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Cylindrical Lévy Processes in the Lévy White Noise Approach

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Abstract

We study the regularity properties of cylindrical Lévy processes and Lévy space-time white noises, by examining their embeddings on the one hand in the spaces of general and tempered (Schwartz) distributions, and on the other hand in weighted Besov spaces. In this manner we analyse when the embedded Lévy object possesses a regularised version in the sense of Itô and Nawata.

Lévy space-time white noises are defined as independently scattered random measures and cylindrical Lévy processes are defined by means of the theory of cylindrical processes. It is shown that Lévy space-time white noises correspond to an entire subclass of cylindrical Lévy processes, which is completely characterised by the characteristic functions of its members. We embed the Lévy space-time white noise, or the corresponding cylindrical Lévy process, in the space of general and tempered distributions and establish that in each case the embedded cylindrical processes are induced by (genuine) Lévy processes in the corresponding space.

We use wavelet analysis to characterise the Lévy measures in weighted Besov spaces. Then we characterise the ranges of such Besov spaces in which $L^2(\mathbb{R}^d)$ is or is not embedded continuously and the embedding is or is not Radonifying. We apply these results, given a cylindrical Lévy process L in $L^2(\mathbb{R}^d)$, to characterise when L is and is not induced by a Lévy process in a given Besov space. These results are then applied to give sharp Besov regularity analysis to two important classes of cylindrical Lévy processes, the canonical stable cylindrical process, and 'hedgehog' processes constructed as a P-a.s. weakly convergent infinite random sum.

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Contents

Dedicated to Jessica and Milo.
Dream big, you can achieve anything you set your mind to.

Chapter 1

Introduction

Stochastic analysis, and in particular the study of stochastic partial differential equations (SPDEs), has in recent years become a vibrant area of mathematics with a large and growing body of work being published and recognised. SPDEs are partial differential equations which are perturbed by a 'noise', the definition of which is by no means trivial. These equations arise naturally in the modelling of randomness in physical and economic processes (see the Introduction of [24] for many examples). To model the noise and give rigorous meaning to the notions of solution and well-posedness and then study these equations has been an active area of research since the early 1970s; see the essay [98] by Zambotti for a well-written brief history of the field.

There are a number of approaches to modelling random perturbations of partial differential equations; usually one follows either a semi-group approach, based on the work by Da Prato and Zabczyk in [24], or a random field approach, originating from the work by Walsh in [94]. Gaussian perturbations are most often modelled either as a cylindrical Brownian motion, corresponding to the former, or a Gaussian space-time white noise as developed in the latter. It is well known in the Gaussian case that both models essentially result in the same dynamics as established by Dalang and Quer-Sardanyons in [26].

Another approach to model such perturbed dynamical systems, e.g. parabolic stochastic partial differential equations, is provided by the recently introduced ambit fields, presented in the monograph [13] by Barndorff-Nielsen, Benth and Veraart, and their relations to SPDE investigated in [12] by the same authors, where they also establish the link between this approach and the random field theory of Walsh.

In many situations, it is natural to assume that the noise in the system has jumps and heavy tails, and to reflect these important features one must use a model of noise more general than Gaussian. Cylindrical Brownian motions can be naturally generalised to cylindrical Lévy processes by exploiting the theory of cylindrical measures and random variables. This was accomplished by Applebaum and Riedle in [6]. In the random field approach, Gaussian space-time white noise is generalised to Lévy space-time white noise as an infinitely divisible random measure, often represented by integrals with respect to Gaussian and Poisson random measures. Both generalisations, cylindrical Lévy processes and Lévy space-time white noises, may be used as models for random perturbations of complex dynamical systems.

These applications can be found for cylindrical Lévy processes for example in the monograph Peszat and Zabczyk [67] or in Kumar and Riedle [56], and for Lévy space-time white noise in Applebaum and Wu [7], Chong [22] and Chong and Kevei [23] among many others.

It is also possible to define Lévy space-time white noise by Lévy additive sheets and generalised random processes (that is, random variables in a space of distributions). Just as the Brownian sheet is the generalisation of a Brownian motion to a multidimensional index set, additive sheets are defined as the corresponding generalisation of an additive process. Adler et al. [2] first defined additive random fields on $\mathbb{R}^d$, and termed them 'Lévy processes' should they be stochastically continuous. In [27], Dalang and Walsh discuss Lévy sheets in $\mathbb{R}^2$. Additive fields with stationary increments are considered by Barndorff-Nielsen and Pedersen in [11] and are called 'homogeneous Lévy sheets'. The study of Lévy white noise as a distribution has been undertaken by Dalang, Humeau, Unser and co-authors, e.g. [9, 25, 31, 32].

Probability theory in infinite-dimensional spaces presents many challenges, particularly outside Hilbert spaces where the underlying geometry interacts with probability in many complex ways. As an example, one may consider the concepts of type and cotype in Banach spaces, see e.g. the monograph [41] by Hytönen et al. The concept of cylindrical probability measures arises naturally in the setting of infinite dimensions. Cylindrical probability measures are finitely-additive set functions whose pushforward under projections onto finite dimensional spaces are genuine probability measures. Their usefulness arises from the fact that the infinite-dimensional analogue of Bochner's Theorem links normalised continuous positive-definite functions with cylindrical probabilities (see [92, p. VI.3]). A simple example is that when generalising a standard normal distribution to an infinite-dimensional Banach space, in this case the analogue of the standard Gaussian measure is a cylindrical probability [72]. Indeed, even in a Hilbert space H the function $\varphi\colon H \to \mathbb{C}$ given by

$$\varphi(h) = \exp\left(-\frac{1}{2}\langle h, Qh\rangle\right)$$

is the characteristic function of a genuine probability measure if and only if the covariance operator Q is nuclear [92, Th. 5.4]. As the identity operator Id does not meet this requirement, this shows that the natural generalisation of the standard Gaussian is a cylindrical probability.

The task of measuring regularity (or smoothness as described by Triebel in Chapter 1 of [91]) of functions and distributions incorporates concepts of continuity, differentiability, integrability and asymptotic growth/decay. The weighted Besov spaces allow all these concepts to be addressed within a single multiparametric family. Besov space analysis has been applied to the study of regularity of sample paths for finite-dimensional Lévy processes [40, 81, 82] and Lévy white noise [9, 25, 31, 32, 93]. The sample path regularity analysis shows the regularity in time, and is a direct generalisation of the well-known result that the sample paths of Brownian motion in $\mathbb{R}^d$ are a.s. Hölder continuous with exponent $< \frac{1}{2}$. This may be stated in functional-analytic terms as $W(\cdot) \in \mathcal{C}^s(\mathbb{R}^d)$ P-a.s. for any $s < \frac{1}{2}$, where $\mathcal{C}^s(\mathbb{R}^d)$ is the Hölder-Zygmund space with index s. As Lévy processes generally have jumps, the Hölder-Zygmund spaces of continuous functions are no longer appropriate and the Besov space scale forms a suitable extension. The Besov spaces $B_s^{p,q}(\mathbb{R}^d)$ are a natural generalisation of Hölder-Zygmund and fractional Sobolev spaces into a single scale [89, 90, 91]. The parameter range is defined for $0 < p, q \leqslant \infty$ and $s, w \in \mathbb{R}$. By way of introduction, the key relations that we have between these spaces are as follows:

- The Hölder-Zygmund space $\mathcal{C}^s(\mathbb{R}^d) = B_s^{\infty,\infty}(\mathbb{R}^d)$ for $s \in \mathbb{R}$ [91, S1.2].
- The fractional Sobolev space $H_s^2(\mathbb{R}^d) = B_s^{2,2}(\mathbb{R}^d)$ for $s \in \mathbb{R}$ [90, S1.3.2]. In particular, $L^2(\mathbb{R}^d) = B_0^{2,2}(\mathbb{R}^d)$.
- Let $k_0 \neq k_1 \in \mathbb{Z}_+$ and $0 < \vartheta < 1$. Then, for $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ the real interpolation space of the Sobolev spaces $(H_{k_0}^p(\mathbb{R}^d), H_{k_1}^p(\mathbb{R}^d))_{\vartheta,q} = B_s^{p,q}(\mathbb{R}^d)$, where $s = k_0(1-\vartheta) + k_1\vartheta$ [90, S1.6.4].
- Let $k \in \mathbb{N}$ and $0 < \vartheta < 1$. Then, for $0 < q \leqslant \infty$ the real interpolation space $(C^k(\mathbb{R}^d), C^0(\mathbb{R}^d))_{\vartheta,q} = B_{k(1-\vartheta)}^{\infty,q}(\mathbb{R}^d)$ [90, S4.4.2(34)].

Furthermore, there are the related Triebel-Lizorkin spaces $F_s^{p,q}(\mathbb{R}^d)$ for the same range of parameters; these spaces do not feature in this work so their relations are only quoted here for completeness.

- $B_s^{p,p\wedge q}(\mathbb{R}^d) \hookrightarrow F_s^{p,q}(\mathbb{R}^d) \hookrightarrow B_s^{p,p\vee q}(\mathbb{R}^d)$ for $0 < p < \infty$, $0 < q \leqslant \infty$ and $s \in \mathbb{R}$ [90, S2.3.2/2]. In particular, $F_s^{p,p}(\mathbb{R}^d) = B_s^{p,p}(\mathbb{R}^d)$.

- The fractional Sobolev space $H^p_s(\mathbb{R}^d) = F^{p,2}_s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 < p < \infty$ [90, S1.3.4/3]. In particular, $L^p(\mathbb{R}^d) = F^{p,2}_0(\mathbb{R}^d)$ and we have $B^{p,p\wedge 2}_0(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow B^{p,p\vee 2}_0(\mathbb{R}^d)$.

- The local Hardy spaces $h^p(\mathbb{R}^d) = F^{p,2}_0(\mathbb{R}^d)$ for $0 < p < \infty$ [89, S2.3.5].

We may explain the meaning of the s parameter by generalising the Laplacian operator Δ on $\mathbb{R}^d$. By the theory of Fourier multipliers, we have for $f \in H^2_2(\mathbb{R}^d)$

$$(\mathrm{Id} - \Delta) f = \mathcal{F}^{-1}\left((1 + |\cdot|^2)\, \mathcal{F} f\right),$$

and thus the fractional powers are given by

$$(\mathrm{Id} - \Delta)^s f = I_{2s} f := \mathcal{F}^{-1}\left((1 + |\cdot|^2)^s\, \mathcal{F} f\right).$$

This definition of $I_s f$ makes sense for all $f \in \mathcal{S}^*(\mathbb{R}^d)$ and $s \in \mathbb{R}$, thus giving an extension of $(\mathrm{Id} - \Delta)^{s/2}$. Then we have the result $I_s B^{p,q}_\sigma(\mathbb{R}^d) = B^{p,q}_{\sigma - s}(\mathbb{R}^d)$ for $0 < p, q \leqslant \infty$ and $s, \sigma \in \mathbb{R}$ [89, S2.3.8]. In this manner, one may interpret the s parameter as a measure of generalised fractional differentiability. The p parameter plays the usual role of a measure of integrability. The cylindrical random variables and processes we shall analyse are not in general integrable over the whole of $\mathbb{R}^d$. Indeed, for the integral

$$\int_{\mathbb{R}^d} (1 + |x|^2)^{\frac{w}{2}}\, \mathrm{d}x$$

to be finite, we require $w < -d$. More generally, the constants are in the weighted space $L^p(\mathbb{R}^d, \omega)$, defined as those $f \in L^0(\mathbb{R}^d)$ such that $\omega f \in L^p(\mathbb{R}^d)$, if $\omega(x) = (1 + |x|^2)^{\frac{w}{2}}$ for any $w < -\frac{d}{p}$, and continuous functions of polynomial growth of order k at infinity are in such spaces if $w < -\frac{d}{p} - k$. With this motivation, we shall study the weighted Besov spaces $B^{p,q}_{s,w}(\mathbb{R}^d)$, which have the simple interpretation that

$$B^{p,q}_{s,w}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}^*(\mathbb{R}^d) \colon (1 + |\cdot|^2)^{\frac{w}{2}} f \in B^{p,q}_{s}(\mathbb{R}^d) \right\}.$$

The discussion above regarding the s parameter and the results presented are unchanged by adding the w parameter, see [91, Ch. 6]. The q parameter in these Besov spaces, whilst of independent interest, shall not play a role in the analysis in this work; see Remark 2.1.2. Furthermore, we will focus on the spaces which are separable and reflexive Banach spaces, and so we shall consider the weighted Besov spaces $B^{p}_{s,w}(\mathbb{R}^d) := B^{p,p}_{s,w}(\mathbb{R}^d)$ for $p > 1$ and $s, w \in \mathbb{R}$.

1.1 Literature Review

The following brief notes are historical in nature and are intended to be neither comprehensive nor critical.

The lecture notes by Walsh from 1986 [94] are widely recognised as the first major monograph in the field of SPDEs, where the random field approach is set out and the theory of Itô integration is extended by the concept of the (worthy) martingale measure. Then in 1992 Da Prato and Zabyczyk published their book [24] detailing the semi-group approach, which naturally treats SPDEs as equations in infinite-dimensional spaces.

One of the first papers analysing SPDEs driven by a Lévy process in a Hilbert space was by Chojnowska-Michalik [20] from 1987, where necessary and sufficient conditions for the existence of mild solutions of Ornstein-Uhlenbeck type were presented. In 1988, Kallianpur and Pérez-Abreu published an analysis [48] of SPDEs on the dual of a nuclear space driven by semimartingales. A decade passed before the next such papers were published; Albeverio, Wu and Zhang [4] analysed SPDEs driven by Poisson white noise with second moments and established existence and uniqueness of their solution, and Mueller [62] studied the heat equation with an α-stable noise constructed as a Poisson random measure, finding sufficient conditions for the short-time existence of solutions.

Then the following decade saw a significant growth in the area, with a number of papers examining SPDEs driven by Lévy noise, either as a random measure (e.g. [3, 7, 10, 17, 60]), a genuine Lévy process in a Hilbert space (e.g. [61, 77, 86]) or as a distribution-valued random variable (e.g. [3, 60, 65])

A specific example of a cylindrical Lévy process appears in the monograph of 2007 by Peszat and Zabczyk [67, Section 7.2]; the authors term this process an *impulsive cylindrical process*, and it is constructed using a Poisson random measure. Further work by Zabczyk and collaborators saw two alternative constructions of cylindrical Lévy processes, namely as an infinite sum of real-valued Lévy processes [19, 68, 69], and as a subordinated cylindrical Brownian motion [18]. A theme arising from both approaches was to consider a cylindrical process on a Hilbert space H as a genuine Lévy process on a larger Hilbert space U such that $H \hookrightarrow U$; a Hilbert-Schmidt embedding has the property of mapping a cylindrical probability measure on H to a Radon probability measure on U; we shall return to this theme in the sequel.

In [18], Brzeźniak and Zabczyk study an Ornstein-Uhlenbeck process driven by a cylindrical Lévy process and analyse the time and space regularity of solutions; in particular they find that solutions do not in general have càdlàg modifications. A similar result is obtained for linear stochastic evolution equations in [19]. However, in [68], Peszat and Zabczyk find conditions for càdlàg versions of solutions of linear stochastic equations, and furthermore the authors present the cylindrical càdlàg property and give conditions under which it holds.

Zhang [99] studies SPDEs driven by an α-stable subordinated cylindrical Brownian motion, and derives the strong Feller property; then, in collaboration with Dong and Xu [30], the strong Feller property and exponential ergodicity of solutions of stochastic Burgers equations are shown. Wang and Rao [96] study the stability of solutions to SPDEs driven by α-stable cylindrical Lévy processes constructed both as a series of real-valued stable Lévy processes and as a subordinated cylindrical Brownian motion; Wang then in [95] gets gradient estimates for linear stochastic evolution equations driven by a cylindrical Lévy process constructed as a series of pure-jump real-valued Lévy processes. Chojnowska-Michalik and Goldys [21] study semilinear stochastic evolution equations driven by a sum of α-stable Lévy processes, and use the embedding of the resultant cylindrical Lévy process in a larger Hilbert space to study convergence to an invariant measure. Liu and Zhai [59] use a series-based α-semistable cylindrical Lévy process to study time regularity of generalised Ornstein-Uhlenbeck processes in Hilbert spaces and give necessary and sufficient conditions for càdlàg and weakly càdlàg modifications. In [57], Li uses a construction of cylindrical Lévy process as a series of α-stable real-valued processes to study fractional SPDEs and show existence and uniqueness of mild solutions.

Fonseca-Mora has, in a series of papers [34, 36, 35], examined cylindrical Lévy processes in the dual of a nuclear space, shown that every cylindrical process has a version which is a genuine Lévy process, and constructed a stochastic integration theory to study the abstract stochastic Cauchy problem, with multiplicative noise, in these spaces.

Herren analysed the Besov regularity of the paths of an α-stable process, for $\alpha \in (1, 2)$, on the unit interval [40]. Schilling then presented extensions of this result to Feller processes on the unit interval and a wider range of Besov space parameters [82], and then in [81] this result is generalised to weighted Besov and Triebel-Lizorkin spaces, which furthermore allows for the process to be considered on the entire half-line. In a series of papers, e.g. [9, 25, 31, 32], Dalang, Humeau, Unser and co-authors have studied the Lévy white noise Z defined as a distribution. Their model of noise is initiated from research on developing sparse statistical models for signal and image processing. Here, Z is defined as a cylindrical random variable in $\mathcal{D}^*(\mathbb{R}^d)$, i.e. a linear and continuous mapping $Z\colon \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega, P)$, with characteristic function

$$\varphi_Z\colon \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}, \qquad \varphi_Z(f) = \exp\left(\int_{\mathbb{R}^d} \psi\big(f(x)\big)\,\mathrm{d}x\right),$$

where $\psi\colon\ \mathbb{R} \to \mathbb{C}$ is defined by

$$\psi(u) := ipu - \tfrac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \big(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_{\mathbb{R}}}(y)\big)\,\nu_0(\mathrm{d}y), \tag{1.1.1}$$

for some constants $p \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$ and a Lévy measure ν_0 on $\mathbb{R}$.

The systematic study of cylindrical Lévy processes was introduced by Applebaum and Riedle in their paper of 2010, [6], based on the theory of cylindrical measures and cylindrical random variables as developed in the 1970s by Badrakian and Schwartz. Riedle then developed the theory in a series of papers and a number of co-authors. In [73] infinitely divisible cylindrical measures are classified in detail. Stochastic integration is then developed, first in [76] cylindrical Lévy processes in a Hilbert space with weak second moments, and then, in collaboration with Jakubowski, in [43] the integration theory was extended to general cylindrical Lévy processes in Hilbert spaces; the class of integrands in both cases is drawn from adapted Hilbert-Schmidt operator-valued stochastic processes. In the joint work with Kosmala [53] the class of integrands is generalised to p-summing operator valued processes. In [74], cylindrical Ornstein-Uhlenbeck processes in Banach spaces are introduced using the theory of stochastic integration of deterministic, operator-valued functions. The stochastic Cauchy problem driven by a stable cylindrical Lévy process is studied in [75], and, in collaboration with Kumar [55, 56], the general weak and mild solution of the stochastic Cauchy problem driven by an additive cylindrical Lévy process in a Hilbert space is derived. Finally, in collaboration with Kosmala, stochastic evolution equations with multiplicative cylindrical Lévy noise are studied in the variational approach in [54] and in the evolution equation approach in [52].

1.2 Outline of Thesis

The main objective of this thesis is to study the regularity properties of cylindrical Lévy processes and Lévy space-time white noises, by examining their embeddings on the one hand in the spaces of general and tempered (Schwartz) distributions, and on the other hand in weighted Besov spaces. In this work rather than studying the regularity of sample paths, we ask a related question: as cylindrical processes only exist in a weak sense on their defining space, can they be understood as arising from genuine stochastic processes in a larger space, and what is the regularity of the functions and distributions in this space? In this manner we analyse when the embedded cylindrical process possesses a regularised version in the sense of Itô and Nawata [42]. This manner of posing the question of regularity fits with the stochastic evolution equation approach, where the driving noise has a distribution which is stationary in time but may have spatial dependency. Furthermore, conditions for when a cylindrical random variable is induced by a (genuine) random variable have applications in the theory of stochastic integration by cylindrical Lévy processes, see for example [43, 53, 76].

We focus our studies herein on two different subsets of cylindrical Lévy processes, the general cylindrical Lévy processes in $L^2(\mathbb{R}^d)$, and the subset of cylindrical Lévy processes corresponding to Lévy space-time white noises. The regularisation of (that is, the existence of a genuine random variable which almost surely gives arise to) a cylindrical random variable in the space of distributions was demonstrated by Itô and Nawata in [42], and this result was applied by Fonseca-Mora to cylindrical Lévy processes in [35].

In order to achieve this, we begin by comparing cylindrical Lévy processes and

Lévy space-time white noises, and characterising when they are equivalent. Then we carry on to characterise the Lévy measures in weighted Besov spaces and the range of these spaces in which $L^2(\mathbb{R}^d)$ is embedded continuously and in such a manner that the images under the embedding of cylindrical probabilities and random variables are (genuine) Radon measures and random variables; in this case we say the embedding is *Radonifying*. It is well known that between Hilbert spaces, the embedding Radonifies every cylindrical random variable if and only if the embedding operator is Hilbert-Schmidt. Furthermore, this may be generalised to embeddings between Banach spaces (subject to moment constraints) by the theory of p-summing operators.

Lévy space-time white noises, defined by means of random measures, do not naturally distinguish the time domain. However, as we compare these with cylindrical processes, which are naturally indexed by time, we break off one coordinate as the time domain. For this purpose, we echo Walsh's definition of a martingale measure in [94] to define Lévy space-time white noise. To differentiate our setting from the various other definitions of Lévy space-time white noises in the literature, we call our model a *Lévy-valued random measure*. This construction allows us to model the aforementioned difference in behaviour of the noise in space versus time.

The comparison of cylindrical Lévy processes and Lévy space-time white noises shows significantly different results vis a vis the Gaussian situation. Only the standard cylindrical Brownian motion corresponds to the Gaussian space-time white noise, see e.g. Kallianpur and Xiong [49], and Gaussian space-time white noise always can be embedded in the space of tempered distributions, see e.g. Gel'fand and Vilenkin [37]. The property of independent scattering for random measures restricts the correspondence between cylindrical processes and space-time noises in the Gaussian setting to the standard case of the identity as the covariance operator. In the non-Gaussian case, it turns out that there is an entire subclass of cylindrical

Lévy processes corresponding to Lévy space-time white noises. We call this subclass *independently scattered cylindrical Lévy processes* according to its defining property. We completely characterise the subclass of independently scattered cylindrical Lévy processes by the particular form of the characteristic function of its members; Theorem 3.3.7 shows that a cylindrical Lévy process in L^p for some $p \geqslant 1$ is independently scattered if and only if its symbol is of the form

$$\begin{aligned}\vartheta_L(f) = i \int_{\mathcal{O}} f(x)\, \gamma(\mathrm{d}x) - \tfrac{1}{2} \int_{\mathcal{O}} f^2(x)\, \Sigma(\mathrm{d}x) \\ + \int_{\mathcal{O}\times\mathbb{R}} \left(e^{if(x)y} - 1 - if(x)y\mathbb{1}_{B_{\mathbb{R}}}(y) \right) \nu(\mathrm{d}x, \mathrm{d}y)\end{aligned}$$

for certain measures γ, Σ and ν.

In order to develop the theory of when a cylindrical Lévy process in $L^2(\mathbb{R}^d)$ is induced by a Lévy process in some Besov space, i.e. the embedded cylindrical process possesses a regularised version, our first task is to characterise the Lévy measures in weighted Besov spaces. In most Banach spaces, an explicit characterisation of Lévy measures is not available. One of the exceptions is Lévy measures on the sequence spaces ℓ^p due to a result by Yurinskii in [97]. This result in general gives separate conditions for necessity and sufficiency of a σ-finite Radon measure on a real separable Banach space to form a Lévy measure. Using the wavelet characterisation of Besov spaces, these results by Yurinskii will enable us to derive the characterisation of Lévy measures on $B^p_{s,w}(\mathbb{R}^d)$ for each $p > 1$ and $s, w \in \mathbb{R}$. Using a definition based on wavelets, for example one may consider the Daubechies wavelets [28], is very convenient as it allows us to naturally develop techniques based on a natural isometry with sequence spaces. We obtain the characterisation, in Theorem 4.1.2, that a σ-finite Borel measure μ is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$ if and only if

(1) for $p \geqslant 2$,

$$\int_{B^p_{s,w}} \left(\|f\|^p_{B^p_{s,w}} \wedge 1\right) \mu(\mathrm{d}f) < \infty,$$

$$\sum_{j,G,m} (\omega^j_m)^p \left(\int_{\|f\|_{B^p_{s,w}} \leqslant 1} [\Psi^{j,G}_m, f]^2 \, \mu(\mathrm{d}f)\right)^{p/2} < \infty;$$

(2) and for $p \in (1,2)$,

$$\int_{B^p_{s,w}} \left(\|f\|^2_{B^p_{s,w}} \wedge 1\right) \mu(\mathrm{d}f) < \infty,$$

$$\sum_{j,G,m} (\omega^j_m)^p \int_0^\infty \left(1 - e^{\int_{\|f\|_{B^p_{s,w}} \leqslant 1} \left(\cos \tau[\Psi^{j,G}_m, f] - 1\right) \mu(\mathrm{d}f)}\right) \frac{\mathrm{d}\tau}{\tau^{1+p}} < \infty.$$

In the expressions above, ω^j_m are weighting constants and $\Psi^{j,G}_m$ are the wavelets used to define $B^p_{s,w}(\mathbb{R}^d)$.

We then explore the theory of Radonifying embeddings applied to the embeddings of $L^2(\mathbb{R}^d)$ into $B^p_{s,w}(\mathbb{R}^d)$ when such continuous embeddings exist. We present sharp ranges of when the embedding of $L^2(\mathbb{R}^d)$ into particular Besov spaces are Radonifying, both generally and subject to moment conditions. The theory of p-Radonifying operators and their link to p-summing operators is due to L. Schwartz (see e.g. [85]); we apply this theory to the embedding operators and then extend by factorisation. We are also able to apply the characterisation we have developed of Lévy measures in Besov spaces to give negative results for Radonification in the cases where we cannot obtain results using the link with p-summing operators, and in this manner we are able to give the sharp ranges for our results as detailed in Theorem 4.2.6.

In Figure 1.1, E_p is the region where $L^2(\mathbb{R}^d)$ is embedded in $B^p_{s,w}(\mathbb{R}^d)$, and R_p is the subset of E_p such that the embedding is Radonifying.

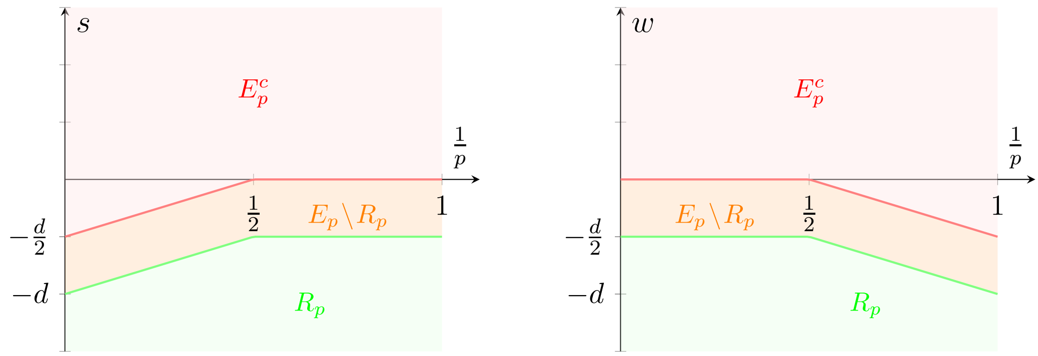

FIGURE 1.1: Triebel diagrams for Radonification

We embed Lévy space-time white noises and, due to the aforementioned correspondence, independently scattered cylindrical Lévy processes, in the space of distributions and tempered distributions. Although the embedding in the former case is possible for all Lévy space-time white noises, the embedding to the space of tempered distribution is restricted to members of a subclass satisfying a certain integrability condition. For both embeddings, we show that the embedded cylindrical Lévy process is induced by a genuine Lévy process in the space of general or tempered distributions, i.e. the embedded cylindrical process possesses a regularised version.

The embedding results enable us to compare the Lévy space-time white noise with the model of Lévy-type noise in the space of distributions. However, it turns out that these two models result in the same object only for Lévy space-time white noises which are additionally assumed to be stationary in space. Similar questions such as the embedding to the space of tempered distributions and the relation to independently scattered infinitely divisible random measures are addressed in Dalang and Humeau [25] and Fageot and Humeau [32] for the Lévy-type noise in the space of distributions. To complete the analysis, we compare Lévy space-time white noises with Lévy additive sheets. We establish the relation between Lévy space-time white noise and additive sheets, which is given by the integration of the Lévy space-time white noise in space, i.e. Lévy space-time white noise can be seen as the weak derivative of a Lévy additive sheet. This relation is established to be one-to-one for Lévy space-time white noise without fixed point of discontinuity in space.

We then turn our attention to the regularisation question in weighted Besov spaces. In the restricted case of independently scattered Lévy processes which are stationary in space, we can combine our results with the work Aziznejad, Fageot and Unser [9] to determine the range of Besov spaces in which a cylindrical Lévy process attains its values. This indicates a potential reasoning for the often observed phenomena of irregular trajectories of solutions of heat equations driven by cylindrical Lévy processes, e.g. in Brzeźniak and Zabczyk [18] and Priola and Zabczyk [69].

For the general case, given a (non-Gaussian) cylindrical Lévy process L in $L^2(\mathbb{R}^d)$, we give sharp results for when L is induced by a Lévy process Y in a Besov space $B^p_{s,w}(\mathbb{R}^d)$. Our technique is to study when the cylindrical Lévy measure μ of L may be extended to a Radon measure which is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$; in this case we are then able to show the existence of the Lévy process Y in $B^p_{s,w}(\mathbb{R}^d)$ such that the finite-dimensional projections of Y and L agree almost surely.

This thesis starts with the preliminary Chapter 2, where we collect the definitions and the fundamental results we shall need regarding weighted Besov spaces, infinitely divisible random measures and additive sheets. Some of these results could not be found in the literature so we present proofs in these cases. We further present the theory of cylindrical Lévy processes as developed by Applebaum and Riedle.

In Chapter 3, we begin by presenting our precise definitions of Lévy-valued random measures and Lévy-valued sheets, recall some known results from the literature and add a few observations particular to our approach. Following on from this, Section 3.3 is devoted to the comparison of cylindrical Lévy processes and Lévy-valued random measures. Our main results here characterise exactly the sub-class of cylindrical Lévy processes which correspond to Lévy-valued random measures.

Chapter 4 presents new results in the study of probability theory in weighted Besov spaces. In Section 4.1, we characterise the Lévy measures in weighted Besov spaces. We then in Section 4.2 present the complete theory of Radonifying embeddings of $L^2(\mathbb{R}^d)$ into weighted Besov spaces $B^p_{s,w}(\mathbb{R}^d)$ in terms of the parameter set (p, s, w).

The final Chapter 5 is dedicated to applications of the theory developed in the previous Chapters. In Section 5.1 we present our first two main results on the embedding of Lévy-valued random measures in the space of distributions and tempered distributions. In Section 5.1.1 we complete the picture by establishing Lévy-valued random measures as the weak derivative of Lévy-valued additive sheets. In Section 5.2 we give a general characterisation of when L^2-cylindrical Lévy processes may be regularised in specific weighted Besov spaces. Finally, we study in depth two important classes of cylindrical Lévy process: in Section 5.3 we analyse the canonical symmetric-α-stable process, and Section 5.4 is devoted to the cylindrical Lévy process representable as an infinite sum of independent one-dimensional Lévy processes, which we call the *hedgehog process*. In both cases we present a full characterisation of the parameter set where the cylindrical process is and is not regularised.

Chapter 2

Preliminaries and Notation

We take $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geqslant 0\}$. All vector spaces are over $\mathbb{R}$. We shall assume throughout the text that we are working in $\mathbb{R}^d$ for a fixed dimension d, and we fix a probability space $(\Omega, \mathcal{A}, P)$. Sequences are referred to by $(x_i)_{i \in I}$; stochastic processes are denoted $(f(i) \colon i \in I)$.

Given a normed space $(U, \|\cdot\|_U)$, we use the notation $B_U := \{f \in U \colon \|f\|_U \leqslant 1\}$ for the closed unit ball in U. For a topological vector space (T, τ), we denote the Borel σ-algebra generated by the open subsets of T by $\mathcal{B}(T)$ and we denote the continuous (topological) dual space by T^*, referred to henceforth simply as the dual. The dual pairing is denoted $\langle t, t^* \rangle_T$ for $t \in T, t^* \in T^*$. For $B \in \mathcal{B}(\mathbb{R}^d)$, we define the δ-ring[1] $\mathcal{B}_b(B) := \{A \in \mathcal{B}(B) \colon A \text{ is relatively compact}\}$. For topological vector spaces T and S we denote the continuous linear operators from T to S by $\mathcal{L}(T, S)$ and $\mathcal{L}(T) := \mathcal{L}(T, T)$.

Given a measure space $(S, \mathcal{A}, \mu)$, the space of μ-equivalence classes of measurable functions $f \colon S \to \mathbb{R}$ is denoted by $L^0(S, \mu)$, and of p-th integrable functions by $L^p(S, \mu)$ for $p > 0$. For a Borel measure μ on S we define the reflected measure μ^- by $\mu^-(A) := \mu(-A)$ for each $A \in \mathcal{B}(S)$. The Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ is denoted by leb. For the case $L^p(\mathbb{R}^d, \text{leb})$ we shall just write $L^p(\mathbb{R}^d)$. We equip $L^0(S, \mu)$ with the topology of convergence in measure (known as convergence in

[1] A δ-ring is a ring that is closed under countable intersections.

probability for the case $L^0(\Omega, P)$), and we equip the spaces $L^p(S,\mu)$ for $p>0$ with their standard metrics and (quasi-)norms, denoted $\|\cdot\|_{L^p(S,\mu)}$ or, where there is no risk of confusion, $\|\cdot\|_{L^p}$. For $p>1$ we define $p'=\frac{p}{p-1}$ to be the conjugate of p with the usual modification for $p\in\{1,\infty\}$.

For $a\in\mathbb{Z}_+\cup\{\infty\}$ and an open set $B\subseteq\mathbb{R}^d$ we denote by $C^a(B)$ the set of real-valued bounded uniformly continuous functions on B with bounded uniformly continuous a-th derivative, where $C(B)=C^0(B)$ denotes the bounded uniformly continuous functions without reference to differentiability, and $a=\infty$ denotes the functions with bounded uniformly continuous derivatives of all orders. Furthermore, $C_c^a(B)$ denotes the subset of $C^a(B)$ with compact support within B.

We shall write $a\lesssim b$ to mean that there exists a positive constant C such that $a\leqslant Cb$. If the constant C depends on the parameters $p_1,\dots,p_n$, we shall also write $C=C(p_1,\dots,p_n)$ and $\lesssim_{p_1,\dots p_n}$. The expression $a\eqsim b$ is equivalent to $a\lesssim b\lesssim a$.

By saying s is a multi-index, we mean $s=(s_1,\dots,s_d)\in\mathbb{Z}_+^d$. For a multi-index s, we define $|s|:=s_1+\cdots+s_d$ and the partial differential operator

$$\partial^s:=\frac{\partial^{|s|}}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}.$$

Let U be a separable topological vector space with separable dual U^*. We define Lévy processes in U in the usual manner, that is, a U-valued process $L=(L(t)\colon t\geqslant 0)$ such that $L(0)=0$; L has independent and stationary increments; and $t\mapsto L(t)$ is continuous in probability. Cylindrical Lévy processes in U are defined (for example in [6, 73, 74]) as a family of continuous operators from U^* to $L^0(\Omega;P)$ such that the d-dimensional projections are Lévy processes in $\mathbb{R}^d$ for each $d\in\mathbb{N}$.

For an open set $\mathcal{O}\subseteq\mathbb{R}^d$ let $\mathcal{D}(\mathcal{O})$ denote $C_c^\infty(\mathcal{O})$ equipped with the inductive topology; that is, $\mathcal{D}(\mathcal{O})$ is the strict inductive limit of the Fréchet spaces $\mathcal{D}(K_i):=\{f\in C^\infty(\mathbb{R}^d):\operatorname{supp}(f)\subseteq K_i\}$ where $\{K_i\}_{i\in\mathbb{N}}$ is a strictly increasing sequence of compact subsets of $\mathcal{O}$ such that $\mathcal{O} = \bigcup_{i\in\mathbb{N}} K_i$. The topology of $\mathcal{D}(K_i)$ is given by the family of seminorms $\|\cdot\|_{\mathcal{D}(K_i),r}$, $r \in \mathbb{Z}_+$ defined by

$$\|f\|_{\mathcal{D}(K_i),r} := \max_{|s|\leqslant r} \sup_{x\in\mathcal{D}(K_i)} |\partial^s f(x)| .$$

The dual space $\mathcal{D}^*(\mathcal{O})$ is called the space of distributions, which we equip with the strong topology, that is the topology generated by the family of seminorms $\{\eta_B\}$, where for each bounded $B \subseteq \mathcal{D}(\mathcal{O})$ we define $\eta_B(f) := \sup_{\varphi\in B} \left|\langle\varphi, f\rangle_{\mathcal{D}(\mathcal{O})}\right|$ for $f \in \mathcal{D}^*(\mathcal{O})$. In these topologies $\mathcal{D}(\mathcal{O})$ and $\mathcal{D}^*(\mathcal{O})$ are reflexive [87, p. 376].

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$, that is $\mathcal{S}(\mathbb{R}^d) := \left\{f \in C^\infty(\mathbb{R}^d)\colon \|f\|_{\mathcal{S}_r} < \infty \text{ for all } r \in \mathbb{Z}_+\right\}$, where the seminorms $\|\cdot\|_{\mathcal{S}_r}$, $r \in \mathbb{Z}_+$, are defined by

$$\|f\|_{\mathcal{S}_r} := \max_{|s|\leqslant r} \sup_{x\in\mathbb{R}^d} (1 + |x|^2)^r |\partial^s f(x)| ,$$

with s a multi-index. With the topology generated by the family of seminorms $(\|\cdot\|_{\mathcal{S}_r})_{r\in\mathbb{Z}_+}$, $\mathcal{S}(\mathbb{R}^d)$ is metrisable, and $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ means $\|f_n - f\|_{\mathcal{S}_r} \to 0$ for each $r \in \mathbb{Z}_+$. Furthermore, $\mathcal{S}(\mathbb{R}^d)$ is a countably Hilbertian nuclear space [49]. The dual space of $\mathcal{S}(\mathbb{R}^d)$ is the space $\mathcal{S}^*(\mathbb{R}^d)$ of tempered distributions, which we shall again equip with the strong topology. With this topology $\mathcal{S}(\mathbb{R}^d)$ is reflexive and separable and densely embedded in $\mathcal{S}^*(\mathbb{R}^d)$ [71, Th.V.14 Cor.1].

2.1 Weighted Besov Spaces

In preparation for our definition of the weighted Besov spaces, we present a brief motivation describing how many familiar function spaces can be embedded in $\mathcal{S}^*(\mathbb{R}^d)$, and thus may be analysed in a consistent fashion. This will allow us to measure the smoothness properties of functions and distributions along several scales.

The spaces $L^p(\mathbb{R}^d)$ for $1 \leqslant p \leqslant \infty$ may be interpreted as subspaces of $\mathcal{S}^*(\mathbb{R}^d)$ in the following manner. For $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$ for some $p \in [1, \infty]$ we define

$$[f, g] := \int_{\mathbb{R}^d} f(x)g(x)\,\mathrm{d}x.$$

We make the usual identification $\langle f, g\rangle_{\mathcal{S}(\mathbb{R}^d)} = [f, g]$ for $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, and then extend to all $g \in \mathcal{S}^*(\mathbb{R}^d)$. In this manner we shall interpret all function spaces considered in this work as subspaces of $S^*(\mathbb{R}^d)$. Furthermore, for any Banach space B in which $\mathcal{S}(\mathbb{R}^d)$ is dense, we interpret $[f, g]$ for $f \in B$ as $\lim_{n\to\infty}[f_n, g]$ for each $g \in \mathcal{S}^*(\mathbb{R}^d)$ such that the limit exists and is finite whenever $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^d)$ is such that $f_n \to f$ in B as $n \to \infty$. In this way we also interpret B^* as a subspace of $S^*(\mathbb{R}^d)$, and furthermore we thus may write $\langle f, g\rangle_B \equiv [f, g] \equiv [g, f]$. However, this interpretation means that we do *not* identify Hilbert spaces with their duals, except in the case of $L^2(\mathbb{R}^d)$.

As we are focused on separable reflexive Banach spaces in this work, we shall use the scale $1 < p < \infty$. We shall define the weighted Besov spaces $B^p_{s,w}(\mathbb{R}^d)$ for $p > 1$ and $s, w \in \mathbb{R}$ in terms of wavelet bases of $L^2(\mathbb{R}^d)$. We summarise the deposition in [88, Se. 1.2.3]. We define subsets $G^j \subseteq \{0, 1\}^d, j \in \mathbb{Z}_+$ as follows:[2]

$$G^j := \begin{cases} \{0, 1\}^d, & \text{if } j = 0, \\ \{G = (G_1, \ldots, G_d) \colon G_i = 1 \text{ for at least one } i\}, & \text{if } j \geqslant 1. \end{cases}$$

Suppose we are given $\left(\Psi_0^G\right)_{G\in G^0} \subseteq C_c(\mathbb{R}^d)$ which form an orthonormal set in $L^2(\mathbb{R}^d)$, which we shall call the *parent wavelets*. Then, for each $j \in \mathbb{Z}_+, G \in G^j$ and $m \in \mathbb{Z}^d$,

[2] So thus we have $G^1 = G^2 = \cdots$.

we define

$$\Psi_m^{j,G}(x) := 2^{jd/2}\Psi_m^G(2^j x) := 2^{jd/2}\Psi_0^G(2^j x - m), \qquad x \in \mathbb{R}^d .$$

It is known that for any $r \in \mathbb{N}$, there exist such parent wavelets $\left(\Psi_0^G\right)_{G\in G^0} \subseteq C_c^r(\mathbb{R}^d)$ such that $\Psi := \{\Psi_m^{j,G} \colon j \in \mathbb{Z}_+, G \in G^j, m \in \mathbb{Z}^d\}$ forms an orthonormal basis in $L^2(\mathbb{R}^d)$ [91, Th. 1.61]; one example is the Daubechies wavelets [28]. We shall call such a Ψ a *wavelet basis of* $L^2(\mathbb{R}^d)$. Henceforth, we shall refer to the wavelet index set

$$\mathbb{W}^d := \left\{(j, G, m) \colon j \in \mathbb{Z}_+, G \in G^j, m \in \mathbb{Z}^d\right\} . \tag{2.1.1}$$

Clearly $\mathbb{W}^d$ is countable. For the purposes of defining the weighted Besov space $B^p_{s,w}(\mathbb{R}^d)$, we shall require a minimum smoothness of the wavelet basis depending on the dimension d and the parameters p and s.

Definition 2.1.1. *Let $p > 1, s \in \mathbb{R}, w \in \mathbb{R}$. A wavelet basis $\Psi = \{\Psi_m^{j,G} \colon (j, G, m) \in \mathbb{W}^d\}$ of $L^2(\mathbb{R}^d)$ is called an* admissible basis of $B^p_{s,w}(\mathbb{R}^d)$ *if $\Psi \subseteq C_c^r(\mathbb{R}^d)$ for some $r \in \mathbb{N}$ satisfying $r > |s|$.*

Our first step shall be to define the *weighted Besov sequence space* $b^p_{s,w}$ for $p > 1, s \in \mathbb{R}, w \in \mathbb{R}$. We introduce the weight constants:

$$\omega_m^j = \omega_m^j(p, s, w) := 2^{j(s-\frac{d}{p}+\frac{d}{2})}(1 + 2^{-2j}\,|m|^2)^{\frac{w}{2}}, \tag{2.1.2}$$

for each $m \in \mathbb{Z}^d$ and $j \in \mathbb{Z}_+$. We define $b^p_{s,w}$ as the vector space of sequences

$$\lambda = \left\{\lambda_m^{j,G} \in \mathbb{R} \colon (j, G, m) \in \mathbb{W}^d\right\}$$

such that

$$\|\lambda\|_{b^p_{s,w}} := \left(\sum_{j\in\mathbb{Z}_+} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^d} \left| 2^{-\frac{jd}{2}} \omega^j_m \lambda^{j,G}_m \right|^p \right)^{1/p} < \infty.$$

For $p > 1$, $(b^p_{s,w}, \|\cdot\|_{b^p_{s,w}})$ forms a Banach space, when $p = 2$ it forms a Hilbert space.

Now let $p > 1$ and $s, w \in \mathbb{R}$. Let Ψ be an admissible basis of $B^p_{s,w}(\mathbb{R}^d)$. The *weighted Besov space* $B^p_{s,w}(\mathbb{R}^d)$ is defined to be

$$B^p_{s,w}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}^*(\mathbb{R}^d) \colon f = \sum_{j\in\mathbb{Z}_+} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^d} \lambda^{j,G}_m 2^{-\frac{jd}{2}} \Psi^{j,G}_m,\ \lambda \in b^p_{s,w} \right\}$$

where the sum is unconditionally convergent in $\mathcal{S}^*(\mathbb{R}^d)$. When this holds, the associated sequence λ is unique and we have

$$\lambda^{j,G}_m = 2^{\frac{jd}{2}} [\Psi^{j,G}_m, f].$$

A consequence of Ψ being an admissible basis of $B^p_{s,w}(\mathbb{R}^d)$ is that the wavelets are of sufficient smoothness to guarantee that they are in $\left(B^p_{s,w}(\mathbb{R}^d)\right)^*$, and so the dual pairing makes sense. As the sums over j, G and m are unconditional in the definitions of both the weighted Besov spaces and the weighted Besov sequence spaces, we will henceforth use the simpler notation $\sum_{j,G,m}$ to mean $\sum_{j\in\mathbb{Z}_+} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^d}$. We may norm $B^p_{s,w}(\mathbb{R}^d)$ by taking $\|f\|_{B^p_{s,w}} = \|f\|_{B^p_{s,w}(\mathbb{R}^d)} := \|\lambda\|_{b^p_{s,w}}$, giving

$$\|f\|_{B^p_{s,w}} = \left(\sum_{j,G,m} (\omega^j_m)^p \left| [\Psi^{j,G}_m, f] \right|^p \right)^{1/p}. \tag{2.1.3}$$

Again, we have that $B^p_{s,w}(\mathbb{R}^d)$ is a Banach space, and a Hilbert space for $p = 2$. We immediately see from this definition that $B^2_{0,0}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, consistently with the relation described in the Introduction.

Remark 2.1.2. *As mentioned in the Introduction, the original definitions of Besov spaces involved another parameter* q*. We shall not use the* q *parameter in this work. This is due to the fact that the embedding theorems available in weighted Besov spaces (see Proposition 3 in [31]) show the continuous embedding of* $B^{p,q}_{s,w}(\mathbb{R}^d)$ *into* $B^{p,p}_{s-\varepsilon,w}(\mathbb{R}^d)$ *for any* $q \in \mathbb{R}$ *and* $\varepsilon > 0$*. The results presented in this work are generally expressed as strict inequalities on the Besov space parameters, and as such are unaffected by the arbitrarily small change in the* s *parameter needed to incorporate any* q *parameter. Thus, the theory herein is developed for the case* $p = q$*, and we shall henceforth define* $B^{p}_{s,w}(\mathbb{R}^d) := B^{p,p}_{s,w}(\mathbb{R}^d)$*.*

2.1.1 The Dual Spaces

The dual spaces for the unweighted Besov spaces are well-known: $\left(B^{p}_{s,0}(\mathbb{R}^d)\right)^* = B^{p'}_{-s,0}(\mathbb{R}^d)$ for $p > 1$ and $s \in \mathbb{R}$ (see e.g. [89, p.179]). We present the generalisation to the weighted case, for which we could find no reference in the literature.

Theorem 2.1.3. *Let* $p > 1$ *and* $s, w \in \mathbb{R}$*. The dual space* $\left(B^{p}_{s,w}(\mathbb{R}^d)\right)^*$ *may be identified with* $B^{p'}_{-s,-w}(\mathbb{R}^d)$*, with the duality given by*

$$\langle f, g\rangle_{B^{p}_{s,w}} = [f, g] = \sum_{j,G,m} [\Psi^{j,G}_{m}, f][\Psi^{j,G}_{m}, g] \tag{2.1.4}$$

where Ψ *is any admissible basis for* $B^{p}_{s,w}(\mathbb{R}^d)$ *(and thus is also an admissible basis for* $B^{p'}_{-s,-w}(\mathbb{R}^d)$*).*

In order to prove this Theorem, we shall first prove some intermediary results about the weighted Besov sequence spaces defined above; we shall then apply the isometry between the weighted Besov sequence spaces and the weighted Besov spaces to complete the proof.

Lemma 2.1.4. *For each $p > 1$ and $s, w \in \mathbb{R}$ the operator $\Upsilon^p_{s,w} \colon b^p_{s,w} \to \ell^p(\mathbb{W}^d)$ defined by*

$$\left(\Upsilon^p_{s,w}\lambda\right)^{j,G}_m := 2^{-\frac{jd}{2}} \omega^j_m \lambda^{j,G}_m \tag{2.1.5}$$

forms an isometric isomorphism. In the expression above, $\omega^j_m = \omega^j_m(p, s, w)$ are the weight constants defined in (2.1.2).

Proof. We take $p > 1$ and $s, w \in \mathbb{R}$ and recall the norm of a sequence $(\lambda^{j,G}_m)_{(j,G,m)\in\mathbb{W}^d}$ is given by

$$\|\lambda\|^p_{b^p_{s,w}} = \sum_{j\in\mathbb{Z}_+} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^d} \left|2^{-\frac{jd}{2}} \omega^j_m \lambda^{j,G}_m\right|^p.$$

As all terms are positive, convergence is unconditional and we see that $\|\lambda\|_{b^p_{s,w}} = \left\|\Upsilon^p_{s,w}\lambda\right\|_{\ell^p(\mathbb{W}^d)}$. As the multipliers $2^{-jd/2}\omega^j_m$ act component-wise and are strictly positive for all $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}^d$, we see that $\Upsilon^p_{s,w}$ is an isometric isomorphism. □

Lemma 2.1.5. *Let $p > 1$ and $s, w \in \mathbb{R}$. The dual space $\left(b^p_{s,w}\right)^*$ may be identified with $b^{p'}_{-s+d,-w}$, with the duality given by*

$$\langle \lambda, \kappa \rangle_{b^p_{s,w}} = \sum_{j,G,m} \lambda^{j,G}_m \kappa^{j,G}_m.$$

Proof. Fix $y \in \ell^{p'}(\mathbb{W}^d)$. Then the map $\lambda \mapsto \langle \Upsilon^p_{s,w}\lambda, y\rangle_{\ell^p(\mathbb{W}^d)}$ is linear and continuous, thus $(\Upsilon^p_{s,w})^* y \in \left(b^p_{s,w}\right)^*$. By examination of each component, we see that

$$\left((\Upsilon^p_{s,w})^* y\right)^{j,G}_m = 2^{-\frac{jd}{2}} \omega^j_m y^{j,G}_m.$$

Taking $\kappa := (\Upsilon^p_{s,w})^* y$ we have

$$\begin{aligned} y^{j,G}_m = \left(\left((\Upsilon^p_{s,w})^* \right)^{-1} \kappa \right)^{j,G}_m &= 2^{\frac{jd}{2}} (\omega^j_m)^{-1} \kappa^{j,G}_m \\ &= 2^{j(-s+\frac{d}{p})} (1 + 2^{-2j} \left| m \right|^2)^{\frac{-w}{2}} \kappa^{j,G}_m \\ &= \left(\Upsilon^{p'}_{-s+d,-w} \kappa \right)^{j,G}_m \end{aligned}$$

thus showing $\kappa \in b^{p'}_{-s+d,-w}$ by the isomorphism. As $(\Upsilon^p_{s,w})^*$ maps $\ell^{p'}(\mathbb{W}^d)$ onto $b^{p'}_{-s+d,-w}$, we conclude $b^{p'}_{-s+d,-w} \subseteq \left(b^p_{s,w} \right)^*$ with the duality as defined.

Now fix $\kappa \in \left(b^p_{s,w} \right)^*$. For each $\lambda \in b^p_{s,w}$ we have $\lambda = (\Upsilon^p_{s,w})^{-1} x$ for some $x \in \ell^p(\mathbb{W}^d)$. Then

$$\langle \lambda, \kappa \rangle_{b^p_{s,w}} = \langle (\Upsilon^p_{s,w})^{-1} x, \kappa \rangle_{b^p_{s,w}} = \langle x, \left((\Upsilon^p_{s,w})^{-1} \right)^* \kappa \rangle_{\ell^p(\mathbb{W}^d)}.$$

The inclusion $b^{p'}_{-s+d,-w} \supseteq \left(b^p_{s,w} \right)^*$ follows by noting that $\left((\Upsilon^p_{s,w})^{-1} \right)^* = \left((\Upsilon^p_{s,w})^* \right)^{-1} = \Upsilon^{p'}_{-s+d,-w}$ component-wise. We may then calculate

$$\begin{aligned} \langle \lambda, \kappa \rangle_{b^p_{s,w}} &= \langle \Upsilon^p_{s,w} \lambda, \Upsilon^{p'}_{-s+d,-w} \kappa \rangle_{\ell^p(\mathbb{W}^d)} \\ &= \sum_{j,G,m} \lambda^{j,G}_m \kappa^{j,G}_m. \end{aligned}$$

To complete the proof we show the operator norm is equal to the Besov space norm, which follows from the isometry between $b^{p'}_{-s+d,-w}$ and $\ell^{p'}(\mathbb{W}^d)$. □

Lemma 2.1.6. *Let $p > 1$ and $s, t, w \in \mathbb{R}$. The space $b^p_{s,w}$ is isometrically isomorphic to the space $b^p_{s+t,w}$ with the isometry given by*

$$D \colon b^p_{s,w} \to b^p_{s+t,w}, \quad (D\kappa)^{j,G}_m = 2^{-jt} \kappa^{j,G}_m.$$

Proof. Let $\kappa \in b^p_{s,w}$; thus we have

$$\left(2^{j(s-\frac{d}{p})}\left(1+2^{-2j}\,|m|^2\right)^{\frac{w}{2}}\kappa^{j,G}_m\right) = \left(2^{j(s+t-\frac{d}{p})}\left(1+2^{-2j}\,|m|^2\right)^{\frac{w}{2}}D\kappa^{j,G}_m\right) \in \ell^{p'}(\mathbb{W}^d).$$

Clearly D is one-to-one and onto. The isometry follows by examination of the formulae for the respective norms. □

Proof of Theorem 2.1.3. First we show (2.1.4) is well-defined. Let $f \in B^p_{s,w}(\mathbb{R}^d)$ and $g \in B^{p'}_{-s,-w}(\mathbb{R}^d)$. Then,

$$\left(2^{j(s-\frac{d}{p}+\frac{d}{2})}(1+2^{-2j}\,|m|^2)^{\frac{w}{2}}[\Psi^{j,G}_m, f]\right)_{j,G,m} \in \ell^p(\mathbb{W}^d)$$

and

$$\left(2^{j(-s-\frac{d}{p'}+\frac{d}{2})}(1+2^{-2j}\,|m|^2)^{\frac{-w}{2}}[\Psi^{j,G}_m, g]\right)_{j,G,m} \in \ell^{p'}(\mathbb{W}^d);$$

thus we see the convergence of (2.1.4) as $\frac{d}{p'} = d - \frac{d}{p}$. By Theorem 6.15 in [91] we have the isometry $I\colon B^p_{s,w}(\mathbb{R}^d) \to b^p_{s,w}$ given by $\left(If\right)^{j,G}_m = 2^{jd/2}[\Psi^{j,G}_m, f]$; combining with Lemma 2.1.6 we obtain an isometry $J\colon B^p_{s,w}(\mathbb{R}^d) \to b^p_{s+d/2,w}$ given by

$$\left(Jf\right)^{j,G}_m = [\Psi^{j,G}_m, f].$$

By Lemma 2.1.5, we have that the dual of $b^p_{s+d/2,w}$ is $b^{p'}_{-s+d/2,-w}$. Applying the same steps as above we obtain the isometry $K\colon B^{p'}_{-s,-w}(\mathbb{R}^d) \to b^{p'}_{-s+d/2,-w}$ given by

$$\left(Kg\right)^{j,G}_m = [\Psi^{j,G}_m, g],$$

which completes the proof. □

2.2 Random Measures and Additive Sheets

We recall the definition of infinitely divisible random measures from the work [70] by Rajput and Rosinski. Instead of general δ-rings, it is sufficient for us to restrict ourselves to the δ-ring $\mathcal{B}_b(\mathcal{O})$ of all relatively compact subsets of the Borel set $\mathcal{O} \in \mathcal{B}(\mathbb{R}^d)$ as the domain of the random measures.

Definition 2.2.1. *A map $M\colon \mathcal{B}_b(\mathcal{O}) \to L^0(\Omega, P)$ is called an* independently scattered random measure *on $\mathcal{B}_b(\mathcal{O})$ if for each collection of disjoint sets $A_1, A_2, \ldots \in \mathcal{B}_b(\mathcal{O})$ the following hold:*

(a) *the random variables $M(A_1)$, $M(A_2), \ldots$ are independent;*

(b) *if $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{B}_b(\mathcal{O})$ then $M\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} M(A_k)$ P-a.s.*

An independently scattered random measure M is called infinitely divisible *if*

(c) *the random variable $M(A)$ is infinitely divisible for each $A \in \mathcal{B}_b(\mathcal{O})$.*

Analogously, an independently scattered random measure is called Gaussian *(or* Poisson*), if $M(A)$ is Gaussian (or Poisson) distributed for each $A \in \mathcal{B}_b(\mathcal{O})$.*

For an arbitrary infinitely divisible independently scattered random measure M on $\mathcal{B}_b(\mathcal{O})$ it is shown in [70] that there exist

(1) a signed measure $\gamma\colon \mathcal{B}_b(\mathcal{O}) \to \mathbb{R}$,

(2) a measure $\Sigma\colon \mathcal{B}_b(\mathcal{O}) \to \mathbb{R}_+$,

(3) a σ-finite measure $\nu\colon \mathcal{B}(\mathcal{O} \times \mathbb{R}) \to [0, \infty]$,

such that for each $A \in \mathcal{B}_b(\mathcal{O})$ the characteristics of $M(A)$ are given by $(\gamma(A), \Sigma(A), \nu_A)$, where the Lévy measure ν_A on $\mathcal{B}(\mathbb{R})$ is defined by $\nu_A(\cdot) := \nu(A \times \cdot)$. For the notion of measures on a ring see e.g. [39]. We call the triple (γ, Σ, ν) the *characteristics of*

M. Furthermore, we may extend the total variation $\|\gamma\|_{\mathrm{TV}}$ of γ and Σ to σ-finite measures on $\mathcal{B}(\mathcal{O})$. In this case, the mapping

$$\lambda\colon \mathcal{B}(\mathcal{O}) \to [0,\infty], \qquad \lambda(A) = \|\gamma\|_{\mathrm{TV}}(A) + \Sigma(A) + \int_{\mathbb{R}} (|y|^2 \wedge 1)\, \nu(A, \mathrm{d}y),$$

defines a σ-finite measure, which is called the *control measure of M*. We note that $\lambda(A) < \infty$ for $A \in \mathcal{B}_b(\mathcal{O})$. The control measure λ is called *atomless* if $\lambda(\{x\}) = 0$ for all $x \in \mathcal{O}$.

Next we present our definition of additive sheets based on the deposition of Dalang and Humeau in [25], which extends [2], and results from Pedersen [66]. For $a, b \in \mathbb{R}^d$ write $a \leqslant b$ if $a_j \leqslant b_j$ for all $j = 1, \dots, d$ and similarly $a < b$, and define boxes $(a, b] := \{s \in \mathbb{R}^d : a < t \leqslant b\}$ and $[a, b] := \{s \in \mathbb{R}^d : a \leqslant t \leqslant b\}$; $[a, b)$ and (a, b) are defined mutatis mutandi. For a function $f\colon \mathbb{R}^d \to \mathbb{R}$, we define the increment of f over $(a, b]$ for a, $b \in \mathbb{R}^d$ with $a < b$ by

$$\Delta_a^b f := \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_k=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_k} f\big(c_1(\varepsilon_1), \dots c_k(\varepsilon_k)\big),$$

where $c_j(0) = b_j$ and $c_j(1) = a_j$. For example, in the case $d = 2$ we have $\Delta_a^b f = f(b_1, b_2) - f(b_1, a_2) - f(a_1, b_2) + f(a_1, a_2)$. Furthermore, we shall extend the terminology of boxes around the origin to all quadrants (with some abuse of notation) and use the convention $(0, x] := \prod_{i=1}^d I_i$ where, for $x = (x_1, \dots, x_d) \in I$, $I_i := (0, x_i]$ when $x_i > 0$ and $I_i := [x_i, 0)$ when $x_i < 0$. In this case, we adapt the calculation of the increments accordingly.

The càdlàg property is generalised to random fields in the following way:

Definition 2.2.2. *A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to be* lamp (limits along monotone paths) *if for every $x \in \mathbb{R}^d$ and any sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ converging to x with either $x_{n,j} < x_j$ or $x_{n,j} \geqslant x_j$ for all $n \in \mathbb{N}$ and $j \in \{1, \dots, d\}$ where $x = (x_1, \dots, x_d)$*

and $x_n = (x_{n,1}, \dots, x_{n,d})$*, the limit* $f(x_n)$ *exists as* $n \to \infty$ *and furthermore* f *is called* right-continuous *if* $f(x_n) \to f(x)$ *as* $n \to \infty$ *for all sequences with* $x \leqslant x_n$ *for all* $n \in \mathbb{N}$*.*

We note that the lamp property is a path-based property, and thus in contrast to random measures we define our sheets as mappings from $\mathbb{R}^d \times \Omega \to \mathbb{R}$.

Definition 2.2.3. *Let* $I \subseteq \mathbb{R}^d$ *with* $0 \in I$*. A real-valued stochastic process* $(X(x) : x \in I)$ *is called an* additive sheet *if the following conditions are satisfied:*

(a) $X(x) = 0$ *a.s. for all* $x = (x_1, \dots, x_d) \in I$ *with* $x_j = 0$ *for some* $j \in \{1, \dots, d\}$*;*

(b) $\Delta_{a_1}^{b_1} X, \dots, \Delta_{a_n}^{b_n} X$ *are independent for disjoint boxes* $(a_1, b_1], \dots, (a_n, b_n] \subseteq I$*;*

(c) X *is continuous in probability;*

(d) *almost all sample paths of* X *are lamp and right-continuous.*

Remark 2.2.4. *For relaxing the requirements in Definition 2.2.3 we refer to [2], e.g. to capture arbitrary initial conditions or sheets which are not continuous in probability. In particular, it is shown that Conditions (a) – (c) gurantee the existence of a lamp and right-continuous modification.*

If $(X(x) : x \in I)$ is an additive sheet then for fixed $x \in I$ the random variable $X(x)$ is infinitely divisible; see Adler [2, Th. 3.1]; let its characteristics be denoted by (p_x, A_x, μ_x). The additive sheet is said to be *natural* if the mapping $x \mapsto p_x$, which is necessarily continuous, is of bounded variation, or equivalently, if there exists an atomless signed measure γ with $p_x = \gamma((0, x])$ for all $x \in I$. The notation of natural additive processes is introduced in Sato [80] for the case $d = 1$.

2.3 Cylindrical Lévy Processes

The concept of cylindrical Lévy processes in Banach spaces is introduced in [6]; this is a natural generalisation of the notion of cylindrical Brownian motion, based on the theory of cylindrical measures and cylindrical random variables.

We begin by defining cylindrical measures and cylindrical random variables. Let U be a topological vector space with separating dual U^* and let $\Gamma \subseteq U^*$. For some $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \Gamma$ we define the projection $\pi_{f_1,\ldots,f_n} : U \to \mathbb{R}^n$ by

$$\pi_{f_1,\ldots,f_n}(u) := \left(\langle u, f_1\rangle_U, \ldots, \langle u, f_n\rangle_U\right).$$

Sets of the form

$$\begin{aligned} Z(f_1, \ldots, f_n; A) &:= \pi^{-1}_{f_1,\ldots,f_n}(A) \\ &= \{u \in U : \left(\langle u, f_1\rangle_U, \ldots, \langle u, f_n\rangle_U\right) \in A\} \end{aligned}$$

for $A \in \mathcal{B}(\mathbb{R}^n)$ are called *cylinder sets* with respect to (U, Γ), and the set of all cylinder sets with respect to (U, Γ) is denoted by $\mathcal{Z}(U, \Gamma)$; we also denote $\mathcal{Z}(U, U^*) =: \mathcal{Z}(U)$. In general $\mathcal{Z}(U, \Gamma)$ is an algebra; in the case that Γ is finite $\mathcal{Z}(U, \Gamma)$ is a σ-algebra. We denote $\mathcal{C}(U) := \sigma(\mathcal{Z}(U))$ and note that $\mathcal{C}(U) \subseteq \mathcal{B}(U)$ with equality holding if and only if U is Polish [92, p.6].

Note that in the case that U is reflexive, cylinder sets in $\mathcal{Z}(U^*)$ can be written in the form

$$Z(u_1, \ldots, u_n; A) = \{f \in U^* : \left(\langle u_1, f\rangle_U, \ldots, \langle u_n, f\rangle_U\right) \in A\}$$

for some $u_1, \ldots, u_n \in U$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

A set function $\lambda : \mathcal{Z}(U) \to [0, \infty]$ is called a *cylindrical measure* if for every finite $\Gamma \subseteq U^*$ the restriction of λ to $\mathcal{Z}(U, \Gamma)$ is a measure. A cylindrical measure λ

is called finite if $\lambda(U) < \infty$ and a *cylindrical probability measure* if $\lambda(U) = 1$.

Definition 2.3.1. *A* cylindrical random variable X *in* U *is a linear and continuous mapping* $X\colon U^* \to L^0(\Omega, P)$. *A* cylindrical process $(X(t)\colon t \in I)$ *is a family of cylindrical random variables indexed by some index set* I.

There is a one-to-one correspondence between cylindrical random variables and cylindrical probability measures: given a cylindrical random variable X on U, then $\lambda\colon \mathcal{Z}(U) \to [0, \infty]$ defined by

$$\lambda\big(Z(f_1, \ldots, f_n; A)\big) := P\big((Xf_1, \ldots, Xf_n) \in A\big)$$

is a cylindrical probability measure, called the *cylindrical distribution* of X; conversely for every cylindrical probability measure λ in $\mathcal{Z}(U)$ there exists a probability space $(\Omega_0, \mathcal{F}_0, P_0)$ and a cylindrical random variable $X : U^* \to L^0(\Omega_0, P_0)$ such that λ is the cylindrical distribution of X [74].

We note that any (genuine) U-valued stochastic process $Y = (Y(t)\colon t \in I)$ induces a cylindrical process in U by the prescription, for each $f \in U^*$ and $\omega \in \Omega$,

$$(Y(t)f)(\omega) := \langle Y(t)(\omega), f\rangle_U$$

which motivates the following

Definition 2.3.2. *A U-valued stochastic process $Y = (Y(t)\colon t \in I)$ is said to* induce *a cylindrical process $X = (X(t)\colon t \in I)$ in U if, for each $t \in I$ and each $f \in U^*$,*

$$X(t)f = \langle Y(t), f\rangle_U \qquad P\text{-a.s.}.$$

Next we introduce the cylindrical Lévy process, following Applebaum and Riedle [6, 74].

Definition 2.3.3. *A cylindrical process* $(L(t) : t \geqslant 0)$ *in* U *is called a* cylindrical Lévy process *if for all* $f_1, \ldots, f_n \in U^*$ *and* $n \in \mathbb{N}$*, the stochastic process* $((L(t)f_1, \ldots, L(t)f_n) : t \geqslant 0)$ *is a Lévy process in* $\mathbb{R}^n$*.*

The following theorem, which is a consequence of Itô's regularisation theorem, reduces the study of cylindrical Lévy processes in $\mathcal{S}^*(\mathbb{R}^d)$ to that of genuine Lévy processes [35, Th. 3.8].

Theorem 2.3.4. *Let* $L = (L(t)\colon t \geqslant 0)$ *be a cylindrical Lévy process in* $\mathcal{D}^*(\mathbb{R}^d)$ *(respectively,* $\mathcal{S}^*(\mathbb{R}^d)$*) such that for every* $T > 0$ *the family* $\{L(t) : t \in [0,T]\}$ *is equicontinuous in probability, equivalently the family of characteristic functions* $\{\varphi_{L(t)}(\cdot) : t \in [0,T]\}$ *is equicontinuous at* 0*. Then there exists a* $\mathcal{D}^*(\mathbb{R}^d)$ *(respectively,* $\mathcal{S}^*(\mathbb{R}^d)$*)-valued, càdlàg Lévy process* $Y = (Y(t)\colon t \geqslant 0)$ *such that* Y *induces* L *and furthermore* Y *is unique up to indistinguishability.*

Now let U be a Banach space with separable dual U^*. In order to present the Lévy-Khintchine formula for a cylindrical Lévy process, we must first give a definition of the cylindrical version of the Lévy measure. In this definition, we must specifically exclude sets containing the origin to avoid consistency issues with finite-dimensional projections. We define the subalgebra $\mathcal{Z}_*(U) \subseteq \mathcal{Z}(U)$ as

$$\mathcal{Z}_*(U) := \{C = Z(f_1, \ldots, f_n; A) \in \mathcal{Z}(U)\colon 0 \notin A\}.$$

Definition 2.3.5. *A set function* $\mu\colon \mathcal{Z}_*(U) \to [0, \infty]$ *is called a* cylindrical Lévy measure *if for all* $f_1, \ldots, f_n \in U^*$ *and* $n \in \mathbb{N}$ *the map*

$$\mu_{f_1,\ldots,f_n}\colon \mathcal{B}(\mathbb{R}^n) \to [0,\infty], \qquad \mu_{f_1,\ldots,f_n}(B) = \mu \circ \pi^{-1}_{f_1,\ldots,f_n}(B \backslash \{0\})$$

defines a Lévy measure on $\mathbb{R}^n$*.*

The characteristic function of a cylindrical Lévy process $(L(t) : t \geqslant 0)$ is given by

$$\varphi_{L(t)} \colon U^* \to \mathbb{C}, \qquad \varphi_{L(t)}(f) = \exp\big(t\vartheta_L(f)\big),$$

for all $t \geqslant 0$. Here, $\vartheta_L \colon U^* \to \mathbb{C}$ is called the (cylindrical) symbol of L, and is of the form

$$\vartheta_L(f) = ia(f) - \tfrac{1}{2}\langle f, Qf\rangle_{U^*} + \int_U \big(e^{i\langle g,f\rangle_U} - 1 - i\langle g, f\rangle_U \, \mathbb{1}_{B_\mathbb{R}}(\langle g, f\rangle_U)\big)\, \mu(\mathrm{d}g),$$

where $a \colon U^* \to \mathbb{R}$ is a continuous mapping with $a(0) = 0$, the mapping $Q \colon U^* \to U^{**}$ is a positive, symmetric operator and μ is a cylindrical Lévy measure on U. We call (a, Q, μ) the *(cylindrical) characteristics of* L.

Furthermore, for each sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq U^*$ which converges in norm to some $f_0 \in U^*$, we have $(|x|^2 \wedge 1)(\mu \circ f_n^{-1})(\mathrm{d}x) \to (|x|^2 \wedge 1)(\mu \circ f_0^{-1})(\mathrm{d}x)$ weakly.

2.3.1 Examples

The following examples are from [6].

Example 2.3.6. A cylindrical Brownian motion is an example of a cylindrical Lévy process, with (cylindrical) characteristics $(0, Q, 0)$, where Q is the covariance operator of the cylindrical Brownian motion.

Example 2.3.7. Let $\zeta \in U^{**}$. Then define the *cylindrical Poisson process* $L = (L(t) \colon t \geqslant 0)$ as, for each $f \in U^*$ and $t \geqslant 0$

$$L(t)f := \langle f, \zeta\rangle_{U^*} N(t)$$

where $N(t)$ is a Poisson process in $\mathbb{R}$ with intensity $\lambda > 0$. Then L is a cylindrical Lévy process with characteristic function

$$\varphi_{L(t)}(f) = \exp\big(\lambda t(e^{i\langle f,\zeta\rangle_{U^*}} - 1)\big).$$

The following important example shows the construction of a cylindrical Lévy process from a series of real-valued Lévy processes [74, Le. 4.2].

Example 2.3.8. Let U be a Hilbert space, $\{e_k\}_{k\in\mathbb{N}} \subseteq U$ an orthonormal basis and let $\{\ell_k\}_{k\in\mathbb{N}}$ be a sequence of independent Lévy processes in $\mathbb{R}$, such that for each $k \in \mathbb{N}$ the characteristics of ℓ_k is (b_k, σ_k^2, ν_k). Then, for each $t \geqslant 0$ and $f \in U^*$ the sum $L(t)f := \sum_{k\in\mathbb{N}} \langle e_k, f\rangle_U \ell_k(t)$ converges $\mathbb{P}$-a.s. if and only if for each $x \in \ell^2(\mathbb{R})$ we have:

1. $\sum_{k\in\mathbb{N}} \mathbb{1}_{B_\mathbb{R}}(x_k) \left| x_k \Big(b_k + \int_{1<|y|\leqslant |x_k|^{-1}} y \, \nu_k(\mathrm{d}y) \Big) \right| < \infty$,
2. $(\sigma_k^2)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{R})$, and
3. $\sum_{k\in\mathbb{N}} \int_\mathbb{R} \big(|x_k y|^2 \wedge 1 \big) \, \nu_k(\mathrm{d}y) < \infty$.

In this case, if the set $\{\varphi_{\ell_k(1)} : k \in \mathbb{N}\}$ is equicontinuous at 0 then $L = (L(t) \colon t \geqslant 0)$ defines a cylindrical Lévy process in U with cylindrical characteristics satisfying, for each $f \in U^*$,

1. $a(f) = \sum_{k\in\mathbb{N}} \langle e_k, f\rangle_U \Big(b_k + \int_\mathbb{R} y \big(\mathbb{1}_{B_\mathbb{R}}(\langle e_k, f\rangle_U y) - \mathbb{1}_{B_\mathbb{R}}(y) \big) \, \nu_k(\mathrm{d}y) \Big)$,
2. $Qf = \sum_{k\in\mathbb{N}} \langle e_k, f\rangle_U \sigma_k^2 e_k$, and
3. $(\mu \circ f^{-1})(\mathrm{d}y) = \sum_{k\in\mathbb{N}} (\nu_k \circ m_{k,f}^{-1})(\mathrm{d}y)$,

where $m_{k,f} : \mathbb{R} \to \mathbb{R} : y \mapsto \langle e_k, f\rangle_U y$.

The support of the cylindrical measure μ of L is in $\bigcup_{k\in\mathbb{N}} \{\beta e_k : \beta \in \mathbb{R}\}$, as $(\ell_k)_{k\in\mathbb{N}}$ are independent, that is to say the measure only has weight on the axes. For this reason, we refer to this process as a *hedgehog cylindrical process*.

Example 2.3.9. The canonical α-stable cylindrical Lévy process in a Banach space U, as detailed in [75], has cylindrical characteristic function given by

$$\varphi_{L(t)} : U^* \to \mathbb{C} : f \mapsto \exp\Big(- t \left\| f \right\|_{U^*}^{\alpha} \Big).$$

The author gives two constructions of a canonical α-stable cylindrical Lévy process. The first is detailed in Lemma 3.1: for U separable and $\alpha \in (0,2)$, let W be a standard cylindrical Brownian motion on U and ℓ an independent $\alpha/2$-stable subordinator in $\mathbb{R}$ with Lévy measure given by

$$\nu(\mathrm{d}y) = \frac{2^{\alpha/2}\alpha/2}{\Gamma(1-\alpha/2)} y^{-\alpha/2-1}\,\mathrm{d}y.$$

Then the prescription $L(t)f := W(\ell(t))f$, for each $f \in U^*$ defines a canonical α-stable cylindrical Lévy process in U.

The second construction, detailed in Lemma 3.3, constructs a canonical α-stable cylindrical Lévy process in $L^{\alpha'}(\mathcal{O})$ from a Lévy random measure on $\mathbb{R}_+ \times \mathcal{O}$, for some $\mathcal{O} \subseteq \mathbb{R}^d$, with characteristics $(0, 0, \nu)$ and $\nu(\mathrm{d}y) = C \left| y \right|^{-\alpha-1}\,\mathrm{d}y$.

In the remainder of this thesis we use the phrase *genuine Lévy process in U* to emphasise the difference between a Lévy process in the space U according to the usual definition, e.g. Definition 4.1 and Definition 14.2 in [67], and a cylindrical Lévy process as defined above.

Chapter 3

Modelling Lévy space-time white noises

In this Chapter, we begin by presenting our precise definitions of Lévy-valued random measures and Lévy-valued sheets, recall some known results from the literature and add a few observations particular to our approach. Following on from this, the rest of this Chapter is devoted to the comparison of cylindrical Lévy processes and Lévy-valued random measures. Our main results here characterise exactly the sub-class of cylindrical Lévy processes which correspond to Lévy-valued random measures.

3.1 Lévy-valued random measures

We define Lévy-valued random measures by extending Definition 2.2.1 to include a dynamical aspect, i.e. a time variable. This extension can be thought of as a similar construction to that of Walsh in [94]. Our construction enforces stationarity in time, whilst allowing for the distribution to depend on the spatial variable.

Definition 3.1.1. *A family $(M(t) : t \geqslant 0)$ of infinitely divisible random measures $M(t)$ on $\mathcal{B}_b(\mathcal{O})$ is called a* Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$ *if, for every*

$A_1, \dots, A_n \in \mathcal{B}_b(\mathcal{O})$ and $n \in \mathbb{N}$, the stochastic process

$$\big((M(t)(A_1), \dots, M(t)(A_n)) : t \geqslant 0\big)$$

is a Lévy process in $\mathbb{R}^n$. We shall write $M(t, A) := M(t)(A)$.

Let $(M(t) : t \geqslant 0)$ be a Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$, and suppose (γ, Σ, ν) and λ are the characteristics and control measure, respectively, of the infinitely divisible random measure $M(1)$. Then, it follows from the stationarity of the increments of the process $(M(t, A) : t \geqslant 0)$ that for each $t \geqslant 0$ the characteristics of the infinitely divisible random measure $M(t, A)$ are given by $(t\gamma, t\Sigma, t\nu)$, and the control measure of $M(t)$ is given by $t\lambda$. We shall refer to (γ, Σ, ν) as the *characteristics of M* and λ as the *control measure of M*.

Our definition above of Lévy-valued random measures assigns a special role to the time domain although this is not necessary for infinitely divisible random measures in general. However, as we will later compare Lévy-valued random measures with cylindrical Lévy processes, which naturally carry a time domain as generalised stochastic processes, we found it more illustrative to have the time domain distinguished. Indeed, the following theorem shows that a Lévy-valued random measure corresponds to an infinitely divisible random measure on space-time, defined as the product space of the time and spatial domains, if the stationarity in the time domain is described by the control measure accordingly. As (up to a multiplicative constant) Lebesgue measure is the unique non-trivial translation-invariant measure on $\mathcal{B}(\mathbb{R})$, this means that the control measure must be of the form $\mathrm{leb} \otimes \lambda_0$.

Proposition 3.1.2.

(a) *Let $M = (M(t) : t \geqslant 0)$ be a Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$. Then, there exists a unique infinitely divisible random measure M' on $\mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O})$ such that $M'((0, t] \times A) = M(t, A)$ for each $t > 0$ and $A \in \mathcal{B}_b(\mathcal{O})$.*

(b) *Each infinitely divisible random measure M' on $\mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O})$ with control measure $\lambda = \mathrm{leb} \otimes \lambda_0$ for a σ-finite measure λ_0 on $\mathcal{B}(\mathcal{O})$ defines by $M(t, A) := M'((0,t] \times A)$ for each $t > 0$ and $A \in \mathcal{B}_b(\mathcal{O})$ a Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$.*

Proof. (a) For fixed $B \in \mathcal{B}_b(\mathcal{O})$, Theorem 3.2 in [80] gives the existence of a unique infinitely divisible random measure M_B on $\mathcal{B}_b(\mathbb{R}_+)$ such that $M_B((0,t]) = M(t, B)$ P-a.s. for each $t \geqslant 0$[1]. Fix $\mathcal{B}_b(\mathbb{R}_+) \ni A = \bigcup_{n\in\mathbb{N}} I_n$, where $\{I_n\}_{n\in\mathbb{N}}$ is a disjoint collection of half-closed intervals with $I_n = (s_n, t_n]$. We see that

$$M_B(A) = \sum_{n\in\mathbb{N}} M_B(I_n) = \sum_{n\in\mathbb{N}} \big(M(t_n, B) - M(s_n, B)\big) \qquad P\text{-a.s.}$$

where the convergence is unconditional. Now, suppose that $\mathcal{B}_b(\mathcal{O}) \ni B = \bigcup_{m\in\mathbb{N}} B_m$ where $\{B_m\}_{m\in\mathbb{N}} \subseteq \mathcal{B}_b(\mathcal{O})$ are disjoint. Then, for A as above, we obtain P-a.s.

$$\begin{aligned} M_B(A) &= \sum_{n\in\mathbb{N}} \big(\sum_{m\in\mathbb{N}} M(t_n, B_m) - \sum_{m\in\mathbb{N}} M(s_n, B_m)\big) \\ &= \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} M_{B_m}(I_n) = \sum_{m\in\mathbb{N}} M_{B_m}(A) \end{aligned} \tag{3.1.1}$$

where again the convergence is unconditional.

Let $\mathcal{R}$ be the semiring of sets of the form $(s,t] \times B$ with $0 \leqslant s < t < \infty$ and B a semi-closed box of the form $B = \prod_{i=1}^d (s_i, t_i] \subseteq \mathcal{O}$. We define a process M' on $\mathcal{R}$ by the prescription

$$M'(A \times B) := M_B(A).$$

To show σ-additivity on $\mathcal{R}$, we shall follow methods used in the proof of Proposition

[1] We note that $M_B(\{t\}) = 0$ P-a.s. for each $t \geqslant 0$ by the stochastic continuity of Lévy processes, and thus whether the endpoints of the intervals are open or closed will not affect $M_B(I_n)$ almost surely.

6.6 in [84]. Suppose that $\{A_n \times B_n\}_{n\in\mathbb{N}} \in \mathcal{R}$ are disjoint and $\bigcup_{n\in\mathbb{N}} A_n \times B_n = A \times B \in \mathcal{R}$. Then clearly $A = \bigcup_{n\in\mathbb{N}} A_n$ and $B = \bigcup_{n\in\mathbb{N}} B_n$; however, the collections $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ may not be disjoint. As in [84, Prop. 6.6] we generate new disjoint families $\{A'_k\}_{k\in\mathbb{N}}$ and $\{B'_\ell\}_{\ell\in\mathbb{N}}$ of the same form as above such that $A = \bigcup_{k\in\mathbb{N}} A'_k$ and $B = \bigcup_{\ell\in\mathbb{N}} B'_\ell$ and furthermore for each $n \in \mathbb{N}$ we have $A_n = \bigcup_{k: A'_k \subseteq A_n} A'_k$ and $B_n = \bigcup_{\ell: B'_\ell \subseteq B_n} B'_\ell$. Then we have P-a.s., by the σ-additivity of M_B and (3.1.1),

$$M'(A \times B) = M_B\Big(\bigcup_{k\in\mathbb{N}} A'_k\Big) = \sum_{k\in\mathbb{N}} M_B(A'_k) = \sum_{k\in\mathbb{N}} \sum_{\ell\in\mathbb{N}} M_{B'_\ell}(A'_k).$$

By the same arguments we see that for each $n \in \mathbb{N}$ we have P-a.s.

$$M'(A_n \times B_n) = \sum\sum_{(k,\ell):\ A'_k \times B'_\ell \subseteq A_n \times B_n} M_{B'_\ell}(A'_k),$$

and thus we obtain P-a.s.

$$\sum_{n\in\mathbb{N}} M'(A_n \times B_n) = \sum_{n\in\mathbb{N}} \sum\sum_{(k,\ell):\ A'_k \times B'_\ell \subseteq A_n \times B_n} M_{B'_\ell}(A'_k) = \sum_{k\in\mathbb{N}} \sum_{\ell\in\mathbb{N}} M_{B'_\ell}(A'_k),$$

and the σ-additivity of M' on $\mathcal{R}$ is shown.

Thus we may apply Theorem 2.15 in [47], considering positive and negative parts separately, to extend M' to an independently scattered random measure on $\mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O})$. Uniqueness follows from [47, Th. 2.2] and infinite divisibility follows immediately from that of M.

(b) By reference to Theorem 3.2 in [80], it is sufficient to observe that the condition on the control measure λ implies the stationarity of the increments of the stochastic processes specified in Definition 3.1.1. □

Remark 3.1.3. *Gaussian space-time white noise is usually defined equivalently to a Gaussian random measure on* $\mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O})$ *in the sense of Definition 2.2.1. Typically,* *one assumes that the measure Σ on $\mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O})$ is either the Lebesgue measure or of the form $\Sigma = \mathrm{leb} \otimes \Sigma_0$ for a σ-finite measure Σ_0 on $\mathcal{B}_b(\mathcal{O})$; see e.g. [49, De. 3.2.2]. Thus, Part (b) of Proposition 3.1.2 shows that our definition of a Lévy-valued random measure naturally extends the class of Gaussian space-time white noises to a Lévy-type setting.*

The relation between random measures and models of Lévy-type noise utilising a Lévy-Itô decomposition seems to be well known. We rigorously formulate this result in our setting:

Proposition 3.1.4. *Let ζ and η be σ-finite Borel measures on $\mathcal{O}$ and let $(U,\mathcal{U},\nu)$ be a σ-finite measure space. Assume that*

(a) $\rho\colon \mathcal{B}_b(\mathcal{O}) \to \mathbb{R}$ *is a signed measure;*

(b) $W\colon \mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O}) \to L^2(\Omega, P)$ *is a Gaussian random measure with characteristics* $(0, \mathrm{leb} \otimes \zeta, 0)$*;*

(c) $N\colon \mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O}) \otimes \mathcal{U} \to L^0(\Omega, P)$ *is Poisson random measure with intensity* $\mathrm{leb} \otimes \eta \otimes \nu$*, independent of W, and with compensated Poisson random measure* $\widetilde{N}$*.*

Then for any functions

(1) $b \in L^2(\mathcal{O}, \zeta)$,

(2) $c\colon \mathcal{O} \times U \to \mathbb{R}$ *with* $\int_{\mathcal{O}\times U} \big(|c(x,y)|^2 \wedge |c(x,y)|\big)\, (\eta \otimes \nu)(dx, dy) < \infty$,

(3) $d : \mathcal{O} \times U \to \mathbb{R}$ *with* $\int_{\mathcal{O}\times U} \big(\, |d(x,y)| \wedge 1\big)\, (\eta \otimes \nu)(dx, dy) < \infty$,

we define a mapping $M'\colon \mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O}) \to L^0(\Omega, P)$ *by*

$$M'(B) = \big(\mathrm{leb} \otimes \rho\big)(B) + \int_B b(x)\, W(\mathrm{d}s, \mathrm{d}x) + \int_{B\times U} c(x,y)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) + \int_{B\times U} d(x,y)\, N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y)\,.$$

Then we obtain a Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$ by the prescription

$$M(t,A) := M'((0,t]\times A) \qquad \textit{for all } A\in\mathcal{B}_b(\mathcal{O}),\, t\geqslant 0.$$

The characteristic function $\varphi_{M(t,A)}\colon \mathbb{R}\to\mathbb{C}$ of $M(t,A)$ is given by

$$\varphi_{M(t,A)}(u) = \exp\Bigg(t\bigg(iu\rho(A) - \tfrac{1}{2}u^2\int_A b^2(x)\,\zeta(\mathrm{d}x) \\ + \int_A\int_U \big(e^{iuc(x,y)} - 1 - iuc(x,y)\big)\,\nu(\mathrm{d}y)\,\eta(\mathrm{d}x) + \int_A\int_U\big(e^{iud(x,y)}-1\big)\,\nu(\mathrm{d}y)\,\eta(\mathrm{d}x)\bigg)\Bigg).$$

Proof. The existence of the Gaussian integral is guaranteed by [94, Th. 2.5] and that of the Poisson integrals by [46, Le. 12.13]. The characteristic function, as stated, of $M'((0,t]\times A)$, see e.g. in [79, Prop. 19.5], shows that M' is an infinitely divisible random measure, and thus applying Proposition 3.1.2 completes the proof. □

Example 3.1.5. The class of α-stable random measures is introduced for example in [78, Se. 3.3]. These can be obtained from Proposition 3.1.4 by defining for $B\in\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$ the random measure

$$M'(B) := \begin{cases} \int_{B\times\mathbb{R}} y\,N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), & \text{if } \alpha\in(0,1], \\ \int_{B\times\mathbb{R}} y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), & \text{if } \alpha\in(1,2), \end{cases}$$

where N is a Poisson random measure on $\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R})$ with intensity $\mathrm{leb}\otimes\mathrm{leb}\otimes\nu_\alpha$, and

$$\nu_\alpha(\mathrm{d}y) = \big(p\alpha y^{-\alpha-1}\mathbb{1}_{(0,\infty)}(y) + q\alpha(-y)^{-\alpha-1}\mathbb{1}_{(-\infty,0)}(y)\big)\,\mathrm{d}y$$

for some $p, q > 0$ satisfying $p + q = 1$; see Balan [10] for this construction[2]. Proposition 3.1.4 guarantees that, by defining $M(t, A) := M'((0, t] \times A)$ for $t \geqslant 0$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$, we obtain a Lévy-valued random measure M on $\mathcal{B}_b(\mathbb{R}^d)$. Direct calculation shows that for $\alpha \neq 1$, the characteristic function of $M(t, A)$ is given by, for $t \geqslant 0$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $u \in \mathbb{R}$,

$$\varphi_{M(t,A)}(u) = \exp\left(t \cdot \text{leb}(A) \cdot \left(i\beta\frac{\alpha}{1-\alpha}u + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_{\mathbb{R}}}(y)\right) \nu_\alpha(\mathrm{d}y)\right)\right),$$

where $\beta := p - q$, and thus we see the characteristics of M are $\left(\beta\frac{\alpha}{1-\alpha}\,\text{leb}, 0, \text{leb} \otimes \nu_\alpha\right)$. The control measure is given by

$$\lambda(A) = \left(\left|\beta\frac{\alpha}{1-\alpha}\right| + \frac{2}{2-\alpha}\right)\text{leb}(A) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^d).$$

For the case $\alpha = 1$, the characteristic function of $M(t, A)$ is given by

$$\varphi_{M(t,A)}(u) = \exp\left(t \cdot \text{leb}(A) \cdot \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_{\mathbb{R}}}(y)\right) \nu_1(\mathrm{d}y)\right)$$

with control measure $\lambda(A) = 2\,\text{leb}(A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$.

Example 3.1.6. Mytnik, in [63], considers a martingale-valued measure $(M(t, A) : t \geqslant 0, A \in \mathcal{B}_b(\mathbb{R}^d))$ in the sense of Walsh [94], such that for any $A \in \mathcal{B}_b(\mathbb{R}^d)$, the process $(M(t, A) : t \geqslant 0)$ is a real-valued α-stable process ($\alpha \in (1, 2)$), with Laplace transform

$$E\left[e^{-uM(t,A)}\right] = e^{-tu^\alpha \cdot \text{leb}(A)}, \qquad t \geqslant 0, u \geqslant 0.$$

[2]For the case $\alpha = 1$ it is required that $p = q = \frac{1}{2}$ as then we obtain:

$$\int_{B\times B_{\mathbb{R}}} y\,\widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) + \int_{B\times B_{\mathbb{R}}^c} y\,N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) = \int_{B\times\mathbb{R}} y\,N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y).$$

By construction, this process forms a Lévy-valued random measure. The author terms M an *α-stable measure without negative jumps.*

Example 3.1.7. Basse-O'Connor and Rosinski in [14, Se. 4] consider an infinitely divisible random measure M on $\mathbb{R} \times V$, for some countably-generated measure space V, which is invariant under translations over $\mathbb{R}$. By Proposition 3.1.2, M defines a Lévy-valued random measure on V, where the generalisation to V is straightforward.

3.2 Lévy-valued additive sheets

Similarly as for infinitely divisible random measures, we introduce Lévy-valued additive sheets by adding a dynamical aspect in the following definition:

Definition 3.2.1. *A family $(X(t,\cdot) : t \geqslant 0)$ of natural, additive sheets $(X(t,x) : x \in \mathbb{R}^d)$ is called a* Lévy-valued additive sheet *if for every $x_1, \dots, x_n \in \mathbb{R}^d$ and $n \in \mathbb{N}$, the stochastic process*

$$\Big(\big(X(t,x_1), \dots, X(t,x_n)\big) : t \geqslant 0\Big)$$

is a Lévy process in $\mathbb{R}^n$.

Remark 3.2.2. *The existence of Lévy-valued additive sheets is shown in Theorem 3.2.4 by construction from certain Lévy-valued random measures.*

The wording 'Lévy-valued additive sheet' is motivated by the following result:

Proposition 3.2.3. *A Lévy-valued additive sheet $(X(t,\cdot) : t \geqslant 0)$ forms a natural additive sheet $(X(z) : z \in \mathbb{R}_+ \times \mathbb{R}^d)$.*

Proof. The domain of definition and Conditions (a), (b) and (d) of Definition 2.2.3 are clearly met. Regarding stochastic continuity, let $(t_n, x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+ \times \mathbb{R}^d$ converging to $(0, x)$. For each $n \in \mathbb{N}$ the random variable $X(1, x_n)$ is infinitely divisible, say with characteristics $(p_{x_n}, V_{x_n}, \mu_{x_n})$. As $X(1,\cdot)$ is a natural, additive sheet, there exists a signed measure γ such that $p_{x_n} = \gamma((0,x_n])$. Since the Lévy process $(X(t,x_n) : t \geqslant 0)$ has stationary increments, it follows that each $X(t,x_n)$ has characteristics $(tp_{x_n}, tV_{x_n}, t\mu_{x_n})$ for every $t \geqslant 0$. Theorem 3.1 in [2] implies that there exist a measure Σ on $\mathcal{B}(\mathbb{R}^d)$ such that $V_{x_n} = \Sigma((0,x_n])$, and a measure ν on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R})$ such that, for each $B \in \mathcal{B}(\mathbb{R})$, the mapping $\nu(\cdot \times B)$ is a measure on $\mathcal{B}(\mathbb{R}^d)$, and $\mu_{x_n} = \nu((0,x_n] \times \cdot)$. Therefore, the Lévy symbol $\vartheta_{X(t_n,x_n)}$ of $X(t_n,x_n)$ is given by, for $u \in \mathbb{R}$,

$$\begin{aligned}\vartheta_{X(t_n,x_n)}(u) = t_n\Big(&iu\gamma((0,x_n]) - \tfrac{1}{2}u^2\Sigma((0,x_n])\\ &+ \int_{(0,x_n]\times\mathbb{R}} \left(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_{\mathbb{R}}}(y)\right)\nu(\mathrm{d}x,\mathrm{d}y)\Big).\end{aligned}$$

As the set $\{x_n : n \in \mathbb{N}\}$ is bounded, there exists a bounded box $I \subseteq \mathbb{R}^d$ containing every box $(0,x_n]$, $n \in \mathbb{N}$. Thus, we obtain for each $u \in \mathbb{R}$ that

$$\left|\vartheta_{X(t_n,x_n)}(u)\right| \leqslant t_n\left(u\,\|\gamma\|_{TV}(I) + \tfrac{1}{2}u^2\Sigma(I) + \int_{I\times\mathbb{R}} \left(u^2y^2 \wedge 1\right)\nu(\mathrm{d}x,\mathrm{d}y)\right).$$

Finiteness of the right side follows from the fact that the measures are finite on I. Therefore, it follows that $X(t_n,x_n) \to 0$ in probability as (t_n,x_n) converges to $(0,x)$. If (t_n,x_n) is an arbitrary sequence converging to (t,x), stationary increments imply for each $c > 0$ that

$$\begin{aligned}&P(|X(t_n,x_n) - X(t,x)| > c)\\ &\quad\leqslant P(|X(t_n,x_n) - X(t,x_n)| > \tfrac{c}{2}) + P(|X(t,x_n) - X(t,x)| > \tfrac{c}{2})\\ &\quad= P(|X(t_n - t,x_n)| > \tfrac{c}{2}) + P(|X(t,x_n) - X(t,x)| > \tfrac{c}{2}).\end{aligned}$$

Consequently, the above established continuity in probability shows the general case, where we have used that $X(t,\cdot)$ is continuous in probability for each $t \geqslant 0$ by definition.

The fact that $X(z)$ is natural can be seen from the form of the characteristic function, where we have $p_z = t\gamma((0,x])$ for $z = (t,x)$. $\square$

We are now able to state the link between Lévy-valued random measures and Lévy-valued additive sheets by formulating a result from Pedersen in [66] in our setting.

Theorem 3.2.4.

(a) *Let $(X(t,\cdot) : t \geqslant 0)$ be a Lévy-valued additive sheet. Then there exists a unique Lévy-valued random measure M on $\mathcal{B}_b(\mathbb{R}^d)$ with atomless control measure λ satisfying $M(t,(0,x]) = X(t,x)$ P-a.s. for each $t \geqslant 0$ and $x \in \mathbb{R}^d$.*

(b) *Let M be a Lévy-valued random measure on $\mathcal{B}_b(\mathbb{R}^d)$ with atomless control measure λ. Then any lamp and right-continuous (as in Definition 2.2.2) modification of the stochastic process $X = (X(t,x) : t \geqslant 0,\, x \in \mathbb{R}^d)$ defined by*

$$X(t,x) := \begin{cases} 0, & \text{if } x_j = 0 \text{ for some } j = 1,\dots,d \\ M\big(t,(0,x]\big), & \text{else} \end{cases}$$

is a Lévy-valued additive sheet[3].

Proof. (a) By the Lévy-Itô decomposition [2, Th. 4.6], we may write

$$X(t,x) = t\gamma((0,x]) + X_g(t,x) + \int_{B_\mathbb{R}} y\,\widetilde{N}((0,t]\times(0,x],\mathrm{d}y) + \int_{B_\mathbb{R}^c} y\,N((0,t]\times(0,x],\mathrm{d}y)$$

almost surely, where X_g is a continuous Gaussian additive sheet and N is an independent Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}$ with intensity measure $\mathrm{d}t\,\nu(\mathrm{d}x,\mathrm{d}y)$.

[3] See Section 2.2 for the meaning of $(0,x]$ for general $x \in \mathbb{R}^d$.

We apply Proposition 3.1.4 and see that the mapping $M' : \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d) \to L^0(\Omega, P)$ defined by

$$M'(B) := (\text{leb} \otimes \gamma)(B) + \int_{B \times B_{\mathbb{R}}} y \, \widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) + \int_{B \times B_{\mathbb{R}}^c} y \, N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y)$$

defines a Lévy-valued random measure on $\mathcal{B}_b(\mathbb{R}^d)$ by $M_0(t, A) := M'((0, t] \times A)$.

We now proceed to define an independently scattered random measure G on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d))$. Let $\mathcal{R}$ be the ring consisting of finite unions of disjoint half-open intervals of the form $(w, z] : w, z \in \mathbb{R}_+ \times \mathbb{R}^d$. On $\mathcal{R}$ we define

$$G(\bigcup_{i=1}^{n} (w_i, z_i]) := \sum_{i=1}^{n} \left(X_g(z_i) - X_g(w_i) \right),$$

and furthermore we set $G(\{0\}) := 0$. The random set function G is clearly additive on $\mathcal{R}$, we now show G is σ-additive. Let $(I_n)_{n \in \mathbb{N}} \subseteq \mathcal{R}$ be a sequence of intervals decreasing to $\emptyset$, by the continuity of X_g we have $G(I_n) \to 0$ P-a.s.. Therefore, we may apply Theorem 2.15 in [47], considering positive and negative parts separately, to extend G to an independently scattered random measure on $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$.

We now define $M(t, A) := M_0(t, A) + G((0, t] \times A)$ for $t \geq 0$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$, which satisfies the statement (a), where the control measure λ is atomless by the stochastic continuity of X. Finally, uniqueness follows from Dynkin's lemma.

(b) It suffices to check that the process $(X(1, x) : x \in \mathbb{R}^d)$ satisfies conditions (b) and (c) of Definition 2.2.3. Independence of increments follows from the independent scattering of the random measure, so it remains to show stochastic continuity. Let $x_n \to x$ in $\mathbb{R}^d$. We have, for fixed $a > 0$,

$$P(|X(1, x) - X(1, x_n)| > a) = P(|M(1, (x_n, x])| > a) \to 0$$

by the atomlessness of λ, as this implies $M(1, \{x\}) = 0$ almost surely for each singleton. □

Remark 3.2.5. *Theorem 3.2.4 and its proof enables us to conclude a converse implication of Proposition 3.1.4. If M is a Lévy-valued random measure M with atomless control measure λ, then it satisfies a Lévy-Itô decomposition of the form*

$$\begin{aligned} M(t,A) &= t\gamma(A) + G\big((0,t]\times A\big) \\ &\quad + \int_{(0,t]\times A\times B_{\mathbb{R}}} y\, \widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y) + \int_{(0,t]\times A\times B_{\mathbb{R}}^{c}} y\, N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), \end{aligned} \tag{3.2.2}$$

where γ is a signed measure on $\mathcal{O}$, G is a Gaussian random measure on $\mathcal{B}_b(\mathbb{R}_+\times\mathcal{O})$ and N is an independent Poisson random measure on $\mathcal{B}_b(\mathbb{R}_+\times\mathcal{O}\times\mathbb{R})$ with compensated part $\widetilde{N}$. The requirement for λ to be atomless is in order that M has no fixed discontinuities.

Furthermore, we see that one does not achieve more generality by allowing an arbitrary measure space $(U,\mathcal{U},\nu)$ in Proposition 3.1.4, as the Poissonian components can be represented as integrals over $\mathbb{R}$.

3.3 Cylindrical Lévy processes

In this section, we establish the correspondence between Lévy-valued random measures and a certain subclass of cylindrical Lévy processes. We analyse the embeddings of Lévy-valued random measures into certain Banach spaces. These embeddings are based on the integration theory for independently scattered infinitely divisible measures developed by Rajput and Rosinski in [70]. Let $\mathcal{O}\subseteq\mathbb{R}^d$ be an open set. The multiplicative relation between the characteristics of the infinitely divisible random measures $M(1)$ and $M(t)$, remarked after Definition 3.1.1, enables us to apply directly the integration theory for infinitely divisible random measures to Lévy-valued random measures $(M(t) : t \geqslant 0)$ on $\mathcal{B}_b(\mathcal{O})$: for a simple function

$$f\colon \mathcal{O} \to \mathbb{R}, \qquad f(x) = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{A_k}(x), \tag{3.3.3}$$

for $\alpha_k \in \mathbb{R}$ and pairwise disjoint sets $A_1, \dots, A_n \in \mathcal{B}_b(\mathcal{O})$, the integral is defined as

$$\int_A f(x)\, M(t, \mathrm{d}x) := \sum_{k=1}^{n} \alpha_k M(t, A \cap A_k) \qquad \text{for all } A \in \mathcal{B}(\mathcal{O}),\ t \geqslant 0. \tag{3.3.4}$$

An arbitrary measurable function $f\colon \mathcal{O} \to \mathbb{R}$ is said to be *M-integrable* if the following hold:

(1) there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ of the form (3.3.3) such that f_n converges pointwise to f λ-a.e., where λ is the control measure of M;

(2) for each $A \in \mathcal{B}(\mathcal{O})$ and $t \geqslant 0$, the sequence $\big(\int_A f_n(x)\, M(t, \mathrm{d}x)\big)_{n\in\mathbb{N}}$ converges in probability.

In this case, the integral of f is defined as

$$\int_A f(x)\, M(t, \mathrm{d}x) := P\!-\lim_{n\to\infty} \int_A f_n(x)\, M(t, \mathrm{d}x). \tag{3.3.5}$$

It is clear, by the stationarity of the increments of Lévy processes, that Condition (2) above holds for all $t \geqslant 0$ if it holds for at least one $t > 0$. Furthermore, Theorem 3.3 in [70] identifies the set of M-integrable functions as the Musielak-Orlicz space

$$L_M(\mathcal{O}, \lambda) := \left\{ f \in L^0(\mathcal{O}, \lambda) : \int_{\mathcal{O}} \Phi_M(|f(x)|, x)\, \lambda(\mathrm{d}x) < \infty \right\},$$

where the modular $\Phi_M : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ is defined as:

$$\Phi_M(u,x) := \sup_{|c|\leqslant 1} |R(cu,x)| + u^2 g(x) + \int_{\mathbb{R}} \left(1 \wedge |uy|^2\right) \rho(x,\mathrm{d}y), \tag{3.3.6}$$

$$\text{with } R(u,x) := ua(x) + u \int_{\mathbb{R}} y\big(\mathbb{1}_{B_{\mathbb{R}}}(uy) - \mathbb{1}_{B_{\mathbb{R}}}(y)\big) \rho(x,\mathrm{d}y), \tag{3.3.7}$$

$$a(x) := \frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}(x), \qquad g(x) = \frac{\mathrm{d}\Sigma}{\mathrm{d}\lambda}(x).$$

Here, (γ, Σ, ν) denotes the characteristics of M. The measure $\rho(x,\cdot)$ is a disintegration of ν over λ, i.e. $\int_{\mathcal{O}\times\mathbb{R}} h(x,y)\,\nu(\mathrm{d}x,\mathrm{d}y) = \int_{\mathcal{O}} \Big(\int_{\mathbb{R}} h(x,y)\,\rho(x,\mathrm{d}y)\Big)\,\lambda(\mathrm{d}x)$ for each measurable function $h\colon \mathcal{O}\times\mathbb{R} \to \mathbb{R}_+$. The space $L_M(\mathcal{O},\lambda)$ is a complete, translation-invariant, linear metric space. Furthermore for all $t \geqslant 0$, the mapping

$$J(t) : L_M(\mathcal{O},\lambda) \to L^0(\Omega, P), \qquad J(t)f = \int_{\mathcal{O}} f(x)\, M(t,\mathrm{d}x), \tag{3.3.8}$$

is continuous [70, Th. 3.3]. Finally, Proposition 2.6 in [70] allows us to immediately state the Lévy symbol of $J(\cdot)f$ as, for $u \in \mathbb{R}$,

$$\begin{aligned}\Psi_{J(\cdot)f}(u) =& iu \int_{\mathcal{O}} f(x)\,\gamma(\mathrm{d}x) - \tfrac{1}{2}u^2 \int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x) \\ &+ \int_{\mathcal{O}\times\mathbb{R}} \Big(e^{iuf(x)y} - 1 - iuf(x)y\,\mathbb{1}_{B_{\mathbb{R}}}(y)\Big)\,\nu(\mathrm{d}x,\mathrm{d}y)\,. \end{aligned}\tag{3.3.9}$$

We are now ready to state our result defining a cylindrical Lévy process from a given Lévy-valued random measure.

Theorem 3.3.1. *Let M be a Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$ with characteristics (γ,Σ,ν) and control measure λ. If U is a Banach space for which U^* is continuously embedded into $L_M(\mathcal{O},\lambda)$, and the simple functions are dense in U^*,* *then*

$$L(t)f := \int_{\mathcal{O}} f(x)\, M(t, \mathrm{d}x) \qquad \text{for all } f \in U^*, \tag{3.3.10}$$

defines a cylindrical Lévy processes L in U. In this case, the characteristics (a, Q, μ) of L is given by

$$\begin{aligned} a(f) &= \int_{\mathcal{O}} f(x)\, \gamma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} f(x)y\big(\mathbb{1}_{B_{\mathbb{R}}}(f(x)y) - \mathbb{1}_{B_{\mathbb{R}}}(y)\big)\, \nu(\mathrm{d}x, \mathrm{d}y), \\ \langle Qf, f\rangle &= \int_{\mathcal{O}} (f(x))^2\, \Sigma(\mathrm{d}x), \qquad\qquad \mu \circ \langle f, \cdot\rangle^{-1} = \nu \circ \chi_f^{-1}, \end{aligned}$$

for each $f \in U^*$, where $\chi_f \colon \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is defined by $\chi_f(x, y) := f(x)y$.

We first prove an intermediate result, which shall be used again in the sequel.

Lemma 3.3.2. *For a Lévy-valued random measure M on $\mathcal{B}_b(\mathcal{O})$ let J be defined by (3.3.8). Then, for any $f_1, \dots, f_n \in L_M(\mathcal{O}, \lambda)$ and $n \in \mathbb{N}$, we have that*

$$\big((J(t)f_1, \dots, J(t)f_n) : t \geqslant 0\big)$$

is a Lévy process in $\mathbb{R}^n$.

Proof. Let f_k for $k = 1, \dots, n$ be simple functions of the form

$$f_k \colon \mathcal{O} \to \mathbb{R}, \qquad f_k(x) = \sum_{j=1}^{m_k} \alpha_{k,j} \mathbb{1}_{A_{k,j}}(x),$$

for $\alpha_{k,j} \in \mathbb{R}$ and $A_{k,j} \in \mathcal{B}_b(\mathcal{O})$ with $A_{k,1}, \dots, A_{k,m_k}$ disjoint for each $k \in \{1, \dots, n\}$. By taking the intersections of all possible permutations of the sets $A_{k,j}$, we can assume that

$$f_k(x) = \sum_{j=1}^{m} \tilde{\alpha}_{k,j}\, \mathbb{1}_{\tilde{A}_j}(x) \quad \text{for all } x \in \mathcal{O},$$

for all $k = 1, \dots, n$, where $\tilde{\alpha}_{k,j} \in \mathbb{R}$ and disjoint sets $\tilde{A}_1, \dots, \tilde{A}_m \in \mathcal{B}_b(\mathcal{O})$ for some $m \in \mathbb{N}$. For each $0 \leqslant t_1 < \cdots < t_n$ we obtain by the definition in (3.3.4) that

$$\begin{aligned} J(t_1)f_1 &= \sum_{j=1}^{m} \tilde{\alpha}_{1,j} M\big(t_1, \tilde{A}_j\big), \\ (J(t_2) - J(t_1))f_2 &= \sum_{j=1}^{m} \tilde{\alpha}_{2,j} \big(M(t_2, \tilde{A}_j) - M(t_1, \tilde{A}_j)\big), \\ &\vdots \\ (J(t_n) - J(t_{n-1}))f_n &= \sum_{j=1}^{m} \tilde{\alpha}_{n,j} \big(M(t_n, \tilde{A}_j) - M(t_{n-1}, \tilde{A}_j)\big). \end{aligned}$$

Independent increments of the Lévy process $\big(M(\cdot, \tilde{A}_1), \dots, M(\cdot, \tilde{A}_m)\big)$ together with independence of $M(t, \tilde{A}_i)$ and $M(t, \tilde{A}_j)$ for all $i, j = 1, \dots, m$ with $i \neq j$ imply that the random variables

$$J(t_1)f_1, \big(J(t_2) - J(t_1)\big)f_2, \dots, \big(J(t_n) - J(t_{n-1})\big)f_n,$$

are independent. This property extends to arbitrary functions $f_1, \dots, f_n \in L_M(\mathcal{O}, \lambda)$ by the definition of the integrals in (3.3.5) as a limit of the integral for simple functions. It follows that the n-dimensional stochastic process $\big((J(t)f_1, \dots, J(t)f_n) : t \geqslant 0\big)$ has independent increments.

Furthermore, if f is a simple function of the form (3.3.3) then

$$J(t)f = \sum_{k=1}^{n} \alpha_k M(t, A_k) \tag{3.3.11}$$

is a Lévy process as it is the sum of independent Lévy processes $M(\cdot, A_k)$. Approximating an arbitrary function $f \in L_M(\mathcal{O}, \lambda)$ by a sequence of simple functions and passing to the limit in (3.3.11) shows that $J(\cdot)f$ is a Lévy process [5, Th. 1.3.7].

Let $f_1, \ldots, f_n$ be arbitrary functions in $L_M(\mathcal{O}, \lambda)$. As $J(\cdot)f$ has stationary increments it follows that $\big((J(t)f_1, \ldots, J(t)f_n) : t \geqslant 0\big)$ has stationary increments by linearity. Furthermore, for each $c > 0$ we have

$$\begin{aligned}
P\big(\,|((J(t)f_1, \ldots, J(t)f_n))| > c\big) &= P\big(\,|J(t)f_1|^2 + \cdots + |J(t)f_n|^2 > c^2\big) \\
&\leqslant \sum_{k=1}^{n} P\left(|J(t)f_k|^2 \geqslant \tfrac{c^2}{n}\right),
\end{aligned}$$

and thus the stochastic continuity of $J(\cdot)f$ implies that of $\big((J(t)f_1, \ldots, J(t)f_n) : t \geqslant 0\big)$. Consequently, the latter is verified as an n-dimensional Lévy process. □

Proof of Theorem 3.3.1. Lemma 3.3.2 shows that L is a cylindrical Lévy process in U. It remains to derive the characteristics of L. For this purpose, let f be a simple function of the form (3.3.3). For each $u \in \mathbb{R}$ we obtain from the definition 3.3.4 that

$$\begin{aligned}
\varphi_{L(t)f}(u) &= \prod_{k=1}^{n} \varphi_{M(t,B_k)}(\alpha_k u) \\
&= \exp\left(t \sum_{k=1}^{n} \Big(i\alpha_k u \gamma(B_k) - \tfrac{1}{2}\alpha_k^2 u^2 \Sigma(B_k) \right. \\
&\qquad \left. + \int_{B_k \times \mathbb{R}} \big(e^{i\alpha_k u y} - 1 - i\alpha_k u y\, \mathbb{1}_{B_{\mathbb{R}}}(y)\big)\, \nu(\mathrm{d}x, \mathrm{d}y)\Big)\right) \\
&= \exp\left(t \Big(iu \int_{\mathcal{O}} f(x)\, \gamma(\mathrm{d}x) - \tfrac{1}{2}u^2 \int_{\mathcal{O}} f^2(x)\, \Sigma(\mathrm{d}x) \right. \\
&\qquad \left. + \int_{\mathcal{O} \times \mathbb{R}} \big(e^{iuf(x)y} - 1 - iuf(x)y\, \mathbb{1}_{B_{\mathbb{R}}}(y)\big)\, \nu(\mathrm{d}x, \mathrm{d}y)\Big)\right).
\end{aligned} \tag{3.3.12}$$

Continuity of $L(t)\colon U^* \to L^0(\Omega, P)$ implies that the characteristic function of $L(t)f$

for simple functions f in (3.3.12) extends to arbitrary functions $f \in U^*$. Consequently, the symbol ϑ_L of L is given by, for $f \in U^*$,

$$\begin{aligned}
&\vartheta_L(f)\\
&= i\int_{\mathcal{O}} f(x)\,\gamma(\mathrm{d}x) - \tfrac{1}{2}\int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} \left(e^{if(x)y} - 1 - if(x)y\,\mathbb{1}_{B_{\mathbb{R}}}(y)\right)\nu(\mathrm{d}x,\mathrm{d}y)\\
&= i\int_{\mathcal{O}} f(x)\,\gamma(\mathrm{d}x) + i\int_{\mathcal{O}\times\mathbb{R}} f(x)y\left(\mathbb{1}_{B_{\mathbb{R}}}(f(x)y) - \mathbb{1}_{B_{\mathbb{R}}}(y)\right)\nu(\mathrm{d}x,\mathrm{d}y)\\
&\qquad - \tfrac{1}{2}\int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x) + \int_{\mathbb{R}} \left(e^{iz} - 1 - iz\mathbb{1}_{B_{\mathbb{R}}}(z)\right)(\nu\circ\chi_f^{-1})(\mathrm{d}z).
\end{aligned}$$

On the other hand, if (a, Q, μ) denotes the characteristics of L, then we obtain for $f \in U^*$ that

$$\begin{aligned}
\vartheta_L(f) &= ia(f) - \tfrac{1}{2}\langle Qf, f\rangle + \int_U \left(e^{i\langle g,f\rangle} - 1 - i\langle g, f\rangle\,\mathbb{1}_{B_{\mathbb{R}}}(\langle g, f\rangle)\right)\mu(\mathrm{d}g)\\
&= ia(f) - \tfrac{1}{2}\langle Qf, f\rangle + \int_{\mathbb{R}} \left(e^{iz} - 1 - iz\,\mathbb{1}_{B_{\mathbb{R}}}(z)\right)(\mu\circ\pi_f^{-1})(\mathrm{d}z).
\end{aligned}$$

Equating the two representations for ϑ_L completes the proof. □

The integration theory developed in [70] and briefly recalled above guarantees that (3.3.10) is well defined for every $f \in L_M$. However, in order to be in the framework of cylindrical Lévy processes we need that the domain of $L(t)$ is the dual of a Banach space (or alternatively is a nuclear space). Since the Musielak-Orlicz space L_M is not in general the dual of a Banach space, for the hypothesis of Theorem 3.3.1 we require the existence of the Banach space U with U^* continuously embedded in L_M. If the control measure λ of M is finite on $\mathcal{O}$, then the following result, which will be needed in the sequel, gives us that $L^2(\mathcal{O}, \lambda)$ is continuously embedded in $L_M(\mathcal{O}, \lambda)$.

Lemma 3.3.3. *Let M be a Lévy-valued random measure on $\mathcal{B}_b(\mathcal{O})$ with finite control measure λ. Then $L^2(\mathcal{O}, \lambda)$ is continuously embedded into $L_M(\mathcal{O}, \lambda)$.*

Proof. Denote the characteristics of M by (γ, Σ, ν). Note, that for arbitrary $g \in L^1(\mathcal{O}, \lambda)$, we have

$$\begin{aligned}\int_{\mathcal{O}} g(x)\, \lambda(\mathrm{d}x) = \int_{\mathcal{O}} g(x)\, \|\gamma\|_{TV}\, (\mathrm{d}x) + \int_{\mathcal{O}} g(x)\, \Sigma(\mathrm{d}x) \\ + \int_{\mathcal{O}\times B_{\mathbb{R}}} g(x)\, |y|^2\, \nu(\mathrm{d}x, \mathrm{d}y) + \int_{\mathcal{O}\times B_{\mathbb{R}}^c} g(x)\, \nu(\mathrm{d}x, \mathrm{d}y).\end{aligned} \tag{3.3.13}$$

Let $f \in L^2(\mathcal{O}, \lambda)$ be given. It follows from (3.3.13) that

$$\int_{\mathcal{O}} |f(x)|^2\, \Sigma(\mathrm{d}x) \leqslant \|f\|^2_{L^2(\mathcal{O},\lambda)} < \infty$$

and

$$\begin{aligned}&\int_{\mathcal{O}\times\mathbb{R}} \left(1\wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad = \int_{\mathcal{O}\times B_{\mathbb{R}}} \left(1\wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) + \int_{\mathcal{O}\times B_{\mathbb{R}}^c} \left(1\wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad \leqslant \|f\|^2_{L^2(\mathcal{O},\lambda)} + \int_{\mathcal{O}\times B_{\mathbb{R}}^c} \left(1\wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad \leqslant \|f\|^2_{L^2(\mathcal{O},\lambda)} + \lambda(\mathcal{O}) < \infty.\end{aligned}$$

As λ is finite, we have $L^2(\mathcal{O}, \lambda) \hookrightarrow L^1(\mathcal{O}, \lambda)$ continuously. Furthermore we obtain, recalling the definition of R from (3.3.7), and (3.3.13), that

$$\begin{aligned}&\int_{\mathcal{O}} |R(|f(x)|, x)|\, \lambda(\mathrm{d}x) \\ &\quad = \int_{\mathcal{O}} \left| |f(x)| \left(a(x) + \int_{\mathbb{R}} y\big(\mathbb{1}_{B_{\mathbb{R}}}(|f(x)|\, y) - \mathbb{1}_{B_{\mathbb{R}}}(y)\big)\, \rho(x, \mathrm{d}y)\right)\right|\, \lambda(\mathrm{d}x) \\ &\quad \leqslant \int_{\mathcal{O}} |f(x)|\, \|\gamma\|_{TV}\, (\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} |f(x)y|\, |\mathbb{1}_{B_{\mathbb{R}}}(|f(x)|\, y) - \mathbb{1}_{B_{\mathbb{R}}}(y)|\, \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad \leqslant \|f\|_{L^1(\mathcal{O},\lambda)} + \|f\|^2_{L^2(\mathcal{O},\lambda)} + \int_{\mathcal{O}\times B_{\mathbb{R}}^c} |f(x)y|\, \mathbb{1}_{B_{\mathbb{R}}}(|f(x)|\, y)\, \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad \leqslant \|f\|_{L^1(\mathcal{O},\lambda)} + \|f\|^2_{L^2(\mathcal{O},\lambda)} + \lambda(\mathcal{O}) < \infty.\end{aligned} \tag{3.3.14}$$

From Theorem 2.7 in [70] we obtain $f \in L_M(\mathcal{O}, \lambda)$ thus showing the stated embedding.

To show that the embedding is continuous, let (f_n) converge to 0 in $L^2(\mathcal{O}, \lambda)$. We firstly show that the functions $(x, y) \mapsto f_n(x)y$ converge to 0 in ν_1-measure where $\nu_1 := \nu\big|_{\mathcal{O}\times B_{\mathbb{R}}^c}$. For given $\varepsilon > 0$ define $M_n := \{(x, y) \in \mathcal{O} \times B_{\mathbb{R}}^c : |f_n(x)y| \geqslant \varepsilon\}$. As ν_1 is a finite measure, there exists a compact set $K \subseteq \mathcal{O}\times B_{\mathbb{R}}^c$ such that $\nu_1(\mathcal{O}\times B_{\mathbb{R}}^c \backslash K) < \frac{\varepsilon}{2}$. Let $C := \sup\{|y| : (x, y) \in K\}$. Define for $n \in \mathbb{N}, x \in \mathcal{O}$ and $y \in \mathbb{R}$ functions $g_n(x, y) := f_n(x)$. Since (f_n) also converges to 0 in $L^1(\mathcal{O}, \lambda)$ it follows from (3.3.13) that (g_n) converges to 0 in $L^1(\mathcal{O} \times B_{\mathbb{R}}^c, \nu)$, and thus in ν_1-measure. Consequently, there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$,

$$\nu_1\big(\{(x, y) \in \mathcal{O} \times B_{\mathbb{R}}^c : |f_n(x)| \geqslant \tfrac{\varepsilon}{C}\}\big) \leqslant \tfrac{\varepsilon}{2}\,.$$

Since $M_n \cap K \subseteq \big\{(x, y) \in \mathcal{O} \times B_{\mathbb{R}}^c : |f_n(x)| \geqslant \frac{\varepsilon}{C}\big\}$, we obtain

$$\nu_1(M_n) = \nu_1(M_n \cap K) + \nu_1(M_n \backslash K) \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \qquad \text{for all } n \geqslant N,$$

which shows the claim.

Since ν_1 is a finite measure, Lebesgue's theorem for dominated convergence in ν_1-measure implies

$$\lim_{n\to\infty} \int_{\mathcal{O}\times B_{\mathbb{R}}^c} |f_n(x)y|\, \mathbb{1}_{B_{\mathbb{R}}}(|f_n(x)|\, y)\, \nu(\mathrm{d}x, \mathrm{d}y) = 0.$$

Similar arguments show that

$$\lim_{n\to\infty} \int_{\mathcal{O}\times B_{\mathbb{R}}^c} \big(1 \wedge |f_n(x)y|^2\big)\, \nu(\mathrm{d}x, \mathrm{d}y) = 0.$$

For each $f \in L^2(\mathcal{O}, \lambda)$, by the definition in (3.3.6) of Φ_M and Lemma 2.8 in [70] we obtain

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi_M(|f(x)|, x)\, \lambda(\mathrm{d}x) \\
&= \int_{\mathcal{O}} \sup_{|c|\leqslant 1} |R(c\,|f(x)|, x)|\, \lambda(\mathrm{d}x) + \int_{\mathcal{O}} |f(x)|^2\, \Sigma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} \left(1 \wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) \\
&\leqslant \int_{\mathcal{O}} \left|R(|f(x)|, x)\right| \lambda(\mathrm{d}x) + 10\, \|f\|^2_{L^2(\mathcal{O},\lambda)} + 9 \int_{\mathcal{O}\times B^c_{\mathbb{R}}} \left(1 \wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y).
\end{aligned}$$

Consequently, it follows from (3.3.14) that (f_n) converges to 0 in $L_M(\mathcal{O}, \lambda)$, which completes the proof. $\square$

It is possible as illustrated in the following example to relax the condition on finiteness of λ, but also the same example shows that there are cases where the finiteness of λ is necessary for any L^p space to be continuously embedded.

Example 3.3.4. We return again to Example 3.1.5; let M be the α-stable random measure for some $\alpha \in (0, 2)$, where now we consider the domain of definition to be $\mathcal{B}_b(\mathcal{O})$ for a general $\mathcal{O} \in \mathcal{B}(\mathbb{R}^d)$. We consider the symmetric case $p = q = \frac{1}{2}$, where the characteristics of M is given by $(0, 0, \mathrm{leb} \otimes \nu_\alpha)$ and the control measure by $\lambda(A) = \frac{2}{2-\alpha}\, \mathrm{leb}(A)$, $A \in \mathcal{B}(\mathcal{O})$. One calculates from (3.3.6) that $L_M(\mathcal{O}, \lambda) = L^\alpha(\mathcal{O}, \mathrm{leb})$; see [10, Le. 4].

Thus, if $\alpha \in (1, 2)$ then we can always choose $F = L^{\alpha'}(\mathcal{O}, \mathrm{leb})$. If $\alpha \in (0, 1]$ and $\mathcal{O}$ is bounded we can choose $F = L^p(\mathcal{O}, \mathrm{leb})$ for any $p > 1$ since $|f(x)|^\alpha \leqslant 1 + |f(x)|^{p'}$. However, if $\mathrm{leb}(\mathcal{O}) = \infty$ and $\alpha \leqslant 1$ then no L^p space is embedded in $L_M(\mathcal{O}, \lambda)$ for $p > 1$.

Assume $\alpha \in (1, 2)$. Then Theorem 3.3.1 implies that (3.3.10) defines a cylindrical Lévy process L in $F = L^{\alpha'}(\mathcal{O}, \mathrm{leb})$, and its symbol is given by

$$\vartheta_L(f) = \int_{\mathcal{O}\times\mathbb{R}} \left(e^{if(x)y} - 1 - if(x)y\mathbb{1}_{B_{\mathbb{R}}}(y)\right) \mathrm{d}x\nu_\alpha(\mathrm{d}y) = -C_\alpha\, \|f\|^\alpha_{L^\alpha(\mathcal{O},\mathrm{leb})},$$

where $C_\alpha = \frac{\Gamma(2-\alpha)}{1-\alpha}\cos\frac{\pi\alpha}{2}$ if $\alpha \neq 1$ and $C_\alpha = \frac{\pi}{2}$ if $\alpha = 1$.

We now turn to the question of which cylindrical Lévy processes induce Lévy-valued random measures. For this purpose we introduce the following:

Definition 3.3.5. *A cylindrical Lévy process $(L(t) : t \geqslant 0)$ in $L^p(\mathcal{O},\zeta)$ for some Borel measure ζ and some $p \geqslant 1$ is called* independently scattered *if for any disjoint sets $A_1,\dots,A_n \in \mathcal{B}_b(\mathcal{O})$ and $n \in \mathbb{N}$, the random variables $L(t)\mathbb{1}_{A_1},\dots,L(t)\mathbb{1}_{A_n}$ are independent for each $t \geqslant 0$.*

Theorem 3.3.6. *An independently scattered cylindrical Lévy process $(L(t) : t \geqslant 0)$ in $L^p(\mathcal{O},\zeta)$ for some Borel measure ζ and some $p \geqslant 1$ defines by*

$$M(t,A) := L(t)\mathbb{1}_A \qquad \text{for all } t \geqslant 0, A \in \mathcal{B}_b(\mathcal{O}), \tag{3.3.15}$$

a Lévy-valued random measure M on $\mathcal{B}_b(\mathcal{O})$.

Proof. For each $t \geqslant 0$, the map $M(t,\cdot)\colon \mathcal{B}_b(\mathcal{O}) \to L^0(\Omega,P)$ is well-defined and $M(t,A)$ is an infinitely divisible random variable for each $A \in \mathcal{B}_b(\mathcal{O})$. Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{B}_b(\mathcal{O})$ such that $A := \bigcup_{k\in\mathbb{N}} A_k \in \mathcal{B}_b(\mathcal{O})$. Then, for each $t \geqslant 0$, by the linearity and continuity of $L(t)$ we have

$$M(t,A) = \lim_{n\to\infty} L(t)\mathbb{1}_{\bigcup_{k=1}^n A_k} = \lim_{n\to\infty}\sum_{k=1}^n L(t)\mathbb{1}_{A_k} = \lim_{n\to\infty}\sum_{k=1}^n M(t,A_k)\,,$$

with the limit in probability and thus almost surely by independence. Clearly, $M(t,\cdot)$ is independently scattered for each $t \geqslant 0$, and $\big(M(\cdot,A_1),\dots,M(\cdot,A_n)\big)$ is a Lévy process for each $A_1,\dots A_n \in \mathcal{B}_b(\mathcal{O})$. ☐

Theorem 3.3.7. *Let $(L(t) : t \geqslant 0)$ be a cylindrical Lévy process in $L^p(\mathcal{O},\zeta)$ for some $p \geqslant 1$. Then L is independently scattered if and only if its symbol is of the* *form*

$$\begin{aligned}\vartheta_L(f) &= i\int_{\mathcal{O}} f(x)\,\gamma(\mathrm{d}x) - \tfrac{1}{2}\int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x)\\ &\quad + \int_{\mathcal{O}\times\mathbb{R}} \left(e^{if(x)y} - 1 - if(x)y\mathbb{1}_{B_{\mathbb{R}}}(y)\right)\,\nu(\mathrm{d}x,\mathrm{d}y)\,, \quad f\in L^{p'}(\mathcal{O},\zeta),\end{aligned} \tag{3.3.16}$$

for a signed measure γ on $\mathcal{B}_b(\mathcal{O})$, a measure Σ on $\mathcal{B}_b(\mathcal{O})$ and a σ-finite measure ν on $\mathcal{B}(\mathcal{O}\times\mathbb{R})$ such that for each $B\in\mathcal{B}_b(\mathcal{O})$, $\nu(B\times\cdot)$ is a Lévy measure on $\mathbb{R}$.

Proof. If L is independently scattered then Theorem 3.3.6 implies that L defines a Lévy-valued random measure M by (3.3.15). Denote the characteristics of M by (γ,Σ,ν) and its control measure by λ. For a simple function f of the form (3.3.3) we obtain

$$L(t)(\mathbb{1}_A\, f) = \sum_{i=1}^n \alpha_i L(t)(\mathbb{1}_{A_i}\,\mathbb{1}_A) = \sum_{i=1}^n \alpha_i M(t, A_i\cap A) = \int_A f(x)\, M(t,\mathrm{d}x)\,. \tag{3.3.17}$$

For an arbitrary function $f\in L^{p'}(\mathcal{O},\zeta)$ let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions converging to f both pointwise ζ-almost everywhere and in $L^{p'}(\mathcal{O},\zeta)$. We note that, as $L(t)\,\mathbb{1}_A = 0$ whenever $\zeta(A)=0$, ζ-null sets have null λ-measure, and thus we have $f_n\to f$ pointwise λ-almost everywhere. Since $L(t)\,\mathbb{1}_A\, f_n \to L(t)\,\mathbb{1}_A\, f$ in probability for each $A\in\mathcal{B}(\mathcal{O})$, it follows from (3.3.17) that $f\in L_M(\mathcal{O},\lambda)$ and $L(t)f = \int_{\mathcal{O}} f(x)\, M(t,\mathrm{d}x)$. We obtain the stated form of the characteristic function of L by (3.3.9).

Conversely, if the Lévy symbol is given by (3.3.16), then this form implies for any disjoint sets $A_1,\dots,A_n\in\mathcal{B}_b(\mathcal{O})$ that

$$\vartheta_L\left(\sum_{k=1}^n u_k\,\mathbb{1}_{A_k}\right) = \sum_{k=1}^n \vartheta_L(u_k\,\mathbb{1}_{A_k}) \qquad \text{for all } u_1,\dots,u_n\in\mathbb{R}\,.$$

Consequently, we obtain for the characteristic function of the random vector $X := (L(1)\,\mathbb{1}_{A_1}, \dots, L(1)\,\mathbb{1}_{A_n})$ for all $u = (u_1, \dots u_n) \in \mathbb{R}^n$, that

$$\begin{aligned}\varphi_X(u) = \varphi_{L(1)}(u_1 \mathbb{1}_{A_1} + \cdots + u_n \mathbb{1}_{A_n}) &= e^{i\Psi_L(u_1 \mathbb{1}_{A_1} + \cdots + u_n \mathbb{1}_{A_n})} \\ &= \varphi_{L(1)\mathbb{1}_{A_1}}(u_1) \cdots \varphi_{L(1)\mathbb{1}_{A_n}}(u_n),\end{aligned}$$

which shows that L is independently scattered. □

Applying Theorem 3.3.6 to a given cylindrical Lévy process L on $L^p(\mathcal{O}, \zeta)$ gives the corresponding Lévy-valued random measure M, say with control measure λ. The first part of the proof of Theorem 3.3.7 shows that $L^{p'}(\mathcal{O}, \zeta)$ is a subspace of $L_M(\mathcal{O}, \lambda)$. The following result guarantees that the embedding is continuous in non-degenerated cases.

Proposition 3.3.8. *Let L be an independently scattered cylindrical Lévy process in $L^p(\mathcal{O}, \zeta)$ for some $p \geqslant 1$ with symbol of the form (3.3.16) and M the corresponding Lévy-valued random measure with control measure λ. If the measures γ, Σ and ν are such that for each $A \in \mathcal{B}_b(\mathcal{O})$ with $\Sigma(A) = 0$ and $\nu(A \times B) = 0$ for each $B \in \mathcal{B}(\mathbb{R})$ bounded away from 0, we have $\|\gamma\|_{\mathrm{TV}}(A) = 0$, then $L^{p'}(\mathcal{O}, \zeta)$ is continuously embedded into $L_M(\mathcal{O}, \lambda)$.*

Proof. By the first part of the proof of Theorem 3.3.7 we have $L^{p'}(\mathcal{O}, \zeta) \subseteq L_M(\mathcal{O}, \lambda)$, and, furthermore, the canonical injection $\iota\colon L^{p'}(\mathcal{O}, \zeta) \to L_M(\mathcal{O}, \lambda)$ is well defined, as the ζ-equivalence class of f is a subset of the λ-equivalence class of f. For each $t \geqslant 0$ we consider the operator $J(t)\colon L_M(\mathcal{O}, \lambda) \to L^0(\Omega, P)$ defined in (3.3.8) and we see that $L(t)$ satisfies the factorisation $L(t) = J(t) \circ \iota$.

For establishing $\ker(J(t)) = \{0\}$, let $f \in L_M(\mathcal{O}, \lambda)$ satisfy $J(t)f = 0$. Then, by considering only the real part of the characteristic function of $J(t)f$, we have for every $u \in \mathbb{R}$

$$-\tfrac{1}{2}u^2 \int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} \big(\cos(uf(x)y) - 1\big)\,\nu(\mathrm{d}x, \mathrm{d}y) = 0.$$

As both terms are non-positive, we obtain that $f = 0$ Σ-a.e. and the function $z(x,y) := f(x)y$ satisfies $z = 0$ ν-a.e. In particular, for the set $A := \{x \in \mathcal{O} : f(x) \neq 0\}$ we have $\Sigma(A) = 0$ and $\nu(A \times B) = 0$ for any $B \in \mathcal{B}(\mathbb{R})$ bounded away from 0. The hypothesis on γ thus leads to $\lambda(A) = 0$, which shows $\ker(J(t)) = \{0\}$.

Let (f_n) be a sequence in $L^{p'}(\mathcal{O}, \zeta)$ converging to $f \in L^{p'}(\mathcal{O}, \zeta)$ and assume that ιf_n converges to some $g \in L_M(\mathcal{O}, \lambda)$. As $\lim_{n\to\infty} J(t)(\iota f_n) = J(t)g$ and $\lim_{n\to\infty} L(t)f_n = L(t)f = J(t)(\iota f)$, we derive $J(t)(g - \iota f) = 0$. Since $J(t)$ is injective, we conclude $g = \iota f$ λ-a.e., and the closed graph theorem implies the continuity of ι. □

Remark 3.3.9. *The foregoing theory can easily be generalised to any space of functions in which the indicator functions to bounded Borel sets in $\mathbb{R}^d$ are dense; e.g. the subset of weighted Besov spaces in which this is the case.*

Example 3.3.10. Peszat and Zabczyk in [67, Se. 7.2] define the impulsive cylindrical process in $L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), \zeta)$ by

$$L(t)f := \int_0^t \int_{\mathcal{O}} \int_{\mathbb{R}} f(x)y\,\widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y),$$

where N is a Poisson random measure on $\mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}$ with intensity $\mathrm{leb} \otimes \zeta \otimes \mu$ for a Lévy measure μ on $\mathcal{B}(\mathbb{R})$; see also [6, Ex. 3.6]. Since its symbol is given by

$$\vartheta_L(f) = \int_{\mathcal{O}} \int_{\mathbb{R}} [e^{if(x)y} - 1 - if(x)y]\,\mu(\mathrm{d}y)\zeta(\mathrm{d}x),$$

Theorem 3.3.7 guarantees that L is independently scattered.

Finally, we note that the class of independently scattered cylindrical Lévy processes is a strict subclass, as the following counter-example shows:

Example 3.3.11. Let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of independent, identically distributed, symmetric α-stable Lévy processes for some $\alpha \in (0,2)$. Thus, each ℓ_k has characteristics $(0,0,\rho)$, with $\rho(\mathrm{d}\beta) = \frac{1}{2}\,\mathbb{1}_{\{\beta\neq 0\}}\,|\beta|^{-1-\alpha}\,\mathrm{d}\beta$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2((0,1),\mathrm{leb})$ such that $e_1 \equiv 1$ (such bases include the standard polynomial and trigonometric bases) and let $(a_k)_{k\in\mathbb{N}} \in \ell^{(2\alpha)/(2-\alpha)}$ with $a_1 = 1$. It follows from Lemma 4.2 and Example 4.5 in [74] that the hedgehog process

$$L(t)f := \sum_{k=1}^{\infty} \langle f, e_k\rangle_{L^2} a_k \ell_k(t), \qquad \text{for all } f \in L^2((0,1),\mathrm{leb}),$$

defines a cylindrical Lévy process L, say with characteristics $(0,0,\mu)$.

Assume for a contradiction that L is independently scattered and fix two disjoint sets $A, B \in \mathcal{B}((0,1))$ with $\mathrm{leb}(A) > 0$ and $\mathrm{leb}(B) > 0$. Thus, $\langle \mathbb{1}_A, e_1\rangle_{L^2} = \mathrm{leb}(A) > 0$ and $\langle \mathbb{1}_B, e_1\rangle_{L^2} = \mathrm{leb}(B) > 0$.

The Lévy measure of the Lévy process $\big((L(t)\,\mathbb{1}_A, L(t)\,\mathbb{1}_B) : t \geqslant 0\big)$ in $\mathbb{R}^2$ is given by $\mu \circ \pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B}$. As $L(1)\,\mathbb{1}_A$ and $L(1)\,\mathbb{1}_B$ are independent, it follows from the uniqueness of the characteristic functions that

$$\mu \circ \pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B} = ((\mu \circ \pi^{-1}_{\mathbb{1}_A}) \otimes \delta_0) + (\delta_0 \otimes \mu \circ \pi^{-1}_{\mathbb{1}_B})),$$

where $\mu \circ \pi^{-1}_{\mathbb{1}_A}$ is the Lévy measure of $(L(t)\,\mathbb{1}_A : t \geqslant 0)$ and $\mu \circ \pi^{-1}_{\mathbb{1}_B}$ is the Lévy measure of $(L(t)\,\mathbb{1}_B : t \geqslant 0)$. It follows in particular that

$$\mu \circ \pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B}\big(\mathbb{R}\backslash\{0\} \times \mathbb{R}\backslash\{0\}\big) = 0. \tag{3.3.18}$$

On the other hand, Lemma 4.2 in [74] implies that

$$\mu \circ \pi^{-1}_{\mathbb{1}_A, \mathbb{1}_B} = \sum_{k=1}^{\infty} (\nu_k \circ r_k^{-1}),$$

where $\nu_k = \rho \circ m_{a_k}^{-1}$, with $m_\gamma \colon \mathbb{R} \to \mathbb{R}$ is defined by $m_\gamma(\beta) = \gamma\beta$ for some $\gamma \in \mathbb{R}$ and $r_k \colon \mathbb{R} \to \mathbb{R}^2$ is defined by $r_k(\beta) = \big(\langle \mathbb{1}_A, e_k\rangle_{L^2}\beta, \langle \mathbb{1}_B, e_k\rangle_{L^2}\beta\big)$. In particular, as we have $a_1 = 1$ we have $\nu_1 = \rho$. It follows from (3.3.18) that

$$0 = \big(\rho \circ r_1^{-1}\big)\big(\mathbb{R}\backslash\{0\} \times \mathbb{R}\backslash\{0\}\big) = \rho(\mathbb{R}\backslash\{0\}) = \infty\,,$$

which results in a contradiction.

Chapter 4

Lévy measures and Radonifying embeddings in Besov spaces

In this chapter, we characterise the Lévy measures on $B^p_{s,w}(\mathbb{R}^d)$ for $p > 1$ and then we characterise when $L^2(\mathbb{R}^d)$ is embedded in $B^p_{s,w}(\mathbb{R}^d)$ such that cylindrical random variables in $L^2(\mathbb{R}^d)$ are induced by (genuine) random variables in $B^p_{s,w}(\mathbb{R}^d)$.

4.1 Characterisation of Lévy measures

Our first task is to define Lévy measures, which is not as straightforward for Banach spaces as for finite-dimensional or Hilbert spaces. Let U be an arbitrary separable Banach space. For an arbitrary finite measure μ on $\mathcal{B}(U)$ define the exponential measure $e(\mu)$ by

$$e(\mu) := e^{-\mu(U)} \sum_{m=0}^{\infty} \frac{1}{m!} \mu^{*m}. \tag{4.1.1}$$

The exponential measure $e(\mu)$ is a compound Poisson distribution with characteristic function

$$\varphi_\mu \colon U^* \to \mathbb{C}, \qquad \varphi_\mu(b^*) = \exp\left(\int_U \left(e^{i\langle u, u^* \rangle_U} - 1 \right) \, \mu(\mathrm{d}u) \right).$$

Whereas in the finite dimensional case, and in Hilbert spaces, the integrability of $|\cdot|^2 \wedge 1$ characterises the Lévy measures, there are no equivalent conditions known in arbitrary Banach spaces for when the same function φ_μ but for a σ-finite measure μ still forms the characteristic function of a probability measure. Thus, Lévy measures are defined implicitly in the following way, see [58]:

Definition 4.1.1. *A σ-finite measure μ on $\mathcal{B}(U)$ is a* Lévy measure *if $\mu(\{0\}) = 0$ and*

$$\varphi_\mu \colon U^* \to \mathbb{C}, \qquad \varphi_\mu(u^*) = \exp\left(\int_U \left(e^{i\langle u,u^*\rangle_U} - 1 - i\langle u, u^*\rangle_U \mathbb{1}_{B_U}(u)\right) \mu(\mathrm{d}u)\right)$$

is the characteristic function of a Radon probability measure, which we shall call $e_S(\mu)$, on $\mathcal{B}(U)$.

Note, that [58, Th. 5.4.8] guarantees that a σ-finite measure μ on $\mathcal{B}(U)$ is a Lévy measure if and only if its symmetrisation $\mu + \mu^-$ is a Lévy measure.

We now apply the results obtained in [97] to characterise the Lévy measures in $B^p_{s,w}(\mathbb{R}^d)$ for $p > 1$ and $s, w \in \mathbb{R}$.

Theorem 4.1.2. *A σ-finite measure μ on $\mathcal{B}(B^p_{s,w}(\mathbb{R}^d))$ with $\mu(\{0\}) = 0$ is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$ for some $p \in (1, \infty)$, $s, w \in \mathbb{R}$ if and only if*

(1) *for $p \geqslant 2$,*

$$\int_{B^p_{s,w}} \left(\|f\|^p_{B^p_{s,w}} \wedge 1\right) \mu(\mathrm{d}f) < \infty, \tag{4.1.2}$$

$$\sum_{j,G,m} (\omega^j_m)^p \left(\int_{\|f\|_{B^p_{s,w}} \leqslant 1} [\Psi^{j,G}_m, f]^2 \, \mu(\mathrm{d}f)\right)^{p/2} < \infty; \tag{4.1.3}$$

(2) *and for* $p \in (1,2)$,

$$\int_{B^p_{s,w}} \left(\|f\|^2_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) < \infty, \tag{4.1.4}$$

$$\sum_{j,G,m} (\omega^j_m)^p \int_0^\infty \left(1 - e^{\int_{\|f\|_{B^p_{s,w}} \leqslant 1} \left(\cos \tau [\Psi^{j,G}_m, f] - 1 \right) \mu(\mathrm{d}f)} \right) \frac{\mathrm{d}\tau}{\tau^{1+p}} < \infty. \tag{4.1.5}$$

In the expressions above, $\omega^j_m = \omega^j_m(p,s,w)$ *are the weight constants defined in* (2.1.2).

Proof. Because of [58, Th. 5.4.8], we can assume that μ is symmetric. Given $0 \leqslant \alpha \leqslant \beta \leqslant \infty$ we define

$$\mu_{\alpha,\beta}(A) := \mu\big(A \cap \{f \in B^p_{s,w}(\mathbb{R}^d) \colon \alpha \leqslant \|f\|_{B^p_{s,w}} < \beta\}\big), \quad A \in \mathcal{B}(B^p_{s,w}(\mathbb{R}^d)).$$

Let $\varepsilon \in [0,1)$ and suppose the Radon measure $e_S(\mu_{\varepsilon,1})$ exists in the sense of Definition 4.1.1. We calculate

$$\begin{aligned} \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \, e_S(\mu_{\varepsilon,1})(\mathrm{d}f) &= \sum_{j,G,m} (\omega^j_m)^p \int_{B^p_{s,w}} \left| [\Psi^{j,G}_m, f] \right|^p e_S(\mu_{\varepsilon,1})(\mathrm{d}f) \\ &= \sum_{j,G,m} (\omega^j_m)^p \int_{\mathbb{R}} |\beta|^p \left(e_S(\mu_{\varepsilon,1}) \circ [\Psi^{j,G}_m, \cdot]^{-1} \right)(\mathrm{d}\beta) \\ &= \sum_{j,G,m} (\omega^j_m)^p E \left| \xi^{j,G}_m \right|^p, \end{aligned} \tag{4.1.6}$$

where $\xi^{j,G}_m$ is a random variable with distribution $e_S(\mu_{\varepsilon,1}) \circ [\Psi^{j,G}_m, \cdot]^{-1} = e_S(\mu_{\varepsilon,1} \circ [\Psi^{j,G}_m, \cdot]^{-1})$ for each $j \in \mathbb{Z}_+, G \in G^j$ and $m \in \mathbb{Z}^d$.

For $p \geqslant 2$, Theorem 1.1 in [29] guarantees

$$E \left| \xi^{j,G}_m \right|^p \eqsim_p \int_{\mathbb{R}} |\beta|^p \left(\mu_{\varepsilon,1} \circ [\Psi^{j,G}_m, \cdot]^{-1} \right)(\mathrm{d}\beta) + \left(\int_{\mathbb{R}} |\beta|^2 \left(\mu_{\varepsilon,1} \circ [\Psi^{j,G}_m, \cdot]^{-1} \right)(\mathrm{d}\beta) \right)^{p/2}.$$

It follows that, for $p \geqslant 2$,

$$\begin{aligned}
&\int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \; e_S(\mu_{\varepsilon,1})(\mathrm{d}f) \\
&\eqsim_p \sum_{j,G,m} (\omega^j_m)^p \left(\int_{\mathbb{R}} |\beta|^p \; (\mu_{\varepsilon,1} \circ [\Psi^{j,G}_m, \cdot]^{-1})(\mathrm{d}\beta) + \left(\int_{\mathbb{R}} |\beta|^2 \; (\mu_{\varepsilon,1} \circ [\Psi^{j,G}_m, \cdot]^{-1})(\mathrm{d}\beta) \right)^{p/2} \right) \\
&= \sum_{j,G,m} (\omega^j_m)^p \left(\int_{B^p_{s,w}} \left|[\Psi^{j,G}_m, f]\right|^p \mu_{\varepsilon,1}(\mathrm{d}f) + \left(\int_{B^p_{s,w}} \left|[\Psi^{j,G}_m, f]\right|^2 \mu_{\varepsilon,1}(\mathrm{d}f) \right)^{p/2} \right) \\
&= \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \; \mu_{\varepsilon,1}(\mathrm{d}f) + \sum_{j,G,m} (\omega^j_m)^p \left(\int_{B^p_{s,w}} \left|[\Psi^{j,G}_m, f]\right|^2 \mu_{\varepsilon,1}(\mathrm{d}f) \right)^{p/2}. \qquad (4.1.7)
\end{aligned}$$

We begin by showing sufficiency. Suppose the conditions in the hypothesis are met. Note that Conditions (4.1.2) and (4.1.4) each imply that $\mu_{\varepsilon,1}(B^p_{s,w}(\mathbb{R}^d))$ is finite. Thus, the exponential measure $e(\mu_{\varepsilon,1})$ defined in (4.1.1) coincides with $e_S(\mu_{\varepsilon,1})$ and Equality (4.1.6) holds.

First we consider the case $p \geqslant 2$. From (4.1.7) we obtain for each $\varepsilon > 0$ that

$$\begin{aligned}
&\int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \; e(\mu_{\varepsilon,1})(\mathrm{d}f) \\
&\qquad \lesssim_p \int_{B^p_{s,w}} \left(\|f\|^p_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) + \sum_{j,G,m} (\omega^j_m)^p \left(\int_{\|f\|_{B^p_{s,w}} \leqslant 1} \left|[\Psi^{j,G}_m, f]\right|^2 \mu(\mathrm{d}f) \right)^{p/2}.
\end{aligned}$$

Thus conditions (4.1.2) and (4.1.3) imply

$$\sup_{\varepsilon \in (0,1)} \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \; e(\mu_{\varepsilon,1})(\mathrm{d}f) < \infty. \qquad (4.1.8)$$

Now, let $g \in \big(B^p_{s,w}(\mathbb{R}^d)\big)^*$. By applying Hölder's inequality twice, Theorem 2.1.3 shows

$$\begin{aligned}
&\int_{\|f\|_{B^p_{s,w}} \leqslant 1} \langle f, g\rangle^2_{B^p_{s,w}} \, \mu(\mathrm{d}f) \\
&= \sum_{j,G,m} \sum_{k,H,n} [\Psi^{j,G}_m, g][\Psi^{k,H}_n, g] \int_{\|f\|_{B^p_{s,w}} \leqslant 1} [\Psi^{j,G}_m, f][\Psi^{k,H}_n, f] \, \mu(\mathrm{d}f) \\
&\leqslant \left(\sum_{j,G,m} [\Psi^{j,G}_m, g] \left(\int_{\|f\|_{B^p_{s,w}} \leqslant 1} [\Psi^{j,G}_m, f]^2 \, \mu(\mathrm{d}f) \right)^{1/2} \right)^2 \\
&= \left(\sum_{j,G,m} (\omega^j_m)^{-1} [\Psi^{j,G}_m, g] \left((\omega^j_m)^2 \int_{\|f\|_{B^p_{s,w}} \leqslant 1} [\Psi^{j,G}_m, f]^2 \, \mu(\mathrm{d}f) \right)^{1/2} \right)^2 \\
&\leqslant \left(\sum_{j,G,m} (\omega^j_m)^{-p'} \left| [\Psi^{j,G}_m, g] \right|^{p'} \right)^{2/p'} \left(\sum_{j,G,m} (\omega^j_m)^p \left(\int_{\|f\|_{B^p_{s,w}} \leqslant 1} [\Psi^{j,G}_m, f]^2 \, \mu(\mathrm{d}f) \right)^{p/2} \right)^{2/p}.
\end{aligned}$$

Since $\sum_{j,G,m} (\omega^j_m)^{-p'} \left| [\Psi^{j,G}_m, g] \right|^{p'} = \|g\|^{p'}_{B^{p'}_{-s,-w}}$, Condition (4.1.3) guarantees

$$\int_{\|f\|_{B^p_{s,w}} \leqslant 1} \langle f, g\rangle^2_{B^p_{s,w}} \, \mu(\mathrm{d}f) \lesssim \|g\|^2_{B^{p'}_{-s,-w}}.$$

Together with (4.1.8) we see that the conditions of Theorem 1.B in [97] are satisfied and hence μ is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$.

Next we consider the case $p \in (1,2)$; in this case, we use the following relation for a real valued symmetric random variable X with characteristic function φ_X and $q \in (0,2)$:

$$E\,|X|^q = c_q \int_0^\infty \frac{1 - \mathrm{Re}(\varphi_X(\tau))}{\tau^{q+1}} \, \mathrm{d}\tau, \tag{4.1.9}$$

where c_q is a constant depending only on q; see for example [50, Th. 11.4.3].

Applying Equality (4.1.9) to a random variable with distribution $e(\mu_{\varepsilon,1})\circ[\Psi_m^{j,G},\cdot]^{-1}$ shows, for each $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}}\, e(\mu_{\varepsilon,1})(\mathrm{d}f) \\
&\quad = c_p \sum_{j,G,m} (\omega^j_m)^p \int_0^\infty \left(1 - e^{\int_{B^p_{s,w}} \left(\cos\tau[\Psi^{j,G}_m, f]-1\right)\mu_{\varepsilon,1}(\mathrm{d}f)}\right)\frac{\mathrm{d}\tau}{\tau^{1+p}} \\
&\quad \leqslant c_p \sum_{j,G,m} (\omega^j_m)^p \int_0^\infty \left(1 - e^{\int_{\{\|f\|_{B^p_{s,w}}\leqslant 1\}} \left(\cos\tau[\Psi^{j,G}_m, f]-1\right)\mu(\mathrm{d}f)}\right)\frac{\mathrm{d}\tau}{\tau^{1+p}}.
\end{aligned}$$

Thus, Condition (4.1.5) guarantees

$$\sup_{\varepsilon\in(0,1)} \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}}\, e(\mu_{\varepsilon,1})(\mathrm{d}f) < \infty. \tag{4.1.10}$$

Now, for $g \in B^{p'}_{-s,-w}(\mathbb{R}^d)$, Condition (4.1.4) implies

$$\int_{\|f\|_{B^p_{s,w}}\leqslant 1} \langle f, g\rangle^2_{B^p_{s,w}}\, \mu(\mathrm{d}f) \leqslant \int_{\|f\|_{B^p_{s,w}}\leqslant 1} \|f\|^2_{B^p_{s,w}}\, \|g\|^2_{B^{p'}_{-s,-w}}\, \mu(\mathrm{d}f) \lesssim \|g\|^2_{B^{p'}_{-s,-w}}.$$

Together with (4.1.10), it follows that the conditions of Theorem 1.B in [97] are satisfied and hence μ is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$.

Now suppose conversely that μ is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$; as before we assume μ is symmetric. Proposition 5.4.1 in [58] implies that we have $\mu\big(B^c_{B^p_{s,w}}\big) < \infty$. Since $\mu_{0,1}$ is a symmetric Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$ according to [58, Cor. 5.4.4], the Radon measure $e_S(\mu_{0,1})$ exists. Corollary 3.3 in [1] then shows, for all $q > 0$, that

$$\int_{B^p_{s,w}} \|f\|^q_{B^p_{s,w}}\, e_S(\mu_{0,1})(\mathrm{d}f) < \infty. \tag{4.1.11}$$

We first consider the case $p \geqslant 2$. Since $e_S(\mu_{0,1})$ exists we can apply (4.1.7) to obtain

$$\int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \, \mu_{0,1}(\mathrm{d}f) + \sum_{j,G,m} (\omega^j_m)^p \left(\int_{B^p_{s,w}} [\Psi^{j,G}_m, f]^2 \, \mu_{0,1}(\mathrm{d}f) \right)^{p/2}$$
$$\eqsim_p \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \, e_S(\mu_{0,1})(\mathrm{d}f) < \infty,$$

which verifies the necessity of Conditions (4.1.2) and (4.1.3) because of (4.1.11).

For the case $p \in (1,2)$, we apply Equality (4.1.9) to a random variable with distribution $e_S(\mu_{0,1}) \circ [\Psi^{j,G}_m, \cdot]^{-1}$ to obtain

$$\sum_{j,G,m} (\omega^j_m)^p \int_0^\infty \left(1 - e^{\int_{\|f\|_{B^p_{s,w}} \leqslant 1} \left(\cos \tau [\Psi^{j,G}_m, f] - 1\right) \mu(\mathrm{d}f)} \right) \frac{\mathrm{d}\tau}{\tau^{1+p}}$$
$$= c_p^{-1} \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \, e_S(\mu_{0,1})(\mathrm{d}f) < \infty,$$

which shows the necessity of Condition (4.1.5) because of (4.1.11). Furthermore, since $B^p_{s,w}(\mathbb{R}^d)$ is isomorphic to $\ell^p(\mathbb{W}^d)$ and the latter is of cotype 2 for $p \in (1,2)$, Theorem 2.2 in [8] directly shows the necessity of condition (4.1.4). □

Remark 4.1.3. *Alternatively, in the proof above for the case of sufficiency when $p \in (1,2)$, we may replace condition (4.1.5) with condition (4.1.3). Indeed, for $\varepsilon \in (0,1)$ we have*

$$\int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \, e(\mu_{\varepsilon,1})(\mathrm{d}f) = \sum_{j,G,m} (\omega^j_m)^p E \left|\xi^{j,G}_m\right|^p$$
$$\leq \sum_{j,G,m} (\omega^j_m)^p \left(E \left|\xi^{j,G}_m\right|^2 \right)^{p/2}$$
$$\lesssim_p \sum_{j,G,m} (\omega^j_m)^p \left(\int_{\mathbb{R}} |\beta|^2 \, (\mu_{\varepsilon,1} \circ [\Psi^{j,G}_m, \cdot]^{-1})(\mathrm{d}\beta) \right)^{p/2}$$
$$\leq \sum_{j,G,m} (\omega^j_m)^p \left(\int_{\|f\|_{B^p_{s,w}} \leq 1} \left|[\Psi^{j,G}_m, f]\right|^2 \, \mu(\mathrm{d}f) \right)^{p/2},$$

thus showing that, for $p \in (1,2)$, condition (4.1.3) implies

$$\sup_{\varepsilon \in (0,1)} \int_{B^p_{s,w}} \|f\|^p_{B^p_{s,w}} \, e(\mu_{\varepsilon,1})(\mathrm{d}f) < \infty.$$

Remark 4.1.4. *We may use Theorem 2.3 in [8] to state another sufficient condition for $p \in [1,2]$. We have $B^p_{s,w}(\mathbb{R}^d)$ is of type p by the isometry with $\ell^p(\mathbb{W}^d)$, and thus a sufficient condition for μ to be a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$ is*

$$\int_{B^p_{s,w}} \left(\|f\|^p_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) < \infty.$$

4.2 Radonifying Embeddings

We continue our analysis of weighted Besov spaces by examining the embedding structure. The space $L^2(\mathbb{R}^d)$ may be considered as representing the central point in the parameter scale of these spaces, as $L^2(\mathbb{R}^d) = B^2_{0,0}(\mathbb{R}^d)$. We shall first present the general theory of continuous embeddings of Besov spaces, and then examine the special case, that the canonical embedding operator ι maps each cylindrical random variable on $L^2(\mathbb{R}^d)$ to a cylindrical random variable on $B^p_{s,w}(\mathbb{R}^d)$ which is induced by a (genuine) random variable.

Definition 4.2.1. *Let F, G be Banach spaces. A continuous linear operator $T\colon G \to F$ is called*

(i) 0-Radonifying *if for each cylindrical random variable X in G, TX is induced by a random variable in F;*

(ii) p-Radonifying *for some $p > 0$ if, for each cylindrical random variable X in G such that $E\,|Xg|^p < \infty$ for each $g \in G$, we have that TX is induced by a random variable in F with strong moments of order p.*

Weighted Besov spaces form various scales according to the parameters p, s and w; we present a general result for their continuous embeddings. The positive results, for when a certain Besov space is continuously embedded in another given Besov space, are well-known; however, we are unaware of any converse results so we develop such converses here.

Proposition 4.2.2. *Let $s_0, s_1, w_0, w_1 \in \mathbb{R}$ and $p_0, p_1 > 1$.*

1. *Suppose $p_0 > p_1$. Then $B^{p_0}_{s_0,w_0}(\mathbb{R}^d) \hookrightarrow B^{p_1}_{s_1,w_1}(\mathbb{R}^d)$ continuously if and only if*

$$s_0 > s_1 \qquad \textit{and} \qquad w_0 - w_1 > d\left(\frac{1}{p_1} - \frac{1}{p_0}\right).$$

2. *Suppose $p_0 \leqslant p_1$. Then $B^{p_0}_{s_0,w_0}(\mathbb{R}^d) \hookrightarrow B^{p_1}_{s_1,w_1}(\mathbb{R}^d)$ continuously if and only if*

$$s_0 - s_1 \geqslant d\left(\frac{1}{p_0} - \frac{1}{p_1}\right) \qquad \textit{and} \qquad w_0 \geqslant w_1.$$

Proof. The continuous embedding in both cases is given by Proposition 3 in [31]. To prove non-inclusion we first note that for any $q > 0$, by a duality argument, if $y = (y^{j,G}_m)_{(j,G,m)\in\mathbb{W}^d}$ is such that $y \notin \ell^q(\mathbb{W}^d)^*$ then there exists $x \in \ell^q(\mathbb{W}^d)$ such that $\sum_{j,G,m} x^{j,G}_m y^{j,G}_m = \infty$.

1. Suppose that $p_0 > p_1$. First we suppose that $w_1 - w_0 \geqslant -d\left(\frac{1}{p_1} - \frac{1}{p_0}\right)$. Take $\alpha = \frac{p_0}{p_0 - p_1}$ and thus $\alpha' = \frac{p_0}{p_1}$, then we have $\alpha p_1(w_1 - w_0) \geqslant -d$. Thus, for any $j \geqslant 0$ we have

$$\sum_{m\in\mathbb{Z}^d} \left(1 + 2^{-2j}|m|^2\right)^{\frac{\alpha p_1(w_1-w_0)}{2}} = \infty$$

(see the proof of [31, Th. 3]) and therefore

$$\sum_{j,G,m} 2^{j\alpha p_1(s_1 - s_0 - \frac{d}{p_1} + \frac{d}{p_0})}\left(1 + 2^{-2j}|m|^2\right)^{\frac{\alpha p_1(w_1-w_0)}{2}} = \infty$$

which means

$$\left(\left(\frac{\omega_m^j(p_1,s_1,w_1)}{\omega_m^j(p_0,s_0,w_0)}\right)^{p_1}\right)_{(j,G,m)\in\mathbb{W}^d} \notin \ell^\alpha(\mathbb{W}^d),$$

where $\omega_m^j(p,s,w) = 2^{j(s-\frac{d}{p}+\frac{d}{2})}(1+2^{-2j}|m|^2)^{\frac{w}{2}}$ are the weight constants defined in (2.1.2). We thus have the existence of $y \in \ell^{\frac{p_0}{p_1}}(\mathbb{W}^d)$ (which clearly we may assume has each term non-negative) such that

$$\sum_{j,G,m}\left(\frac{\omega_m^j(p_1,s_1,w_1)}{\omega_m^j(p_0,s_0,w_0)}\right)^{p_1} y_m^{j,G} = \infty.$$

By the isometry $(\Upsilon_{s_0,w_0}^{p_0})^{-1}\colon \ell^{p_0}(\mathbb{W}^d)\to b_{s_0,w_0}^{p_0}$ (see Lemma 2.1.4) we have $\lambda \in b_{s_0,w_0}^{p_0}$ where

$$\lambda_m^{j,G} = 2^{\frac{jd}{2}}\left(\omega_m^j(p_0,s_0,w_0)\right)^{-1}(y_m^{j,G})^{\frac{1}{p_1}}.$$

We then see that

$$\begin{aligned}\|\lambda\|_{b_{s_1,w_1}^{p_1}}^{p_1} &= \sum_{j,G,m}\left(2^{-\frac{jd}{2}}\omega_m^j(p_1,s_1,w_1)\lambda_m^{j,G}\right)^{p_1}\\ &= \sum_{j,G,m}\left(\frac{\omega_m^j(p_1,s_1,w_1)}{\omega_m^j(p_0,s_0,w_0)}\right)^{p_1} y_m^{j,G} = \infty,\end{aligned}$$

and so $\lambda \notin b_{s_1,w_1}^{p_1}$, which shows that $b_{s_0,w_0}^{p_0} \nsubseteq b_{s_1,w_1}^{p_1}$ which in turn shows $B_{s_0,w_0}^{p_0}(\mathbb{R}^d) \nsubseteq B_{s_1,w_1}^{p_1}(\mathbb{R}^d)$ by the isometry between the weighted Besov spaces and the weighted Besov sequence spaces [91, Th. 6.15].

Now suppose that $s_0 \leqslant s_1$. We define for $(j,G,m)\in\mathbb{W}^d$

$$x_m^{j,G} := \left(\frac{\omega_m^j(p_1,s_1,w_1)}{\omega_m^j(p_0,s_0,w_0)}\right)^{p_1}.$$

We suppose that, for each $j \in \mathbb{Z}_+$, $S_j := \sum_{m\in\mathbb{Z}^d} \left(1 + 2^{-2j}|m|^2\right)^{\frac{\alpha p_1(w_1-w_0)}{2}} < \infty$ (as otherwise there would be nothing to prove). We have by [31, Th. 3] that S_j is asymptotically $\mathcal{O}(2^{jd})$ as $j \to \infty$, and thus

$$\sum_{j,G,m} \left|x_m^{j,G}\right|^\alpha = \sum_{j,G} 2^{\alpha j p_1 (s_1 - s_0 - \frac{d}{p_1} + \frac{d}{p_0})} S_j = \infty$$

as $\alpha p_1\big(s_1 - s_0 - \frac{d}{p_1} + \frac{d}{p_0}\big) \geqslant -d$. Thus we have $\left(x_m^{j,G}\right)_{(j,G,m)\in\mathbb{W}^d} \notin \ell^\alpha(\mathbb{W}^d)$ and the proof of non-inclusion proceeds as above.

2. Let $p_1 \geqslant p_0$. First we take $s_1 - s_0 > d(\frac{1}{p_1} - \frac{1}{p_0})$ so we have $2^{jp_1(s_1-s_0-\frac{d}{p_1}+\frac{d}{p_0})}$ is unbounded as $j \to \infty$. For $(j, G, m) \in \mathbb{W}^d$ we define

$$x_m^{j,G} := \left(\frac{\omega_m^j(p_1, s_1, w_1)}{\omega_m^j(p_0, s_0, w_0)}\right)^{p_1}.$$

Then we have $\left(x_m^{j,G}\right)_{(j,G,m)\in\mathbb{W}^d} \notin \ell^\infty(\mathbb{W}^d)$ and thus have the existence of $y \in \ell^{\frac{p_0}{p_1}}(\mathbb{W}^d)$ such that

$$\sum_{j,G,m} \left(\frac{\omega_m^j(p_1, s_1, w_1)}{\omega_m^j(p_0, s_0, w_0)}\right)^{p_1} y_m^{j,G} = \infty,$$

where we recall the dual of ℓ^p for $p \leqslant 1$ is ℓ^∞. The proof of the non-inclusion follows as above.

Finally, taking $w_0 < w_1$, we again see that $\left(x_m^{j,G}\right)_{(j,G,m)\in\mathbb{W}^d} \notin \ell^\infty(\mathbb{W}^d)$ and the non-inclusion result follows. □

We may now identify the set of weighted Besov spaces containing $L^2(\mathbb{R}^d)$.

Proposition 4.2.3. *Let $p > 1$. We define*

$$E_p := \begin{cases} (-\infty, 0) \times (-\infty, -\frac{d}{p} + \frac{d}{2}), & \text{if } p \in (1,2), \\ (-\infty, 0] \times (-\infty, 0], & \text{if } p = 2, \\ (-\infty, -\frac{d}{2} + \frac{d}{p}) \times (-\infty, 0], & \text{if } p \in (2, \infty). \end{cases} \tag{4.2.12}$$

Then $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ if and only if $(s,w) \in E_p$. When $L^2(\mathbb{R}^d)$ is embedded in $B^p_{s,w}(\mathbb{R}^d)$, the embedding is continuous.

Proof. We recall that $L^2(\mathbb{R}^d) = B^2_{0,0}(\mathbb{R}^d)$. The result then follows by applying Proposition 4.2.2 for the case $p_0 = 2$, $p_1 = p$, $s_0 = 0$, $s_1 = s$, $w_0 = 0$ and $w_1 = w$. □

We now investigate for each $p > 1$ the subset of $(s,w) \in E_p$ such that the canonical embedding operator $\iota\colon L^2(\mathbb{R}^d) \to B^p_{s,w}(\mathbb{R}^d)$ is either p-Radonifying or 0-Radonifying. We begin by defining the p-summing operators.

Definition 4.2.4. *Let F, G be Banach spaces and $p > 0$. A continuous linear operator $T\colon G \to F$ is called* p-summing *if there exists $C > 0$ such that for every finite collection $(g_i)_{i=1}^n \subset G$ we have*

$$\sum_{i=1}^n \|Tg_i\|_F^p \leqslant C^p \sup_{\|y\|_{G^*} \leqslant 1} \sum_{i=1}^n |\langle g_i, y \rangle_G|^p .$$

For each $p > 0$, we have that p-Radonifying operators are p-summing, and 0-Radonifying operators are p-summing for every $p > 0$. Furthermore, for $p > 1$ we have that the classes of p-summing and p-Radonifying operators coincide; see [92, Ch. VI.5]. If F and G are Hilbert spaces, 0-Radonifying operators coincide with Hilbert-Schmidt operators [92, Th. VI.5.2], and thus the class of p-summing operators for all $p > 0$ coincides with the class of Hilbert-Schmidt operators.

We now proceed to study the Radonification properties of embeddings of $L^2(\mathbb{R}^d)$ into weighted Besov spaces.

Theorem 4.2.5. *The embedding $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ for some $p > 1$ is p-Radonifying if and only if*

$$(s, w) \in R_p^{(p)} := (-\infty, -\tfrac{d}{2}) \times (-\infty, -\tfrac{d}{p}). \tag{4.2.13}$$

Proof. The continuous embedding follows as $R_p^{(p)} \subseteq E_p$. Let Ψ be an admissible basis of $B^p_{s,w}(\mathbb{R}^d)$ and $\omega^j_m = \omega^j_m(p, s, w)$ be the weight constants defined in (2.1.2). Note, that

$$\sum_{j,G,m} (\omega^j_m)^p = \sum_{j,G,m} 2^{jp(s-d/p+d/2)}(1 + 2^{-2j} |m|^2)^{pw/2} < \infty \ \Leftrightarrow\ (s, w) \in R_p^{(p)}.$$

This follows from the fact, that the sum over m converges if and only if $w < -\frac{d}{p}$, and in this case is asymptotically $\mathcal{O}(2^{jd})$ as $j \to \infty$ (see the proof of [31, Th. 3]). Thus the total sum converges if and only if $s < -\frac{d}{2}$ and $w < -\frac{d}{p}$.

For any $p > 1$, we obtain for $f_1, \dots, f_n \in L^2(\mathbb{R}^d)$ that

$$\sum_{i=1}^n \|f_i\|^p_{B^p_{s,w}} = \sum_{j,G,m} (\omega^j_m)^p \sum_{i=1}^n \left|\langle f_i, \Psi^{j,G}_m\rangle_{L^2}\right|^p \leqslant \sum_{j,G,m} (\omega^j_m)^p \sup_{\|y\|_{L^2}\leqslant 1} \sum_{i=1}^n |\langle f_i, y\rangle_{L^2}|^p .$$

Thus, the embedding $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ is p-summing if $(s, w) \in R_p^{(p)}$.

Next, let $p \geqslant 2$ and take $f_i = \Psi^{j_i,G_i}_{m_i} \in \Psi$, $i = 1, \dots, n$ to be distinct wavelet basis vectors (i.e. $f_i \neq f_j$ for $i \neq j$). Thus we have $|\langle f_i, y\rangle_{L^2}| \leqslant 1$ for each $i = 1, \dots, n$ and $y \in L^2(\mathbb{R}^d)$ such that $\|y\|_{L^2} \leqslant 1$. In this case we have

$$\sup_{\|y\|_{L^2}\leqslant 1} \sum_{i=1}^n |\langle f_i, y\rangle_{L^2}|^p \leqslant \sup_{\|y\|_{L^2}\leqslant 1} \sum_{j,G,m} \left|\langle \Psi^{j,G}_m, y\rangle_{L^2}\right|^p \leqslant \sup_{\|y\|_{L^2}\leqslant 1} \sum_{j,G,m} \left|\langle \Psi^{j,G}_m, y\rangle_{L^2}\right|^2 = 1.$$

Furthermore, we have

$$\sum_{i=1}^{n} \|f_i\|_{B^p_{s,w}}^p = \sum_{i=1}^{n} (\omega_i)^p,$$

where $\omega_i = \omega_m^j$ when $f_i = \Psi_m^{j,G}$ for some $(j, G, m) \in \mathbb{W}^d$. Thus, if $\sum_{j,G,m} (\omega_m^j)^p = \infty$ then the embedding is not p-summing, and we obtain that the embedding is p-summing if and only if $(s, w) \in R_p^{(p)}$. An application of [92, Th. VI.5.4] completes the proof of the case $p \geqslant 2$.

Necessity of $(s, w) \in R_p^{(p)}$ for $p \in (1, 2)$ follows from Theorem 5.3.1, which gives a counterexample of a cylindrical Lévy process in $L^2(\mathbb{R}^d)$ with p-th weak moments that is not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. for any $(s, w) \notin R_p^{(p)}$. This result, though later in the text, does not rely upon the result that the embedding $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ is not p-Radonifying for $(s, w) \notin R_p^{(p)}$. □

Due to the range of continuous embeddings between the weighted Besov spaces, it is in many cases possible to factorise the embeddings via a Hilbert space which allows for a 0-Radonification result.

Theorem 4.2.6. *The embedding $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ for some $p > 1$ is 0-Radonifying if and only if*

$$(s, w) \in R_p := \begin{cases} (-\infty, -\frac{d}{2}) \times (-\infty, -\frac{d}{p}), & \text{when } p \in (1, 2], \\ (-\infty, -d + \frac{d}{p}) \times (-\infty, -\frac{d}{2}), & \text{when } p \in (2, \infty). \end{cases} \tag{4.2.14}$$

Proof. We show sufficiency by factorising the embedding $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ as follows:

$$L^2(\mathbb{R}^d) \overset{\iota_1}{\hookrightarrow} B^2_{s_1,w_1}(\mathbb{R}^d) \overset{\iota_2}{\hookrightarrow} B^p_{s,w}(\mathbb{R}^d),$$

for some $w_1 < -\frac{d}{2}$ and $s_1 < -\frac{d}{2}$. In this case, Theorem 4.2.5 guarantees that the embedding ι_1 is 2-Radonifying, and thus it is 0-radonifying, since 2-summing operators coincide with 0-Radonifying operators in Hilbert space [92, Th. VI.5.2]. It remains to show that w_1 and s_1 can always be chosen, such that $B^2_{s_1,w_1}(\mathbb{R}^d)$ is continuously embedded into $B^p_{s,w}(\mathbb{R}^d)$, whenever s and w are in the stated ranges.

(i) Let $p \in (1, 2]$. By applying Proposition 4.2.2 we see that ι_2 is a continuous embedding for $w < w_1 - d\left(\frac{1}{p} - \frac{1}{2}\right)$ and $s < s_1$. Thus, by defining $\varepsilon := -w - \frac{d}{p} > 0$, we may take $w_1 = -\frac{\varepsilon}{2} - \frac{d}{2}$ and satisfy all inequalities. A similar argument may be used for s and s_1.

(ii) Let $p > 2$. By Proposition 4.2.2 we see that ι_2 is a continuous embedding for $w < w_1$ and $s < s_1 - d\left(\frac{1}{2} - \frac{1}{p}\right)$. We proceed by similar arguments as above.

The necessity for $p = 2$ follows from Theorem 4.2.5 since 0-Radonifying and p-Radonifying operators between Hilbert spaces coincide. To show the necessity for $p \neq 2$, we shall refer to Theorem 5.3.1 and Proposition 5.4.12, which provide counterexamples of cylindrical Lévy processes in $L^2(\mathbb{R}^d)$ that are not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ for any $(s, w) \notin R_p$. Although these negative results referred to are later in the text, they are not based upon this Theorem. □

Remark 4.2.7. *Comparing the results in Theorem 4.2.5 and Theorem 4.2.6, we see that when* $p \in (1, 2]$*, we have* $R_p = R_p^{(p)}$*. Outside these ranges, we have instead that* $R_p \subsetneq R_p^{(p)}$

We summarise the Radonification regions in Figures 4.1 and 4.2 plotting s and w against $\frac{1}{p}$, which follows naturally from the ranges specified. We refer to diagrams plotting s and w against $\frac{1}{p}$ as Triebel diagrams[1].

[1]Plots of s and w against $\frac{1}{p}$ were described in [9] as 'Triebel diagrams' and used to illustrate various properties of the scales of Besov spaces.

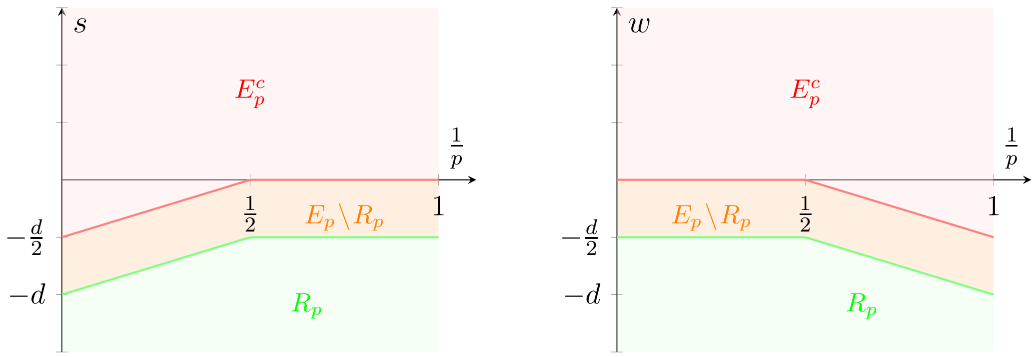

FIGURE 4.1: Triebel diagrams for 0-Radonification

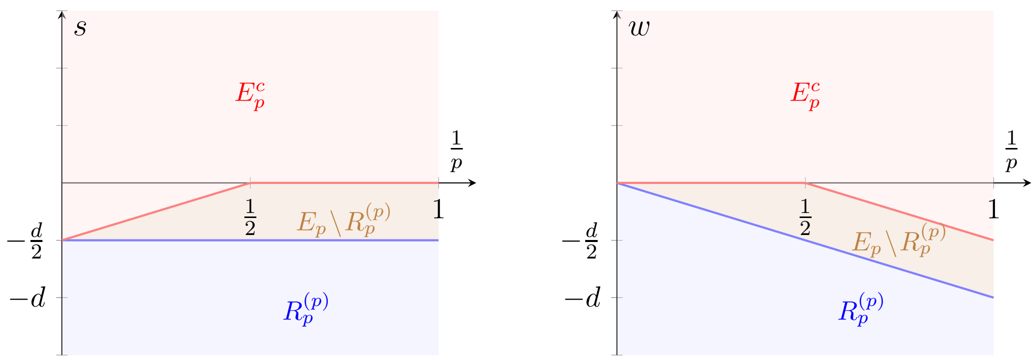

FIGURE 4.2: Triebel diagrams for p-Radonification

Chapter 5

Regularisation of Lévy space-time noises

In this final Chapter we apply the results developed in the prior Chapters to study when embeddings of the Lévy noises, modelled either as Lévy-valued random measures or cylindrical Lévy processes, give rise to genuine Lévy processes in larger spaces.

In Section 5.1 we present our first two main results on the embedding of independently scattered cylindrical Lévy processes in the space of distributions and tempered distributions. In Section 5.2 we give a general characterisation of when L^2-cylindrical Lévy processes may be regularised in specific weighted Besov spaces.

Finally, we study in depth two important classes of cylindrical Lévy process: in Section 5.3 we analyse the canonical symmetric-α-stable process, and Section 5.4 is devoted to the hedgehog process. In both cases we present a full characterisation of the parameter set where the cylindrical process is and is not regularised.

5.1 Lévy-valued random measures

In this section, we embed the Lévy-valued random measure from Definition 3.1.1 into the spaces of distributions and of tempered distributions. By the correspondence detailed in Theorem 3.3.1 and Proposition 3.3.6, this is equivalent to the embeddings of independently scattered cylindrical Lévy processes. These embeddings are based on the integration theory for Lévy-valued random measures as examined in Section 3.3.

We fix an open set $\mathcal{O} \subset \mathbb{R}^d$ and recall the space $\mathcal{D}(\mathcal{O})$ of infinitely differentiable functions with compact support, and the dual space $\mathcal{D}^*(\mathcal{O})$ of distributions, as defined in Chapter 2. As locally integrable functions and measures are identified with distributions, we proceed analogously to relate a Lévy-valued random measure M on $\mathcal{B}_b(\mathcal{O})$ to a distribution-valued process. For this purpose, we define for each $t \geqslant 0$ the integral mapping

$$J_{\mathcal{D}}(t) : \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P)\,, \qquad J_{\mathcal{D}}(t)f = \int_{\mathcal{O}} f(x)\, M(t, \mathrm{d}x). \tag{5.1.1}$$

We recall from Section 3.3 the definition of the integral with respect to M in (3.3.5) and the Musielak-Orlicz space $L_M(\mathcal{O}, \lambda)$ of M-integrable functions, based on the modular Φ_M defined in (3.3.6). In the proof of Theorem 5.1.1 below we show that $\mathcal{D}(\mathcal{O})$ is continuously embedded in $L_M(\mathcal{O}, \lambda)$, and thus the mapping $J_{\mathcal{D}}(t)$ is well-defined.

Theorem 5.1.1. *For a Lévy-valued random measure M on $\mathcal{B}_b(\mathcal{O})$ let $J_{\mathcal{D}}$ be defined by (5.1.1). Then there exists a genuine Lévy process $(Y(t) : t \geqslant 0)$ in $\mathcal{D}^*(\mathcal{O})$ satisfying*

$$J_{\mathcal{D}}(t)f = \langle f, Y(t)\rangle_{\mathcal{D}(\mathcal{O})} \qquad P\text{-a.s. for all } f \in \mathcal{D}(\mathcal{O}),\ t \geqslant 0.$$

Proof. We first show that the space $\mathcal{D}(K)$ is continuously embedded in $L_M(\mathcal{O},\lambda)$ for each compact $K \subseteq \mathcal{O}$. Trivially, the space $\mathcal{D}(K)$ is continuously embedded in $L^\infty(K,\lambda)$. As $K \in \mathcal{B}_b(\mathcal{O})$, the control measure λ is finite on K, and it follows that $L^\infty(K,\lambda)$ is continuously embedded in $L^2(K,\lambda)$. The latter is continuously embedded in $L_M(K,\lambda)$ by Lemma 3.3.3. Because whenever $\operatorname{supp}(f) \subseteq K$ we have

$$\int_{\mathcal{O}} \Phi_M(|f(x)|, x)\,\lambda(\mathrm{d}x) = \int_K \Phi_M(|f(x)|, x)\,\lambda(\mathrm{d}x),$$

it follows that $\mathcal{D}(K)$ is continuously embedded in $L_M(\mathcal{O},\lambda)$. As $\mathcal{D}(\mathcal{O})$ is the inductive limit of $\{\mathcal{D}(K_i)\}_{i\in\mathbb{N}}$, we thus conclude that $\mathcal{D}(\mathcal{O})$ is continuously embedded in $L_M(\mathcal{O},\lambda)$.

Let $\iota\colon \mathcal{D}(\mathcal{O}) \to L_M(\mathcal{O},\lambda)$ be the continuous embedding. We recall the continuous integral mapping $J\colon L_M(\mathcal{O},\lambda) \to L^0(\Omega,P)$ defined in (3.3.8). Then the mapping $J_{\mathcal{D}}(t)\colon \mathcal{D}(\mathcal{O}) \to L^0(\Omega,P)$ can be represented as $J_{\mathcal{D}}(t) = J(t)\circ\iota$ for each $t \geqslant 0$, showing that $J_{\mathcal{D}}(t)$ is continuous. Lemma 3.3.2 shows that $J_{\mathcal{D}}$ is a cylindrical Lévy process in $\mathcal{D}^*(\mathcal{O})$. Furthermore, since $J_{\mathcal{D}}(t)$ is continuous, and $\mathcal{D}(\mathcal{O})$ is nuclear [87, Th. 51.5] and ultrabornological [64, p.447], Theorem 3.8 in [35] implies the existence of the $\mathcal{D}^*(\mathcal{O})$-valued Lévy process Y. □

In the second part of this section, we embed the Lévy-valued random measure into the space of tempered distribution $\mathcal{S}^*(\mathbb{R}^d)$, again recalling the definition from Chapter 2. We define for each $t \geqslant 0$ the integral mapping

$$J_{\mathcal{S}}(t) : \mathcal{S}(\mathbb{R}^d) \to L^0(\Omega,P)\,, \qquad J_{\mathcal{S}}(t)f = \int_{\mathbb{R}^d} f(x)\,M(t,\mathrm{d}x). \tag{5.1.2}$$

Clearly, the mapping $J_{\mathcal{S}}(t)$ is only well defined if $\mathcal{S}(\mathbb{R}^d)$ is embedded in $L_M(\mathbb{R}^d,\lambda)$. The following theorem gives an equivalent condition for this.

Theorem 5.1.2. *Let M be a Lévy-valued random measure on $\mathcal{B}_b(\mathbb{R}^d)$ with control measure λ. Then the following are equivalent:*

(a) *$\mathcal{S}(\mathbb{R}^d)$ is continuously embedded in $L_M(\mathbb{R}^d, \lambda)$;*

(b) *there exists an $r > 0$ such that the function $x \mapsto (1 + |x|^2)^{-r}$ is in $L_M(\mathbb{R}^d, \lambda)$.*

In this case, the mapping $J_{\mathcal{S}}(t)$ as specified in (5.1.2) is well-defined and continuous for each $t \geqslant 0$. Furthermore, there exists a genuine Lévy process $(Y(t) : t \geqslant 0)$ in $\mathcal{S}^*(\mathbb{R}^d)$ satisfying

$$J_{\mathcal{S}}(t)f = \langle f, Y(t) \rangle_{\mathcal{S}(\mathbb{R}^d)} \qquad P\text{-a.s. for all } f \in \mathcal{S}(\mathbb{R}^d),\ t \geqslant 0.$$

Proof. We begin by showing the implication (b) $\Rightarrow$ (a), for which we suppose there exists $r > 0$ such that $x \mapsto \left(1 + |x|^2\right)^{-r}$ is in $L_M(\mathbb{R}^d, \lambda)$. For each $f \in \mathcal{S}(\mathbb{R}^d)$ there exists $K > 0$ such that $(1 + |x|^2)^r |f(x)| \leqslant K$ for all $x \in \mathbb{R}^d$. Since $\Phi_M(\cdot, x)$ is monotone for each $x \in \mathbb{R}^d$ according to [70, Le. 3.1], we have

$$\Phi_M(|f(x)|, x) \leqslant \Phi_M(K(1 + |x|^2)^{-r}, x) \qquad \text{for each } x \in \mathbb{R}^d,$$

which implies $f \in L_M(\mathbb{R}^d, \lambda)$.

Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^d)$ be a sequence converging to 0 in $\mathcal{S}(\mathbb{R}^d)$. As the convergence is uniform in x, we have the existence of another $K > 0$ such that $(1+|x|^2)^r |f_n(x)| \leqslant K$ for all $x \in \mathbb{R}^d$ and for all $n \in \mathbb{N}$. For fixed $x \in \mathbb{R}^d$ we have $\Phi_M(|f_n(x)|, x) \to \Phi_M(0, x) = 0$ by continuity [70, Le. 3.1], and as $\int_{\mathbb{R}^d} \Phi_M(K(1+|x|^2)^{-r}, x)\, \lambda(\mathrm{d}x) < \infty$, Lebesgue's theorem for dominated convergence implies

$$\int_{\mathbb{R}^d} \Phi_M(|f_n(x)|, x)\, \lambda(\mathrm{d}x) \to 0 \qquad \text{as } n \to \infty,$$

which completes the proof of the implication (b) $\Rightarrow$ (a).

Conversely, suppose $\mathcal{S}(\mathbb{R}^d)$ is continuously embedded in $L_M(\mathbb{R}^d, \lambda)$. Thus, the identity mapping $\iota\colon \mathcal{S}(\mathbb{R}^d) \to L_M(\mathbb{R}^d, \lambda)$ is continuous. Then, there exists a neighbourhood

$$U(0; k, \delta) := \left\{ f \in \mathcal{S}(\mathbb{R}^d) : \|f\|_{\mathcal{S}_k} < \delta \right\},$$

for some $k \in \mathbb{N}$ and $\delta > 0$ such that ι maps $U(0; k, \delta)$ into the open unit ball of $L_M(\mathbb{R}^d, \lambda)$. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^d)$ be any sequence such that $\|f_n\|_{\mathcal{S}_k} \to 0$. Then, (f_n) is eventually in $U(0; k, \delta)$ and thus (ιf_n) is eventually in the unit ball and so is bounded in $L_M(\mathbb{R}^d, \lambda)$. By Proposition 4 of [44, p. 41] we have the continuity of ι in the semi-norm $\|\cdot\|_{\mathcal{S}_k}$, and thus we may extend ι by continuity to the completion of $\mathcal{S}(\mathbb{R}^d)$ in this semi-norm. We thus obtain the integrability condition by observing that the $C^\infty(\mathbb{R}^d)$ mapping $x \mapsto (1 + |x|^2)^r$ has finite semi-norm $\|\cdot\|_{\mathcal{S}_k}$ for $r \leqslant -k$.

As in the proof of Theorem 5.1.1, an application of Lemma 3.3.2 and Theorem 3.8 in [35] establishes the existence of the Lévy process Y in $\mathcal{S}(\mathbb{R}^d)$. □

Remark 5.1.3. *In Kabanava [45], it is shown that a Radon measure ζ can be identified with a tempered distribution in $\mathcal{S}^*(\mathbb{R}^d)$ if and only if there is a real number r such that $x \mapsto (1 + |x|^2)^r$ is integrable over $\mathbb{R}^d$ with respect to ζ. Our condition for the mapping $J_{\mathcal{S}}$ in Theorem 5.1.2 is analogous.*

Remark 5.1.4. *By Proposition 3.1.2, we may also view the Lévy-valued random measure M as an infinitely divisible random measure M' on $\mathcal{B}_b(\mathbb{R}_+ \times \mathcal{O})$, and define the integral mapping*

$$J'_{\mathcal{D}}\colon \mathcal{D}((0,\infty) \times \mathcal{O}) \to L^0(\Omega, P), \qquad J'_{\mathcal{D}} f = \int_{(0,\infty)\times\mathcal{O}} f(t,x)\, M'(\mathrm{d}t, \mathrm{d}x).$$

Analogously to Theorem 5.1.1 we obtain an infinitely divisible random variable Y' in $\mathcal{D}^*((0,\infty) \times \mathcal{O})$ satisfying $\langle f, Y' \rangle = J'_{\mathcal{D}} f$ for all $f \in \mathcal{D}((0,\infty) \times \mathcal{O})$. Similarly, *under the conditions of Theorem 5.1.2, we may consider the operator J' on the space $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$.*

5.1.1 Weak derivative of a Lévy-valued random measure

As an immediate application of the embedding, we establish the relation between a Lévy-valued random measure and a Lévy-valued additive sheet. For this purpose, we introduce a stochastic integral of deterministic functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with respect to a Lévy-valued additive sheet. Instead of following the standard approach starting with simple functions and extending the integral operator by continuity, we utilise the correspondence between Lévy-valued additive sheets and Lévy valued random measures, established in Theorem 3.2.4, and refer to the integration for the latter developed in Rajput and Rosinksi [70] as presented in Section 5.1. For a Lévy-valued additive sheet $(X(t,x) : t \geqslant 0, x \in \mathbb{R}^d)$ let M denote the corresponding Lévy-valued random measure on $\mathcal{B}_b(\mathbb{R}^d)$ with control measure λ. Then we define for all $f \in L_M(\mathbb{R}^d, \lambda)$, $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geqslant 0$:

$$\int_A f(x)\, \mathrm{d}X(t,x) := \int_A f(x)\, M(t, \mathrm{d}x). \tag{5.1.3}$$

Let $(X(t,x) : x \in \mathbb{R}^d, t \geqslant 0)$ be a Lévy-valued additive sheet and $\mathcal{O} \subseteq \mathbb{R}^d$ be open. Then the definition in (5.1.3) allows us to define the same operator $J_{\mathcal{D}}(t)$ as introduced in (5.1.1) for a Lévy-valued additive sheet X:

$$J_{\mathcal{D}}(t)\colon \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P), \quad J_{\mathcal{D}}(t)f = \int_{\mathcal{O}} f(x)\, \mathrm{d}X(t,x). \tag{5.1.4}$$

Theorem 5.1.1 guarantees that $J_{\mathcal{D}}$ is well-defined and even more, induces a genuine Lévy process Y in $\mathcal{D}^*(\mathcal{O})$. We define the operator

$$I_{\mathcal{D}}(t)\colon \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P), \qquad I_{\mathcal{D}}(t)(f) = \int_{\mathcal{O}} f(x)X(t,x)\,\mathrm{d}x\,. \tag{5.1.5}$$

The mapping $I_{\mathcal{D}}$ is well defined because of the lamp property of $X(t,\cdot)$ for each $t \geqslant 0$ and as each $f \in \mathcal{D}(\mathcal{O})$ has compact support in $\mathcal{O}$. Lebesgue's dominated convergence theorem shows that $I_{\mathcal{D}}$ is continuous, as every convergent sequence in $\mathcal{D}(\mathcal{O})$ is bounded and uniformly compactly supported.

The following result establishes the relation that, if we neglect the embedding by the operators $I_{\mathcal{D}}$ and $J_{\mathcal{D}}$, we have that M may be viewed as the weak derivative of X. This is in accordance with classical measure theory; if we adapt notions, the relation $M(t,(0,x]) = X(t,x)$ derived in Theorem 3.2.4 gives that X is the cumulative distribution function of the random measure M.

Theorem 5.1.5. *For a Lévy-valued additive sheet $(X(t,\cdot) : t \geqslant 0)$ and an open set $\mathcal{O} \subseteq \mathbb{R}^d$ let $I_{\mathcal{D}}$ be defined by (5.1.5). Then there exists a stochastic process $(V(t) : t \geqslant 0)$ in $\mathcal{D}^*(\mathcal{O})$ satisfying*

$$I_{\mathcal{D}}(t)f = \langle f, V(t)\rangle_{\mathcal{D}(\mathcal{O})} \qquad P\text{-a.s. for all } f \in \mathcal{D}(\mathcal{O}),\ t \geqslant 0.$$

Furthermore, we have the equality

$$(-1)^d I_{\mathcal{D}}(t)(\dot{f}) = J_{\mathcal{D}}(t)(f) \qquad P\text{-a.s. for all } f \in \mathcal{D}(\mathcal{O}), \tag{5.1.6}$$

where $\dot{f} = \frac{\partial^d}{\partial x_1 \cdots \partial x_d} f$ and $J_{\mathcal{D}}(t)$ denotes the operator in (5.1.4).

Proof. We show that, for each $f \in \mathcal{D}(\mathcal{O})$, the process $(I_{\mathcal{D}}(t)f : t \geqslant 0)$ has a càdlàg modification. First we consider a sequence $(t_n)_{n\in\mathbb{N}}$ decreasing monotonically to some $t \geqslant 0$. Let K be the support of f. Then, as $(t_n)_{n\in\mathbb{N}}$ is bounded, there exists a $C > 0$

such that $t_n \in [t, t+C]$ for each n. The lamp property of X implies that X is bounded on the compact set $[t, t+C] \times K$. Thus, since $X(t_n, x)$ converges to $X(t, x)$ in probability for each $x \in \mathcal{O}$, Lebesgue's dominated convergence theorem (for a stochastically convergent sequence) implies

$$P - \lim_{n\to\infty} \int_{\mathcal{O}} f(x) X(t_n, x)\, \mathrm{d}x = \int_{\mathcal{O}} f(x) X(t, x)\, \mathrm{d}x.$$

A similar argument establishes that the left limits exists.

The existence of the stochastic process V follows from Theorem 3.2 in [34] (as $\mathcal{D}(\mathcal{O})$ is nuclear [87, Th. 51.5] and ultrabornological [64, p.447]).

To show (5.1.6) we use ideas from [25]. By the fundamental theorem of calculus, as f has compact support,

$$f(x) = (-1)^d \int_{\mathcal{O}} \dot{f}(y) \mathbb{1}_{\{y \geqslant x\}}\, \mathrm{d}y \qquad \text{for all } x \in \mathcal{O}.$$

(We recall that for $a, b \in \mathbb{R}^d$ we write $a \geqslant b$ if $a_j \geqslant b_j$ for all $j = 1, \ldots, d$.) By utilising an analogue of Fubini's theorem for Lévy-valued random measures, as detailed below, we obtain

$$\begin{aligned} J_{\mathcal{D}}(t) f = \int_{\mathcal{O}} f(x)\, X(t, \mathrm{d}x) &= \int_{\mathcal{O}} \left((-1)^d \int_{\mathcal{O}} \dot{f}(y) \mathbb{1}_{\{y \geqslant x\}}\, \mathrm{d}y \right) X(t, \mathrm{d}x) \\ &= (-1)^d \int_{\mathcal{O}} \left(\int_{\mathcal{O}} \mathbb{1}_{\{y \geqslant x\}}\, X(t, \mathrm{d}x) \right) \dot{f}(y)\, \mathrm{d}y \\ &= (-1)^d \int_{\mathcal{O}} X(t, y) \dot{f}(y)\, \mathrm{d}y\,. \end{aligned}$$

We now show the analogue of Fubini's theorem to complete the proof. Let M denote the Lévy-valued random measure corresponding to X according to Theorem

3.2.4. The Lévy-Itô decomposition (3.2.2) yields that M admits the decomposition

$$M(t,A) = t\gamma(A) + G(t,A) + M_c(t,A) + M_p(t,A) \quad \text{for all } A \in \mathcal{B}_b(\mathbb{R}^d) \text{ and } t \geqslant 0.$$

Here, γ is a signed measure, G is a pure Gaussian Lévy-valued random measure with characteristics $(0, \Sigma, 0)$, and

$$M_c(t,A) := \int_0^t \int_{A\times B_{\mathbb{R}}} y\, \widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y), \qquad M_p(t,A) := \int_0^t \int_{A\times B_{\mathbb{R}}^c} y\, N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y).$$

The classic Fubini theorem may be applied to γ. The Lévy-valued random measure M_p is a finite random sum and the Fubini result holds trivially.

For G and the compensated Poisson Lévy-valued random measure M_c we apply Theorem 2.6 in [94]. We note that $(G(t,\cdot) + M_c(t,\cdot) : t \geqslant 0)$ forms a martingale-valued measure. Furthermore, $G + M_c$ meets the orthogonality condition of the cited Theorem by the pairwise independence of the processes $(G(t,A) : t \geqslant 0)$, $(M_c(t,A) : t \geqslant 0)$, $(G(t,B) : t \geqslant 0)$ and $(M_c(t,B) : t \geqslant 0)$ whenever $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ are disjoint. The covariance process is given by $Q_t(A,B) = t\big(\Sigma(A\cap B) + \int_{(A\cap B)\times B_{\mathbb{R}}} |y|^2\, \nu(\mathrm{d}x, \mathrm{d}y)\big)$. As the required dominating measure K, one can choose $K(A\times B\times (0,t]) = t\lambda(A\cap B)$. The required integrability condition follows as f is compactly supported and bounded. □

Remark 5.1.6. *According to Proposition 3.2.3, a Lévy-valued additive sheet X defines a natural additive sheet $(X(t,x) : t \geqslant 0,\, x \in \mathbb{R}^d)$. Due to its lamp trajectories, we can define the mapping*

$$I'_{\mathcal{D}} \colon \mathcal{D}((0,\infty)\times\mathbb{R}^d) \to L^0(\Omega, P), \qquad I'_{\mathcal{D}}(f) = \int_{(0,\infty)\times\mathbb{R}^d} f(t,x) X(t,x)\, \mathrm{d}t\, \mathrm{d}x.$$

On the other side, one can conclude as in Theorem 3.2.4 or by [66, Th. 4.1], that there exists an infinitely divisible random measure M' on $\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$ satisfying

$M((0,z]) = X(z)$ for all $z \in \mathbb{R}_+ \times \mathbb{R}^d$. Thus, as in Remark 5.1.4, we can define

$$J'_{\mathcal{D}}\colon \mathcal{D}((0,\infty)\times\mathbb{R}^d) \to L^0(\Omega,P), \quad J'_{\mathcal{D}}(g) = \int_{(0,\infty)}\int_{\mathbb{R}^d} g(t,x)\, M'(\mathrm{d}t,\mathrm{d}x).$$

One can conclude as in the proof of Theorem 5.1.5 that there exists a genuine random variable W in $\mathcal{D}^*((0,\infty)\times\mathbb{R}^d)$ satisfying

$$\langle g, W\rangle = I'_{\mathcal{D}}(g) \qquad \text{for all } g \in \mathcal{D}((0,\infty)\times\mathbb{R}^d).$$

Furthermore, we have the equality

$$(-1)^d I'_{\mathcal{D}}(\dot{g}) = J'_{\mathcal{D}}(g) \qquad \text{for all } g \in \mathcal{D}((0,\infty)\times\mathbb{R}^d).$$

5.1.2 Besov regularity of stationary Lévy white noise

Let M be a Lévy-valued random measure on $\mathcal{B}_b(\mathbb{R}^d)$ with characteristics (γ,Σ,ν) and $J_{\mathcal{D}}(t)$ the corresponding operator defined in (5.1.1) for $t \geqslant 0$. By comparing the Lévy symbol in (3.3.9) with (1.1.1) it follows that, for fixed $t \geqslant 0$, the mapping $J_D(t)$ is a Lévy white noise as defined in [25], if and only if

$$\gamma = p\cdot \mathrm{leb}, \qquad \Sigma = \sigma^2\cdot\mathrm{leb}, \qquad \nu = \mathrm{leb}\otimes\nu_0,$$

for some $p \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$ and a Lévy measure ν_0 on $\mathbb{R}$. It follows that $M(t,A) \overset{\mathscr{D}}{=} M(t,B)$ for any sets $A,B \in \mathcal{B}_b(\mathbb{R}^d)$ with $\mathrm{leb}(A) = \mathrm{leb}(B)$. In this case, we call M *stationary in space.*

Dalang and Humeau have shown in [25] that a Lévy white noise in $\mathcal{D}^*(\mathbb{R}^d)$ with Lévy symbol (1.1.1) takes values in $\mathcal{S}^*(\mathbb{R}^d)$ P-a.s. if and only if

$$\int_{B_{\mathbb{R}}^c} |y|^{\varepsilon}\, \nu_0(\mathrm{d}y) < \infty \quad \text{for some } \varepsilon > 0\,.$$

This result is seemingly different to our Theorem 5.1.2. However, as Lévy-valued random measures are not necessarily stationary in space, our condition is more complex. For example, even in the pure Gaussian case with characteristics $(0, \Sigma, 0)$, the measure Σ must be tempered; cf. Remark 5.1.3. For the case of a Lévy-valued random measure which is stationary in space, these results are in fact equivalent, as shown in the following result. For simplicity, we consider the symmetric non-Gaussian case.

Theorem 5.1.7. *Let $M = \{M(t, A) : t \geqslant 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ be a Lévy-valued random measure with characteristics of the form $(0, 0, \mathrm{leb} \otimes \nu)$ such that ν is symmetric. Then $\mathcal{S}(\mathbb{R}^d) \subseteq L_M(\mathbb{R}^d)$, so that every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is M-integrable if and only if there exists $\varepsilon > 0$ such that*

$$\int_{B_{\mathbb{R}}^c} |y|^{\varepsilon}\, \nu(\mathrm{d}y) < \infty.$$

Proof. First, we assume that $\mathcal{S}(\mathbb{R}^d) \subseteq L_M(\mathbb{R}^d)$. By Theorem 5.1.2 we have the existence of $r > 0$ such that $x \mapsto (1 + |x|^2)^{-r} \in L_M(\mathbb{R}^d, \lambda)$. Thus, by [70, Th. 2.7] we have

$$\int_{\mathbb{R}^d \times \mathbb{R}} (1 \wedge (1 + |x|^2)^{-2r}\, |y|^2)\, \mathrm{d}x\nu(\mathrm{d}y) < \infty,$$

which implies

$$\int_{\mathbb{R}^d} \int_{(1+|x|^2)^r<|y|} \nu(\mathrm{d}y)\,\mathrm{d}x < \infty,$$

and therefore by Fubini's Theorem we have

$$\int_{\mathbb{R}} R(y)\,\nu(\mathrm{d}y) < \infty,$$

where $R(y) := \mathrm{leb}(\{x : (1+|x|^2)^r < |y|\})$ for $y \neq 0$ and $R(0) := 0$. This leads finally to the result that

$$\int_{B_{\mathbb{R}}^c} (|y|^{1/r} - 1)^{1/2}\,\nu(\mathrm{d}y) < \infty,$$

which indicates moments of order $\frac{1}{2r}$ for M.

To show the converse, suppose that $\int_{B_{\mathbb{R}}^c} |y|^{\varepsilon}\,\nu(\mathrm{d}y) < \infty$. Take $r = \frac{1}{2\varepsilon}$. Then clearly we have

$$\int_{B_{\mathbb{R}}^c} (|y|^{1/r} - 1)^{1/2}\,\nu(\mathrm{d}y) = \int_{\mathbb{R}} R(y)\,\nu(\mathrm{d}y) = \int_{\mathbb{R}^d} \int_{(1+|x|^2)^r<|y|} \nu(\mathrm{d}y)\,\mathrm{d}x < \infty.$$

Thus, by properties of Lévy measures (and further assuming that $r > \frac{d}{2}$) we see that

$$\int_{\mathbb{R}^d\times\mathbb{R}} (1 \wedge (1+|x|^2)^{-2r}\,|y|^2)\,\mathrm{d}x\nu(\mathrm{d}y) < \infty,$$

which in the symmetric case is sufficient to apply [70, Th. 2.7] to show $x \mapsto (1+|x|^2)^{-r} \in L_M(\mathbb{R}^d, \lambda)$. An application of Theorem 5.1.2 completes the proof. □

Regularity of (independently scattered) Lévy white noises in terms of Besov spaces is studied in [9]. These results can be applied to a Lévy-valued random measure if it is additionally assumed to be stationary in space, i.e. which can be considered as a Lévy white noise in the above sense. We illustrate such an application in the following example.

Let M be the α-stable random measure, $\alpha \in (0,2)$, described in Example 3.1.5. For simplicity we consider the symmetric case, i.e. $p = q = \frac{1}{2}$. As the characteristics of M is given by $(0, 0, \mathrm{leb} \otimes \nu_\alpha)$, it follows that M is stationary in space. Thus, for a fixed time $t \geqslant 0$, the mapping $J_{\mathcal{D}}(t)$ or, equivalently the random variable $Y(t)$, where Y denotes the Lévy process derived in Theorem 5.1.1, can be considered as a Lévy white noise in $\mathcal{D}^*(\mathbb{R}^d)$. Furthermore, since $\int_{\mathbb{R}} \left(|y|^\varepsilon \wedge |y|^2 \right) \nu_\alpha(\mathrm{d}y) < \infty$ for $\varepsilon < \alpha$, we have that $Y(t)$ is in $\mathcal{S}^*(\mathbb{R}^d)$ P-a.s. By applying the results from [9] we obtain the following: for $p \in (0,2) \cup 2\,\mathbb{N} \cup \{\infty\}$ and for all $t \geqslant 0$, we have, P-a.s.:

$$\text{if } s < d\left(\frac{1}{p \vee \alpha} - 1\right) \text{ and } w < -\frac{d}{p \wedge \alpha} \text{ then } Y(t) \in B^p_{s,w}(\mathbb{R}^d), \tag{5.1.7}$$

$$\text{if } s > d\left(\frac{1}{p \vee \alpha} - 1\right) \text{ or } w > -\frac{d}{p \wedge \alpha} \text{ then } Y(t) \notin B^p_{s,w}(\mathbb{R}^d). \tag{5.1.8}$$

Furthermore, a modification of Y is a Lévy process in any Besov space satisfying (5.1.7), since its characteristic function is continuous in 0, guaranteeing stochastic continuity.

5.2 Cylindrical Lévy processes in L^2

We analyse cylindrical Lévy processes in $L^2(\mathbb{R}^d)$, which we shall identify with its dual; we first embed these processes in the space of tempered distributions, and then determine when they arise from a Lévy process in a (larger) Besov space.

Definition 5.2.1. *Let H be a Hilbert space continuously and densely embedded in a topological vector space V. A cylindrical Lévy process L in H is said to be induced* *by a Lévy process Y in V if*

$$L(t)f = \langle Y(t), f\rangle_V \quad P\text{-a.s. for all } f \in V^* \text{ and } t \geqslant 0.$$

5.2.1 Regularisation in $\mathcal{S}^*$

We now examine cylindrical Lévy processes in $L^2(\mathbb{R}^d)$. As $\mathcal{S}^*(\mathbb{R}^d)$ is the dual of a nuclear space, we are able to give the following universal result for the embedding.

Theorem 5.2.2. *Every cylindrical Lévy process L in $L^2(\mathbb{R}^d)$ is induced by a Lévy process Y in $\mathcal{S}^*(\mathbb{R}^d)$ and Y is unique up to indistinguishability.*

Proof. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is embedded continuously and densely in $L^2(\mathbb{R}^d)$; indeed, the dense inclusion follows from [33, Th. 8.17], and continuity can be seen by, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\|\varphi\|_{L^2}^2 = \int_{\mathbb{R}^d} (1+|x|^2)^{-2d}(1+|x|^2)^{2d} |\varphi(x)|^2 \, \mathrm{d}x \le \|\varphi\|_{\mathcal{S}_d}^2 \int_{\mathbb{R}^d} (1+|x|^2)^{-2d} \, \mathrm{d}x < \infty.$$

Let ι be the canonical embedding of $L^2(\mathbb{R}^d)$ in $\mathcal{S}^*(\mathbb{R}^d)$ and define

$$\widetilde{L}(t)\colon \mathcal{S}(\mathbb{R}^d) \to L^0(\Omega), \qquad \widetilde{L}(t)\varphi = L(t)(\iota^*\varphi). \tag{5.2.9}$$

It follows that $\widetilde{L} := (\widetilde{L}(t) : t \ge 0)$ is a cylindrical Lévy process in $\mathcal{S}^*(\mathbb{R}^d)$. Since $\mathcal{S}(\mathbb{R}^d)$ is a barrelled nuclear space, the family $(\widetilde{L}(t)\colon t \in [0,T])$ of linear maps from $\mathcal{S}(\mathbb{R}^d)$ into $L^0(\Omega)$ is equicontinuous for every $T > 0$; see the argument in [35, Cor. 3.11]. Thus, Theorem 3.8 in [35] enables us to conclude that there exists a $\mathcal{S}^*(\mathbb{R}^d)$-valued, càdlàg Lévy process $Y = (Y(t))_{t\ge 0}$ such that Y induces $\widetilde{L}$ and Y is unique up to indistinguishability. ∎

5.2.2 Regularisation in weighted Besov spaces

The literature dealing with the Besov localisation of Gaussian and Lévy processes is primarily concerned with the path properties of finite-dimensional processes, or analogously with white noise, conceived as a distribution-valued generalised random variable which represents the weak derivative in space and time of a finite-dimensional process. In this work we generalise from a distribution-valued random variable to a distribution-valued process by considering the time variable as separate; in this manner we are primarily assessing the spatial regularity of the process.

We shall now examine conditions on p, s and w (depending on L) such that L is induced by a process Y taking values in $B^p_{s,w}(\mathbb{R}^d)$. With reference to Definition 5.2.1, we shall only consider Besov spaces which contain $L^2(\mathbb{R}^d)$. This will allow us to develop the mathematical theory without the complication arising in the case that the cylindrical Lévy process L may have a non-trivial kernel and thus may be induced by a process in a Besov space which does not contain the whole of $L^2(\mathbb{R}^d)$.

Cylindrical Brownian motions

Let $W = (W(t) \colon t \geq 0)$ be a standard cylindrical Brownian motion in $L^2(\mathbb{R}^d)$. We may use the correspondence identified in Chapter 3 between independently scattered cylindrical Lévy processes and distribution-valued white noises to state a result for W. We recall the characteristic function of $W(1)$ is given by

$$f \mapsto \exp\left(-\frac{1}{2}\|f\|^2_{L^2}\right) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d} |f(x)|^2 \, \mathrm{d}x\right).$$

This means that we may apply Proposition 3.4 in [9] and conclude that, given $p > 1$ and $s, w \in \mathbb{R}$, W is induced by a Brownian motion in $B^p_{s,w}(\mathbb{R}^d)$ if and only if

$$s < -\frac{d}{2} \qquad \text{and} \qquad w < -\frac{d}{p}.$$

For a general cylindrical Brownian motion $W = (W(t) \colon t \geq 0)$ in $L^2(\mathbb{R}^d)$, let the covariance operator of W be $Q \in \mathcal{L}\big(L^2(\mathbb{R}^d)\big)$, with reproducing kernel Hilbert space H_Q. Suppose $p > 1$ and $w, s \in \mathbb{R}$ are such that $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$, then W is induced by a $B^p_{s,w}(\mathbb{R}^d)$-valued Brownian motion if and only if the injection $H_Q \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ is γ-Radonifying in the sense of Chapter 9 in [41].

Non-Gaussian cylindrical Lévy processes

The cylindrical Lévy process $L = (L(t) : t \geqslant 0)$ on $L^2(\mathbb{R}^d)$ defines by $L'(t)(b^*) := L(t)(\iota^* b^*)$ a cylindrical Lévy process $L' = (L'(t) : t \geqslant 0)$ on $B^p_{s,w}(\mathbb{R}^d)$ where $\iota : L^2(\mathbb{R}^d) \to B^p_{s,w}(\mathbb{R}^d)$ is the canonical embedding for $(s, w) \in E_p$ and $p > 1$.

Let μ be the cylindrical Lévy measure of the the cylindrical Lévy process L on $L^2(\mathbb{R}^d)$; see [73, Th. 3.4]. The cylindrical Lévy measure $\tilde{\mu}$ of L' is given by $\mu \circ \iota^{-1}$. We shall study the case when $\tilde{\mu}$ extends to a Radon measure on $B^p_{s,w}(\mathbb{R}^d)$. In this case, we will mildly abuse notation and simply refer to this Radon extension as μ where the context allows no confusion. The starting point for our analysis shall be to examine this Radon extension using the results previously developed.

The following result demonstrates that, in the non-Gaussian case, regularisation of the cylindrical Lévy process results from the extension of the cylindrical Lévy measure, and furthermore allows us to concentrate in the sequel on symmetric cylindrical Lévy processes.

Theorem 5.2.3. *Let L be a cylindrical Lévy process in $L^2(\mathbb{R}^d)$ with no Guassian component and let $p > 1$ and $(s, w) \in E_p$. Let $L^S := L - L_c$ be the symmetrisation of L, where L_c is an independent copy of L. Then L is induced by a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ if and only if the cylindrical Lévy measure of L^S has a Radon extension which is a Lévy measure on $B^p_{s,w}(\mathbb{R}^d)$.*

Proof. The 'only if' implication is clear. To establish the converse, denoting the cylindrical Lévy measure of L by μ, we have that L^S has cylindrical Lévy measure

$\mu + \mu^-$. Let $\tilde{\mu}_S$ denote the Radon extension of $\mu + \mu^-$ on $B^p_{s,w}(\mathbb{R}^d)$; by assumption $\tilde{\mu}_S$ is a Lévy measure. As we have $\mu \leqslant \tilde{\mu}_S$ on each cylinder set of $B^p_{s,w}(\mathbb{R}^d)$, by [74, Th. 3.4] we have that μ has a Radon extension on $B^p_{s,w}(\mathbb{R}^d)$ which is a Lévy measure.

We now fix $t > 0$ and apply Theorem 5.6 in [74] to the function $f\colon [0,t] \to \mathcal{L}(L^2(\mathbb{R}^d), B^p_{s,w}(\mathbb{R}^d))$ given by $f(u) = \iota$ for all $u \in [0,t]$, and we conclude that f is stochastically integrable with respect to L. (This Theorem actually requires that the function $a\colon L^2(\mathbb{R}^d) \to \mathbb{R}$ as defined in Equation 3.1 of [74] is weak*-weakly sequentially continuous, however a careful analysis of the proof indicates that in a reflexive Banach space such as $B^p_{s,w}(\mathbb{R}^d)$, this requirement is not necessary.)

We then see that the process

$$Y = \left(Y(t) := \int_{[0,t]} f(u)\, \mathrm{d}L(u) \colon t \geqslant 0 \right)$$

forms a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ which induces L. □

In the sequel we shall make use of the following technique. It is possible to express the tests in the hypothesis of Corollary 5.2.5 as limits of finite-dimensional projections. As the sums defining membership of a given Besov space are required to be unconditionally convergent, we may take any convenient ordering of the terms. For any enumeration of the countable set of indices j, G and m, we denote a sum over the first n terms in this enumeration by $\sum^n_{j,G,m}$, and Ψ_k refers to the wavelet $\Psi^{j_k,G_k}_{m_k}$ corresponding to the k-th term in this enumeration. We define for each $n \in \mathbb{N}$ the projection $P_n \in \mathcal{L}(B^p_{s,w}(\mathbb{R}^d))$ onto the subspace spanned by the first n elements in the enumeration of Ψ, that is

$$P_n f := \sum_{j,G,m}^{n} [\Psi^{j,G}_m, f] \Psi^{j,G}_m, \qquad f \in B^p_{s,w}(\mathbb{R}^d). \tag{5.2.10}$$

Theorem 5.2.4. *Let μ be a cylindrical Lévy measure on $B^p_{s,w}(\mathbb{R}^d))$ for some $p > 1$ and $(s,w) \in E_p$. Then μ extends to a σ-finite measure on $B^p_{s,w}(\mathbb{R}^d)$ if*

$$\lim_{R\to\infty} \lim_{n\to\infty} \mu\big(\{f : \|P_n f\|_{B^p_{s,w}} > R\}\big) = 0.$$

Proof. We shall apply the Theorem on [38, p.327]. This Theorem gives conditions for when a cylindrical probability measure extends to a Radon probability measure on a Banach space with a separable dual. In order to apply this Theorem in our setting, a careful study of the proof indicates that the Theorem may be applied to any finite cylindrical measure satisfying the continuity condition

$$\lim_{k\to\infty} \mu\big(\{f\colon [f, f^*_{1,k}] < a_1, \dots, [f, f^*_{m,k}] < a_m\}\big) = \mu\big(\{f\colon [f, f^*_1] < a_1, \dots, [f, f^*_m] < a_m\}\big) \tag{5.2.11}$$

for μ-almost all $a_1, \dots, a_m$ whenever $\left\|f^*_{i,k} - f^*_i\right\|_{B^{p'}_{-s,-w}} \to 0$ for each $i = 1, \dots, m$.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an (unconditional) Schauder basis of $B^p_{s,w}(\mathbb{R}^d)$ with coordinate functionals $\{e^*_k\}_{k\in\mathbb{N}}$ such that $\|e^*_k\|_{B^{p'}_{-s,-w}(\mathbb{R}^d)} = 1$ for each $k \in \mathbb{N}$. We consider the cylindrical measure $\mu_{1,1}$ defined by

$$\mu_{1,1}(C) = \mu\left(C \cap \{f\colon\ |[f, e^*_1]| > 1\}\right)$$

for each cylinder set C in $\mathcal{Z}(B^p_{s,w}(\mathbb{R}^d), \{e^*_k\}_{k\in\mathbb{N}})$. Clearly the set of functionals $\{e^*_k\}_{k\in\mathbb{N}}$ is separating, and thus $\mathcal{Z}(B^p_{s,w}(\mathbb{R}^d), \{e^*_k\}_{k\in\mathbb{N}})$ generates $\mathcal{B}(B^p_{s,w}(\mathbb{R}^d))$. By the properties of cylindrical Lévy measures, $\mu_{1,1}$ is a finite cylindrical measure on $\mathcal{Z}(B^p_{s,w}(\mathbb{R}^d), \{e^*_k\}_{k\in\mathbb{N}})$. As μ satisfies $\lim_{k\to\infty}(|\beta|^2 \wedge 1)(\mu\circ\pi^{-1}_{e^*_1, f^*_{1,k}, \dots, f^*_{m,k}})(\mathrm{d}\beta) = (|\beta|^2 \wedge 1)(\mu \circ \pi^{-1}_{e^*_1, f^*_1, \dots, f^*_m})(\mathrm{d}\beta)$ weakly due to Lemma 4.4 in [73], we see that (5.2.11) is satisfied for $\mu_{1,1}$ as each finite-dimensional projection only takes weight on $|\beta| > 1$. Thus we may apply [38] and extend $\mu_{1,1}$ to a finite Radon measure $\tilde{\mu}_{1,1}$ on

$B^p_{s,w}(\mathbb{R}^d)$. Clearly $\tilde{\mu}_{1,1}$ is supported on $\{f\colon\ |[f,e_1^*]|>1\}$, and for each cylinder set $C\in\mathcal{Z}(B^p_{s,w}(\mathbb{R}^d),\{e_k^*\}_{k\in\mathbb{N}})$ we have $\tilde{\mu}_{1,1}(C)=\mu(C\cap\{f\colon\ |[f,e_1^*]|>1\})$.

Next, for each $n\in\mathbb{N}$ we construct cylindrical measures $\mu_{1,n+1}$ by

$$\mu_{1,n+1}(C)=\mu\left(C\cap\left\{f\colon \frac{1}{n+1}<|[f,e_1^*]|\leqslant\frac{1}{n}\right\}\right),\qquad C\in\mathcal{Z}(B^p_{s,w}(\mathbb{R}^d),\{e_k^*\}_{k\in\mathbb{N}}).$$

Applying the same argument as above (using an easy rescaling), we obtain a sequence of finite Radon measures $\{\tilde{\mu}_{1,n}\}_{n\in\mathbb{N}}$ with pairwise disjoint support. We define the measure $\tilde{\mu}_1$ by

$$\tilde{\mu}_1(A)=\sum_{n=1}^{\infty}\tilde{\mu}_{1,n}(A),\qquad A\in\mathcal{B}(B^p_{s,w}(\mathbb{R}^d)).$$

By this construction, $\tilde{\mu}_1$ forms a σ-finite Radon measure on $B^p_{s,w}(\mathbb{R}^d)$ supported on $B_1:=\{f\colon\ |[f,e_1^*]|\neq 0\}$.

We next repeat the procedure on the subspace $\{f\colon\ |[f,e_1^*]|=0\}$. We start by defining, for $C\in\mathcal{Z}(B^p_{s,w}(\mathbb{R}^d),\{e_k^*\}_{k\in\mathbb{N}})$,

$$\mu_{2,1}(C)=\mu\left(C\cap\{f\colon\ |[f,e_1^*]|=0,|[f,e_2^*]|>1\}\right),$$

and, for $n\in\mathbb{N}$,

$$\mu_{2,n+1}(C)=\mu\left(C\cap\left\{f\colon\ |[f,e_1^*]|=0,\frac{1}{n+1}<|[f,e_2^*]|\leqslant\frac{1}{n}\right\}\right).$$

We in this manner obtain a σ-finite Radon measure $\tilde{\mu}_2$ supported on $B_2:=\{f\colon\ |[f,e_1^*]|=0,|[f,e_2^*]|\neq 0\}$. We then continue this procedure to create the set of measures $\{\tilde{\mu}_k\}_{k\in\mathbb{N}}$, where for each $k\in\mathbb{N}$, $\tilde{\mu}_k$ is supported on $B_k:=\{f\colon\ |[f,e_1^*]|=0,\dots,\left|[f,e_{k-1}^*]\right|=$

$0, |[f, e_k^*]| \neq 0\}$. We observe that the set $\{B_k\}_{k\in\mathbb{N}}$ is pairwise disjoint, and

$$B^p_{s,w}(\mathbb{R}^d) = \{0\} \cup \bigcup_{k=1}^{\infty} B_k.$$

We now finally define the measure $\tilde{\mu}$ on $B^p_{s,w}(\mathbb{R}^d)$ by setting $\tilde{\mu}(\{0\}) = 0$ and

$$\tilde{\mu}(A) = \sum_{k=1}^{\infty} \tilde{\mu}_k(A), \qquad A \in \mathcal{B}(B^p_{s,w}(\mathbb{R}^d)).$$

As $\tilde{\mu}$ is a sum of σ-finite measures with pairwise disjoint support, it follows that $\tilde{\mu}$ is σ-finite. Let $C \in \mathcal{Z}_*(B^p_{s,w}(\mathbb{R}^d), \{e_k^*\}_{k\in\mathbb{N}})$, say $C = \{f\colon ([f, e_{k_1}^*], \ldots, [f, e_{k_m}^*]) \in K\}$, with $0 \notin K$, as the Lévy measure $\mu \circ \pi^{-1}_{e_{k_1}^*,\ldots,e_{k_m}^*}$ is only defined on such sets. Then we have

$$\tilde{\mu}(C) = \sum_{k=1}^{k_m} \tilde{\mu}_k(C) = \sum_{k=1}^{k_m} \mu(C \cap B_k) = \mu(C).$$

The representation as a finite sum holds as cylinder sets are all defined on the same finite group of functionals and so $\tilde{\mu}$ forms an extension of the restriction of μ to $\mathcal{Z}_*(B^p_{s,w}(\mathbb{R}^d), \{e_k^*\}_{k\in\mathbb{N}})$. As $\tilde{\mu}$ is a σ-finite measure on $\mathcal{B}(B^p_{s,w}(\mathbb{R}^d))$, we have by uniqueness of extensions that $\tilde{\mu}$ forms an extension of μ. □

We now present a Corollary to Theorem 4.1.2 which characterises when a cylindrical Lévy process in $L^2(\mathbb{R}^d)$ is induced by a (genuine) Lévy process in some given weighted Besov space.

Corollary 5.2.5. *Let L be a cylindrical Lévy process in $L^2(\mathbb{R}^d)$ with no Gaussian component and with cylindrical Lévy measure μ. Let $p > 1$ and $(s, w) \in E_p$. Assume that μ extends to a σ-finite measure on $B^p_{s,w}(\mathbb{R}^d)$. Then L is induced by a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if and only if for any admissible basis Ψ of $B^p_{s,w}(\mathbb{R}^d)$ we have*

1. *for* $p \geqslant 2$,

$$\lim_{n\to\infty} \int_{B^p_{s,w}} \left(\|P_n f\|^p_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) < \infty, \tag{5.2.12}$$

$$\lim_{n\to\infty} \sum_{j,G,m}^{n} (\omega^j_m)^p \left(\int_{B^p_{s,w}} \mathbb{1}_{B_\mathbb{R}} \left(\|P_n f\|_{B^p_{s,w}} \right) \left| [\Psi^{j,G}_m, f] \right|^2 \mu(\mathrm{d}f) \right)^{p/2} < \infty;$$

2. *for* $p \in (1,2)$,

$$\lim_{n\to\infty} \int_{B^p_{s,w}} \left(\|P_n f\|^2_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) < \infty, \tag{5.2.13}$$

$$\lim_{n\to\infty} \sum_{j,G,m}^{n} (\omega^j_m)^p \int_0^\infty \left(1 - e^{\int_{B^p_{s,w}} \mathbb{1}_{B_\mathbb{R}} \left(\|P_n f\|_{B^p_{s,w}} \right) \left(\cos \tau [\Psi^{j,G}_m, f] - 1 \right) \mu(\mathrm{d}f)} \right) \frac{\mathrm{d}\tau}{\tau^{1+p}} < \infty.$$

In the expressions above, $\omega^j_m = \omega^j_m(p,s,w)$ *are the weight constants defined in* (2.1.2).

Proof. By Theorem 5.2.3 we may assume that L and μ are symmetric. Then necessity and sufficiency of the conditions is clear from Theorem 4.1.2. □

5.3 Canonical cylindrical α-stable processes

In this section, we apply our previous results to investigate the regularity of some typical examples of cylindrical Lévy processes as they often appear in the literature; see e.g. [18, 69]. A cylindrical Lévy process $L = (L(t) : t \geqslant 0)$ in $L^2(\mathbb{R}^d)$ is called *canonical* α*-stable* for some $\alpha \in (0,2)$ if its characteristic function is of the form

$$\varphi_{L(t)}(u) = \exp(-t\,\|u\|^\alpha_{L^2}), \qquad u \in L^2(\mathbb{R}^d).$$

The existence of a cylindrical distribution with this characteristic function is guaranteed by Bochner's theorem for cylindrical measures; see [92, Prop. IV.4.2]; two possible explicit constructions can be found in [75].

Theorem 5.3.1. *Let L be a canonical α-stable cylindrical process in $L^2(\mathbb{R}^d)$ for some $\alpha \in (0,2)$. Then L is induced by a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. for some $p > 1$ and $(s,w) \in E_p$ if and only if*

$$s < -\frac{d}{2} \qquad \textit{and} \qquad w < -\frac{d}{p}.$$

Proof. Let Ψ be an admissible basis of $B^p_{s,w}(\mathbb{R}^d)$; we recall Ψ forms an orthonormal basis of $L^2(\mathbb{R}^d)$. Lemma 2.4 in [75] shows that the Lévy measure $\nu_n := \mu \circ \pi^{-1}_{\Psi_1,\dots\Psi_n}$ of any n-dimensional projection is given by

$$\nu_n(B) = \frac{\alpha}{c_\alpha} \int_{S^n} \lambda_n(\mathrm{d}\xi) \int_0^\infty \mathbb{1}_B(r\xi) r^{-1-\alpha}\, \mathrm{d}r \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^n),$$

where λ_n is uniformly distributed on the sphere $S^n = \{\xi \in \mathbb{R}^n \colon\ |\xi| = 1\}$ with

$$\lambda_n(S^n) = r_n := \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})} \qquad \text{and} \qquad c_\alpha = \begin{cases} -\alpha\cos(\frac{\alpha\pi}{2})\Gamma(-\alpha), & \text{if } \alpha \neq 1, \\ \frac{\alpha\pi}{2}, & \text{if } \alpha = 1. \end{cases}$$

First we show sufficiency of the conditions. For $p \in (1,2]$ this follows directly from the fact that the embedding of $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ is 0-Radonifying, see Part (i) of Theorem 4.2.6. For $p > 2$, we will establish the Borel extension of μ by the result in Theorem 5.2.4 and then apply Corollary 5.2.5. We choose an enumeration of the indices j, G and m. We recall the projections P_n as defined in (5.2.10), and we define

$$\begin{aligned}
\Sigma_n &:= \int_{B^p_{s,w}} \left(\|P_n f\|^p_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) \\
&= \int_{\mathbb{R}^n} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|\beta^{j,G}_m\right|^p \wedge 1 \right) (\mu \circ \pi^{-1}_{\Psi_1,\dots,\Psi_n})(\mathrm{d}\beta) \\
&= \frac{\alpha}{c_\alpha} \int_{S^n} \int_0^\infty \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|r\xi^{j,G}_m\right|^p \wedge 1 \right) r^{-1-\alpha}\, \mathrm{d}r\, \lambda_n(\mathrm{d}\xi) \\
&= \frac{p}{c_\alpha(p-\alpha)} \int_{S^n} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda_n(\mathrm{d}\xi). \qquad (5.3.14)
\end{aligned}$$

Letting $\lambda^1_n := \frac{1}{r_n}\lambda_n$, Jensen's inequality implies

$$\begin{aligned}
\Sigma_n &= \frac{pr_n}{c_\alpha(p-\alpha)} \int_{S^n} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda^1_n(\mathrm{d}\xi) \\
&\leqslant \frac{pr_n}{c_\alpha(p-\alpha)} \left(\int_{S^n} \sum_{j,G,m}^{n} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \lambda^1_n(\mathrm{d}\xi) \right)^{\alpha/p}.
\end{aligned}$$

By Lemma A.2 in [75] we have $\int_{S^n} \left|\xi^{j,G}_m\right|^p \lambda^1_n(\mathrm{d}\xi) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})}$ and thus

$$\begin{aligned}
\Sigma_n &\leqslant \frac{pr_n}{c_\alpha(p-\alpha)} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})} \right)^{\alpha/p} \\
&= \frac{p}{c_\alpha(p-\alpha)} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})} \left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})} \right)^{\alpha/p} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \right)^{\alpha/p}.
\end{aligned}$$

Since $\frac{\Gamma(x+\alpha)}{\Gamma(x)} = x^\alpha\big(1 + \mathcal{O}(x^{-1})\big)$ as $x \to \infty$ [15, Prop. 2.1.3], we conclude that Σ_n converges to a finite limit as $n \to \infty$ since $\sum_{j,G,m}(\omega^j_m)^p < \infty$ given $s < -\frac{d}{2}$ and

$w < -\frac{d}{p}$ (see the proof of [31, Th. 3]). Next we define

$$\begin{aligned}\Upsilon_n &:= \sum_{j,G,m}^{n} (\omega_m^j)^p \left(\int_{B_{s,w}^p} \mathbb{1}_{B_\mathbb{R}} \left(\|P_n f\|_{B_{s,w}^p} \right) \left|[\Psi_m^{j,G}, f]\right|^2 \mu(\mathrm{d}f) \right)^{p/2} \\ &= \sum_{j,G,m}^{n} (\omega_m^j)^p \left(\int_{\mathbb{R}^n} \mathbb{1}_{B_\mathbb{R}} \left(\sum_{i,H,l}^{n} (\omega_l^i)^p \left|\beta_l^{i,H}\right|^p \right) \left|\beta_m^{j,G}\right|^2 (\mu \circ \pi_{\Psi_1,\dots,\Psi_n}^{-1})(\mathrm{d}\beta) \right)^{p/2} \\ &= \sum_{j,G,m}^{n} (\omega_m^j)^p \left(\frac{\alpha}{c_\alpha} \int_{S^n} \int_0^\infty \mathbb{1}_{B_\mathbb{R}} \left(\sum_{i,H,l}^{n} (\omega_l^i)^p \left|r\xi_l^{i,H}\right|^p \right) \left|r\xi_m^{j,G}\right|^2 r^{-1-\alpha} \,\mathrm{d}r \,\lambda_n(\mathrm{d}\xi) \right)^{p/2} \\ &= \left(\frac{\alpha}{c_\alpha(2-\alpha)} \right)^{p/2} \sum_{j,G,m}^{n} (\omega_m^j)^p \left(\int_{S^n} \left|\xi_m^{j,G}\right|^2 \left(\sum_{i,H,l}^{n} (\omega_l^i)^p \left|\xi_l^{i,H}\right|^p \right)^{(\alpha-2)/p} \lambda_n(\mathrm{d}\xi) \right)^{p/2}.\end{aligned}$$

Applying first Jensen's inequality and then Hölder's inequality, we obtain

$$\begin{aligned}\Upsilon_n &\leqslant \left(\frac{\alpha}{c_\alpha(2-\alpha)} \right)^{p/2} r_n^{p/2} \sum_{j,G,m}^{n} (\omega_m^j)^p \int_{S^n} \left|\xi_m^{j,G}\right|^p \left(\sum_{i,H,l}^{n} (\omega_l^i)^p \left|\xi_l^{i,H}\right|^p \right)^{(\alpha-2)/2} \lambda_n^1(\mathrm{d}\xi) \\ &= \left(\frac{\alpha}{c_\alpha(2-\alpha)} \right)^{p/2} r_n^{p/2} \int_{S^n} \left(\sum_{j,G,m}^{n} (\omega_m^j)^p \left|\xi_m^{j,G}\right|^p \right)^{\alpha/2} \lambda_n^1(\mathrm{d}\xi) \\ &\leqslant \left(\frac{\alpha}{c_\alpha(2-\alpha)} \right)^{p/2} r_n^{p/2} \left(\int_{S^n} \sum_{j,G,m}^{n} (\omega_m^j)^p \left|\xi_m^{j,G}\right|^p \lambda_n^1(\mathrm{d}\xi) \right)^{\alpha/2} \\ &= \left(\frac{\alpha}{c_\alpha(2-\alpha)} \right)^{p/2} r_n^{p/2} \left(\sum_{j,G,m}^{n} (\omega_m^j)^p \right)^{\alpha/2} \left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})} \right)^{\alpha/2}.\end{aligned}$$

Since by properties of the Gamma function we have

$$r_n^{p/2} \left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})} \right)^{\alpha/2} = \left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})} \right)^{p/2} \left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})} \right)^{\alpha/2} = \mathcal{O}(1) \qquad \text{as } n \to \infty,$$

it follows that Υ_n has a finite limit as $n \to \infty$ as $\sum_{j,G,m}^{n} (\omega_m^j)^p$ does. We recall the definition (5.2.10) of the projection P_n onto the subspace spanned by the first n

elements in the enumeration of Ψ. It follows that we have

$$\|P_n f\|^p_{B^p_{s,w}} = \sum_{j,G,m}^{n} (\omega^j_m)^p \left|[\Psi^{j,G}_m, f]\right|^p, \qquad f \in B^p_{s,w}(\mathbb{R}^d).$$

Let $R > 0$. It follows that

$$\begin{aligned}
\mu\big(\{f : \|P_n f\|^p_{B^p_{s,w}} > R^p\}\big) &= \int_{\mathbb{R}^n} \mathbb{1}_{\left\{\sum^n_{j,G,m}(\omega^j_m)^p \left|\beta^{j,G}_m\right|^p > R^p\right\}} (\mu \circ \pi^{-1}_{\Psi_1,\dots,\Psi_n})(\mathrm{d}\beta) \\
&= \frac{\alpha}{c_\alpha} \int_{S^n} \int^\infty_{R(\sum^n_{j,G,m}(\omega^j_m)^p \left|\xi^{j,G}_m\right|^p)^{-1/p}} r^{-1-\alpha} \,\mathrm{d}r \, \lambda_n(\mathrm{d}\xi) \\
&= R^{-\alpha} c^{-1}_\alpha \int_{S^n} \left(\sum^n_{j,G,m} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda_n(\mathrm{d}\xi) \\
&= R^{-\alpha} \frac{p-\alpha}{p} \Sigma_n.
\end{aligned}$$

As $\Sigma_n \to \Sigma_\infty < \infty$ as $n \to \infty$, we see we have $\lim_{R\to\infty} \lim_{n\to\infty} \mu\big(\{f : \|P_n f\|_{B^p_{s,w}} > R\}\big) = 0$ and we may thus apply Theorem 5.2.4 to show the extension of μ to a σ-finite measure on $B^p_{s,w}(\mathbb{R}^d)$. As we have shown $\lim_{n\to\infty} \Sigma_n < \infty$ and $\lim_{n\to\infty} \Upsilon_n < \infty$, Corollary 5.2.5 verifies the sufficiency for these values of w and s and $p > 2$.

We now show the necessity of the conditions in the hypothesis. First we consider the case $p \geqslant 2$. We define $A_n := \sum^n_{j,G,m} (\omega^j_m)^p$ and, applying Jensen's inequality to the concave sum $\left(\sum^n_{j,G,m} A^{-1}_n (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p\right)^{\alpha/p}$ we obtain from (5.3.14), using again [75, Lem. A.2],

$$\begin{aligned}
\Sigma_n &= \frac{pA^{\alpha/p}_n}{c_\alpha(p-\alpha)} \int_{S^n} \left(\sum^n_{j,G,m} A^{-1}_n (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda_n(\mathrm{d}\xi) \\
&\geqslant \frac{pA^{\alpha/p}_n}{c_\alpha(p-\alpha)} \int_{S^n} \sum^n_{j,G,m} A^{-1}_n (\omega^j_m)^p \left|\xi^{j,G}_m\right|^\alpha \lambda_n(\mathrm{d}\xi) \\
&= \frac{pA^{\alpha/p}_n}{c_\alpha(p-\alpha)} r_n \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})} = \frac{pA^{\alpha/p}_n}{c_\alpha(p-\alpha)}.
\end{aligned}$$

For $s \geqslant -\frac{d}{2}$ or $w \geqslant -\frac{d}{p}$ we have $A_n \to \infty$ as $n \to \infty$ (see the proof of [31, Th. 3]). Thus we have shown that $\Sigma_n \to \infty$ as $n \to \infty$, so Condition 1 of Corollary 5.2.5 is not met for these values of w and s, and the necessity is shown for the case $p \geqslant 2$.

For $p \in (1, 2)$ we define

$$
\begin{aligned}
\Lambda_n &:= \int_{B^p_{s,w}} \left(\|P_n f\|^2_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) \\
&= \int_{\mathbb{R}^n} \left(\left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|\beta^{j,G}_m\right|^p \right)^{2/p} \wedge 1 \right) (\mu \circ \pi^{-1}_{\Psi_1,\dots\Psi_n})(\mathrm{d}\beta) \\
&= \frac{\alpha}{c_\alpha} \int_{S^n} \int_0^\infty \left(\left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|r\xi^{j,G}_m\right|^p \right)^{2/p} \wedge 1 \right) r^{-1-\alpha}\, \mathrm{d}r \lambda_n(\mathrm{d}\xi) \\
&= \frac{2}{c_\alpha(2-\alpha)} \int_{S^n} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda_n(\mathrm{d}\xi).
\end{aligned}
$$

For $p \geqslant \alpha$, we can conclude as above for Σ_n that $\Lambda_n \to \infty$ as $n \to \infty$ and Condition 2 of Corollary 5.2.5 is not met for these values of w and s, which shows the necessity. For $p < \alpha$, applying Jensen's inequality this time to $\int_{S^n} \left(\sum^n_{j,G,m} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda^1_n(\mathrm{d}\xi)$ gives us

$$
\begin{aligned}
\Lambda_n &= \frac{2r_n}{c_\alpha(2-\alpha)} \int_{S^n} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \right)^{\alpha/p} \lambda^1_n(\mathrm{d}\xi) \\
&\geqslant \frac{2r_n}{c_\alpha(2-\alpha)} \left(\int_{S^n} \sum_{j,G,m}^{n} (\omega^j_m)^p \left|\xi^{j,G}_m\right|^p \lambda^1_n(\mathrm{d}\xi) \right)^{\alpha/p} \\
&= \frac{2r_n}{c_\alpha(2-\alpha)} \left(\sum_{j,G,m}^{n} (\omega^j_m)^p \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+p}{2})} \right)^{\alpha/p}.
\end{aligned}
$$

Since $\frac{\Gamma(x+\alpha)}{\Gamma(x)} = x^\alpha\big(1 + \mathcal{O}(x^{-1})\big)$ as $x \to \infty$, see [15, Prop. 2.1.3], and $\sum_{j,G,m} (\omega^j_m)^p = \infty$, it follows $\Lambda_n \to \infty$ as $n \to \infty$. Applying Corollary 5.2.5 completes the proof. □

Remark 5.3.2. *Recall that R_p denotes the (s, w)-plane where the embeddings $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ are 0-Radonifying. For $p \leqslant 2$, Theorem 5.3.1 states that L is induced by* *a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if and only if $(s,w) \in R_p$. However, for $p > 2$ the Theorem gives a stronger result, in that the region of the (s,w)-plane where L is induced by a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. is a proper superset of R_p.*

Remark 5.3.3. *This result is somewhat surprising, in that there is no dependency on α for the regularisation. Indeed, the same result holds for the standard cylindrical Brownian motion in $L^2(\mathbb{R}^d)$. This contrasts with the result (5.1.7) for the independently scattered case. However in this case, for $\alpha \in (1,2)$, the Musielak-Orlicz space $L_M(\mathbb{R}^d, \mathrm{leb})$, which is the domain of integrable functions, is $L^{\alpha'}(\mathbb{R}^d)$. This means that the independently scattered cylindrical Lévy process which this result holds for is a cylindrical process in $L^{\alpha'}(\mathbb{R}^d)$. It can be shown that the embedding $L^{\alpha'}(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ is 0-Radonifying for $s < d\left(\frac{1}{p\vee\alpha} - 1\right)$ and $w < -\frac{d}{p\wedge\alpha}$, which is consistent with (5.1.7). In this manner, the regularisation is seen to be a property of the domain of definition of the cylindrical Lévy process in this case.*

5.4 Hedgehog cylindrical Lévy process

In this section, we apply our previous results to investigate the regularity of some typical examples of cylindrical Lévy processes as they often appear in the literature; see e.g. [18, 69]

In this section let L be a cylindrical Lévy process in $L^2(\mathbb{R}^d)$ of the form

$$L(t)f = \sum_{k=1}^{\infty} [f, e_k] a_k \ell_k \qquad \text{for all } f \in L^2(\mathbb{R}^d),\, t \geqslant 0, \tag{5.4.15}$$

where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $(\ell_k)_{k\in\mathbb{N}}$ are identically distributed and independent symmetric real-valued Lévy processes with characteristics $(0,0,\rho)$ for a Lévy measure $\rho \neq 0$ in $\mathbb{R}$. By Theorem 5.2.3 it is sufficient for our analysis to focus on the symmetric case. The sequence $(a_k)_{k\in\mathbb{N}}$ is real-valued and satisfies

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(|a_k c_k \beta|^2 \wedge 1 \right) \rho(\mathrm{d}\beta) < \infty \tag{5.4.16}$$

for each $(c_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{R})$. This condition guarantees that the sum in (5.4.15) converges P-a.s. in $\mathbb{R}$; see [74, Lem. 4.2]. To avoid redundancies in the representation, we assume $a_k \neq 0$ for all $k \in \mathbb{N}$.

The support of the cylindrical measure μ of L is in $\bigcup_{k\in\mathbb{N}}\{\beta e_k : \beta \in \mathbb{R}\}$, as $(\ell_k)_{k\in\mathbb{N}}$ are independent, that is to say the measure only has weight on the axes. For this reason, we refer to this process as a *hedgehog cylindrical process.*

We first present further Corollaries to Theorem 4.1.2 and Remark 4.1.4 tailored to this setting.

Corollary 5.4.1. *Let L be a cylindrical Lévy process of the form (5.4.15). Let $p > 1$ and $(s, w) \in E_p$, and furthermore suppose the orthonormal basis used in (5.4.15) satisfies $\{e_k\}_{k\in\mathbb{N}} \subseteq B^{p'}_{-s,-w}(\mathbb{R}^d)$. Then L is induced by a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if and only if*

1. *for $p \geqslant 2$,*

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\|a_k e_k\|^p_{B^p_{s,w}} |\beta|^p \wedge 1 \right) \rho(\mathrm{d}\beta) < \infty,$$

$$\sum_{j,G,m} (\omega^j_m)^p \left(\sum_{k=1}^{\infty} \left| [\Psi^{j,G}_m, a_k e_k] \right|^2 \int_{|\beta| \leqslant \|a_k e_k\|^{-1}_{B^p_{s,w}}} \beta^2 \, \rho(\mathrm{d}\beta) \right)^{p/2} < \infty;$$

2. *for $p \in (1, 2)$,*

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\|a_k e_k\|^2_{B^p_{s,w}} |\beta|^2 \wedge 1 \right) \rho(\mathrm{d}\beta) < \infty,$$

$$\sum_{j,G,m} (\omega^j_m)^p \int_0^{\infty} \left(1 - e^{\sum_{k=1}^{\infty} \int_{\beta \leqslant \|a_k e_k\|^{-1}_{B^p_{s,w}}} (\cos \tau [\Psi^{j,G}_m, a_k e_k]\beta - 1) \, \rho(\mathrm{d}\beta)} \right) \frac{\mathrm{d}\tau}{\tau^{1+p}} < \infty.$$

Proof. Let Ψ be an admissible basis for $B^p_{s,w}(\mathbb{R}^d)$. Lemma 4.2 in [74] and Lemma 3.10 in [51] show the Lévy measure μ of L extends to a σ-finite Borel measure on $L^2(\mathbb{R}^d)$ with projection on the n-th partial sum given by

$$(\mu \circ \pi^{-1}_{e_1,\dots,e_n})(\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n) = \sum_{k=1}^{n} (\rho \circ m_{a_k})(\mathrm{d}\beta_k) \tag{5.4.17}$$

where $m_{a_k}\colon \mathbb{R} \to \mathbb{R}$ is defined by $m_{a_k}(\beta) = a_k\beta$. We henceforth identify μ with its pushforward measure on $B^p_{s,w}(\mathbb{R}^d)$ under the canonical injection.

For $p > 1$ we define $q := p \vee 2$ and calculate for each $n \in \mathbb{N}$, applying (5.4.17),

$$\begin{aligned}
&\int_{B^p_{s,w}} \left(\left(\sum_{j,G,m} \left| \omega^j_m \sum_{k=1}^{n} [\Psi^{j,G}_m, e_k][e_k, f] \right|^p \right)^{q/p} \wedge 1 \right) \mu(\mathrm{d}f) \\
&\int_{\mathbb{R}^n} \left(\left(\sum_{j,G,m} \left| \omega^j_m \sum_{k=1}^{n} [\Psi^{j,G}_m, e_k]\beta_k \right|^p \right)^{q/p} \wedge 1 \right) (\mu \circ \pi^{-1}_{e_1 \dots e_n})(\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n) \\
&= \sum_{k=1}^{n} \int_{\mathbb{R}} \left(\left(\sum_{j,G,m} \left| \omega^j_m [\Psi^{j,G}_m, e_k] a_k \beta \right|^p \right)^{q/p} \wedge 1 \right) \rho(\mathrm{d}\beta) \\
&= \sum_{k=1}^{n} \int_{\mathbb{R}} \left(\|a_k e_k\|^q_{B^p_{s,w}} |\beta|^q \wedge 1 \right) \rho(\mathrm{d}\beta).
\end{aligned}$$

By taking the limit as $n \to \infty$ we obtain

$$\int_{B^p_{s,w}} \left(\|f\|^q_{B^p_{s,w}} \wedge 1 \right) \mu(\mathrm{d}f) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\|a_k e_k\|^q_{B^p_{s,w}} |\beta|^q \wedge 1 \right) \rho(\mathrm{d}\beta),$$

which shows μ satisfies the first conditions in (1) or (2) in Theorem 4.1.2.

For $p \geqslant 2$ we calculate, for $(i,H,l) \in \mathbb{W}^d$ and $n \in \mathbb{N}$ that

$$\int_{\mathbb{R}^n} \mathbb{1}_{B_\mathbb{R}} \left(\sum_{j,G,m} (\omega_m^j)^p \left| \sum_{k=1}^n [\Psi_m^{j,G}, e_k]\beta_k \right|^p \right) \left| \sum_{k=1}^n [\Psi_l^{i,H}, e_k]\beta_k \right|^2 (\mu \circ \pi_{e_1,\dots,e_n}^{-1})(\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n)$$
$$= \sum_{k=1}^n \int_{\mathbb{R}} \mathbb{1}_{B_\mathbb{R}} \left(\sum_{j,G,m} (\omega_m^j)^p \left| [\Psi_m^{j,G}, e_k] a_k \beta \right|^p \right) \left| [\Psi_l^{i,H}, e_k] a_k \beta \right|^2 \rho(\mathrm{d}\beta)$$
$$= \sum_{k=1}^n \int_{|\beta| \leqslant \|a_k e_k\|_{B_{s,w}^p}^{-1}} \left| [\Psi_l^{i,H}, a_k e_k] \beta \right|^2 \rho(\mathrm{d}\beta).$$

Since Lebesgue's dominated convergence theorem shows

$$\sum_{j,G,m} (\omega_m^j)^p \left(\int_{\|f\|_{B_{s,w}^p} \leqslant 1} [\Psi_m^{j,G}, f]^2 \, \mu(\mathrm{d}f) \right)^{p/2}$$
$$= \sum_{j,G,m} (\omega_m^j)^p \left(\sum_{k=1}^\infty \left| [\Psi_m^{j,G}, a_k e_k] \right|^2 \int_{|\beta| \leqslant \|a_k e_k\|_{B_{s,w}^p}^{-1}} \beta^2 \, \rho(\mathrm{d}\beta) \right)^{p/2},$$

applying Theorem 4.1.2 shows that μ is a Lévy measure on $B_{s,w}^p(\mathbb{R}^d)$. Following the method of the proof of Corollary 5.2.5 completes the proof of Part (1).

Part (2) follows by a similar calculation and from applying Theorem 4.1.2 and the methods of the proof of Corollary 5.2.5. □

Corollary 5.4.2. *Let $p \in [1,2]$ and $(s,w) \in E_p$. A cylindrical Lévy process L of the form (5.4.15) with $\{e_k\}_{k\in\mathbb{N}} \subseteq B_{-s,-w}^{p'}(\mathbb{R}^d)$ is induced by a Lévy process in $B_{s,w}^p(\mathbb{R}^d)$ P-a.s. if*

$$\sum_{k=1}^\infty \int_{\mathbb{R}} \left(\|a_k e_k\|_{B_{s,w}^p}^p |\beta|^p \wedge 1 \right) \rho(\mathrm{d}\beta) < \infty.$$

Proof. By Remark 4.1.4 it suffices to show

$$\int_{B_{s,w}^p(\mathbb{R}^d)} \left(\|f\|_{B_{s,w}^p}^p \wedge 1 \right) \mu(\mathrm{d}f) < \infty,$$

which follows using the same calculation as in the proof of Corollary 5.4.1. ☐

To characterise the Besov membership of a hedgehog process L we introduce some indices in terms of the Lévy measure of the real-valued Lévy processes ℓ_k in the representation (5.4.15). For this purpose, let ρ be a Lévy measure in $\mathbb{R}$ and define for $q \in \mathbb{R}_+$

$$\overline{\tau}^{(q)} := \inf_{\tau \geqslant 0} \left\{ \limsup_{\xi \downarrow 0} \xi^{-\tau} \int_{B_{\mathbb{R}}^c} \left(\xi^q \left|\beta\right|^q \wedge 1\right) \rho(\mathrm{d}\beta) = \infty \right\}, \tag{5.4.18}$$

$$\underline{\tau}^{(q)} := \inf_{\tau \geqslant 0} \left\{ \liminf_{\xi \downarrow 0} \xi^{-\tau} \int_{B_{\mathbb{R}}^c} \left(\xi^q \left|\beta\right|^q \wedge 1\right) \rho(\mathrm{d}\beta) = \infty \right\}. \tag{5.4.19}$$

In all definitions above we apply the convention $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. It is easy to see that $\overline{\tau}^{(q)} \leqslant \underline{\tau}^{(q)} \leqslant q$ when $\rho \neq 0$. The Examples following Theorem 5.4.5 show calculations of these indices in a number of standard situations.

The following Proposition establishes a simple interpretation of $\overline{\tau}^{(q)}$. We recall that a Lévy process with Lévy measure ρ has finite p-th moments if and only if $\int_{B_{\mathbb{R}}^c} |\beta|^p \, \rho(\mathrm{d}\beta) < \infty$.

Proposition 5.4.3. *For a Lévy measure $\rho \neq 0$ on $\mathbb{R}$ define*

$$p_{\max} := \sup \left\{ p > 0 \colon \int_{B_{\mathbb{R}}^c} |\beta|^p \, \rho(\mathrm{d}\beta) < \infty \right\}.$$

If $p_{\max} > 0$ then $\overline{\tau}^{(q)} = p_{\max} \wedge q$ and thus $(p_{\max} \wedge q) \leqslant \underline{\tau}^{(q)} \leqslant q$ for all $q \geqslant 0$.

Remark 5.4.4. *The index $p_{\max}$ is the same as defined in [9] and has a close relationship with the index β_0 introduced by Schilling in [83] and known as the Blumenthal-Getoor index at zero. In particular, we have the relationship*

$$\beta_0 = p_{\max} \wedge 2.$$

Proof of Proposition 5.4.3. To demonstrate this, we shall consider the following indices:

$$\overline{\tau}_1^{(q)} := \sup\left\{\tau \geqslant 0\colon \limsup_{\xi\downarrow 0} \xi^{-\tau} \int_{1<|\beta|\leqslant\xi^{-1}} \xi^q\,|\beta|^q\,\rho(\mathrm{d}\beta) < \infty\right\} \qquad \text{for } q \in \mathbb{R}_+,$$
$$\overline{\tau}_2 := \sup\left\{\tau > 0\colon \limsup_{\xi\downarrow 0} \xi^{-\tau} \int_{|x|>\xi^{-1}} \rho(\mathrm{d}\beta) < \infty\right\}.$$

We define a finite measure $\overline{\rho} := \rho\big|_{B_{\mathbb{R}}^c}$; clearly we may replace ρ with $\overline{\rho}$ in the definitions of $\overline{\tau}_1^{(q)}$ and $\overline{\tau}_2$. By Markov's inequality we have, for $\xi < 1$ and $p < p_{\max}$,

$$\overline{\rho}\big(\{|\beta| > \xi^{-1}\}\big) \leqslant \xi^p \int_{\mathbb{R}} |\beta|^p\,\overline{\rho}(\mathrm{d}\beta) = \xi^p \int_{B_{\mathbb{R}}^c} |\beta|^p\,\rho(\mathrm{d}\beta),$$

thus showing $\overline{\tau}_2 \geqslant p_{\max}$. On the other side, for $\tau < \overline{\tau}_2$ there exists a constant $C > 0$ such that

$$\overline{\rho}\big(\{|\beta| > t\}\big) \leqslant Ct^{-\tau} \qquad \text{for all } t \geqslant 1.$$

The tail formula for the integral shows for $0 < p < \tau$ that

$$\int_{B_{\mathbb{R}}^c} |\beta|^p\,\rho(\mathrm{d}\beta) = \int_{\mathbb{R}} |\beta|^p\,\overline{\rho}(\mathrm{d}\beta) = \int_0^\infty \overline{\rho}\big(\{|\beta|^p > t\}\big)\,\mathrm{d}t \leqslant \overline{\rho}(\mathbb{R}) + C\int_1^\infty t^{-\frac{\tau}{p}}\,\mathrm{d}t < \infty,$$

which enables us to conclude $p_{\max} \geqslant \overline{\tau}_2$ and hence $\overline{\tau}_2 = p_{\max}$.

Next choose some $\tau < \overline{\tau}_2 \wedge q$. Then Fubini's theorem implies

$$\begin{aligned}
\int_{1<|\beta|\leqslant\xi^{-1}} |\beta|^q \, \overline{\rho}(\mathrm{d}\beta) &= \int_0^\infty \overline{\rho}\big(\{|\beta|^q \, \mathbb{1}_{1<|\beta|\leqslant\xi^{-1}} > t\}\big)\,\mathrm{d}t \\
&= \int_0^1 \overline{\rho}\big(\{1 < |\beta| \leqslant \xi^{-1}\}\big)\,\mathrm{d}t + \int_1^{\xi^{-q}} \overline{\rho}\big(\{t^{\frac{1}{q}} < |\beta| \leqslant \xi^{-1}\}\big)\,\mathrm{d}t \\
&\leqslant \overline{\rho}\big(\{1 < |\beta| \leqslant \xi\}\big) + \int_1^{\xi^{-q}} \overline{\rho}\big(\{t^{\frac{1}{q}} < |\beta|\}\big)\,\mathrm{d}t \\
&\leqslant \overline{\rho}(\mathbb{R}) + C\int_1^{\xi^{q}} t^{-\frac{\tau}{q}}\,\mathrm{d}t \lesssim 1 + \xi^{\tau-q}.
\end{aligned}$$

It follows $\xi^{q-\tau}\int_{1<|\beta|\leqslant\xi^{-1}} |\beta|^q \, \rho(\mathrm{d}\beta) < \infty$ for all $\xi < 1$, implying $\tau \leqslant \overline{\tau}_1^{(q)}$. As $\tau < \overline{\tau}_2 \wedge q$ is arbitrary we have shown that $\overline{\tau}_1^{(q)} \geqslant \overline{\tau}_2 \wedge q$. As we have $\overline{\tau}^{(q)} = \overline{\tau}_1^{(q)} \wedge \overline{\tau}_2$, we thus conclude $\overline{\tau}^{(q)} = p_{\max} \wedge q$. □

The following result gives conditions such that a hedgehog process is induced by a Lévy process in a certain Besov space. The critical value will be the parameters $\overline{\tau}^{(\min\{p,2\})}$ and $\underline{\tau}^{(\max\{p,2\})}$.

Theorem 5.4.5. *Let L be defined by (5.4.15) and let $p > 1$ and $(s, w) \in E_p$. Define*

$$q_{\min} := \inf\left\{ q > 0\colon \sum_{k=1}^\infty \|a_k e_k\|^q_{B^p_{s,w}} < \infty \right\}.$$

Then,

(a) *L is induced by a Lévy process Y in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if one of the following is satisfied:*

  (i) *$(s, w) \in R_p$;*

  (ii) *$(s, w) \in R_p^c$ and $q_{\min} < \overline{\tau}^{(\min\{p,2\})}$.*

(b) *L is not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if:*

$$(s, w) \in R_p^c \text{ and } q_{\min} > \underline{\tau}^{(\max\{p,2\})}.$$

Proof. Part (a): the first alternative condition follows from the 0-Radonification in Theorem 4.2.6. To show the second alternative, we note that $B^p_{s,w}(\mathbb{R}^d)$ is of type $\min\{p,2\}$ by isometry with $\ell^p(\mathbb{W}^d)$ (see (2.1.5)). Then, by Proposition 7.1.16 in [41], if $q \leqslant \min\{p,2\}$ is such that $E\,|\ell_1|^q < \infty$ then $\big(\,\|a_k e_k\|_{B^p_{s,w}}\big)_{k\in\mathbb{N}} \in \ell^q(\mathbb{R})$ implies that $\sum_{k\in\mathbb{N}} a_k e_k \ell_k(1)$ converges in $B^p_{s,w}(\mathbb{R}^d)$ in q-th mean and thus converges strongly P-almost surely. Finally we note that, by Proposition 5.4.3, ℓ_1 has moments of order $\overline{\tau}^{(q)}$ for any $q \leqslant 2$.

Part (b): the first condition shows that Radonification does not apply. By Corollary 5.4.1, L is not induced by a Lévy process in $B^p_{s,w}(\mathbb{R}^d)$ if

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \big(\,\|a_k e_k\|^{\max\{2,p\}}_{B^p_{s,w}}\,|\beta|^{\max\{2,p\}} \wedge 1\big)\,\rho(\mathrm{d}\beta) = \infty. \tag{5.4.20}$$

Due to the hypothesis, we can choose $q > \underline{\tau}^{(\max\{2,p\})}$ such that $\big(\,\|a_k e_k\|_{B^p_{s,w}}\big)_{k\in\mathbb{N}} \notin \ell^q(\mathbb{R})$. It follows that there exists a constant K_0 such that, for large enough k, we have

$$\int_{B^c_{\mathbb{R}}} \big(\,\|a_k e_k\|^{\max\{2,p\}}_{B^p_{s,w}}\,|\beta|^{\max\{2,p\}} \wedge 1\big)\,\rho(\mathrm{d}\beta) \geqslant K_0\,\|a_k e_k\|^q_{B^p_{s,w}}\,,$$

which establishes (5.4.20). □

Remark 5.4.6. *We may conclude that $(s,w) \in R_p$ implies that $q_{\min} \leqslant \underline{\tau}^{(\max\{2,p\})}$, as otherwise Part (a) and (b) of Theorem 5.4.5 would contradict. This equality can also be proven analytically. Furthermore, one can show that $(s,w) \in E_p$ implies that $\big(\,\|e_k\|_{B^p_{s,w}}\big)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ for any orthonormal basis $(e_k)_{k\in\mathbb{N}}$.*

The first two examples we present show that, in the case each ℓ_k has moments of all orders, the critical summability is of a particularly simple form.

Example 5.4.7. Let L be a cylindrical Lévy process of the form (5.4.15) with $\rho = \delta_1$; thus each of the ℓ_k is a Poisson process with unit intensity. We have

$(a_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. As ρ has moments of all orders we have $\overline{\tau}^{(q)} = \underline{\tau}^{(q)} = q$ for each $q \in \mathbb{R}_+$. Thus, the critical summability needs to satisfy $q_{\min} < p \wedge 2$ for inclusion and $q_{\min} > p \vee 2$ for exclusion.

Example 5.4.8. Let $\rho(\mathrm{d}\beta) = \mathbb{1}_{\beta\neq 0} |\beta|^{-\zeta} e^{-|\beta|} \mathrm{d}\beta$ for some $\zeta \in (0,3)$, this gives rise to tempered stable processes. For $\zeta = 1$ this gives the symmetric Gamma process and for $\zeta = \frac{3}{2}$ this gives the symmetric inverse Gaussian process. For $q > \zeta - 1$ we have

$$\int_{\mathbb{R}} |\beta|^{q-\zeta} e^{-|\beta|} \mathrm{d}\beta = 2\Gamma(q - \zeta + 1) < \infty;$$

and we conclude that ρ has moments of all orders and thus again we have $\overline{\tau}^{(q)} = \underline{\tau}^{(q)} = q$ for each $q \in \mathbb{R}_+$. Thus, the critical summability again needs to satisfy $q_{\min} < p \wedge 2$ for inclusion and $q_{\min} > p \vee 2$ for exclusion.

Next we examine the symmetric-α-stable case, where the limits on moments comes into play.

Example 5.4.9. Let $\rho(\mathrm{d}\beta) = \mathbb{1}_{\beta\neq 0} |\beta|^{-1-\alpha} \mathrm{d}\beta$ for some $\alpha \in (0,2)$; by Example 4.5 in [74] we have $(a_k)_{k\in\mathbb{N}} \in \ell^{2\alpha/(2-\alpha)}(\mathbb{R})$. Then $\overline{\tau}^{(q)} = q \wedge \alpha$ for each $q \leqslant 2$ and $\underline{\tau}^{(q)} = \alpha$ for each $q \geqslant 2$.

Let $p > 1$ and $(s,w) \in E_p$. In this case, we obtain the following dichotomy in the critical regime $(s,w) \in R_p^c \cap E_p$:

(a) L is induced by a Lévy process Y in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if $q_{\min} < p \wedge \alpha$;

(b) L is not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if $q_{\min} > \alpha$.

The following example gives a construction whereby $\underline{\tau}^{(q)} \neq \overline{\tau}^{(q)}$.

Example 5.4.10. Let $\alpha_1 \in (1,2)$ and $\alpha_2 \in (\alpha_1, 2)$. Let ρ be given by

$$\rho(\mathrm{d}\beta) = \sum_{k=0}^{\infty} \left(\mathbb{1}_{(2k,2k+1]}(\beta) \, |\beta|^{-1-\alpha_1} \, \mathrm{d}\beta + \mathbb{1}_{(2k+1,2k+2]}(\beta) \, |\beta|^{-1-\alpha_2} \, \mathrm{d}\beta \right).$$

Then it is straightforward to see that $\overline{\tau}^{(q)} = q \wedge \alpha_1$ for each $q \leqslant 2$ and $\underline{\tau}^{(q)} = \alpha_2$ for each $q \geqslant 2$. Let $p > 1$ and $s, w \in E_p$. In this case, we obtain for $(s,w) \in R_p^c \cap E_p$:

(a) L is induced by a Lévy process Y in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if $q_{\min} < p \wedge \alpha_1$;

(b) L is not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s. if $q_{\min} > \alpha_2$.

Example 5.4.11. Let $\alpha \in (0,2)$ and let ρ be given by

$$\rho(\mathrm{d}\beta) = \mathbb{1}_{\beta \neq 0} \, |\beta|^{-1-\alpha} \, v(|\beta|)\mathrm{d}\beta,$$

where v is a slowly varying function; see e.g. Definition 1.2.1 in [16]. An application of Proposition 1.5.10 in [16] shows that $\overline{\tau}^{(q)} = q \wedge \alpha$ for each $q \leqslant 2$. However, it is known, see [16, p.16], that there exist slowly varying functions v such that $\liminf_{\beta \to \infty} v(\beta) = 0$ and $\limsup_{\beta \to \infty} v(\beta) = \infty$. Thus, we cannot in general improve on the bound $q \wedge \alpha \leqslant \underline{\tau}^{(q)} \leqslant q$ for this class of processes, which form a subclass of the subexponential Lévy processes.

5.4.1 Hedgehog process defined on wavelet basis

We may further analyse the symmetric-α-stable case by selecting an admissible wavelet basis of $B^p_{s,w}(\mathbb{R}^d)$ as the orthonormal basis of $L^2(\mathbb{R}^d)$, as in this case we may directly calculate the summability of the basis norms. This will allow us to construct counterexamples which show that, for $p > 2$, the region of the (s,w)-plane for which the embedding $L^2(\mathbb{R}^d) \hookrightarrow B^p_{s,w}(\mathbb{R}^d)$ is 0-Radonifying is exactly R_p.

Let $\Psi = \{\Psi_m^{j,G} \colon (j,G,m) \in \mathbb{W}^d\}$ be an admissible basis for $B^p_{s,w}(\mathbb{R}^d)$ for some $p > 1$ and $(s,w) \in E_p$ and let $(\ell_m^{j,G})_{(j,G,m)\in\mathbb{W}^d}$ be a family of independent identically distributed canonical α-stable processes for some $\alpha \in (0,2)$, i.e. $\rho(\mathrm{d}x) = \mathbb{1}_{x\neq 0} \left|x\right|^{-1-\alpha} \mathrm{d}x$. We consider a cylindrical Lévy process L of the form

$$L(t)f = \sum_{j,G,m} [f, \Psi_m^{j,G}] a_m^{j,G} \ell_m^{j,G} \qquad \text{for all } f \in L^2(\mathbb{R}^d),\ t \geqslant 0. \tag{5.4.21}$$

As in Example 5.4.9, Condition (5.4.16) is satisfied if $\left(a_m^{j,G}\right)_{j,G,m} \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{W}^d)$.

The following Proposition allows us to determine sharp boundaries for each $p > 1$ of the 0-Radonification region R_p of the (s,w) plane in the parameter space defining the weighted Besov spaces.

Proposition 5.4.12. *Let $p > 2$ and $(s,w) \in E_p \backslash R_p$. Then for any $\alpha \in (0,2)$ there exists a sequence $\left(a_m^{j,G}\right)_{j,G,m} \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{W}^d)$ such that L as constructed in (5.4.21) is not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s..*

We first state an intermediate result on the summability of the Besov space weights.

Lemma 5.4.13. *Let $\omega_m^j = \omega_m^j(p,s,w)$ be the wavelet weight constants for $B^p_{s,w}(\mathbb{R}^d)$ for some $p > 0$, $s < \frac{d}{p} - \frac{d}{2}$ and $w < 0$. Then $\left(\omega_m^j\right)_{j,G,m} \in \ell^k(\mathbb{W}^d)$ for some $k > 0$ if and only if*

$$k > \max\left\{-\frac{d}{w}, \frac{2dp}{2d - dp - 2ps}\right\}.$$

Proof. We must assess the convergence of

$$\sum_{j\geqslant 0} 2^{jk(s-\frac{d}{p}+\frac{d}{2})} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^d} (1 + 2^{-2j}\left|m\right|^2)^{\frac{kw}{2}}.$$

We first consider $S_j := \sum_{m\in\mathbb{Z}^d}(1+2^{-2j}|m|^2)^{\frac{kw}{2}}$. We have $S_j < \infty$ for each j if and only if $kw < -d$ (see the proof of [31, Th. 3]) which gives the first term in the maximum above, recalling that $w < 0$. If $S_j < \infty$ for each j, then S_j is asymptotically $\mathcal{O}(2^{jd})$ as $j \to \infty$ according to [31, Th. 3], and thus

$$\sum_{j\geqslant 0} 2^{jk(s-\frac{d}{p}+\frac{d}{2})} \sum_{G\in G^j} S_j = 2^d S_0 + (2^d-1)\sum_{j\geqslant 1} 2^{jk(s-\frac{d}{p}+\frac{d}{2})} S_j,$$

which is finite if and only if $k(s-\frac{d}{p}+\frac{d}{2}) < -d$. As $k > 0$ and $s - \frac{d}{p}+\frac{d}{2} < 0$ this condition is equivalent to $k > \frac{-d}{s-\frac{d}{p}+\frac{d}{2}}$, which completes the proof. □

Proof of Proposition 5.4.12. We note that $\left\|\Psi_m^{j,G}\right\|_{B^p_{s,w}} = \omega_m^j$, where $\omega_m^j = \omega_m^j(p,s,w)$ are the weight constants for $B^p_{s,w}(\mathbb{R}^d)$, and we have $\left(\omega_m^j\right)_{j,G,m} \in \ell^\infty(\mathbb{W}^d)$ as $(s,w) \in E_p$.

For $0 < q < \frac{2\alpha}{2-\alpha}$, we have that the sum

$$\sum_{j,G,m} \left\|a_m^{j,G}\Psi_m^{j,G}\right\|^q_{B^p_{s,w}(\mathbb{R}^d)} = \sum_{j,G,m} \left|a_m^{j,G}\omega_m^j\right|^q$$

is finite for every $\left(a_m^{j,G}\right)_{j,G,m} \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{W}^d)$ if and only if

$$\left(\left|\omega_m^j\right|^q\right)_{j,G,m} \in \left(\ell^{\frac{2\alpha}{q(2-\alpha)}}(\mathbb{W}^d)\right)^* = \ell^{\frac{2\alpha}{2\alpha+\alpha q-2q}}(\mathbb{W}^d). \tag{5.4.22}$$

If we further assume that $p > 2$, we have that $(s,w) \in E_p$ implies $s < \frac{d}{p} - \frac{d}{2}$ and $w \leqslant 0$. For the case $w = 0$, we note that $\sum_{j,G,m}(\omega_m^j)^k = \infty$ for any $k > 0$, and thus there exists $\left(a_m^{j,G}\right)_{j,G,m} \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{W}^d)$ such that $\sum_{j,G,m}\left|a_m^{j,G}\omega_m^j\right|^\alpha = \infty$. We continue to consider the case $w < 0$. By applying Lemma 5.4.13, we see that (5.4.22) is satisfied if and only if

$$\frac{2\alpha q}{2\alpha+\alpha q-2q} > \max\left\{-\frac{d}{w}, \frac{2dp}{2d-dp-2ps}\right\}.$$

As $q < \frac{2\alpha}{2-\alpha}$, we have $2\alpha + \alpha q + 2q > 0$, and thus we have

$$\frac{2\alpha q}{2\alpha + \alpha q - 2q} > -\frac{d}{w} \Leftrightarrow q > \frac{2\alpha d}{2d - \alpha d - 2\alpha w}, \tag{5.4.23}$$

where we note that $2d - \alpha d - 2\alpha w > 0$ as $\alpha < 2$ and $w < 0$. Furthermore, as $s < \frac{d}{p} - \frac{d}{2}$ we have $2d - dp - 2ps > 0$ and so

$$\frac{2\alpha q}{2\alpha + \alpha q - 2q} > \frac{2dp}{2d - dp - 2ps} \Leftrightarrow q > \frac{\alpha dp}{\alpha d - \alpha dp - \alpha ps + dp} \tag{5.4.24}$$

where we have $\alpha d - \alpha dp - \alpha ps + dp > dp(1 - \frac{\alpha}{2}) > 0$. Taking $q = \alpha$, we see that there exists $\left(a_m^{j,G}\right)_{j,G,m} \in \ell^{\frac{2\alpha}{2-\alpha}}(\mathbb{W}^d)$ such that $\sum_{j,G,m} \left|a_m^{j,G}\omega_m^j\right|^\alpha = \infty$ when either $w \geqslant -\frac{d}{2}$ by (5.4.23), or $s \geqslant -d + \frac{d}{p}$ by (5.4.24).

By referring to the conditions shown in Example 5.4.9, it follows that if L is constructed using such a sequence $\left(a_m^{j,G}\right)_{j,G,m}$, the summability index $q_{\min}$ as defined in Theorem 5.4.5 has $q_{\min} > \alpha$, and so L is not induced by a process in $B^p_{s,w}(\mathbb{R}^d)$ P-a.s.. □

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