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A Generalization of the Increasing Generalized Failure Rate Unimodality Condition

Stefanos Leonardos[1*], Constandina Koki[2] and Costis Melolidakis[3]

[1*]Department of Informatics, King's College London.
[2]Mathematics Institute, University of Warwick.
[3]Department of Mathematics, National and Kapodistrian University of Athens.

*Corresponding author(s). E-mail(s): stefanos.leonardos@kcl.ac.uk;
Contributing authors: constandina.koki@warwick.ac.uk;
cmelol@math.uoa.gr;

Abstract

Purpose: In this paper, we study unimodality conditions for distributions that describe markets with stochastic demand. Such conditions naturally emerge in the analysis of game-theoretic models of market competition (Cournot games) and supply chain coordination (Stackelberg games). **Methods:** We express the price elasticity of expected demand in terms of the *mean residual life* (MRL) function of the demand distribution and characterize optimal prices or equivalently, points of unitary elasticity, as fixed points of the MRL function. This leads to economic interpretable conditions on the demand distribution under which such fixed points exist and are unique. **Results:** We find that markets with increasing price elasticity of expected demand that eventually become elastic correspond to distributions with *decreasing generalized mean residual life* (DGMRL) and finite second moment. DGMRL distributions strictly generalize the widely used *increasing generalized failure rate* (IGFR) distributions. We further elaborate on the relationship of the two classes, link their limiting behavior at infinity and examine moment and closure properties of DGMRL distributions that are important in economic applications. **Conclusions:** The DGMRL unimodality condition is useful in the analysis of optimal decisions under uncertainty in settings that are not covered by the widely-used IGFR condition; thus, it can be of broader interest to the game-theory and operations research literature.

Keywords: Price elasticity of expected demand, decreasing generalized mean residual life, increasing generalized failure rate, unimodality, fixed points, revenue maximization, stochastic demand

1 Introduction

Game-theoretic models of market competition are predominantly based on the principle of utility maximization by participating agents, competing sellers and buyers. To make meaningful economic predictions through such models, it is often desirable, if not necessary, to ensure analytical tractability of the involved utility and revenue functions [6, 27, 58]. Accordingly, a large strand of the operations research literature is concerned with the study of economically interpretable "unimodality" conditions, i.e., conditions that ensure "well-behaved" utility functions, and, hence, unique equilibrium or optimal strategies for the agents, [12, 15, 28, 34].

When agents' decisions are made under uncertainty, unimodality conditions refer to properties of the probability distribution of the underlying source of uncertainty. The most widely used unimodality condition in this line of research is the *increasing generalized failure* (IGFR) property [32, 56]. A probability distribution with cumulative distribution function F and probability density function f is said to have the IGFR property if its *generalized failure rate* (GFR), $g(x) := \frac{xf(x)}{1-F(x)}$, is non-decreasing in x for all x such that $F(x) < 1$. IGFR distributions include most distributions that are commonly used in economic applications [2, 47].[1]

The IGFR unimodality condition naturally emerges in the context of revenue maximization under stochastic demand [31, 33]. Specifically, in a single item market, when the seller posts a price p and the buyer's reservation price, α, is randomly drawn from a distribution F, then the seller's expected demand is $\mathbb{E}\left(D\left(p \mid \alpha\right)\right) = 1 - F\left(p\right)$ and, hence, its expected revenue is $p(1 - F(p))$. Here, $D\left(p \mid \alpha\right)$ is the demand at price p given that the buyer's realized type is α. In this context, the IGFR condition has a clear economic interpretation. In particular, the GFR function, $\mathrm{g}\left(x\right)$, corresponds to the price elasticity of demand and, hence, the assumptions that $\mathrm{g}\left(x\right)$ is increasing and eventually exceeds 1 capture the economic intuition of increasing and eventually elastic demand. As a result, the seller's optimal price, p^*, coincides with the point of unitary price elasticity, $g(p^*) = 1$, [32].

However, when applied to game-theoretic settings, an important shortcoming of the IGFR unimodality condition is that it concerns the *instantaneous* or *local* behavior of the underlying probability distribution. Thus, when sellers' revenue (utility) maximization requires information about the whole range of the demand distribution, then the IGFR condition may fail to provide a meaningful unimodality condition.

Such problems can be captured by the following common abstraction. A seller is selling to a market with stochastic *demand level*, α, that is realized after the seller sets their price. In this case, α no longer describes the demand for a single item, but rather the demand for multiple items (whole market).

[1]The GFR function was introduced in economic applications by [52], who used it to model income distributions. It was further studied in the same context by [3] and [4] who provided an alternative definition of the IGFR property without requiring the existence of a density.

Accordingly, the seller's expected *revenue function* is given by

$$R(p) = p\mathbb{E}(D(p \mid \alpha)) \tag{1}$$

where the expectation, $\mathbb{E}$, is taken over the distribution of α or, equivalently, over the seller's belief about it. The seller's objective is to determine the optimal price p^* that maximizes $R(p)$. By differentiating $R(p)$, the seller's first order condition can be written as

$$p = -\frac{\mathbb{E}(D(p \mid \alpha))}{\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}(D(p \mid \alpha))} \tag{2}$$

Given that $\varepsilon(p) := -\frac{\mathrm{d}\mathbb{E}(D(p|\alpha))/\mathbb{E}(D(p|\alpha))}{\mathrm{d}p/p}$ is the price elasticity of expected demand [57], the solutions of (2) correspond to the points of unitary price elasticity of expected demand.

However, even if we assume that $D(p \mid \alpha)$ is continuous and naturally non-increasing in p, equation (2) may have a single, multiple, or even no solutions. As argued above, in this setting, the IGFR condition does not directly apply to yield a unimodality condition since the expression in eq. (2) requires information about the whole range of the distribution, i.e., evaluation of the conditional expectation and its derivative, and not only about its local behavior at the current demand level.[2]

The above abstract formulation captures a wide range of game-theoretic models with potentially multiple sellers and buyers that naturally emerge in operations research and economic problems. Such problems involve equilibrium uniqueness in horizontal quantity competition (Cournot games) [12, 28, 29, 34], equilibrium prices in markets with bandwagon effects, i.e., markets in which demand leads to more demand [36], and supply chain coordination in vertical markets (Stackelberg games) with demand uncertainty [15, 35, 38, 44] among others.

Model and Results

Motivated by the shortcomings of the IGFR property to simplify the sellers' pricing problem in these game-theoretic settings, we seek to formulate an alternative unimodality condition on the distribution of the random demand that will yield a unimodal revenue function in equation (1) and, as a result, a unique solution to equation (2).

To achieve this, we first express the first order condition, equation (2), in terms of the *mean residual life* (MRL) function of the underlying demand distribution. The MRL function is defined as $\mathrm{m}(p) := \mathbb{E}(\alpha - p \mid \alpha > p)$ whenever $F(p) < 1$ (cf. equation (3)), see [30, 51]. This allows us to show that solutions

[2] A technical restriction of the IGFR condition is that it is confined to distributions that are defined over connected intervals [32]. This limits the real-life economic applications that can be studied under the IGFR condition, for example by excluding settings in which sellers maintain beliefs (demand distributions) over disjoint intervals that correspond to low, modal, and high (extreme) demand realizations.

of the first order condition of the seller's revenue function are precisely solutions of the fixed point equation $p = \mathrm{m}(p)$ for $p > 0$, cf. Lemma 2. This implies that a sufficient condition for the unimodality of the seller's expected revenue function is that the MRL function of the associated stochastic demand has a unique fixed point.

As a result, our aim is to study fixed points of the MRL function and derive conditions under which such fixed points exist and are unique. To study this problem, we introduce the *generalized mean residual life* (GMRL) function, $\ell(p) := \mathrm{m}(p)/p$, cf. equation (6). In the current context, the GMRL function has an important economic interpretation: it is the inverse of the price elasticity of expected demand, cf. equation (7). It follows that fixed points, $p^* = \mathrm{m}(p^*)$, of the MRL function which maximize the seller's expected revenue correspond to prices with unitary price elasticity, i.e., $\ell(p^*) = 1$. If the expected demand has increasing price elasticity and eventually becomes elastic, then such a fixed point exists and is unique. In terms of the demand distribution this is equivalent to saying that $\ell(p)$ is decreasing, i.e., that it has the *decreasing generalized mean residual life* (DGMRL) property, and that it eventually becomes less than 1.This is the main result of Section 2 which is formulated in Theorem 1.

An immediate implication of Theorem 1 is that markets with increasing price elasticity of expected demand can be modelled via DGMRL distributions. Recall that when demand uncertainty corresponds to the buyer's valuation for a single product unit, increasingly elastic markets are described by distributions with *increasing generalized failure rate* (IGFR), see [33, 56]. This provides a natural motivation to study the relationship between IGFR and DGMRL distributions and compare their properties. In Theorem 2, we provide an alternative proof to the well-known fact (see [4, 24]) that DGMRL distributions generalize the IGFR distributions and establish that the converse is also true if the MRL function is log-convex. A commonly used distribution that is DGMRL but not IGFR is the Birnbaum-Saunders distribution for specific values of its parameters, cf. Example 1. In contrast to IGFR and IFR distributions, we also find that if a distribution is DGMRL, then its logarithmic transformation does not necessarily satisfy the more restrictive *decreasing MRL* (DMRL) property, cf. Example 4.

We next turn to the study of moments of DGMRL distributions. In a result that is similar in flavor to Theorem 2 of [32], we show that the moments of DGMRL distributions with unbounded support are linked to their limiting behavior at infinity (cf. Theorems 3 and 4). Specifically, if the GMRL function tends to $c \geq 0$ as $p \to +\infty$, then for any $n > 0$, its $(n+1)$-th moment is finite if and only if $c < 1/n$. This implies, that markets with increasing and eventually elastic demand, i.e., for which $\ell(p) < 1$ for every p sufficiently large, correspond to DGMRL distributions with finite second moment.

Finally, we examine closure properties of DGMRL and DMRL distributions that are useful in economic modelling, and compare our findings with [47] and [2]. Such properties capture settings in which sellers update their information about the demand distribution or aggregate different demands. In mathematical terms, these updates are expressed via increasing or decreasing transformations and convolutions (Theorem 5, Corollary 1) and scale transformations or truncations, (Theorem 6). We conclude the paper with a discussion of the current results along with open questions in Section 5.

Other related works

Related unimodality and elasticity conditions are studied in [6, 25, 41] and, in a spirit more similar to ours, in [2, 32, 56] and [27]. The DGMRL condition that is analyzed in the current paper, has been first identified as a useful unimodality property in the context of Nash equilibrium uniqueness in horizontal Cournot competition [34] and of Stackelberg equilibrium uniqueness in vertical markets with multiple competing retailers [35, 37].

The MRL and GMRL functions have been studied by [20] and [18] and more recently by [1] in the context of reliability and statistics with scarce references to game-theoretic applications. However, in economics and operations research, the MRL and GMRL functions naturally arise in pricing or inventory problems under demand uncertainty. A non-exhaustive list of the former includes [11, 13, 21, 48, 49] and more recently [9, 10, 42]. Concerning optimal inventory decisions, [43, 53, 54], and references cited therein, study the tail of the distribution of the source of uncertainty, see e.g., [54], Lemma 1 and [53], equation (2). The DGMRL condition leads to a succinct formulation and, in some cases, a refinement of these results.

Finally, under various perspectives, demand uncertainty in supply chains (Stackelberg games) has been studied in [9, 10] for linear demand functions[3] and by [15, 39, 57] and [38] for general distributions (i.e., beyond the linear model), but typically, under the more restrictive IFR and absolute continuity assumptions on the demand distribution. By contrast, the current analysis only requires that F is continuous. This is satisfied as long as the distribution of the random demand is atomless, i.e., as long as there do not exist single points with positive probability, even if the distribution is supported over disjoint intervals. In technical terms, this means that our results do not require F to be absolutely continuous, i.e., to have a density $f = F'$. In fact, our analysis extends even to singular distributions, see [36].

2 The DGMRL Unimodality Condition

2.1 Preliminaries

Recall from Section 1 that we are concerned with solutions of equation (2), $p = -\frac{\mathbb{E}(D(p|\alpha))}{\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}(D(p|\alpha))}$, which describe the first order conditions for the maximization of the revenue function in equation (1), $R(p) = p\mathbb{E}(D(p \mid \alpha))$. In the following

[3] Both papers study the trade-off between generality of technical assumptions on the demand distribution and limitations on the demand curve. Their results offer novel perspectives on the micro foundations of the linear demand model, e.g., as a good approximation of various demand curves, and justify its use in a wider than previously thought spectrum of economic modelling.

exposition, we focus on the particular instantiation of the additive demand model introduced by [45], with the common assumption of linear deterministic component, studied (among others) in [10, 22, 48] and [37]. In Section 2.3, we will show how this analysis readily generalizes to more general demand functions (as e.g., the ones used in [12, 40] and in references cited therein).

Specifically, let $D(p \mid \alpha) = (\alpha - p)_+$, where α denotes the random demand level. We assume that α is a non-negative random variable with continuous cumulative distribution function (cdf) F, tail $\bar{F} := 1 - F$ and finite expectation[4], $\mathbb{E}\alpha < +\infty$. For the support of α, let $L := \sup\{p \geq 0, F(p) = 0\} \geq 0$ and $H := \inf\{p \geq 0 : F(p) = 1\} \leq +\infty$. Using this notation, (2) can be expressed in terms of the *mean residual life* (MRL) function, defined as

$$\mathrm{m}(p) := \begin{cases} \mathbb{E}(\alpha - p \mid \alpha > p) = \dfrac{1}{\bar{F}(p)} \displaystyle\int_p^{+\infty} \bar{F}(u)\,\mathrm{d}u, & \text{if } p < H \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The term MRL stems from the widespread use of this function in reliability theory, see, e.g., [51] or [30]. In the current context, equation (2) can be conveniently expressed in terms of the MRL function $\mathrm{m}(p)$. To see this, we will use that under mild analytical assumptions on $D(p \mid \alpha)$, we have that $\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}(D(p \mid \alpha)) = \mathbb{E}\left(\frac{\partial}{\partial p} D(p \mid \alpha)\right)$, [17]. However, in the specific case that $D(p \mid \alpha) = (\alpha - p)_+$, this can be derived in a straightforward way as shown in Lemma 1.

Lemma 1 *If α is a non-negative random variable with finite expectation $\mathbb{E}\alpha < +\infty$ and continuous distribution function F, then $\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}(\alpha - p)_+ = \mathbb{E}\left(\frac{\partial(\alpha - p)_+}{\partial p}\right) = -\bar{F}(p)$ for any $p > 0$.*

Proof Let $K_h(\alpha) := -\frac{1}{h}\left[(\alpha - p - h)_+ - (\alpha - p)_+\right]$ and take $h > 0$. Then, $K_h(\alpha) = \mathbf{1}_{\{\alpha > p+h\}} + \dfrac{\alpha - p}{h}\mathbf{1}_{\{p < \alpha \leq p+h\}}$ and therefore $\lim_{h \to 0+} K_h(\alpha) = \mathbf{1}_{\{\alpha > p\}}$. Since $0 \leq K_h(\alpha) \leq 1$ for all α, the dominated convergence theorem implies that $\lim_{h \to 0+} \mathbb{E}(K_h(\alpha)) = P(\alpha > p)$. In a similar fashion, one may show that $\lim_{h \to 0-} \mathbb{E}(K_h(\alpha)) = P(\alpha \geq p)$. Since the distribution of α is non-atomic, $P(\alpha > p) = P(\alpha \geq p)$ and hence, $\lim_{h \to 0} \mathbb{E}(K_h(\alpha)) = \bar{F}(p)$. By the definition of $K_h(\alpha)$, it follows that $\lim_{h \to 0} \mathbb{E}(K_h(\alpha)) = -\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}(\alpha - p)_+$, as claimed. □

Note that, if additionally, α is an absolutely continuous random variable with $F' = f$ almost everywhere, for some density function f, one can easily verify that the derivative $\mathrm{m}'(p)$ exists and is given by

$$\mathrm{m}'(p) = \mathrm{h}(p)\,\mathrm{m}(p) - 1 \tag{4}$$

[4]For one of our results, Theorem 1, we will also require that $\mathbb{E}\alpha^2$ is also finite. However, unless stated otherwise, we do not make this assumption.

where $\mathrm{h}(p) = f(p)/\bar{F}(p)$ denotes the *hazard rate function* of α, see e.g., [8]. In any case, using Lemma 1, the following formulation of (2) is now immediate.

Lemma 2 *In the linear demand case, $D(p \mid \alpha) = (\alpha - p)_+$, the seller's first order condition, (2), can be written as*

$$p = \mathrm{m}(p) \tag{5}$$

where $\mathrm{m}(p)$ denotes the MRL function of the demand distribution.

Proof Since $(\alpha - p)_+$ is non-negative, we may write

$$\mathbb{E}(\alpha - p)_+ = \int_0^\infty P\left((\alpha - p)_+ > u\right) \mathrm{d}u = \int_p^\infty \bar{F}(u)\,\mathrm{d}u,$$

for $0 \le p < H$, see [7]. Using (3), we thus, have $\mathbb{E}(\alpha - p)_+ = \mathrm{m}(p)\,\bar{F}(p)$ and (2) takes the form $p = \mathrm{m}(p)$. □

To study this equation, we introduce the *generalized mean residual life* (GMRL) function, $\ell(p) := \mathrm{m}(p)/p$, for $0 < p < H$, cf. (6), which corresponds to the inverse of the price elasticity of expected demand. It follows that prices p^* with unitary price elasticity which maximize the seller's expected revenue, satisfy $\ell(p^*) = 1$ or equivalently $p^* = \mathrm{m}(p^*)$. Under the assumption that F is absolutely continuous, with $F' = f$, (2) takes the form $p\mathrm{h}(p) = 1$, for $p < H$, where $\mathrm{h}(p) := f(p)/\bar{F}(p)$ is the hazard rate function of α.

2.2 Unimodality of the Seller's Revenue Function

Our goal in this Section is to establish necessary and sufficient conditions for the unimodality of the seller's revenue function. This is the statement of Theorem 1, which crucially relies on the expression of the price elasticity of expected demand via the *generalized mean residual life (GMRL)* function which is derived next.

From the seller's revenue maximization perspective, we are interested in conditions for the existence and uniqueness of solutions of (5). Based on the previous analysis in Section 2.1, solutions of (2) are precisely solutions of the fixed point equation $p = \mathrm{m}(p)$ for $p > 0$. Thus, our aim will be to study fixed points of the MRL function. To study this problem, we define the *generalized mean residual life (GMRL)* function

$$\ell(p) := \frac{\mathrm{m}(p)}{p} = \frac{1}{p\bar{F}(p)} \int_p^{+\infty} \bar{F}(u)\,\mathrm{d}u \tag{6}$$

for all $0 < p < H$. We say that a random variable D has the *DGMRL property*, if $\ell(p)$ is non-increasing in p for $0 < p < H$. While the MRL function at a point p expresses the expected additional demand given that current demand has reached (or exceeded) the threshold p, the GMRL function expresses the corresponding expected additional demand as a percentage of the current demand. From an economic perspective, $\ell(p)$ has an appealing interpretation, since it is the inverse of the price elasticity of the *expected* demand, $\varepsilon(p) := -p \cdot \frac{\mathrm{d}}{\mathrm{d}p}\mathbb{E}(D(p \mid \alpha))/\mathbb{E}(D(p \mid \alpha))$,

$$\ell(p) = \frac{\mathrm{m}(p)}{p} = \left(\frac{\bar{F}(p)}{\mathrm{m}(p)\bar{F}(p)} \cdot p\right)^{-1} = \varepsilon(p)^{-1} \tag{7}$$

Thus, demand distributions with the DGMRL property precisely capture markets of goods with increasing price elasticity of expected demand. Moreover, together with (5), (7) implies that the seller's revenue is maximized at prices p^* with unitary price elasticity of *expected* demand. In non-trivial, realistic problems, demand eventually becomes elastic, see also [32]. Accordingly, let $p_1 := \sup\{p \geq 0 : \ell(p) \geq 1\}$ and assume that $p_1 < +\infty$ or equivalently that the price elasticity of expected demand, eventually becomes greater than 1. For a continuous distribution F with finite expectation $\mathbb{E}\alpha$, such that $F(0) = 0$, we have that $\mathrm{m}(0) = \mathbb{E}\alpha > 0$ and hence, $p_1 > 0$. Combining the above, we obtain necessary and sufficient conditions for the unimodality of the seller's revenue function $R(p)$, or equivalently for the existence and uniqueness of a solution of (5).

Theorem 1 *Suppose that α is a random variable with continuous distribution F, $F(0) = 0$, and finite expectation, such that $p_1 < +\infty$. The seller's revenue function $R(p) = p\mathbb{E}(\alpha - p)_+$ is maximized at all points p^* with unitary elasticity of expected demand, i.e., at all points p^* that satisfy $\ell(p) = 1$ or equivalently, $p^* = \mathrm{m}(p^*)$. If $\ell(p)$ is strictly decreasing, then a fixed point p^* exists and is unique.*

Proof To establish the first part, it remains to check that any point satisfying (5) corresponds to a maximum under the assumption that $\ell(p)$ is strictly decreasing. Clearly, $\ell(p)$ is continuous and since $\mathrm{m}(0) = \mathbb{E}\alpha < +\infty$, we have that $\lim_{p\to 0^+} \ell(p) = +\infty$. Hence, for values of p close to 0, demand is inelastic and the seller's revenue increases as prices increase. However, the limiting behavior of $\ell(p)$ as p approaches H from the left may vary, depending on whether H is finite or not. If H is finite, i.e., if the support of α is bounded, then $\lim_{p\to H-} \ell(p) = 0$. Hence, in this case, demand eventually becomes elastic and a critical point $p^* \in (0, H)$ that maximizes $R(p)$ always exists. The assumption that $\ell(p)$ is strictly decreasing, establishes the uniqueness of p^*. If $H = +\infty$, then an optimal solution p^* may not exist because the limiting behavior of $\mathrm{m}(p)$, as $p \to +\infty$, may vary, see e.g., the Pareto distribution in Example 3. However, under the assumption that $\ell(p)$ is strictly decreasing and that $p_1 < +\infty$, such a critical p^* exists and is unique. □

Remark 1 The assumption $p_1 < +\infty$ is equivalent to the condition that the distribution of α has finite second moment. Indeed, as we show in Theorem 3, if the support of α is unbounded, and $\ell(p)$ is decreasing, then, $\lim_{p\to+\infty} \ell(p) < 1$ if and only if $\mathbb{E}\alpha^2$ is finite. The assumption of strict monotonicity eliminates intervals with $\mathrm{m}(p) = p$, in which multiple consecutive solutions occur. However, it may be relaxed to weak monotonicity without significant loss of generality. This relies on the explicit characterization of distributions with MRL functions that contain linear segments which is given in Proposition 10 of [20]. Namely, $\mathrm{m}\,(p) = p$ on some interval $J = [a, b] \subseteq [L, H]$ if and only if $\bar{F}\,(p)\,p^2 = \bar{F}\,(a)\,a^2$ for all $p \in J$. If J is unbounded, this implies that α has the Pareto distribution on J with shape parameter 2. In this case, $\mathbb{E}\alpha^2 = +\infty$, see Example 3, which is precluded by the requirement that $p_1 < +\infty$. Hence, to replace strict by weak monotonicity, it suffices to exclude distributions that contain intervals $J = [a, b] \subseteq [L, H]$ with $b < +\infty$ in their support, for which $\bar{F}\,(p)\,p^2 = \bar{F}\,(a)\,a^2$ for all $p \in J$.

2.3 General Demand Functions

While in the above presentation we used the linear demand model, the results readily extend to more general demand forms. In particular, consider the functional form (which includes the linear or constantly elastic cases)

$$p(Q) = \begin{cases} \alpha - \beta Q^{n+1}, & \text{if } n \neq -1, \\ \alpha - \beta \log Q, & \text{if } n = -1, \end{cases}$$

that is used in [12, 40] and references cited therein. In this formulation, the quantity, Q, is the free variable (and thus, the interpretation of the problem changes accordingly). These functions constitute instantiations of the more general form

$$p(Q) = \alpha - g(Q),$$

where $g(Q)$ is an arbitrary function (a similar formulation holds if we choose price as the free variable, but in this part we stick without loss of generality with Q as the variable of choice to be in line with the referenced papers). In this case, using again (3), it is straightforward to verify that the seller's revenue function becomes $R(Q) = Q \cdot f(Q) = Qm(g(Q))\bar{F}(g(Q))$. Thus, assuming that α is absolutely continuous (to simplify the calculations), the derivative of the seller's revenue is

$$\begin{aligned} \frac{d}{dQ} Q\mathrm{m}(g(Q))\bar{F}(g(Q)) &= -Q\mathrm{m}(g(Q))f(g(Q))g'(Q) + \mathrm{m}(g(Q))\bar{F}(g(Q)) + \\ &\quad + Q\mathrm{m}'(g(Q))g'(Q)\bar{F}(Q) \\ &= -Q\mathrm{m}(g(Q))f(g(Q))g'(Q) + \mathrm{m}(g(Q))\bar{F}(g(Q)) + \\ &\quad + Q\left[\mathrm{m}(g(Q))\mathrm{h}(g(Q)) - 1\right]g'(Q)\bar{F}(Q) \\ &= \bar{F}(g(Q))\left[\mathrm{m}(g(Q)) - Qg'(Q)\right] \end{aligned}$$

where we used equation (4) for the second equality. This leads to the first order condition

$$Qg'(Q) = \mathrm{m}(g(Q)),$$

that can be viewed as *generalized* fixed point equation. For instance, if $g(Q) = Q$, we recover the previous analysis (with quantity instead of price as the decision variable) of the linear model. Similarly, if $g(Q) = bQ^n$ for $n > 0$ as in [12, 40], then we derive the equation $nbQ^n = \mathrm{m}(bQ^n)$ which after a variable transformation $bQ^n \to x$ leads to the (scaled) fixed point equation $x = \frac{1}{n}\mathrm{m}(x)$.

Extension of the current results to more general demand function remains an interesting open questions. In this direction, the framework of [50] provides a promising starting point to study, for instance, transformations related to the inverse function, $g^{-1}(Q)$.

3 Properties of DGMRL Distributions

Section 2 motivates the study of DGMRL distributions as class of distributions that arise naturally in a seller's pricing optimization problem when the seller is facing a linear stochastic demand. It turns out, that the class of DGMRL distributions is general enough to include as a subclass the IFR, DMRL and IGFR distributions that are widely used in revenue management applications. This statement along with several analytical and closure properties of the DGMRL distributions are established next.

For the remaining part, let $X \sim F$ be a non-negative random variable, with support in L, H as in Section 1, continuous distributions function F, tail $\bar{F} := 1 - F$ and finite expectation $\mathbb{E}X < +\infty$. Let $\mathrm{m}\,(x)$ denote the MRL function of X, as defined in (3), and $\ell\,(x)$ denote the GMRL function of X, as defined in (6). We say that distribution X has the *decreasing MRL* (DMRL) property, or simply that X is DMRL, if $\mathrm{m}\,(p)$ is non-increasing in p for $p < H$.

3.1 The DGMRL and IGFR Classes of Distributions

To compare the IGFR and DGMRL classes, we restrict attention to non-negative, absolutely continuous random variables. We, then have

Theorem 2 *If X is a non-negative, absolutely continuous random variable, with $\mathbb{E}X < +\infty$, then*
(i) If X is IGFR, then X is DGMRL.
(ii) If X is DGMRL and $\mathrm{m}\,(x)$ *is* log-*convex, then X is IGFR.*

Part (i) of Theorem 3, has already been stated by [4] and [24]. To derive an alternative proof of part (i) and to establish part (ii) of Theorem 2, we will use the notions of stochastic orderings, see [51] or [5]. Let X_i be random variables with distribution, failure rate and MRL functions denoted by F_i, h_i and m_i respectively, for $i = 1, 2$. X_1 is said to be smaller than X_2 in the *usual stochastic order*, denoted by $X_1 \preceq_{\mathrm{st}} X_2$, if $F_2\,(x) \le F_1\,(x)$ for all $x \in \mathbb{R}$. Similarly, X_1 is said to be smaller than X_2 in the *failure or hazard rate order*, denoted by $X_1 \preceq_{\mathrm{hr}} X_2$, if $h_2\,(x) \le h_1\,(x)$ for all $x \in \mathbb{R}$. Finally, X_1 is said to be smaller than X_2 in the *mean residual life order*, denoted by $X_1 \preceq_{\mathrm{mrl}} X_2$, if $\mathrm{m}_1\,(x) \le \mathrm{m}_2\,(x)$ for all $x \in \mathbb{R}$.

Proof Proof of Theorem 2. By Theorem 1 of [32], X is IGFR if and only if $X \preceq_{\mathrm{hr}} \lambda X$ for all $\lambda \geq 1$. By Theorem 2.A.1 of [51], if $X \preceq_{\mathrm{hr}} \lambda X$, then $X \preceq_{\mathrm{mrl}} \lambda X$. Now, $\mathrm{m}_{\lambda X}(x) = \lambda \cdot \mathrm{m}(x/\lambda)$. Hence, for $\lambda \geq 1$, $X \preceq_{\mathrm{mrl}} \lambda X$ is by definition equivalent to $\mathrm{m}(x) \leq \mathrm{m}_{\lambda X}(x)$ for all $x > 0$, which in turn is equivalent to $\ell(x) \leq \ell(x/\lambda)$ for all $x > 0$. As this holds for any $\lambda \geq 1$, the last inequality is equivalent to $\ell(x)$ being decreasing, i.e., to X being DGMRL.

To prove the second part of the Theorem, it suffices to show that $\mathrm{m}(x)/\mathrm{m}_{\lambda X}(x)$ is increasing in x, for $0 < x < H$ and all $\lambda \geq 1$. Indeed, if this is the case, Theorem 2.A.2 of [51] implies that $X \preceq_{\mathrm{mrl}} \lambda X$ for all $\lambda \geq 1$ is equivalent to $X \preceq_{\mathrm{hr}} \lambda X$ for all $\lambda \geq 1$, which as we have seen, is equivalent to X being IGFR. Since $\mathrm{m}_{\lambda X}(x) = \lambda \mathrm{m}(x/\lambda)$ and $\mathrm{m}(x)$ is differentiable, $\mathrm{m}(x)/\mathrm{m}_{\lambda X}(x)$ is increasing in $x \in (0, H)$ for all $\lambda \geq 1$ if and only if $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{m}(x)}{\lambda \mathrm{m}(x/\lambda)}\right) \geq 0$, for all $\lambda \geq 1$, i.e., if and only if $\frac{\mathrm{m}'(x)}{\mathrm{m}(x)} \geq \frac{\mathrm{m}'(x/\lambda)}{\lambda \mathrm{m}(x/\lambda)}$, for all $\lambda \geq 1$. This is equivalent to $\frac{\mathrm{d}}{\mathrm{d}x}\log(\mathrm{m}(x))$ being increasing in x, i.e., to $\mathrm{m}(x)$ being log-convex. □

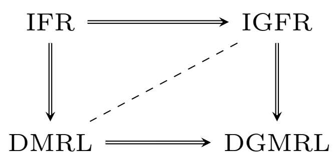

Fig. 1 Relationship between the IFR, IGFR, DMRL and DGMRL classes of distributions. The IFR property implies the IGFR and DMRL properties, which in turn imply the DGMRL property. All inclusions are proper. The DMRL property neither implies nor is implied by the IGFR property. Finally, Theorem 2-(ii) provides a condition under which a DGMRL distribution is also IGFR.

Based on the proof of Theorem 2, a DGMRL random variable X is not IGFR if there exists $\lambda \geq 1$ such that X is smaller than λX in the mean residual life order but not in the hazard rate order. Although more involved, the present derivation of part (i) utilizes the characterization of both IGFR and DGMRL in terms of stochastic orderings – $\preceq_{\mathrm{hr}}$ for IGFR and $\preceq_{\mathrm{mrl}}$ for DGMRL – and thus, points to the sufficiency condition of part (ii). Specifically, in view of the proof of part (i), the proof of part (ii) reduces to finding conditions, under which, the mean residual life order implies the hazard rate order. Such conditions are provided in Theorem 2.A.2 of [51]. However, as [51] already point out, the condition of log-convexity is restrictive and indeed there are many distributions with log-concave MRL function that are nevertheless IGFR. Hence, it would be of interest to obtain part (ii) of Theorem 2 under a more general condition.

Conceptually, the GFR and GMRL functions differ in the same sense that the FR and MRL functions do. Namely, while the GFR function at a point x provides information about the instantaneous behavior of the distribution just after point x, the GMRL function provides information about the entire expected behavior of the distribution after point x. As the IGFR is trivially implied by the IFR property, the same holds for the DGMRL and DMRL properties. The relationships between all four classes are shown in Figure 1. The IGFR property does not imply, nor is implied by the DMRL property. However, the former seems more inclusive than the latter, cf. [1], Table 3 and [2], Table 1. Conversely, DMRL distributions that are not IGFR can be

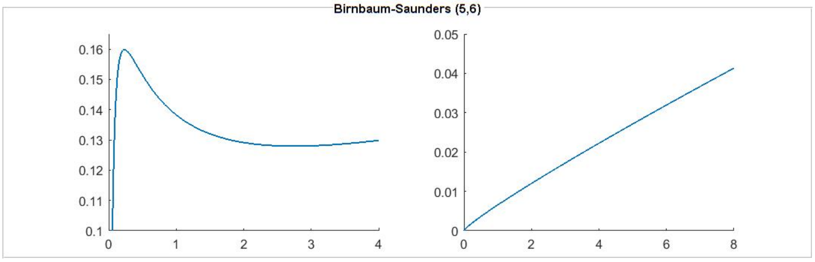

Fig. 2 Birnbaum-Saunders distribution for $a = 6, \beta = 5$. The GFR function (left panel) is not monotone increasing in contrast to the price elasticity of expected demand (right panel) which is the inverse of the GMRL function.

constructed by considering random variables without a connected support. This relies on the observation that if a distribution X is IGFR, then its support must be an interval, see [33]. However, it remains unclear whether or not the DMRL property implies the IGFR property when attention is restricted to absolutely continuous random variables with connected support. A commonly used distribution that is DGMRL but not IGFR is the Birnbaum-Saunders distribution.

Example 1 (Birnbaum-Saunders distribution) The Birnbaum-Saunders (BS) distribution, which is extensively used in reliability applications, see [23], provides an example of a random variable which is DGMRL but not IGFR for certain values of its parameters. The pdf of X is

$$f(x) = \frac{1}{2ax\sqrt{2\pi}} \left(\sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}} \right) \exp\left(-\frac{1}{2a^2} \left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}} \right)^2 \right), \qquad \text{for } x > 0,$$

where $a > 0$ is the shape parameter and $\beta > 0$ is the scale parameter. In particular, let $X \sim$ BS with parameters $a = 6$ and $\beta = 5$. Using the formula for $f(x)$, Figure 2 can be obtained numerically. Implementing the BS distribution for different β and γ, shows that, unlike other distribution families, as e.g., the Gamma or Beta, the shapes of the GFR and GMRL functions of the BS distribution depend largely on the exact values of its parameters. For different values of its parameters, the BS distribution has either increasing or bathtub-shaped (first decreasing and then increasing) MRL function, [55].

Mixtures of DGMRL distributions over disjoint intervals

As mentioned above, IGFR random variables must have a connected support. Under certain circumstances, this property poses restrictive limitations in economic modelling. For instance, when a seller is uncertain about the exact support of the demand, their belief can be naturally expressed as a mixture of two or more distributions over disjoint intervals. In this case, even if each individual distribution is IGFR, their mixture is certainly not. In this respect, the DGMRL property is more promising since mixtures of IGFR distributions may still be DGMRL. However, in general, different mixtures of IGFR distributions may or may not be DGMRL even if the only difference is in the mixing weights. Such a case is illustrated in Example 2.

Example 2 (Mixture of Uniform distributions on disjoint intervals) Let $U(L, H)$ denotes the uniform distribution on (L, H) and let $X_1 \sim U(1, 2)$ with cdf F_1 and $X_2 \sim U(3, 4)$ with cdf F_2. Further, let X_λ with cdf $F_\lambda = \lambda F_1 + (1-\lambda) F_2$ for $\lambda \in (0, 1)$ describe the seller's belief about the demand. Both X_1, X_2 are IFR, hence IGFR, DMRL and DGMRL.

The support of X_λ is not connected, hence X_λ is not IGFR for $0 < \lambda < 1$. Contrarily, the GMRL ℓ_λ of X_λ is given by

$$\ell_\lambda(x) = \begin{cases} \lambda \ell_1(x) + (1-\lambda)\ell_2(x), & 0 < x \le 1 \\ \dfrac{\lambda(2-x)\ell_1(x) + (1-\lambda)\ell_2(x)}{\lambda(2-x) + (1-\lambda)}, & 1 \le x \le 2 \\ \ell_2(x), & 2 \le x < 4 \end{cases}$$

Hence, $\ell_\lambda(x)$ is decreasing for $x \notin [1, 2]$. For $x \in [1, 2]$, a direct substitution shows that $\ell_{1/4}(x)$ is decreasing over $[1, 2]$, hence $X_{1/4}$ is DGMRL, while $\ell_{3/4}(x)$ is first decreasing and then increasing, as shown in Figure 3 and hence $X_{3/4}$ is not DGMRL.

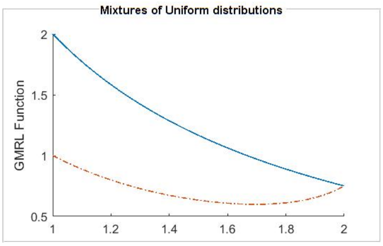

Fig. 3 The GMRL function of X_λ for $\lambda = 1/4$ (solid) and $\lambda = 3/4$ (dotted).

The derivation of necessary and/or sufficient conditions under which such mixtures retain the DGMRL property, i.e., the derivation of closure properties under mixtures of the DGMRL class of distributions, remains an interesting open question. In a related study that may prove useful in this direction, [46] confirm that mixtures of standard IFR (and hence DGMRL) distributions, e.g., exponential, may not be DGMRL (it may be bathtub-shaped), and derive sufficient conditions under which asymptotical monotonicity is retained.

3.2 Limiting Behavior & Moments of DGMRL Distributions

The moments of DGMRL distributions with unbounded support are closely linked with the limiting behavior of the GMRL function $\ell(x)$, as $x \to +\infty$.

Theorem 3 *Let X be a non-negative DGMRL random variable with $\mathbb{E}X < +\infty$ and $\lim_{x \to +\infty} \ell(x) = c$. If $\beta > 0$, then $c < \frac{1}{\beta}$, if and only if $\mathbb{E}X^{\beta+1} < +\infty$. In particular, $c = 0$ if and only if $\mathbb{E}X^{\beta+1} < +\infty$ for every $\beta > 0$.*

For the proof of Theorem 3, we utilize the theory of regularly varying distributions, see [16, 20] and [19]. First, observe that if X is a non-negative random variable, then by a simple change of variable, one may rewrite[5] $\ell(x)$ in (6) as $\ell(x) = \int_1^{+\infty} \frac{\bar{F}(ux)}{\bar{F}(x)}\,\mathrm{d}u$. Since we have assumed that $\mathbb{E}X < +\infty$, $\ell(x)$ is well defined. We say that $\bar{F}$ is *regularly varying at infinity* with exponent $\rho \in \mathbb{R}$, if $\bar{F}(ux)/\bar{F}(x) \to u^\rho$ for all $u \geq 0$ as $x \to +\infty$. In this case, we write $\bar{F} \in \mathcal{RV}(\rho)$. If $\bar{F}(ux)/\bar{F}(x) \to \infty$ for $0 < u < 1$ and $\bar{F}(ux)/\bar{F}(x) \to 0$ for $u > 1$ as $x \to +\infty$, then we say that $\bar{F}$ is *rapidly varying at infinity with exponent* $-\infty$ or simply that $\bar{F}$ is *rapidly varying*, in symbols $\bar{F} \in \mathcal{RV}(-\infty)$. If $\bar{F} \in \mathcal{RV}(\rho)$ with $\rho \in \mathbb{R}$, then we can write $\bar{F}$ as $\bar{F}(u) = u^\rho Z(u)$, where Z is regularly varying at infinity with exponent $\rho = 0$. In this case, we say that Z is *slowly varying at infinity* and write $Z \in \mathcal{SV}$. [16], see Section VIII.8, shows that if $Z(u) > 0$ and $Z \in \mathcal{SV}$, then the integral $\int_0^{+\infty} u^\rho Z(u)\,\mathrm{d}u$ is convergent for $\rho < -1$ and divergent for $\rho > -1$. We are now ready to prove Theorem 3.

Proof Proof of Theorem 3. Let $c > 0$. Then, the convergence of $\ell(x)$ to some $c \in (0, +\infty)$ is equivalent to $\bar{F}$ being regularly varying at infinity with exponent $-1 - \frac{1}{c}$, in symbols $\bar{F} \in \mathcal{RV}\left(-1 - \frac{1}{c}\right)$, see Proposition 11(b) of [20]. Hence, there exists a function $Z \in \mathcal{SV}$, such that $\bar{F}(x) = x^{-1-\frac{1}{c}} Z(x)$. Since X is non-negative, this implies that for any $\beta > 0$, we may write $\mathbb{E}X^{\beta+1} = \int_0^{+\infty} (\beta+1)\, u^\beta \bar{F}(u)\,\mathrm{d}u = (\beta+1) \int_0^{+\infty} u^{\beta-1-\frac{1}{c}} Z(u)\,\mathrm{d}u$. Using [16], the latter integral converges for $\beta < \frac{1}{c}$ and diverges for $\beta > \frac{1}{c}$. For $c = \frac{1}{\beta}$, we employ the approach of [32] and compare X with a random variable $Y \sim \mathrm{Pareto}(1, \beta+1)$, where 1 is the location parameter and $\beta+1$ the shape parameter. In this case $\mathrm{m}_Y(x) = x/\beta$ and $\mathbb{E}Y^{\beta+1} = +\infty$, which may be used to conclude that $\mathbb{E}X^{\beta+1} = +\infty$ as well. To see this, observe that since $\ell(x)$ is decreasing to $1/\beta$ by assumption, we have that $\mathrm{m}_X(x) \geq x/\beta = \mathrm{m}_Y(x)$ and hence $Y \preceq_{\mathrm{mrl}} X$. Moreover, $\frac{\mathrm{m}_Y(x)}{\mathrm{m}_X(x)} = \frac{1}{\beta} \cdot \frac{1}{\ell(x)}$, which by assumption increases in x for all $x > 0$. This implies that Y is smaller than X in the hazard rate order, see Theorem 2.A.2 of [51], and hence also in the usual stochastic order, i.e., $Y \preceq_{\mathrm{st}} X$. Hence, $\mathbb{E}X^{\beta+1} \geq \mathbb{E}Y^{\beta+1} = +\infty$.

If $c = 0$, then $\bar{F}(x)$ is rapidly varying with exponent $-\infty$, i.e., $\bar{F} \in \mathcal{RV}(-\infty)$, see Proposition 11(c) of [20]. It is known, see [14], that all moments of rapidly varying distributions are finite. Conversely, if $\mathbb{E}X^{\beta+1} < +\infty$ for every $\beta > 0$, then it is a straightforward implication that $c = 0$. □

Theorem 3 is comparable to Theorem 2 of [32], which states an analogous result for IGFR distributions. Theorem 4 establishes the link between the two.

Theorem 4 *Let X be an absolutely continuous, non-negative random variable with unbounded support and $\mathbb{E}X < +\infty$. If $\lim_{x\to+\infty} \mathrm{g}(x)$ exists and is equal to κ with $\kappa > 1$ (possibly infinite), then*

$$\lim_{x\to+\infty} \ell(x) = 1/(\kappa - 1) \tag{8}$$

[5] By differentiating this expression, provided that $F' = f$ almost everywhere, one obtains an alternative straightforward proof that IGFR implies DGMRL.

Proof Since $\mathbb{E}X < +\infty$, both $\lim_{x\to+\infty}\int_x^{+\infty}\bar{F}(u)\,\mathrm{d}u$ and $\lim_{x\to+\infty}x\bar{F}(x)$ are equal to 0. To compute $\lim_{x\to+\infty}\ell(x)$ we use L'Hôpital's rule. We have that $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{+\infty}\bar{F}(u)\,\mathrm{d}u = -\bar{F}(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\bar{F}(x)\right) = \bar{F}(x)(1-\mathrm{g}(x))$. Hence, under the assumption that $\lim_{x\to+\infty}\mathrm{g}(x) = \kappa$, we conclude that

$$\lim_{x\to+\infty}\ell(x) = \lim_{x\to+\infty}\frac{1}{\mathrm{g}(x)-1} = \frac{1}{\kappa-1}.$$

□

The inverse relationship in the limiting behavior of $\ell(x)$ and $\mathrm{g}(x)$ in (8) is similar in flavor to equation (2.1) of [8]. In the case that $\kappa < +\infty$, Theorem 2 of [32] restricted to $n > 1$, follows from Theorems 2 and 3, and equation (8). This approach also covers the case $n = \kappa$, which is not considered in the proof by [32]. As for IGFR distributions, the Pareto distribution provides a limiting case between decreasing and increasing GMRL distributions, since it is the unique distribution with constant GMRL function.

Example 3 (Pareto distribution) Let X be Pareto distributed with pdf $f(x) = kL^kx^{-(k+1)}\mathbf{1}_{\{L\le x\}}$, and parameters $L > 0$ and $k > 1$ (for $0 < k \le 1$ we get $\mathbb{E}X = +\infty$, which contradicts the basic assumptions of our model). To simplify, let $L = 1$, so that $f(x) = kx^{-k-1}\mathbf{1}_{\{1\le x\}}$, $\bar{F}(x) = x^{-k}\mathbf{1}_{\{1\le x\}}$, and $\mathbb{E}X = \frac{k}{k-1}$. The mean residual life of X is given by $\mathrm{m}(x) = \frac{x}{k-1} + \frac{k}{k-1}(1-x)_+$ and, hence, is decreasing for $x < 1$ and increasing for $x \ge 1$. However, the GMRL function $\ell(x) = \frac{1}{k-1}$ is decreasing for $0 < x < 1$ and constant for $x \ge 1$, hence, X is DGMRL. Similarly, for $1 \le x$ the failure (hazard) rate $\mathrm{h}(x) = kx^{-1}$ is decreasing, but the generalized failure rate $\mathrm{g}(x) = k$ is constant and, hence, X is IGFR. In this case, the seller's payoff function, (1), becomes

$$R(x) = x\mathrm{m}(x)\bar{F}(x) = \begin{cases} x\left(\dfrac{k}{k-1} - x\right), & \text{if } 0 \le x < 1 \\ \dfrac{x^{2-k}}{(k-1)}, & \text{if } x \ge 1, \end{cases}$$

which diverges as $x \to +\infty$, for $k < 2$ and remains constant for $k = 2$. In particular, for $k \le 2$, the second moment of X is infinite, i.e., $\mathbb{E}X^2 = +\infty$, and also $\lim_{x\to+\infty}\ell(x) = \frac{1}{k-1} \ge 1$ and $\lim_{x\to+\infty}\mathrm{g}(x) = k \le 2$, which agrees with Theorem 3. On the other hand, for $k > 2$, there exists a unique fixed point $x^* = \frac{k}{2(k-1)}$, as expected.

Example 3 provides a motivation to further explore the relationship between the properties of a DGMRL distribution X and its logarithmic transformation, $\log(X)$. Specifically, similar to X being IGFR if and only if $\log(X)$ is IFR, cf. Theorem 1 of [32], it is natural to ask whether X is DGMRL if and only $\log(X)$ is DMRL. The intuition behind this conjecture is straightforward: the Pareto distribution is the boundary case for the DGMRL class (has constant GMRL function), and $\log(X)$, which, in this case, is the exponential distribution is the boundary case for the DMRL class, i.e., $\log(X)$ has constant MRL function. Moreover, a property similar to the one stated in

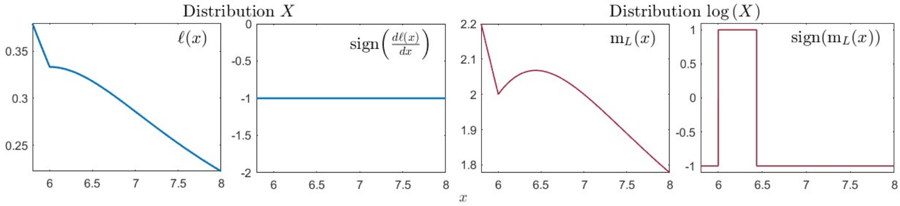

Fig. 4 The GMRL function, $\ell(x)$, of the distribution X of Example 4 (first panel) and the MRL function, $\mathrm{m}_L(x)$, of the transformed distribution $\log(X)$ (third panel). The second and fourth panels show the sign of the derivatives of $\ell(x)$ and $m_L(x)$ respectively. It can be seen that X is DGMRL, but $\log X$ is not DMRL.

Theorem 2 holds for IFR and DMRL distributions: IFR implies DMRL, and DMRL together with concavity of the MRL function imply the IFR property, cf. [51]. However, contrary to the above intuition, the following example shows that this is not true.

Example 4 (X DGMRL does not imply that $\log(X)$ is DMRL) Consider a non-negative random variable X. Let $F_L(x) := P(\log(X) \le x) = F(e^x)$ and $f_L(x) := e^x f(e^x)$ denote the cumulative distribution and probability density function, respectively, of the transformed random variable $\log(X)$. Then, the MRL function, $\mathrm{m}_L(x)$, of $\log(X)$ is given by

$$\begin{aligned}\mathrm{m}_L(x) &= \frac{1}{\bar{F}_L(x)}\int_x^{+\infty} \bar{F}_L(u)\,\mathrm{d}u = \frac{1}{\bar{F}(e^x)}\int_x^{+\infty} \bar{F}\left(e^u\right)\mathrm{d}u \\ &= \frac{1}{\bar{F}(e^x)}\int_{e^x}^{+\infty} \frac{1}{u}\bar{F}(u)\,\mathrm{d}u\,. \end{aligned} \tag{9}$$

Since $1/u$ is strictly decreasing, equation (9) trivially suggests that $\mathrm{m}_L(x) \le \ell(x)$ for any $x > 0$. However, this does not provide a way to express $\mathrm{m}_L(x)$ in terms of $\ell(x)$.

By contrast, to find a counterexample for the statement that X *is DGMRL if and only if* X *is DMRL*, it suffices to find a distribution function, F, such that $\ell(x) = \frac{1}{x\bar{F}(x)}\int_x^{+\infty}\bar{F}(u)\,\mathrm{d}u$ is decreasing in $x > 0$, but $\mathrm{m}_L(x) = \frac{1}{\bar{F}(x)}\int_x^{+\infty}\frac{1}{u}\bar{F}(u)\,\mathrm{d}u$ is not. To see that such a function exists, let

$$s(x) := \left(1 - \frac{1}{3}x + \frac{1}{33}x^2\right)e^{-x}, \quad \text{for } x \ge 6,$$

and let $\bar{F}(x) := s(x)/s(6)$ for $x \ge 6$, and $F(x) = 1$ for $x \in [0, 6)$. It holds that $\bar{F}(0) = 1$, $\lim_{x\to+\infty} F(x) = 0$ and $\int_0^\infty \bar{F}(u)\,\mathrm{d}u = 8$ which imply that $\bar{F}$ is a valid survival function of a continuous, non-negative random variable with finite expectation. The functions $\ell(x)$ and $\mathrm{m}_L(x)$ are shown in Figure 4. It can be seen that X is DGMRL, however, $\log(X)$ is not DMRL. The monotonicity of both functions, $\ell(x)$ and $\mathrm{m}_L(x)$ outside the illustrated intervals has been verified numerically and analytically (using Matlab and Mathematica).[6]

[6] Our experiments (not presented here) suggest that the current counterexample is sensitive to even minor changes in the coefficients of the polynomial in $s(x)$. Thus, the relationship between properties of X and $\log(X)$ is worth further exploring.

4 Closure Properties of the DGMRL Class of Distributions

[47] and [2] study closure properties of the IFR and IGFR classes under operations that involve continuous transformations, truncations, and convolutions. Such operations are important in economic applications, as they can be used to model changes or updates in the seller's beliefs (transformations and truncations) or aggregation of demands from different markets (convolutions). Resembling the IFR when compared to the IGFR class, the DMRL class exhibits better closure properties than the DGMRL class.

Theorem 5 *Let X be a non-negative, absolutely continuous, DMRL random variable and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing, concave and differentiable function. Then, $Y := \phi(X)$ is DMRL.*

Proof Let F denote the cdf of X, f its pdf and h its hazard rate. Then, for $y > 0$, $F_Y(y) = F\left(\phi^{-1}(y)\right)$ and $f_Y(y) = f\left(\phi^{-1}(y)\right)\frac{1}{\phi'(\phi^{-1}(y))}$, where ϕ^{-1} denotes the inverse of ϕ. Hence $\mathrm{m}_Y(y) = \left(\bar{F}\left(\phi^{-1}(y)\right)\right)^{-1} \cdot \int_y^{+\infty}\bar{F}\left(\phi^{-1}(u)\right)\mathrm{d}u = \left(\bar{F}\left(\phi^{-1}(y)\right)\right)^{-1} \cdot \int_{\phi^{-1}(y)}^{+\infty}\bar{F}(u)\,\phi'(u)\,\mathrm{d}u$. By (4), and since $\mathrm{h}_Y(y) = \mathrm{h}\left(\phi^{-1}(y)\right) \cdot \frac{1}{\phi'(\phi^{-1}(y))}$, we conclude that $\mathrm{m}'_Y(y) = \mathrm{h}\left(\phi^{-1}(y)\right) \cdot \left(\bar{F}\left(\phi^{-1}(y)\right)\right)^{-1} \cdot \int_{\phi^{-1}(y)}^{+\infty}\bar{F}(u)\frac{\phi'(u)}{\phi'(\phi^{-1}(y))}\mathrm{d}u - 1$. Concavity of ϕ implies that for $u > \phi^{-1}(y)$, $\frac{\phi'(u)}{\phi'(\phi^{-1}(y))} \leq 1$. Thus, $\mathrm{m}'_Y(y) \leq \mathrm{h}\left(\phi^{-1}(y)\right)\mathrm{m}\left(\phi^{-1}(y)\right) - 1 = \mathrm{m}'\left(\phi^{-1}(y)\right) \leq 0$, since $\mathrm{m}(y)$ is decreasing by assumption. □

Hence, the class of absolutely continuous, DMRL random variables is closed under strictly increasing, differentiable and concave transformations. By Theorem 5, it is immediate that

Corollary 1 *Let X be a non-negative, absolutely continuous, DMRL random variable. Then,*
(i) for any $\alpha > 0$ and $\beta \in \mathbb{R}$, $\alpha X + \beta$ is DMRL, (i.e., the DMRL class is closed under positive scale transformations and shifting).
(ii) for any $0 < \alpha \leq 1$, X^α is DMRL.

More results about the DMRL class can be found in [1, 30] and [51]. Turning to the DGMRL class, it is straightforward (thus omitted) to show that the DGMRL property is preserved under positive scale transformations and left truncations. For a random variable X with support in-between L and H, and any $\alpha \in (L, H)$, the left truncated random variable X_α is defined as $X_\alpha = X\mathbf{1}_{\{X \geq \alpha\}}$.

Theorem 6 *Let X be a DGMRL random variable with support in-between L and H with $0 \leq L < H \leq +\infty$. Then,*
(i) for any $\lambda > 0$, the random variable λX is DGMRL (i.e., the DGMRL class is closed under positive scale transformations).
(ii) for any $\alpha \in (L, H)$, the left truncated random variable X_α has the same GMRL function as X on (α, H). In particular, the DGMRL class is closed under left truncations.

In Proposition 1, [2] establish that IGFR distributions are closed under right truncations as well. It remains unclear whether DGMRL distributions are also closed under right truncations or not. On the other hand, as expected, the DGMRL class inherits some closure counterexamples from the IGFR class. [2] illustrate that the IGFR property is not preserved under shifting and convolutions. Both of their examples establish the same conclusions for the DGMRL property, as shown below.

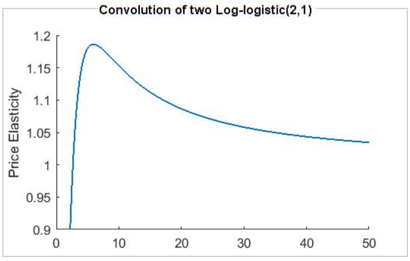

Fig. 5 The price elasticity (inverse of the GMRL function) for the convolution of 2 standard Log-logistic($k = 2$) random variables.

Using their notation, the GMRL function of the Pareto distribution of the second kind (Lomax distribution) is $\ell(x) = \frac{1}{k-1}\left(\frac{B-A}{x} + 1\right)$, for $x \geq A$, where A denotes the location parameter. Hence, when $A = 0$ (i.e., no shift) or $A < B$, the GMRL is decreasing, whereas, for $A > B$, the GMRL function is increasing. Similar to the behavior exhibited by the GFR function, the GMRL function is constant for $A = B$, and, in particular for $A = B = 1$, which corresponds to the standard Pareto distribution. To show that the IGFR class is not closed under convolution, [2] consider the sum of two log-logistic distributions. The log-logistic distribution is IGFR, and, hence, DGMRL. Using their formula for F, one may establish numerically that the price elasticity $\varepsilon(p) = \ell(p)^{-1}$ is first increasing and then decreasing, as can be seen in Figure 5.

5 Discussion and Conclusions

In this paper, we studied a novel unimodality condition for equilibrium uniqueness in markets with stochastic demand. We expressed the price elasticity of expected demand in terms of the mean residual life (MRL) function of the demand distribution and characterized the seller's optimal prices as fixed points of the MRL function. This led to a novel description of markets with increasingly elastic demand in terms of the properties of the underlying demand distribution and in turn, to a novel unimodality condition. Namely, the seller's optimal price in a stochastic market exists and is unique if the demand distribution has the *decreasing generalized mean residual life* (DGMRL) property and finite second moment.

The GMRL function, and hence, the DGMRL unimodality condition, naturally arise in the analysis of game-theoretic models of horizontal market competition (Cournot games) [28, 34], supply chain coordination (Stackelberg games) [26, 35, 37] or inventory optimization under uncertainty, see, e.g., [43, 53, 54]. Importantly, these problems are not covered by the widely used *increasing generalized failure rate* (IGFR) distributions, [32], because, in these cases, revenue (utility) maximization requires information for the global (tail) rather than the local (pointwise) behavior of the distribution of the underlying source of uncertainty. Motivated by the fact that DGMRL distributions strictly generalize IGFR distributions, we then studied properties of DGMRL distributions.

Our results also open several directions for future research. These include the extension of the current analysis to more general demand functions, i.e., the strengthening of the analysis in Section 2.3, e.g. via the framework of [50], the derivation of less restrictive conditions for which DGMRL distributions are IGFR, i.e., for which Theorem 2-(ii) holds, the derivation of conditions under which the DMRL property implies the IGFR property, the study of closure properties of the DGMRL class under mixtures of distributions, and, importantly, from a game-theoretic perspective, the study of the MRL function and the location or properties of its fixed points, i.e., seller's optimal prices.

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