammar of this thesis no end!

Contents

Chapter 1

Introduction

Let F be a number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O} = \mathcal{O}_F$. Let $\mathfrak{p}$ be a prime of F, lying above a rational prime p, and let H be a finite abelian extension of F such that $\mathfrak{p}$ splits completely in H. In 1981, Tate proposed the Brumer–Stark conjecture, [22, Conjecture 5.4], stating the existence of $\mathfrak{p}$-unit u in H, the Brumer–Stark unit. This unit has $\mathfrak{P}$-order equal to the value of a partial zeta function at 0 for a prime $\mathfrak{P}$, of H, above $\mathfrak{p}$. Since the unit u is only non-trivial when F is totally real and H is totally complex containing a complex multiplication (CM) subfield, we assume this throughout this thesis. The Brumer–Stark conjecture is a refinement of a result of Gross, [16, Proposition 3.8], which proves the existence of an element satisfying the same property regarding its $\mathfrak{P}$-order. However, in this result, the element is in the group of $\mathfrak{p}$-units tensored over $\mathbb{Z}$ with $\mathbb{Q}$, rather than being a genuine $\mathfrak{p}$-unit. Recent work of Dasgupta–Kakde in [11] has shown that the Brumer–Stark conjecture holds away from 2.

There have been three formulas conjectured for the Brumer–Stark units. In [8, Definition 3.18], Dasgupta constructed explicitly, in terms of the values of Shintani zeta functions at $s = 0$, an element $u_1 \in F_{\mathfrak{p}}^*$. Dasgupta conjectured, in [8, Conjecture 3.21], that this unit is equal to the Brumer–Stark unit. This equality has recently been shown to be correct up to a root of unity by Dasgupta–Kakde in [10]. The key ingredient in the proof of the above theorem is Dasgupta–Kakde's proof of the p-part of the integral Gross–Stark conjecture. The other two formulas are cohomological in nature and were conjectured by Dasgupta–Spieß in [13] and [14], we denote these formulas by u_2 and u_3, respectively. In [13, Conjecture 6.1] and [14, Remark 4.5], respectively, u_2 and u_3 are conjectured to be equal to the Brumer–Stark unit. In this thesis, we give a complete account of these formulas before considering the progress we have made on problems related to these formulas.

The first result of this thesis is that u_1 is equal to u_3 when F is a cubic extension of $\mathbb{Q}$ (i.e., when $n = 3$). The equality of u_1 and u_3, for any totally real field F, was conjectured by Dasgupata–Spieß in [14, Remark 4.5]. They also proved the case when F is a quadratic field (i.e., when $n = 2$) in [14, Theorem 4.4]. This first result has been attempted previously by Tsosie in [24]. However, as we show in the appendix, we find a counterexample to the statement of [24, Lemma 2.1.3]; this lemma is necessary for his work. The statement concerns having a nice translation property of Shintani sets, for more details see Statement A.1.1 in the appendix. The main contribution of this first result is the methods we develop to recover some control of the translation properties of Shintani sets. We note that currently we have no way to extend the arguments used in this proof to allow us to apply this method to work with totally real fields of any degree. The translation properties of Shintani sets has not previously been studied and leads to a surprisingly simple conjecture. However, we are unable to prove this conjecture. We note this conjecture in the appendix. We remark that this conjecture appears to contain the additional information which would allow us to extend the proof presented in Chapter 7 to work for any totally real field, rather than only for cubic extensions. We give more detail on this in the appendix.

The second main result of this thesis is that u_1, u_2 and u_3 are all equal to each other, for any totally real number field F. This result is joint work with Samit Dasgupta. The approach of this proof is very different to that used by the author for the prior proof that $u_1 = u_3$ when F is a cubic extension. Firstly, it is possible to show that u_2 is equal to u_3 by direct calculation. More precisely, we write each of the cohomological expressions explicitly and then show that these two elements are equal. For the proof that u_1 is equal to u_2, we show that each of u_1 and u_2 satisfy a strong enough functorial property to force them to be equal. Namely, we show that they each satisfy a norm compatibility property. This result supersedes the earlier result of $u_1 = u_3$ when $n = 3$, although its proof is very different in style to the proof of our first result. An immediate consequence of our main result is that each of the formulas for the Brumer–Stark unit are correct up to a root of unity. More precisely, that [13, Conjecture 6.1] and [14, Remark 4.5] hold up to a root of unity. This follows applying our result that $u_1 = u_2 = u_3$ to Dasgupta–Kakde's proof, in [10], that u_1 is equal to the Brumer–Stark unit up to a root of unity.

In [14, Conjecture 3.1], Dasgupta–Spieß conjecture a cohomological formula for the principal minors and the characteristic polynomial of the Gross regulator matrix associated to a totally odd character of the totally real field F. The diagonal terms of the Gross regulator matrix are defined via the Brumer–Stark units. Let χ be a chosen totally odd character. Then, the diagonal terms are expressed via the ratio of the p-adic logarithm and the $\mathfrak{p}$-order of the χ^{-1} component of the Brumer–Stark unit. By considering [14, Conjecture 3.1] for the 1×1 principal minors, Dasgupa–Spieß conjectured a formula for this value. This formula is a specialisation of their formula u_3. It follows from our main result that Conjecture 3.1 in [14] holds for the 1×1 principal minors. We note that there is no root of unity ambiguity here due to the presence of $\log_p$ and norm maps in the definition of the Gross regulator matrix which removes roots of unity.

Finally, we consider the root of unity ambiguity in the result of Dasgupta–Kakde which proves that u_1 is equal to $u_{\mathfrak{p}}$ up to a root of unity. We show that these formulas are in fact correct up to a 2-power root of unity. In particular, we prove this for u_2 and use the equality of the formulas to obtain this result for u_1 and u_3. The key result that allows us to prove this theorem is the recent work of Bullach–Burns–Daoud–Seo in [2, Theorem B] where they prove the minus part of the eTNC away from 2 for finite CM extensions of totally real fields. It follows from [3, Corollary 4.3] of Burns that the integral Gross–Stark conjecture away from 2 is implied by [2, Theorem B]. We show that the integral Gross–Stark conjecture away from 2 is strong enough to imply that each of the formulas are correct up to a 2-power root of unity.

Chapter 2

Background and Conjectures

2.1 The analytic class number formula

L-functions are a central object of study in modern number theory. In particular, there is a focus on showing relations between special values of these analytic functions and arithmetic objects. The classical example of this type of relation is the analytic class number formula. Let F be a number field with ring of integers $\mathcal{O}_F$. One can define an analogue of the Riemann zeta function for F by using the norm of integral ideals, of $\mathcal{O}_F$, in place of the natural numbers in the Riemann zeta function, i.e., we define

$$\zeta_F(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})^s}, \quad s \in \mathbb{C},\ \mathrm{Re}(s) > 1.$$

Here the sum is over all the non-zero integral ideals of $\mathcal{O}_F$. This is the Dedekind zeta function for F. As with the Riemann zeta function, $\zeta_F(s)$ has a meromorphic continuation to all of $\mathbb{C}$, with only a simple pole at $s = 1$. We remark that these objects are purely analytic in nature and that when $F = \mathbb{Q}$ the Dedekind zeta function is equal to the Riemann zeta function. For a number field F, the class number of F is the order of the class group, which is defined by the quotient of the group of fractional ideals of $\mathcal{O}_F$ by the principal ideals of $\mathcal{O}_F$. The class group measures the failure of unique factorisation into primes and is thus a fundamental object of study. We denote the class number of F by h_F. The analytic class number formula provides a precise and remarkable relationship between the leading term at $s = 1$ of $\zeta_F(s)$ and the class number of F.

Theorem 2.1.1 (The analytic class number formula)**.** *Let F be a number field with $[F : \mathbb{Q}] = r_1 + 2r_2$, where r_1 and r_2 denote, respectively, the number of real and pairs of imaginary embeddings of F. Then,*

$$\lim_{s \to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} R_F h_F}{\omega_F \sqrt{|D_F|}},$$

where R_F, h_F, ω_F and D_F are, respectively, the regulator, class number, number of roots of unity *and discriminant of F.*

We have defined h_F above and will define R_F after the next theorem. The analytic class number formula was proved for quadratic fields by Dirichlet and in general by Dedekind. As with the Riemann zeta function, ζ_F satisfies a functional equation. In particular, we have for $s \in \mathbb{C}$,

$$\Lambda_F(s) = \Lambda_F(1-s) \quad \text{where} \quad \Lambda_F(s) = \mid D_F \mid^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_F(s).$$

Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, where $\Gamma(s)$ is the Gamma function. It follows from the functional equation for ζ_F (with $s = 0$) and the analytic class number formula, that the coefficient of the leading term in the Taylor expansion of ζ_F, denoted by $\zeta_F^*(0)$, is given by

$$\zeta_F^*(0) = \lim_{s \to 0} s^r \zeta_F(s) = -\frac{h_F R_F}{\omega_F}$$

where $r = r_1 + r_2 - 1$. This value r is also of significance; it is the rank of the group $\mathcal{O}_F^*$ of units of F. This follows from the following famous theorem of Dirichlet.

Theorem 2.1.2 (Dirichlet's unit theorem)**.** *The group $\mathcal{O}_F^*$ of units of a number field F is isomorphic to $W \times \mathbb{Z}^r$, where W is a finite cyclic group consisting of all the roots of unity in F and $r = r_1 + r_2 - 1$.*

The proof of this theorem is where the definition of R_F, the regulator of F, first appears. Since this definition appears throughout this thesis we shall recall it here. We fix the order of the embeddings $\sigma_1, \dots, \sigma_{r_1+r_2}$ of F such that σ_i is a real embedding if $1 \le i \le r_1$ and a complex embedding if $r_1 + 1 \le i \le r_1 + r_2$. Define the Dirichlet regulator $l : \mathcal{O}_F^* \to \mathbb{R}^{r_1+r_2}$ such that for $\alpha \in \mathcal{O}_F^*$ we have $l(\alpha) = (l_1(\alpha), \dots, l_{r_1+r_2}(\alpha))$ where

$$l_i(\alpha) = \begin{cases} \log(\mid \sigma_i(\alpha) \mid) & \text{if } 1 \le i \le r_1, \\ 2\log(\mid \sigma_i(\alpha) \mid) & \text{if } r_1 + 1 \le i \le r_1 + r_2. \end{cases}$$

The image of $\mathcal{O}_F^\times$ is an r-dimensional lattice in $\mathbb{R}^{r_1+r_2}$. As before we have $r = r_1 + r_2 - 1$. Letting $u_1, \dots, u_r \in \mathcal{O}_F^*$ be units such that the set $\{l(u_1), \dots, l(u_r)\}$ is a $\mathbb{Z}$-basis of the lattice $l(\mathcal{O}_F^*)$, we define

$$R_F = \mid \det(l_i(u_j))_{i,j=1,\dots,r} \mid.$$

We note that changing the choice of basis will only change the sign of the determinant and thus R_F is well defined.

2.2 The Gross–Stark conjecture

As the Dedekind zeta function extends the definition of the Riemann zeta function to arbitrary number fields, we now want to extend the definition of the Dedekind zeta function to work with extensions of arbitrary number fields. We are interested in finite abelian extensions of number fields H/F. The Galois group of H/F, which we denote $G = \mathrm{Gal}(H/F)$, is the group of automorphisms of H that fix the base field F. Let R denote a finite set of places of F containing the infinite places of F and those that are ramified in H. Let $\chi : G \to \mathbb{C}^*$ be any character of G. As usual, we view χ also as a multiplicative map on the semigroup of integral fractional ideals of F by defining $\chi(\mathfrak{q}) = \chi(\sigma_\mathfrak{q})$ if $\mathfrak{q}$ is unramified in H and $\chi(\mathfrak{q}) = 0$ if $\mathfrak{q}$ is ramified in H. Here $\sigma_\mathfrak{q}$ is the image of the ideal $\mathfrak{q}$ under the Artin map of class field theory. We can thus associate to any such χ the Artin L-function

$$L_R(\chi, s) = \sum_{(\mathfrak{a},R)=1} \frac{\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} = \prod_{\mathfrak{q}\notin R} \frac{1}{1-\chi(\mathfrak{q})\mathrm{N}\mathfrak{q}^{-s}}, \quad s \in \mathbb{C},\ \mathrm{Re}(s) > 1.$$

Here, the sum is over all non-zero integral ideals of $\mathcal{O}_F$ that are coprime R, i.e., the ideals that are coprime to each prime ideal in R. The product is over all prime ideals of $\mathcal{O}_F$ that are not contained in R. Here and from now on we write $\mathrm{N} = \mathrm{N}_{F/\mathbb{Q}}$. Similar to the Dedekind zeta function, if χ is non-trivial, we can analytically continue $L_R(\chi, s)$ to a holomorphic function on all of $\mathbb{C}$. Write $\overline{F}$ for the algebraic closure of F. We now let

$$\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}^*$$

be a character of the absolute Galois group of F. Fix a rational prime p. We fix embeddings $\overline{\mathbb{Q}} \subset \mathbb{C}$ and $\overline{\mathbb{Q}} \subset \mathbb{C}_p$, so χ may be viewed as taking values in $\mathbb{C}$ or $\mathbb{C}_p$. We let H denote the fixed field of the kernel of χ. We now give the construction of the p-adic L-function. Write P for the set of primes of F lying above p and let $R_P = R \cup P$. Partition P as $S_p \cup R_1$, where S_p denotes the subset of primes that split completely in H and R_1 the set of remaining primes of P. Let

$$\omega : \mathrm{Gal}(F(\mu_{2p})/F) \to (\mathbb{Z}/2p\mathbb{Z})^* \to \mu_{2(p-1)}$$

denote the Teichmüller character. For $n \in \mathbb{Z}_{>0}$, we have let μ_n denote the cyclic group of n^{th} roots of unity. There is a p-adic meromorphic function

$$L_p(\chi\omega, \cdot) : \mathbb{Z}_p \to \mathbb{C}_p$$

uniquely determined by the interpolation property

$$L_p(\chi\omega, k) = L_{R_P}(\chi\omega^k, k) \quad \text{for} \quad k \in \mathbb{Z}_{\geq 0}.$$

We refer to this function as the p-adic L-function. Under the Leopoldt conjecture, the p-adic L-function has a simple pole at $s = 1$ when $\chi = \omega^{-1}$. The existence of this function was shown independently by Deligne–Ribet [15] and Cassou-Noguès [4]. It follows from the functional equation of $L_{R_P}(\chi\omega^k, s)$ that $L_p(\chi\omega, \cdot)$ is the zero function unless F is totally real and χ is totally odd. We say F is totally real if $r_2 = 0$ and fix this choice of F. To define the notion of totally odd we first note that we can define the **sign** of χ as the tuple, $\mathrm{sign}(\chi) = (r_f)_{f\in\mathrm{Hom}(F,\mathbb{R})} \in \{0,1\}^{\#\mathrm{Hom}(F,\mathbb{R})}$, such that

$$\chi(a\mathcal{O}_F) = \prod_{f\in\mathrm{Hom}(F,\mathbb{R})} \mathrm{sign}(f(a))^{r_f} \quad \text{for all } a \equiv 1 \pmod{\mathfrak{f}}.$$

Here $\mathfrak{f}$ is the conductor of H/F. We then say χ is totally odd if $\mathrm{sign}(\chi) = (1,\ldots,1)$. For the remainder of this section we assume that F is totally real and χ is totally odd. We note that, in this case, H is a finite cyclic CM extension of F.

The Gross–Stark conjecture was stated by Gross in [16] and gives a relation between the leading term of a p-adic L-function, twisted by χ, and an algebraic invariant called Gross's regulator. Let $r_\chi = \#S_p$. We refer to this quantity as the rank of the conjecture.

Conjecture 2.2.1 (Gross–Stark conjecture)**.** *We have*

$$\frac{L_p^{(r_\chi)}(\chi,0)}{r_\chi! L_R(\chi,0)} = \mathcal{R}_p(\chi) \prod_{\mathfrak{p}\in R_1} (1-\chi(\mathfrak{p})),$$

where $\mathcal{R}_p(\chi)$ is a certain regulator of p-units of H, namely Gross's regulator which we define below.

Remark 2.2.2. *Conjecture 2.2.1 was first proved in the $r_\chi = 1$ case, assuming Leopoldt's conjecture and a technical condition, by Dasgupta–Darmon–Pollack in [9]. Both of these assumptions were later removed by Ventullo in [25]. The case of arbitrary rank was proved by Dasgupta–Kakde–Ventullo in [12].*

Definition 2.2.3. *For each prime $\mathfrak{p} \in S_p$, we define the group*

$$U'_{\mathfrak{p}} = \{u \in H^* : |\, u\,|_{\mathfrak{P}} = 1 \textit{ if } \mathfrak{P} \textit{ does not divide } \mathfrak{p}\}.$$

Here $\mathfrak{P}$ ranges over all finite and archimedean places of H; in particular, each complex conjugation in H acts as an inversion on $U'_{\mathfrak{p}}$. We remark that the standard notation for the above group is $U_{\mathfrak{p}}$. However, we require this notation later for a different object, to avoid confusion we have denoted the above group with a prime. We then write

$$\begin{aligned} U'_{\mathfrak{p},\chi} &\coloneqq (U'_{\mathfrak{p}} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}})^{\chi^{-1}} \\ &= \{u \in U'_{\mathfrak{p}} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}} \mid \sigma\cdot u = \chi^{-1}(\sigma)\cdot u \text{ for all } \sigma \in \mathrm{Gal}(H/F)\}. \end{aligned}$$

The Galois equivariant strengthening of Dirichlet's unit theorem, by Herbrand (see Chapter I, §3, §4 of [23]), implies that

$$\dim_{\overline{\mathbb{Q}}}(U'_{\mathfrak{p},\chi}) = \begin{cases} 1 & \text{if } \mathfrak{p} \in S_p, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u_{\mathfrak{p},\chi}$ denote any generator (i.e., non-zero element) of the $\overline{\mathbb{Q}}$-vector space $U_{\mathfrak{p},\chi}$. Consider the continuous homomorphisms

$$o_{\mathfrak{p}} := \mathrm{ord}_{\mathfrak{p}} : F_{\mathfrak{p}}^* \to \mathbb{Z}, \tag{2.1}$$

$$l_{\mathfrak{p}} := \log_p \circ \mathrm{Norm}_{F_{\mathfrak{p}}/\mathbb{Q}_p} : F_{\mathfrak{p}}^* \to \mathbb{Z}_p. \tag{2.2}$$

Here, $\log_p : \mathbb{Q}_p^* \to \mathbb{Z}_p$ denotes Iwasawa's p-adic logarithm and $\mathrm{Norm}_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ is the p-adic norm map on $F_{\mathfrak{p}}$. Suppose we choose for each $\mathfrak{p} \in S_p$, a prime $\mathfrak{P}_{\mathfrak{p}}$ of H lying above $\mathfrak{p}$. Then, for $\mathfrak{p}, \mathfrak{q} \in S_p$ we have

$$U'_{\mathfrak{p}} \subset H \subset H_{\mathfrak{P}_{\mathfrak{q}}} \cong F_{\mathfrak{q}}.$$

The isomorphism holds since $\mathfrak{p}$ splits completely in H. We can thus evaluate $o_{\mathfrak{q}}$ and $l_{\mathfrak{q}}$ on elements of $U_{\mathfrak{p}}$, and extend by linearity to maps

$$o_{\mathfrak{q}}, l_{\mathfrak{q}} : U'_{\mathfrak{p},\chi} \to \mathbb{C}_p.$$

Note that $o_{\mathfrak{q}}(U'_{\mathfrak{p},\chi})$ is non-zero if $\mathfrak{p} = \mathfrak{q}$ and is zero for $\mathfrak{p} \neq \mathfrak{q}$. Define the ratio

$$\mathscr{L}_{\mathrm{alg}}(\chi)_{\mathfrak{p},\mathfrak{q}} = -\frac{l_{\mathfrak{q}}(u_{\mathfrak{p},\chi})}{o_{\mathfrak{p}}(u_{\mathfrak{p},\chi})},$$

which is clearly independent of the choice of $u_{\mathfrak{p},\chi} \in U'_{\mathfrak{p},\chi}$. **Gross's regulator**, $\mathscr{R}_p(\chi)$, is the determinant of the $(\#S_p) \times (\#S_p)$ matrix whose entries are given by these values:

$$\mathscr{R}_p(\chi) := \det(\mathscr{M}_p(\chi)), \quad \text{where} \quad \mathscr{M}_p(\chi) := (\mathscr{L}_{\mathrm{alg}}(\chi)_{\mathfrak{p},\mathfrak{q}})_{\mathfrak{p},\mathfrak{q} \in S_p}.$$

We refer to $\mathscr{M}_p(\chi)$ as the **Gross regulator matrix**. More generally, for any subset $J \subset S_p$, the principal minor of $\mathscr{M}_p(\chi)$ corresponding to J is defined by

$$\mathscr{R}_p(\chi)_J := \det(\mathscr{L}_{\mathrm{alg}}(\chi)_{\mathfrak{p},\mathfrak{q}})_{\mathfrak{p},\mathfrak{q} \in J}.$$

We note that both $\mathscr{R}_p(\chi)$ and $\mathscr{R}_p(\chi)_J$ are independent of all choices. In particular, we note that for each prime $\mathfrak{q} \in S_p$, the maps $l_{\mathfrak{q}}$ and $o_{\mathfrak{q}}$ depend on the choice of a prime $\mathfrak{P}_q$ of H lying above $\mathfrak{q}$. If, rather than $\mathfrak{P}_q$, one chooses $\sigma(\mathfrak{P}_q)$ for some $\sigma \in G$, then this scales $l_{\mathfrak{q}}$ and $o_{\mathfrak{q}}$ by $\chi(\sigma)$. Hence the diagonal entries are unchanged by this choice. Furthermore, this choice multiplies the $\mathfrak{q}$th row of $\mathscr{M}_p(\chi)$ by $\chi(\sigma)^{-1}$ and the $\mathfrak{q}$th column of $\mathscr{M}_p(\chi)$ by $\chi(\sigma)$. It follows that both $\mathscr{R}_p(\chi)$ and $\mathscr{R}_p(\chi)_J$ are independent of these choices.

Let $J \subset S_p$ be nonempty. Dasgupta–Spieß have constructed, via group cohomology, a formula which they conjecture, in [14, Conjecture 3.1], to be equal to the value $\mathscr{R}_p(\chi)_J$. If we take $J = \{\mathfrak{p}\}$ for some $\mathfrak{p} \in S_p$, then the value of $\mathscr{R}_p(\chi)_{\mathfrak{p}}$ is the diagonal entry at $\mathfrak{p}$ of the Gross regulator matrix, i.e.,

$$\mathscr{R}_p(\chi)_{\mathfrak{p}} = \mathscr{L}_{\mathrm{alg}}(\chi)_{\mathfrak{p},\mathfrak{p}} = -\frac{l_{\mathfrak{p}}(u_{\mathfrak{p},\chi})}{o_{\mathfrak{p}}(u_{\mathfrak{p},\chi})}.$$

2.3 The Brumer–Stark units

We now consider an arbitrary finite abelian field extension H/F. As in the previous section, fix a prime $\mathfrak{p}$ of F, above the rational prime p, such that $\mathfrak{p}$ splits completely in H. As before, we let R denote a finite set of places of F such that R contains the archimedean places, $\mathfrak{p} \notin R$, and R contains the places that are ramified in H. The conjectures we wish to consider now are integral rather then p-adic. I.e., rather then considering the leading term of a p-adic L-function we want to consider the leading term of an Artin L-function. Furthermore we want the value we consider to be an integer. Thus, we must define a modified L-function. Let T denote a finite set of places of F disjoint from R such that T contains two primes of different residue characteristic or one prime of residue characteristic larger than $[F : \mathbb{Q}] + 1$. We always assume that this condition holds. Let χ be a character of $G = \mathrm{Gal}(H/F)$. We then define the R-depleted, T-smoothed Artin L-function of χ,

$$L_{R,T}(\chi, s) = L_R(\chi, s) \prod_{\mathfrak{q} \in T} (1 - \chi(\mathfrak{q})\mathrm{N}\mathfrak{q}^{1-s}), \quad s \in \mathbb{C},\ \mathrm{Re}(s) > 1.$$

If χ is nontrivial, then the function $L_{R,T}(\chi, s)$ can be analytically continued to a holomorphic function on all of $\mathbb{C}$. This follows from the equivalent result for $L_R(\chi, s)$. We can extend χ so that we can consider $\chi : \mathbb{C}[G] \to \mathbb{C}$. One can package together the Artin L-functions into a Stickelberger element $\Theta_{R,T}(s)$ which lives in the group ring $\mathbb{C}[G]$. This element is defined by the property that if we specialise it to a character χ of G, then we get the modified Artin L-function $L_{R,T}(\chi^{-1}, s)$, i.e.,

$$\chi(\Theta_{R,T}(s)) = L_{R,T}(\chi^{-1}, s).$$

An important theorem of Deligne–Ribet [15] and Cassou-Nogués [4] states that the value of the Stickelberger element at 0, which we denote $\Theta_{R,T} = \Theta_{R,T}(0)$, is in fact contained in the integral group ring $\mathbb{Z}[G]$.

We also need to modify the class group we are considering. The T-smoothed ray class group of H, which we denote as $\mathrm{Cl}^T(H)$, is defined to be the quotient of the group of fractional ideals of $\mathcal{O}_H$ which are coprime to primes in T, by the principal ideals of $\mathcal{O}_H$ which are generated by elements which are congruent to 1 modulo primes of $\mathcal{O}_H$ above primes of T. The following conjecture, stated by Tate in [22], is known as the Brumer–Stark conjecture.

Conjecture 2.3.1 (Brumer–Stark conjecture)**.** *We have*

$$\Theta_{R,T} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(H)).$$

Conjecture 2.3.1 provides another remarkable relation between an analytic object, $\Theta_{R,T}$, and an algebraic invariant, $\mathrm{Cl}^T(H)$. This time we have these objects associated to a field extension H/F. We note that Conjecture 2.3.1 is a generalisation of Stickelberger's theorem. Let $K/\mathbb{Q}$ be an arbitrary abelian extension of $\mathbb{Q}$. Stickelberger's theorem and its generalisations provide an ideal in the group ring $\mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]$ which annihilates the class group of K. Conjecture 2.3.1 extends this to considering arbitrary abelian extensions H/F. We note that Conjecture 2.3.1 is only nontrivial in the case that F is totally real and H is totally complex containing a CM subfield. We have this assumption for the remainder of this thesis.

Recall we have let $\mathfrak{p}$ be a prime of F that splits completely in H. We further assume that $\mathfrak{p}$ is not contained in the union of R and T. Choose a prime $\mathfrak{P}$ of H above $\mathfrak{p}$. Let σ be an element of the Galois group G. We then define the partial zeta function $\zeta_{R,T}(\sigma)$ to be the σ^{-1} component of $\Theta_{R,T}$, such that

$$\Theta_{R,T} = \sum_{\sigma \in G} \zeta_{R,T}(\sigma)[\sigma^{-1}].$$

For an element $\sigma \in G$ we write $[\sigma] \in \mathbb{Z}[G]$ for the group ring element. When the notation is clear we will drop the brackets. We note that by our assumption on T we have $\zeta_{R,T}(\sigma) \in \mathbb{Z}$. We can also define $\zeta_{R,T}(\sigma)$ as the special value of an L-function. Firstly, we define the following.

Definition 2.3.2. *For $\sigma \in G$, we define the **partial zeta function***

$$\zeta_R(\sigma, s) = \sum_{\substack{(\mathfrak{a},R)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \mathrm{N}\mathfrak{a}^{-s}, \quad s \in \mathbb{C},\ Re(s) > 1. \tag{2.3}$$

Here the sum is over all non-zero integral ideals $\mathfrak{a} \subset \mathcal{O}_F$ that are relatively prime to the elements of R and whose associated Frobenius element $\sigma_{\mathfrak{a}} \in G$ is equal to σ.

Note that the series (2.3) converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$, regular outside $s = 1$. The zeta functions associated to the sets of primes R and $S = R \cup \{\mathfrak{p}\}$ are related to each other by the formula

$$\zeta_S(\sigma, s) = (1 - \mathrm{N}\mathfrak{p}^{-s})\zeta_R(\sigma, s).$$

If K is a finite abelian extension of F and $\sigma \in \mathrm{Gal}(K/F)$ we use the notation $\zeta_R(K/F, \sigma, s)$ for the partial zeta function defined as above but with the equality $\sigma_{\mathfrak{a}} = \sigma$ being viewed in $\mathrm{Gal}(K/F)$. We then define the partial zeta function associated to the sets R and T by the group ring equation

$$\sum_{\sigma \in G} \zeta_{R,T}(\sigma, s)[\sigma] = \prod_{\eta \in T}(1 - [\sigma_\eta]\mathrm{N}\eta^{1-s}) \sum_{\sigma \in G} \zeta_R(\sigma, s)[\sigma]. \tag{2.4}$$

Here we define $\zeta_{R,T}(\sigma, s)$ to be equal to the σ component on the right hand side of (2.4), after expanding the product and sum. We then have $\zeta_{R,T}(\sigma) = \zeta_{R,T}(\sigma, 0) \in \mathbb{Z}$. I.e., the partial zeta function associated to σ is equal to the σ^{-1} component of the Stickelberger element. Conjecture 2.3.1 implies that the ideal

$$\mathfrak{P}^{\Theta_{R,T}} = \prod_{\sigma \in G} \sigma^{-1}(\mathfrak{P})^{\zeta_{R,T}(\sigma)}$$

is a principal ideal (u) generated by an element u which is congruent to 1 modulo all primes of $\mathcal{O}_H$ above primes of T. From now on we write this as $u \equiv 1 \pmod{T}$. We further conjecture that u can be chosen so that its image under complex conjugation is equal to its inverse. It is this statement that was originally proposed by Tate in [22]. Thus our statement of Conjecture 2.3.1 is slightly weaker than the original statement of Tate. We now give the formulation of the Brumer–Stark conjecture due to Gross. We note that this statement follows from Conjecture 2.3.1.

Conjecture 2.3.3 (Conjecture 7.4, [17]). *Let $\mathfrak{P}$ be a prime in H above $\mathfrak{p}$. There exists an element $u_T \in U'_{\mathfrak{p}}$ such that $u_T \equiv 1 \pmod{T}$, and for all $\sigma \in G$, we have*

$$\mathrm{ord}_{\mathfrak{P}}(u_T^{\sigma}) = \zeta_{R,T}(H/F,\sigma,0). \tag{2.5}$$

Our assumption on T implies that there are no non-trivial roots of unity in H that are congruent to 1 modulo T. Furthermore, recalling the definition of $U'_{\mathfrak{p}}$ from Definition 2.2.3 and noting that $|u_v| = 1$ for all the infinite places we see that the $\mathfrak{p}$-unit, if it exists, is unique. Note also that our u_T is actually the inverse of the u in [17, Conjecture 7.4]. The conjectural element $u_T \in U'_{\mathfrak{p}}$ satisfying Conjecture 2.3.3 is called the Brumer–Stark unit for the data $(S,T,H,\mathfrak{P})$. Throughout this thesis, for ease of notation, we have $T = \{\lambda\}$ for an appropriate choice of λ. In particular, we choose λ such that $\mathrm{N}\lambda = l$ for a prime number $l \in \mathbb{Z}$ and $l \geq n+2$. Recall that we have denoted $n = [F:\mathbb{Q}]$. It will be convenient for us to work with the following element of $H^*[G]$. We define

$$u_{\mathfrak{p}} = \sum_{\sigma \in G} u_T^{\sigma} \otimes [\sigma^{-1}] \in H^*[G].$$

Throughout this thesis we will write $u_{\mathfrak{p}}(\sigma) = u_T^{\sigma}$ for the σ^{-1} component of $u_{\mathfrak{p}}$. We will also refer to $u_{\mathfrak{p}}$ as the Brumer–Stark unit, it will always be clear from context if we are referring to $u_{\mathfrak{p}}$ or u_T.

The Brumer–Stark conjecture (Conjecture 2.3.1) has recently been proved, away from 2, by Dasgupta–Kakde in [11]. In particular, they prove the following theorem.

Theorem 2.3.4 (Theorem 1.2, [11]). *We have*

$$\Theta_{R,T} \in \mathrm{Ann}_{\mathbb{Z}[G][1/2]}\left(\mathrm{Cl}^T(H) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]\right).$$

This theorem is proved by applying Ribet's method. One of the key parts of [11] is the use of group ring valued Hilbert modular forms to show the existence of the required cohomology class. It follows, from the paragraph before Conjecture 2.3.3, that Theorem 2.3.4 implies the existence of a unit $u_T \in U'_{\mathfrak{p}} \otimes \mathbb{Z}[\frac{1}{2}]$ satisfying equation (2.5). Equivalently, Theorem 2.3.4 implies the existence of a unit $u_T \in U'_{\mathfrak{p}}$ that satisfies equation (2.5) up to multiplication by a power of 2. I.e. for each $\sigma \in G$, we have

$$\mathrm{ord}_{\mathfrak{P}}(u_T^{\sigma}) = 2^k \zeta_{R,T}(H/F,\sigma,0),$$

for some $k \in \mathbb{Z}_{\geq 0}$.

There have been three formulas conjectured for the Brumer–Stark unit $u_{\mathfrak{p}}$ in $F_{\mathfrak{p}}^{*} \otimes \mathbb{Z}[G]$. The first by Dasgupta in [8] is a p-adic analytic formula which we denote by u_1. The other two formulas were defined by Dasgupta–Spieß in [13] and [14], as in the introduction, we denote the elements given by these formulas by u_2 and u_3, respectively. Both these formulas are cohomological in nature and are defined using an Eisenstein cocycle, we give more details on this in §3.5. Each of the elements, u_1, u_2 and u_3, are conjectured to be equal to the Brumer–Stark unit, $u_{\mathfrak{p}}$. These conjectures are due to Dasgupta in [8] and Dasgupta–Spieß in [13] and [14], for u_1, u_2 and u_3 respectively. We combine these three conjectures in the following.

Conjecture 2.3.5. *Let $i = 1, 2, 3$ then*

$$u_i = u_{\mathfrak{p}}.$$

Recent work of Dasgupta–Kakde in [10] has proved this conjecture for u_1 up to a root of unity under some mild assumptions. In particular, they have proved the following theorem.

Theorem 2.3.6 (Theorem 1.6, [10])**.** *Let p denote the rational prime below $\mathfrak{p}$. Suppose that p is odd,*

$$\begin{aligned} &\text{there exists } \mathfrak{q} \in S \text{ where } \mathfrak{q} \text{ is a prime of } F \text{ that is unramified in } H \\ &\text{and whose associated Frobenius } \sigma_{\mathfrak{q}} \text{ is a complex conjugation in } H, \end{aligned} \tag{2.6}$$

and

$$H \cap F(\mu_{p^\infty}) \subset H^+, \text{ the maximal totally real subfield of } H. \tag{2.7}$$

Then, Conjecture 2.3.5 for u_1 holds up to multiplication by a root of unity in $F_{\mathfrak{p}}^{*}$, i.e.,

$$u_1 = u_{\mathfrak{p}} \text{ in } (F_{\mathfrak{p}}^{*}/\mu(F_{\mathfrak{p}}^{*})) \otimes \mathbb{Z}[G].$$

We have $\mu(F_{\mathfrak{p}}^{*})$ for the group of roots of unity of $F_{\mathfrak{p}}^{*}$.

The key ingredient in the proof of the above theorem is Dasgupta–Kakde's proof of the p-part of the integral Gross–Stark conjecture. We give this result in the next section. We note here the work of Darmon–Pozzi–Vonk [7] which proves a version of Conjecture 2.3.5 in the setting that F is a real quadratic field and the rational prime p is inert in F. Their work also used deformations of p-adic modular forms and their associated Galois representations. However, this thesis, and the work of Dasgupta–Kakde's, uses "horizontal" tame deformations while Darmon-Pozzi-Vonk use "vertical" p-adic towers.

The main result of this thesis is the following theorem. This result was conjectured by Dasgupta–Spieß in [13] and [14].

Theorem 2.3.7. *One has,*

$$u_1 = u_2 = u_3.$$

Remark 2.3.8. *Theorem 2.3.7 and Theorem 2.3.6 imply that we have* $u_2 = u_3 = u_{\mathfrak{p}}$ *in* $(F_{\mathfrak{p}}^*/\mu(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G]$*, i.e., in addition to Theorem 2.3.6 holding for* u_1 *it also holds for* u_2 *and* u_3*. We remark that by following the arguments of [8] and using Proposition 6.3 in [13] one can show that Theorem 2.3.6 holds for* u_2 *without using Theorem 2.3.7.*

Remark 2.3.9. *Theorem 2.3.7 would also follow from a proof of Conjecture 2.3.5. However, all current approaches to this conjecture have a root of unity ambiguity, whereas in Theorem 2.3.7 we have no such ambiguity.*

We prove Theorem 2.3.7 in two stages. Initially, we show that $u_2 = u_3$ in §8.1. Then, we show that $u_1 = u_2$ in §8.2 and this completes the proof of the theorem. This theorem is joint work with Dasgupta. Prior to this, we prove in Chapter 7 that $u_1 = u_3$ when F is of degree three. The approach of this proof is very different to the work in Chapter 8.

The final result of this thesis, which we prove in Chapter 9, is the following theorem.

Theorem 2.3.10. *Let* p *denote the rational prime below* $\mathfrak{p}$*. Suppose that* p *is odd, and (2.6) and (2.7) hold. Then, Conjecture 2.3.5 for* u_2 *holds up to multiplication by a 2-power root of unity in* $F_{\mathfrak{p}}^*$*, i.e.,*

$$u_2 = u_{\mathfrak{p}} \text{ in } (F_{\mathfrak{p}}^*/\mu_2(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G].$$

We write $\mu_2(F_{\mathfrak{p}}^*)$ *for the group of 2-power roots of unity of* $F_{\mathfrak{p}}^*$*.*

This theorem is another step towards a proof of Conjecture 2.3.5 by reducing the root of unity ambiguity to a 2-power root of unity. We note that we do not require any additional assumptions to those used in Theorem 2.3.6.

2.4 The integral Gross–Stark conjecture

The integral Gross–Stark conjecture or, as it is also known, Gross's tower of fields conjecture, is an integral version of the Gross–Stark conjecture (Conjecture 2.2.1). Gross first stated this conjecture in [17]. In this conjecture, we consider a tower of fields $L/H/F$, as before F is totally real. We take H and L to be finite abelian extensions of F that are CM fields such that L contains H. Write $\mathfrak{g} = \mathrm{Gal}(L/F)$. Recall that $S = R \cup \{\mathfrak{p}\}$ where $\mathfrak{p}$ splits completely in H/F. The integral Gross–Stark conjecture gives a relationship between Brumer–Stark $\mathfrak{p}$-units and the Stickelberger element, $\Theta_{S,T}^{L/F}$, for L/F. Here T is as in §2.3. Denote by

$$\mathrm{rec}_{\mathfrak{p}} : F_{\mathfrak{p}}^* \to \mathbb{A}_F^* \to \mathfrak{g}$$

the local component of the reciprocity map of class field theory. Since $H \subset H_{\mathfrak{P}} \cong F_{\mathfrak{p}}$, we can evaluate $\mathrm{rec}_{\mathfrak{p}}$ on H^*. Note that if $x \in H^*$ then $\mathrm{rec}_{\mathfrak{p}}(x) \in \mathrm{Gal}(L/H)$. Let I denote the relative augmentation ideal associated to $\mathfrak{g}$ and G, i.e., the kernel of the canonical projection

$$\mathrm{Aug}_G^{\mathfrak{g}} : \mathbb{Z}[\mathfrak{g}] \twoheadrightarrow \mathbb{Z}[G].$$

The key point of the following conjecture is that it is integral, as opposed to the Gross–Stark conjecture which is p-adic. This integrality is one of the things that makes the following conjecture so difficult to prove.

Conjecture 2.4.1 (Integral Gross–Stark conjecture). *Define*

$$\mathrm{rec}_G(u_{\mathfrak{p}}) = \sum_{\sigma \in G} (\mathrm{rec}_{\mathfrak{p}}(u_{\mathfrak{p}}(\sigma)) - 1)\tilde{\sigma}^{-1} \in I/I^2,$$

where $\tilde{\sigma} \in \mathfrak{g}$ *is any lift of* $\sigma \in G$ *and* $u_{\mathfrak{p}} = \sum_{\sigma \in G} u_{\mathfrak{p}}(\sigma) \otimes \sigma^{-1}$ *is the Brumer–Stark unit. Then*

$$\mathrm{rec}_G(u_{\mathfrak{p}}) \equiv \Theta_{S,T}^{L/F} \pmod{I^2}, \tag{2.8}$$

in I/I^2.

To see how this conjecture can provide more information about the Brumer–Stark unit we first consider the σ component of equation (2.8). Then, since $I/I^2 \cong \mathfrak{g}$ via the isomorphism $\sigma - 1 \mapsto \sigma$, one can see that Conjecture 2.4.1 implies

$$\mathrm{rec}_{\mathfrak{p}}(u_{\mathfrak{p}}(\sigma)) = \prod_{\substack{\tau \in \mathfrak{g} \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,T}(L/F, \tau^{-1})}. \tag{2.9}$$

Taking the inverse of $\mathrm{rec}_{\mathfrak{p}}$ on both sides of the above equation allows us to gain more information about the unit $u_{\mathfrak{p}}$. In particular it gives us the value of $u_{\mathfrak{p}}(\sigma) \in F_{\mathfrak{p}}^*/\ker(\mathrm{rec}_{\mathfrak{p}})$. Let $\mathfrak{f}$ denote the conductor of H/F. To gain some more precise information one can apply equation (2.9) with $L = K$ for every $H \supset K \subset H_{\mathfrak{f}\mathfrak{p}^\infty}$. Here we define $H_{\mathfrak{f}\mathfrak{p}^\infty}$ to be the union of the narrow ray class fields $H_{\mathfrak{f}\mathfrak{p}^m}$ for each $m \in \mathbb{Z}_{\geq 1}$. The local reciprocity map at $\mathfrak{p}$ induces an isomorphism

$$\mathrm{rec}_{\mathfrak{p}} : F_{\mathfrak{p}}^*/\widehat{E_+(\mathfrak{f})_{\mathfrak{p}}} \cong \mathrm{Gal}(H_{\mathfrak{f}\mathfrak{p}^\infty}/H),$$

where we write $E_+(\mathfrak{f})_{\mathfrak{p}}$ for the totally positive $\mathfrak{p}$-units of F which are congruent to 1 (mod $\mathfrak{f}$). Then $\widehat{E_+(\mathfrak{f})_{\mathfrak{p}}}$ denotes the closure of $E_+(\mathfrak{f})_{\mathfrak{p}}$ in $F_{\mathfrak{p}}^*$. Thus we can use (2.9) to give the value of $u_{\mathfrak{p}}(\sigma)$ in $F_{\mathfrak{p}}^*/\widehat{E_+(\mathfrak{f})_{\mathfrak{p}}}$. In [8] Dasgupta develops the methods of horizontal Iwasawa theory to further refine this kernal and shows that the p-part of Conjecture 2.4.1 implies Theorem 2.3.6. It is these horizontal methods that we will use in Chapter 9 of this thesis to show that Theorem 2.3.10 follows from the combination of the l-part of Conjecture 2.4.1 for every odd prime l.

The p-part of Conjecture 2.4.1, when p is odd, has recently been proved by Dasgupta–Kakde [10]. Recall that $\mathfrak{p}$ lies above p. We give the statement of their theorem below. As with Dasgupta–Kakde's proof of the prime-to-2 part of the Brumer–Stark conjecture, the approach is to apply Ribet's method again working with group ring valued Hilbert modular forms.

Theorem 2.4.2 (Theorem 1.4, [10]). *Let* p *be an odd prime and suppose that* $\mathfrak{p}$ *lies above* p. *The integral Gross–Stark conjecture (Conjecture 2.4.1) holds in* $(I/I^2) \otimes \mathbb{Z}_p$.

Remark 2.4.3. *Recent work of Bullach–Burns–Daoud–Seo in [2, Theorem B] has proved the minus-part of the eTNC away from* 2*, for finite abelian CM extensions of totally real fields. Burns has proved in [3, Corollary 4.3] that [2, Theorem B] implies the integral Gross–Stark conjecture. It follows from this that the* l*-part of Conjecture 2.4.1 holds for all primes* $l \neq 2$*. Thus, the following theorem holds.*

Theorem 2.4.4. *Let* l *be an odd prime. The integral Gross–Stark conjecture (Conjecture 2.4.1) holds in* $(I/I^2) \otimes \mathbb{Z}_l$*.*

The above theorem is crucial in proving the final result of this thesis, i.e., for the proof of Theorem 2.3.10.

Chapter 3

Shintani Zeta Functions and the Eisenstein Cocycle

3.1 Notation

Recall that we let F be a totally real field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O} = \mathcal{O}_F$. Let $E = E_F = \mathcal{O}_F^*$ denote the group of global units. More generally, for a finite set S of non-archimedean places of F, we denote by $E_S = E_{F,S}$ the group of S-units of F. We define

$$\overline{S} = \{\mathfrak{q} \text{ prime of } F : \mathfrak{q} \mid q \text{ where, for some } \mathfrak{r} \in S,\ \mathfrak{r} \mid q\}. \tag{3.1}$$

We also let H/F be a totally complex extension containing a CM subfield. Write E_+ for the totally positive units of F. Let $\mathfrak{f}$ denote the conductor of the extension H/F. We write $E_+(\mathfrak{f})$ for the totally positive units of F which are congruent to 1 (mod $\mathfrak{f}$). Write $G_\mathfrak{f}$ for the narrow ray class group of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$ and suppose that $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1$ (mod $\mathfrak{f}$) and π totally positive. We write $\mathbb{O} = \mathcal{O}_\mathfrak{p} - \pi\mathcal{O}_\mathfrak{p} \subset F_\mathfrak{p}^*$.

Define $\mathbb{A} = \mathbb{A}_F$ as the adele ring of F. We define

$$\widehat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p,$$

where the second equality follows from the Chinese remainder theorem. For a $\mathbb{Q}$-vector space W, fix the notation $W_{\widehat{\mathbb{Z}}} = W \otimes_\mathbb{Z} \widehat{\mathbb{Z}} = W \otimes_\mathbb{Q} \mathbb{Q} \otimes_\mathbb{Z} \widehat{\mathbb{Z}} = W \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q}$. For an abelian group A and prime number l, we put $A_l = A \otimes_\mathbb{Z} \mathbb{Q}_l$.

For a place v of F, we put $U_v = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ if $v \mid \infty$ and $U_v = \mathcal{O}_v^*$ if v is finite. For a set S of places of F, we let $\mathbb{A}^S$ denote the S-adeles. We also define $U^S = \prod_{v \notin S} U_v$, and $U_S = \prod_{v \in S} U_v$. We shall also use the notation $F^S = (\mathbb{A}_F^S \times U_S) \cap F^*$.

Finally, we note that if we have a function $f : X \to Z$ and $X \subseteq Y$ then we can extend f to a function $f_! : Y \to Z$ by defining

$$f_!(y) = \begin{cases} f(y) & \text{if } y \in X \\ 0 & \text{if } y \in Y - X. \end{cases}$$

We call this function the extension of f to Y by 0.

3.2 Shintani zeta functions

Shintani zeta functions are a crucial ingredient in each of the three constructions we are studying. The first step in defining these modified zeta functions considers the work of Shintani, initially developed in his paper [19], and the definitions of Shintani cones and domains. We establish the necessary notation here.

Let R_∞ denote the set of infinite places of F. For each $v \in R_\infty$, we write $\sigma_v : F \to \mathbb{R}$ and fix the order of these embeddings. We can then embed F into $\mathbb{R}^n$ by $x \mapsto (\sigma_v(x))_{v \in R_\infty}$. We note that F^* acts on $\mathbb{R}^n$ with $x \in F$ acting by multiplication by $\sigma_v(x)$ on the v-component of any vector in $\mathbb{R}^n$. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ denote the positive elements of $\mathbb{R}$. For linearly independent $v_1, \dots, v_r \in \mathbb{R}_+^n$, define the simplicial cone

$$C(v_1, \dots, v_r) = \left\{ \sum_{i=1}^{r} c_i v_i \in \mathbb{R}_+^n : c_i > 0 \right\}.$$

Definition 3.2.1. *A* ***Shintani cone*** *is a simplicial cone $C(v_1, \dots, v_r)$ generated by elements $v_i \in F \cap \mathbb{R}_+^n$. A* ***Shintani set*** *is a subset of $\mathbb{R}_+^n$ that can be written as a finite disjoint union of Shintani cones.*

We now give a Lemma of Dasgupta relating to the intersection of two Shintani sets.

Lemma 3.2.2 (Lemma 3.14, [8])**.** *The intersection of two Shintani sets is a Shintani set. For two Shintani sets D and D' there exists only a finite number of $\epsilon \in E_+$ such that $\epsilon D \cap D'$ is nonempty.*

We now give the definition for **Shintani zeta functions**. Write $\mathfrak{f}$ for the conductor of the extension H/F. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\overline{T}$, and let D be a Shintani set. For each compact open $U \subseteq \mathcal{O}_\mathfrak{p}$, define, for $\mathrm{Re}(s) > 1$,

$$\zeta_R(\mathfrak{b}, D, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in F \cap D,\ \alpha \in U \\ (\alpha, R) = 1,\ \alpha \in \mathfrak{b}^{-1} \\ \alpha \equiv 1 \pmod{\mathfrak{f}}}} \mathrm{N}\alpha^{-s}.$$

Here the sum is over elements of F which are relatively prime to R, congruent to 1 modulo $\mathfrak{f}$, and contained in D, U and $\mathfrak{b}^{-1}$. We define $\zeta_{R,T}(\mathfrak{b}, D, U, s)$ in analogy with (2.4), i.e., by the group ring equation

$$\sum_{\sigma_{\mathfrak{b}}\in G}\zeta_{R,T}(\mathfrak{b},D,U,s)[\sigma_{\mathfrak{b}}]=\prod_{\eta\in T}(1-[\sigma_\eta]\mathrm{N}\eta^{1-s})\sum_{\sigma_{\mathfrak{b}}\in G}\zeta_R(\mathfrak{b},D,U,s)[\sigma_{\mathfrak{b}}]. \tag{3.2}$$

It follows from Shintani's work in [19] that the function $\zeta_{R,T}(\mathfrak{b},D,U,s)$ has a meromorphic continuation to $\mathbb{C}$. We now want to define conditions on the set of primes T and the Shintani set D to allow our Shintani zeta functions to be integral at $s=0$.

Definition 3.2.3. *A prime ideal η of F is called **good** for a Shintani cone C if*

- $\mathrm{N}\eta$ *is a rational prime l; and*
- *the cone C may be written $C=C(v_1,\dots,v_r)$ with $v_i\in\mathcal{O}$ and $v_i\notin\eta$.*

*We also say that η is **good** for a Shintani set D if D can be written as a finite disjoint union of Shintani cones for which η is good.*

Definition 3.2.4. *The set T is **good** for a Shintani set D if D can be written as a finite disjoint union of Shintani cones $D=\cup C_i$, so that for each cone C_i, there are at least two primes in T that are good for C_i (necessarily of different residue characteristic by our earlier assumption) or one prime $\eta\in T$ that is good for C_i such that $\mathrm{N}\eta\geq n+2$.*

Remark 3.2.5. *Given any Shintani set D, it is possible to choose a set of primes T such that T is good for D. In fact, all but a finite number of prime ideals are good for a given Shintani set.*

We can now note the required property to allow our Shintani zeta functions to be integral at zero. This follows from the following proposition of Dasgupta.

Proposition 3.2.6 (Proposition 3.12, [8])**.** *If the set of primes T contains a prime η that is good for a Shintani cone C and $\mathrm{N}\eta=l$, then*

$$\zeta_{R,T}(\mathfrak{b},C,U,0)\in\mathbb{Z}[1/l].$$

Furthermore, the denominator of $\zeta_{R,T}(\mathfrak{b},C,U,0)$ is at most $l^{n/(l-1)}$.

As is noted by Dasgupta at the top of p.15 in [8], the corollary below follows easily from Proposition 3.2.6.

Corollary 3.2.7. *If the set of primes T is good for a Shintani set D, then*

$$\zeta_{R,T}(\mathfrak{b},D,U,0)\in\mathbb{Z}.$$

We define a $\mathbb{Z}$-valued measure $\nu(\mathfrak{b},D)$ on $\mathcal{O}_{\mathfrak{p}}$ by

$$\nu(\mathfrak{b},D,U):=\zeta_{R,T}(\mathfrak{b},D,U,0), \tag{3.3}$$

for $U \subseteq \mathcal{O}_\mathfrak{p}$ compact open.

We are mostly interested in a particular type of Shintani set, one which is a fundamental domain for the action of $E_+(\mathfrak{f})$.

Definition 3.2.8. *We call a Shintani set D a **Shintani domain** if D is a fundamental domain for the action of $E_+(\mathfrak{f})$ on $\mathbb{R}^n_+$. That is, when*

$$\mathbb{R}^n_+ = \bigcup_{\epsilon \in E_+(\mathfrak{f})} \epsilon D \quad \textit{(disjoint union)}.$$

The existence of such domains follows the work of Shintani, in particular from [19, Proposition 4]. We note here some simple equalities which follow from the definitions, more details are given in §3.3 of [8]. Recall we have written $G_\mathfrak{f}$ for the narrow ray class group of conductor $\mathfrak{f}$. We have let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$, and have $\mathfrak{p}^e = (\pi)$ with $\pi \equiv 1 \pmod{\mathfrak{f}}$ and π totally positive. Let $\mathscr{D}$ be a Shintani domain and recall that we have defined $\mathbb{O} = \mathcal{O}_\mathfrak{p} - \pi\mathcal{O}_\mathfrak{p}$. Then,

$$\nu(\mathfrak{b}, \mathscr{D}, \mathbb{O}) = \zeta_{S,T}(H/F, \mathfrak{b}, 0) = 0, \quad \text{and} \quad \nu(\mathfrak{b}, \mathscr{D}, \mathcal{O}_\mathfrak{p}) = \zeta_{R,T}(H/F, \mathfrak{b}, 0).$$

We now give two technical definitions which are necessary for the definition of Dasgupta's explicit formula and recall a useful lemma which is used repeatedly in the proof of our later results. We also generalise to working with $V \subseteq E_+$ rather than with $E_+(\mathfrak{f})$.

Definition 3.2.9. *Let $V \subseteq E_+$ be a finite index free subgroup of rank $n-1$. We call a Shintani set D a **Colmez domain** for V if D is a fundamental domain for the action of V on $\mathbb{R}^n_+$. That is, when*

$$\mathbb{R}^n_+ = \bigcup_{\epsilon \in V} \epsilon D \quad \textit{(disjoint union)}.$$

We note that in the definition of a Colmez domain we allow ourselves to work with $V = E_+(\mathfrak{f})$. Thus the definition includes Shintani domains.

Proposition 3.2.10. *Let $V \subseteq E_+$ be a finite index free subgroup of rank $n-1$. Let D and D' be Colmez domains for V. We may write D and D' as finite disjoint unions of the same number of simplicial cones*

$$D = \bigcup_{i=1}^{d} C_i, \quad D' = \bigcup_{i=1}^{d} C'_i, \tag{3.4}$$

with $C'_i = \epsilon_i C_i$ for some $\epsilon_i \in V$, $i = 1, \ldots, d$.

Proof. [8, Proposition 3.15] proves this result when $V = E_+(\mathfrak{f})$. The proof of this proposition is analogous. □

A decomposition as in (3.4) is called a **simultaneous decomposition** of the Colmez domains (D, D').

Definition 3.2.11. *Let (D, D') be a pair of Colmez domains. A set T is **good** for the pair (D, D') if there is a simultaneous decomposition as in (3.4) such that for each cone C_i, there* *are at least two primes in T that are good for C_i, or there is one prime $\eta \in T$ that is good for C_i such that* $\mathrm{N}\eta \geq n+2$.

Definition 3.2.12. *Let D be a Colmez domain. If $\beta \in F^*$ is totally positive, then T is β-**good** for D if T is good for the pair $(D, \beta^{-1}D)$.*

The following lemma is used throughout the remainder of this thesis.

Lemma 3.2.13 (Lemma 3.20, [8]). *Let D be a Shintani set and U a compact open subset of $\mathcal{O}_{\mathfrak{p}}$. Let $\mathfrak{b}$ be a fractional ideal of F, and let $\beta \in F^*$ be totally positive so that $\beta \equiv 1 \pmod{\mathfrak{f}}$ and* $\mathrm{ord}_{\mathfrak{p}}(\beta) \geq 0$. *Suppose that $\mathfrak{b}$ and β are relatively prime to R and that $\mathfrak{b}$ is also relatively prime to $\overline{T}$. Let $\mathfrak{q} = (\beta)\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\beta)}$. Then*

$$\zeta_{R,T}(\mathfrak{b}\mathfrak{q}, D, U, 0) = \zeta_{R,T}(\mathfrak{b}, \beta D, \beta U, 0).$$

We end this section with a lemma of Colmez which allows us to give an explicit Colmez domain. Let α be, up to a sign, one of the standard basis vectors of $\mathbb{R}^n$ then we note that its ray $(\alpha\mathbb{R}_+)$ is preserved by the action of $\mathbb{R}^n_+$. We define $\overline{C}_\alpha(v_1, \ldots, v_r)$ to be the union of the cone $C(v_1, \ldots, v_r)$ with the boundary cones that are brought into the interior of the cone by a small perturbation by α, i.e., the set whose characteristic function is given by

$$\mathbb{1}_{\overline{C}_\alpha(v_1,\ldots,v_r)}(x) = \lim_{h \to 0^+} \mathbb{1}_{C(v_1,\ldots,v_r)}(x + h\alpha). \tag{3.5}$$

Throughout this paper, we use the notation

$$[x_1 \mid \ldots \mid x_{n-1}] = (1, x_1, x_1x_2, \ldots, x_1 \ldots x_{n-1}).$$

Let $x_1, \ldots, x_n \in F$. We define the sign map $\delta : F^n \to \{-1, 0, 1\}$ such that

$$\delta(x_1, \ldots, x_n) = \mathrm{sign}(\det(\omega(x_1, \ldots, x_n))), \tag{3.6}$$

where $\omega(x_1, \ldots, x_n)$ denotes the $n \times n$ matrix whose columns are the images of the x_i in $\mathbb{R}^n$. Note that we have the convention that $\mathrm{sign}(0) = 0$. The lemma below is equivalent to [6, Lemma 2.2], rather then using equivalence classes, we write the lemma in terms of the perturbation defined above.

Lemma 3.2.14 (Lemma 2.2, [6]). *Let α be, up to a sign, one of the standard basis vectors of $\mathbb{R}^n$. Let $\varepsilon_1, \ldots, \varepsilon_{n-1} \in E_+$ such that $V = \langle \varepsilon_1, \ldots, \varepsilon_{n-1} \rangle \subset E_+$ is a free subgroup of rank $n-1$ and finite index. Suppose that for all $\tau \in S_{n-1}$ we have*

$$\delta([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) = \mathrm{sign}(\tau).$$

Then the Shintani set

$$D = \bigcup_{\tau\in S_{n-1}} \overline{C}_\alpha([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]),$$

is a Colmez domain for V.

3.3 Continuous maps

For topological spaces X and Y, let $C(X,Y)$ denote the set of continuous maps $X \to Y$. If R is a topological ring we let $C_c(X,R)$ denote the subset of $C(X,R)$ of continuous maps with compact support. If we consider Y (resp. R) with the discrete topology then we shall also write $C^0(X,Y)$ (resp. $C_c^0(X,R)$) instead of $C(X,Y)$ (resp. $C_c(X,Y)$).

Assume now that X is a totally disconnected topological Hausdorff space and A a locally profinite group. We define subgroups $C^\diamond(X,A) \subseteq C(X,A)$ and $C_c^\diamond(X,A) \subseteq C_c(X,A)$ by

$$C^\diamond(X,A) = C^0(X,A) + \sum_K C(X,K),$$

$$C_c^\diamond(X,A) = C_c^0(X,A) + \sum_K C_c(X,K),$$

where the sums are taken over all compact open subgroups K of A. So $C_c^\diamond(X,A)$ is the subgroup of $C_c(X,A)$ generated by locally constant maps with compact support $X \to A$ and by continuous maps with compact support $X \to K \subseteq A$ for some compact open subgroup $K \subseteq A$. Similarly, $C^\diamond(X,A)$ is the subgroup of $C(X,A)$ generated by locally constant maps $X \to A$ and by continuous maps $X \to K \subseteq A$ for some compact open K.

The following notation is used in the formulation of u_2. Given two arbitary finite, disjoint sets Σ_1, Σ_2 of places of F and a locally profinite group A, we put

$$\mathscr{C}_?(\Sigma_1, A)^{\Sigma_2} = C_?((\mathbb{A}_F^{\Sigma_2})^*/U^{\Sigma_1\cup\Sigma_2}, A).$$

where $? \in \{\diamond, c, 0\}$. We note that in the notation above, ? is displayed as a subscript. However, when $? \in \{\diamond, 0\}$, this is viewed as superscript. We use this notation below as well. For a set of places S, we denote by U^S the subgroup of $\mathbb{A}_F^*$ of ideles $(x_v)_v$ with local components $x_v = 1$ if $v \in S$ and $x_v > 0$ if $v \mid \infty$, and x_v is a local unit if $v \notin S \cup R_\infty$.

We also introduce the following generalisation of the above notation, for S_1, S_2 disjoint sets of places of F

$$\mathscr{C}_?(S_1, S_2, A) = C_?(\prod_{\mathfrak{p}\in S_1} F_\mathfrak{p} \times (\mathbb{A}_F^{S_1})^*/U^{S_1\cup S_2}, A).$$

If S_3 is an additional disjoint set of places, we also define

$$\mathscr{C}_?(S_1, S_2, A)^{S_3} = C_?(\prod_{\mathfrak{p}\in S_1} F_\mathfrak{p} \times (\mathbb{A}_F^{S_1\cup S_3})^*/U^{S_1\cup S_2\cup S_3}, A).$$

3.4 Measures

Note that the construction in this section is a generalisation of that given in §2 of [13]. We wish to attach to a homomorphism $\mu : C_c(X,\mathbb{Z}) \to \mathbb{Z}[G]$ an $A \otimes \mathbb{Z}[G]$-valued measure on X for any abelian group A and finite abelian group G. We write the group operation of A multiplicatively. Firstly, by tensoring μ with the identity we obtain a homomorphism

$$\mu_A : C_c(X,\mathbb{Z}) \otimes (A \otimes \mathbb{Z}[G]) \cong C_c^0(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G].$$

To write this map explicitly, we first note that the isomorphism (and its inverse) $C_c(X,\mathbb{Z}) \otimes (A \otimes \mathbb{Z}[G]) \cong C_c^0(X, A \otimes \mathbb{Z}[G])$ are given by

$$f \otimes \alpha \mapsto \alpha \cdot f \text{ with inverse } g \mapsto \sum_{\alpha \in A \otimes \mathbb{Z}[G]} (\alpha \otimes g_\alpha),$$

where $g_\alpha(x) = 1$ if $g(x) = \alpha$ and 0 otherwise. Here we have $f \in C_c(X,\mathbb{Z})$, $\alpha \in A \otimes \mathbb{Z}[G]$ and $g \in C_c^0(X, A \otimes \mathbb{Z}[G])$. Thus, we define the homomorphism μ_A as

$$\mu_A(g) = \sum_{\alpha \in A \otimes \mathbb{Z}[G]} \left(\sum_{\sigma \in G} \sum_{\tau \in G} \alpha_\tau^{\mu_\sigma(g_\alpha)} \otimes \sigma\tau \right),$$

where $\alpha = \sum_{\tau \in G} \alpha_\tau \otimes \tau$, $\mu(g_\alpha) = \sum_{\sigma \in G} \mu_\sigma(g_\alpha)[\sigma]$ and g_α is as defined before. Thus, if A is profinite we can consider the homomorphism

$$\mu_A \coloneqq \varprojlim_K \mu_{A/K} : \varprojlim_K C_c(X, A/K \otimes \mathbb{Z}[G]) \to \varprojlim_K A/K \otimes \mathbb{Z}[G] = A \otimes \mathbb{Z}[G],$$

where K ranges over the open subgroups of A. Since $C_c(X, A \otimes \mathbb{Z}[G]) \subseteq \varprojlim_K C_c(X, A/K \otimes \mathbb{Z}[G])$, we see that μ_A extends canonically to a homomorphism $C_c(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]$ (which we denote by μ_A as well). For a general A (not necessarily profinite), we have seen that μ induces a homomorphism $C_c(X, K \otimes \mathbb{Z}[G]) \to K \otimes \mathbb{Z}[G]$ for every compact open subgroup $K \subset A$. Combining these maps, we see that μ induces a canonical homomorphism $\mu_A : C_c^\diamond(X, A \otimes \mathbb{Z}[G]) \to A \otimes \mathbb{Z}[G]$. Define the set of $A \otimes \mathbb{Z}[G]$-valued measures on X to be

$$\operatorname{Meas}(X, A \otimes \mathbb{Z}[G]) = \operatorname{Hom}(C_c^\diamond(X, A \otimes \mathbb{Z}[G]), A \otimes \mathbb{Z}[G]).$$

The map $\mu \mapsto \mu_A$ defines a homomorphism $\operatorname{Hom}(C_c(X, \mathbb{Z}[G]), \mathbb{Z}) \to \operatorname{Meas}(X, A \otimes \mathbb{Z}[G])$.

For the formulas of interest we will not consider quite this measure but two simpler measures. The measure we constructed above is a generalisation that includes each of the specialisations we require.

For u_2, we have $\mu \in \operatorname{Hom}(C_c(X,\mathbb{Z}),\mathbb{Z})$ rather than in $\operatorname{Hom}(C_c(X,\mathbb{Z}),\mathbb{Z}[G])$. We include

$\mathrm{Hom}(C_c(X,\mathbb{Z}),\mathbb{Z})$ into $\mathrm{Hom}(C_c(X,\mathbb{Z}),\mathbb{Z}[G])$ under the map

$$\iota_1 : \mathrm{Hom}(C_c(X,\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(C_c(X,\mathbb{Z}),\mathbb{Z}[G]),$$

such that for $f \in C_c(X,\mathbb{Z})$

$$\iota_1(\mu)(f) = \mu(f)[\mathrm{id}].$$

For u_3, we want to have a measure on A rather than on $A \otimes \mathbb{Z}[G]$. We include $C_c^{\diamond}(X,A)$ into $C_c^{\diamond}(X, A \otimes \mathbb{Z}[G])$ via the map

$$\iota_2 : C_c^{\diamond}(X,A) \to C_c^{\diamond}(X, A \otimes \mathbb{Z}[G])$$

such that for $x \in X$

$$\iota_2(f)(x) = f(x) \otimes 1.$$

3.5 Eisenstein cocycles

We now define the Eisenstein cocycle. In our study of the cohomological constructions for u_2 and u_3 we require a few variations on the Eisenstein cocycle. We define these variations at the end of this section.

Let $\mathcal{O}_{F,S_p}$ denote the ring of S_p integers of F. Recall that S_p is the set of primes of F above p that split completely in H. Note that $S_p \neq \varnothing$ since $\mathfrak{p} \in S_p$. For any fractional ideal $\mathfrak{b} \subset F$ relatively prime to S, we let $\mathfrak{b}_{S_p} = \mathfrak{b} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,S_p}$ denote the $\mathcal{O}_{F,S_p}$-module generated by $\mathfrak{b}$. Let

$$U \subset F_{S_p} \coloneqq \prod_{\mathfrak{q} \in S_p} F_{\mathfrak{q}}$$

be a compact open subset. Let D be a Shintani set. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the Shintani L-function

$$\mathfrak{L}_R(D,\mathfrak{b},U,s) = (\mathrm{N}\mathfrak{b})^{-s} \sum_{\substack{\xi \in D \cap \mathfrak{b}_{S_p}^{-1},\ \xi \in U \\ (\xi,R)=1}} \frac{\mathrm{rec}_{H/F}((\xi))^{-1}}{\mathrm{N}\xi^s} \in \mathbb{C}[G]. \tag{3.7}$$

Here $\mathrm{rec}_{H/F}$ denotes the Artin reciprocity map for the extension H/F. It follows from work of Shintani, in [19], that the L-function in (3.7) has a meromorphic continuation to $\mathbb{C}$. Furthermore, after fixing $D, \mathfrak{b}$ and s, the values $\mathfrak{L}_R(D,\mathfrak{b},U,s)$ form a distribution on F_{S_p} in the sense that, for disjoint compact open sets $U_1, U_2 \subset F_{S_p}$, we have

$$\mathfrak{L}_R(D,\mathfrak{b},U_1 \cup U_2,s) = \mathfrak{L}_R(D,\mathfrak{b},U_1,s) + \mathfrak{L}_R(D,\mathfrak{b},U_2,s).$$

Let λ be a prime of F such that $\mathrm{N}\lambda = l$ for a prime number $l \in \mathbb{Z}$ and $l \geq n+2$. We assume that no primes in S have residue characteristic equal to l. We then define the smoothed Shintani

L-function

$$\mathfrak{L}_{R,\lambda}(D,\mathfrak{b},U,s) \coloneqq \mathfrak{L}_R(D,\mathfrak{b}\lambda^{-1},U,s) - \mathrm{rec}_{H/F}(\lambda)^{-1} l^{1-s}\mathfrak{L}_R(D,\mathfrak{b},U,s).$$

It follows from the work in [5, §4.4] that the following proposition holds.

Proposition 3.5.1. *For a compact open subset $U \subset F_{S_p}$,*

$$\mathfrak{L}_{R,\lambda}(D,\mathfrak{b},U,0) \in \mathbb{Z}[G].$$

Let F_+^* denote the group of totally positive elements of F. Let $E_{S_p,+}$ denote the group of totally positive units in $\mathcal{O}_{F,S_p}$ which we view as a subgroup of F_+^*. Let $x_1,\dots,x_n \in F_+^*$. We recall the definition of $\overline{C}_{e_1}(x_1,\dots,x_n)$ from (3.5) and the definition of $\delta(x_1,\dots,x_n)$ from (3.6). The following proposition follows directly from [5, Theorem 1.6].

Proposition 3.5.2. *Let $x_1,\dots,x_n \in E_{S_p,+}$. For a compact open subset $U \subset F_{S_p}$, let*

$$\mu_{\mathfrak{b},\lambda}(x_1,\dots,x_n)(U) \coloneqq \delta(x_1,\dots,x_n)\mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}(x_1,\dots,x_n),\mathfrak{b},U,0).$$

Then $\mu_{\mathfrak{b},\lambda}$ is an $E_{S_p,+}$-invariant homogeneous $(n-1)$-cocycle yielding a class

$$\kappa_{\mathfrak{b},\lambda} \coloneqq [\mu_{\mathfrak{b},\lambda}] \in H^{n-1}(E_{S_p,+},\mathrm{Hom}(C_c(F_{S_p},\mathbb{Z}),\mathbb{Z}[G])).$$

Remark 3.5.3. *$\mu_{\mathfrak{b},\lambda}(x_1,\dots,x_n)$ is viewed as an element of* $\mathrm{Hom}(C_c(F_{S_p},\mathbb{Z}),\mathbb{Z}[G])$ *via the following canonical integration pairing*

$$(f,\mu) \mapsto \int_{F_R} f(t)d\mu(t) \coloneqq \lim_{\|\mathcal{V}\|\to 0}\sum_{V\in\mathcal{V}} f(t_V)\mu(V)$$

where the limit is over increasingly finer covers $\mathcal{V}$ of the support of f by compact open subgroups $V \subseteq F_{S_p}$ and $t_V \in V$ is any element of V.

We define the Eisenstein cocycle associated to λ by

$$\kappa_\lambda = \sum_{i=1}^{h} \mathrm{rec}_{H/F}(\mathfrak{b}_i)^{-1}\kappa_{\mathfrak{b}_i,\lambda} \in H^{n-1}(E_{S_p,+},\mathrm{Hom}(C_c(F_{S_p},\mathbb{Z}),\mathbb{Z}[G])).$$

Here $\{\mathfrak{b}_1,\dots,\mathfrak{b}_h\}$ is a set of integral ideals representing the narrow class group of $\mathcal{O}_{F,S_p}$, i.e., the group of fractional ideals of $\mathcal{O}_{F,S_p}$ modulo the group of fractional principal ideals generated by totally positive elements of F.

This construction is adapted from the construction of Dasgupta–Spieß given in §2 of [14]. We adapt their construction simply by replacing χ with $\mathrm{rec}_{H/F}^{-1}$. Thus our construction can be specialised to theirs by applying χ^{-1}. For more details on this construction, see §2 of [14]. The reason for using $\mathrm{rec}_{H/F}^{-1}$ rather than $\mathrm{rec}_{H/F}$ is to make our formulation of u_3 consistent with $u_{\mathfrak{p}}$,

u_1 and u_2. We expand further on this at the start of §6.1. We now give a variation on this Eisenstein cocycle.

Let $E_+(\mathfrak{f})_\mathfrak{p}$ denote the group of $\mathfrak{p}$-units of F which are congruent to 1 $\pmod{\mathfrak{f}}$. We note that $E_+(\mathfrak{f})_\mathfrak{p}$ is free of rank n. For $x_1, \ldots, x_n \in E_+(\mathfrak{f})_\mathfrak{p}$, a fractional ideal $\mathfrak{b}$ coprime to S and l, and compact open $U \subset F_\mathfrak{p}$, we put

$$\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}(x_1, \ldots, x_n)(U) \coloneqq \delta(x_1, \ldots, x_n)\zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}(x_1, \ldots, x_n), U, 0),$$

where δ is defined as in (3.6). We recall the definition of the Shintani zeta function from (3.2) and the Shintani set $\overline{C}_{e_1}(x_1, \ldots, x_n)$ from (3.5). Then, $\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}$ is a homogeneous $(n-1)$-cocycle on $E_+(\mathfrak{f})_\mathfrak{p}$ with values in the space of $\mathbb{Z}$-distributions on $F_\mathfrak{p}$. This follows from Theorem 2.6 of [5]. Hence, we define a class

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \coloneqq [\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}] \in H^{n-1}(E_+(\mathfrak{f})_\mathfrak{p}, \mathrm{Hom}(C_c(F_\mathfrak{p}, \mathbb{Z}), \mathbb{Z})).$$

Here the $\nu^{\mathfrak{p}}_{\mathfrak{b},\lambda}$ is being viewed as an element of $\mathrm{Hom}(C_c(F_\mathfrak{p}, \mathbb{Z}), \mathbb{Z})$ via the integration pairing from Remark 3.5.3. We also consider

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda} = \sum_{[\mathfrak{b}] \in G_\mathfrak{f}/\langle \mathfrak{p} \rangle} \mathrm{rec}_{H/F}(\mathfrak{b})^{-1} \omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \in H^{n-1}(E_+(\mathfrak{f})_\mathfrak{p}, \mathrm{Hom}(C_c(F_\mathfrak{p}, \mathbb{Z}), \mathbb{Z}[G])),$$

where the sum ranges over a system of representatives of $G_\mathfrak{f}/\langle \mathfrak{p} \rangle$. This construction is adapted from the construction of $\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda}$ in §3.3 of [14].

We now consider the final variant of the Eisenstein cocycle we require. We do not give the definition in full generality since the construction is much longer. The ideas required are all similar to those of the constructions before. We end this section with a proposition which contains the information, in the cases we require, for this construction for our later applications.

We write W for F considered as a $\mathbb{Q}$-vector space, and $W_\infty \coloneqq W \otimes_\mathbb{Q} \mathbb{R}$. As before, let λ be a prime of F such that $\mathrm{N}\lambda = l$ for a prime number $l \in \mathbb{Z}$ and $l \geq n+2$. We assume that no primes in S have residue characteristic equal to l. We also let $W_l = W \otimes_\mathbb{Q} \mathbb{Q}_l$.

We define $\phi_\lambda \in C_c(W_l, \mathbb{Z})$ by $\phi_\lambda = \mathbb{1}_{\mathcal{O}_F \otimes \mathbb{Z}_l} - l\mathbb{1}_{\lambda \otimes \mathbb{Z}_l}$, i.e.,

$$\phi_\lambda(v) = \begin{cases} 1 & \text{if } v \in (\mathcal{O}_F \otimes \mathbb{Z}_l) - (\lambda \otimes \mathbb{Z}_l), \\ 1-l & \text{if } v \in \lambda \otimes \mathbb{Z}_l, \\ 0 & \text{if } v \in W_l - (\mathcal{O}_F \otimes \mathbb{Z}_l). \end{cases} \tag{3.8}$$

By fixing an ordering of the infinite places, $v \in R_\infty$, we fix an identification $W_\infty \cong \mathbb{R}^n$ such that $x \cdot e_v = \sigma_v(x) e_v$ for all $v \in R_\infty$ and $x \in F^*$. Here we write e_v for the standard basis element of $\mathbb{R}^n$ which has value 1 at the v component. From now on we fix a choice $v \in R_\infty$ such that

$$\text{the image of basis element } \widetilde{e_v} \in \mathbb{R}^n \text{ is } e_1. \tag{3.9}$$

We define $F^{l,v}$ to be F viewed as a (diagonally embedded) subset of $\mathbb{A}_F^{S_l\cup\{v\}}$. The following proposition is a special case of the much more general construction given in [13]. In order to give this we first introduce some notation. If D is a Shintani set and $\Phi \in C_c(W_{\widehat{\mathbb{Z}}}, \mathbb{Z})$ then we define the Dirichlet series

$$L(D, \Phi; s) = \sum_{v \in W \cap D} \Phi(v) N(v)^{-s}. \tag{3.10}$$

It is known to converge for $\mathrm{Re}(s) > 1$ and extend to the whole complex plane except for possibly a simple pole at $s = 0$. Moreover, if D and Φ are as given in the following proposition then $L(D, \Phi; s)$ is holomorphic. We remark that the set S does not appear in the definition of this Dirichlet series. In the following proposition we will decorate the L-function with λ since the choice of Φ incorporates λ into it.

Proposition 3.5.4. *Let $\omega_1, \dots, \omega_n \in F^{l,v}$. For a map $\phi \in C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z})$, let*

$$\mathrm{Eis}^0_{F,\lambda}(\omega_1, \dots, \omega_n)(\phi) = \delta(\omega_1, \dots, \omega_n) L_\lambda(\overline{C}_{e_1}(\omega_1, \dots, \omega_n), \Phi; 0),$$

where $\Phi = \phi \otimes \phi_\lambda$. Then, $\mathrm{Eis}^0_{F,\lambda}$ is an $F^{l,v}$-homogeneous $(n-1)$-cocycle yielding a class

$$\mathrm{Eis}^0_{F,\lambda} \in H^{n-1}(F^{l,v}, \mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z}), \mathbb{Z})(\delta)).$$

Proof. This proposition follows the combination of [13, Definiton 4.5] and [13, Lemma 5.1]. □

In the above proposition we have the following notation. For a subgroup $H \subseteq F^{l,v}$ and an H-module M, let $M(\delta) = M \otimes \mathbb{Z}(\delta)$. Thus, $M(\delta)$ is the group M with H-action given by $x \cdot m = \delta(x) x m$ for $x \in H$ and $m \in M$.

3.6 Colmez subgroups

In the definitions for the Eisenstein cocycle and its variants, the sign map δ appears. Recall that we define E_+ to be the group of totally positive units of F. For the explicit calculations we later perform it is convenient if we can work with a subgroup $V \subseteq E_+$ free of rank $n-1$ such that $V = \langle g_1, \dots, g_{n-1} \rangle$ and that we can choose $g_n = \pi$ such that

- $\langle g_1, \dots, g_{n-1} \rangle \subseteq E_+(\mathfrak{f})$ is a finite index subgroup free of rank $n-1$, and
- for $\tau \in S_n$ we have $\delta([g_{\tau(1)} \mid \dots \mid g_{\tau(n-1)}]) = \mathrm{sign}(\tau)$.

We refer to such subgroups as Colmez subgroups. We define

$$\mathrm{Log} : \mathbb{R}^n_+ \to \mathbb{R}^n, \quad (x_1, \dots, x_n) \mapsto (\log(x_1), \dots, \log(x_n)).$$

We remark that the map Log is the Dirichlet regulator on E_+. We defined the full Dirichlet regulator after Theorem 2.1.2 in §2.1. For $z \in \mathbb{R}^n$ we write $z = (z_1, \dots, z_n)$. Let $\mathscr{H} \subset \mathbb{R}^n$ be the hyperplane defined by $\mathrm{Tr}(z) = 0$, where $\mathrm{Tr}(z) = \sum_{i=1}^{n} z_i$. Then, Dirichlet proved that, $\mathrm{Log}(E_+)$ is a lattice in $\mathscr{H}$. If $z \in \mathbb{R}_+^n$ and $\mathrm{Log}(z) \in \mathbb{R}^n$ is not an element of $\mathscr{H}$, then we define the projection

$$z_{\mathscr{H}} = (z_1 \dots z_n)^{-\frac{1}{n}} \cdot z. \tag{3.11}$$

We have that $\mathrm{Log}(z_{\mathscr{H}}) \in \mathscr{H}$. Note that z and $z_{\mathscr{H}}$ lie on the same ray in $\mathbb{R}_+^n$. For any $M > 0$ and $i = 0, 1, \dots, n-1$, write $l_i(M)$ for the element of $\mathscr{H}$ which has value M in the $(i+1)$ place and $-M/2$ in the other places. We endow $\mathbb{R}^{n-1}$ with the sup-norm. We denote by $B(x, r)$ the ball centred at x of radius r.

The following lemma, which builds on Lemma 2.1 of [6], allows us to find a collection of possible subsets $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$ such that we get a nice sign property that allows us to explicitly calculate the Eisenstein cocycle more easily.

Lemma 3.6.1. *There exists $R_1 > 0$ such that for all $R > R_1$, $M > K_1(R)$ (where $K_1(R)$ is some constant we define which depends only on R). We have the following: For $i = 1, \dots, n-1$ let $g_i \in E_+$ and $g_n = g_\pi \in \pi_{\mathscr{H}} E_+$ such that $\mathrm{Log}(g_i) \in B(l_i(M), R)$ and $\mathrm{Log}(g_\pi) \in B(l_0(M), R)$. Then*

- *$\langle g_1, \dots, g_{n-1} \rangle \subseteq E_+$ is a finite index subgroup, and*
- *For $\tau \in S_n$ we have $\delta([g_{\tau(1)} \mid \dots \mid g_{\tau(n-1)}]) = \mathrm{sign}(\tau)$.*

Proof. This proof largely follows the ideas of Colmez in his proof of Lemma 2.1 in [6]. First, note that both $\mathrm{Log}(E_+)$ and $\mathrm{Log}(\pi_{\mathscr{H}} E_+)$ are lattices inside $\mathscr{H}$. There exists a constant $R_1 := R(E_+, \pi)$ such that for all $M > 0$ and any $r > R(E_+, \pi)$, there exist $g_1, \dots, g_{n-1} \in E_+$ and $g_\pi \in \pi_{\mathscr{H}} E_+$ such that $\mathrm{Log}(g_i) \in B(l_i(M), r)$ for $i = 1, \dots, n-1$ and $\mathrm{Log}(g_\pi) \in B(l_0(M), r)$. The existence of R_1 follows from Dirichlet's unit theorem and, in particular, the non-vanishing of the regulator of a number field. Since the $l_i(M)$ form a basis of $\mathscr{H}$, the $\mathrm{Log}(g_i)$ form a free family of finite index in $\mathrm{Log}(E_+)$. This is only if M is large enough relative to r, say $M > k(r)$.

Now take M satisfying

i) $M \geq 2(n-1)^4 r$,

ii) $M > (n-1)^2 \log(n!)$,

iii) $M > k(r)$.

For simplicity, let $K_1(r) = \max(2(n-1)^4 r, (n-1)^2 \log(n!), k(r))$ so that we only require $M > K_1(r)$.

Let $\Delta = \det([g_1 \mid \dots \mid g_{n-1}])$. Put $E_i = \exp(M(1 - \frac{i-2}{n-1}))$ and $F_i = \exp(-M(\frac{i-1}{n-1}))$. Hence, the matrix given by $[g_1 \mid \ldots \mid g_{n-1}]$ is written

$$\begin{pmatrix} 1 & \beta_{1,2}F_2 & \beta_{1,3}F_3 & \ldots & \beta_{1,n}F_n \\ 1 & \beta_{2,2}E_2 & \beta_{2,3}E_3 & \ldots & \beta_{2,n}E_n \\ 1 & \beta_{3,2}F_2 & \beta_{3,3}E_3 & \ldots & \beta_{3,n}E_n \\ 1 & \beta_{4,2}F_2 & \beta_{4,3}F_3 & \ldots & \beta_{3,n}E_n \\ & & \ldots & & \\ 1 & \beta_{n,2}F_2 & \beta_{n,3}F_3 & \ldots & \beta_{n,n}E_n \end{pmatrix},$$

where by i),

$$e^{\frac{-M}{2(n-1)^3}} < \beta_{i,j} < e^{\frac{M}{2(n-1)^3}}.$$

Expand Δ and isolate the diagonal term; using the bounds we defined previously we obtain

$$\mid \Delta - e^{\frac{nM}{2}} \prod_{i=2}^{n} \beta_{i,i} \mid \leq (n! - 1) e^{\frac{M}{2(n-1)^2}} e^{M(\frac{n}{2} - \frac{n}{n-1})}$$

and therefore

$$\Delta \geq e^{\frac{nM}{2}} (e^{\frac{-M}{2(n-1)^2}} - (n! - 1) e^{(\frac{M}{2(n-1)^2} - \frac{nM}{n-1})}) > 0$$

according to ii). We show the other required sign properties in the same way. □

After fixing a choice of generators $E_+ = \langle \varepsilon_1, \ldots, \varepsilon_{n-1} \rangle$. Any element $\varepsilon \in E_+$ can then be written uniquely as

$$\varepsilon = \prod_{i=1}^{n-1} \varepsilon_i^{a_i}.$$

We then define, for $i = 1, \ldots, n-1$, the map $\iota_i : E_+ \to E_+$ such that if ε is as above then $\iota_i(\varepsilon) = \varepsilon_i^{a_i}$.

Lemma 3.6.2. *There exists*

1. $R_f, R_g > R_1$,
2. $M_f > K_1(R_f)$ *and*
3. $M_g > K_1(R_g)$,

such that we have the following. Firstly, for $i = 1, \ldots, n-1$ *we can choose* $f_i, g_i \in E_+$ *such that* $\mathrm{Log}(f_i) \in B(l_i(M_f), R)$ *and* $\mathrm{Log}(g_i) \in B(l_i(M_g), R)$. *Furthermore, after writing*

$$V_f = \langle f_1, \ldots f_{n-1} \rangle \quad \text{and} \quad V_g = \langle g_1, \ldots g_{n-1} \rangle$$

we have that $[E_+ : V_f]$ *is coprime to* $[E_+ : V_g]$.

Proof of Lemma 3.6.2. We firstly choose the $f_i \in E_+$ via Lemma 3.6.1 and let $V_f = \langle f_1, \ldots f_{n-1} \rangle$. I.e., we have $\mathrm{Log}(f_i) \in B(l_i(M_f), R_f)$ for some $R_f > R_1$ and $M_f > K_1(R_f)$. We can then choose generators $\langle \delta_1, \ldots, \delta_{n-1} \rangle$ such that for $i = 1, \ldots, n-1$ we have $f_i = \delta_i^{a_i}$ with $a_i > 0$. We note here that $[E_+ : V_f] = \prod_{i=1}^{n-1} a_i$

For ease of notation, let $a = \prod_{i=1}^{n-1} a_i$. For $i = 1, \ldots, n-1$ there exists $R_{g,i} > 0$ and $M_{g,i} > 0$ such that for all $M > M_{g,i}$ there exists $\alpha \in E_+$ with $\mathrm{Log}(\alpha) \in B(l_i(M), R_{g,i})$ and, the positive integer $\iota_i(\alpha)$ is coprime to a. This existence follows from the properties of a lattice. Here ι_i is the map that gives the δ_i component.

Now let $R_g = \max(R_{g,1}, \ldots, R_{g,n-1})$ and choose any $M_g > K_1(R_g)$. Then we can choose, for $i = 1, \ldots, n-1$, units $g_i \in E_+$ with $\mathrm{Log}(g_i) \in B(l_i(M_g), R_g)$ and, the positive integer $\iota_i(g_i)$ is coprime to a. Let $V_g = \langle g_1, \ldots, g_{n-1} \rangle$, the result follows. □

3.7 1-cocycles attached to homomorphisms

Let $g : F_{\mathfrak{p}}^* \to A$ be a continuous homomorphism where A is a locally profinite group. We want to define a cohomology class $c_g \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, A))$ attached to g. We define an $F_{\mathfrak{p}}^*$-action on $C_c(F_{\mathfrak{p}}^*, \mathbb{Z})$ by $(xf)(y) = f(x^{-1}y)$. The following definition is due to Spieß and first appears in Lemma 2.11 of [20]. This definition is crucial in making the constructions of Dasgupta–Spieß's cohomological formulas work. We also remark that the definition is unusual in that it appears as though the cocycle z_g should be a coboundary. However, it may not be a coboundary since g does not necessarily extend to a continuous function on $F_{\mathfrak{p}}$.

Definition 3.7.1. *Let $g : F_{\mathfrak{p}}^* \to A$ be a continuous homomorphism, where A is a locally profinite group. Let $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ such that $f(0) = 1$. We define c_g to be the class of the cocycle $z_{f,g} : F_{\mathfrak{p}}^* \to C_c(F_{\mathfrak{p}}, A)$ where $z_{f,g}(x) =$ "$(1-x)(g \cdot f)$", or more precisely*

$$z_{f,g}(x)(y) = (xf)(y) \cdot g(x) + ((f - xf) \cdot g)(y) \tag{3.12}$$

for $x \in F_{\mathfrak{p}}^*$ and $y \in F_{\mathfrak{p}}$.

The second term in (3.12) is allowed to be evaluated at $0 \in F_{\mathfrak{p}}$ since we can extend continuously the function from $F_{\mathfrak{p}}^*$ to $F_{\mathfrak{p}}$ as

$$(f - xf)(0) = 0.$$

Definition 3.7.1 defines an element $c_g := [z_{f,g}] \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, K))$ for any continuous homomorphism $g : F_{\mathfrak{p}}^* \to K$ and any $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ with $f(0) = 1$. We note that the class is independent of the choice of $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$ with $f(0) = 1$. In particular, we can consider the class $c_{\mathrm{id}} \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}))$.

For the results we want to show, Definition 3.7.1 is all that we require. For more information on these objects, see §3.2 of [13] and §3.1 of [14].

Chapter 4

The Multiplicative Integral Formula (u_1)

In this chapter we will consider the explicit $\mathfrak{p}$-adic formula constructed by Dasgupta in [8]. We begin by reviewing the defintion of this constructed unit. This formula makes use of the Shintani domains which we gave the definition of in §3.2. Recall that Shintani domains are a fundamental domain for the action of $E_+(\mathfrak{f})$ on $\mathbb{R}^n_+$. In the second part of this chapter we will let $V \subset E_+(\mathfrak{f})$ be a free, finite index subgroup, of rank $n-1$. Recall n is the degree of F over $\mathbb{Q}$. We note that there is no torsion in the group E_+ since it contains only the totally positive units of the totally real field F. We then consider the effect on Dasgupta's formula when we move to considering a fundamental domain for the action of V (a Colmez domain) on $\mathbb{R}^n_+$ in place of the Shintani domain. We refer to this process as "transferring to a subgroup". Working with the formulas, after transferring to a subgroup V, will be crucial when we compare the formulas.

4.1 The definition of u_1

Definition 4.1.1. *Let I be an abelian topological group that may be written as an inverse limit of discrete groups*

$$I = \varprojlim I_\alpha.$$

Let $\psi_\alpha : I \to I_\alpha$ denote the projection map. Denote the group operation on I multiplicatively. For each $i \in I_\alpha$, we define

$$U_i = \{x \in I \mid \psi_\alpha(x) = i\}.$$

Note that U_i is an open subset of I. Suppose that G is a compact open subset of a quotient of $\mathbb{A}^*_F$. Let $f : G \to I$ be a continuous map, and let μ be a $\mathbb{Z}$-valued measure of G. We define the

multiplicative integral, *written with a cross through the integration sign, by*

$$\times\!\!\!\!\!\!\int_G f(x)d\mu(x) = \varprojlim \prod_{i\in I_\alpha} i^{\mu(f^{-1}(U_i))} \in I.$$

We remark that this definition of a measure is consistent with the definition we give in §3.4. As before, let λ be a prime of F such that $\mathrm{N}\lambda = l$ for a prime number $l \in \mathbb{Z}$ and $l \geq n+2$. We assume that no primes in S have residue characteristic equal to l. The first definition we make towards the formula is that of an element of $E_+(\mathfrak{f})$. We refer to the element constructed here as the error term of u_1. After the definition, we check that it is well defined.

Definition 4.1.2. *Let $\mathscr{D}$ be a Shintani domain, and assume that λ is π-good for $\mathscr{D}$. Define the* ***error term***

$$\epsilon(\mathfrak{b}, \mathscr{D}, \pi) \coloneqq \prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathscr{D}\cap\pi^{-1}\mathscr{D}, \mathbb{O}_\mathfrak{p})}. \tag{4.1}$$

By Lemma 3.2.2, only finitely many of the exponents in (4.1) are nonzero. Corollary 3.2.7 and the assumption that λ is π-good for $\mathscr{D}$ imply that the exponents are integers. We recall from (3.3) that the measure is defined as

$$\nu(\mathfrak{b}, \epsilon\mathscr{D} \cap \pi^{-1}\mathscr{D}, \mathbb{O}_\mathfrak{p}) = \zeta_{R,\lambda}(\mathfrak{b}, \epsilon\mathscr{D} \cap \pi^{-1}\mathscr{D}, \mathbb{O}_\mathfrak{p}, 0).$$

We are now ready to write down Dasgupta's conjectural formula. We note that for any Shintani domain $\mathscr{D}$ we can always choose a prime λ that is π-good for $\mathscr{D}$. We note that all but a finite number of primes will satisfy this property. Henceforth, we shall assume that we are in this case. We now give the main definition of this section.

Definition 4.1.3. *Let $\mathscr{D}$ be a Shintani domain, and assume that λ is π-good for $\mathscr{D}$. Define*

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D}) \coloneqq \epsilon(\mathfrak{b}, \mathscr{D}, \pi)\pi^{\zeta_{R,\lambda}(H/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b}, \mathscr{D}, x) \in F_\mathfrak{p}^*.$$

As our notation suggests, we have the following proposition.

Proposition 4.1.4 (Proposition 3.19, [8])**.** *The element $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D})$ does not depend on the choice of generator π of $\mathfrak{p}^e$.*

Dasgupta made the following conjecture concerning his construction. For this conjecture we recall the definition of $\overline{\lambda}$ from (3.1).

Conjecture 4.1.5 (Conjecture 3.21, [8])**.** *Let e be the order of $\mathfrak{p}$ in $G_\mathfrak{f}$, and suppose that $\mathfrak{p}^e = (\pi)$ with π totally positive and $\pi \equiv 1 \pmod{\mathfrak{f}}$. Let $\mathscr{D}$ be a Shintani domain, and let λ be π-good for $\mathscr{D}$. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\overline{\lambda}$. We have the following.*

1. *The element $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D}) \in F_\mathfrak{p}^*$ depends only on the class of $\mathfrak{b} \in G_\mathfrak{f}/\langle\mathfrak{p}\rangle$ and no other choices, including the choice of $\mathscr{D}$, and hence may be denoted $u_{\mathfrak{p},\lambda}(\sigma_\mathfrak{b})$, where $\sigma_\mathfrak{b} \in \mathrm{Gal}(H/F)$.*

2. *The element* $u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}})$ *lies in* $U_{\mathfrak{p}}$, *and* $u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}}) \equiv 1 \pmod{\lambda}$.

3. *Shimura reciprocity law: For any fractional ideal* $\mathfrak{a}$ *of* F *prime to* S *and to* $\overline{\lambda}$, *we have*

$$u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{ab}}) = u_{\mathfrak{p},\lambda}(\sigma_{\mathfrak{b}})^{\sigma_{\mathfrak{a}}}.$$

As we noted in §2.3, this conjecture has been proved up to a root of unity, under some mild assumptions, by Theorem 2.3.6. We want to define the formula over $F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ to match with the cohomological constructions. We thus make the following definition.

Definition 4.1.6. *We define*

$$u_1 = \sum_{\mathfrak{b} \in G_{\mathfrak{f}}/\langle \mathfrak{p} \rangle} u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}) \otimes \sigma_{\mathfrak{b}}^{-1} \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G].$$

4.2 Transferring to a subgroup

Let V be a finite index subgroup of $E_+(\mathfrak{f})$, free of rank $n-1$. Recall that π is totally positive, congruent 1 modulo $\mathfrak{f}$ and such that $(\pi) = \mathfrak{p}^e$ where e is the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$. Let $\mathcal{D}_V'$ be a Shintani set which is a fundamental domain for the action of V on $\mathbb{R}_+^n$. As before, we shall refer to such Shintani sets as Colmez domains.

We define

$$u_1(V,\sigma) = u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}_V') \coloneqq \prod_{\epsilon \in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b}, \epsilon\mathcal{D}_V' \cap \pi^{-1}\mathcal{D}_V', \mathcal{O}_{\mathfrak{p}}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{D}_V', \mathcal{O}_{\mathfrak{p}}, 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, d\nu(\mathfrak{b}, \mathcal{D}_V', x), \qquad (4.2)$$

and write $u_1(V) = \sum_{\sigma \in G} u_1(V,\sigma) \otimes \sigma^{-1}$. At this point we have not shown that this definition makes sense. In fact, it does not make sense for all possible fundamental domains. In Proposition 4.2.2 we show that for the particular choice of domain we require, the definition above is sensible. We note that $u_1(V,\sigma)$ depends on the choice of $\mathcal{D}_V$ used, we consider this choice in the comparison result below.

Proposition 4.2.1. *Let* $\mathcal{K}$ *and* $\mathcal{K}'$ *be two Colmez domains for* V, *and* λ *a prime of* F *such that* λ *is* π*-good for* $\mathcal{K}$ *and* $\mathcal{K}'$. *If* λ *is also good for* $(\mathcal{K}, \mathcal{K}')$, *then* $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{K}) = u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{K}')$.

Proof. Theorem 5.3 of [8] proves this result when $V = E_+(\mathfrak{f})$. The proof of this proposition is analogous. □

Let $V \subset E_+(\mathfrak{f})$ be a finite index subgroup, free of rank $n-1$. The following proposition shows the relation between $u_1(\sigma)$ and $u_1(V,\sigma)$.

Proposition 4.2.2. *Let* $\mathcal{D}$ *be a Shintani domain for* $E_+(\mathfrak{f})$. *Let* V *be a free, finite index, subgroup of* $E_+(\mathfrak{f})$ *of rank* $n-1$, *such that* $E_+(\mathfrak{f})/V \cong \mathbb{Z}/b_1 \times \cdots \times \mathbb{Z}/b_{n-1}$ *with* $b_1, \ldots, b_{n-1} > M$, *where* $M = M(\pi, g_1, \ldots, g_{n-1})$ *is some constant that depends on* $g_1, \ldots, g_{n-1}$ *and* π *(up to multiplication* *by an element of $E_+(\mathfrak{f})$ which we define later). Here, we have chosen $g_1, \dots, g_{n-1}$ to be a $\mathbb{Z}$-basis for $E_+(\mathfrak{f})$ such that $g_1^{b_1}, \dots, g_{n-1}^{b_{n-1}}$ is a $\mathbb{Z}$-basis for V. We now define*

$$\mathcal{D}_V := \bigcup_{j_1=0}^{b_1-1} \cdots \bigcup_{j_{n-1}=0}^{b_{n-1}-1} g_1^{j_1} \dots g_{n-1}^{j_{n-1}} \mathcal{D}.$$

Then,

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D}_V) = u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D})^{[E_+(\mathfrak{f}):V]}.$$

Remark 4.2.3. *The proof of Proposition 4.2.2 builds on the work of Tsosie and we follow the strategy in his proof of Proposition 2.1.4 in [24]. We are required to alter the proof as we find a counterexample to the statement of Lemma 2.1.3 in [24], which is used in his proof. In the appendix we give this counterexample explicitly. It is possible to prove Proposition 4.2.2 without our additional assumption that $b_1, \dots, b_{n-1} > M$. However, the proof becomes lengthier. Since our strategy is to make V small enough to satisfy other properties, we do not lose anything by including this simplifying assumption.*

Proof of Proposition 4.2.2. By a result of Colmez in §2 of [6] (p. 372), we have

$$[E_+(\mathfrak{f}) : V]\zeta_\lambda(\mathfrak{b}, \mathcal{D}, U, s) = \zeta_\lambda(\mathfrak{b}, \mathcal{D}_V, U, s).$$

This immediately implies that

$$\pi^{[E_+(\mathfrak{f}):V]\zeta_{R,\lambda}(\mathfrak{b},\mathcal{D},\mathcal{O}_\mathfrak{p},0)} = \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathcal{D}_V,\mathcal{O}_\mathfrak{p},0)}$$

and

$$\left(\times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b}, \mathcal{D}, x) \right)^{[E_+(\mathfrak{f}):V]} = \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b}, \mathcal{D}_V, x).$$

It remains to show that

$$\left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_\mathfrak{p},0)} \right)^{[E_+(\mathfrak{f}):V]} = \prod_{\epsilon\in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathcal{D}_V\cap\pi^{-1}\mathcal{D}_V,\mathcal{O}_\mathfrak{p},0)}.$$

We now consider $\pi^{-1}\mathcal{D}$. By multiplying π by an appropriate element of $E_+(\mathfrak{f})$, we can assume

$$\pi^{-1}\mathcal{D} \subset \bigcup_{i_1=0}^{\alpha_1} \cdots \bigcup_{i_{n-1}=0}^{\alpha_{n-1}} g_1^{i_1} \dots g_{n-1}^{i_{n-1}} \mathcal{D},$$

for some $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{Z}_{>1}$. If we further impose that $g_1^{-1} \dots g_{n-1}^{-1} \pi^{-1}\mathcal{D}$ is not fully contained in the positive translates of $\mathcal{D}$ and, for each i, choosing the minimal α_i, then the required element of $E_+(\mathfrak{f})$ is chosen uniquely. Since the formula is independent of the choice of π we are allowed this assumption. Now, let $M = M(\pi, g_1, \dots, g_{n-1}) = \max(\alpha_1, \dots, \alpha_{n-1})$. Since we have assumed

$b_i > M$, for each $i = 1, \ldots, n-1$, it is easy to see that

$$\pi^{-1}\mathscr{D}_V \subset \bigcup_{k_1=0}^{1} \cdots \bigcup_{k_{n-1}=0}^{1} g_1^{k_1 b_1} \ldots g_{n-1}^{k_{n-1} b_{n-1}} \mathscr{D}_V.$$

For ease of notation, we have, for a Shintani set D, the notation $\nu(D) = \zeta_{R,\lambda}(\mathfrak{b}, D, \mathscr{O}_{\mathfrak{p}}, 0)$. We now calculate

$$\prod_{\epsilon \in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b}, \epsilon\mathscr{D}_V \cap \pi^{-1}\mathscr{D}_V, \mathscr{O}_{\mathfrak{p}}, 0)} = \prod_{i=1}^{n-1} g_i^{S_i}, \text{ where } S_i = b_i \left(\sum_{k_j=0}^{1} \right)_{j \neq i} \nu(g_i^{b_i} (\prod_{j \neq i} g_j^{b_j k_j}) \mathscr{D}_V \cap \pi^{-1}\mathscr{D}_V). \quad (4.3)$$

Here, we have the notation

$$\left(\sum_{k_j=0}^{1} \right)_{j \neq i} = \sum_{k_1=0}^{1} \cdots \sum_{k_{i-1}=0}^{1} \sum_{k_{i+1}=0}^{1} \cdots \sum_{k_{n-1}=0}^{1}.$$

To make the notation clearer, we note that

$$S_1 = b_1 \sum_{k_2=0}^{1} \cdots \sum_{k_{n-1}=0}^{1} \nu(g_1^{b_1} (\prod_{j=2}^{n-1} g_j^{b_j k_j}) \mathscr{D}_V \cap \pi^{-1}\mathscr{D}_V).$$

Consider the power above g_1 in (4.3). Substituting the domain $\mathscr{D}_V$ for its definition as a union of translates of $\mathscr{D}$, on each side of the intersection, and expanding unions and inverting the elements on the right-hand side of the intersection, we have

$$S_1 = b_1 \left(\sum_{k_j=0}^{1} \right)_{j=2}^{n-1} \left(\sum_{c_l=0}^{b_l-1} \sum_{a_l=0}^{b_l-1} \right)_{l=1}^{n-1} \nu(g_1^{b_1+c_1-a_1} (\prod_{j=2}^{n-1} g_j^{b_j k_j + c_j - a_j}) \mathscr{D} \cap \pi^{-1}\mathscr{D}).$$

Since $1 - b_i \leq c_i - a_i \leq b_i - 1$, it is possible to rewrite our sums and deduce that the power above g_1 is equal to

$$S_1 = b_1 \left(\sum_{k_j=0}^{1} \right)_{j=2}^{n-1} \left(\sum_{m_l=1-b_l}^{b_l-1} \right)_{l=1}^{n-1} \prod_{l=1}^{n-1} (b_l - \mid m_l \mid) \nu(g_1^{b_1+m_1} (\prod_{j=2}^{n-1} g_j^{b_j k_j + m_j}) \mathscr{D} \cap \pi^{-1}\mathscr{D}).$$

The terms in the sum are only non-zero when $0 \leq b_1 + m_1 \leq \alpha_1$ and for $j = 2, \ldots, n-1$, when

$$\begin{cases} 0 \leq m_j \leq \alpha_j & \text{if } k_j = 0 \\ 0 \leq b_j + m_j \leq \alpha_j & \text{if } k_j = 1. \end{cases}$$

We now apply this to our sums, working term by term. For the m_1 sum we shift the index of the summand by b_1. We now expand the k_2 sum out. For the $k_2 = 1$ part we shift the index of the m_2 sum by b_2. Thus, we see that the power above g_1 in (4.3) is equal to

$$b_1 \sum_{m_1=1}^{\alpha_1} \left(\sum_{k_j=0}^{1} \right)_{j=3}^{n-1} \left(\sum_{m_l=1-b_l}^{b_l-1} \right)_{l=3}^{n-1} (m_1 \prod_{l=3}^{n-1} (b_l - \mid m_l \mid)) \left(\sum_{m_2=0}^{\alpha_2} (b_2 - m_2) + \sum_{m_2=1}^{\alpha_2} m_2 \right) \nu(g_1^{m_1} g_2^{m_2} (\prod_{j=2}^{n-1} g_j^{b_j k_j + m_j}) \mathcal{D} \cap \pi^{-1} \mathcal{D}).$$

Cancelling the m_2 terms in the sums then gives that the power above g_1 in (4.3) is in fact

$$b_1 b_2 \sum_{m_1=1}^{\alpha_1} \sum_{m_2=0}^{\alpha_2} \left(\sum_{k_j=0}^{1} \right)_{j=3}^{n-1} \left(\sum_{m_l=1-b_l}^{b_l-1} \right)_{l=3}^{n-1} (m_1 \prod_{l=3}^{n-1} (b_l - \mid m_l \mid)) \nu(g_1^{m_1} g_2^{m_2} (\prod_{j=2}^{n-1} g_j^{b_j k_j + m_j}) \mathcal{D} \cap \pi^{-1} \mathcal{D}).$$

Continuing to work term by term for $j = 3, \ldots, n-1$, and noting that $[E_+(\mathfrak{f}) : V] = b_1 \ldots b_{n-1}$, we are able to deduce that

$$S_1 = [E_+(\mathfrak{f}) : V] \sum_{m_1=1}^{\alpha_1} \sum_{m_2=0}^{\alpha_2} \cdots \sum_{m_{n-1}=0}^{\alpha_{n-1}} m_1 \nu(g_1^{m_1} \ldots g_{n-1}^{m_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D}).$$

Similarly, the power above g_i in (4.3), for $i = 2, \ldots, n-1$, is equal to

$$[E_+(\mathfrak{f}) : V] \sum_{m_i=1}^{\alpha_i} \left(\sum_{m_j=0}^{\alpha_j} \right)_{j \neq i} m_i \nu(g_1^{m_1} \ldots g_{n-1}^{m_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D}).$$

Thus,

$$\prod_{\epsilon \in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b}, \epsilon \mathcal{D}_V \cap \pi^{-1} \mathcal{D}_V, \mathcal{O}_{\mathfrak{p}}, 0)} = \left(\prod_{i=1}^{n-1} g_i^{S_i'} \right)^{[E_+(\mathfrak{f}):V]},$$

where

$$S_i' = \sum_{m_i=1}^{\alpha_i} \left(\sum_{m_j=0}^{\alpha_j} \right)_{j \neq i} m_i \nu(g_1^{m_1} \ldots g_{n-1}^{m_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D}).$$

It remains for us to consider the error term for $u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathcal{D})$. We calculate

$$\begin{aligned}
\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b}, \epsilon \mathcal{D} \cap \pi^{-1} \mathcal{D}, \mathcal{O}_{\mathfrak{p}}, 0)} &= \prod_{m_1=0}^{\alpha_1} \cdots \prod_{m_{n-1}=0}^{\alpha_{n-1}} (g_1^{m_1} \ldots g_{n-1}^{m_{n-1}})^{\nu(g_1^{m_1} \ldots g_{n-1}^{m_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D})} \\
&= \prod_{i=1}^{n-1} g_i^{S_i'}.
\end{aligned}$$

This completes the result. □

Chapter 5

The Cohomological Element u_2

In this section we consider the cohomological formula conjectured by Dasgupta–Spieß in [13]. We begin by recalling the construction of this element. In the construction of this element we require a map which we will denote by Δ_*. This map is fairly technical in its construction and we require the explicit form of the map for our later work. We consider this map in detail in §5.2. A key element of the formula is a generator of the homology group $H_n(E_{\mathfrak{p},+},\mathbb{Z})$, denoted by $\eta_\mathfrak{p}$. Let $V \subseteq E_{\mathfrak{p},+}$ be a free, finite index subgroup, free of rank n. We then consider the effect on the formula of Dasgupta–Spieß when replacing $\eta_\mathfrak{p}$ with a generator of $H_n(V_\mathfrak{p},\mathbb{Z})$. As in Chapter 4, we refer to this as "transferring to a subgroup".

5.1 The definition of u_2

Throughout this section we use the notation established in §3.3 for continuous maps. Let $E_{+,\mathfrak{p}}$ denote the group of totally positive $\mathfrak{p}$-units of F. We first note that by Dirichlet's unit theorem the homology group $H_n(E_{\mathfrak{p},+},\mathbb{Z})$ is free abelian of rank 1. Let $\eta_\mathfrak{p}$ be a generator of $H_n(E_{\mathfrak{p},+},\mathbb{Z})$.

Let $\mathscr{F}$ be a fundamental domain for the action of $F^*/E_{\mathfrak{p},+}$ on $(\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p}$, then $\mathbb{1}_\mathscr{F}$ is an element of $H^0(E_{\mathfrak{p},+},C(\mathscr{F},\mathbb{Z})) \cong (C(\mathscr{F},\mathbb{Z}))^{E_{\mathfrak{p},+}}$. Taking the cap product then gives $\mathbb{1}_\mathscr{F} \cap \eta_\mathfrak{p} \in H_n(E_{\mathfrak{p},+},C(\mathscr{F},\mathbb{Z}))$, since $C(\mathscr{F},\mathbb{Z}) \otimes \mathbb{Z} \cong C(\mathscr{F},\mathbb{Z})$. We now define $\vartheta^\mathfrak{p} \in H_n(F^*,\mathscr{C}_c(\varnothing,\mathbb{Z})^\mathfrak{p})$ as the homology class corresponding to $\mathbb{1}_\mathscr{F} \cap \eta_\mathfrak{p}$ under the isomorphism

$$H_n(E_{\mathfrak{p},+},C(\mathscr{F},\mathbb{Z})) \cong H_n(F^*,C_c((\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p},\mathbb{Z})), \tag{5.1}$$

that is induced by $C_c((\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p},\mathbb{Z}) \cong \mathrm{Ind}_{E_{\mathfrak{p},+}}^{F^*} C(\mathscr{F},\mathbb{Z})$.

We now follow the construction of Dasgupta–Spieß as given in §6 of [13]. Cap and cup products are a crucial element of Dasgupta–Spieß's formula. For the definitions of these products, refer to Chapter 6 of [1]. Since the local norm residue symbol for H/F at $\mathfrak{p}$ is trivial we omit it from the reciprocity map, i.e., we consider the homomorphism

$$\mathrm{rec}_{H/F}^{\mathfrak{p}} : (\mathbb{A}_F^{\mathfrak{p}})^* \to G \hookrightarrow \mathbb{Z}[G]^*, \quad x = (x_v)_{v \neq \mathfrak{p}} \mapsto \prod_{v \neq \mathfrak{p}} \mathrm{rec}_{H/F,v}(x).$$

Let $R' = R - R_\infty$. We can view $\mathrm{rec}_{H/F}^{\mathfrak{p}}$ as an element of $H^0(F^*, \mathscr{C}(R', \mathbb{Z}[G])^{\mathfrak{p}})$ and denote by

$$\rho_{H/F} \in H_n(F^*, \mathscr{C}_c(R', \mathbb{Z}[G])^{\mathfrak{p}})$$

its image under the following map,

$$H^0(F^*, \mathscr{C}^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}}) \to H_n(F^*, \mathscr{C}_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}}), \quad \psi \mapsto \psi \cap \vartheta^{\mathfrak{p}}.$$

The cap product here is induced by the map

$$\mathscr{C}^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}} \times \mathscr{C}_c(\varnothing, \mathbb{Z})^{\mathfrak{p}} \to \mathscr{C}_c^\diamond(R', \mathbb{Z}[G])^{\mathfrak{p}}, \quad (\psi, \phi) \mapsto \psi \cdot \phi, \tag{5.2}$$

where $\psi \cdot \phi$ denotes the function $xU^{R' \cup \mathfrak{p}} \mapsto \psi(xU^{R' \cup \mathfrak{p}})\phi(xU^{\mathfrak{p}})$.

For a locally profinite abelian group A, the bilinear map $\otimes : A \times \mathbb{Z}[G] \to A \otimes \mathbb{Z}[G]$ induces a bilinear map

$$C_c^\diamond(F_{\mathfrak{p}}, A) \times \mathscr{C}_c(R', \mathbb{Z}[G])^{\mathfrak{p}} \to \mathscr{C}_c^\diamond(\mathfrak{p}, R', A \otimes \mathbb{Z}[G]), \quad (f, g) \mapsto f \otimes g.$$

This then induces a cap-product pairing

$$H^1(F^*, C_c^\diamond(F_{\mathfrak{p}}, A)) \times H_n(F^*, \mathscr{C}_c(R', \mathbb{Z}[G])^{\mathfrak{p}}) \to H_{n-1}(F^*, \mathscr{C}_c^\diamond(\mathfrak{p}, R', A \otimes \mathbb{Z}[G])).$$

In particular, we can consider

$$c_{\mathrm{id}} \cap \rho_{H/F} \in H_{n-1}(F^*, \mathscr{C}_c^\diamond(\mathfrak{p}, R', F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])),$$

where c_{id} is as defined in Definition 3.7.1. Now choose $v \in R_\infty$ such that (3.9) holds. Write $R_\infty^v = R_\infty - \{v\}$. Recall that we write W for F considered as a $\mathbb{Q}$-vector space. In [13, §5.3], Dasgupta–Spieß define the following map.

$$\Delta_* : H_{n-1}(F^*, \mathscr{C}_c^\diamond(\mathfrak{p}, R', F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])) \to H_{n-1}(F^{l,v}, C_c^\diamond(W_{\widehat{\mathbb{Z}}^\lambda}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G])(\delta)).$$

The explicit definition of Δ_* is too long to give conveniently here. We study this map in the next section. In §5.2, we show the results related to Δ_* that we require by writing the map in a completely explicit way.

Now the canonical pairing, where we recall the definition of $\mu_{F_{\mathfrak{p}}^*}$ from §3.4,

$$\mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z}), \mathbb{Z}) \times C_c^\diamond(W_{\widehat{\mathbb{Z}}^\lambda}, F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]) \to F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G], \quad (\mu, f) \mapsto \mu_{F_{\mathfrak{p}}^*}(f),$$

induces via cap-product a pairing

$$\cap : H^{n-1}(F^{l,v}, \mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z}), \mathbb{Z})(\delta)) \times H_{n-1}(F^{l,v}, C_c^\diamond(W_{\widehat{\mathbb{Z}}^\lambda}, F_\mathfrak{p}^* \otimes \mathbb{Z}[G])(\delta)) \to F_\mathfrak{p}^* \otimes \mathbb{Z}[G]. \tag{5.3}$$

Recall the Eisenstein cocycle, $\mathrm{Eis}^0_{F,\lambda} \in H^{n-1}(F^{l,v}, \mathrm{Hom}(C_c(W_{\widehat{\mathbb{Z}}^\lambda}, \mathbb{Z}), \mathbb{Z})(\delta))$, from Proposition 3.5.4. Applying (5.3) with the Eisenstein cocycle $\mathrm{Eis}^0_F = \mathrm{Eis}^0_{F,\lambda}$ and $\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F})$, we obtain an element $u_2 = u_{S,\lambda} \in F_\mathfrak{p}^* \otimes \mathbb{Z}[G]$ such that

$$u_2 = u_{S,\lambda} = \sum_{\sigma \in G} u_2(\sigma) \otimes [\sigma^{-1}] = \mathrm{Eis}^0_F \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H/F}). \tag{5.4}$$

Dasgupta–Spieß then conjecture [13, Conjecture 6.1] that the element u_2 is equal to the Brumer–Stark unit, $u_\mathfrak{p}$. We end this section by stating the results that Dasgupta–Spieß have shown concerning their cohomological construction.

Proposition 5.1.1 (Proposition 6.3, [13])**.** *The formula u_2 has the following properties.*

a) For $\sigma \in G$, we have $\mathrm{ord}_\mathfrak{p}(u_2(\sigma)) = \zeta_{S,\lambda}(\sigma, 0)$.

b) Let L/F be an abelian extension with $L \supseteq H$ and put $\mathfrak{g} = \mathrm{Gal}(L/F)$. Assume that L/F is unramified outside S and that $\mathfrak{p}$ splits completely in L. Then we have

$$u_2(\sigma) = \prod_{\tau \in \mathfrak{g}, \tau|_H = \sigma} u_2(L/F, \tau).$$

c) Let $\mathfrak{r}$ be a nonarchimedean place of F with $\mathfrak{r} \notin S \cup \overline{\lambda}$. Then we have

$$u_2(S \cup \{\tau\}, \sigma) = u_2(S, \sigma) u_2(S, \sigma_\mathfrak{r} \sigma)^{-1}.$$

d) Assume that H has a real archimedean place $w \nmid v$. Then $u_2(\sigma) = 1$ for all $\sigma \in G$.

e) Let L/F be a finite abelian extension of F containing H and unramified outside S. Then we have

$$\mathrm{rec}_\mathfrak{p}(u_2(\sigma)) = \prod_{\substack{\tau \in \mathrm{Gal}(L/F) \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,\lambda}(K/F, \tau^{-1}, 0)}.$$

5.2 The map Δ_*

In this section, we consider the map Δ_*. We begin by defining it in the case we require and then give a series of propositions which allows us to calculate the map explicitly in the next section. For more information and the more general construction we refer readers to §5.3 of Dasgupta–Spieß's paper [13].

Throughout this section we let $A = F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G]$ to ease notation. We also note the following definition which will be used throughout. For sets X_1, X_2 and a map $\psi : X_1 \times X_2 \to A$,

$$\mathrm{Supp}(X_1, X_2, \psi) := \{x_1 \in X_1 \mid \exists x_2 \in X_2 \text{ with } (x_1, x_2) \in \mathrm{supp}(\psi)\}, \tag{5.5}$$

where $\mathrm{supp}(\psi)$ is the support of ψ. This set is the image of $\mathrm{supp}(\psi)$ under the projection $X_1 \times X_2 \to X_1$. The following simple proposition is used repeatedly in this construction. Since we require the explicit isomorphism given by this proposition, we include a proof of this proposition.

Proposition 5.2.1. *Let X_1, X_2 be totally disconnected topological Hausdorff spaces, with X_1 discrete. Let A be a locally profinite group. Then the map*

$$C_c(X_1, \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^{\diamond}(X_2, A) \to C_c^{\diamond}(X_1 \times X_2, A),$$

$$f \otimes g \mapsto ((x_1, x_2) \mapsto f(x_1) \cdot g(x_2))$$

is an isomorphism.

Proof. We calculate that the inverse map as follows. Let $Y_1 = \mathrm{Supp}(X_1, X_2, \psi)$ be a subset of X_1 defined in (5.5). We note that Y_1 is finite since ψ has compact support. Then define the map

$$\psi \mapsto \sum_{y \in Y_1} \mathbb{1}_y \otimes_{\mathbb{Z}} \psi(y, \cdot) \in C_c(X_1, \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^{\diamond}(X_2, A).$$

It is clear that this map is the inverse of the map in the statement of the proposition and this completes the proof. □

Corollary 5.2.2. *Let S_1, S_2 be finite disjoint sets of finite places and let $S_3 \subseteq R_\infty$ be a set of infinite places. Then there exists an isomorphism,*

$$\mathscr{C}_c^{\diamond}(S_1, S_2, A) \to C(F_{S_3}^*/U_{S_3}, \mathbb{Z}) \otimes \mathscr{C}_c^{\diamond}(S_1, S_2, A)^{S_3}. \tag{5.6}$$

Proof. Let $F_{S_3}^* = \prod_{v \in S_3} F_v^*$. Since we have

$$\prod_{\mathfrak{p} \in S_1} F_{\mathfrak{p}} \times (\mathbb{A}_F^{S_1})^*/U^{S_1 \cup S_2} = \left(\prod_{\mathfrak{p} \in S_1} F_{\mathfrak{p}} \times (\mathbb{A}_F^{S_1 \cup S_3})^*/U^{S_1 \cup S_2 \cup S_3} \right) \times F_{S_3}^*/U_{S_3}$$

and $F_{S_3}^*/U_{S_3}$ is finite, we are able to apply Proposition 5.2.1 and this gives us that the map is an isomorphism and also allows us to write the map explicitly. Let $\psi \in \mathscr{C}_c^{\diamond}(S_1, S_2, A)$ and write

$$Y_1 = \mathrm{Supp}(F_{S_3}^*/U_{S_3}, \prod_{\mathfrak{p} \in S_1} F_{\mathfrak{p}} \times (\mathbb{A}_F^{S_1 \cup S_3})^*/U^{S_1 \cup S_2 \cup S_3}, \psi).$$

We note that Y_1 is finite since ψ has compact support. Then the inverse map is

$$\psi \mapsto \sum_{y \in Y_1} \mathbb{1}_y \otimes_{\mathbb{Z}} \psi(y, \cdot) \in C(F_{S_3}^*/U_{S_3}, \mathbb{Z}) \otimes \mathscr{C}_c^{\diamond}(S_1, S_2, A)^{S_3}.$$

□

We now fix continuous homomorphisms $\delta_w : F_w^* \to \{\pm 1\} = \mathbb{Z}^*$ for every $w \in R_\infty$ to be the sign map. We also put $F_{R_\infty^v} = \prod_{w \in R_\infty^v} F_w$ and define

$$\delta_{R_\infty^v} : F_{R_\infty^v}^* \to \{\pm 1\} \quad \text{such that} \quad (x_w)_{w \in R_\infty^v} \mapsto \prod_{w \in R_\infty^v} \delta_w(x_w).$$

We recall the following notation from the end of §3.5. For a subgroup $H \subseteq F_{R_\infty^v}^*$ and an H-module M, we define $M(\delta_{R_\infty^v}) = M \otimes \mathbb{Z}(\delta_{R_\infty^v})$. Thus, $M(\delta_{R_\infty^v})$ is M but the H-action is given by $x \cdot m = \delta_{R_\infty^v}(x) x m$ for $x \in H$ and $m \in M$. By tensoring the ($F_{R_\infty^v}^*$-equivariant) homomorphism

$$C(F_{R_\infty^v}^*/U_{R_\infty^v}, \mathbb{Z}) \to \mathbb{Z}(\delta_{R_\infty^v}), \quad f \mapsto \sum_{x \in F_{R_\infty^v}^*/U_{R_\infty^v}} \delta_{R_\infty^v}(x) f(x) \tag{5.7}$$

with $\mathrm{id}_{\mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{R_\infty^v}}$ we obtain, via Corollary 5.2.2, an $(\mathbb{A}_F^{\mathfrak{p}})^*$-equivariant map

$$\mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A) \to \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v}). \tag{5.8}$$

We now calculate this map explicitly.

Proposition 5.2.3. *Let $\psi \in \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)$. The image of ψ under (5.8) is given by*

$$\sum_{y \in Y_1} \delta_{R_\infty^v}(y) \psi(y, \cdot),$$

where $\psi(y, \cdot) \in \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{R_\infty^v}$.

Proof. The result follows from Corollary 5.2.2 and (5.7). □

Before giving the second map that we require we recall that we have defined the notation, for S a finite set of primes, $F^S = (\mathbb{A}_F^S \times U_S) \cap F^*$. We now consider the following proposition.

Proposition 5.2.4. *We have*

$$H_{n-1}(F^*, \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v})) \cong H_{n-1}(F^{\overline{\lambda} \cup v}, \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{\overline{\lambda}, \infty}(\delta)), \tag{5.9}$$

where δ is as defined in (3.6). Furthermore, if we write

$$\Psi = [g_1 \mid \dots \mid g_{n-1}] \otimes \psi \in H_{n-1}(F^*, \mathscr{C}_c^{\diamond}(\{\mathfrak{p}\}, R', A)^{R_\infty^v}(\delta_{R_\infty^v}))$$

then the image of Ψ under the isomorphism in (5.9) is

$$\sum_{f\in F^*/F^{\overline{\lambda}\cup v}} f\cdot[g_1\mid\dots\mid g_{n-1}]\otimes\delta_v(f_v)\psi(f_{\overline{\lambda}\cup v},\cdot),$$

where $f_{\overline{\lambda}\cup v}$ is the image of f in $\prod_{w\in\overline{\lambda}\cup v}F_w^*$ and $\psi(f_{\overline{\lambda}\cup v},\cdot)\in\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty}(\delta)$.

Proof. It is clear to see that

$$\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{R_\infty^v}(\delta_{R_\infty^v})\cong\operatorname{Ind}_{(\mathbb{A}_F^{\overline{\lambda}\cup\mathfrak{p},\infty})^*}^{(\mathbb{A}_F^{R_\infty^v\cup\mathfrak{p}})^*}\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty}(\delta)\cong\operatorname{Ind}_{(\mathbb{A}_F^{v\cup\overline{\lambda}})^*}^{(\mathbb{A}_F)^*}\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty}(\delta).$$

Thus, by weak approximation, we have

$$\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{R_\infty^v}(\delta_{R_\infty^v})\cong\operatorname{Ind}_{F^{\overline{\lambda}\cup v}}^{F^*}\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty}(\delta).$$

It follows that the required isomorphism holds. The explicit description of the map follows simply by tracing through the definitions. □

The last map we need to construct before giving the definition of Δ_* is the $(\mathbb{A}_F^{\overline{\lambda},\infty})^*$ -equivariant map

$$\Delta_{S'}^{\overline{\lambda}}:\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty}\to C_c^\diamond(\mathbb{A}_F^{\overline{\lambda},\infty},A)\cong C_c^\diamond(W_{\widehat{\mathbb{Z}}^\lambda},A),\tag{5.10}$$

where we have the notation $S'=R'\cup\{\mathfrak{p}\}$ and note that $\mathbb{A}_F^{\overline{\lambda},\infty}\cong W_{\widehat{\mathbb{Z}}^\lambda}$. There exist canonical homomorphisms

$$C_c^\diamond(F_\mathfrak{p}\times\prod_{\mathfrak{q}\in R'}F_\mathfrak{q}^*,A)\otimes\mathscr{C}_c(\varnothing,\mathbb{Z})^{S'\cup\overline{\lambda},\infty}\to\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty},\tag{5.11}$$

$$C_c^\diamond(\prod_{\mathfrak{q}\in S'}F_\mathfrak{q},A)\otimes C_c(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty},\mathbb{Z})\to C_c^\diamond(\mathbb{A}_F^{\overline{\lambda},\infty},A).\tag{5.12}$$

By Proposition 5.2.1, the first map, (5.11), is an isomorphism. Let $\mathcal{J}^{S'\cup\overline{\lambda}}$ denote the ring of ideals of F which are coprime to $S'\cup\overline{\lambda}$. Since $(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}$ is isomorphic to $\mathcal{J}^{S'\cup\overline{\lambda}}$, the ring $\mathscr{C}_c^0(\varnothing,\mathbb{Z})^{S'\cup\overline{\lambda},\infty}$ can be identified with the group ring $\mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}]$. We define (5.10) as the tensor product $\Delta_{S'}^{\overline{\lambda}}=i\otimes I^{S\cup\overline{\lambda}}$ where $i:C_c^\diamond(F_\mathfrak{p}\times\prod_{\mathfrak{q}\in R'}F_\mathfrak{q}^*,A)\to C_c^\diamond(\prod_{\mathfrak{q}\in S'}F_\mathfrak{q},A)$ is the inclusion map induced by extension by 0 (as defined in §3.1) and $I^{S'\cup\overline{\lambda}}:\mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}]\to C_c(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty},\mathbb{Z})$. Here $I^{S'\cup\overline{\lambda}}$ maps a fractional ideal $\mathfrak{a}\in\mathcal{J}^{S'\cup\overline{\lambda}}$ to the characteristic function of $\widehat{\mathfrak{a}}^{S'\cup\overline{\lambda}}=\mathfrak{a}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_\mathfrak{r})$ which we denote by $\operatorname{char}(\mathfrak{a}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_\mathfrak{r}))$.

Proposition 5.2.5. *If $\psi\in\mathscr{C}_c^\diamond(\{\mathfrak{p}\},R',A)^{\overline{\lambda},\infty}$, then*

$$\Delta_{S'}^{\overline{\lambda}}(\psi)=\sum_{z\in Z}\psi(z,\cdot)_!\odot\operatorname{char}\left(\prod_{w\ \textit{finite}}\mathfrak{q}_w^{\operatorname{ord}_w(z_w)}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_\mathfrak{r})\right)\in C_c^\diamond(\mathbb{A}_F^{\overline{\lambda},\infty},A),$$

where $Z = \operatorname{Supp}((\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}, F_{\mathfrak{p}} \times \prod_{\mathfrak{q}\in R'} F_{\mathfrak{q}}^*, \psi)$ *and we have the notation that for functions* $f : X_1 \to A$ *and* $g : X_2 \to \mathbb{Z}$*, we have the function* $f \odot g : X_1 \times X_2 \to A$ *such that* $(f \odot g)(x_1, x_2) = f(x_1)g(x_2)$*.*

Proof. Let $\psi \in \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', A)^{\overline{\lambda},\infty}$ and define Z as above. Then the image of ψ under the inverse map of the isomorphism (5.11) is

$$\psi \mapsto \sum_{z\in Z} \psi(z,\cdot) \otimes \mathbb{1}_z.$$

We now calculate the effect of $I^{S'\cup\overline{\lambda}}$. First note that the isomorphism

$$\mathcal{J}^{S'\cup\overline{\lambda}} \to (\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}$$

is given by

$$\mathfrak{m} \mapsto \prod_{\mathfrak{q}\in S^{\mathfrak{m}}} \pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})},$$

where $S_{\mathfrak{m}}$ is the set of places that divide $\mathfrak{m}$ and $\mathfrak{m}(\mathfrak{q})$ is the integer such that the fractional ideal $\mathfrak{q}^{-\mathfrak{m}(\mathfrak{q})}\mathfrak{m}$ is coprime to $\mathfrak{q}$ and $\pi_{\mathfrak{q}}$ is the uniformiser associated to the prime ideal $\mathfrak{q}$. We then view the image as an element of $(\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty}$ by imposing that at the places away from $S_{\mathfrak{m}}$ the value is 1. Thus, when we identify $\mathscr{C}_c^0(\varnothing,\mathbb{Z})^{S'\cup\overline{\lambda},\infty}$ with $\mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}]$, we map

$$\phi \mapsto \sum_{\mathfrak{m}\in\mathcal{J}^{S'\cup\overline{\lambda}}} \phi(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}} \pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathfrak{m} \in \mathbb{Z}[\mathcal{J}^{S'\cup\overline{\lambda}}].$$

Applying $I^{S'\cup\overline{\lambda}}$, we have

$$\sum_{\mathfrak{m}\in\mathcal{J}^{S'\cup\overline{\lambda}}} \phi(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}} \pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathfrak{m} \mapsto \sum_{\mathfrak{m}\in\mathcal{J}^{S\cup\overline{\lambda}}} \phi(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}} \pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathbb{1}_{\mathfrak{m}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{r}})}.$$

Returning to ψ, we have that under the map $\Delta_S^{\overline{\lambda}}$

$$\begin{aligned}\psi &\mapsto \sum_{z\in Z} \psi(z,\cdot)_! \otimes \sum_{\mathfrak{m}\in\mathcal{J}^{S'\cup\overline{\lambda}}} \mathbb{1}_z(\prod_{\mathfrak{q}\in S_{\mathfrak{m}}} \pi_{\mathfrak{q}}^{\mathfrak{m}(\mathfrak{q})})\mathbb{1}_{\mathfrak{m}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{r}})} \\ &= \sum_{z\in Z} \psi(z,\cdot)_! \otimes \operatorname{char}\left(\prod_{v \text{ finite}} \mathfrak{q}_v^{\operatorname{ord}_v(z_v)}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}} \mathcal{O}_{\mathfrak{r}})\right).\end{aligned}$$

Lastly, the image of the above under the map (5.12) is

$$\sum_{z\in Z} \psi(z,\cdot)_! \odot \operatorname{char}\left(\prod_{v \text{ finite}} \mathfrak{q}_v^{\operatorname{ord}_v(z_v)}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}} \mathcal{O}_{\mathfrak{r}})\right).$$

□

We are now able to define Δ_* via the composition

$$\begin{aligned}\Delta_*: H_{n-1}(F^*, \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', F_\mathfrak{p}^* \otimes \mathbb{Z}[G])) &\xrightarrow{(5.8)_*} H_{n-1}(F^*, \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', F_\mathfrak{p}^* \otimes \mathbb{Z}[G])^{R_\infty^v}(\delta_{R_\infty^v}))\\ &\xrightarrow{(5.9)} H_{n-1}(F^{\overline{\lambda}\cup v}, \mathscr{C}_c^\diamond(\{\mathfrak{p}\}, R', F_\mathfrak{p}^* \otimes \mathbb{Z}[G])^{\overline{\lambda},\infty}(\delta))\\ &\xrightarrow{(5.10)_*} H_{n-1}(F^{\overline{\lambda}\cup v}, C_c^\diamond(W_{\overline{\mathbb{Z}}^\lambda}, F_\mathfrak{p}^* \otimes \mathbb{Z}[G])(\delta)).\end{aligned}$$

5.3 Transferring to a subgroup

Let V be a finite index free subgroup of E_+ of rank $n-1$. Let $\eta_{\mathfrak{p},V}$ be a generator of $H_n(V\oplus\langle\pi\rangle, \mathbb{Z})$. Let $\mathscr{F}_V$ be a fundamental domain for the action of $F^*/(V\oplus\langle\pi\rangle)$ on $(\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p}$. Then, $\mathbb{1}_{\mathscr{F}_V}$ is an element of $H^0(V\oplus\langle\pi\rangle, C(\mathscr{F}_V,\mathbb{Z})) \cong (C(\mathscr{F}_V,\mathbb{Z}))^{V\oplus\langle\pi\rangle}$. Taking the cap product then gives $\mathbb{1}_{\mathscr{F}_V} \cap \eta_{\mathfrak{p},V} \in H_n(V\oplus\langle\pi\rangle, C(\mathscr{F}_V,\mathbb{Z}))$, since $C(\mathscr{F}_V,\mathbb{Z})\otimes\mathbb{Z} \cong C(\mathscr{F}_V,\mathbb{Z})$. We now define $\vartheta_V^\mathfrak{p} \in H_n(F^*, \mathscr{C}_c(\varnothing,\mathbb{Z})^\mathfrak{p})$ as the homology class corresponding to $\mathbb{1}_{\mathscr{F}_V}\cap\eta_{\mathfrak{p},V}$ under the isomorphism

$$H_n(V\oplus\langle\pi\rangle, C(\mathscr{F}_V,\mathbb{Z})) \cong H_n(F^*, C_c((\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p},\mathbb{Z})) \tag{5.13}$$

that is induced by $C_c((\mathbb{A}_F^\mathfrak{p})^*/U^\mathfrak{p},\mathbb{Z}) \cong \mathrm{Ind}_{V\oplus\langle\pi\rangle}^{F^*} C(\mathscr{F}_V,\mathbb{Z})$. As before, we view $\mathrm{rec}_{H/F}^\mathfrak{p}$ as an element of $H^0(F^*, \mathscr{C}(R',\mathbb{Z}[G])^\mathfrak{p})$ and denote by

$$\rho_{H/F,V} \in H_n(F^*, \mathscr{C}_c(R',\mathbb{Z}[G])^\mathfrak{p})$$

its image under the following map

$$H^0(F^*, \mathscr{C}^\diamond(R',\mathbb{Z}[G])^\mathfrak{p}) \to H_n(F^*, \mathscr{C}_c^\diamond(R',\mathbb{Z}[G])^\mathfrak{p}), \quad \psi\mapsto\psi\cap\vartheta_V^\mathfrak{p},$$

where the cap product is induced by the map (5.2). We then define

$$u_2(V) = \mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}}\cap\rho_{H/F,V}).$$

Proposition 5.3.1. *Let V be a free, finite index, subgroup of E_+ of rank $n-1$. Then*

$$u_2(V) = \sum_{\sigma\in G} u_2(\sigma)^{[V:E_+]}\otimes\sigma^{-1},$$

where

$$u_2 = \sum_{\sigma\in G} u_2(\sigma)\otimes\sigma^{-1}.$$

Proof. We mimic the proof of Theorem 1.5 in [5]. General properties of group cohomology (see pp. 112-114, [1]) yield the following commutative diagram.

$$\begin{array}{ccccc} H^0(V \oplus \langle \pi \rangle, C(\mathcal{F}_V, \mathbb{Z})) & \times & H_n(V \oplus \langle \pi \rangle, \mathbb{Z}) & \xrightarrow{\cap} & H_n(F^*, C_c((\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}}, \mathbb{Z})) \\ \text{res}\uparrow & & \downarrow\text{cores} & & \downarrow\text{id} \\ H^0(E_{+,\mathfrak{p}}, C(\mathcal{F}, \mathbb{Z})) & \times & H_n(E_{+,\mathfrak{p}}, \mathbb{Z}) & \xrightarrow{\cap} & H_n(F^*, C_c((\mathbb{A}_F^{\mathfrak{p}})^*/U^{\mathfrak{p}}, \mathbb{Z})) \end{array} \tag{5.14}$$

Note that in the above, the cap-products in the top and bottom rows include applying the isomorphisms (5.1) and (5.13), respectively. By Proposition 9.5 in §3 of [1], we have the following identities,

$$\text{cores}(\eta_{\mathfrak{p},V}) = [E_{+,\mathfrak{p}} : V]\eta_{\mathfrak{p}},$$
$$\text{res}(\mathbb{1}_{\mathcal{F}}) = \mathbb{1}_{\mathcal{F}_V}.$$

Applying these identities with diagram (5.14) gives

$$\vartheta_V^{\mathfrak{p}} = [E_{+,\mathfrak{p}} : V]\vartheta^{\mathfrak{p}}.$$

It follows that the proposition holds. □

5.4 Explicit expression for u_2

Let $V \subseteq E_+$ be a free, finite index subgroup of rank $n-1$ such that if $V = \langle \varepsilon_1, \ldots, \varepsilon_{n-1} \rangle$ the ε_i and π satisfy Lemma 3.6.1. For ease of notation we write $\varepsilon_n = \pi$. We now calculate explicitly the value of $u_2(V) = \text{Eis}_F^0 \cap \Delta_*(c_{\text{id}} \cap \rho_{H/F,V})$.

Following Spieß [21, Remark 2.1(c)], we choose the following generator for $H_n(V \oplus \langle \pi \rangle, \mathbb{Z})$,

$$\eta_{\mathfrak{p},V} = (-1) \sum_{\tau \in S_n} \text{sign}(\tau)[\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n)}] \otimes 1.$$

We note here that this choice is consistent with that made in the proof of [14, Proposition 4.6]. We then calculate

$$\vartheta_V^{\mathfrak{p}} = \mathbb{1}_{\mathcal{F}_V} \cap \eta_{\mathfrak{p},V} = (-1) \sum_{\tau \in S_n} \text{sign}(\tau)[\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n)}] \otimes \mathbb{1}_{\mathcal{F}_V}.$$

Using this description of $\vartheta_V^{\mathfrak{p}}$ that we computed, we have

$$\rho_{H/F,V} = (-1) \sum_{\tau \in S_n} \text{sign}(\tau)[\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n)}] \otimes (\text{rec}_{H/F}^{\mathfrak{p}} \cdot \mathbb{1}_{\mathcal{F}_V}),$$

where $\mathrm{rec}^{\mathfrak{p}}_{H/F} \cdot \mathbb{1}_{\mathscr{F}_V}$ is as defined in (5.2). It then follows that $c_{\mathrm{id}} \cap \rho_{H/F,V}$ is equal to

$$c_{\mathrm{id}} \cap \rho_{H/F,V} = (-1)^n(-1) \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \\ \Big((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p}}_{H/F} \cdot \mathbb{1}_{\mathscr{F}_V}) \Big).$$

We note that we have the action $(x \cdot f)(y) = f(yx^{-1})$ for a continuous map f and unit x. We also recall the definition of $z_{\mathrm{id}} = z_{\mathbb{1}_{\mathcal{O}_{\mathfrak{p}}},\mathrm{id}}$ from §3.7. Then $z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \in C_c^{\diamond}(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*)$. By changing the sign of our choice of the generator $\eta_{\mathfrak{p},V}$, if necessary, we can remove the factor of $(-1)^n$ in the above. We now apply the map Δ_* to this quantity. In §5.2 Δ_* is defined via the composition of three maps, namely $(5.8)_*$, (5.9) and $(5.8)_*$. By Proposition 5.2.3, we have that the image of $c_{\mathrm{id}} \cap \rho_{H/F,V}$ under $(5.8)_*$ is given by

$$(-1)^{n+1} \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \\ \left((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes \sum_{y \in Y_{\tau(n)}} \delta_{R_\infty^v}(y)(\mathrm{rec}^{\mathfrak{p}}_{H/F}(y,\cdot) \cdot \mathbb{1}_{\mathscr{F}_V}(y,\cdot)) \right), \quad (5.15)$$

where

$$Y_{\tau(n)} = \mathrm{Supp}(F^*_{R_\infty^v}/U_{R_\infty^v}, F_{\mathfrak{p}} \times (\mathbb{A}_F^{\mathfrak{p} \cup R_\infty^v})^*/U^{S \cup R_\infty^v}, \psi_{\tau(n)}).$$

Here, for ease of notation we have written

$$\psi_{\tau(n)} = (\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p}}_{H/F} \cdot \mathbb{1}_{\mathscr{F}_V}).$$

It is now convenient for us to make a choice for $\mathscr{F}_V$. Let G_V denote the group of fractional ideals of $\mathcal{O}_{F,\mathfrak{p}}$ modulo the group of fractional principal ideals generated by elements of V, where $\mathcal{O}_{F,\mathfrak{p}}$, as we defined in §3.5, denotes the ring of $\mathfrak{p}$ integers of F. Let $\{\mathfrak{b}_1, \dots, \mathfrak{b}_h\}$ be a set of integral ideals prime to $R' \cup \overline{\lambda}$ representing G_V. We may then choose

$$\mathscr{F}_V = \{b_1 U^{\mathfrak{p}}, \dots, b_h U^{\mathfrak{p}}\}$$

where $b_1, \dots, b_h \in (\mathbb{A}_F^{\mathfrak{p}})^*$ are ideles whose associated fractional $\mathcal{O}_{F,\mathfrak{p}}$-ideals are $\mathfrak{b}_1 \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}, \dots,$ $\mathfrak{b}_h \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$. Thus, for $i = 1, \dots, h$ we can choose that the b_i are totally positive and prime to $R' \cup \overline{\lambda}$. This description of $\mathscr{F}_V$ is similar to a construction given in [14] on page 14. From this description of $\mathscr{F}_V$ we have that $Y_{\tau(n)}$ is trivial for all n. Thus, (5.15) is equal to

$$(-1)^{n+1} \sum_{\tau \in S_n} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}] \otimes \Big((\varepsilon_{\tau(1)} \dots \varepsilon_{\tau(n-1)}) \cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)}) \otimes (\mathrm{rec}^{\mathfrak{p} \cup R_\infty^v}_{H/F} \cdot \mathbb{1}^{R_\infty^v}_{\mathscr{F}_V}) \Big). \quad (5.16)$$

We now apply (5.9). By Proposition 5.2.4, we have that the image of (5.16) under (5.9) is equal to

$$(-1)^{n+1}\sum_{\tau\in S_n}\sum_{f\in F^*/F^{\overline{\lambda}\cup v}}\mathrm{sign}(\tau)f[\varepsilon_{\tau(1)}\mid\ldots\mid\varepsilon_{\tau(n-1)}]\otimes$$
$$\Big((\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})\otimes\delta_v(f_v)(\mathrm{rec}_{H/F}^{\mathfrak{p}\cup R_\infty^v}(f_{\overline{\lambda}\cup v},\cdot)\cdot\mathbb{1}_{\mathscr{F}_V}^{R_\infty^v}(f_{\overline{\lambda}\cup v},\cdot))\Big).\quad(5.17)$$

By our choice of $\mathscr{F}_V$, we have that the sum over $F^*/F^{\overline{\lambda}\cup v}$ is also trivial. Hence, (5.17) is equal to

$$(-1)^{n+1}\sum_{\tau\in S_n}\mathrm{sign}(\tau)[\varepsilon_{\tau(1)}\mid\ldots\mid\varepsilon_{\tau(n-1)}]\otimes\Big((\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})\otimes(\mathrm{rec}_{H/F}^{\mathfrak{p}\cup\overline{\lambda}\cup\infty}\cdot\mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda}\cup\infty})\Big).\quad(5.18)$$

We can now finish calculating the effect of Δ_* on $c_{\mathrm{id}}\cap\rho_{H/F,V}$ by applying (5.10) to (5.18) and using Proposition 5.2.5 to calculate that $\Delta_*(c_{\mathrm{id}}\cap\rho_{H/F,V})$ is equal to

$$(-1)^{n+1}\sum_{\tau\in S_n}\mathrm{sign}(\tau)[\varepsilon_{\tau(1)}\mid\ldots\mid\varepsilon_{\tau(n-1)}]\otimes$$
$$\Bigg((\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})\otimes\sum_{z\in Z}(\mathrm{rec}_{H/F}^{\mathfrak{p}\cup\overline{\lambda}\cup\infty}(z,\cdot)\cdot\mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda}\cup\infty}(z,\cdot))$$
$$\odot\mathrm{char}\Bigg(\prod_{w\text{ finite}}\mathfrak{q}_w^{\mathrm{ord}_w(z_w)}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{r}})\Bigg)\Bigg),\quad(5.19)$$

where

$$Z=\mathrm{Supp}((\mathbb{A}_F^{S'\cup\overline{\lambda},\infty})^*/U^{S'\cup\overline{\lambda},\infty},F_{\mathfrak{p}}\times\prod_{\mathfrak{q}\in R'}F_{\mathfrak{q}}^*,\psi_{\tau(n)}).$$

Here, we have $\psi_{\tau(n)}=(\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})\otimes(\mathrm{rec}_{H/F}^{\mathfrak{p}\cup\overline{\lambda}\cup\infty}\cdot\mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda}\cup\infty})$. We now apply the measure Eis_F^0 to $\Delta_*(c_{\mathrm{id}}\cap\rho_{H/F,V})$. Recall that the measure is applied as defined in (5.3). We write $\mu_{F_{\mathfrak{p}}^*}$ for the measure with values in $F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G]$ induced from the Eisenstein series Eis_F^0. We now consider the function $(\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})$. For ease of notation we define, for $\tau\in S_n$ and $z\in Z$,

$$\phi_{\tau(n),z}=(\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})\otimes\sum_{z\in Z}(\mathrm{rec}_{H/F}^{\mathfrak{p}\cup\overline{\lambda}\cup\infty}(z,\cdot)\cdot\mathbb{1}_{\mathscr{F}_V}^{\overline{\lambda}\cup\infty}(z,\cdot))$$
$$\odot\mathrm{char}\Bigg(\prod_{w\text{ finite}}\mathfrak{q}_w^{\mathrm{ord}_w(z_w)}(\prod_{\mathfrak{r}\notin S'\cup\overline{\lambda}}\mathcal{O}_{\mathfrak{r}})\Bigg)\Bigg).$$

We are then able to calculate, after recalling from §3.7 that we can choose $z_{\mathrm{id}}=z_{\mathbb{1}_{\mathcal{O}_{\mathfrak{p}}},\mathrm{id}}$,

$$((\varepsilon_{\tau(1)}\ldots\varepsilon_{\tau(n-1)})\cdot z_{\mathrm{id}}(\varepsilon_{\tau(n)})=\begin{cases}\mathbb{1}_{\mathbb{O}}\cdot\mathrm{id}_{F_{\mathfrak{p}}^*}+\mathbb{1}_{\pi\mathcal{O}_{\mathfrak{p}}}\cdot\pi & \text{if }\tau(n)=n,\\ \mathbb{1}_{\pi\mathcal{O}_{\mathfrak{p}}}\cdot\varepsilon_{\tau(n)} & \text{if }\tau(n)\neq n.\end{cases}\quad(5.20)$$

To calculate the measure, we first note that $F_{\mathfrak{p}}^*\cong\langle\pi\rangle\oplus\mathbb{O}$ and that $\mathbb{O}\cong\lim_{m\to\infty}\mathbb{O}/1+\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}}$. By

(5.20), we are able to calculate the value at π and $\mathbb{O}$ separately. We now give some additional notation which we require. Let $m \geq 0$ and $\alpha \in \mathbb{O}/1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p}$. We then write $U_\alpha = \alpha(1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})$. For $\sigma \in G$, we define the following maps

$$\phi_{n,z}^{\pi\otimes\sigma^{-1}} : W_{\widehat{\mathbb{Z}}^\lambda} \to \mathbb{Z}, \text{ and } \phi_{n,z}^{U_\alpha\otimes\sigma^{-1}} : W_{\widehat{\mathbb{Z}}^\lambda} \to \mathbb{Z},$$

such that

$$\phi_{n,z}^{\pi\otimes\sigma^{-1}}(x) = \begin{cases} 1 & \text{if } \phi_{n,z}(x) = \pi\otimes\sigma^{-1}, \\ 0 & \text{else,} \end{cases} \quad \text{and } \phi_{n,z}^{U_\alpha\otimes\sigma^{-1}}(x) = \begin{cases} 1 & \text{if } \phi_{n,z}(x) \in U_\alpha\otimes\sigma^{-1}, \\ 0 & \text{else.} \end{cases}$$

For $\tau \in S_n$ with $\tau(n) \neq n$, we also define

$$\phi_{n,z}^{\varepsilon_{\tau(n)}\otimes\sigma^{-1}} : W_{\widehat{\mathbb{Z}}^\lambda} \to \mathbb{Z}$$

such that

$$\phi_{\tau(n),z}^{\varepsilon_{\tau(n)}\otimes\sigma^{-1}}(x) = \begin{cases} 1 & \text{if } \phi_{\tau(n),z}(x) = \varepsilon_{\tau(n)}\otimes\sigma^{-1}, \\ 0 & \text{else.} \end{cases}$$

The construction of the measure given in §3.4 now allows us to calculate

$$\begin{aligned} &\mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = (-1)^{n+1}(-1)^{(n-1)(n-1)} \\ &\sum_{z\in Z}\sum_{\sigma\in G}\left(\sum_{\substack{\tau\in S_n\\ \tau(n)=n}} \mathrm{sign}(\tau) \lim_{m\to\infty}\left(\sum_{\alpha\in\mathbb{O}/(1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})} \mathrm{Eis}_\tau^0(\phi_{n,z}^{U_\alpha\otimes\sigma^{-1}})(\alpha\otimes\sigma^{-1})\right) + \mathrm{Eis}_\tau^0(\phi_{n,z}^{\pi\otimes\sigma^{-1}})(\pi\otimes\sigma^{-1})\right. \\ &\left. + \sum_{\substack{\tau\in S_n\\ \tau(n)\neq n}} \mathrm{sign}(\tau)\mathrm{Eis}_\tau^0(\phi_{\tau(n),z}^{\varepsilon_{\tau(n)}\otimes\sigma^{-1}})(\varepsilon_{\tau(n)}\otimes\sigma^{-1})\right). \end{aligned} \tag{5.21}$$

For ease of notation we have written $\mathrm{Eis}_\tau^0 = \mathrm{Eis}_F^0([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])$. Let $\phi \in \{\phi_{n,z}^{U_\alpha\otimes\sigma^{-1}}, \phi_{n,z}^{\pi\otimes\sigma^{-1}}, \phi_{n,z}^{\varepsilon_{\tau(n)}\otimes\sigma^{-1}}\}$. Then, by Proposition 3.5.4 we have that

$$\mathrm{Eis}_F^0([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])(\phi) = \delta([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])L_\lambda(\overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]), \Phi; 0)$$

where $\Phi = \phi \otimes \phi_\lambda$ and ϕ_λ is as defined in (3.8). We will decorate Φ to match with the notation used for ψ. For example we write $\Phi_{n,z}^{U_\alpha\otimes\sigma^{-1}} = \phi_{n,z}^{U_\alpha\otimes\sigma^{-1}} \otimes \phi_\lambda$. For ease of notation, we let $\overline{C}_\tau = \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}])$. Recall from (3.10) that for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we have

$$L_\lambda(\overline{C}_\tau, \Phi; s) = \sum_{v\in W\cap\overline{C}_\tau} \Phi(v)\mathrm{N}v^{-s}.$$

For $i = 1, \dots, n$ we define

$$\mathscr{B}_i \coloneqq \bigcup_{\substack{\tau \in S_n \\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]). \tag{5.22}$$

We also write $\mathscr{B} = \mathscr{B}_n$. Since we have chosen the ε_i as in Lemma 3.6.1 we have

$$\operatorname{sign}(\tau)\delta([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]) = 1. \tag{5.23}$$

Applying (5.23) and the definition in (5.22) to (5.21), we have, after noting $(-1)^{n+1}(-1)^{(n-1)(n-1)} = 1$,

$$\begin{aligned}\operatorname{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = \\ \sum_{z \in Z} \sum_{\sigma \in G} \left(\lim_{m \to \infty} \left(\sum_{\alpha \in \mathbb{O}/(1+\mathfrak{p}^m \mathscr{O}_\mathfrak{p})} L_\lambda(\mathscr{B}, \Phi_{n,z}^{U_\alpha \otimes \sigma^{-1}}; 0)(\alpha \otimes \sigma^{-1}) \right) + L_\lambda(\mathscr{B}, \Phi_{n,z}^{\pi \otimes \sigma^{-1}}; 0)(\pi \otimes \sigma^{-1}) \right. \\ \left. + \sum_{i=1}^{n-1} L_\lambda(\mathscr{B}_i, \Phi_{i,z}^{\varepsilon_i \otimes \sigma^{-1}}; 0)(\varepsilon_i \otimes \sigma^{-1}) \right).\end{aligned} \tag{5.24}$$

We now calculate each term in the above expression, beginning with the limit term. Fix $m \geq 0$ and $\sigma \in G$. Let $\alpha \in \mathbb{O}/1 + \mathfrak{p}^m \mathscr{O}_\mathfrak{p}$. We also let $\mathfrak{b}$ be a fractional ideal of F, coprime to $S \cup \overline{\lambda}$, and such that $\sigma_\mathfrak{b} = \sigma$. For this we need to find the elements $z \in Z$ such that $\phi_{n,z}^{U_\alpha \otimes \sigma^{-1}}$ is not trivial. For this we require that for some $x \in \prod_{\mathfrak{q} \in R'} F_\mathfrak{q}^*$,

$$\sigma^{-1} = (\operatorname{rec}_{H/F}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}(z, x) \cdot \mathbb{1}_{\mathscr{F}_V}^{\mathfrak{p} \cup \overline{\lambda} \cup \infty}(z, x)).$$

By the definition of $\mathscr{F}_V$ and the reciprocity map, the above equation is nontrivial only if $z \in \mathscr{F}_V$ and $\prod_{v \text{ finite}} \mathfrak{q}_v^{-\operatorname{ord}_v(z_v)} = \mathfrak{b}^{-1}(\alpha)\mathfrak{p}^{\operatorname{ord}_\mathfrak{p}(\alpha)}$ for some $(\alpha) \in P^{\mathfrak{f},1}$. Recall that we have $\mathfrak{f}$ as the conductor of H/F and define $P^{\mathfrak{f},1} = \{(\alpha) \mid \alpha \in F_+^*,\ \alpha \equiv 1 \pmod{\mathfrak{f}}\}$. By the description of $\mathscr{F}_V$, we note that for each $\sigma \in G$ there is a unique $z \in Z$ that satisfies the above equation. Since $\prod_{v \text{ finite}} \mathfrak{q}_v^{\operatorname{ord}_v(z_v)} = \mathfrak{b}^{-1}(\alpha)\mathfrak{p}^{-\operatorname{ord}_\mathfrak{p}(\alpha)}$ for $(\alpha) \in P^{\mathfrak{f},1}$, we have that $(\alpha)\mathfrak{p}^{-\operatorname{ord}_\mathfrak{p}(\alpha)}$ must be coprime to $\mathfrak{p} \cup \overline{R} \cup \overline{\lambda}$ since z and $\mathfrak{b}^{-1}$ are. Thus, for all $r \in (\alpha)\mathfrak{p}^{-\operatorname{ord}_\mathfrak{p}(\alpha)}$ we have $r^{-1} \in \prod_{\mathfrak{r} \notin S' \cup \overline{\lambda}} \mathscr{O}_\mathfrak{r}$. Thus, we have

$$F \cap \mathfrak{b}^{-1}(\alpha)\mathfrak{p}^{-\operatorname{ord}_\mathfrak{p}(\alpha)}(\prod_{\mathfrak{r} \notin S' \cup \overline{\lambda}} \mathscr{O}_\mathfrak{r}) = \mathfrak{b}^{-1}.$$

We now define, for a Shintani set A, $U \subseteq F_\mathfrak{p}$ compact open, fractional ideal $\mathfrak{b}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$

$$L_{R,\lambda}(\mathfrak{b}, A, U, s) \coloneqq \sum_{\substack{x \in W \cap A,\ x \in U, \\ x \in \mathfrak{b}^{-1},\ (x,R)=1}} \phi_\lambda(x) \mathrm{N}(x)^{-s}.$$

We thus have

$$\lim_{m\to\infty}\left(\sum_{\alpha\in\mathbb{O}/(1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})} L_\lambda(\mathscr{B},\Phi_{n,z}^{U_\alpha\otimes\sigma_\mathfrak{b}^{-1}};0)(\alpha\otimes\sigma^{-1})\right) = \lim_{m\to\infty}\left((\prod_{\alpha\in\mathbb{O}/(1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})}\alpha^{L_{R,\lambda}(\mathfrak{b},\mathscr{B},U_\alpha;0)}\otimes\sigma_\mathfrak{b}^{-1})\right)$$
$$= (\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ dL_{R,\lambda}(\mathfrak{b},\mathscr{B},x;0))\otimes\sigma_\mathfrak{b}^{-1},$$

where the multiplicative integral is as defined in Definition 4.1.1. We can apply similar, and in fact easier, calculations for the other terms in (5.24) and thus deduce the explicit expression for $u_2(V)$,

$$\mathrm{Eis}_F^0\cap(\Delta_*(c_{\mathrm{id}}\cap\rho_{H/F,V})) = \sum_{\sigma_\mathfrak{b}\in G}\left((\prod_{i=1}^{n-1}\varepsilon_i^{L_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathcal{O}_\mathfrak{p};0)})\pi^{L_{R,\lambda}(\mathfrak{b},\mathscr{B},\pi\mathcal{O}_\mathfrak{p};0)}\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ dL_{R,\lambda}(\mathfrak{b},\mathscr{B},x;0))\otimes\sigma_\mathfrak{b}^{-1}\right). \quad (5.25)$$

Chapter 6

The Cohomological Element u_3

In this section we consider the second cohomological formula conjectured by Dasgupta–Spieß. This is conjectured in [14]. We begin by recalling the construction of this element. As with the other cohomological formula of Dasgupta–Spieß(u_2), a key element of the formula is a generator of a homology group. In this case it is a generator of $H_{n+r-1}(E_{S_p,+}, \mathbb{Z})$. As was done in Chapter 5, we consider the effect on this formula when $E_{S_p,+}$ is replaced by a free, finite index subgroup, of rank n.

6.1 The definition of u_3

In [14], Dasgupta–Spieß give two equivalent constructions for their formula. Since we require each of them in the later chapters we give both here. We denote them by u_3 and u_3'. In this thesis we give slightly different constructions to those given in [14], namely our $u_3 = u_3(DS)^{\#}$, and similarly for u_3'. Here $\#$ denotes the involution on $\mathbb{Z}[G]$ given by $g \mapsto g^{-1}$ for $g \in G$, and $u_3(DS)$ is the construction in [14]. This is done by modifying the definitions of κ_λ and $\omega_{\mathfrak{f},\lambda}^{\mathfrak{p}}$ in §3.5. The key adjustment we have is to use $\mathrm{rec}_{H/F}((\eta))^{-1}$ rather than $\mathrm{rec}_{H/F}((\eta))$ in (3.7).

We begin with u_3. Recall that in §3.7 and §3.5 we have defined the following objects:

$$c_g \in H^1(F_{\mathfrak{p}}^*, C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*)) \quad \text{and} \quad \kappa_\lambda \in H^{n-1}(E_{S_p,+}, \mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), F_{\mathfrak{p}} \otimes \mathbb{Z}[G])).$$

Let $r = \#S_p$. We now consider $H_{n+r-1}(E_{S_p,+}, \mathbb{Z})$. By Dirichlet's unit theorem, $E_{S_p,+}$ is a free abelian group of rank n. Hence, $H_{n+r-1}(E_{S_p,+}, \mathbb{Z}) \cong \mathbb{Z}$. We are thus able to choose a generator $\vartheta \in H_{n+r-1}(E_{S_p,+}, \mathbb{Z})$. We recall that $F_{S_p}^* = \prod_{\mathfrak{q} \in S_p} F_{\mathfrak{q}}^*$ and label the elements of S_p by $\mathfrak{p}, \mathfrak{p}_2 \ldots, \mathfrak{p}_r$. We now define a class

$$c_{\mathrm{id},\mathfrak{p}} \in H^r(F_{S_p}^*, C_c(F_{S_p}, F_{\mathfrak{p}}^*))$$

by

$$c_{\mathrm{id},\mathfrak{p}} = c_{\mathrm{id}} \cup c_{o_{\mathfrak{p}_2}} \cup \cdots \cup c_{o_{\mathfrak{p}_r}}.$$

Here the cup product is induced by the canonical map

$$C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*) \otimes \cdots \otimes C_c(F_{\mathfrak{p}}, F_{\mathfrak{p}}^*) \to C_c(F_{S_p}, F_{\mathfrak{p}}^*)$$

defined by

$$\bigotimes_{\mathfrak{q} \in S_p} f_{\mathfrak{q}} \mapsto \left((x_{\mathfrak{q}})_{\mathfrak{q} \in S_p} \mapsto \prod_{\mathfrak{q} \in S_p} f_{\mathfrak{q}}(x_{\mathfrak{q}}) \right).$$

Definition 6.1.1. *Let $\vartheta \in H_{n+r-1}(E_{S_p,+}, \mathbb{Z})$ be a generator. Then we define*

$$u_3 = c_{id,\mathfrak{p}} \cap (\kappa_\lambda \cap \vartheta) \in F_{\mathfrak{p}}^* \otimes \mathbb{Z}[G].$$

Adapted from [14, Conjecture 3.1] we have the following conjecture.

Conjecture 6.1.2. *We have $u_3 = u_{\mathfrak{p}}$*

We now give the definition for u_3'. Recall that in §3.5 we have defined

$$\omega_{\mathfrak{f},\lambda}^{\mathfrak{p}} \in H^{n-1}(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z}[G])).$$

Definition 6.1.3. *Let $\vartheta' \in H_n(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z})$ be a generator. Then, we define*

$$u_3' := c_{id} \cap (\omega_{\mathfrak{f},\lambda}^{\mathfrak{p}} \cap \vartheta'). \tag{6.1}$$

The following Proposition follows from [14, Proposition 3.6].

Proposition 6.1.4. *We have $u_3 = u_3'$.*

6.2 Transferring to a subgroup

Let $\vartheta_V' \in H_n(V \oplus \langle \pi \rangle, \mathbb{Z})$ be a generator. For $x_1, \ldots, x_n \in V \oplus \langle \pi \rangle$ and compact open $U \subset F_{\mathfrak{p}}$ we put

$$\nu_{\mathfrak{b},\lambda,V}^{\mathfrak{p}}(x_1, \ldots, x_n)(U) := \delta(x_1, \ldots, x_n) \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}(x_1, \ldots, x_n), U, 0).$$

As before, it follows from Theorem 2.6 of [5] that $\nu_{\mathfrak{b},\lambda,V}^{\mathfrak{p}}$ is a homogeneous $(n-1)$-cocycle on $V \oplus \langle \pi \rangle$ with values in the space of $\mathbb{Z}$-distribution on $F_{\mathfrak{p}}$. Hence, we obtain a class

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} := [\nu_{\mathfrak{b},\lambda,V}^{\mathfrak{p}}] \in H^{n-1}(V \oplus \langle \pi \rangle, \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z})).$$

We then define

$$u_3'(V) = c_{\mathrm{id}} \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda,V}^{\mathfrak{p}} \cap \vartheta_V').$$

The next proposition shows the relation between u_3' and $u_3'(V)$.

Proposition 6.2.1. *Let V be a free, finite index, subgroup of $E_+(\mathfrak{f})$ of rank $n-1$. Then, we have*

$$c_{id} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta'_V) = (c_{id} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \cap \vartheta'))^{[E_+(\mathfrak{f}):V]}. \tag{6.2}$$

Proof. This proof is adapted from the proof of Proposition 2.1.4 in [24]. We mimic the proof of Theorem 1.5 in [5]. For ease of notation, in the following diagrams we write

$$\mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z}) = \mathrm{Hom}(C_c(F_{\mathfrak{p}}, \mathbb{Z}), \mathbb{Z}).$$

General properties of group cohomology (see pp. 112-114, [1]) yield the following commutative diagrams.

$$\begin{array}{ccccc} H^{n-1}(V, \mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z})) & \times & H_n(V \oplus \langle \pi \rangle, \mathbb{Z}) & \xrightarrow{\cap} & H_1(V \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z})) \\ \mathrm{res}\uparrow & & \downarrow\mathrm{cores} & & \downarrow\mathrm{cores} \\ H^{n-1}(E_+(\mathfrak{f}), \mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z})) & \times & H_n(E_+(\mathfrak{f}) \oplus \langle \pi \rangle, \mathbb{Z}) & \xrightarrow{\cap} & H_1(E_+(\mathfrak{f}) \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z})) \end{array} \tag{6.3}$$

and

$$\begin{array}{ccccc} H^1(F^\times_{\mathfrak{p}}, C_c(F_{\mathfrak{p}}, F^*_{\mathfrak{p}})) & \times & H_1(V \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z})) & \xrightarrow{\cap} & F^*_{\mathfrak{p}} \\ \downarrow\mathrm{id} & & \downarrow\mathrm{cores} & & \downarrow\mathrm{id} \\ H^1(F^\times_{\mathfrak{p}}, C_c(F_{\mathfrak{p}}, F^*_{\mathfrak{p}})) & \times & H_1(E_+(\mathfrak{f}) \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, \mathbb{Z})) & \xrightarrow{\cap} & F^*_{\mathfrak{p}}. \end{array} \tag{6.4}$$

By Proposition 9.5 in §3 of [1], we have following identities,

$$\mathrm{cores}(\vartheta'_V) = [E_+(\mathfrak{f}) : V]\vartheta',$$
$$\mathrm{res}(\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda}) = \omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}.$$

Diagram (6.3) gives the equality

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \cap \mathrm{cores}(\vartheta'_V) = \mathrm{cores}(\mathrm{res}(\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda}) \cap \vartheta'_V).$$

The identities above then show that

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \cap [E_+(\mathfrak{f}) : V]\vartheta' = \mathrm{cores}(\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta'_V).$$

Applying diagram (6.4) to the above equality gives us the result. We note the the factor $[V : E_+(\mathfrak{f})]$ becomes a power due to the multiplicative nature of the formula. □

Let $V \subseteq E_+(\mathfrak{f})$ be a finite index subgroup free of rank $n-1$. We now note the relation between $u'_3(V)$ and $u_3(V)$. We first give the additional notation required to define $u_3(V)$. Let $V \subseteq E_+$ be a free finite index subgroup of rank $n-1$. Write

$$V_{S_p} = V \oplus \langle \pi_1, \ldots, \pi_r \rangle.$$

Let $\vartheta_V \in H_{n+r-1}(V_{S_p}, \mathbb{Z})$ be a generator. For $x_1, \dots, x_n \in V_{S_p}$ and compact open $U \subset F_{\mathfrak{p}}$ we put

$$\mu_{\chi,\mathfrak{b},\lambda,V}(x_1,\dots,x_n)(U) \coloneqq \delta(x_1,\dots,x_n)\mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}(x_1,\dots,x_n),\mathfrak{b},U,0).$$

As before, it follows from [5, Theorem 2.6] that $\mu_{\chi,\mathfrak{b},\lambda,V}$ is a homogeneous $n-1$-cocycle on V_{S_p} with values in the space of $\mathbb{Z}$-distribution on F_{S_p}. Hence, we obtain a class

$$\kappa_{\lambda,V} \coloneqq \sum_{i=1}^{h} \mathrm{rec}(\mathfrak{b}_i)^{-1}[\mu_{\mathfrak{b},\lambda,V}] \in H^{n-1}(V_{S_p}, \mathrm{Hom}(C_c(F_{S_p},\mathbb{Z}),\mathbb{Z}[G])).$$

We then define

$$u_3(V) = c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\lambda,V} \cap \vartheta_V).$$

Proposition 6.2.2. *Let V be a free, finite index subgroup of $E_+(\mathfrak{f})$, of rank $n-1$. We now let V' be any free, finite index subgroup of E_+, of rank $n-1$ such that $V' \subseteq V$ and $[E_+ : V'] = [E_+(\mathfrak{f}) : V]$. Then,*

$$c_{id} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta'_V) = c_{id,\mathfrak{p}} \cap (\kappa_{\lambda,V'} \cap \vartheta_{V'}),$$

i.e., $u_3(V') = u'_3(V)$.

Proof. This proposition follows from the proof of [14, Proposition 3.6]. □

Following from this proposition, we have a simple corollary.

Corollary 6.2.3. *Let V be a free, finite index, subgroup of E_+ of rank $n-1$. Then, for each $\sigma \in G$,*

$$u_3(V,\sigma) = u_3(\sigma)^{[E_+:V]}.$$

6.3 Explicit expression for u_3

Let $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle \subseteq E_+$ where $\varepsilon_1, \dots, \varepsilon_{n-1}$ and $\pi = \varepsilon_n$ are chosen to satisfy Lemma 3.6.1. Write $\varepsilon_n = \pi$. As before we have the notation $V_{S_p} = \langle \varepsilon_1, \dots, \varepsilon_{n-1}, \pi_1, \dots, \pi_r \rangle$. Here we have $\pi = \pi_1$. As in (5.22), for $i = 1, \dots, n$ we define

$$\mathscr{B}_i \coloneqq \bigcup_{\substack{\tau \in S_n \\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]).$$

As before, we write $\mathscr{B} = \mathscr{B}_n$. We now calculate explicitly the value of u_3. We begin by firstly calculating the value of $c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V)$. For ease of notation we let $\varepsilon_{n+i-1} = \pi_i$, for $i = 1, \dots, r$. We choose the following generator for $H_{n+r-1}(V_{S_p}, \mathbb{Z})$,

$$\vartheta_V = (-1)^{(n-1)(n+r-1)}(-1)^r \sum_{\tau \in S_{n+r-1}} \mathrm{sign}(\tau)[\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n+r-1)}] \otimes 1.$$

This choice is stated by Spieß in [21, Remark 2.1(c)] and is consistent with the choice we made for η_V at the start of §5.4. We can now calculate, after noting $(-1)^{(n-1)(n+r-1)}(-1)^r(-1)^{(n+r-1)(n-1)} = (-1)^r$,

$$\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V = (-1)^r \sum_{\tau \in S_{n+r-1}} \operatorname{sign}(\tau)\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) \otimes [\varepsilon_{\tau(n)} \mid \ldots \mid \varepsilon_{\tau(n+r-1)}].$$

Recall from §3.7 that we can choose, as a representative of c_{id}, the inhomogeneous 1-cocycle $z_{\mathrm{id}} = z_{\mathbb{1}_{\pi\mathcal{O}_\mathfrak{p}},\mathrm{id}}$, i.e., we take $f = \mathbb{1}_{\pi\mathcal{O}_\mathfrak{p}}$ in Definition 3.7.1. One can easily compute, as is done by Dasgupta–Spieß in the proof of [14, Proposition 4.6], that for $i = 1, \ldots, n+r-1$ and $i \neq n$, we have

$$\varepsilon_i^{-1} z_{\mathrm{id}}(\varepsilon_i) = \mathbb{1}_{\pi\mathcal{O}_\mathfrak{p}} \cdot \varepsilon_i, \tag{6.5}$$

and

$$\pi^{-1} z_{\mathrm{id}}(\pi) = \mathbb{1}_{\mathbb{O}} \cdot \mathrm{id}_{F_\mathfrak{p}^*} + \mathbb{1}_{\mathcal{O}_\mathfrak{p}} \cdot \pi. \tag{6.6}$$

Returning to our main calculation, we have

$$\begin{aligned} c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V) = (-1)^r(-1)^{r^2} \sum_{\tau \in S_{n+r-1}} \int_{F_{S_p}} & c_{\mathrm{id},\mathfrak{p}}([\varepsilon_{\tau(n)} \mid \ldots \mid \varepsilon_{\tau(n+r-1)}])(x) \\ & d(\operatorname{sign}(\tau)([\varepsilon_{\tau(n)} \mid \ldots \mid \varepsilon_{\tau(n+r-1)}])\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]))(x). \end{aligned}$$

Now we note that for $i = 2, \ldots, r$ we have that $c_{o_{\mathfrak{p}_i}}(\varepsilon_j) = 0$ unless $i = j$. Hence, we only get non-zero terms when $\tau(k) = k$ for $k = n+1, \ldots, n+r-1$. Therefore, since $(-1)^r(-1)^{r^2} = 1$, we have

$$\begin{aligned} c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V) = \sum_{\tau \in S_n} \int_{F_{S_p}} & c_{\mathrm{id},\mathfrak{p}}([\varepsilon_{\tau(n)} \mid \varepsilon_{n+1} \mid \ldots \mid \varepsilon_{n+r-1}])(x) \\ & d(\operatorname{sign}(\tau)([\varepsilon_{\tau(n)} \mid \varepsilon_{n+1} \mid \ldots \mid \varepsilon_{n+r-1}])\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]))(x). \end{aligned}$$

Then, since for $i = 2, \ldots, r$ and $\tau \in S_n$ we can calculate $(\varepsilon_{\tau(n)}\varepsilon_{n+1} \ldots \varepsilon_{n+i-1})^{-1} \cdot c_{o_{\mathfrak{p}_i}}(\varepsilon_i) = \mathbb{1}_{\mathcal{O}_{\mathfrak{p}_i}}$, we have

$$\begin{aligned} c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V) = \sum_{\tau \in S_n} \int_{F_\mathfrak{p}} & c_{\mathrm{id}}([\varepsilon_{n+1} \mid \ldots \mid \varepsilon_{n+r-1}])(x) \\ & d(\operatorname{sign}(\tau)\varepsilon_{\tau(n)}\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]))(x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}). \end{aligned}$$

We recall the definition of $\kappa_{\mathfrak{b},\lambda,V}$ from the start of §6.1. Since we have chosen V and π through Lemma 3.6.1, we can note that, for $\tau \in S_n$ and a compact open $U \subseteq \mathcal{O}_\mathfrak{p}$, we have by definition that,

$$\operatorname{sign}(\tau)\kappa_{\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) = \mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]), \mathfrak{b}, U, 0),$$

where $\mathfrak{L}$ is as defined in (3.7). Thus,

$$c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V) = \sum_{\tau \in S_n} \int_{F_\mathfrak{p}} \varepsilon_{\tau(n)}^{-1} z_{\mathrm{id}}(\varepsilon_{\tau(n)})(x) \\ d(\mathfrak{L}_{R,\lambda}(\overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \dots \mid \varepsilon_{\tau(n-1)}]), \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)).$$

Applying (6.5) and (6.6), and piecing together the appropriate Shintani sets, we further deduce

$$c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V) = \times\!\!\!\!\!\!\int_{\mathbb{O}} x \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)) \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}} \pi \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)) \\ \prod_{i=1}^{n-1} \times\!\!\!\!\!\!\int_{\pi\mathcal{O}_\mathfrak{p}} \varepsilon_i \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}_i, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)). \quad (6.7)$$

We note the switch to products here as after integrating we are in the multiplicative group $F_\mathfrak{p}^*$. Considering the first two terms on the right hand side of (6.7), it is clear that

$$\times\!\!\!\!\!\!\int_{\mathbb{O}} x \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)) \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}} \pi \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)) \\ = \pi^{\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, \mathcal{O}_{S_p}, 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)),$$

where $\mathcal{O}_{S_p} = \prod_{j=1}^{r} \mathcal{O}_{\mathfrak{p}_j} \subset F_{S_\mathfrak{p}}$. We now consider the product on the right hand side of (6.7). It is straight forward to see that

$$\prod_{i=1}^{n-1} \times\!\!\!\!\!\!\int_{\pi\mathcal{O}_\mathfrak{p}} \varepsilon_i \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}_i, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)) = \prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathcal{B}_i, \mathfrak{b}, \pi\mathcal{O}_\mathfrak{p} \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)}.$$

Hence,

$$c_{\mathrm{id},\mathfrak{p}} \cap (\kappa_{\mathfrak{b},\lambda,V} \cap \vartheta_V) \\ = (\prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathcal{B}_i, \mathfrak{b}, \pi\mathcal{O}_\mathfrak{p} \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)}) \pi^{\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, \mathcal{O}_{S_p}, 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)).$$

Thus, we have

$$u_3(V) = \sum_{k=1}^{h} \mathrm{rec}_{H/F}(\mathfrak{b}_k)^{-1} \Bigg((\prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathcal{B}_i, \mathfrak{b}_k, \pi\mathcal{O}_\mathfrak{p} \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)}) \pi^{\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}_k, \mathcal{O}_{S_p}, 0)} \\ \times\!\!\!\!\!\!\int_{\mathbb{O}} x \ d(\mathfrak{L}_{R,\lambda}(\mathcal{B}, \mathfrak{b}_k, x \times \prod_{j=2}^{r} \mathcal{O}_{\mathfrak{p}_j}, 0)) \Bigg).$$

6.4 Explicit expression for u_3'

For later calculations we also require an explicit expression of u_3'. Let V be a free, finite index subgroup of $E_+(\mathfrak{f})$ of rank $n-1$ such that $V = \langle \varepsilon_1, \ldots, \varepsilon_{n-1}\rangle$ where $\varepsilon_1, \ldots, \varepsilon_{n-1}$ and $\pi = \varepsilon_n$ are chosen to satisfy Lemma 3.6.1. For $i = 1, \ldots, n$ write

$$\mathcal{B}_i \coloneqq \bigcup_{\substack{\tau \in S_n \\ \tau(n)=i}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]).$$

Let $\mathcal{B} = \mathcal{B}_n$.As in §5.4, we choose the following generator for $H_n(E_+(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z})$,

$$\vartheta_V' = (-1) \sum_{\tau \in S_n} \operatorname{sign}(\tau)[\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n)}] \otimes 1.$$

This choice is stated by Spieß in [21, Remark 2.1(c)]. We can now calculate

$$\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta_V' = (-1)^{n(n-1)}(-1) \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \operatorname{sign}(\tau)\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) \otimes [\varepsilon_i].$$

We recall the definition of $\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}$ from §3.5. For $\tau \in S_n$ and a compact open $U \subseteq \mathcal{O}_{\mathfrak{p}}$, we have, by definition, that

$$\operatorname{sign}(\tau)\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]) = \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]), U, 0). \tag{6.8}$$

Returning to our main calculation, using (6.8), we have

$$c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta_V') = \sum_{i=1}^{n} \sum_{\substack{\tau \in S_n \\ \tau(n)=i}} \int_{F_{\mathfrak{p}}} z_{\mathrm{id}}(\varepsilon_i)(x)\ d(\varepsilon_i \zeta_{R,\lambda}(\mathfrak{b}, \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]), x, 0)).$$

Applying (6.5) and (6.6), and piecing together the appropriate Shintani sets, we further deduce

$$\begin{aligned} c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta_V') = &\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0)) \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}} \pi\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0)) \\ &\prod_{i=1}^{n-1} \times\!\!\!\!\!\!\int_{\pi\mathcal{O}_{\mathfrak{p}}} \varepsilon_i\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}_i, x, 0)). \end{aligned} \tag{6.9}$$

We change to products rather then sums here as after integrating we are in the multiplicative group $F_{\mathfrak{p}}^*$. It is clear that we can then write

$$\begin{aligned} u_3'(V, \sigma) &= c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta_V') \\ &= \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}_i, \pi\mathcal{O}_{\mathfrak{p}}, 0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, \mathcal{O}_{\mathfrak{p}}, 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d(\zeta_{R,\lambda}(\mathfrak{b}, \mathcal{B}, x, 0))(x). \end{aligned}$$

6.5 A formula for the principal minors of the Gross regulator matrix

As we noted in the Introduction, the construction we give for u_3 is a generalisation of the construction by Dasgupta–Spieß in [14], for the diagonal entry of Gross's regulator matrix. In [14] a formula is given for the minors of the Gross regulator matrix. The simplest case of this is the diagonal entries. In this section, we give this construction and note how our results allow for the application of Theorem 2.3.6 and Theorem 2.3.7 to prove that Dasgupta–Spieß's formula for the diagonal entries holds.

As in §2.2, we let $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}$ be a totally odd character. We recall that we have fixed embeddings $\overline{\mathbb{Q}} \subset \mathbb{C}$ and $\overline{\mathbb{Q}} \subset \mathbb{C}_p$, so χ may be viewed as taking values in $\mathbb{C}$ or $\mathbb{C}_p$. As in §2.2, we let H denote the fixed field of the kernel of χ.

Applying χ^{-1} to the measures κ_λ and $\omega^{\mathfrak{p}}_{\mathfrak{f},\lambda}$, defined in §3.5, gives the measures we require for these constructions. To define this precisely, we first let k denote the cyclotomic field generated by the values of χ. Now let $\mathfrak{P}$ be the prime of k above p corresponding to the embeddings $k \subset \overline{\mathbb{Q}} \subset \mathbb{C}_p$. Let D be a Shintani set and $U \subseteq \mathcal{O}_{\mathfrak{p}}$ be compact open. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define, similar to $\mathfrak{L}$ in (3.7), the Shintani L-function

$$L_R(D, \chi, \mathfrak{b}, U, s) = (\mathrm{N}\mathfrak{b})^{-s} \sum_{\substack{\xi \in D \cap \mathfrak{b}^{-1}_{S_p},\ \xi \in U \\ (\xi, R)=1}} \frac{\chi((\xi))}{\mathrm{N}\xi^s}.$$

Similar to $\mathfrak{L}_{R,\lambda}$, we define

$$L_{R,\lambda}(D, \chi, \mathfrak{b}, U, s) \coloneqq L_R(D, \chi, \mathfrak{b}\lambda^{-1}, U, s) - \chi(\lambda) l^{1-s} L_R(D, \chi, \mathfrak{b}, U, s).$$

Let $x_1, \ldots, x_n \in E_{S_p,+}$. For a compact open subset $U \subset F_{S_p}$, let

$$\mu_{\chi,\mathfrak{b},\lambda}(x_1, \ldots, x_n)(U) \coloneqq \delta(x_1, \ldots, x_n) L_{R,\lambda}(\overline{C}_{e_1}(x_1, \ldots, x_n), \chi, \mathfrak{b}, U, 0).$$

Let K be a finite extension of $\mathbb{Q}_p$ which contains all the values of the character χ. Then $\mu_{\chi,\mathfrak{b},\lambda}$ is an $E_{S_p,+}$-invariant homogeneous $(n-1)$-cocycle yielding a class

$$\kappa_{\chi,\mathfrak{b},\lambda} \coloneqq [\mu_{\chi,\mathfrak{b},\lambda}] \in H^{n-1}(E_{S_p,+}, \mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), K)).$$

Here we are using Remark 3.5.3. We define the Eisenstein cocycle associated to λ and χ by

$$\kappa_{\chi,\lambda} = \sum_{i=1}^{h} \chi(\mathfrak{b}_i) \kappa_{\chi,\mathfrak{b}_i,\lambda} \in H^{n-1}(E_{S_p,+}, \mathrm{Hom}(C_c(F_{S_p}, \mathbb{Z}), K)).$$

Using §3.7, we can define elements $c_{o_\mathfrak{p}}, c_{l_\mathfrak{p}} \in H^1(F^*_\mathfrak{p}, C_c(F_\mathfrak{p}, K))$. The homomorphisms $o_\mathfrak{p}$ and $l_\mathfrak{p}$ are as defined in (2.1) and (2.2). As before, we label the elements of S_p by $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ and let

$J \subset S_p$. We now define classes

$$c_o, c_{l,J} \in H^r(F_{S_p}^*, C_c(F_{S_p}, K))$$

by

$$c_o = c_{o_{\mathfrak{p}_1}} \cup \cdots \cup c_{o_{\mathfrak{p}_r}},$$
$$c_{l,J} = c_{g_1} \cup \cdots \cup c_{g_r},$$

where
$$g_i = \begin{cases} l_{\mathfrak{p}_i} & \text{if } i \in J, \\ o_{\mathfrak{p}_i} & \text{if } i \notin J. \end{cases}$$

Definition 6.5.1 (Equation (17), [14]). *Let $\vartheta \in H_{n+r-1}(E_{S_p,+}, \mathbb{Z})$ be a generator. Then for any subset $J \subset S_p$, we define*

$$\mathscr{R}_p(\chi)_{J,an} \coloneqq (-1)\frac{c_{l,J} \cap (\kappa_{\chi,\lambda} \cap \vartheta)}{c_o \cap (\kappa_{\chi,\lambda} \cap \vartheta)}. \tag{6.10}$$

The "*an*" notation here is only used to distinguish the formula $\mathscr{R}_p(\chi)_{J,\mathrm{an}}$ from the algebraic quantity $\mathscr{R}_p(\chi)_J$. Dasgupta–Spieß conjectured that their formula $\mathscr{R}_p(\chi)_{J,\mathrm{an}}$ is in fact equal to $\mathscr{R}_p(\chi)_J$.

Conjecture 6.5.2 (Conjecture 3.1, [14]). *For each subset $J \subset S_p$, we have $\mathscr{R}_p(\chi)_J = \mathscr{R}_p(\chi)_{J,an}$.*

We now give the second formulation that Dasgupta–Spieß give for $\mathscr{R}_p(\chi)_{J,\mathrm{an}}$. We are first required to generalise our definition of the Shintani zeta function. Let $\mathfrak{b}$ be a fractional ideal of F relatively prime to S and $\overline{\lambda}$, and let D be a Shintani set. For each compact open $U \subseteq F_J$, define, for $\mathrm{Re}(s) > 1$,

$$\zeta_R^J(\mathfrak{b}, D, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in F \cap D,\ \alpha \in U \\ (\alpha, R)=1,\ \alpha \in \mathfrak{b}^{-1} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,J} \\ \alpha \equiv 1 \pmod{\mathfrak{f}}}} \mathrm{N}\alpha^{-s}.$$

We define $\zeta_{R,\lambda}^J(\mathfrak{b}, D, U, s)$ in analogy with (2.4). Let $E_+(\mathfrak{f})_J$ denote the group of J-units of F which are congruent to 1 (mod $\mathfrak{f}$). We note that $E_+(\mathfrak{f})_J$ is free of rank $n+j-1$. For $x_1, \ldots, x_n \in E_+(\mathfrak{f})_J$, a fractional ideal $\mathfrak{b}$ coprime to S and l, and compact open $U \subset F_\mathfrak{p}$, we put

$$\nu_{\mathfrak{b},\lambda}^J(x_1, \ldots, x_n)(U) \coloneqq \delta(x_1, \ldots, x_n)\zeta_{R,\lambda}^J(\mathfrak{b}, \overline{C}_{e_1}(x_1, \ldots, x_n), U, 0)$$

where δ and $\overline{C}_{e_1}(x_1, \ldots, x_n)$ are defined as in (3.6) and (3.5), respectively. Then, $\nu_{\mathfrak{b},\lambda}^J$ is a homogeneous $(n-1)$-cocycle on $E_+(\mathfrak{f})_J$ with values in the space of $\mathbb{Z}$-distributions on F_J. This follows from Theorem 2.6 of [5]. Hence, we have defined a class

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda}^J \coloneqq [\nu_{\mathfrak{b},\lambda}^J] \in H^{n-1}(E_+(\mathfrak{f})_J, \mathrm{Hom}(C_c(F_\mathfrak{p}, \mathbb{Z}), \mathbb{Z})).$$

Here the $\nu_{\mathfrak{b},\lambda}^{J}$ is being viewed as an element of $\mathrm{Hom}(C_c(F_{\mathfrak{p}},\mathbb{Z}),\mathbb{Z})$ via the integration pairing from Remark 3.5.3. We also consider

$$\omega_{\mathfrak{f},\lambda}^{J} = \sum_{[\mathfrak{b}]\in G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle} \chi(\mathfrak{b})\omega_{\mathfrak{f},\mathfrak{b},\lambda}^{J} \in H^{n-1}(E_{+}(\mathfrak{f})_{J}, \mathrm{Hom}(C_c(F_{\mathfrak{p}},\mathbb{Z}),\mathbb{Z}[G])),$$

where the sum ranges over a system of representatives of $G_{\mathfrak{f}}/\langle\mathfrak{p}\rangle$. This construction is adapted from the construction of $\omega_{\mathfrak{f},\lambda}^{J}$ in §3.3 of [14]. Write $J = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_j\}$. We then define

$$c_o^J = c_{o_{\mathfrak{p}_1}} \cup\cdots\cup c_{o_{\mathfrak{p}_j}} \quad \text{and} \quad c_l^J = c_{l_{\mathfrak{p}_1}} \cup\cdots\cup c_{l_{\mathfrak{p}_j}}.$$

Proposition 6.5.3 (Proposition 3.6, [14])**.** *Let $\vartheta' \in H_n(E_{+}(\mathfrak{f})_{\mathfrak{p}},\mathbb{Z})$ be a generator. Then, we have*

$$\mathscr{R}_p(\chi)_{\mathfrak{p},an} = (-1)\frac{c_{l_{\mathfrak{p}}} \cap (\omega_{\chi,\lambda}^{\mathfrak{p}} \cap \vartheta')}{c_{o_{\mathfrak{p}}} \cap (\omega_{\chi,\lambda}^{\mathfrak{p}} \cap \vartheta')},$$

i.e., we have a second formula for $\mathscr{R}_p(\chi)_{\mathfrak{p},an}$.

Using the main result of this thesis (Theorem 2.3.7), we can prove the following theorem.

Theorem 6.5.4. *Conjecture 6.5.2 holds in the case $\#J = 1$.*

Proof. Let $\mathfrak{p} \in S_p$ such that $J = \{\mathfrak{p}\}$. By Theorem 2.3.6 of Dasgupta–Kakde, we have

$$u_1 = u_{\mathfrak{p}} \text{ in } (F_{\mathfrak{p}}^{*}/\mu(F_{\mathfrak{p}}^{*}))\otimes\mathbb{Z}[G].$$

Recall that $\mu(F_{\mathfrak{p}}^{*})$ denotes the roots of unity of $F_{\mathfrak{p}}^{*}$. Theorem 2.3.7 then gives that $u_3 = u_1$. By specialising to χ and applying the maps $l_{\mathfrak{p}}$ and $o_{\mathfrak{p}}$ respectively, we have the result. Here we have used the fact that $l_{\mathfrak{p}}$ and $o_{\mathfrak{p}}$ are trivial on $\mu(F_{\mathfrak{p}}^{*})$. □

Chapter 7

Comparing the Formulas for Cubic Fields

In this chapter we present work from [18] in which we show that $u_1 = u_3$ when F is a cubic field. This result was proved by Dasgupta–Spießin [14, Theorem 4.4] when F is a quadratic field. We note that we change the notation of the proof slightly in this thesis to focus more on the units, whereas in [18] the paper is written to focus more on the diagonal entries of the Gross regulator matrix. In this chapter, we prove the following theorem.

Theorem 7.0.1. *Suppose that F is a totally real field with $[F : \mathbb{Q}] = 3$. Then,*

$$u_1 = u_3.$$

We note that this theorem has been attempted previously by Tsosie in [24]. However, as we show in the appendix, we find a counterexample to the statement of a lemma necessary for his proof, namely, [24, Lemma 2.1.3]. The statement concerns having a nice translation property of Shintani sets, for more detail see Statement A.1.1 in the appendix. The main contribution of this chapter is the methods we develop to recover some control of the translation properties of Shintani sets. This is done in §7.1. We spend the majority of this chapter proving the following theorem.

Theorem 7.0.2. *Suppose that F is a totally real field of degree 3. Let $\sigma \in G$ and let $V \subseteq E_+(\mathfrak{f})$ for a good choice of V (we make this choice precise in §7.1). Then,*

$$u_1(V, \sigma) = u_3'(V, \sigma).$$

We show at the end of §7.2 that Theorem 7.0.2 implies the result of this chapter, Theorem 7.0.1.

7.1 Choosing a Colmez domain

We are required to make a good choice of our free, finite index subgroup $V \subset E_+(\mathfrak{f})$. We recall the definition of a Colmez domain we gave in Definition 3.2.9. We initially follow the ideas of Colmez in [6]. Here, the choice of V is used to give a nice Colmez domain $\mathscr{D}_V$. However, we need to use our choice of V to give us both the existence of a suitable Colmez domain $\mathscr{D}_V$, and to give us some control over the translation of $\mathscr{D}_V$. This approach was not used in [24]. Instead, they used a stronger statement, [24, Lemma 2.1.3]. However, we find a counterexample to this statement. This counterexample is given in the appendix of this thesis. We therefore require a new approach. In this section, we need to restrict to the case when F is a field of degree 3, i.e., we assume $n = 3$ henceforth. Note that in this case $E_+(\mathfrak{f})$ is free of rank 2. The main aim of this section is to prove the following proposition. We remark that currently we have not been able to prove such a proposition for $n > 3$. Thus we have to restrict to the case $n = 3$.

Proposition 7.1.1. *Let $\pi \in F_+$. Then, there exists $\varepsilon_1, \varepsilon_2, \omega \in E_+(\mathfrak{f})$ such that*

1) $\langle \varepsilon_1, \varepsilon_2 \rangle \subseteq E_+(\mathfrak{f})$ *is a finite index subgroup, free of rank 2,*

2) $\delta([\varepsilon_1 \mid \varepsilon_2]) = -\delta([\varepsilon_2 \mid \varepsilon_1]) = 1,$

3) $\delta([\varepsilon_1 \mid \omega\pi]) = -\delta([\omega\pi \mid \varepsilon_1]) = \delta([\varepsilon_2 \mid \omega\pi]) = -\delta([\omega\pi \mid \varepsilon_2]) = 1,$

4) $\omega^{-1}\pi^{-1} \in C([\varepsilon_1 \mid \varepsilon_2]) \cup C([\varepsilon_2 \mid \varepsilon_1]) \cup C(1, \varepsilon_1\varepsilon_2).$

Recall the definition of δ from (3.6). The choices we make using Proposition 7.1.1 allow us to form a nice Colmez domain, and in the process of choosing $\varepsilon_1, \varepsilon_2, \omega$ we also allow ourselves to have some control over the translation of $\mathscr{D}_V$. We note that the hardest part of this proposition is being able to have *3)* and *4)* simultaneously. We recall from §3.6 the definitions of Log, $\mathscr{H}$ and $z_{\mathscr{H}}$ for $z \in \mathbb{R}^3_+$ but with $n = 3$ rather than of arbitrary value. As in §3.6, for any $M > 0$ and $i = 0, 1, 2$, write $l_i(M)$ for the element of $\mathscr{H}$ which has value M in the $(i+1)$ place and $-M/2$ in the other places. We endow $\mathbb{R}^3$ with the sup-norm. We denote by $B(x, r)$ the ball centred at x of radius r.

Note that if we choose $R > R'_1 := \max(1, R(E_+(\mathfrak{f}), \pi))$ in Lemma 3.6.1, then $K_1(R) = \max(2^5 R, k(R))$. The proof of Lemma 3.6.1, when $n = 3$, also gives the following corollary.

Corollary 7.1.2. *Let $R > R'_1$ and $M > 2^5 R$. For $i = 1, 2$, let $g_i \in E_+(\mathfrak{f})$ and $g_\pi \in \pi_{\mathscr{H}} E_+(\mathfrak{f})$ such that* $\operatorname{Log}(g_i) \in B(l_i(M), R) \neq \varnothing$ *and* $\operatorname{Log}(g_\pi) \in B(l_0(M), R) \neq \varnothing$. *Then*

- $\delta([g_1 \mid g_2]) = -\delta([g_2 \mid g_1]) = 1,$
- $\delta([g_1 \mid g_\pi]) = -\delta([g_\pi \mid g_1]) = \delta([g_2 \mid g_\pi]) = -\delta([g_\pi \mid g_2]) = -1.$

In considering this corollary, rather than Lemma 3.6.1, we only lose the condition that the group, generated by g_1, g_2, is free of rank 2. For later use, we let $K'_1(R) = 2^5 R$.

We need to define a projection that depends on elements $g_1, g_2 \in E_+(\mathfrak{f})$ that generate a free group of rank 2 and acts on $(\mathbb{R}^3_+/\sim)$. Here, $x \sim y$ if $\exists \gamma \in \mathbb{R}_+$ such that $x = \gamma y$. We define below $\varphi_{(g_1,g_2)} : (\mathbb{R}^3_+/\sim) \to \mathbb{R}^2$ such that

i) $\varphi_{(g_1,g_2)}(g_1) = (1,0)$ and $\varphi_{(g_1,g_2)}(g_2) = (0,1)$,

ii) for $\alpha, \beta \in \mathbb{R}^3_+$, $\varphi_{(g_1,g_2)}(\alpha\beta) = \varphi_{(g_1,g_2)}(\alpha) + \varphi_{(g_1,g_2)}(\beta)$.

Write $g_1 = (g_1(1), g_1(2), g_1(3))$ and $g_2 = (g_2(1), g_2(2), g_2(3))$. Let $\alpha \in \mathbb{R}^3_+/\sim$ and write $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. We recall from (3.11) the notation $\alpha_{\mathscr{H}} = (\alpha_1\alpha_2\alpha_3)^{-\frac{1}{3}} \cdot \alpha$. Write $\alpha_{\mathscr{H}} = (\alpha_{\mathscr{H},1}, \alpha_{\mathscr{H},2}, \alpha_{\mathscr{H},3})$, we define

$$\varphi_{(g_1,g_2)}(\alpha) := \left(\frac{\log(\alpha_{\mathscr{H},2})\log(g_2(1)) - \log(\alpha_{\mathscr{H},1})\log(g_2(2))}{\log(g_2(1))\log(g_1(2)) - \log(g_2(2))\log(g_1(1))}, \frac{\log(\alpha_{\mathscr{H},2})\log(g_1(1)) - \log(\alpha_{\mathscr{H},1})\log(g_1(2))}{\log(g_1(1))\log(g_2(2)) - \log(g_1(2))\log(g_2(1))} \right). \tag{7.1}$$

Choosing $\langle g_1, g_2 \rangle \subseteq E_+(\mathfrak{f})$ to be of finite index, combined with Dirichlet's unit theorem, gives that the denominators in (7.1) are non-zero and the terms are therefore well defined. This is equivalent to the fact that $\{\mathrm{Log}(g_1), \mathrm{Log}(g_2)\}$ is a basis for $\mathscr{H}$ over $\mathbb{R}$. The idea for the function $\varphi_{(g_1,g_2)}$ comes from the following. We take $\mathrm{Log}(\alpha)$ and then project it onto the hyperplane $\mathscr{H}$ (this is the same as choosing $\alpha_{\mathscr{H}}$). We write the element of $\mathscr{H}$ in terms of the basis $\{\mathrm{Log}(g_1), \mathrm{Log}(g_2)\}$. It is clear from the definition that we have the properties i) and ii) as required.

Now consider $g_1, g_2 \in E_+(\mathfrak{f})$ that satisfy the first two properties of Lemma 3.6.1. We define

$$D(g_1, g_2) = \overline{C}_{e_1}([g_1 \mid g_2]) \cup \overline{C}_{e_1}([g_2 \mid g_1]). \tag{7.2}$$

Since we assume g_1, g_2 satisfy the second property of Lemma 3.6.1, Lemma 3.2.14 gives that $D(g_1, g_2)$ is a Colmez domain for $\langle g_1, g_2 \rangle$. Additionally, we let $\overline{D}(g_1, g_2)$ be the union of $C([g_1 \mid g_2]) \cup C([g_2 \mid g_1])$ with all of their boundary cones. Then, $D(g_1, g_2) \subset \overline{D}(g_1, g_2)$ and they only differ on some of the boundary cones. Consider $\varphi_{(g_1,g_2)}(\overline{D}(g_1, g_2))$. Write

$$\mathscr{C}_1(g_1, g_2) = \varphi_{(g_1,g_2)}(C(1, g_1) \cup C(1) \cup C(g_1)),$$
$$\mathscr{C}_2(g_1, g_2) = \varphi_{(g_1,g_2)}(C(1, g_2) \cup C(1) \cup C(g_2)).$$

Thus, $\varphi_{(g_1,g_2)}(\overline{D}(g_1, g_2))$ is bounded by $\mathscr{C}_1 \cup \mathscr{C}_2 \cup ((0,1) + \mathscr{C}_1) \cup ((1,0) + \mathscr{C}_2)$. We note that $\mathscr{C}_1$ and $\mathscr{C}_2$ are smooth lines in $\mathbb{R}^2$ with an increasing or decreasing derivative. Our next aim is to calculate the derivatives of $\mathscr{C}_1$ and $\mathscr{C}_2$ at their endpoints. For $i = 1, 2$ and $t \in [0,1]$, let $L_i(t)$ be the line from $(1,1,1)$ to $(g_i(1), g_i(2), g_i(3))$. We now calculate the projection of the line $L_i(t)$

under the map $z \mapsto z_{\mathscr{H}}$. Explicitly, we have, for $t \in [0,1]$,

$$L_i(t)_{\mathscr{H}} = \left(\left(\frac{(1+t(g_i(1)-1))^2}{(1+t(g_i(2)-1))(1+t(g_i(3)-1))} \right)^{\frac{1}{3}}, \left(\frac{(1+t(g_i(2)-1))^2}{(1+t(g_i(1)-1))(1+t(g_i(3)-1))} \right)^{\frac{1}{3}}, \left(\frac{(1+t(g_i(3)-1))^2}{(1+t(g_i(1)-1))(1+t(g_i(2)-1))} \right)^{\frac{1}{3}} \right).$$

All the terms in brackets lie in $\mathbb{R}$. We take the cube root in $\mathbb{R}$ so that $L_i(t) \in \mathbb{R}^3$. We define $\mathscr{C}_i(t) = \varphi_{(g_1,g_2)}(L_i(t)) = (x_i(t), y_i(t))$ and using our formula for $L_i(t)_{\mathscr{H}}$, we calculate

$$x_i(t) = \frac{\log\left(\frac{(1+t(g_i(2)-1))^2}{(1+t(g_i(1)-1))(1+t(g_i(3)-1))}\right)\log(g_2(1)) - \log\left(\frac{(1+t(g_i(1)-1))^2}{(1+t(g_i(2)-1))(1+t(g_i(3)-1))}\right)\log(g_2(2))}{3(\log(g_2(1))\log(g_1(2)) - \log(g_2(2))\log(g_1(1)))},$$

$$y_i(t) = \frac{\log\left(\frac{(1+t(g_i(2)-1))^2}{(1+t(g_i(1)-1))(1+t(g_i(3)-1))}\right)\log(g_1(1)) - \log\left(\frac{(1+t(g_i(1)-1))^2}{(1+t(g_i(2)-1))(1+t(g_i(3)-1))}\right)\log(g_1(2))}{3(\log(g_1(1))\log(g_2(2)) - \log(g_1(2))\log(g_2(1)))}.$$

Let $l \geq 1$ be an integer. For $i = 1,2$ and $t \in [0,1]$, let $L_{i,l}(t)$ be the line from $(1,1,1)$ to $(g_i(1)^l, g_i(2)^l, g_i(3)^l)$. Similar to before, we write $\mathscr{C}_{i,l}(t) = \varphi_{(g_1,g_2)}(L_{i,l}(t)) = (x_{i,l}(t), y_{i,l}(t))$. We calculate $\frac{dy_{i,l}(t)}{dx_{i,l}(t)}(t=0)$ and $\frac{dy_{i,l}(t)}{dx_{i,l}(t)}(t=1)$ for $i = 1,2$ and $l \geq 1$.

Lemma 7.1.3. *We have*

$$\frac{dy_{i,l}(t)}{dx_{i,l}(t)}(t=0) = (-1)\frac{(2g_i(2)^l - g_i(1)^l - g_i(3)^l)\log(g_1(1)) - (2g_i(1)^l - g_i(2)^l - g_i(3)^l)\log(g_1(2))}{(2g_i(2)^l - g_i(1)^l - g_i(3)^l)\log(g_2(1)) - (2g_i(1)^l - g_i(2)^l - g_i(3)^l)\log(g_2(2))},$$

and

$$\frac{dy_{i,l}(t)}{dx_{i,l}(t)}(t=1) = (-1)\frac{(2g_i(2)^{-l} - g_i(1)^{-l} - g_i(3)^{-l})\log(g_1(1)) - (2g_i(1)^{-l} - g_i(2)^{-l} - g_i(3)^{-l})\log(g_1(2))}{(2g_i(2)^{-l} - g_i(1)^{-l} - g_i(3)^{-l})\log(g_2(1)) - (2g_i(1)^{-l} - g_i(2)^{-l} - g_i(3)^{-l})\log(g_2(2))}.$$

Proof. The calculation is long but straightforward. L'Hôpital's rule is required in both calculations. □

In Lemma 7.1.4, we show that under conditions on the units g_1, g_2, we have some control over the derivatives of the curves $\mathscr{C}_{1,l}(t)$ and $\mathscr{C}_{2,l}(t)$ at $t = 0$ and $t = 1$ for large enough l. We then show in Lemma 7.1.5 that there exist units as in Lemma 3.6.1 which satisfy these conditions.

Lemma 7.1.4. *Let g_1, g_2 be as above. Assume further that*

- $g_1(2) > g_1(1)^{-2} > g_1(1)^{-1} > 1$, *and*
- $g_2(1) < g_2(2) < 1$.

Then, we have the limits

1)
$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=0) = (-1)\frac{2\log(g_1(1)) + \log(g_1(2))}{2\log(g_2(1)) + \log(g_2(2))} > 0,$$

2)
$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=1) = (-1)\frac{-\log(g_1(1)) + \log(g_1(2))}{-\log(g_2(1)) + \log(g_2(2))} < 0,$$

3)
$$\lim_{l\to\infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=0) = (-1)\frac{-\log(g_1(1)) + \log(g_1(2))}{-\log(g_2(1)) + \log(g_2(2))} < 0,$$

4)
$$\lim_{l\to\infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=1) = (-1)\frac{\log(g_1(1)) + 2\log(g_1(2))}{\log(g_2(1)) + 2\log(g_2(2))} > 0.$$

Proof. We first note that since $g_1, g_2 \in E_+(\mathfrak{f})$ we have $g_i(3) = g_i(1)^{-1}g_i(2)^{-1}$. We work with each statement individually. Considering *1)*, we have

$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=0) = \lim_{l\to\infty}(-1)$$
$$\frac{(2g_1(2)^l - g_1(1)^l - g_1(1)^{-l}g_1(2)^{-l})\log(g_1(1)) - (2g_1(1)^l - g_1(2)^l - g_1(1)^{-l}g_1(2)^{-l})\log(g_1(2))}{(2g_1(2)^l - g_1(1)^l - g_1(1)^{-l}g_1(2)^{-l})\log(g_2(1)) - (2g_1(1)^l - g_1(2)^l - g_1(1)^{-l}g_1(2)^{-l})\log(g_2(2))}.$$

Dividing the numerator and denominator by $g_1(2)^l$, we see that

$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=0) =$$
$$\lim_{l\to\infty}(-1)\frac{(2 - \left(\frac{g_1(1)}{g_1(2)}\right)^l - \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l)\log(g_1(1)) - (2\left(\frac{g_1(1)}{g_1(2)}\right)^l - 1 - \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l)\log(g_1(2))}{(2 - \left(\frac{g_1(1)}{g_1(2)}\right)^l - \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l)\log(g_2(1)) - (2\left(\frac{g_1(1)}{g_1(2)}\right)^l - 1 - \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l)\log(g_2(2))}.$$

Since $g_1(2) > g_1(1)^{-2} > g_1(1)^{-1} > 1$, the fractions $\left(\frac{g_1(1)}{g_1(2)}\right)^l, \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l \to 0$. Hence,

$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=0) = (-1)\frac{2\log(g_1(1)) + \log(g_1(2))}{2\log(g_2(1)) + \log(g_2(2))}.$$

This value is greater than 0 as, from the conditions we assume, $2\log(g_1(1)) + \log(g_1(2)) > 0$ and $2\log(g_2(1)) + \log(g_2(2)) < 0$, thus giving *1)*.

For *2)*, we have

$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=1) = \lim_{l\to\infty}(-1)$$
$$\frac{(2g_1(2)^{-l} - g_1(1)^{-l} - g_1(1)^l g_1(2)^l)\log(g_1(1)) - (2g_1(1)^{-l} - g_1(2)^{-l} - g_1(1)^l g_1(2)^l)\log(g_1(2))}{(2g_1(2)^{-l} - g_1(1)^{-l} - g_1(1)^l g_1(2)^l)\log(g_2(1)) - (2g_1(1)^{-l} - g_1(2)^{-l} - g_1(1)^l g_1(2)^l)\log(g_2(2))}.$$

Multiplying the numerator and denominator by $g_1(1)^{-l} g_1(2)^{-l}$, we see that

$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=1) = \lim_{l\to\infty}(-1)$$
$$\frac{(2\left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l - \left(\frac{g_1(1)^{-2}}{g_1(2)}\right)^l - 1)\log(g_1(1)) - (2\left(\frac{g_1(1)^{-2}}{g_1(2)}\right)^l - \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l - 1)\log(g_1(2))}{(2\left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l - \left(\frac{g_1(1)^{-2}}{g_1(2)}\right)^l - 1)\log(g_2(1)) - (2\left(\frac{g_1(1)^{-2}}{g_1(2)}\right)^l - \left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l - 1)\log(g_2(2))}.$$

Since $g_1(2) > g_1(1)^{-2} > g_1(1)^{-1} > 1$, the fractions $\left(\frac{g_1(1)^{-1}}{g_1(2)^2}\right)^l, \left(\frac{g_1(1)^{-2}}{g_1(2)}\right)^l \to 0$. Hence,

$$\lim_{l\to\infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=1) = (-1)\frac{-\log(g_1(1)) + \log(g_1(2))}{-\log(g_2(1)) + \log(g_2(2))}.$$

From the conditions we assume, $-\log(g_1(1)) + \log(g_1(2)) > 0$ and $-\log(g_2(1)) + \log(g_2(2)) > 0$. Hence, we get the correct sign.

For *3)*, consider $\lim_{l\to\infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=0)$ and multiply the numerator and denominator of the corresponding fraction by $g_2(1)^l g_2(2)^l$. Since $g_2(1)^l, g_2(2)^l \to 0$, we see that

$$\lim_{l\to\infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=0) = (-1)\frac{-\log(g_1(1)) + \log(g_1(2))}{-\log(g_2(1)) + \log(g_2(2))}.$$

From the conditions we assume, $-\log(g_1(1)) + \log(g_1(2)) > 0$ and $-\log(g_2(1)) + \log(g_2(2)) > 0$. Hence, we get the correct sign.

Finally, for *4)*, consider $\lim_{l\to\infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=1)$ and multiply the numerator and denominator of the corresponding fraction by $g_2(1)^l$. Since $g_2(1)^l, g_2(2)^l \to 0$, we see that

$$\lim_{l\to\infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=1) = (-1)\frac{-\log(g_1(1)) - 2\log(g_1(2))}{-\log(g_2(1)) - 2\log(g_2(2))} = (-1)\frac{\log(g_1(1)) + 2\log(g_1(2))}{\log(g_2(1)) + 2\log(g_2(2))}.$$

From the conditions we assume, $\log(g_1(1)) + 2\log(g_1(2)) > 0$ and $\log(g_2(1)) + 2\log(g_2(2)) < 0$. Hence, we get the correct sign.

□

We now show that it is possible to find elements that satisfy the properties in the statement of Lemma 7.1.4. Note that in Lemma 7.1.5 we do not show that g_1, g_2 generate a finite index subgroup in $E_+(\mathfrak{f})$. After the proof of the lemma, we choose r and M to be large enough so that the conditions of Lemma 3.6.1 are satisfied as well.

Lemma 7.1.5. *There exists $R_2 > 0$ such that, for all $R > R_2$ and $M > K_2(R)$ (where $K_2(R)$ is some constant we define which depends only on R), we have the following. For $i = 1, 2$, there exists $g_i \in E_+(\mathfrak{f})$ such that $\mathrm{Log}(g_i) \in B(l_i(M), R)$ and if we write $g_i = (g_i(1), g_i(2), g_i(3))$,*

i) $g_1(2) > g_1(1)^{-2} > g_1(1)^{-1} > 1$,

ii) $g_2(1) < g_2(2) < 1$.

Proof. We only give the proof for g_1 since the proof for g_2 is similar and easier. Recall that $l_1(M) = (-M/2, M, -M/2)$. Since $\mathrm{Log}(E_+(\mathfrak{f}))$ is a lattice inside $\mathscr{H}$, we are able to fix $R_2 > 0$ such that if $R > R_2$ then, for all $M > 0$, there exists $x = (x_1, x_2, x_3) \in E_+(\mathfrak{f})$ such that

- $\mathrm{Log}(x) \in B(l_1(M), R)$,
- $\log(x_1) + \frac{M}{2} > 0$,
- $\log(x_2) - M > 0$.

We let $K_2(R) = 2R$ and impose that $M > K_2(R)$. With this assumption we then have, in addition to the properties above, $\log(x_1) < 0$. The result now follows by noting that *i)* is equivalent to

i′) $\log(g_1(2)) > -2\log(g_1(1)) > -\log(g_1(1)) > 0$.

∎

We fix $r > \max(R_1', R_2, 1)$ and $M_1 > \max(K_1(r), K_2(r), 4K_1'(r))$. We choose $g_1, g_2 \in E_+(\mathfrak{f})$ such that, for $i = 1, 2$, $\mathrm{Log}(g_i) \in B(l_i(M_1), r)$ and satisfies *i)* and *ii)* in the statement of Lemma 7.1.5, respectively. We remark that the reason for taking $4K_1'(r)$ rather than simply $K_1'(r)$ will not be apparent until Lemma 7.1.8. The choices we make here are henceforth fixed. For clarity, we note that under these conditions we have, by Lemma 3.6.1 and Lemma 7.1.5, the existence of $g_1, g_2 \in E_+(\mathfrak{f})$ such that

- $\langle g_1, g_2 \rangle \subseteq E_+(\mathfrak{f})$ is a finite index subgroup, free of rank 2,
- $\delta([g_1 \mid g_2]) = -\delta([g_2 \mid g_1]) = 1$,
- $g_1(2) > g_1(1)^{-2} > g_1(1)^{-1} > 1$,
- $g_2(1) < g_2(2) < 1$.

We fix this choice of g_1 and g_2 for the remainder of the chapter. We now show that when choosing our subgroup V, we are allowed to raise our current choices to positive powers. This enables us to make use of the controls we obtained in Lemma 7.1.4.

Proposition 7.1.6. *For all $l \geq 1$, we have*

1) $\langle g_1^l, g_2^l \rangle \subseteq E_+(\mathfrak{f})$ *is a finite index subgroup, free of rank 2,*

2) $\delta([g_1^l \mid g_2^l]) = -\delta([g_2^l \mid g_1^l]) = 1$.

Proof. Since $\langle g_1, g_2 \rangle$ is free of rank 2 and finite index, we must have that $\langle g_1^l, g_2^l \rangle$ is also free of rank 2 and finite index. Let $i = 1, 2$ and since $\mathrm{Log}(g_i) \in B(l_i(M_1), r)$, we have $\mathrm{Log}(g_i^l) \in B(l_i(M_1 l), rl)$. Thus, $rl \geq r > R_1'$ and $lM_1 > 2^5 rl$. By the work immediately following the statement of Proposition 7.1.1, we therefore get that *2)* holds as well. □

We are now able to use our choices to control the curves $\mathscr{C}_{1,l}(t)$ and $\mathscr{C}_{2,l}(t)$.

Corollary 7.1.7. *There exists $L_1 > 0$ such that for any $l > L_1$,*

i) $y_{1,l}(t) \geq 0$,

ii) $x_{2,l}(t) \leq 0$,

iii) $0 \leq x_{1,l}(t) \leq l$,

iv) $0 \leq y_{2,l}(t) \leq l$,

for all $t \in [0, 1]$.

Proof. By Lemma 7.1.4, there exists $L_1 > 0$ such that for all $l > L_1$

$$\frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t = 0) > 0, \quad \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t = 1) < 0,$$
$$\frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t = 0) < 0, \quad \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t = 1) > 0.$$

We recall the definition of $D(g_1^l, g_2^l)$ from (7.2) and note that from *2)* in Proposition 7.1.6 we have the sign properties required to show that $D(g_1^l, g_2^l)$ forms a fundamental domain for the action of $\langle g_1^l, g_2^l \rangle$ on $\mathbb{R}^3_+$. This follows from [6, Lemma 2.2]. From this we deduce two key properties. Firstly, we have

$$\mathscr{C}_{1,l} \cap ((0, l) + \mathscr{C}_{1,l}) = \varnothing \quad \text{and} \quad \mathscr{C}_{2,l} \cap ((l, 0) + \mathscr{C}_{2,l}) = \varnothing.$$

Secondly, the curves $\mathscr{C}_{1,l}$ and $\mathscr{C}_{2,l}$ can only intersect at the endpoints. More precisely, we have

$$\begin{aligned} \mathscr{C}_{1,l} \cap \mathscr{C}_{2,l} &= \{(0, 0)\}, \\ ((0, l) + \mathscr{C}_{1,l}) \cap \mathscr{C}_{2,l} &= \{(0, l)\}, \\ \mathscr{C}_{1,l} \cap ((l, 0) + \mathscr{C}_{2,l}) &= \{(l, 0)\}, \\ ((0, l) + \mathscr{C}_{1,l}) \cap ((l, 0) + \mathscr{C}_{2,l}) &= \{(l, l)\}. \end{aligned}$$

Henceforth, we choose $l > L_1$. Note that the map $\varphi_{(g_1,g_2)}$ is equivalent to taking a projection followed by the Log map, followed by a base change. It therefore maps straight lines in $\mathbb{R}^3_+$, which are not contained in rays, to continuous strictly convex curves in $\mathbb{R}^2$. We have strictly convex curves as we can never obtain straight lines in $\mathbb{R}^2$ from straight lines in $\mathbb{R}^3_+$ that are not contained in rays. More precisely, let $\gamma(t)$, $t \in [0,1]$, be any straight line of finite length in $\mathbb{R}^3_+$ where $\gamma(0)$ and $\gamma(1)$ are not both lying on the same ray. Then, we have

$$\{\varphi_{(g_1,g_2)}(\gamma(0)) + k(\varphi_{(g_1,g_2)}(\gamma(1)) - \varphi_{(g_1,g_2)}(\gamma(0))) \mid k \in [0,1]\} \cap \{\varphi_{(g_1,g_2)}(\gamma(t)) \mid t \in [0,1]\}$$
$$= \{\varphi_{(g_1,g_2)}(\gamma(0)), \varphi_{(g_1,g_2)}(\gamma(1))\}.$$

We first show *ii)*. Proceeding by contradiction, we suppose that $x_{2,l}(T) > 0$ for some $T \in [0,1]$. Since $\mathscr{C}_{2,l}$ is strictly convex and contains the points $(0,0)$ and $(0,l)$, we deduce that $x_{2,l}(t) \geq 0$ for all $t \in [0,1]$. Since

$$\frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=0) < 0 \quad \text{and} \quad \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t=1) > 0,$$

there exist $T_1, T_2 \in [0,1]$ such that $y_{2,l}(T_1) < 0$ and $y_{2,l}(T_2) > l$.

Consider $\mathscr{C}_{1,l}$. Since $\frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=0) > 0$ and $\mathscr{C}_{1,l}$ is strictly convex, we must have that $y_{1,l}(t) \leq 0$ for all $t \in [0,1]$. Note that if we had $y_{1,l}(t) > 0$ for some $t \in [0,1]$ then $\mathscr{C}_{1,l}$ and $\mathscr{C}_{2,l}$ would intersect on at least one point other than $(0,0)$.

We now consider the curve $(0,l)+\mathscr{C}_{1,l}$. Since we have $\mathscr{C}_{2,l} \cap ((0,l)+\mathscr{C}_{1,l}) = \{(0,l)\}$, $y_{1,l}(t) \leq 0$ for all $t \in [0,1]$ and the existence of T_1, there exists $K \in [0,1]$ such that

- $l + y_{1,l}(K) < 0$,
- $x_{1,l}(K) = 0$, and
- $x_{1,l}(t) \leq 0$ for all $t \in [0,K]$.

These three conditions imply that $\mathscr{C}_{1,l} \cap ((0,l)+\mathscr{C}_{1,l}) \neq \varnothing$ which is a contradiction. This gives a contradiction to the existence of $T \in [0,1]$ such that $x_{2,l}(T) > 0$. Hence, we have that $x_{2,l}(t) \leq 0$ for all $t \in [0,1]$ and so *ii)* holds.

To prove *i)* we again work by contradiction and suppose that $y_{1,l}(T) < 0$ for some $T \in [0,1]$. As before, we deduce that $y_{1,l}(t) \leq 0$ for all $t \in [0,1]$. Since

$$\frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=0) > 0 \quad \text{and} \quad \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t=1) < 0,$$

there exist $T_1, T_2 \in [0,1]$ such that $x_{1,l}(T_1) < 0$ and $x_{1,l}(T_2) > l$. As before, we consider the curve $(0,l)+\mathscr{C}_{1,l}$. Using a similar argument as above, we are able to show that $\mathscr{C}_{1,l} \cap ((0,l)+\mathscr{C}_{1,l}) \neq \varnothing$. This contradiction then gives us that *i)* holds.

From what we deduced about the derivatives and the fact that the first two statements hold, it is clear that *iii)* and *iv)* must also hold. □

The results of Corollary 7.1.7, combined with the fact that $\mathscr{C}_{1,l}$ and $\mathscr{C}_{2,l}$ are strictly convex curves, gives us that the image of $\mathscr{C}_{1,l} \cup \mathscr{C}_{2,l} \cup ((0,l) + \mathscr{C}_{1,l}) \cup ((l,0) + \mathscr{C}_{2,l})$ is always in a similar form to the following example. Note that in the image below we choose an example where we can take $l = 1$. Throughout the following proofs, one should try to keep the image below in mind. We give more details on the explicit choices and calculations needed to form this image in the appendix. Although the image appears to show that the lines $\mathscr{C}_{1,l}$ and $(l,0) + \mathscr{C}_{2,l}$ overlap in the bottom right corner, this in fact does not happen. This only appears in the diagram due to the fixed thickness of the lines.

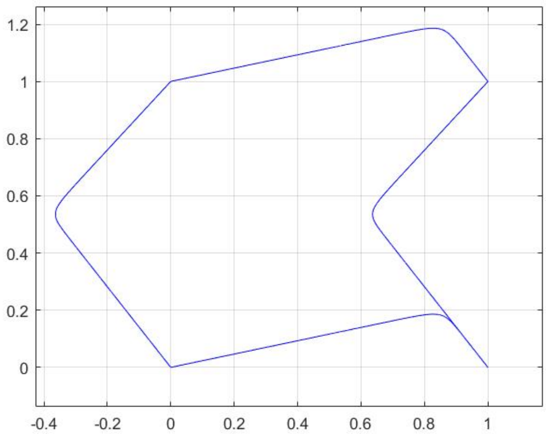


Figure 7.1: A Colmez domain chosen as in Corollary 7.1.7.

Using the corollary above, the next lemma shows that we are now able to find an element of $\pi_{\mathscr{H}}^{-1} E_+(\mathfrak{f})$ which satisfies properties similar to *3)* and *4)* of Proposition 7.1.1. Note that the element we find in the next lemma directly gives rise to an element which satisfies *3)* and *4)* of Proposition 7.1.1.

Lemma 7.1.8. *There exists $L_2 > 0$ such that for all $l > \max(L_1, L_2)$, there exists $\alpha \in \pi_{\mathscr{H}}^{-1} E_+(\mathfrak{f})$ such that*

- $\alpha \in C([g_1^l \mid g_2^l]) \cup C([g_2^l \mid g_1^l]) \cup C(1, g_1^l g_2^l)$,
- $\operatorname{Log}(\alpha) \in B(-l_0(lM_1), 4lr)$.

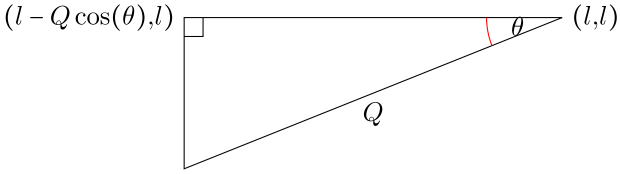
Proof. We assume that $l > L_1$. By Lemma 7.1.4, we have the limit

$$d_2 = \lim_{l \to \infty} \frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t = 1) > 0.$$

Then, there exists $L_2' > 0$ such that for all $l > L_2'$,

$$\frac{dy_{2,l}(t)}{dx_{2,l}(t)}(t = 1) > \frac{d_2}{2}.$$

Let $\theta = \arctan(d_2/2) > 0$, and for $Q > 0$ define $T(\theta, Q, (l,l))$ to be the triangle drawn below.



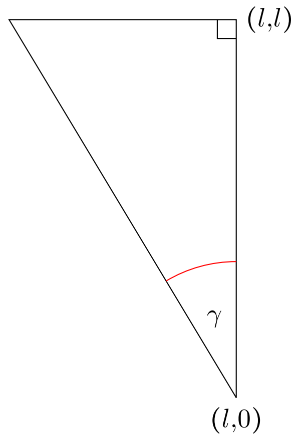
We choose Q big enough such that for all $l > 0$, there exists $\alpha \in \pi_{\mathscr{H}}^{-1} E_+(\mathfrak{f}) \cap T(\theta, Q, (l,l))$. As seen in the proof of Lemma 3.6.1, the existence of such a Q follows from Dirichlet's unit theorem and, in particular, the non-vanishing of the regulator of a number field. The next idea of the proof is to make l big enough such that the triangle $T(\theta, Q, (l,l))$ is guaranteed to be contained inside $C([g_1^l \mid g_2^l]) \cup C([g_2^l \mid g_1^l]) \cup C(1, g_1^l g_2^l)$. The triangle is chosen such that, for all $l > L_2'$, it lies to the left of the curve $(l,0) + \mathscr{C}_{2,l}$. Again by Lemma 7.1.4, we have the limit

$$d_1 = \lim_{l \to \infty} \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t = 1) < 0.$$

Then, there exists $L_2'' > 0$ such that for all $l > L_2''$,

$$0 > \frac{dy_{1,l}(t)}{dx_{1,l}(t)}(t = 1) > \frac{d_1}{2}.$$

Note that the first inequality above follows from our assumption that $l > L_1$. Let $\gamma = -\arctan(d_1/2) > 0$ and define $T(\gamma, (l,l))$ to be the triangle drawn below.



Note that for all $l > \max(L_2', L_2'')$, we have $T(\theta, Q, (l,l)) \cap T(\gamma, (l,l)) \subset C([g_1^l \mid g_2^l]) \cup C([g_2^l \mid g_1^l]) \cup C(1, g_1^l g_2^l)$. Since the size of $T(\theta, Q, (l,l))$ is fixed, there exists L_2''' such that for all $l > L_2'''$, $T(\theta, Q, (l,l)) \subset T(\gamma, (l,l))$. Thus, if we choose $\widetilde{L_2} = \max(L_2', L_2'', L_2''')$, then for $l > \widetilde{L_2}$, there exists $\alpha \in \pi_{\mathscr{H}}^{-1} E_+(\mathfrak{f})$ such that $\alpha \in C([g_1^l \mid g_2^l]) \cup C([g_2^l \mid g_1^l]) \cup C(1, g_1^l g_2^l)$. Since $\mathrm{Log}(g_i) \in B(l_i(M_1), r)$, we have $\mathrm{Log}(g_1^l g_2^l) \in B(-l_0(lM_1), 2lr)$. The size of the triangle $T(\theta, Q, (l,l))$ is fixed and always has a point at (l,l). It is therefore clear that for l big enough (say $l > \widetilde{L_2}'$) the pre-image of the triangle before the change of basis is contained in $B(-l_0(lM_1), 4lr)$. Note that we achieved this by simply doubling the radius of the ball. We finish by setting $L_2 = \max(\widetilde{L_2}, \widetilde{L_2}')$ to ensure that we obtain all the required conditions. □

We are now ready to prove the proposition we stated at the start of this section.

Proof of Proposition 7.1.1. Let $l > \max(L_1, L_2)$, and write $\varepsilon_i = g_i^l$ for $i = 1,2$. By Proposition 7.1.6, we get *1)* and *2)* in Proposition 7.1.1. By Lemma 7.1.8, there exists $\alpha \in \pi_{\mathscr{H}}^{-1} E_+(\mathfrak{f})$ such that

- $\alpha \in C([g_1^l \mid g_2^l]) \cup C([g_2^l \mid g_1^l]) \cup C(1, g_1^l g_2^l)$,
- $\operatorname{Log}(\alpha) \in B(-l_0(lM_1), 4lr)$.

We then define $\omega = \alpha^{-1}\pi_{\mathscr{H}}^{-1} \in E_+(\mathfrak{f})$. Since $\alpha = \pi_{\mathscr{H}}^{-1}\omega^{-1} = k \cdot \pi^{-1}\omega^{-1}$ for some $k \in \mathbb{R}_{>0}$, in the second equality we consider the elements as vectors in $\mathbb{R}_+^3$. Hence, we have

$$\omega^{-1}\pi^{-1} \in C([g_1^l \mid g_2^l]) \cup C([g_2^l \mid g_1^l]) \cup C(1, g_1^l g_2^l) \subset \overline{C}_{e_1}([\varepsilon_1 \mid \varepsilon_2]) \cup \overline{C}_{e_1}([\varepsilon_2 \mid \varepsilon_1]).$$

Thus, we obtain *4)* of the proposition. Now, let $g_\pi = \alpha^{-1} = \pi_{\mathscr{H}}\omega$. Then,

$$\operatorname{Log}(g_\pi) \in B(l_0(lM_1), 4lr).$$

Since $M_1 > 4K_1'(r) = 4 \cdot 2^5 r$, we have $lM_1 > K_1'(4lr)$. Thus, by Lemma 3.6.1, we obtain *3)*. This completes the proof of the proposition. □

We fix the choice of $\varepsilon_1, \varepsilon_2$ and, for ease of notation, write $\pi = \omega\pi$, as is prescribed by Proposition 7.1.1. We assume, in addition to the properties given by Proposition 3.6.1, that $\langle \varepsilon_1, \varepsilon_2 \rangle \cong \mathbb{Z}/b_1\mathbb{Z} \times \mathbb{Z}/b_1\mathbb{Z}$ with b_1, b_2 large enough to satisfy the conditions required in Proposition 4.2.2. This is achieved by simply choosing a larger l than in the proof of Proposition 7.1.1, if required. Let

$$\mathscr{B} := \overline{C}_{e_1}([\varepsilon_1 \mid \varepsilon_2]) \cup \overline{C}_{e_1}([\varepsilon_2 \mid \varepsilon_1]).$$

By *2)* of Proposition 7.1.1 and Lemma 3.2.14, this is a Colmez domain for $\langle \varepsilon_1, \varepsilon_2 \rangle$. We also define

$$\mathscr{B}_1 := \overline{C}_{e_1}([\varepsilon_2 \mid \pi]) \cup \overline{C}_{e_1}([\pi \mid \varepsilon_2]),$$
$$\mathscr{B}_2 := \overline{C}_{e_1}([\varepsilon_1 \mid \pi]) \cup \overline{C}_{e_1}([\pi \mid \varepsilon_1]).$$

Then by *3)* of Proposition 7.1.1, $\mathscr{B}_1$ is a fundamental domain for the action of $\langle \varepsilon_2, \pi \rangle$ on $\mathbb{R}_+^3$ and $\mathscr{B}_2$ is a fundamental domain for the action of $\langle \varepsilon_1, \pi \rangle$ on $\mathbb{R}_+^3$. We are now ready to show that, through our choice of $\varepsilon_1, \varepsilon_2$ and π, we can obtain control over the π^{-1} translate of $\mathscr{B}$.

Proposition 7.1.9. *With the choice of π fixed as before, we have*

$$\pi^{-1}\mathscr{B} \subset \bigcup_{k_1=0}^{1} \bigcup_{k_2=0}^{2} \varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}.$$

Remark 7.1.10. *The purpose of the careful choice of ε_1 and ε_2 is to obtain this proposition. In [24], a stronger statement than this is used, [24, Lemma 2.1.3]. However, as stated before, we obtain a counterexample to this. This counterexample is given explicitly in the appendix. We note that in the appendix we also give a conjecture which predicts the existence of units such that a statement similar to [24, Lemma 2.1.3] can hold.*

Proof of Proposition 7.1.9. The result of Proposition 7.1.9 follows from the proof of containments below.

i) $\pi^{-1}C(1,\varepsilon_1) \subset \mathscr{B} \cup \varepsilon_1\mathscr{B} \cup \varepsilon_2\mathscr{B} \cup \varepsilon_1\varepsilon_2\mathscr{B}$,

ii) $\pi^{-1}C(1,\varepsilon_2) \subset \mathscr{B} \cup \varepsilon_2\mathscr{B}$.

It is enough to show i) and ii) since there are no holes in $\bigcup_{k_1=0}^{1}\bigcup_{k_2=0}^{2}\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}$. Thus, if we can show that the boundary of $\overline{\mathscr{B}}$ lies in $\bigcup_{k_1=0}^{1}\bigcup_{k_2=0}^{2}\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}$, then we are done. The combination of i) and ii) gives us exactly this.

We begin with i). We consider the curves under our map $\varphi_{(g_1,g_2)}$. Throughout this proof, we refer to the positive second coordinate as "up", the positive first coordinate as "right", and similarly for "down" and "left". Since π^{-1} is chosen to be in the interior of $\mathscr{B}$, and by Corollary 7.1.7, we must have that $\varphi_{(g_1,g_2)}(\pi^{-1})$ lies above $\mathscr{C}_{1,l}$ in $\mathbb{R}_2$. Since the curve $\mathscr{C}_{1,l}$ is strictly convex, as defined before, we see that the curve

$$\varphi_{(g_1,g_2)}(\pi^{-1}) + \mathscr{C}_{1,l} \quad \text{lies above} \quad \bigcup_{k\in\mathbb{Z}}((kl,0) + \mathscr{C}_{1,l}).$$

By *2)* of Proposition 7.1.1, $\mathscr{B}$ forms a fundamental domain. From this, it follows that $\mathscr{C}_{1,l}$ must lie between $\bigcup_{k\in\mathbb{Z}}((0,kl) + \mathscr{C}_{2,l})$ and $\bigcup_{k\in\mathbb{Z}}((l,kl) + \mathscr{C}_{2,l})$. Hence,

$$\bigcup_{k\in\mathbb{Z}}((0,kl) + \mathscr{C}_{2,l}) \text{ is to the left of } \varphi_{(g_1,g_2)}(\pi^{-1}) + \mathscr{C}_{1,l} \text{ is to the left of } \bigcup_{k\in\mathbb{Z}}((2l,kl) + \mathscr{C}_{2,l}).$$

At this point, we have shown that

$$\pi^{-1}C(1,\varepsilon_1) \subset \bigcup_{k_2\geq 0}\varepsilon_2^{k_2}(\mathscr{B} \cup \varepsilon_1\mathscr{B}).$$

Now, suppose that $\pi^{-1}C(1,\varepsilon_1) \cap \varepsilon_2^2(\mathscr{B} \cup \varepsilon_1\mathscr{B}) \neq \varnothing$. This means that after moving back to $\mathbb{R}^2$ we see that there exists a point on $\mathscr{C}_{1,l}$ whose value in the second component is greater than 1. Consider the cone $C(1,\pi^{-1}\varepsilon)$. By *3)* of Proposition 7.1.1, we have that $\mathscr{B}_2$ is well defined, and thus $\pi^{-1}\mathscr{B}_2$ is also well defined. Hence, in $\mathbb{R}^2$ we must have that $\varphi_{(g_1,g_2)}(C(1,\pi^{-1}\varepsilon))$ is above $\mathscr{C}_{1,l}$ but also passes below $\varphi_{(g_1,g_2)}(\pi^{-1})$. Yet, since there exists a point on $\mathscr{C}_{1,l}$ whose second component has value greater than 1, the curve $\varphi_{(g_1,g_2)}(C(1,\pi^{-1}\varepsilon))$ cannot be strictly convex. This gives us a contradiction. Hence, i) holds.

For ii), we use similar methods as above to deduce that

$$\bigcup_{k\in\mathbb{Z}}((kl,0)+\mathscr{C}_{1,l}) \quad \text{is below} \quad \varphi_{(g_1,g_2)}(\pi^{-1})+\mathscr{C}_{2,l} \quad \text{is below} \quad \bigcup_{k\in\mathbb{Z}}((kl,2l)+\mathscr{C}_{1,l}).$$

Using Corollary 7.1.7, we have

$$\pi^{-1}C(1,\varepsilon_2)\subset\bigcup_{k_1\leq 0}\varepsilon_1^{k_1}(\mathscr{B}\cup\varepsilon_2\mathscr{B}).$$

As before, we then use *3)* of Proposition 7.1.1 to deduce that $C(1,\varepsilon_2)\cap\pi^{-1}C(1,\varepsilon_2)=\varnothing$. This allows us to conclude. □

Remark 7.1.11. *We remark here that for some choices of π, ε_1 and ε_2 we have the stronger inclusion*

$$\pi^{-1}\mathscr{B}\subset\bigcup_{k_1=0}^{1}\bigcup_{k_2=0}^{1}\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}.$$

In the next section, we need to divide into these two cases. In this section we include examples of how each case can look to aid the reader when considering our proofs.

7.2 Explicit calculations

Let $V=\langle\varepsilon_1,\varepsilon_2\rangle$, where $\varepsilon_1,\varepsilon_2$ are as chosen before and write $\varepsilon_3=\pi$. Before continuing we are required to choose an auxiliary prime λ such that

- λ is π-good for $\mathscr{B}$ and $\mathscr{D}_V$, where $\mathscr{D}_V$ is as defined in Proposition 4.2.2,
- λ is good for $(\mathscr{D}_V,\mathscr{B})$.

In [8] (after Definition 3.16), Dasgupta notes that given a Shintani domain D all but finitely many prime ideals η of F, with $\mathrm{N}\eta$ prime, are π-good for D. In particular, Dasgupta notes that the set of such primes has Dirichlet density 1. Again in [8] (after the proof of Theorem 5.3), Dasgupta notes that for any pair of Shintani domain (D,D') all but finitely many prime ideals η of F, with $\mathrm{N}\eta$ prime, are good for D.

It follows that there are an infinite number of primes λ which satisfy the properties written above. Note that moving from a Shintai domain to a Colmez domain does not cause any issues here. Hence, such a choice of λ is always possible. We fix this choice of λ henceforth. Proposition 4.2.1 implies

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{B})=u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}_V).$$

To prove Theorem 7.0.2, we show, for our choice of V and therefore $\mathscr{B}$, that

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{B}))=c_{\mathrm{id}}\cap(\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}\cap\vartheta'_V).$$

We show the above equality by explicitly calculating each side. In §6.4, we calculated that

$$c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta'_V) = \prod_{i=1}^{2} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathbb{O}_{\mathfrak{p}},0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B},x,0))(x).$$

We recall the definition

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{B})) = \prod_{\epsilon\in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)} \pi^{\zeta_{R,\lambda}(H/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b},\mathscr{B},x) \in F_{\mathfrak{p}}^*.$$

Thus, since $\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B},\mathbb{O}_{\mathfrak{p}},0) = \zeta_{R,\lambda}(H/F,\mathfrak{b},0)$, it only remains for us to prove the equality

$$\prod_{i=1}^{2} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathbb{O}_{\mathfrak{p}},0)} = \prod_{\epsilon\in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)}.$$

By Proposition 7.1.9, we have

$$\prod_{\epsilon\in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)} = \varepsilon_1^{\sum_{k_2=0}^{2} \zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)} \varepsilon_2^{\sum_{k_2=1}^{2}\sum_{k_1=0}^{1} k_2\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)}.$$

Thus, it remains for us to show that the following two equalities hold.

$$\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B}_1,\pi\mathbb{O}_{\mathfrak{p}},0) = \sum_{k_2=0}^{2} \zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0), \tag{7.3}$$

$$\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B}_2,\pi\mathbb{O}_{\mathfrak{p}},0) = \sum_{k_2=1}^{2}\sum_{k_1=0}^{1} k_2\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0). \tag{7.4}$$

We begin by considering the left hand side and note that for $i = 1, 2$ by Proposition 3.2.13,

$$\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathbb{O}_{\mathfrak{p}},0) = \zeta_{R,\lambda}(\mathfrak{b},\pi^{-1}\mathscr{B}_i,\mathbb{O}_{\mathfrak{p}},0).$$

It is useful for our remaining calculations to make explicit the boundary cones that are contained in $\mathscr{B}$, $\mathscr{B}_1$ and $\mathscr{B}_2$. To achieve this, we first define

$$\mathscr{B}' = C(1) \cup C(1,\varepsilon_1) \cup C(1,\varepsilon_2) \cup C(1,\varepsilon_1\varepsilon_2) \cup C(1,\varepsilon_1,\varepsilon_1\varepsilon_2) \cup C(1,\varepsilon_2,\varepsilon_1\varepsilon_2).$$

By Lemma 3.2.13 and the fact that $\mathscr{B}$ and $\mathscr{B}'$ are equal up to translation of the boundary cones by $E_+(\mathfrak{f})$, we note that for any $k_1, k_2 \in \{0,1,2\}$ we have

$$\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_{\mathfrak{p}},0) = \zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}'\cap\pi^{-1}\mathscr{B}',\mathbb{O}_{\mathfrak{p}},0).$$

Here we are also making use of the fact that in Proposition 7.1.9 we made no assumptions about the boundary cones of $\mathscr{B}$. Thus, we henceforth assume that $\mathscr{B} = \mathscr{B}'$. We now consider $\mathscr{B}_1$ and

$\mathscr{B}_2$. For $a, b, c \in \{0,1\}$, we define the Shintani sets

$$\mathscr{B}_1'(a,b) = C(\pi^a) \cup C(\pi^b, \varepsilon_2\pi^b) \cup C(1,\pi) \cup C(1,\varepsilon_2\pi) \cup C(1,\varepsilon_2,\varepsilon_2\pi) \cup C(1,\pi,\varepsilon_2\pi),$$
$$\mathscr{B}_2'(a,b) = C(\pi^a) \cup C(\pi^b, \varepsilon_1\pi^b) \cup C(1,\pi) \cup C(1,\varepsilon_1\pi) \cup C(1,\varepsilon_1,\varepsilon_1\pi) \cup C(1,\pi,\varepsilon_1\pi).$$

By the definition of $\mathscr{B}_i$, for $i = 1, 2$, there exists $a_i, b_i \in \{0,1\}$ such that $\mathscr{B}_i$ and $\mathscr{B}_i'(a_i, b_i)$ are equal up to a translation of the boundary cones by $E_+(\mathfrak{f})$. Thus, by Lemma 3.2.13 we have the equalities

$$\zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_1, \mathcal{O}_\mathfrak{p}, 0) = \zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_1'(a_1,b_1), \mathcal{O}_\mathfrak{p}, 0),$$

and

$$\zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_2, \mathcal{O}_\mathfrak{p}, 0) = \zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_2'(a_2,b_2), \mathcal{O}_\mathfrak{p}, 0).$$

Henceforth, we assume that $a_i = b_i = 1$ for $i = 1, 2$ and write $\mathscr{B}_i = \mathscr{B}_i'(1,1)$ for $i = 1, 2$. The proof of our main result in all other cases follows with exactly the same ideas and the calculations are almost identical. Hence, we fix the choices of $\mathscr{B}$, $\mathscr{B}_1$ and $\mathscr{B}_2$ we have made. Note that we can make the same choice of $\mathscr{B}$ in all cases. We now recall that from this point on we assumed

$$\mathscr{B} = C(1) \cup C(1,\varepsilon_1) \cup C(1,\varepsilon_2) \cup C(1,\varepsilon_1\varepsilon_2) \cup C(1,\varepsilon_1,\varepsilon_1\varepsilon_2) \cup C(1,\varepsilon_2,\varepsilon_1\varepsilon_2),$$
$$\mathscr{B}_1 = C(\pi) \cup C(\pi,\varepsilon_2\pi) \cup C(1,\pi) \cup C(1,\varepsilon_2\pi) \cup C(1,\varepsilon_2,\varepsilon_2\pi) \cup C(1,\pi,\varepsilon_2\pi),$$
$$\mathscr{B}_2 = C(\pi) \cup C(\pi,\varepsilon_1\pi) \cup C(1,\pi) \cup C(1,\varepsilon_1\pi) \cup C(1,\varepsilon_1,\varepsilon_1\pi) \cup C(1,\pi,\varepsilon_1\pi).$$

With these choices, we now show that the equalities (7.3) and (7.4) hold. We begin with the following simple lemma.

Lemma 7.2.1. *We have the following inclusions*

$$\pi^{-1}\mathscr{B}_1 \subset \mathscr{B} \cup \varepsilon_2\mathscr{B},$$
$$\pi^{-1}\mathscr{B}_2 \subset \bigcup_{k_1=0}^{1}\bigcup_{k_2=0}^{1} \varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}.$$

Proof. We begin by considering $\mathscr{B}_1$. By definition we have that $\pi^{-1}\mathscr{B}_1$ is bounded by the cones

$$C(1), C(\pi^{-1}), C(\varepsilon_2), C(\varepsilon_2\pi^{-1}), C(1,\varepsilon_2), C(1,\pi^{-1}), C(\varepsilon_2,\varepsilon_2\pi^{-1}), C(\pi^{-1},\varepsilon_2\pi^{-1}).$$

Note that not all of the above cones are contained in $\pi^{-1}\mathscr{B}_1$. By the definition of $\mathscr{B}$ and the fact that $\pi^{-1} \in \mathscr{B}$, we see that all of the following Shintani cones are contained in $\mathscr{B} \cup \varepsilon_2\mathscr{B}$,

$$C(1), C(\pi^{-1}), C(\varepsilon_2), C(\varepsilon_2\pi^{-1}), C(1,\varepsilon_2), C(1,\pi^{-1}), C(\varepsilon_2,\varepsilon_2\pi^{-1}).$$

It remains for us to show that $C(\pi^{-1},\varepsilon_2\pi^{-1}) \subset \mathscr{B} \cup \varepsilon_2\mathscr{B}$. Since $C(\pi^{-1},\varepsilon_2\pi^{-1})$ and

$C(\varepsilon_1\pi^{-1}, \varepsilon_1\varepsilon_2\pi^{-1})$ are boundary cones for $\pi^{-1}\mathscr{B}$, Proposition 7.1.9 gives the inclusions

$$C(\pi^{-1}, \varepsilon_2\pi^{-1}) \subset \bigcup_{k_1=0}^{1} \bigcup_{k_2=0}^{2} \varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B},$$
$$C(\varepsilon_1\pi^{-1}, \varepsilon_1\varepsilon_2\pi^{-1}) \subset \bigcup_{k_1=0}^{1} \bigcup_{k_2=0}^{2} \varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}.$$

These inclusions together imply that

$$C(\pi^{-1}, \varepsilon_2\pi^{-1}) \subset \bigcup_{k_2=0}^{2} \varepsilon_2^{k_2}\mathscr{B}.$$

If we write $\varphi_{(g_1,g_2)}(\pi^{-1}) = (a, b)$ then, by the choices made in Lemma 7.1.8, we see that $b < l$. Hence, by Corollary 7.1.7, the curve $\varphi_{(g_1,g_2)}(C(\pi^{-1}, \varepsilon_2\pi^{-1})) = \varphi_{(g_1,g_2)}(\pi^{-1}) + \mathscr{C}_{2,l}$ lies strictly below the curve $(0, 2l) + \mathscr{C}_{2,l}$, while still being contained in $\bigcup_{k_2=0}^{2} \varepsilon_2^{k_2}\mathscr{B}$. Hence, we have $C(\pi^{-1}, \varepsilon_2\pi^{-1}) \subset \mathscr{B} \cup \varepsilon_2\mathscr{B}$. This gives us the result for $\mathscr{B}_1$.

The proof of the result for $\mathscr{B}_2$ is almost identical. As before, we use Proposition 7.1.9 to deal with the cone $C(\pi^{-1}, \varepsilon_1\pi^{-1})$. □

Using the above lemma, we deduce

$$\zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_1, \mathcal{O}_\mathfrak{p}, 0) = \zeta_{R,\lambda}(\mathfrak{b}, (\pi^{-1}\mathscr{B}_1 \cap \mathscr{B}) \cup \varepsilon_2^{-1}(\pi^{-1}\mathscr{B}_1 \cap \varepsilon_2\mathscr{B}), \mathcal{O}_\mathfrak{p}, 0)$$

and

$$\begin{aligned}\zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_2, \mathcal{O}_\mathfrak{p}, 0) = \zeta_{R,\lambda}(\mathfrak{b}, (\pi^{-1}\mathscr{B}_2 \cap \mathscr{B}) \cup \varepsilon_1^{-1}(\pi^{-1}\mathscr{B}_2 \cap \varepsilon_1\mathscr{B}), \mathcal{O}_\mathfrak{p}, 0)\\ + \zeta_{R,\lambda}(\mathfrak{b}, \pi^{-1}\mathscr{B}_2 \cap (\varepsilon_2\mathscr{B} \cup \varepsilon_1\varepsilon_2\mathscr{B}), \mathcal{O}_\mathfrak{p}, 0).\end{aligned}$$

We now need to consider two possible cases. It is possible that the final zeta function in the sum above is 0. This will happen when, as noted in Remark 7.1.11, we have the stronger inclusion

$$\pi^{-1}\mathscr{B} \subset \bigcup_{k_1=0}^{1} \bigcup_{k_2=0}^{1} \varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B},$$

rather than that which is written in the statement of Proposition 7.1.9. We note that in this case, the sums on the right hand side of (7.3) and (7.4) become

$$\sum_{k_2=0}^{1} \zeta_{R,\lambda}(\mathfrak{b}, \varepsilon_1\varepsilon_2^{k_2}\mathscr{B} \cap \pi^{-1}\mathscr{B}, \mathcal{O}_\mathfrak{p}, 0)$$

and

$$\sum_{k_1=0}^{1} \zeta_{R,\lambda}(\mathfrak{b}, \varepsilon_1^{k_1}\varepsilon_2\mathscr{B} \cap \pi^{-1}\mathscr{B}, \mathcal{O}_\mathfrak{p}, 0),$$

respectively.

In the following proposition, we need to divide the proof into two cases to deal with this possibility. In the case of the stronger inclusion, the following proposition completes the proof of Theorem 7.0.2. We refer to the case of the stronger inclusion as Case 1 and the other as Case 2. We now include two pictures showing how Case 1 and Case 2 can arise in the example from before by making different choices of π. Note that we can choose π up to a factor of $E_+(\mathfrak{f})$. Both these diagrams are calculated by making explicit choices. As before we give more details in the appendix. In each of the diagrams the blue lines are boundary cones of the translates of $\mathscr{B}$ required in each case, and the red lines are the boundary cones of $\pi^{-1}\mathscr{B}$ for each choice of π. We note as before that although the image appears to show that some of the lines overlap, this does not happen. This only occurs in the diagram due to the fixed thickness of the lines.

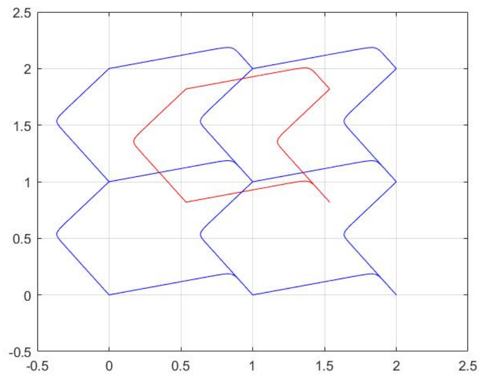


Figure 7.2: Case 1

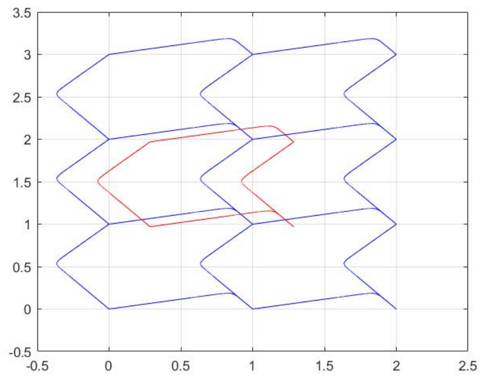


Figure 7.3: Case 2

Remark 7.2.2. *Figure 7.3, which concerns Case 2, is not chosen by the methods outlined in Lemma 7.1.8. The reason for this is that the calculations necessary to draw the figures work poorly when working with subgroups $V \subset E_+(\mathfrak{f})$ of large index. Thus, for the units we chose for the figures, Lemma 7.1.8 cannot give rise to an element π so that we are in Case 2. However, to give the reader an idea of how this case would look we find a choice of π^{-1} that lies in the Colmez domain and is close to the region that Lemma 7.1.8 gives to contain π^{-1}. Note that when working with subgroups $V \subset E_+(\mathfrak{f})$ of large index, we are not able to guarantee that there exists a choice of π^{-1} in the region given by Lemma 7.1.8 such that we always land in Case 1. Hence, we must continue to work with both cases.*

Proposition 7.2.3. *In Case 1, we have*

$$\zeta_{R,\lambda}(\mathfrak{b},(\pi^{-1}\mathscr{B}_1\cap\mathscr{B})\cup\varepsilon_2^{-1}(\pi^{-1}\mathscr{B}_1\cap\varepsilon_2\mathscr{B}),\mathcal{O}_\mathfrak{p},0)=\sum_{k_2=0}^{1}\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathcal{O}_\mathfrak{p},0),$$
$$\zeta_{R,\lambda}(\mathfrak{b},(\pi^{-1}\mathscr{B}_2\cap\mathscr{B})\cup\varepsilon_1^{-1}(\pi^{-1}\mathscr{B}_2\cap\varepsilon_1\mathscr{B}),\mathcal{O}_\mathfrak{p},0)=\sum_{k_1=0}^{1}\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathcal{O}_\mathfrak{p},0).$$

In Case 2, we have

$$\zeta_{R,\lambda}(\mathfrak{b},(\pi^{-1}\mathscr{B}_1\cap\mathscr{B})\cup\varepsilon_2^{-1}(\pi^{-1}\mathscr{B}_1\cap\varepsilon_2\mathscr{B}),\mathcal{O}_\mathfrak{p},0)=\sum_{k_2=0}^{2}\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathcal{O}_\mathfrak{p},0),$$
$$\zeta_{R,\lambda}(\mathfrak{b},(\pi^{-1}\mathscr{B}_2\cap\mathscr{B})\cup\varepsilon_1^{-1}(\pi^{-1}\mathscr{B}_2\cap\varepsilon_1\mathscr{B}),\mathcal{O}_\mathfrak{p},0)=\sum_{k_1=0}^{1}\sum_{k_2=1}^{2}\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathcal{O}_\mathfrak{p},0).$$

Proof. We first calculate

$$\sum_{k_2=0}^{2}\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathcal{O}_\mathfrak{p},0)=\zeta_{R,\lambda}(\mathfrak{b},\bigcup_{k_2=0}^{2}\varepsilon_1^{-1}\varepsilon_2^{-k_2}(\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B}),\mathcal{O}_\mathfrak{p},0),$$
$$\sum_{k_1=0}^{1}\sum_{k_2=1}^{2}\zeta_{R,\lambda}(\mathfrak{b},\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathcal{O}_\mathfrak{p},0)=\zeta_{R,\lambda}(\mathfrak{b},\bigcup_{k_1=0}^{1}\bigcup_{k_2=1}^{2}\varepsilon_1^{-k_1}\varepsilon_2^{-k_2}(\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B}),\mathcal{O}_\mathfrak{p},0).$$

Thus, if we can show the following equalities of Shintani sets

$$(\pi^{-1}\mathscr{B}_1\cap\mathscr{B})\cup\varepsilon_2^{-1}(\pi^{-1}\mathscr{B}_1\cap\varepsilon_2\mathscr{B})=\bigcup_{k_2=0}^{2}\varepsilon_1^{-1}\varepsilon_2^{-k_2}(\varepsilon_1\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B}),\tag{7.5}$$
$$(\pi^{-1}\mathscr{B}_2\cap\mathscr{B})\cup\varepsilon_1^{-1}(\pi^{-1}\mathscr{B}_2\cap\varepsilon_1\mathscr{B})=\bigcup_{k_1=0}^{1}\bigcup_{k_2=1}^{2}\varepsilon_1^{-k_1}\varepsilon_2^{-k_2}(\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}\cap\pi^{-1}\mathscr{B}),\tag{7.6}$$

then we are done. To show the above, we need to calculate each side in terms of explicit Shintani cones. We begin by showing (7.5). Recall that we defined the following

$$\mathscr{B}=C(1)\cup C(1,\varepsilon_1)\cup C(1,\varepsilon_2)\cup C(1,\varepsilon_1\varepsilon_2)\cup C(1,\varepsilon_1,\varepsilon_1\varepsilon_2)\cup C(1,\varepsilon_2,\varepsilon_1\varepsilon_2),$$
$$\pi^{-1}\mathscr{B}_1=C(1)\cup C(1,\varepsilon_2)\cup C(1,\pi^{-1})\cup C(\pi^{-1},\varepsilon_2)\cup C(\pi^{-1},\varepsilon_2,\varepsilon_2\pi^{-1})\cup C(1,\varepsilon_2,\pi^{-1}).$$

Let $\alpha\in C(\pi^{-1},\varepsilon_2\pi^{-1})\cap C(\varepsilon_2,\varepsilon_1\varepsilon_2)$, we then have

$$\pi^{-1}\mathscr{B}_1\cap\mathscr{B}=C(1)\cup C(1,\varepsilon_2)\cup C(1,\pi^{-1})\cup C(\pi^{-1},\varepsilon_2)\cup C(\varepsilon_2,\pi^{-1},\alpha)\cup C(1,\varepsilon_2,\pi^{-1})$$

and

$$\pi^{-1}\mathscr{B}_1\cap\varepsilon_2\mathscr{B}=C(\varepsilon_2,\alpha)\cup C(\varepsilon_2,\alpha,\varepsilon_2\pi^{-1}).$$

We can now explicitly write the left hand side of (7.5). In particular, we have

$$
\begin{aligned}
&(\pi^{-1}\mathscr{B}_1 \cap \mathscr{B}) \cup \varepsilon_2^{-1}(\pi^{-1}\mathscr{B}_1 \cap \varepsilon_2\mathscr{B}) \\
&\qquad = C(1) \cup C(1,\varepsilon_2) \cup C(1,\pi^{-1}) \cup C(\pi^{-1},\varepsilon_2) \cup C(\varepsilon_2,\pi^{-1},\alpha) \cup C(1,\varepsilon_2,\pi^{-1}) \\
&\qquad\qquad \cup C(1,\varepsilon_2^{-1}\alpha) \cup C(1,\varepsilon_2^{-1}\alpha,\pi^{-1}).
\end{aligned}
$$

We now consider the right hand side of (7.5). Suppose that we are in Case 1. Then, the right hand side of (7.5) becomes $\varepsilon_1^{-1}(\varepsilon_1\mathscr{B}\cap\pi^{-1}\mathscr{B})\cup\varepsilon_1^{-1}\varepsilon_2^{-1}(\varepsilon_1\varepsilon_2\mathscr{B}\cap\pi^{-1}\mathscr{B})$. Let $\beta \in C(\varepsilon_1,\varepsilon_1\varepsilon_2)\cap C(\pi^{-1},\pi^{-1}\varepsilon_1)$. We can then calculate

$$
\begin{aligned}
&\varepsilon_1\mathscr{B} \cap \pi^{-1}\mathscr{B} = \\
&\quad C(\beta) \cup C(\beta,\varepsilon_1\varepsilon_2) \cup C(\beta,\varepsilon_1\pi^{-1}) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_1\pi^{-1}) \cup C(\varepsilon_1\varepsilon_2,\beta,\varepsilon_1\pi^{-1}) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_1\alpha,\varepsilon_1\pi^{-1})
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_1\varepsilon_2\mathscr{B} \cap \pi^{-1}\mathscr{B} &= C(\varepsilon_1\varepsilon_2) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_2\beta) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_1\alpha) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_1\varepsilon_2\pi^{-1}) \\
&\qquad \cup C(\varepsilon_1\varepsilon_2,\varepsilon_1\alpha,\varepsilon_1\varepsilon_2\pi^{-1}) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_2\beta,\varepsilon_1\varepsilon_2\pi^{-1}).
\end{aligned}
$$

Using the fact that $\beta \in C(\varepsilon_1,\varepsilon_1\varepsilon_2)$, we have

$$
\begin{aligned}
&(\varepsilon_1\mathscr{B} \cap \pi^{-1}\mathscr{B}) \cup \varepsilon_2^{-1}(\varepsilon_1\varepsilon_2\mathscr{B} \cap \pi^{-1}\mathscr{B}) \\
&= C(\varepsilon_1) \cup C(\varepsilon_1,\varepsilon_1\varepsilon_2) \cup C(\varepsilon_1,\varepsilon_1\pi^{-1}) \cup C(\varepsilon_1\pi^{-1},\varepsilon_1\varepsilon_2) \cup C(\varepsilon_1\varepsilon_2,\varepsilon_1\pi^{-1},\varepsilon_1\alpha) \cup C(\varepsilon_1,\varepsilon_1\varepsilon_2,\varepsilon_1\pi^{-1}) \\
&\qquad \cup C(\varepsilon_1,\varepsilon_1\varepsilon_2^{-1}\alpha) \cup C(\varepsilon_1,\varepsilon_1\varepsilon_2^{-1}\alpha,\varepsilon_1\pi^{-1}).
\end{aligned}
$$

By multiplying the above by ε_1^{-1}, it is then clear that (7.5) holds in Case 1. The proof of (7.5) in Case 2 is very similar. The extra calculations which arise from being in Case 2 are very similar to those which we deal with in our proof of (7.6) in Case 2.

We now consider (7.6). In Case 1, the proof is symmetric to the proof of (7.5) in Case 1. So it only remains to show (7.6) when we are in Case 2. Let $\alpha \in C(\pi^{-1},\varepsilon_1\pi^{-1}) \cap C(\varepsilon_2,\varepsilon_1\varepsilon_2)$ and $\beta \in C(\pi^{-1},\varepsilon_1\pi^{-1}) \cap C(\varepsilon_1\varepsilon_2,\varepsilon_1^2\varepsilon_2)$. Using similar calculations as before, we deduce

$$
\begin{aligned}
&(\pi^{-1}\mathscr{B}_2 \cap \mathscr{B}) \cup \varepsilon_1^{-1}(\pi^{-1}\mathscr{B}_2 \cap \varepsilon_1\mathscr{B}) \\
&\quad = C(1) \cup C(1,\varepsilon_2) \cup C(1,\varepsilon_1) \cup C(1,\varepsilon_1^{-1}\beta) \cup C(1,\pi^{-1}) \cup C(\pi^{-1},\varepsilon_1) \cup C(\pi^{-1},\varepsilon_1\varepsilon_2) \\
&\quad\quad \cup C(1,\varepsilon_2,\varepsilon_1^{-1}\beta) \cup C(1,\pi^{-1},\varepsilon_1^{-1}\beta) \cup C(1,\pi^{-1},\varepsilon_1) \cup C(\varepsilon_1,\pi^{-1},\varepsilon_1\varepsilon_2) \cup C(\pi^{-1},\alpha,\varepsilon_1\varepsilon_2).
\end{aligned}
$$

We are able to calculate that the same is also true for $\bigcup_{k_1=0}^{1}\bigcup_{k_2=1}^{2}\varepsilon_1^{-k_1}\varepsilon_2^{-k_2}(\varepsilon_1^{k_1}\varepsilon_2^{k_2}\mathscr{B}' \cap \pi^{-1}\mathscr{B}')$ and thus we complete the proof.

□

This final proposition completes the proof of Theorem 7.0.2.

Proposition 7.2.4. *If we are in Case 2, then*

$$\zeta_{R,\lambda}(\mathfrak{b},(\varepsilon_2\mathscr{B}\cup\varepsilon_1\varepsilon_2\mathscr{B})\cap\mathscr{B}_2,\mathscr{O}_\mathfrak{p},0)=\zeta_{R,\lambda}(\mathfrak{b},(\varepsilon_1\varepsilon_2\mathscr{B}\cup\varepsilon_2^2\mathscr{B})\cap\pi^{-1}\mathscr{B},\mathscr{O}_\mathfrak{p},0).$$

Proof. Lemma 3.2.13 implies that it is enough to show the following equality of Shintani sets

$$(\varepsilon_2\mathscr{B}\cup\varepsilon_1\varepsilon_2\mathscr{B})\cap\mathscr{B}_2=\varepsilon_2^{-1}((\varepsilon_1\varepsilon_2\mathscr{B}\cup\varepsilon_2^2\mathscr{B})\cap\pi^{-1}\mathscr{B}).$$

Again letting $\alpha\in C(\pi^{-1},\varepsilon_1\pi^{-1})\cap C(\varepsilon_2,\varepsilon_1\varepsilon_2)$ and $\beta\in C(\pi^{-1},\varepsilon_1\pi^{-1})\cap C(\varepsilon_1\varepsilon_2,\varepsilon_1^2\varepsilon_2)$, we are able to calculate that each side of the above equation is equal to

$$C(\varepsilon_1\varepsilon_2)\cup C(\alpha,\varepsilon_1\varepsilon_2)\cup C(\varepsilon_1\varepsilon_2,\beta)\cup C(\alpha,\varepsilon_1\varepsilon_2,\beta).$$

This concludes the result. □

We end this chapter by proving Theorem 7.0.1. The key step is to note that since we have shown

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D}_V)=c_{\mathrm{id}}\cap(\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V}\cap\vartheta'_V),$$

then by Proposition 4.2.2 and Proposition 6.2.1, we have

$$u_{\mathfrak{p},\lambda}(\mathfrak{b},\mathscr{D})=\gamma_{[E_+(\mathfrak{f}):V]}(c_{\mathrm{id}}\cap(\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda}\cap\vartheta')),$$

where $\gamma_{[E_+(\mathfrak{f}):V]}$ is a root of unity of order $[E_+(\mathfrak{f}):V]$. To prove Theorem 7.0.1, it is thus enough for us to find two free subgroups, $V,V'\subseteq E_+(\mathfrak{f})$, such that they are small enough to use in our work for Theorem 7.0.2 and such that $\gcd([E_+(\mathfrak{f}):V],[E_+(\mathfrak{f}):V'])=1$.

Proof of Theorem 7.0.1. When we choose g_1 and g_2, we do so such that $\mathrm{Log}(g_i)\in B(l_i(M_1),r)$ where r and M_1 are as we write after Lemma 7.1.5. Note that there is no upper bound on these choices. It is therefore clear that if we allow r and M_1 to be large enough, we can choose g_1,g_2 and g_1',g_2' such that

- $\langle g_1,g_2\rangle$ and $\langle g_1',g_2'\rangle$ are free of rank 2,
- g_1,g_2 and g_1',g_2' satisfy the properties of Lemma 7.1.5, and
- $[E_+(\mathfrak{f}):\langle g_1,g_2\rangle]$ and $[E_+(\mathfrak{f}):\langle g_1',g_2'\rangle]$ are coprime.

Next, we raise g_1,g_2 by a large power l in Corollary 7.1.7 and Lemma 7.1.8. Again, the only condition on l is that it is greater than a fixed lower bound. Hence, we can choose l and l' such that they are coprime to each other and to

$$[E_+(\mathfrak{f}):\langle g_1,g_2\rangle][E_+(\mathfrak{f}):\langle g_1',g_2'\rangle].$$

We then get $V = \langle g_1^l, g_2^l \rangle$ and $V' = \langle (g_1')^{l'}, (g_2')^{l'} \rangle$. Following our work for Theorem 7.0.2, we then see that

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D}_V) = c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta'_V) \quad \text{and} \quad u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D}_{V'}) = c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V'} \cap \vartheta'_{V'}).$$

Hence,

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D}) = \gamma_{[E_+(\mathfrak{f}):V]}(c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \cap \vartheta')),$$

and

$$u_{\mathfrak{p},\lambda}(\mathfrak{b}, \mathscr{D}) = \gamma_{[E_+(\mathfrak{f}):V']}(c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda} \cap \vartheta')).$$

In the above, $\gamma_{[E_+(\mathfrak{f}):V]}$ is a root of order $[E_+(\mathfrak{f}) : V]$ and $\gamma_{[E_+(\mathfrak{f}):V']}$ is a root of order $[E_+(\mathfrak{f}) : V']$. Our choice of V and V' gives that $\gcd([E_+(\mathfrak{f}) : V], [E_+(\mathfrak{f}) : V']) = 1$. By the above equations we see that $\gamma_{[E_+(\mathfrak{f}):V]} = \gamma_{[E_+(\mathfrak{f}):V']}$. Since the orders of these roots of unity are coprime we can deduce that, in fact, $\gamma_{[E_+(\mathfrak{f}):V]} = \gamma_{[E_+(\mathfrak{f}):V']} = 1$. Thus we have the result. □

Chapter 8

Comparing the Formulas for General Fields

In this chapter, we prove the main result of this thesis. In particular we prove Theorem 2.3.7. The work we present in this chapter is joint with Dasgupta. We prove that $u_1 = u_2 = u_3$ for any totally real field F. We show this in two steps. We firstly show that $u_2 = u_3$. This proof does not offer much insight into the formulas as it is largely computational in nature. We then present the proof which shows that $u_1 = u_2$. The proof here, compared to the proof of $u_2 = u_3$, is much more elegant; rather than working completely explicitly with the formulas, we show that they each satisfy a functorial property which we then show is strong enough to imply that they must in fact be equal. More precisely, we show that u_1 and u_2 satisfy a norm compatibility relation.

8.1 Proof that u_2 is equal to u_3

In this section, we begin our proof of the main result of this thesis, namely Theorem 2.3.7. We prove the following theorem.

Theorem 8.1.1. *We have*

$$u_2 = u_3.$$

Proof. Let V be finite index subgroup of E_+ free of rank $n-1$. We show that

$$u_2(V) = u_3(V).$$

By Proposition 5.3.1 and Proposition 6.2.1, we have that for each $\sigma \in G$,

$$u_2(V,\sigma) = u_2(\sigma)^{[E_+:V]} \quad \text{and} \quad u_3(V,\sigma) = u_3(\sigma)^{[E_+:V]}.$$

Then, by working with subgroups of coprime orders, as in the proof of Theorem 7.0.1, we have the result of the theorem. We now recall the explicit calculations for $u_2(V)$ and $u_3(V)$ as given in §5.4 and §6.3, respectively.

$$u_2(V) = \mathrm{Eis}_F^0 \cap (\Delta_*(c_{\mathrm{id}} \cap \rho_{H/F,V})) = \sum_{\sigma_{\mathfrak{b}} \in G} \left((\prod_{i=1}^{n-1} \varepsilon_i^{L_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathbb{O}_{\mathfrak{p}};0)}) \pi^{L_{R,\lambda}(\mathfrak{b},\mathscr{B},\pi\mathbb{O}_{\mathfrak{p}};0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, dL_{R,\lambda}(\mathfrak{b},\mathscr{B},x;0)) \otimes \sigma_{\mathfrak{b}}^{-1} \right),$$

and

$$u_3(V) = \sum_{k=1}^{h} \mathrm{rec}_{H/F}(\mathfrak{b}_k)^{-1} \left((\prod_{i=1}^{n-1} \varepsilon_i^{\mathfrak{L}_{R,\lambda}(\mathscr{B}_i,\mathfrak{b}_k,\pi\mathbb{O}_{\mathfrak{p}} \times \prod_{j=2}^{r} \mathbb{O}_{\mathfrak{p}_j},0)}) \pi^{\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b}_k,\mathbb{O}_{S_p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \, d(\mathfrak{L}_{R,\lambda}(\mathscr{B},\mathfrak{b}_k, x \times \prod_{j=2}^{r} \mathbb{O}_{\mathfrak{p}_j},0)) \right).$$

Let $i = 1, \ldots, n$ and $k = 1, \ldots, h$. We now note that for $U \subseteq \mathbb{O}_{\mathfrak{p}}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned} \mathfrak{L}_R(\mathscr{B}_i, \mathfrak{b}_k, U \times \prod_{j=2}^{r} \mathbb{O}_{\mathfrak{p}_j}, s) &= (\mathrm{N}\mathfrak{b}_k)^{-s} \sum_{\substack{\xi \in \mathscr{B}_i \cap (\mathfrak{b}_k)_{S_p}^{-1},\ \xi \in U \times \prod_{j=2}^{r} \mathbb{O}_{\mathfrak{p}_j} \\ (\xi, R)=1}} \frac{\mathrm{rec}_{H/F}((\xi))^{-1}}{\mathrm{N}\xi^s} \\ &= (\mathrm{N}\mathfrak{b}_k)^{-s} \sum_{\substack{\xi \in F \cap \mathscr{B}_i,\ \xi \in U \\ \xi \in \mathfrak{b}_k^{-1}\ (\xi,R)=1}} \frac{\mathrm{rec}_{H/F}((\xi))^{-1}}{\mathrm{N}\xi^s}. \end{aligned} \tag{8.1}$$

We write

$$\mathfrak{L}_{R,\lambda}(\mathscr{B}_i, \mathfrak{b}_k, U \times \prod_{j=2}^{r} \mathbb{O}_{\mathfrak{p}_j}, 0) = \sum_{\sigma \in G} \mathfrak{L}_{R,\lambda}(\sigma, \mathscr{B}_i, \mathfrak{b}_k, U, 0) \otimes \sigma^{-1}$$

where $\mathfrak{L}_{R,\lambda}(\sigma, \mathscr{B}_i, \mathfrak{b}_k, U, 0) \in \mathbb{Z}$. We recall that in §3.4 we have defined that for $L = \sum_{\sigma \in G} a_\sigma \otimes \sigma \in \mathbb{Z}[G]$ we have, for $\alpha \in F_{\mathfrak{p}}^*$,

$$\alpha^L = \sum_{\sigma \in G} \alpha^{a_\sigma} \otimes \sigma^{-1}.$$

It then follows from the definitions of $\mathfrak{L}_{R,\lambda}$ and $L_{R,\lambda}$, and the calculation in (8.1), that for $\alpha \in F_{\mathfrak{p}}^*$,

$$\sum_{k=1}^{h} \sum_{\sigma_{\mathfrak{b}} \in G} \alpha^{\mathfrak{L}_{R,\lambda}(\sigma_{\mathfrak{b}},\mathscr{B}_i,\mathfrak{b}_k,U,0)} \otimes \sigma_{\mathfrak{b}_i}^{-1} \sigma_{\mathfrak{b}}^{-1} = \sum_{\sigma_{\mathfrak{b}} \in G} \alpha^{L_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,U;0)} \otimes \sigma_{\mathfrak{b}}^{-1}.$$

This completes the proof that $u_2(V) = u_3(V)$. □

8.2 Proof that u_1 is equal to u_2

In this section, we complete the proof of Theorem 2.3.7. We show the following theorem.

Theorem 8.2.1. *For all $\sigma \in G$, we have*

$$u_1(\sigma) = u_2(\sigma).$$

Here we have no assumptions on the degree of F. The approach of this proof is very different to that used in Chapter 7 and §8.1. Rather than explicitly calculating the error terms, we show a strong enough functorial property for each of the formulas which forces them to be equal. In particular, we prove a norm compatibility property for each of the formulas u_1 and u_2. This approach allows us to avoid the complexities arising from working explicitly with Shintani sets and their translates, as we saw in Chapter 7.

Let $H \subset H'$ be two finite abelian extensions of F in which $\mathfrak{p}$ splits completely. Let $\mathfrak{f}\mathfrak{f}'$ be the conductor of H'/F where, as before, $\mathfrak{f}$ is the conductor of H/F. Let $\sigma \in G$. Write $u_1(\sigma, H)$ and $u_2(\sigma, H)$ for σ components of the formulas u_1 and u_2, for the extension H/F and Galois group element σ. We show for $i = 1$ in §8.3, and for $i = 2$ in §8.4, that

$$u_i(\sigma, H) = \prod_{\substack{\tau \in G' \\ \tau|_H = \sigma}} u_i(\tau, H'). \tag{8.2}$$

We refer to (8.2) as norm compatibility for u_i. For now we assume that the above equality holds for $i = 1, 2$. We prove this in the following sections of this chapter. We now prove the following proposition, note that here the congruences are taken multiplicatively.

Proposition 8.2.2. *We have*

$$u_1(\sigma, H) \equiv u_2(\sigma, H) \pmod{E_+(\mathfrak{f})}.$$

Proof. Let V be a free, finite index subgroup of $E_+(\mathfrak{f})$ of rank $n-1$ satisfying the conditions given in the statement of Proposition 4.2.2. We then let V' be a free, finite index subgroup of E_+ of rank $n-1$, contained in V, such that $[E_+ : V'] = [E_+(\mathfrak{f}) : V]$. Furthermore, we can choose V' such that if $V' = \langle \varepsilon'_1, \dots, \varepsilon'_{n-1} \rangle$ then the ε'_i with π' satisfy Lemma 3.6.1. By Theorem 8.1.1 and Proposition 6.2.2, we have

$$u_2(V', \sigma) = u_3(V', \sigma) = u'_3(V, \sigma).$$

We now recall from §6.4 the explicit description of $u'_3(V, \sigma)$,

$$\begin{aligned} u'_3(V, \sigma) &= c_{\mathrm{id}} \cap (\omega^{\mathfrak{p}}_{\mathfrak{f},\mathfrak{b},\lambda,V} \cap \vartheta'_V) \\ &= \prod_{i=1}^{n-1} \varepsilon_i^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B}_i,\pi\mathbb{O}_{\mathfrak{p}},0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B},\mathbb{O}_{\mathfrak{p}},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d(\zeta_{R,\lambda}(\mathfrak{b}, \mathscr{B}, x, 0))(x). \end{aligned}$$

In (4.2), we defined

$$u_1(V,\sigma) = \prod_{\epsilon \in V} \epsilon^{\zeta_{R,\lambda}(\mathfrak{b},\epsilon\mathscr{B}\cap\pi^{-1}\mathscr{B},\mathbb{O}_\mathfrak{p},0)} \pi^{\zeta_{R,\lambda}(\mathfrak{b},\mathscr{B},\mathbb{O}_\mathfrak{p},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu(\mathfrak{b},\mathscr{B},x),$$

where

$$\mathscr{B} = \bigcup_{\tau \in S_{n-1}} \overline{C}_{e_1}([\varepsilon_{\tau(1)} \mid \ldots \mid \varepsilon_{\tau(n-1)}]).$$

Thus, $u_1(V,\mathscr{B},\mathfrak{b}) \equiv u_3'(V,\sigma) \pmod{E_+(\mathfrak{f})}$ and hence $u_1(V,\mathscr{B},\mathfrak{b}) \equiv u_2(V,\sigma) \pmod{E_+(\mathfrak{f})}$. Working with coprime choices for V allows us to complete the proof of this proposition. □

Assuming that (8.2) holds, we can give the proof of the main theorem of this section.

Proof of Theorem 8.2.1. From Proposition 8.2.2, we have that for each $\tau \in G'$,

$$u_1(\tau, H') \equiv u_2(\tau, H') \pmod{E_+(\mathfrak{f}\mathfrak{f}')}.$$

Our assumption that (8.2) holds then gives that for each $\sigma \in G$,

$$u_1(\sigma, H) \equiv u_2(\sigma, H) \pmod{E_+(\mathfrak{f}\mathfrak{f}')}.$$

Repeating this for enough field extensions $H'/H/F$ shows that

$$u_1(\sigma, H) = u_2(\sigma, H).$$

This completes the proof. □

8.3 Norm compatibility for u_1

To work with the definition for $u_1(\sigma, H')$, we introduce some additional notation. The reciprocity map identifies $\mathrm{Gal}(H'/H)$ with

$$\{\beta \in (\mathcal{O}_F/\mathfrak{f}\mathfrak{f}')^* \mid \beta \equiv 1 \pmod{\mathfrak{f}}\}/E_+(\mathfrak{f})_\mathfrak{p}. \tag{8.3}$$

We let $\mathscr{D}_\mathfrak{f}$ be a Shintani domain for $E_+(\mathfrak{f})$ and define

$$\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma \mathscr{D}_\mathfrak{f},$$

where the union is over a set of representatives $\{\gamma\}$ for $E_+(\mathfrak{f}\mathfrak{f}')$ in $E_+(\mathfrak{f})$. Let e' be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}\mathfrak{f}'}$, and suppose that $\mathfrak{p}^{e'} = (\pi')$ with π' totally positive and $\pi' \equiv 1 \pmod{\mathfrak{f}\mathfrak{f}'}$. We can choose π' such that $\pi' = \pi^\alpha$ for some $\alpha \geq 1$. We then define $\mathbb{O}' = \mathcal{O}_\mathfrak{p} - \pi'\mathcal{O}_\mathfrak{p}$.

Let B denote a set of totally positive elements of $\mathcal{O}_F$ which are relatively prime to S and $\overline{\lambda}$ and whose images in $(\mathcal{O}_F/\mathfrak{f}\mathfrak{f}')^*$ are a set of distinct representatives for (8.3).

The following theorem is stated by Dasgupta in [8, Theorem 7.1]. For completeness, we include a proof of this theorem here.

Theorem 8.3.1 (Theorem 7.1, [8])**.** *We have*

$$u_1(\sigma_{\mathfrak{b}}, \mathscr{D}_{\mathfrak{f}}) = \prod_{\beta \in B} u_1(\sigma_{\mathfrak{b}(\beta)}, \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}).$$

The strategy of this proof is to explicitly calculate the product over $\beta \in B$ and show that it is equal to $u_1(\sigma_{\mathfrak{b}}, \mathscr{D}_{\mathfrak{f}})$. The key to these calculations is to use translation properties of Shintani sets. We begin by introducing some additional notation that will be used for the proof of this theorem. For a subset A, of equivalence classes of (8.3), we let $\nu_A(\mathfrak{b}, \mathscr{D}, U) = \zeta^A_{R,\lambda}(\mathfrak{b}, \mathscr{D}, U, 0)$, where ζ^A_R is the zeta function

$$\zeta^A_R(\mathfrak{b}, \mathscr{D}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in \mathfrak{b}^{-1} \cap \mathscr{D},\ \alpha \in U \\ \alpha \in A,\ (\alpha, R) = 1}} \mathrm{N}\alpha^{-s}.$$

This definition extends to $\zeta^A_{R,\lambda}$ as in (2.4). Throughout this section we will use the following simple equality,

$$\nu_{\{\pi^{-1}\}}(\mathfrak{b}, \mathscr{D}, U) = \nu_{\{1\}}(\mathfrak{b}, \pi\mathscr{D}, \pi U).$$

This follows from Lemma 3.2.13. Let $\beta \in B$. We recall the following definition,

$$u_1(\sigma_{\mathfrak{b}(\beta)}, \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}) = \epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi')(\pi')^{\zeta_{R,\lambda}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)} \times\!\!\!\!\!\!\int_{\mathbb{O}'} x \; d\nu(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, x).$$

It is clear from the definition of B that Theorem 8.3.1 follows from the following theorem.

Theorem 8.3.2. *Let $\beta \in B$. We then have*

$$u_1(\sigma_{\mathfrak{b}(\beta)}, \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}) = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_B(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \right) \pi^{\nu_B(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}}, \mathcal{O}_{\mathfrak{p}})} \times\!\!\!\!\!\!\int_{\mathbb{O}} x \; d\nu_B(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}}, x).$$

The proof of Theorem 8.3.2 is largely an exercise in explicit calculation. We begin by considering the multiplicative integral in $u_1(\sigma_{\mathfrak{b}(\beta)}, \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'})$.

Lemma 8.3.3. *We have*

$$\begin{aligned}&\times\!\!\!\!\!\!\int_{\mathbb{O}'} x\ d\nu(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x)\\&\quad=\left(\prod_{i=1}^{\alpha-1}\pi^{i\nu(\mathfrak{b}(\beta),\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\left(\prod_{i=0}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f}\mathfrak{f}')}\epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'}\cap\pi^{-i}\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},\mathbb{O})}\right)\\&\qquad\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_A(\mathfrak{b}(\beta),\gamma\beta^{-1}\mathfrak{D}_{\mathfrak{f}},\mathbb{O})}\right)\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_B(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}},x).\end{aligned}$$

Proof. Since $\pi' = \pi^\alpha$ and $\mathbb{O}' = \mathcal{O}_\mathfrak{p} - \pi'\mathcal{O}_\mathfrak{p}$, we have $\mathbb{O}' = \bigcup_{i=0}^{\alpha-1}\pi^i\mathbb{O}$. Then

$$\times\!\!\!\!\!\!\int_{\mathbb{O}'} x\ d\nu(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x) = \prod_{i=0}^{\alpha-1}\times\!\!\!\!\!\!\int_{\pi^i\mathbb{O}} x\ d\nu(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x).$$

For ease of notation we define $I(\beta)$ to be equal to the multiplicative integral in $u_1(\sigma_{\mathfrak{b}(\beta)},\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'})$. By a change of variables and then factoring out π^i, we have

$$\begin{aligned}I(\beta) &= \left(\prod_{i=1}^{\alpha-1}\pi^{i\nu(\mathfrak{b}(\beta),\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\prod_{i=0}^{\alpha-1}\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i x)\\&= \left(\prod_{i=1}^{\alpha-1}\pi^{i\nu(\mathfrak{b}(\beta),\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\prod_{i=0}^{\alpha-1}\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\pi^{-i}\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x).\end{aligned}$$

We now note that we can write, for $i = 1,\ldots,\alpha-1$,

$$\pi^{-i}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\epsilon\in E_+(\mathfrak{f}\mathfrak{f}')}(\epsilon\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'}\cap\pi^{-i}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'}).$$

Then,

$$\begin{aligned}&\prod_{i=0}^{\alpha-1}\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\pi^{-i}\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x)\\&=\prod_{i=0}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f}\mathfrak{f}')}\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'}\cap\pi^{-i}\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x)\\&=\left(\prod_{i=0}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f}\mathfrak{f}')}\epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'}\cap\pi^{-i}\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},\mathbb{O})}\right)\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_A(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x)\end{aligned}$$

where $A = \{1,\pi^{-1},\ldots,\pi^{\alpha-1}\}$. Then, since $\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'} = \bigcup_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma\mathfrak{D}_\mathfrak{f}$, we can write

$$\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_A(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}\mathfrak{f}'},x) = \left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_A(\mathfrak{b}(\beta),\gamma\beta^{-1}\mathfrak{D}_{\mathfrak{f}},\mathbb{O})}\right)\times\!\!\!\!\!\!\int_{\mathbb{O}} x\ d\nu_{\langle A,E\rangle}(\mathfrak{b}(\beta),\beta^{-1}\mathfrak{D}_{\mathfrak{f}},x)$$

where $E = E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')$. Thus we have, noting that $B = \langle A, E\rangle$,

$$I(\beta) = \left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\left(\prod_{i=0}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}\cap\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\mathbb{O})}\right)$$

$$\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta),\gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}\right)⨍_{\mathbb{O}} x\, d\nu_B(\mathfrak{b}(\beta),\beta^{-1}\mathscr{D}_{\mathfrak{f}},x).$$

□

We now consider the powers of π given in the definition of $u_1(\sigma_{\mathfrak{b}(\beta)},\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'})$ and arising in the result of Lemma 8.3.3. Recall that $\pi' = \pi^\alpha$.

Lemma 8.3.4. *We have*

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\pi^{\alpha\zeta_{R,\lambda}(H_{\mathfrak{f}\mathfrak{f}'}/F,\mathfrak{b}(\beta),0)} = \pi^{\nu_B(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}.$$

Proof. Since $\pi^i\mathbb{O} = \pi^i\mathcal{O}_{\mathfrak{p}} - \pi^{i+1}\mathcal{O}_{\mathfrak{p}}$ we have, by a telescope argument,

$$\sum_{i=1}^{\alpha-1} i\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O}) = -(\alpha-1)\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^\alpha\mathcal{O}_{\mathfrak{p}}) + \sum_{i=1}^{\alpha-1}\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathcal{O}_{\mathfrak{p}}).$$

Recalling the definition of $\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}$ we also note that for $i = 0,\ldots,\alpha-1$, we have

$$\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathcal{O}_{\mathfrak{p}}) = \nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\pi^i\mathcal{O}_{\mathfrak{p}}).$$

Thus, we can calculate, using the fact that $\zeta_{R,\lambda}(H_{\mathfrak{f}\mathfrak{f}'}/F,\mathfrak{b}(\beta),0) = \nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\mathcal{O}_{\mathfrak{p}})$,

$$\begin{aligned}
&\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathbb{O})}\right)\pi^{\alpha\zeta_{R,\lambda}(H_{\mathfrak{f}\mathfrak{f}'}/F,\mathfrak{b}(\beta),0)}\\
&=\left(\prod_{i=1}^{\alpha-1} \pi^{\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^i\mathcal{O}_{\mathfrak{p}})}\right)\pi^{-(\alpha-1)\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\pi^\alpha\mathcal{O}_{\mathfrak{p}})}\pi^{\alpha\nu(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}\mathfrak{f}'},\mathcal{O}_{\mathfrak{p}})}\\
&=\left(\prod_{i=1}^{\alpha-1} \pi^{\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\pi^i\mathcal{O}_{\mathfrak{p}})}\right)\pi^{-(\alpha-1)\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\pi^\alpha\mathcal{O}_{\mathfrak{p}})}\pi^{\alpha\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}.
\end{aligned}$$

Let $i = 1,\ldots,\alpha$. By Lemma 3.2.13, we have that,

$$\nu_E(\mathfrak{b}(\beta),\mathscr{D}_{\mathfrak{f}},\pi^i\mathcal{O}_{\mathfrak{p}}) = \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}(\beta),\pi^{-i}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}}).$$

We can then write $\pi^i \mathscr{D}_\mathfrak{f} = \bigcup_{\delta \in E_+(\mathfrak{f})} \delta \mathscr{D}_\mathfrak{f} \cap \pi^{-i} \mathscr{D}_\mathfrak{f}$. Then

$$\begin{aligned}\nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \pi^{-i}\mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p}) &= \sum_{\delta \in E_+(\mathfrak{f})} \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \delta\mathscr{D}_\mathfrak{f} \cap \pi^{-i}\mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p}) \\ &= \sum_{\delta \in E_+(\mathfrak{f})} \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \mathscr{D}_\mathfrak{f} \cap \delta\pi^{-i}\mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p}) \\ &= \nu_{E,\{\pi^{-i}\}}(\mathfrak{b}, \mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p}).\end{aligned}$$

Remarking that $\{\pi^{-\alpha}\} = \{1\}$ then allows us to use the above calculations to deduce that

$$\left(\prod_{i=1}^{\alpha-1} \pi^{i\nu(\mathfrak{b}(\beta), \mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi^i \mathbb{O})}\right) \pi^{\alpha\zeta_{R,\lambda}(H_{\mathfrak{f}\mathfrak{f}'}/F, \mathfrak{b}(\beta), 0)} = \prod_{i=0}^{\alpha-1} \pi^{\nu_{E,\{\pi^{-i}\}}(\mathfrak{b}(\beta), \mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p})} = \pi^{\nu_{\langle A,E\rangle}(\mathfrak{b}(\beta), \mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p})}.$$

Noting again that $B = \langle A, E\rangle$ completes the proof. □

We now consider the error term in the definition of $u_1(\sigma_{\mathfrak{b}(\beta)}, \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'})$ and the products of elements of $E_+(\mathfrak{f})$ which arise in Lemma 8.3.3. Considering Lemma 8.3.3 and Lemma 8.3.4, we can see that to prove Theorem 8.3.2 it is enough to prove the following proposition.

Proposition 8.3.5. *We have*

$$\mathrm{Err}(\beta) = \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_B(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_\mathfrak{f} \cap \pi^{-1}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p})}$$

where

$$\mathrm{Err}(\beta) = \epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi')\left(\prod_{i=0}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})}\right) \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathbb{O})}\right).$$

For clarity, we perform the calculations required for this proposition in a few lemmas.

Lemma 8.3.6. *We have*

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_E(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathbb{O}_\mathfrak{p})}\right)\left(\prod_{i=1}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_\mathfrak{f} \cap \pi^{-i}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathbb{O})}\right).$$

Proof. Considering the definition of $\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}$, we can calculate

$$\begin{aligned}
&\epsilon(\mathfrak{b}(\beta), \beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \pi') \\
&= \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_{\mathfrak{p}})} \\
&= \prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_{\mathfrak{p}})} \\
&= \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathscr{O}_{\mathfrak{p}})} \right) \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_{\mathfrak{p}})} \right).
\end{aligned} \tag{8.4}$$

Similarly, we have

$$\begin{aligned}
&\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathscr{O}_{\mathfrak{p}})} \\
&\qquad = \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu(\mathfrak{b}(\beta), \gamma\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathscr{O}_{\mathfrak{p}})} \right) \left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_E(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathscr{O}_{\mathfrak{p}})} \right).
\end{aligned} \tag{8.5}$$

We can also calculate, for $i = 1, \ldots, \alpha - 1$,

$$\begin{aligned}
&\prod_{\epsilon \in E_+(\mathfrak{f}\mathfrak{f}')} \epsilon^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}\mathfrak{f}'}, \mathbb{O})} \\
&\qquad = \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O}) + \nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right) \\
&\qquad\qquad \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}} \cap \pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})}.
\end{aligned} \tag{8.6}$$

We now note the following equalities, both of which hold via telescope sum arguments.

1.
$$\begin{aligned}
&\prod_{i=1}^{\alpha-1} \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{-\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right) \left(\prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu_A(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})} \right) \\
&\qquad = \prod_{\gamma \in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')} \gamma^{\nu(\mathfrak{b}(\beta), \gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}}, \mathbb{O})}
\end{aligned}$$

2.

$$\prod_{i=1}^{\alpha-1}\left(\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu_{\{\pi^{-i}\}}(\mathfrak{b}(\beta),\gamma\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}\right)$$
$$=\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu(\mathfrak{b}(\beta),\gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})-\nu_{\{\pi^{-\alpha}\}}(\mathfrak{b}(\beta),\gamma\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}$$
$$=\prod_{\gamma\in E_+(\mathfrak{f})/E_+(\mathfrak{f}\mathfrak{f}')}\gamma^{\nu(\mathfrak{b}(\beta),\gamma\beta^{-1}\mathscr{D}_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})-\nu(\mathfrak{b}(\beta),\gamma\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}.$$

Combining these two equalites with the calculations in (8.4), (8.5) and (8.6) gives the result. □

If $\alpha = 1$ then Lemma 8.3.6 is equivalent to Proposition 8.3.5 and thus we are finished in the case $\alpha = 1$. Henceforth, we assume that $\alpha > 1$.

Lemma 8.3.7. *If $\alpha > 1$, then*

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{\nu_B(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}\right)\left(\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}\right)$$
$$\prod_{i=1}^{\alpha-1}\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{-\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})}$$
$$\prod_{i=2}^{\alpha-1}\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}.$$

Proof. For $i = 2,\dots,\alpha$, we have

$$\pi^i\mathscr{D}_{\mathfrak{f}} = \bigcup_{\delta\in E_+(\mathfrak{f})}\pi^{-1}\delta\mathscr{D}_{\mathfrak{f}}\cap\pi^{-i}\mathscr{D}_{\mathfrak{f}}.$$

Thus, applying this to the result of Lemma 8.3.6, we have

$$\mathrm{Err}(\beta) = \left(\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{\nu_E(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}\right)\left(\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}\right)$$
$$\prod_{i=1}^{\alpha-1}\left(\prod_{\epsilon\in E_+(\mathfrak{f})}\epsilon^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}\right.$$
$$\left.\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_{\langle E,\{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-i}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})}\right).$$

Remarking that $\prod_{\delta\in E_+(\mathfrak{f})}\delta^{\nu_{\langle E,\{\pi^{-1}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}},\mathbb{O})} = 1$, since $\epsilon\beta^{-1}\pi^{-1}\mathscr{D}_{\mathfrak{f}}\cap\pi^{-1}\beta^{-1}\mathscr{D}_{\mathfrak{f}} = \varnothing$, gives the result of this lemma. □

If $\alpha = 2$, it is straightforward to see that Lemma 8.3.7 is equivalent to Proposition 8.3.5. Thus we are also done in the case $\alpha = 2$. We henceforth assume that $\alpha > 2$. From Lemma 8.3.7, one can see that to prove Proposition 8.3.5 it is enough for us to show

$$1 = \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-1}\mathscr{D}_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathscr{O}_\mathfrak{p})} \prod_{i=1}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\mathscr{D}_\mathfrak{f} \cap \pi^{-1}\beta^{-1}\mathscr{D}_\mathfrak{f}, \pi\mathscr{O}_\mathfrak{p})} \prod_{i=2}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-1}\mathscr{D}_\mathfrak{f} \cap \pi^{-i}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathbb{O})}. \quad (8.7)$$

To do this, we first show the following lemma.

Lemma 8.3.8. *We have that for $j = 1, \dots, \alpha - 1$ the right hand side of (8.7) is equal to $e(j)$, where we define*

$$e(j) = \left(\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathscr{O}_\mathfrak{p})} \right) \prod_{i=j}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-(j-1)}\mathscr{D}_\mathfrak{f} \cap \pi^{-j}\beta^{-1}\mathscr{D}_\mathfrak{f}, \pi\mathscr{O}_\mathfrak{p})} \prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}\mathscr{D}_\mathfrak{f} \cap \pi^{-i}\beta^{-1}\mathscr{D}_\mathfrak{f}, \mathbb{O})}.$$

Note that for $j = \alpha - 1$ the last product is empty. We also remark that it is implicit in the statement of this lemma that $e(1) = \cdots = e(\alpha - 1)$.

Proof. We prove this by induction. The case $j = 1$ holds trivially. We now assume it holds for j and prove the result for $j + 1$, i.e., we show $e(j) = e(j + 1)$. To do this we note that for $i = j + 2, \dots, \alpha$, we have

$$\pi^{-i}\mathscr{D}_\mathfrak{f} = \bigcup_{\kappa \in E_+(\mathfrak{f})} \pi^{-(j+1)}\kappa\mathscr{D}_\mathfrak{f} \cap \pi^{-i}\mathscr{D}_\mathfrak{f}.$$

Thus, $e(j)$ is equal to the product of the following elements,

$$\left(\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}\mathcal{D}_\mathfrak{f} \cap \pi^{-(j+1)}\beta^{-1}\mathcal{D}_\mathfrak{f}, \mathcal{O}_\mathfrak{p})}\right)\left(\prod_{\kappa \in E_+(\mathfrak{f})} \kappa^{\nu_E(\mathfrak{b}(\beta), \kappa\beta^{-1}\pi^{-(j+1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-\alpha}\beta^{-1}\mathcal{D}_\mathfrak{f}, \mathcal{O}_\mathfrak{p})}\right) \tag{8.8}$$

$$\prod_{i=j}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})} \tag{8.9}$$

$$\prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}\mathcal{D}_\mathfrak{f} \cap \pi^{-(j+1)}\beta^{-1}\mathcal{D}_\mathfrak{f}, \mathbb{O})} \tag{8.10}$$

$$\prod_{i=j+2}^{\alpha-1} \prod_{\kappa \in E_+(\mathfrak{f})} \kappa^{\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \kappa\beta^{-1}\pi^{-(j+1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-i}\beta^{-1}\mathcal{D}_\mathfrak{f}, \mathbb{O})}. \tag{8.11}$$

We remark that the first bracketed term in (8.8), and (8.11) are already products in $e(j+1)$. We now consider (8.10) and calculate that it is equal to

$$\left(\prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-(i-1)}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})}\right)$$
$$\left(\prod_{i=j+1}^{\alpha-1} \prod_{\delta \in E_+(\mathfrak{f})} \delta^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}\mathcal{D}_\mathfrak{f} \cap \pi^{-(j+1)}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})}\right). \tag{8.12}$$

We now consider the way the terms in (8.12) interact with (8.9). Multiplying (8.12) by (8.9) gives

$$\left(\prod_{i=j+1}^{\alpha-1} \prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-i}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-j}\mathcal{D}_\mathfrak{f} \cap \pi^{-(j+1)}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})}\right)$$
$$\left(\prod_{\epsilon \in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E, \{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta), \epsilon\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})}\right). \tag{8.13}$$

The first term in (8.13) is the term we were missing from $e(j+1)$. Thus it only remains to show that the second bracketed term in (8.8) multiplied by the second bracketed term in (8.13) is equal to one. This is shown by the following calculation,

$$\begin{aligned}
&\prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-j}\mathcal{D}_\mathfrak{f} \cap \pi^{-(j+1)}\beta^{-1}\mathcal{D}_\mathfrak{f}, \mathcal{O}_\mathfrak{p})} \\
&= \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})} \\
&= \prod_{\delta \in E_+(\mathfrak{f})} \delta^{\nu_{\langle E, \{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta), \delta\beta^{-1}\pi^{-(j-1)}\mathcal{D}_\mathfrak{f} \cap \pi^{-j}\beta^{-1}\mathcal{D}_\mathfrak{f}, \pi\mathcal{O}_\mathfrak{p})},
\end{aligned}$$

we can thus deduce that

$$e(j) = e(j+1)$$

as claimed. This completes the proof of the lemma. □

We are now ready to prove Proposition 8.3.5.

Proof of Proposition 8.3.5. We consider $e(\alpha-1)$. From Lemma 8.3.8, we have that $e(\alpha-1)$ is equal to the right hand side of (8.7). Then

$$e(\alpha-1) = \left(\prod_{\delta\in E_+(\mathfrak{f})} \delta^{\nu_E(\mathfrak{b}(\beta),\delta\beta^{-1}\pi^{-(\alpha-1)}\mathfrak{D}_{\mathfrak{f}}\cap\pi^{-\alpha}\beta^{-1}\mathfrak{D}_{\mathfrak{f}},\mathcal{O}_{\mathfrak{p}})}\right) \prod_{\epsilon\in E_+(\mathfrak{f})} \epsilon^{-\nu_{\langle E,\{\pi^{-(\alpha-1)}\}\rangle}(\mathfrak{b}(\beta),\epsilon\beta^{-1}\pi^{-(\alpha-2)}\mathfrak{D}_{\mathfrak{f}}\cap\pi^{-(\alpha-1)}\beta^{-1}\mathfrak{D}_{\mathfrak{f}},\pi\mathcal{O}_{\mathfrak{p}})}.$$

Since $\{\pi\} = \{\pi^{-(\alpha-1)}\}$, it is clear that

$$e(\alpha-1) = 1.$$

This completes the proof of Proposition 8.3.5 and thus proves Theorem 8.3.1. □

8.4 Norm compatibility for u_2

We are now able to give the theorem that completes our proof of Theorem 8.2.1 and consequently Theorem 2.3.7. We recall the definition

$$u_2 = \sum_{\sigma\in G} u_2(\sigma)\otimes[\sigma^{-1}] = \mathrm{Eis}_F^0 \cap \Delta_*(c_{\mathrm{id}}\cap\rho_{H/F}).$$

Write $u_2(\sigma) = u_{S,\lambda,\sigma}$. In this section, we prove the following theorem.

Theorem 8.4.1. *We have for any $\sigma\in G$,*

$$u_{S,\lambda,\sigma,H} = \prod_{\substack{\tau\in G' \\ \tau|_H=\sigma}} u_{S,\lambda,\tau,H'}.$$

Remark 8.4.2. *This theorem has been proved by Dasgupta–Spieß in Proposition 5.1.1. We include the proof for completeness. We note also that the proof of the norm compatibility for u_2 is much simpler than that for u_1. This is a result of the additional structure we have due to the cohomological nature of the construction.*

Proof of Theorem 8.4.1. We consider the natural map

$$\psi : F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G'] \to F_{\mathfrak{p}}^*\otimes\mathbb{Z}[G]$$
$$\sum_{\tau\in G'} n_\tau\otimes[\tau] \mapsto \sum_{\sigma\in G}\Big(\prod_{\substack{\tau\in G' \\ \tau|_H=\sigma}} n_\tau\Big)\otimes[\sigma].$$

Then, we begin by considering the action of ψ on the element $u_{S,\lambda,H'}$,

$$\psi(u_{S,\lambda,H'}) = \sum_{\sigma\in G}\Big(\prod_{\substack{\tau\in G'\\ \tau|_H=\sigma}} u_{S,\lambda,\tau,H'}\Big)\otimes[\sigma].$$

We can also consider the action of ψ on the cohomological description of $u_{S,\lambda,H'}$,

$$\begin{aligned}\psi(u_{S,\lambda,H'}) &= \psi(\mathrm{Eis}^0_F \cap \Delta_*(c_{\mathrm{id}} \cap \rho_{H'/F}))\\ &= \mathrm{Eis}^0_F \cap \psi_*\Delta_*(c_{\mathrm{id}} \cap \rho_{H'/F})\\ &= \mathrm{Eis}^0_F \cap \Delta_*(c_{\mathrm{id}} \cap \psi_*\rho_{H'/F}).\end{aligned}$$

The only equality of note here is the final one. This follows since we can commute ψ_* with Δ_*. This clearly follows from the calculations done in §5.2 and §5.4. Finally, since $\psi_*\rho_{H'/F} = \rho_{H/F}$, we have the result. □

Chapter 9

The Root of Unity Ambiguity

In this chapter, we prove that the formulas for the Brumer–Stark units hold up to a 2-power root of unity. We prove Theorem 2.3.10. In particular, we show this result for u_2, i.e., we prove, that under some mild assumptions,

$$u_2 = u_{\mathfrak{p}} \text{ in } (F_{\mathfrak{p}}^*/\mu_2(F_{\mathfrak{p}}^*)) \otimes \mathbb{Z}[G]$$

where we write $\mu_2(F_{\mathfrak{p}}^*)$ for the group of 2-power roots of unity of $F_{\mathfrak{p}}^*$. As we noted in §2.3 and §2.4, the key result required for our proof of this result is the l-part of the integral Gross–Stark conjecture (Theorem 2.4.4). As noted in Remark 2.4.3, this theorem follows from the recent work of Bullach–Burns–Daoud–Seo in [2, Theorem B] which proves the minus-part of the eTNC away from 2, for finite abelian CM extensions of totally real fields.

9.1 Equality of the formula up to a 2-power root of unity

As before, we let $\mathfrak{f}$ be the conductor of the extension H/F and write $E_+(\mathfrak{f})$ for the totally positive units of F which are congruent to 1 modulo $\mathfrak{f}$. Let $\mathfrak{g}$ denote the product of the finite primes in S that do not divide $\mathfrak{f}\mathfrak{p}$. Then we define $H_S \coloneqq H_{(\mathfrak{f}\mathfrak{p}\mathfrak{g})^\infty}$. Here, $H_{(\mathfrak{f}\mathfrak{p}\mathfrak{g})^\infty}$ is the union of the narrow ray class fields $H_{\mathfrak{f}^a\mathfrak{p}^b\mathfrak{g}^c}$ for all positive integers a, b, c. For $v \mid \mathfrak{f}\mathfrak{g}$, let $U_{v,\mathfrak{f}}$ denote the group of elements of $\mathcal{O}_v^*$ which are congruent to 1 modulo $\mathfrak{f}\mathcal{O}_v^*$. In particular, $U_{v,\mathfrak{f}} = \mathcal{O}_v^*$ for $v \mid \mathfrak{g}$. Let $\mathcal{U}_{\mathfrak{f}\mathfrak{g}} = \prod_{v \mid \mathfrak{f}\mathfrak{g}} U_{v,\mathfrak{f}}$.

Proposition 9.1.1 (Proposition 3.4, [8])**.** *Conjecture 2.4.1 is equivalent to the existence of an element $u_\lambda \in U_{\mathfrak{p}}$ with $u_\lambda \equiv 1 \pmod{\lambda}$ and*

$$(u_\lambda^{\sigma_{\mathfrak{b}}}, 1) = \pi^{\zeta_{R,T}(H/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}\times\mathcal{U}_{\mathfrak{f}\mathfrak{g}}/\overline{E_+(\mathfrak{f})}} x \, d\mu(\mathfrak{b}, x)$$

in $(F_{\mathfrak{p}}^* \times \mathcal{U}_{\mathfrak{f}\mathfrak{g}})/\overline{E_+(\mathfrak{f})}$ for all fractional ideals $\mathfrak{b}$ relatively prime to S.

Since the full strength of the integral Gross–Stark conjecture (Conjecture 2.4.1) has not yet been proved, we work with the following corollary. This corollary instead uses the l-part of the integral Gross–Stark conjecture.

Corollary 9.1.2. *Theorem 2.4.4 is equivalent to the existence of an element $u_\lambda \in U_\mathfrak{p}$ with $u_\lambda \equiv 1 \pmod{\lambda}$ and*

$$(u_\lambda^{\sigma_\mathfrak{b}}, 1) = \pi^{\zeta_{R,\lambda}(H/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbb{O}\times\mathcal{U}_{\mathfrak{f}\mathfrak{g}}/\overline{E_+(\mathfrak{f})}} x \, d\mu(\mathfrak{b}, x)$$

in $((F_\mathfrak{p}^* \times \mathcal{U}_{\mathfrak{f}\mathfrak{g}})/\overline{E_+(\mathfrak{f})}) \otimes \mathbb{Z}_l$ for all fractional ideals $\mathfrak{b}$ relatively prime to S.

Define

$$D(\mathfrak{f}, \mathfrak{g}) = \{x \in F_\mathfrak{p}^* : (x, 1) \in \overline{E_+(\mathfrak{f})} \subset F_\mathfrak{p}^* \times \mathcal{U}_{\mathfrak{f}\mathfrak{g}}\}.$$

Dasgupta notes in [8] that Proposition 9.1.1 may be interpreted as stating that Conjecture 2.4.1 is equivalent to a formula for the image of u_λ in $F_\mathfrak{p}^*/D(\mathfrak{f}, \mathfrak{g})$. Similarly, Corollary 9.1.2 states that Theorem 2.4.4 is equivalent to a formula for the image of u_λ in $(F_\mathfrak{p}^*/D(\mathfrak{f}, \mathfrak{g})) \otimes \mathbb{Z}_l$. The reciprocity map of class field theory induces an isomorphism

$$\mathrm{rec}_S : (F_\mathfrak{p}^* \times \mathcal{U}_{\mathfrak{f}\mathfrak{g}})/\overline{E_+(\mathfrak{f})_\mathfrak{p}} \cong \mathrm{Gal}(H_S/H).$$

As before, we have defined $E_+(\mathfrak{f})_\mathfrak{p}$ as the group of totally positive $\mathfrak{p}$-units congruent to 1 modulo $\mathfrak{f}$.

Proposition 9.1.3. *Assume Conjecture 2.4.1. Let $\sigma \in G$. The construction, $u_2(\sigma)$, is equal to the Brumer–Stark unit, $u_\mathfrak{p}(\sigma)$, in $F_\mathfrak{p}^*/D(\mathfrak{f}, \mathfrak{g})$, i.e.,*

$$u_2(\sigma) \equiv u_\mathfrak{p}(\sigma) \pmod{D(\mathfrak{f}, \mathfrak{g})}.$$

Proof. We consider the unit $u_2(\sigma)$ and apply rec_S to $(u_2(\sigma), 1)$, then by *e)* of Proposition 5.1.1 we have

$$\mathrm{rec}_S((u_2(\sigma), 1)) = \prod_{\substack{\tau\in\mathrm{Gal}(H_S/F),\\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,\lambda}(H_S/F,\tau^{-1},0)} = \mathrm{rec}_S((u_\mathfrak{p}(\sigma), 1))$$

where the second equality follows from (2.9) which, as we noted in §2.4, follows from Conjecture 2.4.1. Thus, we have the result. □

Again, since the full strength of the integral Gross–Stark conjecture (Conjecture 2.4.1) has not yet been proved, we work with the following corollary. It is clear that we have the following corollary which gives the weaker result obtainable by using the l-part of the integral Gross–Stark conjecture (Theorem 2.4.4).

Corollary 9.1.4. *Let $\sigma \in G$. The construction, $u_2(\sigma)$, is equal to the Brumer–Stark unit, $u_\mathfrak{p}(\sigma)$, in $(F_\mathfrak{p}^* \otimes \mathbb{Z}_l)/D(\mathfrak{f}, \mathfrak{g})$.*

Let $\mathfrak{q}$ be a prime of F that is unramified in H and whose associated Frobenius $\sigma_{\mathfrak{q}}$ is a complex conjugation in H.

Lemma 9.1.5. *Let l be a rational prime, and $m \in \mathbb{Z}_{\geq 1}$. There exists a finite set of prime ideals $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_s\}$ in the narrow ray class of $\mathfrak{q}$ modulo $\mathfrak{f}$ such that the group $D(\mathfrak{f}, \mathfrak{r}_1 \ldots \mathfrak{r}_s)$ does not contain $\mu'_{l^m}(F^*_{\mathfrak{p}})$. Moreover, $\mu'_{l^m}(F^*_{\mathfrak{p}})$ is the set of non-trivial roots of unity of order l^m.*

Proof. We follow the ideas in the proof of Lemma 5.17 in [8]. Let $\varepsilon \in \mu'_{l^m}$ then there exists a prime, $\mathfrak{r}$, of F such that $\mathfrak{r}$ is in the narrow ray class of $\mathfrak{q}$ modulo $\mathfrak{f}$ and such that ε is not congruent to 1 modulo $\mathfrak{r}$.

Suppose now that $\varepsilon \in D(\mathfrak{f}, \mathfrak{r})$. Then by the definition of $D(\mathfrak{f}, \mathfrak{r})$, and in particular the definition of $\mathcal{U}_{\mathfrak{f}\mathfrak{r}}$, we see that $\varepsilon \equiv 1 \pmod{\mathfrak{r}}$. This contradicts our choice of $\mathfrak{r}$. Letting the $\mathfrak{r}_i$ consist of such an ideal prime $\mathfrak{r}$, for each element $\varepsilon \in \mu'_{l^m}$, completes the proof. □

The following corollary is stated as a remark in [13]. We include the proof for completeness.

Corollary 9.1.6 (Remark 6.4 (c), [13])**.** *Suppose $\mathfrak{q} \in S$. Let $\mathfrak{r}$ be a nonarchimedean place of F with $\mathfrak{r} \notin S \cup \overline{\lambda}$ and $\mathfrak{r}$ in the narrow ray class of $\mathfrak{q}$ modulo $\mathfrak{f}$. Put $S' = S \cup \{\mathfrak{r}\}$. Then, we have*

$$u_2(S', \sigma) = u_2(S, \sigma)^2.$$

Proof. From *c)* of Proposition 5.1.1, we have $u_2(S', \sigma) = u_2(S, \sigma) u_2(S, \sigma_{\mathfrak{r}}\sigma)^{-1}$. Applying *c)* again and writing $S'' := S - \{\mathfrak{q}\}$, we deduce

$$\begin{aligned} u_2(S', \sigma) &= (u_2(S'', \sigma) u_2(S'', \sigma_{\mathfrak{q}}\sigma)^{-1})(u_2(S'', \sigma_{\mathfrak{r}}\sigma)^{-1} u_2(S'', \sigma_{\mathfrak{q}}\sigma_{\mathfrak{r}}\sigma)) \\ &= (u_2(S'', \sigma) u_2(S'', \sigma_{\mathfrak{q}}\sigma)^{-1})(u_2(S'', \sigma_{\mathfrak{q}}\sigma)^{-1} u_2(S'', \sigma)) \\ &= u_2(S, \sigma)^2. \end{aligned}$$

□

We are now able to prove the main theorem of this chapter.

Proof of Theorem 2.3.10. The proof of this proposition follows the ideas of the proof of Theorem 5.18 in [8]. We begin by noting that the roots of unity in $F^*_{\mathfrak{p}}$ have order that divides $p^a(\mathrm{N}\mathfrak{p} - 1)$ for some $a \in \mathbb{Z}_{\geq 0}$. Let l^m be an odd prime power that exactly divides $p^a(\mathrm{N}\mathfrak{p} - 1)$. We show that

$$u_2(\sigma) = u_{\mathfrak{p}}(\sigma) \text{ in } F^*_{\mathfrak{p}} / \mu_{p^a(\mathrm{N}\mathfrak{p}-1)/l^m}(F^*_{\mathfrak{p}}). \tag{9.1}$$

Repeating this for each such odd prime power, we have the result. Fix such a prime power l^m. Let $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_s\}$ be a finite set of prime ideals as in Lemma 9.1.5, and let $\mathfrak{r}$ be one of the $\mathfrak{r}_i$. It follows from

$$\zeta_{R \cup \{\mathfrak{r}\}}(H/F, \sigma, s) = \zeta_R(H/F, \sigma, s) - \mathrm{N}\mathfrak{r}^{-s} \zeta_R(H/F, \sigma\sigma_{\mathfrak{r}}^{-1}, s)$$

that the Brumer–Stark units attached to S and $S \cup \{\mathfrak{r}\}$ are related by

$$u_{\mathfrak{p}}(S \cup \{\mathfrak{r}\}, \sigma) = \frac{u_{\mathfrak{p}}(S, \sigma)}{u_{\mathfrak{p}}(S, \sigma\sigma_{\mathfrak{r}}^{-1})} = u_{\mathfrak{p}}(S, \sigma)^2,$$

where this last equation follows from the fact that complex conjugation acts as inversion on $U'_{\mathfrak{p}}$. Thus, if we let $S' \coloneqq S \cup \{\mathfrak{r}_1, \dots, \mathfrak{r}_s\}$, then we inductively obtain

$$u_{\mathfrak{p}}(S', \sigma) = u_{\mathfrak{p}}(S, \sigma)^{2^s}. \tag{9.2}$$

Applying Corollary 9.1.6 inductively, we also have

$$u_2(S', \sigma) = u_2(S, \sigma)^{2^s}. \tag{9.3}$$

We showed in Corollary 9.1.4 that $u_2(S', \sigma) \equiv u_{\mathfrak{p}}(S', \sigma) \pmod{D(\mathfrak{f}, \mathfrak{r}_1 \dots \mathfrak{r}_s)}$ in $F_{\mathfrak{p}}^* \otimes \mathbb{Z}_l$. By our choice of the $\mathfrak{r}_i$, we have that μ'_{l^m} is not contained in $D(\mathfrak{f}, \mathfrak{r}_1 \dots \mathfrak{r}_s)$. Since the roots of unity in $\mathbb{Z}_l$ are of order $l-1$, tensoring $F_{\mathfrak{p}}^*$ by $\mathbb{Z}_l$ does not add in any additional roots of unity of order l^m. It then follows from Theorem 2.3.6 and Corollary 9.1.4 that (9.1) holds. As we noted above, repeating this for each prime power which exactly divides $p^a(\mathrm{N}\mathfrak{p} - 1)$ gives us the result. □

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Appendix A

Appendix

A.1 Translating Shintani domains

Overcoming the lack of a nice translation property for Shintani domains in §7.1 is the main work of Chapter 7. In this appendix, we first provide an explicit counterexample which shows why this work is necessary. We then show the calculations which give rise to the figures. These figures demonstrate our method to overcome this counterexample, namely, Figure 7.1, Figure 7.2, and Figure 7.3. We begin by finding a counterexample to the following statement of Tsosie in [24]. The statement below is given for F of any degree $n > 1$. We provide a counterexample when F is a cubic field as this is the case we work with in Chapter 7.

Statement A.1.1. *Let V be a finite index subgroup of $E_+(\mathfrak{f})$ and let $\epsilon_1, \dots, \epsilon_{n-1}$ be a $\mathbb{Z}$-basis for V. Furthermore, let $\mathfrak{D}$ be a fundamental domain for the action of V on $\mathbb{R}^n_+$ and $\pi^{-1} \in \mathfrak{D}$. Then for $\epsilon = \prod_{i=1}^{n-1} \epsilon_i^{m_i}$,*

$$\epsilon\mathfrak{D} \cap \pi^{-1}\mathfrak{D} = \varnothing$$

unless $m_i \in \{0, 1\}$, $1 \le i \le n-1$.

We note that in general there appears to be no bounds that can be put on the set which the m_i's are allowed to be in to make this statement hold. However, we do not provide explicit evidence for this here.

Remark A.1.2. *It is straightforward to show that this statement holds when F is of degree 2. It is for this reason that Dasgupta–Spieß's proof that $u_1 = u_3$, in the case F is of degree 2, is much shorter.*

The computations used to find our counterexample below are done using Magma. Let F be the number field with defining polynomial $2x^3 - 4x^2 - x + 1$ over $\mathbb{Q}$. F is then a totally real number field of degree 3. We define

$$H = F(\sqrt{-2}).$$

H is then totally complex. It is also a degree 2 extension of F so H is a CM extension of F. We note that the extension H/F is abelian. Now, choose $y \in F$ such that we can write

$$F = \mathbb{Q}(y).$$

Let $\mathfrak{f}$ be the conductor of H/F. We calculate, as the generators of $E_+(\mathfrak{f})$, the elements $g_1 = -96y^2 + 152y + 113$ and $g_2 = 160y^2 + 32y - 31$, i.e., we have

$$\langle -96y^2 + 152y + 113\ ,\ 160y^2 + 32y - 31 \rangle = E_+(\mathfrak{f}).$$

We choose as our rational prime $p = 113$. We make this choice as there are two primes of F above 113 and both of them split completely in H. We choose $\mathfrak{p} \mid p$, a prime ideal of F that splits completely in H. We find that the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$ is 2. We choose an element π to satisfy the following

- π is totally positive,
- $\pi \equiv 1 \pmod{\mathfrak{f}}$,
- $(\pi) = \mathfrak{p}^2$,
- $\pi^{-1} \in \overline{C}_{e_1}([g_1 \mid g_2]) \cup \overline{C}_{e_1}([g_2 \mid g_1])$.

In particular, we choose $\pi = 192y^2 - 488y + 177$. Let $\mathfrak{D} = \overline{C}_{e_1}([g_1 \mid g_2]) \cup \overline{C}_{e_1}([g_2 \mid g_1])$ and note that this is a Shintani domain. With these choices, we calculate that $\pi^{-1}\mathfrak{D} \cap g_1 g_2^{-1}\mathfrak{D} \neq \varnothing$ and $\pi^{-1}\mathfrak{D} \cap g_2^{-1}\mathfrak{D} \neq \varnothing$. This completes our counterexample to Statement A.1.1. Furthermore, the curved nature of the domains, as illustrated further with Figure A.1 below, gives a good reason as to why results bounding where $\pi^{-1}\mathfrak{D}$ is contained should not be possible without considerable work.

To make our example clearer, we include below a plot of $\mathfrak{D} \cup g_1\mathfrak{D} \cup g_2\mathfrak{D} \cup g_1 g_2\mathfrak{D}$ (in blue) and $\pi^{-1}\mathfrak{D}$ (in red) under the map $\varphi_{(g_1,g_2)}$ (Figure A.1). This plot is drawn using MATLAB. Notice that the boundary of $\pi^{-1}\mathfrak{D}$ falls outside that of $\mathfrak{D} \cup g_1\mathfrak{D} \cup g_2\mathfrak{D} \cup g_1 g_2\mathfrak{D}$. As we remarked with the other diagrams, although the image appears to show that some of the lines overlap, this does not happen. This only appears in the diagram due to the fixed thickness of the lines.

We now make note of the calculations we made to obtain Figure 7.1, Figure 7.2, and Figure 7.3. We continue to hold all of the choices made thus far in this appendix. We define

$$\varepsilon_1 = g_1^{-3} g_2^4 \quad \text{and} \quad \varepsilon_2 = g_1^{-5}.$$

These choices are found using Magma so that ε_1 and ε_2 satisfy the conditions in Lemma 7.1.5. We find that when considering Corollary 7.1.7, we can choose $l = 1$ to satisfy the conditions given, i.e., ε_1 and ε_2 are already good enough to obtain Corollary 7.1.7. Using MATLAB, we plot Figure 7.1. We define

$$\pi_1 = g_1^{-6} g_2^2 \pi \quad \text{and} \quad \pi_2 = g_1^{-6} g_2 \pi,$$

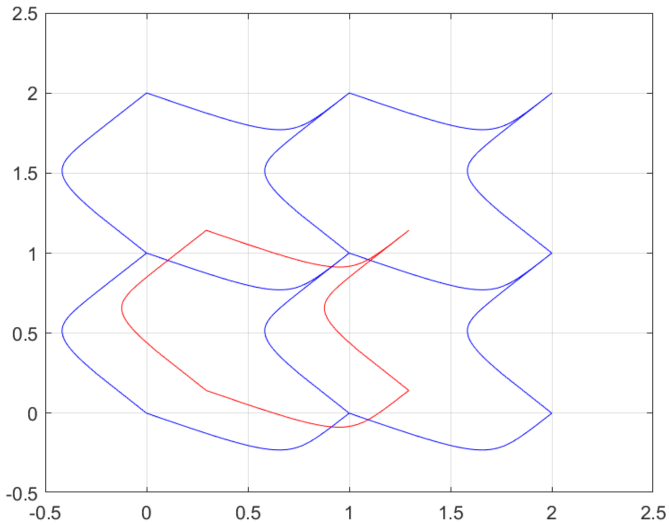

Figure A.1: A counterexample

where π is as defined before. Using π_1 as our choice of π, and using MATLAB, we plot Figure 7.2 which shows Case 1. Similarly, using π_2 as our choice of π, Figure 7.3 shows Case 2.

We end this appendix by giving a conjecture regarding these Shintani domains in a more general setting. The proving of this conjecture should allow one to give a direct proof that $u_1 = u_3$ in the same style as in Chapter 7.

Let $1, x_1, \ldots, x_{n-1} \in \mathbb{R}^n_+$ be linearly independent vectors where $1 = (1, \ldots, 1)$. Then

$$\mathscr{D} \coloneqq \bigcup_{\tau \in S_{n-1}} \overline{C}_{e_1}([x_{\tau(1)} \mid \ldots \mid x_{\tau(n-1)}])$$

is a fundamental domain for the action of $E = \langle x_1, \ldots, x_{n-1} \rangle$ on $\mathbb{R}^n_+$.

Conjecture A.1.3. *For all $R > 0$ and $0 < r < R/2$, there exists $y_1, \ldots, y_{n-1} \in E$ such that we have the following.*

1. *$V = \langle y_1, \ldots, y_{n-1} \rangle$ is a subgroup of E, free of rank $n-1$.*
2. *If we write*

$$\mathscr{B} = \bigcup_{\tau \in S_{n-1}} \overline{C}_{e_1}([y_{\tau(1)} \mid \ldots \mid y_{\tau(n-1)}]),$$

then

- *for all $y \in \mathscr{B}$ there exists $z \in B(y, R)$ such that $B(z, r) \subseteq B(y, R) \cap \mathscr{B}$ and*
- *for all $z \in \mathscr{B}$ such that $B(z, r) \subseteq \mathscr{B}$ we have*

$$z\mathscr{B} \cap (y_1^{\alpha_1} \ldots y_{n-1}^{\alpha_{n-1}})\mathscr{B} = \varnothing$$

unless $\alpha_j \in \{0, 1\}$ for $j = 1, \ldots, n-1$.

Remark A.1.4. *We first note that 2. in Conjecture A.1.3 is equivalent to*

$$z\mathscr{B} \subseteq \bigcup_{\alpha_1=0}^{1} \cdots \bigcup_{\alpha_{n-1}=0}^{1} (y_1^{\alpha_1} \dots y_{n-1}^{\alpha_{n-1}})\mathscr{B}.$$

Furthermore, Conjecture A.1.3 is trivial when $n = 2$. *Even with* $n = 3$, *we currently are not able to prove a statement as strong as this conjecture. Instead we prove Proposition 7.1.9.*