Addendum to "On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded"

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Abstract. We show that the necessity part of the main result of [**1**] can be easily derived from its predecessor [**2**, Theorem 4.1] and its sufficiency part.

Let the notation be as in [**1**]. The main result of that paper is the following theorem.

Theorem 1 ([**1**, Theorem 1.3]). *Let $p(\cdot) : \mathbb{R}^d \to [1, \infty]$ be a measurable function satisfying $1 < p_- \le p_+ < \infty$. Then $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ if and only if for every $q \in (1, \infty)$, there exists a number $\Theta_{p(\cdot),q} \in (0, 1)$ such that for every $\theta \in (0, \Theta_{p(\cdot),q}]$ the variable exponent $r(\cdot)$ defined by*

$$\frac{1}{p(x)} = \frac{\theta}{q} + \frac{1-\theta}{r(x)}, \quad x \in \mathbb{R}^d, \tag{1}$$

belongs to $\mathcal{B}_M(\mathbb{R}^d)$.

This is an improvement of an earlier result [**2**, Theorem 4.1] that claimed the existence of $q \in (1, \infty)$ and $\theta \in (0, 1)$ such that $r(\cdot)$ defined by (1) belongs to $\mathcal{B}_M(\mathbb{R}^d)$. The proof of [**1**, Theorem 1.3] did not depend on that result. The aim of this addendum is to show that the (more difficult) necessity part of the above theorem can be easily derived from its predecessor [**2**, Theorem 4.1] and its sufficiency part.

Proof. Take any $q \in (1, \infty)$. It follows from [**2**, Theorem 4.1] (see also [**1**, Theorem 1.2]) that there exist numbers $q_1 \in (1, \infty)$ and $\theta_1 \in (0, 1)$ such that the variable exponent $r_1(\cdot)$ defined by

$$\frac{1}{p(x)} = \frac{\theta_1}{q_1} + \frac{1-\theta_1}{r_1(x)}, \quad x \in \mathbb{R}^d, \tag{2}$$

belongs to $\mathcal{B}_M(\mathbb{R}^d)$. Take any positive $\theta_2 < \min\left\{\frac{q}{q_1}, 1 - \frac{1}{q_1}\right\} < 1$ and define q_2 by

$$\frac{1}{q_1} = \frac{\theta_2}{q} + \frac{1-\theta_2}{q_2}\,. \tag{3}$$

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Then

$$\frac{1}{q_2} = \left(\frac{1}{q_1} - \frac{\theta_2}{q}\right)(1-\theta_2)^{-1} > 0,$$
$$\frac{1}{q_2} = \left(\frac{1}{q_1} - \frac{\theta_2}{q}\right)(1-\theta_2)^{-1} < \frac{1}{q_1}(1-\theta_2)^{-1} < \frac{1}{q_1}\left(\frac{1}{q_1}\right)^{-1} = 1.$$

Hence $q_2 \in (1,\infty)$.

Substituting (3) into (2), we get

$$\frac{1}{p(x)} = \frac{\theta_1\theta_2}{q} + \frac{\theta_1(1-\theta_2)}{q_2} + \frac{1-\theta_1}{r_1(x)} = \frac{\theta}{q} + \frac{1-\theta}{r(x)}\,, \tag{4}$$

where $\theta := \theta_1\theta_2$ and

$$\frac{1}{r(x)} := \frac{\theta_1(1-\theta_2)(1-\theta_1\theta_2)^{-1}}{q_2} + \frac{(1-\theta_1)(1-\theta_1\theta_2)^{-1}}{r_1(x)}\,. \tag{5}$$

Clearly, $\theta_1(1-\theta_2)(1-\theta_1\theta_2)^{-1} > 0$ and $(1-\theta_1)(1-\theta_1\theta_2)^{-1} > 0$. Since

$$\frac{\theta_1(1-\theta_2)}{1-\theta_1\theta_2} + \frac{1-\theta_1}{1-\theta_1\theta_2} = \frac{\theta_1-\theta_1\theta_2+1-\theta_1}{1-\theta_1\theta_2} = 1,$$

(5) can be rewritten in the following form

$$\frac{1}{r(x)} = \frac{\theta_0}{q_2} + \frac{1-\theta_0}{r_1(x)}\,, \qquad \theta_0 := \theta_1(1-\theta_2)(1-\theta_1\theta_2)^{-1} \in (0,1).$$

Since $r_1(\cdot)$ belongs to $\mathcal{B}_M(\mathbb{R}^d)$, it follows from the sufficiency part of the theorem that $r(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$. In view of (4), this completes the proof for any positive $\Theta_{p(\cdot),q} < \theta_1 \min\left\{\frac{q}{q_1}, 1-\frac{1}{q_1}\right\}$. □

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