



King's Research Portal

DOI:

[10.1515/gmj-2023-2002](https://doi.org/10.1515/gmj-2023-2002)

Document Version

Peer reviewed version

[Link to publication record in King's Research Portal](#)

Citation for published version (APA):

Shargorodsky, E., Diening, L., & Karlovych, O. (2023). Addendum to "On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded". *Georgian Mathematical Journal*, 30(2), 211-212. <https://doi.org/10.1515/gmj-2023-2002>

Citing this paper

Please note that where the full-text provided on King's Research Portal is the Author Accepted Manuscript or Post-Print version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version for pagination, volume/issue, and date of publication details. And where the final published version is provided on the Research Portal, if citing you are again advised to check the publisher's website for any subsequent corrections.

General rights

Copyright and moral rights for the publications made accessible in the Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognize and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Research Portal

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Addendum to “On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded”

Lars Diening, Oleksiy Karlovykh, and Eugene Shargorodsky

ABSTRACT. We show that the necessity part of the main result of [1] can be easily derived from its predecessor [2, Theorem 4.1] and its sufficiency part.

Let the notation be as in [1]. The main result of that paper is the following theorem.

THEOREM 1 ([1, Theorem 1.3]). *Let $p(\cdot) : \mathbb{R}^d \rightarrow [1, \infty]$ be a measurable function satisfying $1 < p_- \leq p_+ < \infty$. Then $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ if and only if for every $q \in (1, \infty)$, there exists a number $\Theta_{p(\cdot), q} \in (0, 1)$ such that for every $\theta \in (0, \Theta_{p(\cdot), q}]$ the variable exponent $r(\cdot)$ defined by*

$$\frac{1}{p(x)} = \frac{\theta}{q} + \frac{1-\theta}{r(x)}, \quad x \in \mathbb{R}^d, \quad (1)$$

belongs to $\mathcal{B}_M(\mathbb{R}^d)$.

This is an improvement of an earlier result [2, Theorem 4.1] that claimed the existence of $q \in (1, \infty)$ and $\theta \in (0, 1)$ such that $r(\cdot)$ defined by (1) belongs to $\mathcal{B}_M(\mathbb{R}^d)$. The proof of [1, Theorem 1.3] did not depend on that result. The aim of this addendum is to show that the (more difficult) necessity part of the above theorem can be easily derived from its predecessor [2, Theorem 4.1] and its sufficiency part.

PROOF. Take any $q \in (1, \infty)$. It follows from [2, Theorem 4.1] (see also [1, Theorem 1.2]) that there exist numbers $q_1 \in (1, \infty)$ and $\theta_1 \in (0, 1)$ such that the variable exponent $r_1(\cdot)$ defined by

$$\frac{1}{p(x)} = \frac{\theta_1}{q_1} + \frac{1-\theta_1}{r_1(x)}, \quad x \in \mathbb{R}^d, \quad (2)$$

belongs to $\mathcal{B}_M(\mathbb{R}^d)$. Take any positive $\theta_2 < \min \left\{ \frac{q}{q_1}, 1 - \frac{1}{q_1} \right\} < 1$ and define q_2 by

$$\frac{1}{q_1} = \frac{\theta_2}{q} + \frac{1-\theta_2}{q_2}. \quad (3)$$

2020 *Mathematics Subject Classification.* 46E30, 42B25.

Key words and phrases. Variable Lebesgue space, Hardy-Littlewood maximal operator.

Then

$$\begin{aligned}\frac{1}{q_2} &= \left(\frac{1}{q_1} - \frac{\theta_2}{q} \right) (1 - \theta_2)^{-1} > 0, \\ \frac{1}{q_2} &= \left(\frac{1}{q_1} - \frac{\theta_2}{q} \right) (1 - \theta_2)^{-1} < \frac{1}{q_1} (1 - \theta_2)^{-1} < \frac{1}{q_1} \left(\frac{1}{q_1} \right)^{-1} = 1.\end{aligned}$$

Hence $q_2 \in (1, \infty)$.

Substituting (3) into (2), we get

$$\frac{1}{p(x)} = \frac{\theta_1 \theta_2}{q} + \frac{\theta_1(1 - \theta_2)}{q_2} + \frac{1 - \theta_1}{r_1(x)} = \frac{\theta}{q} + \frac{1 - \theta}{r(x)}, \quad (4)$$

where $\theta := \theta_1 \theta_2$ and

$$\frac{1}{r(x)} := \frac{\theta_1(1 - \theta_2)(1 - \theta_1 \theta_2)^{-1}}{q_2} + \frac{(1 - \theta_1)(1 - \theta_1 \theta_2)^{-1}}{r_1(x)}. \quad (5)$$

Clearly, $\theta_1(1 - \theta_2)(1 - \theta_1 \theta_2)^{-1} > 0$ and $(1 - \theta_1)(1 - \theta_1 \theta_2)^{-1} > 0$. Since

$$\frac{\theta_1(1 - \theta_2)}{1 - \theta_1 \theta_2} + \frac{1 - \theta_1}{1 - \theta_1 \theta_2} = \frac{\theta_1 - \theta_1 \theta_2 + 1 - \theta_1}{1 - \theta_1 \theta_2} = 1,$$

(5) can be rewritten in the following form

$$\frac{1}{r(x)} = \frac{\theta_0}{q_2} + \frac{1 - \theta_0}{r_1(x)}, \quad \theta_0 := \theta_1(1 - \theta_2)(1 - \theta_1 \theta_2)^{-1} \in (0, 1).$$

Since $r_1(\cdot)$ belongs to $\mathcal{B}_M(\mathbb{R}^d)$, it follows from the sufficiency part of the theorem that $r(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$. In view of (4), this completes the proof for any positive $\Theta_{p(\cdot), q} < \theta_1 \min \left\{ \frac{q}{q_1}, 1 - \frac{1}{q_1} \right\}$. \square

Funding. This work was supported by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P. (Portuguese Foundation for Science and Technology) within the scope of the project UIDB/00297/2020 (Centro de Matemática e Aplicações). Lars Diening was also funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB 1283/2 2021 – 317210226.

References

- [1] L. Diening, O. Karlovykh, and E. Shargorodsky. On interpolation of reflexive variable Lebesgue spaces on which the Hardy–Littlewood maximal operator is bounded. *Georgian Math. J.*, 29(3):347–352, 2022.
- [2] A. Y. Karlovich and I. M. Spitkovsky. Pseudodifferential operators on variable Lebesgue spaces. In *Operator theory, pseudo-differential equations, and mathematical physics*, volume 228 of *Oper. Theory Adv. Appl.*, pages 173–183. Birkhäuser/Springer Basel AG, Basel, 2013.

UNIVERSITÄT BIELEFELD, FAKULTÄT FÜR MATHEMATIK, POSTFACH 10 01 31, D-33501 BIELEFELD, GERMANY

Email address: `lars.diening@uni-bielefeld.de`

CENTRO DE MATEMÁTICA E APLICAÇÕES, DEPARTAMENTO DE MATEMÁTICA, FACULDADE DE CIÊNCIAS E TECNOLOGIA, UNIVERSIDADE NOVA DE LISBOA, QUINTA DA TORRE, 2829-516 CAPARICA, PORTUGAL

Email address: `oyk@fct.unl.pt`

DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON, STRAND, LONDON WC2R 2LS, UNITED KINGDOM AND TECHNISCHE UNIVERSITÄT DRESDEN, FAKULTÄT MATHEMATIK, 01062 DRESDEN, GERMANY

Email address: `eugene.shargorodsky@kcl.ac.uk`