Special Spinors and Homogeneous Geometries

Hofmann, Jordan

Awarding institution:
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Jordan Ariel Mackay Hofmann

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Department of Mathematics

King's College London

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Abstract

In this thesis we examine homogeneous spaces from the viewpoint of spin geometry, with a particular focus on the existence (or non-existence) of special spinor fields and their corresponding geometric structures. Recalling that a reductive homogeneous space $M = G/H$ has an elegant description of its geometrically relevant bundles (e.g. the tangent bundle, frame bundle, bundles of tensors and differential forms, etc.) as homogeneous bundles associated to the H-principal bundle $G \to G/H$, the natural objects of study are the (G-)invariant sections of these bundles. Under certain topological conditions on the isotropy representation, there exists a G-invariant spin structure and associated spinor bundle on $M = G/H$ (see [DKL22, Prop. 1.3]), and we shall be interested in the invariant sections of the latter. By working at the origin $o = eH \in G/H$, finding invariant objects can be reduced to a purely representation-theoretic problem, which we approach using various results from classical invariant theory, among other methods.

Chapter 3 is devoted to the exposition of [AHL23] (joint work with I. Agricola and M.-A. Lawn), where we have obtained a classification of the invariant spinors on the nine realizations of the sphere as a Riemannian homogeneous space. Partial results for a few of the simpler cases have appeared in, or may be deduced from, [Wan89], however a full classification and description of the invariant spinors and their related geometric structures has before now not been attempted. In each case we give an explicit basis for the space of invariant spinors, using the realization of the spin representation in terms of exterior forms, and describe the differential equations they satisfy (e.g. Killing, generalized Killing, etc.). Notably, we construct (to our knowledge) the first examples of generalized Killing spinors whose associated endomorphism field has four distinct eigenvalues. Where relevant, we also explore the relationships between the invariant spinors and certain invariant tensors and differential forms (and their related G-structures); these are compared with known results from the literature.

Chapter 4 presents the work [Hof22], which deals mainly with invariant spinors on homogeneous 3-Sasakian spaces, $(M = G/H, g, \xi_i, \eta_i, \varphi_i)_{i=1}^3$. The dimensions of the spaces of invariant forms of degree ≤ 3 on these spaces have appeared already in [DOP20]. We build on this to obtain a complete description of the invariant φ_1-(anti-)holomorphic differential forms of all degrees, as well as an explicit description of the space of invariant spinors, which to the author's knowledge has never appeared beyond the isolated case of $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ treated in [AHL23]. We show that the invariant spinors are spanned by the Clifford products of invariant differential forms with a certain invariant Killing spinor. It is well-known that a simply-connected 3-Sasakian manifold of dimension $4n-1$ admits $n+1$ linearly independent Killing spinors [Bär93], and a partial construction of these spinors as sections of certain subbundles E_i^-, $i = 1, 2, 3$ of the spinor bundle is given in [FK90], however this description is incomplete for spaces of dimension > 19. We complete this description in the homogeneous case, giving an explicit basis for the space of Killing spinors carried by a homogeneous 3-Sasakian space. It follows from our result that any Killing spinor on a homogeneous 3-Sasakian space is invariant.

Chapters 5 and 6 are based on joint work with I. Agricola. We consider 3-(α, δ)-Sasaki spaces, which can be viewed as deformations of 3-Sasakian spaces [AD20]. The first half of the chapter contains a novel examination of the behaviour of certain Killing spinors on 3-Sasakian spaces under such deformations; we give a detailed proof of the new spinorial field equation satisfied by the *deformed Killing spinors* on the resulting 3-(α, δ)-Sasaki space. The second half of the chapter studies the dual compact/non-compact pairs of homogeneous 3-(α, δ)-Sasaki spaces described in [ADS21, Remark 3.1.1c]. We modify the dualization construction of Kath in [Kat00] to obtain an identification of the spinor bundles for these dual pairs, and show that there is a natural correspondence between deformed Killing spinors on the two spaces.

Statement of Originality

The work presented in this thesis is my own except where explicitly noted otherwise, and all joint work is clearly indicated on page 8. Parts of this thesis have previously appeared in [AHL23, Hof22].

Breakdown of Joint Work

Chapter 3 is based on [AHL23], which is joint work with Prof. Dr. habil. Ilka Agricola and Dr. Marie-Amélie Lawn. The idea to study spinors using the exterior forms approach was due to Dr. Lawn, and the body of the paper was written by me, with the exception of: the introduction, abstract, Section 4.1, Remark 5.9, and certain parts of the preliminaries section, which were written by Dr. Lawn; and Remark 5.3, the paragraph before Proposition 3.4, and the paragraph after Proposition 4.7, which were written by Prof. Agricola. These parts are not included directly in this thesis (rather, they are cited as needed). Prof. Agricola and Dr. Lawn also proposed extensive revisions to the initial drafts and provided valuable references and historical insights on the topic.

Chapters 5 and 6 are based on joint work with Prof. Dr. habil. Ilka Agricola. The idea to study deformations of Killing spinors in the 3-(α, δ)-Sasaki setting and their associated dual spinors was due to Prof. Agricola, and the calculations and writing were carried out by me, with extensive revisions of initial drafts proposed by Prof. Agricola.

1 Introduction

The existence of special spinor fields on a Riemannian manifold efficiently encodes a great deal of geometric information, and is relevant, for example, to the study of immersion theory, Einstein metrics, holonomy theory, and G-structures, among others [Fri98, Wan89, Bär93, Fri80, ACFH15]. The most extensively investigated special spinors are the real Riemannian Killing spinors, i.e. those satisfying the differential equation $\nabla^g_X \psi = \pm\frac{1}{2} X \cdot \psi$ for any vector field X, whose existence places strong constraints on the geometry of the underlying manifold. Indeed, it was shown by Friedrich in [Fri80] that Killing spinors are eigenspinors realizing the lower bound for eigenvalues of the Dirac operator, and that any manifold carrying such spinors is Einstein with scalar curvature $R = n(n-1)$. The classification of complete simply connected manifolds with real Killing spinors was subsequently accomplished by Bär in [Bär93], where it was shown that they correspond to parallel spinors (or equivalently, to a reduction of holonomy) on the metric cone. Comparing with Wang's classification of geometries carrying parallel spinors [Wan89], one sees that Killing spinors are in fact somewhat rare and, beyond isolated cases in dimensions 6 and 7, are carried only by Riemannian manifolds (M^{2n-1}, g) such that the holonomy of the metric cone $(M \times \mathbb{R}, g + r^2 dr)$ reduces to a subgroup of $\mathrm{SU}(n)$ or $\mathrm{Sp}(n/2)$. These are precisely the round spheres, Einstein-Sasakian and 3-Sasakian manifolds (see e.g. [BG99]), and these spaces play an outsized role in spin geometry. Importantly, 3-Sasakian manifolds may be considered in full generality from the perspective of spin geometry due to Kuo's result that they admit a reduction of the structure group of the tangent bundle to the simply-connected subgroup $\{1\} \times \mathrm{Sp}(n-1)$ of $\mathrm{SO}(4n-4)$ and hence are necessarily spin [Kuo70]. Similarly, it is well-known that simply-connected Einstein-Sasakian manifolds and spheres of any dimension are spin (see e.g. [LM89, BG99]). Chapters 3 and 4 of this thesis are devoted to the study of these three classes of manifolds from the spinorial viewpoint, with a particular focus on the extent to which their global geometric structures are determined by their spinors. Chapters

5 and 6 examine the behaviour of Killing spinors under deformations of 3-Sasakian metrics to 3-(α, δ)-Sasaki metrics, and how the resulting *deformed Killing spinors* behave under the duality between positive ($\alpha\delta > 0$) and negative ($\alpha\delta < 0$) homogeneous 3-(α, δ)-Sasaki spaces fibering over Wolf spaces.

Spheres:

The spinorial properties of round spheres have been studied for some time now. Strikingly, their spinor bundles may be trivialized by a basis of Riemannian Killing spinors for either of the constants $\pm\frac{1}{2}$ (see e.g. [CGLS86, Prop. 11, Cor. 2], the latter of which is credited to J. Rawnsley by the authors of that paper), and for an explicit construction in stereographic coordinates of these Killing spinors, see e.g. Example 2 on p. 37 of [BFGK91]. This starkly contrasts with the behaviour of the tangent bundles of these spheres, which fail to admit even a single non-vanishing vector field in even dimensions (due to the Hairy Ball Theorem), emphasizing the power of the spin geometry approach to capture geometric information otherwise unavailable via the usual tensorial approach to geometry. Indeed, spheres constitute one of the most basic classes of spin manifolds, however, apart from Killing spinors on round spheres, very little is known about them from the perspective of spin geometry. Moroianu and Semmelmann approached this issue in [MS14a, MS14b] by investigating, for the cases of round spheres and Einstein manifolds, the so-called *generalized Killing equation*: $\nabla^g_X \psi = A(X) \cdot \psi$ for any vector field X, where $A \in \mathrm{Sym}^2(TM)$ denotes a symmetric endomorphism field. They showed that in certain dimensions any generalized Killing spinor on a round sphere is in fact a Killing spinor (i.e. the endomorphism A is a multiple of the identity), and gave a description of the case where the endomorphism A has precisely two distinct eigenvalues. In Chapter 3 we approach the issue from a different angle, focusing instead on non-round metrics. Specifically, we specialize to the case of homogeneous spheres, which Montgomery and Samelson showed in [MS43] are limited

Lie group	Manifold	Isotropy Subgroup	Invariant Spin Structure ([DKL22])
$\mathrm{SO}(n)$	S^{n-1}	$\mathrm{SO}(n-1)$	No
$\mathrm{U}(n)$	S^{2n-1}	$\mathrm{U}(n-1)$	No
$\mathrm{SU}(n)$	S^{2n-1}	$\mathrm{SU}(n-1)$	Yes
$\mathrm{Sp}(n)$	S^{4n-1}	$\mathrm{Sp}(n-1)$	Yes
$\mathrm{Sp}(n)\,\mathrm{Sp}(1)$	S^{4n-1}	$\mathrm{Sp}(n-1)\,\mathrm{Sp}(1)$	n even
$\mathrm{Sp}(n)\,\mathrm{U}(1)$	S^{4n-1}	$\mathrm{Sp}(n-1)\,\mathrm{U}(1)$	n even
G_2	S^6	$\mathrm{SU}(3)$	Yes
$\mathrm{Spin}(7)$	S^7	G_2	Yes
$\mathrm{Spin}(9)$	S^{15}	$\mathrm{Spin}(7)$	Yes

Table 1.1: Homogeneous Spheres and Invariant Spin Structures

to a relatively short list of possibilities (see Table 1.1).

The invariant spin structures on homogeneous spheres were classified by Daura Serrano, Kohn, and Lawn in [DKL22] (see the final column of Table 1.1 above), and we build upon this using representation-theoretic arguments to obtain a classification of the invariant spinors carried by these spaces and a description of the related geometries. Additionally, we note that many of these invariant spinors are generalized Killing spinors, which are interesting from the perspective of immersion theory [Fri98]. Our findings are summarized in the following theorem:

Theorem. *The dimension of the space of invariant spinors for each realization of the sphere as a homogeneous space is given in Table 1.2. For the realizations admitting non-trivial invariant spinors, we find:*

(1) A pair of linearly independent generalized Killing spinors with two eigenvalues on $(S^{2n-1} = \mathrm{SU}(n)/\,\mathrm{SU}(n-1), g_{a,b})$, and a related invariant α-Sasakian structure for $\alpha = \frac{\sqrt{an}}{2b\sqrt{n-1}}$;

(2) A $2n$-dimensional space of invariant spinors on $(S^{4n-1} = \mathrm{Sp}(n)/\,\mathrm{Sp}(n-1), g_{\vec{a}})$, expressed in terms of the structure tensors of the invariant 3-Sasakian structure;

  - For $n = 2$, four linearly independent invariant generalized Killing spinors with four eigenvalues;

(3) A generalized Killing spinor with two eigenvalues on $(S^7 = \mathrm{Sp}(2)\,\mathrm{Sp}(1)/\,\mathrm{Sp}(1)\,\mathrm{Sp}(1), g_{a,b})$;

(4) An invariant α-Sasakian structure on $(S^{4n-1} = \mathrm{Sp}(n)\,\mathrm{U}(1)/\,\mathrm{Sp}(n-1)\,\mathrm{U}(1), g_{a,b,c})$ for $\alpha = \frac{a}{b\Omega} = \frac{a}{2c\Omega}$, together with a pair of linearly independent invariant spinors not associated to the α-Sasakian structure and which, for $n > 2$ are not generalized Killing spinors;

  - For $n = 2$, a pair of linearly independent invariant generalized Killing spinors with three eigenvalues;

(5) An invariant Killing spinor (resp. a pair of linearly independent invariant Killing spinors) on the round sphere $S^6 = \mathrm{G}_2\,/\,\mathrm{SU}(3)$ (resp. the round sphere $S^7 = \mathrm{Spin}(7)/\,\mathrm{G}_2$);

(6) An invariant spinor on $(S^{15} = \mathrm{Spin}(9)/\,\mathrm{Spin}(7), g_{a,b})$ satisfying a differential equation depending on the 3-form determined by the spinor via the squaring construction.

Finally, we investigate these spheres within the context of non-integrable geometries and adapted connections. Roughly speaking, non-integrable geometries are given by G-structures which the Levi-Civita connection fails to preserve under parallel transport, and it is well-known that such structures are closely related to the existence of special spinors in low dimensions (see e.g.

[FKMS97, Iva04, CS07, ACFH15]). As such, the Levi-Civita connection is poorly adapted to the G-geometry, necessitating instead the use of other, more compatible, connections. In this vein, the existence of a so-called *characteristic connection*, i.e. a metric G-connection with totally skew-symmetric torsion tensor, has by now been thoroughly studied for many different groups G (see e.g. [FI02, Iva04, Agr06]). Similarly, for homogeneous spaces, a particularly useful choice is the *Ambrose-Singer connection*, whose parallel sections correspond to G-invariant objects; indeed this connection coincides with the Levi-Civita connection in the case of symmetric spaces, allowing us to view homogeneous geometries as a torsion analogue of symmetric geometries. For each of the cases in Table 1.1 we find the Ambrose-Singer connection (by explicitly calculating its torsion tensor), determine the torsion type, and discuss the relationship of its parallel spinors with the non-integrable geometries in Table 1.2.

Lie group	$\dim_{\mathbb{C}} \Sigma_{\text{inv}}$	Notable Spinors	Geometric Structures
$\mathrm{SO}(n)$	0		
$\mathrm{U}(n)$	0		
$\mathrm{SU}(n)$	2	generalized Killing	α-Sasakian ($\alpha = \frac{\sqrt{an}}{2b\sqrt{n-1}}$)
$\mathrm{Sp}(n)$	$2n$	deformed Killing	3-(α,δ)-Sasaki
$\mathrm{Sp}(n)\,\mathrm{Sp}(1)$	1 ($n=2$), 0 ($n \neq 2$)	generalized Killing ($n=2$)	co-calibrated G_2 ($n=2$)
$\mathrm{Sp}(n)\,\mathrm{U}(1)$	2 (n even), 0 (n odd)	generalized Killing ($n=2$)	α-Sasakian ($\frac{a}{b\Omega} = \frac{a}{2c\Omega} = \alpha$)
G_2	2	Killing	nearly Kähler
$\mathrm{Spin}(7)$	1	Killing	nearly parallel G_2
$\mathrm{Spin}(9)$	1		

Table 1.2: Invariant Spinors and Geometric Structures on Homogeneous Spheres

Throughout the chapter we shall refer to *Hermitian* and *quaternionic* spheres; we would like to clarify that this refers not to Hermitian or quaternionic structures on the spheres themselves but rather the fact that they are realized as homogeneous spaces via the action of a group on a Hermitian or quaternionic vector space.

3-Sasakian Manifolds:

In Chapter 4 we shed new light on the correspondence between (3-)Sasakian structures and Killing spinors by giving an explicit construction of the former in terms of the latter:

Theorem. Let (M, g) be a Riemannian spin manifold carrying a pair ψ_1, ψ_2 of Killing spinors (resp. four Killing spinors $\psi_1, \psi_2, \psi_3, \psi_4$) for the same Killing number $\lambda \in \{\frac{1}{2}, \frac{-1}{2}\}$. If the vector field ξ_{ψ_1,ψ_2} defined by the equation

$$g(\xi_{\psi_1,\psi_2}, X) := \Re\langle \psi_1, X \cdot \psi_2 \rangle$$

for all $X \in TM$ has locally constant non-zero length (resp. if the vector fields ξ_{ψ_1,ψ_2}, ξ_{ψ_3,ψ_4} are orthogonal and have locally constant non-zero lengths), then this vector field determines a Sasakian structure on M (resp. these vector fields determine a 3-Sasakian structure on M). Conversely, any Einstein-Sasakian (resp. 3-Sasakian) structure on a simply-connected manifold arises by this construction.

Using a new argument valid in all dimensions, this theorem generalizes previous results of Friedrich and Kath in dimensions 5 and 7 [FK88, FK89, FK90], which were proved by employing certain special spinorial properties occuring in these dimensions.

In the latter sections of Chapter 4 we concern ourselves mainly with 3-Sasakian manifolds, which initially appeared over fifty years ago in [Kuo70, Udr69], among others. Notable milestones in the subject include Konishi's construction of 3-Sasakian structures on certain principal SO(3)-bundles over quaternionic Kähler manifolds of positive scalar curvature [Kon75], and the result that, if the Reeb vector fields are complete, then the leaf space of the associated 3-dimensional foliation is a quaternionic Kähler orbifold [BGM94]. Indeed, these results show that 3-Sasakian manifolds lie between quaternionic Kähler geometries below and hyperKähler geometries above, emphasizing the fact that they provide natural examples of interesting odd dimensional quaternionic geometries.

Previous work on 3-Sasakian manifolds from the spinorial perspective includes e.g. [FK90, AF10], which give a thorough and elegant accounting of the situation in dimension 7, however until now very little is known about these spaces in higher dimensions. In Chapters 4.2, 4.3, and 4.4 of this thesis we provide, in arbitrary dimension, a detailed spinorial picture of the homogeneous 3-Sasakian spaces, which were classified in [BGM94]. Using Kuo's $\mathrm{Sp}(n-1)$-reduction together with the description of these spaces in terms of 3-*Sasakian data* in [DOP20], we apply our new invariant theoretic approach developed in [AHL23, Section 4.1] to give a classification of the invariant spinors:

Theorem. For a homogeneous 3-Sasakian manifold $(M^{4n-1} = G/H, g, \xi_i, \eta_i, \varphi_i)$, the space of invariant spinors forms an algebra under the wedge product, and is isomorphic to the algebra of invariant φ_1-anti-holomorphic differential forms,

$$\Sigma_{\mathrm{inv}} \simeq \Lambda^{0,\bullet}_{\mathrm{inv}}(T^*_{\mathbb{C}}M).$$

Furthermore, this algebra is generated by the forms $y_1 := \frac{1}{\sqrt{2}}(\xi_2 + i\xi_3)$ and $\omega := -\frac{1}{2}(\Phi_2|_{\mathcal{H}} + i\Phi_3|_{\mathcal{H}})$, where $\Phi_i := g(-, \varphi_i(-))$.

Finally, we use this to give a complete description of the invariant Killing spinors on homogeneous 3-Sasakian spaces. Indeed, a partial construction of the Killing spinors on 3-Sasakian manifolds is given in [FK90] as sections of certain rank two subbundles of the spinor bundle, however this description can produce at most six linearly independent Killing spinors and is thus incomplete for spaces of dimension > 19. In Chapter 4.4 we resolve this issue in the homogeneous setting, obtaining the following result:

Theorem. If $n \geq 2$ then the space of invariant Killing spinors on a simply-connected homogeneous 3-Sasakian manifold $(M^{4n-1} = G/H, g, \xi_i, \eta_i, \varphi_i)$ has a basis given by

$$\psi_k := \omega^{k+1} - i(k+1)y_1 \wedge \omega^k, \qquad -1 \leq k \leq n-1,$$

where we use the conventions $\omega^{-1} = 0$ and $\omega^0 = 1$. If $n = 1$ then the space of invariant Killing spinors has a basis given by 1, y_1. Furthermore, if $(M, g) \not\cong (S^{4n-1}, g_{\text{round}})$ then any Killing spinor is invariant.

As a consequence, we deduce explicit formulas for the Killing spinors which recover the homogeneous 3-Sasakian strucuture via the preceding construction.

3-(α, δ)-Sasaki Manifolds:

In Chapters 5 and 6 we investigate the spinorial properties of 3-(α, δ)-Sasaki manifolds–a new class of almost 3-contact manifolds introduced by Agricola and Dileo in [AD20], which encapsulate in a single framework the notion of 3-(α-)Sasakian structures and the quaternionic Heisenberg group, among others. In general 3-(α, δ)-Sasaki manifolds are not Einstein (see [AD20, Prop. 2.3.3]) and hence do not admit Riemannian Killing spinors, however by viewing such structures as deformations of 3-Sasakian structures by the two real parameters α, δ, we prove in Chapter 5 that 3-(α, δ)-Sasaki manifolds admit spinors satisfying a certain *deformed Killing equation*:

Theorem. Any 3-(α, δ)-Sasaki manifold $(M, g, \varphi_i, \xi_i, \eta_i)$ carries at least two linearly independent spinors satisfying

$$\nabla^g_X \psi = \frac{\alpha}{2} X \cdot \psi + \frac{\alpha - \delta}{2} \sum_{p=1}^{3} \eta_p(X) \Phi_p \cdot \psi \qquad \text{for all } X \in TM.$$

Next we focus on the homogeneous case; specifically, the duality between compact and non-compact homogeneous 3-(α, δ)-Sasaki spaces fibering over a symmetric base (see [ADS21, Remark

3.1.1]). Their construction exploits the duality between compact and non-compact symmetric spaces to obtain a notion of duality for the total spaces of the fibrations, and is similar to Kath's T-duality construction for Riemannian/pseudo-Riemannian pairs [Kat00]. In Chapter 6 we build upon Kath's construction to examine the relationship between the deformed Killing spinors of a compact/non-compact homogeneous 3-(α, δ)-Sasaki dual pair, and find that they correspond in a one-to-one manner:

Theorem. Suppose that M and M' are a dual pair of homogeneous 3-(α, δ)-Sasaki spaces of dimension $4n-1$, and identify the spinor modules $\Sigma \cong \Sigma' \cong \Lambda^\bullet \mathbb{C}^{2n-1}$. If ψ is an invariant spinor satisfying the deformed Killing equation

$$\nabla^g_X \psi = \frac{\alpha}{2} X \cdot \psi + \frac{\alpha - \delta}{2} \sum_{i=1}^{3} \eta_i(X) \Phi_i \cdot \psi \qquad \text{for all } X \in TM,$$

then the corresponding spinor ψ' on M' is also invariant and satisfies the corresponding deformed Killing equation

$$\nabla^{g'}_X \psi' = \frac{\alpha'}{2} X \cdot \psi' + \frac{\alpha' - \delta'}{2} \sum_{i=1}^{3} \eta'_i(X) \Phi'_i \cdot \psi' \qquad \text{for all } X \in TM,$$

where $\alpha' := \alpha$, $\delta' := -\delta$.

This result generalizes Kath's correspondence between Killing spinors on T-dual pairs [Kat00], and emphasizes that our notion of deformed Killing spinors is a natural one. Finally, we examine in detail the situation in dimension 7, and determine the behaviour under the duality construction of the canonical and auxiliary spinors ψ_0, $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$ introduced in [AF10, AD20].

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2 Preliminaries

In this chapter we give basic definitions and background related to the spin representation, spinors on homogeneous spaces, and (homogeneous) Sasakian, 3-Sasakian, and 3-(α, δ)-Sasaki structures. For a thorough introduction to these topics, among others, we recommend [LM89, BFGK91, BGM94, BG99, Fri00, BG08, AD20, DOP20].

2.1 The Spin Representation via Exterior Forms

Throughout this thesis we shall make use of the realization of the spin representation in terms of exterior forms. This realization is well-known in the context of representation theory, however its application to spin geometry and the study of spinors has not yet been widely adopted outside of [CGLS86, Wan89]. For a detailed description of this construction we refer the reader to [GW09, Chapter 6.1.2] (beware their different convention for the Clifford relation), and for its application to spinorial calculations on homogeneous spaces see [AHL23]. We briefly recall here the basic definitions and properties insofar as they relate to this work.

Let $(V = \mathbb{R}^{2n-1}, g)$ be the standard Euclidean inner product space and $\{e_1, \dots, e_{2n-1}\}$ the usual orthonormal basis. Letting $\varphi\colon V \to V$ denote the almost complex structure on $(\mathbb{R}e_1)^\perp$ given by

$$\varphi(e_{2j}) = e_{2j+1}, \quad \varphi(e_{2j+1}) = -e_{2j},$$

the complexification of V can be written as a direct sum

$$V^{\mathbb{C}} = \mathbb{C}_0 \oplus L \oplus L', \tag{2.1}$$

where $\mathbb{C}_0 := \mathbb{C}e_1$ and L (resp. L') denotes the space of φ-holomorphic (resp. φ-anti-holomorphic) vectors. Explicitly, these spaces are given by

$$L := \mathrm{span}_{\mathbb{C}}\{x_j := \frac{1}{\sqrt{2}}(e_{2j} - ie_{2j+1})\}_{j=1}^{n-1}, \quad L' := \mathrm{span}_{\mathbb{C}}\{y_j := \frac{1}{\sqrt{2}}(e_{2j} + ie_{2j+1})\}_{j=1}^{n-1}. \tag{2.2}$$

Letting $u_0 := ie_1$, we define an action of V on the algebra $\Sigma := \Lambda^\bullet L'$ of φ-anti-holomorphic forms via

$$u_0 := -\,\mathrm{Id}|_{\Sigma^{\mathrm{even}}} + \mathrm{Id}|_{\Sigma^{\mathrm{odd}}}, \quad x_j \cdot \eta := i\sqrt{2}\; x_j \lrcorner \eta, \quad y_j \cdot \eta := i\sqrt{2}\; y_j \wedge \eta, \tag{2.3}$$

where Σ^{even} and Σ^{odd} denote the even and odd graded parts of $\Sigma = \Lambda^\bullet L'$. Recalling the definition of the complex Clifford algebra,

$$\mathbb{C}l(V^{\mathbb{C}}, g^{\mathbb{C}}) := T(V^{\mathbb{C}})/(v \otimes w + w \otimes v = -2g^{\mathbb{C}}(v,w)1),$$

one easily verifies using the identities (5.43) in [GW09] that the action (2.3) descends to a representation of $\mathbb{C}l(V^{\mathbb{C}}, g^{\mathbb{C}})$ on Σ. Solving for the real orthonormal basis vectors $e_1, \dots, e_{2n-1}$ in (2.2) gives

$$e_1 = -iu_0, \quad e_{2j} = \frac{1}{\sqrt{2}}(x_j + y_j), \quad e_{2j+1} = \frac{-i}{\sqrt{2}}(y_j - x_j), \qquad \forall\, j = 1, \dots, n-1,$$

and the corresponding action on Σ (i.e. the *Clifford multiplication*) is given by

$$e_1 = i\,\mathrm{Id}|_{\Sigma^{\mathrm{even}}} - i\,\mathrm{Id}|_{\Sigma^{\mathrm{odd}}}, \quad e_{2j} \cdot \eta = i(x_j \lrcorner \eta + y_j \wedge \eta), \quad e_{2j+1} \cdot \eta = (y_j \wedge \eta - x_j \lrcorner \eta), \tag{2.4}$$

for all $\eta \in \Sigma$. We also note that there is an equivalent realization of this representation in terms of Kronecker products. Indeed, if we define subspaces $U_j := \Lambda^\bullet \mathbb{C}y_j = \mathbb{C}1 \oplus \mathbb{C}y_j$ of the spinor module Σ then we have

$$\Sigma = \Lambda^\bullet L' = \Lambda^\bullet(\mathbb{C}y_1 \oplus \cdots \oplus \mathbb{C}y_l) \cong U_1 \wedge \cdots \wedge U_l,$$

giving the vector space isomorphism

$$\Sigma \cong U_1 \otimes \cdots \otimes U_l.$$

Explicitly, this isomorphism is given by identifying $y_{j_1} \wedge \cdots \wedge y_{j_p} \in \Lambda^\bullet L'$ with $u_1 \otimes \cdots \otimes u_l \in$

$U_1 \otimes \cdots \otimes U_l$, where

$$u_j := \begin{cases} y_j & \text{if } j \in \{j_1, \ldots, j_p\}, \\ 1 & \text{if } j \notin \{j_1, \ldots, j_p\}. \end{cases}$$

Under this identification we have

$$\mathrm{End}(\Sigma) \cong \mathrm{End}(U_1) \otimes \cdots \otimes \mathrm{End}(U_l).$$

With the choice of ordered bases $\{1, y_j\}$ for the U_j, the representation (2.3) of $\mathbb{C}l(V^{\mathbb{C}}, g^{\mathbb{C}})$ is realized by the Kronecker products

$$\begin{aligned} u_0 &\mapsto -H \otimes \cdots \otimes H, \\ x_j &\mapsto i\sqrt{2}\; H \otimes \cdots \otimes H \otimes \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{j\text{th place}} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}, \\ y_j &\mapsto i\sqrt{2}\; H \otimes \cdots \otimes H \otimes \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{j\text{th place}} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}, \end{aligned}$$

where $H := \mathrm{diag}[1, -1]$. The corresponding operators associated to the real orthonormal basis $e_1, \ldots, e_n$ are

$$\begin{aligned} e_1 &\mapsto i\; H \otimes \cdots \otimes H, \\ e_{2j} &\mapsto i\; H \otimes \cdots \otimes H \otimes \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{j\text{th place}} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}, \\ e_{2j+1} &\mapsto \; H \otimes \cdots \otimes H \otimes \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{j\text{th place}} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}. \end{aligned}$$

Remark 2.1.1. In odd dimensions there is another inequivalent irreducible representation of the Clifford algebra, obtained by replacing the action of u_0 in (2.3) with $u_0 := \mathrm{Id}|_{\Sigma^{\mathrm{even}}} - \mathrm{Id}|_{\Sigma^{\mathrm{odd}}}$. The corresponding operator for the real orthonormal basis vector is

$$e_{2l+1} = -i\,\mathrm{Id}|_{\Sigma^{\mathrm{even}}} + i\,\mathrm{Id}|_{\Sigma^{\mathrm{odd}}},$$

and, in the Kronecker product setting,

$$u_0 \mapsto H \otimes \cdots \otimes H, \quad e_{2l+1} \mapsto -iH \otimes \cdots \otimes H.$$

In this thesis we will always use the representation described in (2.3).

Remark 2.1.2. It is possible to similarly define the spin representation for even dimensional spaces by deleting the $\mathbb{C}_0$ factor in the decomposition (2.1) and the corresponding operators u_0, e_1 in (2.3) and (2.4).

2.2 Spinors on Homogeneous Spaces

Let $M = G/H$ be a reductive homogeneous space for a semisimple group G, and fix a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus_\perp \mathfrak{m}$ which is orthogonal with respect to the Killing form on $\mathfrak{g}$. In this section we review the construction of some geometrically relevant bundles as homogeneous bundles associated to the projection $G \to G/H$. For a more detailed introduction to reductive homogeneous spaces we refer to [Arv03], and for examples illustrating the process of finding invariant spinors we recommend [BFGK91, Chapters 4.5, 5.4].

Letting $\pi\colon G \to G/H$ denote the projection map, the tangent space T_oM at the origin $o := eH$ is naturally identified with $\mathfrak{m} \cong \ker d_e\pi$. The other tangent spaces are therefore obtained by displacements of $\mathfrak{m}$ under the isometries in G, leading to the realization of the tangent bundle as a homogeneous bundle via

$$TM = G \times_{\mathrm{Ad}|_H} \mathfrak{m},$$

where $\pi\colon G \to G/H$ is viewed as a principal H-bundle. The natural representation of $H = \mathrm{Stab}_G(o)$ on T_oM (by letting $h \in H$ act via $dh_o\colon T_oM \to T_oM$) is called the *isotropy representation*, and it is isomorphic to the restricted adjoint representation $\mathrm{Ad}_H\colon H \to \mathrm{GL}(\mathfrak{m})$, $h \mapsto \mathrm{Ad}(h)|_{\mathfrak{m}}$. Under this identification, an invariant Riemannian metric on M correponds to a inner product $g\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ with the property that $\mathrm{Ad}|_H(H) \subseteq \mathrm{SO}(\mathfrak{m}, g) \subseteq GL(\mathfrak{m})$; for an invariant metric, the oriented frame bundle is then given as a homogeneous bundle via

$$P_{\mathrm{SO}} = G \times_{\mathrm{Ad}|_H} \mathrm{SO}(\mathfrak{m}, g).$$

Suppose now that there exists a lift of the isotropy representation to the spin group, i.e. a group homomorphism $\widetilde{\mathrm{Ad}|_H}$ such that the diagram in Figure 2.1 commutes. Such a lift induces a spin

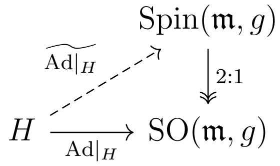

Figure 2.1: Homogeneous Spin Structure

structure and spinor bundle as homogeneous bundles via

$$P_{\mathrm{Spin}} := G \times_{\widetilde{\mathrm{Ad}|_H}} \mathrm{Spin}(\mathfrak{m}, g), \qquad \Sigma M := P_{\mathrm{Spin}} \times_\sigma \Sigma = G \times_{\sigma \circ \widetilde{\mathrm{Ad}|_H}} \Sigma, \tag{2.5}$$

where $\sigma\colon \mathrm{Spin}(\mathfrak{m}) \to \mathrm{Aut}(\Sigma)$ denotes the spin representation. Furthermore, for a connected isotropy group H, this was shown to be the unique G-invariant spin structure on $M = G/H$ by Daura Serrano, Kohn, and Lawn in [DKL22] (see also the earlier works [CG88, HS90], which consider certain special cases). For the cases in Table 1.1 which don't admit invariant spin structures, we consider instead (and without further mention) the double coverings described in the closing remarks of [DKL22], which do admit invariant spin structures. We note that the corresponding group actions on the sphere are non-effective, however the calculations in Chapter 3 are done at the Lie algebra level and are therefore not affected.

In Chapters 4, 5, and 6 of this thesis we shall mainly be concerned with 3-Sasakian and 3-(α, δ)-Sasaki spaces, which are necessarily spin due to Kuo's reduction of the structure group of the tangent bundle to the (simply-connected) symplectic group of the horizontal distribution [Kuo70, Thm. 5]. For a simply-connected homogeneous 3-Sasakian or 3-(α, δ)-Sasaki space, denote by $\mathfrak{m} = \mathfrak{m}_{\mathcal{V}} \oplus \mathfrak{m}_{\mathcal{H}}$ the splitting into vertical and horizontal distributions. Invariance of the structure tensors implies that the image of H under the isotropy representation is contained in the above reduction, i.e.

$$\mathrm{Ad}|_H(H) \subseteq \{1\} \times \mathrm{Sp}(\mathfrak{m}_{\mathcal{H}}) \subseteq \mathrm{SO}(\mathfrak{m}),$$

and one therefore obtains a (unique) lifting of the isotropy representation and the associated G-invariant spin structure as above. Throughout the thesis we will always use this invariant spin structure when considering simply-connected homogeneous 3-Sasakian and 3-(α, δ)-Sasaki spaces.

In light of the associated bundle construction of the spinor bundle in (2.5), spinors are identified with H-equivariant maps $\varphi\colon G \to \Sigma$, i.e. maps satisfying

$$\varphi(gh) = \Delta \circ \widetilde{\mathrm{Ad}|_H}(h^{-1}) \cdot \varphi(g) \quad \forall g \in G, h \in H. \tag{2.6}$$

The *G-invariant spinors* correspond precisely to the constant H-equivariant maps $\varphi\colon G \to \Sigma$, and we denote by $\Sigma_{\mathrm{inv}} \subseteq \Sigma M$ the subbundle of such spinors. Equivalently, it follows from (2.6) that invariant spinors correspond to trivial subrepresentations of $\sigma \circ \widetilde{\mathrm{Ad}|_H}\colon H \to \mathrm{GL}(\Sigma)$.

One may similarly realize the bundles of k-tensors and differential k-forms on M as homogeneous bundles via

$$\otimes^k TM = G \times_{(\mathrm{Ad}|_H)^{\otimes k}} \mathfrak{m}^{\otimes k}, \qquad \Lambda^k TM = G \times_{\Lambda^k(\mathrm{Ad}|_H)} \Lambda^k \mathfrak{m},$$

and invariant sections then correspond to trivial H-subrepresentations of $\mathfrak{m}^{\otimes k}$ and $\Lambda^k\mathfrak{m}$ respectively. The representation theoretic problem of finding trivial subrepresentations is approached in this thesis using results from classical invariant theory, together with computer calculations in LiE ([LCL88]) for certain cases involving the exceptional Lie groups.

Finally, we have the following definition which is valid for *any* spin manifold (not necessarily homogeneous):

Definition 2.2.1. A spinor ψ on a Riemannian spin manifold (M, g) is called a *(Riemannian) Killing spinor* for the constant $\lambda \in \mathbb{C}$ if it satisfies

$$\nabla^g_X \psi = \lambda X \cdot \psi \quad \text{ for all } X \in TM.$$

It is a remarkable result of Friedrich that the existence of a non-trivial Killing spinor on a connected spin manifold (M^n, g) implies that the metric is Einstein and has scalar curvature $R = 4n(n-1)\lambda^2$ (see [Fri80, BFGK91]). We shall refer interhangeably to Riemannian Killing spinors and Killing spinors, and, unless otherwise stated, we will only consider *real* Killing spinors (i.e. those with $\lambda \in \mathbb{R}$) in this thesis. By rescaling the metric if necessary, one can assume that any (non-parallel) real Killing spinor has $\lambda = \pm\frac{1}{2}$.

2.3 Sasakian and 3-Sasakian Structures

Let us briefly define Sasakian and 3-Sasakian structures and discuss some of their properties, following the exposition in [BGM94].

Definition 2.3.1. A Sasakian structure on a Riemannian manifold (M, g) is a unit length Killing vector field ξ such that the endomorphism field $\varphi := -\nabla^g \xi$ satisfies

$$(\nabla^g_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X$$

for all $X, Y \in TM$ (where ∇^g denotes the Levi-Civita connection). It is customary to denote a Sasakian structure by $(M, g, \xi, \eta, \varphi)$, where $\eta := \xi^\flat$. The *vertical* and *horizontal* distributions are defined by

$$\mathcal{V} := \mathbb{R}\xi, \qquad \mathcal{H} := \ker \eta,$$

and the vector field ξ is called the *Reeb vector field.* The *fundamental* 2-*form* is defined by

$$\Phi(X, Y) := g(X, \varphi(Y)) \qquad \forall X, Y \in TM.$$

Here we collect several basic properties of Sasakian manifolds:

Proposition 2.3.2. *(Based on [BGM94, Prop. 2.2]). If $(M, g, \xi, \eta, \varphi)$ is a Sasakian manifold, then*

$$\varphi^2 = -\operatorname{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \operatorname{Im}(\varphi) \subseteq \mathcal{H}, \quad d\eta = 2\Phi,$$
$$0 = N_\varphi(X, Y) := [\varphi(X), \varphi(Y)] + \varphi^2[X, Y] - \varphi[\varphi(X), Y] - \varphi[X, \varphi(Y)] + d\eta(X, Y)\xi,$$
$$0 = g(\varphi(X), Y) + g(X, \varphi(Y)), \quad g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in TM$.

In a similar spirit, we have the notion of a 3-Sasakian structure, which consists of three orthogonal Sasakian structures whose Reeb vector fields satisfy the relations of the imaginary quaternions under the Lie bracket:

Definition 2.3.3. A 3-Sasakian structure on a Riemannian manifold (M, g) consists of three Sasakian structures $(g, \xi_i, \eta_i, \varphi_i)$, $i = 1, 2, 3$ such that the Reeb vector fields ξ_i, $i = 1, 2, 3$ are orthogonal and satisfy

$$[\xi_i, \xi_j] = 2\xi_k$$

for any even permutation (i, j, k) of $(1, 2, 3)$. It is customary to denote a 3-Sasakian structure by $(M, g, \xi_i, \eta_i, \varphi_i)$, omitting the "$i = 1, 2, 3$". The *vertical* and *horizontal* distributions are defined by

$$\mathcal{V} := \mathrm{span}_{\mathbb{R}}\{\xi_i\}_{i=1,2,3}, \qquad \mathcal{H} := \cap_{i=1,2,3} \ker(\eta_i),$$

and the vector fields ξ_i, $i = 1, 2, 3$ are called the *Reeb vector fields.* The *fundamental* 2*-forms* are defined by

$$\Phi_i(X, Y) := g(X, \varphi_i(Y)) \qquad \forall X, Y \in TM.$$

In addition to the identities in Proposition 2.3.2, the tensors defining a 3-Sasakian structure satisfy certain "pseudo-quaternionic" compatibility relations:

Proposition 2.3.4. *(Based on [BGM94, Eqn. (2.4)] and [AD20, Eqn. (1.5)]). If* $(M, g, \xi_i, \eta_i, \varphi_i)$ *is a 3-Sasakian manifold, then*

$$\varphi_i = \varphi_j \circ \varphi_k - \eta_k \otimes \xi_j = -\varphi_k \circ \varphi_j + \eta_j \otimes \xi_k$$
$$\varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k, \qquad \eta_i = \eta_j \circ \varphi_k = -\eta_k \circ \varphi_j,$$

for any even permutation (i, j, k) *of* $(1, 2, 3)$*.*

Remark 2.3.5. Just as any two of $i, j, k \in \mathbb{H}$ generate the third, one sees that any two Sasakian structures with orthogonal Reeb vector fields generate a 3-Sasakian structure (see e.g. [FK90, p. 556]).

For calculations on 3-Sasakian manifolds, we will often exploit a particularly nice choice of local frame:

Definition 2.3.6. Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-Sasakian manifold. A local frame $e_1, \ldots, e_{4n-1}$ of TM is called *adapted* if

$$e_i = \xi_i, \qquad e_{4p+i} = \varphi_i(e_{4p}), \qquad \text{for all } i = 1, 2, 3, \quad p = 1, \ldots, n-1.$$

For 3-Sasakian manifolds we recall that there is a particularly useful choice of metric connection adapted to the geometry: the so-called *canonical connection* (see [AD20, Section 4], where this connection is introduced for the more general class of 3-(α, δ)-Sasaki manifolds). We review its main properties here:

Theorem 2.3.7. *(Based on [AD20, Section 4]). For a 3-Sasakian manifold* $(M, g, \xi_i, \eta_i, \varphi_i)$*,* *the canonical connection ∇ is the unique metric connection with skew torsion such that*

$$\nabla_X \varphi_i = -2(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \qquad \text{for all } X \in TM.$$

The derivatives of the other structure tensors are

$$\nabla_X \xi_i = -2(\eta_k(X)\xi_j - \eta_j(X)\xi_k), \quad \nabla_X \eta_i = -2(\eta_k(X)\eta_j - \eta_j(X)\eta_k),$$

and the torsion 3-form is given by $T = \sum_{i=1}^3 \eta_i \wedge d\eta_i$.

Finally, we recall from [FK90] certain subbundles of the spinor bundle which will be highly relevant for our purposes:

Theorem 2.3.8. *(Based on [FK90, Thm. 1] and Theorem 1 in [BFGK91, Chapter 4.2]). If $(M, g, \xi_i, \eta_i, \varphi_i)$ is a 3-Sasakian manifold (resp. a simply-connected Einstein-Sasakian manifold, by allowing only $i = 1$), then the bundles*

$$E_i^\pm := \{\psi \in \Sigma M\colon (\pm 2\varphi_i(X) + \xi_i \cdot X - X \cdot \xi_i) \cdot \psi = 0 \quad \forall X \in TM\}, \qquad i = 1, 2, 3$$

have bases consisting of Riemannian Killing spinors for the constants $\mp\frac{1}{2}$. With respect to the Clifford algebra representation described in Chapter 2.1 (i.e. the representation with $u_0 \cdot \eta^\pm = \mp\eta^\pm$), the ranks of these bundles are given in Table 2.1.

$\dim(M)$	$\mathrm{rank}(E_i^+)$	$\mathrm{rank}(E_i^-)$
1 (mod 4)	1	1
3 (mod 4)	0	2

Table 2.1: Ranks of the Bundles $E_i^\pm$

2.4 Homogeneous 3-Sasakian Spaces

First, we recall Boyer, Galicki, and Mann's classification of homogeneous 3-Sasakian spaces:

Theorem 2.4.1. *(Based on [BGM94, Thm. C]). The homogeneous 3-Sasakian spaces $(M^{4n-1} = G/H, g)$ are precisely*

$$S^{4n-1} \cong \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}, \quad \mathbb{RP}^{4n-1} \cong \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)\times\mathbb{Z}_2}, \quad \frac{\mathrm{SU}(n+1)}{\mathrm{S}(\mathrm{U}(n-1)\times\mathrm{U}(1))},$$
$$\frac{\mathrm{SO}(n+3)}{\mathrm{SO}(n-1)\times\mathrm{Sp}(1)}, \quad \frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}_7},$$

where the permissible values of n are as follows:

G	$\mathrm{Sp}(n)$	$\mathrm{SU}(n+1)$	$\mathrm{SO}(n+3)$
n	$n \geq 1$	$n \geq 2$	$n \geq 4$

This classification was obtained by proving that any homogeneous 3-Sasakian space fibers over a Wolf space with a finite list of possibilities for the fiber, and then using the classification of Wolf spaces in [Wol65]. Recently, a new proof of the classification was obtained in [GRS23] using root systems of complex simple Lie algebras to construct homogeneous 3-Sasakian spaces. Previously, and also from the algebraic point of view, the invariant connections on homogeneous 3-Sasakian spaces were studied in detail in [DOP20]. Importantly, they gave a characterization of these spaces in terms of purely Lie theoretic data called *3-Sasakian data*:

Theorem 2.4.2. *(Based on [DOP20, Def. 4.1, Thm. 4.3]). Let $M^{4n-1} = G/H$ be a homogeneous space with connected isotropy group H, satisfying the following properties:*

(i) $\mathfrak{g}$ *is compact;*

(ii) $\mathfrak{g}$ *is simple and there is a $\mathbb{Z}_2$-graded decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h}$;*

(iii) There exists an $\mathfrak{h}^{\mathbb{C}}$-module U of complex dimension $2(n-1)$ such that $\mathfrak{g}_1^{\mathbb{C}} \cong \mathbb{C}^2 \otimes U$ as a module for $\mathfrak{g}_0^{\mathbb{C}} \cong \mathfrak{sp}(1)^{\mathbb{C}} \oplus \mathfrak{h}^{\mathbb{C}}$, where $\mathbb{C}^2$ is the standard representation of $\mathfrak{sp}(1)^{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$.

Then there is a homogeneous 3-Sasakian structure $(\xi_i, \eta_i, \varphi_i)$ on $M = G/H$ determined by the tensors

$$\xi_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \xi_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \varphi_i = \frac{1}{2}\,\mathrm{ad}(\xi_i)|_{\mathfrak{sp}(1)} + \mathrm{ad}(\xi_i)|_{\mathfrak{g}_1},$$
$$g := -\frac{1}{4(n+1)}\kappa|_{\mathfrak{sp}(1)\times\mathfrak{sp}(1)} - \frac{1}{8(n+1)}\kappa|_{\mathfrak{g}_1\times\mathfrak{g}_1},$$

where κ denotes the Killing form of $\mathfrak{g}$. Furthermore, the Nomizu map of the Levi-Civita connection ∇^g is given by

$$\Lambda^g(X)Y = \begin{cases} \frac{1}{2}[X,Y]_{\mathfrak{m}} & \text{if } X, Y \in \mathfrak{sp}(1) \text{ or } X, Y \in \mathfrak{g}_1, \\ 0 & \text{if } X \in \mathfrak{sp}(1),\ Y \in \mathfrak{g}_1, \\ [X,Y]_{\mathfrak{m}} & \text{if } X \in \mathfrak{g}_1,\ Y \in \mathfrak{sp}(1), \end{cases} \tag{2.7}$$

where subscript $\mathfrak{m}$ denotes projection onto the reductive complement $\mathfrak{m} := \mathfrak{sp}(1) \oplus \mathfrak{g}_1$.

Indeed, they proved that all simply-connected homogeneous 3-Sasakian spaces can be constructed from 3-Sasakian data, and they gave an explicit description of the data in each case which will be extremely useful for our purposes.

Finally, we recall the result of Agricola, Dileo, and Stecker that, in the homogeneous case, the Nomizu map of the canonical connection (see Theorem 2.3.7) takes a simple form:

Proposition 2.4.3. *(Based on [ADS21, Prop. 4.2.1]). For a homogeneous 3-Sasakian space, the Nomizu map* Λ *of the canonical connection* ∇ *is given by*

$$\Lambda(X) = \begin{cases} -\operatorname{ad}(X) & \text{if } X \in \mathcal{V}, \\ 0 & \text{if } X \in \mathcal{H}. \end{cases} \tag{2.8}$$

2.5 3-(α,δ)-Sasaki Structures

In this section we recall the basic definitions and properties of 3-(α,δ)-Sasaki manifolds. These structures generalize 3-Sasakian structures (which are recovered by setting $\alpha = \delta = 1$), however we shall take a different viewpoint than that presented in Chapter 2.3 in order to emphasize the role played by the real constants α, β; we will follow instead the notation and exposition laid out in the foundational paper [AD20].

Definition 2.5.1. (Almost contact, almost contact metric, normal, contact metric, Reeb vector field, horizontal and vertical spaces, fundamental 2-form, α-Sasakian). An *almost contact* manifold $(M^{2n-1}, \xi, \eta, \varphi)$ consists of an odd-dimensional manifold M^{2n-1} together with a vector field ξ (the *Reeb vector field*), a 1-form η, and an endomorphism field $\varphi \in \operatorname{End}(TM)$ satisfying

$$\varphi^2 = -\operatorname{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

The tensors ξ, η give rise to a decomposition $TM = \mathcal{V} \oplus \mathcal{H}$ into *vertical* and *horizontal* spaces

$$\mathcal{V} := \operatorname{span}_{\mathbb{R}}\{\xi\}, \quad \mathcal{H} := \ker \eta,$$

and it is well-known (see e.g. [Bla10, Chapter 4.1]) that there exists a Riemannian metric g which is compatible with the almost contact structure in the following sense:

$$g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X)\eta(Y) \quad \text{for all } X, Y \in TM. \tag{2.9}$$

By considering separately the horizontal and vertical directions, we note that the compatibility equation (2.9) simply encodes the fact that the metric g is φ-invariant in the horizontal directions and renders $\mathcal{H} \perp \mathcal{V}$ and $||\xi||^2 = 1$. The *fundamental 2-form* is defined by $\Phi := g(\cdot, \varphi(\cdot))$. An *almost contact metric* manifold $(M, g, \xi, \eta, \varphi)$ consists of an almost contact manifold (M, ξ, η, φ) together with a compatible metric g, and, if the additional condition

$$d\eta = 2\alpha\Phi, \qquad \alpha \in \mathbb{R} \setminus \{0\}$$

is satisfied, this is called an *α-contact metric manifold* (or simply a *contact metric manifold* in the case $\alpha = 1$). An almost contact manifold (M, ξ, η, φ), is called *normal* if the (modified) Nijenhuis tensor

$$N_\varphi(X,Y) := [\varphi(X), \varphi(Y)] + \varphi^2[X,Y] - \varphi[\varphi(X), Y] - \varphi[X, \varphi(Y)] + d\eta(X,Y)\xi, \quad X, Y \in TM$$

vanishes. A normal α-contact metric manifold is called *α-Sasakian* (or simply *Sasakian* if $\alpha = 1$).

Remark 2.5.2. It follows immediately from [BG99, Prop. 2.1.2, Prop. 2.1.3] that a Sasakian manifold (in the sense of Definition 2.3.1) is a 1-Sasakian manifold in the sense of the preceding definition, so the notion of α-Sasakian structures is truly a generalization of Sasakian structures.

Next, we define the corresponding "deformed" objects generalizing 3-Sasakian manifolds. By Remark 2.3.5, 3-Sasakian manifolds are determined by just a pair of Sasakian structures, so any generalization of α-Sasakian structures to the 3-contact setting should depend only on a pair of real parameters, which we call α, δ.

Definition 2.5.3. (Almost 3-contact, almost 3-contact metric, 3-(α,δ)-Sasaki). In a similar spirit to the previous definition, an *almost 3-contact manifold* $(M^{4n-1}, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ consists of a manifold M of dimension $4n-1$ together with three almost contact structures $(\xi_i, \eta_i, \varphi_i)$ satisfying

$$\begin{aligned} &\varphi_i = \varphi_j \circ \varphi_k - \eta_k \otimes \xi_j = -\varphi_k \circ \varphi_j + \eta_j \otimes \xi_k, \\ &\varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k, \\ &\eta_i = \eta_j \circ \varphi_k = -\eta_k \circ \varphi_j, \end{aligned}$$

for all even permutations (i, j, k) of $(1, 2, 3)$. There is again a splitting $TM = \mathcal{V} \oplus \mathcal{H}$ into *vertical* and *horizontal* spaces defined by

$$\mathcal{V} := \mathrm{span}_{\mathbb{R}}\{\xi_i\}_{i=1,2,3}, \quad \mathcal{H} := \cap_{i=1,2,3} \ker(\eta_i).$$

The horizontal restrictions $\varphi_i|_{\mathcal{H}}$ satisfy the defining relations of the quaternions,

$$\varphi_i|_{\mathcal{H}} \circ \varphi_j|_{\mathcal{H}} = \varphi_k|_{\mathcal{H}}$$

for all even permutations (i, j, k) of $(1, 2, 3)$, so in particular this defines a quaternionic contact structure. The action of $\varphi_1, \varphi_2, \varphi_3$ in the vertical directions is determined by

$$\varphi_i(\xi_j) = \xi_k$$

for all even permutations (i, j, k) of $(1, 2, 3)$. Notably, it is known from [Kuo70] that any almost 3-contact manifold $(M, \xi_i, \eta_i, \varphi_i)$ admits a metric g which is compatible in the sense of (2.9) with all three almost contact structures, and the data $(M, g, \xi_i, \eta_i, \varphi_i)$ is called an *almost 3-contact metric manifold*. A *3-(α, δ)-Sasaki manifold* is an almost 3-contact metric manifold $(M, g, \xi_i, \eta_i, \varphi_i)$ satisfying the additional equation

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$$

for all even permutations (i, j, k) of $(1, 2, 3)$.

Remark 2.5.4. Clearly this notion contains the 3-Sasakian spaces ($\alpha = \delta = 1$), however it also contains many other interesting classes such as the Einstein 3-α-Sasakian spaces ($\alpha = \delta$), parallel spaces ($\delta = 2\alpha$), degenerate spaces ($\delta = 0$), and a second Einstein metric ($\delta = (2n+1)\alpha$, where $\dim M := 4n - 1$).

Remark 2.5.5. While 3-(α, δ)-Sasaki manifolds do not in general admit Riemannian Killing spinors, one expects to recover the Killing spinors in the 3-Sasakian case ($\alpha = \delta = 1$) as members of a family satisfying a deformed Killing spinor equation. Using a modified version of the argument in [FK90] we will prove in Chapter 5 the existence of such spinors in all dimensions.

Next we recall that, while it is impossible in general to find a nice connection parallelizing all the structure tensors of a 3-(α, δ)-Sasaki manifold, there is nonetheless a good choice available, generalizing the canonical connection of a 3-Sasakian manifold (see Theorem 2.3.7):

Theorem 2.5.6. *(Based on Theorem 4.1.1 and the discussion in Section 4 of [AD20]). Let*

$(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-(α,δ)-Sasaki manifold. Then M admits a unique metric connection ∇ with skew torsion such that, for some function $\beta \in C^\infty(M)$,

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \Gamma(TM).$$

Furthermore the function β is determined by $\beta = 2(\delta - 2\alpha)$ and the torsion of ∇ is the 3-form

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta - \alpha)\eta_1 \wedge \eta_2 \wedge \eta_3.$$

The derivatives of the structure tensors are given by

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k), \quad \nabla_X \xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k), \quad \nabla_X \eta_i = \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k),$$

and, in particular, are parallel in the horizontal directions.

The connection ∇ from Theorem 2.5.6 is called the *canonical connection* of the 3-(α,δ)-Sasaki structure. It is an important member of the class of metric connections with parallel skew torsion, which were studied in detail in [CMS21].

Finally, we review the algebraic description from [ADS21] of homogeneous 3-(α,δ)-Sasaki spaces fibering over Wolf spaces in terms of *generalized 3-Sasakian data*, generalizing the 3-Sasakian data recalled above in Theorem 2.4.2. Indeed, by omitting the compactness requirement in the definition of 3-Sasakian data (correspondingly, omitting assumption (i) in Theorem 2.4.2), they define the notion of *generalized 3-Sasakian data* and prove:

Theorem 2.5.7. *(Based on [ADS21, Thm. 3.1.1 and Prop. 4.2.2]). Let $M^{4n-1} = G/H$ be a homogeneous space with connected isotropy group H, satisfying the following properties:*

- *(i) $\mathfrak{g}$ is simple and there is a $\mathbb{Z}_2$-graded decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h}$;*
- *(ii) There exists an $\mathfrak{h}^{\mathbb{C}}$-module U of complex dimension $2(n-1)$ such that $\mathfrak{g}_1^{\mathbb{C}} \cong \mathbb{C}^2 \otimes U$ as a module for $\mathfrak{g}_0^{\mathbb{C}} \cong \mathfrak{sp}(1)^{\mathbb{C}} \oplus \mathfrak{h}^{\mathbb{C}}$, where $\mathbb{C}^2$ is the standard representation of $\mathfrak{sp}(1)^{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C})$.*

Suppose also that $\alpha\delta > 0$ if G is compact and $\alpha\delta < 0$ if G is non-compact. Then there is a homogeneous 3-(α,δ)-Sasaki structure $(\xi_i, \eta_i, \varphi_i)$ on $M = G/H$ determined by the tensors

$$\xi_1 := \delta \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \xi_2 := \delta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 := \delta \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \varphi_i = \frac{1}{2\delta}\operatorname{ad}(\xi_i)|_{\mathfrak{sp}(1)} + \frac{1}{\delta}\operatorname{ad}(\xi_i)|_{\mathfrak{g}_1},$$
$$g := -\frac{1}{4\delta^2(n+1)}\kappa|_{\mathfrak{sp}(1)\times\mathfrak{sp}(1)} - \frac{1}{8\alpha\delta(n+1)}\kappa|_{\mathfrak{g}_1\times\mathfrak{g}_1},$$

where κ denotes the Killing form of $\mathfrak{g}$. Furthermore, the Nomizu map Λ^g of the Levi-Civita connection ∇^g is given by

$$\Lambda^g(X)Y = \begin{cases} \frac{1}{2}[X,Y]_{\mathfrak{m}} & \text{if } X,Y \in \mathfrak{sp}(1) \text{ or } X,Y \in \mathfrak{g}_1, \\ (1-\frac{\alpha}{\delta})[X,Y]_{\mathfrak{m}} & \text{if } X \in \mathfrak{sp}(1),\ Y \in \mathfrak{g}_1, \\ \frac{\alpha}{\delta}[X,Y]_{\mathfrak{m}} & \text{if } X \in \mathfrak{g}_1,\ Y \in \mathfrak{sp}(1), \end{cases} \tag{2.10}$$

where subscript $\mathfrak{m}$ denotes projection onto the reductive complement $\mathfrak{m} := \mathfrak{sp}(1) \oplus \mathfrak{g}_1$.

It is clear that taking G compact and substituting $\alpha = \delta = 1$ above recovers the result from [DOP20] for the homogeneous 3-Sasakian case (cf. Theorem 2.4.2). Furthermore, for homogeneous 3-(α,δ)-Sasaki manifolds fibering over Wolf spaces, a simple expression for the Nomizu map of the canonical connection (recalled in Theorem 2.5.6) is available:

Proposition 2.5.8. *(Based on [ADS21, Prop. 4.2.1]). If $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is a homogeneous 3-(α,δ)-Sasaki manifold fibering over a Wolf space, then the Nomizu map of the canonical connection is given by*

$$\Lambda(X) = \begin{cases} \frac{\delta-2\alpha}{\delta}\,\mathrm{ad}(X) & \text{if } X \in \mathcal{V}, \\ 0 & \text{if } X \in \mathcal{H}. \end{cases} \tag{2.11}$$

2.6 Invariant Metric Connections on Homogeneous Spaces

By applying translations, any invariant connection on G/H is uniquely determined by its value at the origin $o := eH$, and a similar principle applies to invariant spinorial connections. Under the identification $\mathfrak{m} \cong T_o(G/H)$, it follows from a result of Nomizu in [Nom54], later generalized by Wang in [Wan58], that an invariant metric connection corresponds to the data of an $\mathrm{Ad}(H)$-equivariant *Nomizu map*

$$\Lambda\colon \mathfrak{m} \to \mathfrak{so}(\mathfrak{m}). \tag{2.12}$$

Explicitly, the relationship between the Nomizu map Λ and the covariant derivative ∇ associated to the connection is

$$(\nabla_{\widehat{X}}\omega)_o = \Lambda(X)\omega_o, \quad X \in \mathfrak{m},$$

for any invariant tensor (or invariant differential form) ω, where $\widehat{X}$ is the fundamental vector field associated to $X \in \mathfrak{m}$ and the action of $\Lambda(X) \in \mathfrak{so}(\mathfrak{m})$ on ω_o is the natural one (see [ANT23, Chapter 6] for a modern treatment of the topic). Moreover, by [KN69, Prop. 2.3], the torsion and curvature tensors of ∇ are given at the origin by

$$T_o(X,Y) = \Lambda(X)Y - \Lambda(Y)X - [X,Y]_{\mathfrak{m}}, \tag{2.13}$$

$$R_o(X,Y) = [\Lambda(X),\Lambda(Y)] - \Lambda([X,Y]_{\mathfrak{m}}) - \mathrm{ad}([X,Y]_{\mathfrak{h}}), \tag{2.14}$$

for all $X, Y \in \mathfrak{m}$. Composing (2.12) with the Lie algebra isomorphism $\mathfrak{spin}(\mathfrak{m}) \cong \mathfrak{so}(\mathfrak{m})$ gives the Nomizu map

$$\widetilde{\Lambda}\colon \mathfrak{m} \to \mathfrak{spin}(\mathfrak{m})$$

associated to the spin lift $\widetilde{\nabla}$ of ∇, and the covariant derivative at the origin of an invariant spinor ψ can be similarly described via

$$(\widetilde{\nabla}_{\widehat{X}})_o\psi = \widetilde{\Lambda}(X)\cdot\psi_o,$$

where the action of $\widetilde{\Lambda}(X)$ on ψ_0 is via the spin representation.

For an invariant Riemannian metric g, the Nomizu map $\Lambda^g\colon \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ of the Levi-Civita connection is given by

$$\Lambda^g(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y), \quad \forall X, Y \in \mathfrak{m}, \tag{2.15}$$

where the symmetric $(2,0)$-tensor U is determined by

$$2g(U(X,Y),Z) = g([Z,X]_{\mathfrak{m}},Y) + g(X,[Z,Y]_{\mathfrak{m}}). \tag{2.16}$$

For a proof of this fact we refer to [Nom54, Thm. 13.1], noting that there is a sign error in Equation (13.1).

Another geometrically significant invariant connection is the *Ambrose-Singer connection*, sometimes called the canonical connection, whose horizontal distribution is generated by left translations of $\mathfrak{m} \subset \mathfrak{g}$. Such a connection is unique after fixing a reductive complement $\mathfrak{m}$. In this thesis we shall always refer to this as the Ambrose-Singer connection, and reserve the term canonical connection for the distinguished metric connection on 3-(α,δ)-Sasaki manifolds recalled in Theorem 2.5.6. The Ambrose-Singer connection has Nomizu map identically equal to zero,

$$\Lambda^{\mathrm{AS}} \equiv 0,$$

and it parallelizes all invariant tensors [KN69, Prop. 2.7]. Noting that the Ambrose-Singer connection coincides with the Levi-Civita connection if and only if its torsion tensor vanishes, it is evident from (2.13) that they coincide precisely when the underlying space is symmetric. One sees furthermore that the Ambrose-Singer torsion is totally skew-symmetric (i.e. a 3-form) if and only if g is a naturally reductive metric,

$$g([X,Y]_{\mathfrak{m}}, Z) + g(Y, [X,Z]_{\mathfrak{m}}) = 0, \quad \forall X, Y, Z \in \mathfrak{m}.$$

Fixing notation, the Levi-Civita and Ambrose-Singer connections, their corresponding Nomizu maps, and their torsion tensors will be denoted by ∇^g, ∇^{AS}, Λ^g, Λ^{AS}, and T^g, T^{AS} respectively. By abuse of notation we shall denote the corresponding spinorial connections also by ∇^g, ∇^{AS}, and the associated spinorial Nomizu maps by $\widetilde{\Lambda}^g$, $\widetilde{\Lambda}^{\mathrm{AS}}$. Any other connections used will be introduced in the relevant sections.

2.7 Metric Connections with Torsion

Let (M^n, g) be an n-dimensional Riemannian manifold. In certain situations it will be advantageous to consider metric connections other than the Levi-Civita connection which are better adapted to the geometry at hand. Such connections are uniquely determined by their torsion tensor, and for a detailed introduction to the subject we refer to [Agr06]. The space of possible torsion tensors is given by

$$\mathcal{T} := \{T \in TM^{\otimes 3} \colon T(X,Y,Z) + T(Y,X,Z) = 0\},$$

and it splits as an $\mathrm{O}(n)$-representation into three inequivalent irreducible submodules,

$$\mathcal{T} \simeq \mathcal{T}_{\mathrm{vec}} \oplus \mathcal{T}_{\mathrm{skew}} \oplus \mathcal{T}_{\mathrm{CT}},$$

called *torsion classes*. Metric connections with torsion in these three spaces are called *vectorial*, *totally skew-symmetric*, and *cyclic traceless* respectively.

For a metric connection ∇, the *difference tensor* is defined by

$$A(X,Y) := \nabla_X Y - \nabla^g_X Y.$$

We note that the space

$$\mathcal{A}^g := \{A \in TM^{\otimes 3} \colon A(X,Y,Z) + A(X,Z,Y) = 0\}$$

of possible difference tensors is isomorphic to $\mathcal{T}$ as $\mathrm{O}(n)$-representations via

$$T(X,Y,Z) = A(X,Y,Z) - A(Y,X,Z), \tag{2.17}$$
$$A(X,Y,Z) = \frac{1}{2}(T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y)). \tag{2.18}$$

Let ∇ be a metric connection with torsion $T \in \mathcal{T}$ and difference tensor $A \in \mathcal{A}^g$. With respect to an arbitrary orthonormal frame $e_1, \dots, e_n$, we define the trace

$$c_{12}(A) := \sum_{i=1}^{n} A(e_i, e_i, -).$$

The images of the torsion classes under the isomorphism (2.18) are given in [TV83, Chap. 3] by

$$\begin{aligned}
\mathcal{A}^g_{\text{vec}} &= \{A \in \mathcal{A}^g \colon A(X,Y,Z) = g(X,Y)g(V,Z) - g(X,Z)g(V,Y),\ V \in TM\},\\
\mathcal{A}^g_{\text{skew}} &= \{A \in \mathcal{A}^g \colon A(X,Y,Z) + A(Y,X,Z) = 0\},\\
\mathcal{A}^g_{\text{CT}} &= \{A \in \mathcal{A}^g \colon \mathfrak{S}_{X,Y,Z} A(X,Y,Z) = 0,\ c_{12}(A) = 0\},
\end{aligned}$$

as well as explicit formulas for the projections of A onto each class,

$$A_{\text{vec}}(X,Y,Z) = g(X,Y)\phi(Z) - g(X,Z)\phi(Y), \tag{2.19}$$
$$A_{\text{skew}}(X,Y,Z) = \frac{1}{3}\mathfrak{S}_{X,Y,Z} A(X,Y,Z), \tag{2.20}$$
$$A_{\text{CT}}(X,Y,Z) = A(X,Y,Z) - A_{\text{vec}}(X,Y,Z) - A_{\text{skew}}(X,Y,Z), \tag{2.21}$$

where $\phi(v) := \frac{1}{n-1} c_{12}(A)(v)$ for all $v \in TM$. Formulas for the projections of T onto the three torsion classes may then be easily deduced using (2.17). We shall make frequent use of the formulas in this section to calculate the torsion type of the Ambrose-Singer connection for each case in Table 1.1.

2.8 Matrix Lie Algebras

Let us fix notation related to matrix Lie algebras. We will use $E^{(n)}_{i,j}$ (resp. $F^{(n)}_{i,j}$) throughout to denote the elementary skew-symmetric $n \times n$ matrix (resp. the elementary symmetric $n \times n$ matrix),

$$E^{(n)}_{i,j} = \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \quad & i & j & \quad \end{array} \\ \begin{bmatrix} & & \vdots & \\ & & -1 & \cdots \\ \cdots & 1 & & \\ & \vdots & & \end{bmatrix} \end{array}, \quad F^{(n)}_{i,j} = \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \quad & i & j & \quad \end{array} \\ \begin{bmatrix} & & \vdots & \\ & & 1 & \cdots \\ \cdots & 1 & & \\ & \vdots & & \end{bmatrix} \end{array}.$$

We also adopt the convention that $F^{(n)}_{i,i}$ is the diagonal matrix with 1 in the (i,i) position and zeros elsewhere. The following commutator relations will be used extensively throughout the thesis for calculations involving matrix Lie algebras.

$$[E^{(n)}_{i,j}, E^{(n)}_{k,l}] = \begin{cases} E^{(n)}_{j,l} & \text{if } i = k, \\ 0 & \text{if } i,j,k,l \text{ distinct}, \end{cases}$$

$$[E^{(n)}_{i,j}, F^{(n)}_{k,l}] = \begin{cases} F^{(n)}_{j,l} & \text{if } i = k,\ j \neq l,\ k \neq l, \\ 2(F^{(n)}_{j,j} - F^{(n)}_{i,i}) & \text{if } i = k,\ j = l,\ k \neq l, \\ (\delta_{i,k} F^{(n)}_{j,k} - \delta_{j,k} F^{(n)}_{i,k}) & \text{if } k = l, \\ 0 & \text{if } i,j,k,l \text{ distinct}, \end{cases}$$

$$[F^{(n)}_{p,q}, F^{(n)}_{r,s}] = \begin{cases} -E^{(n)}_{q,s} & \text{if } p = r,\ q \neq s,\ p \neq q,\ r \neq s, \\ (-\delta_{q,r} E^{(n)}_{p,r} - \delta_{p,r} E^{(n)}_{q,r}) & \text{if } p \neq q,\ r = s, \\ 0 & \text{if } p = q,\ r = s, \\ 0 & \text{if } p,q,r,s \text{ distinct}, \end{cases}$$

$$[\lambda_1 F^{(n)}_{p,q}, \lambda_2 F^{(n)}_{r,s}] = \begin{cases} \lambda_3 F^{(n)}_{q,s} & \text{if } p = r,\ q \neq s,\ p \neq q,\ r \neq s, \\ 2\lambda_3 (F^{(n)}_{p,p} + F^{(n)}_{q,q}) & \text{if } p = r,\ q = s,\ p \neq q, \\ \lambda_3 (\delta_{q,r} F^{(n)}_{p,r} + \delta_{p,r} F^{(n)}_{q,r}) & \text{if } p \neq q,\ r = s, \\ 2\delta_{p,r} \lambda_3 F^{(n)}_{p,p} & \text{if } p = q,\ r = s, \\ 0 & \text{if } p,q,r,s \text{ distinct}, \end{cases}$$

where $(\lambda_1, \lambda_2, \lambda_3)$ is an even permutation of the imaginary quaternions (i, j, k). Note that this doesn't cover all possible cases, however the rest can be deduced from above using skew-symmetry (resp. symmetry) of the matrices $E_{i,j}^{(n)}$ (resp. $F_{i,j}^{(n)}$). We shall use B_0 to denote the bilinar form on the space of matrices (of the appropriate size, depending on context) given by

$$B_0(X_1, X_2) := -\Re \operatorname{tr}(X_1 X_2). \tag{2.22}$$

After fixing an invariant inner product on $\mathfrak{m}$, an orthonormal basis will be denoted by $e_1, \ldots, e_{\dim \mathfrak{m}}$, and the shorthand $e_{i_1,\ldots,i_p} := e_{i_1} \wedge \cdots \wedge e_{i_p}$ for differential forms will be used.

3

Invariant Spinors on Homogeneous Spheres

This chapter contains joint work with Prof. Dr. habil. Ilka Agricola and Dr. Marie-Amélie Lawn which has appeared, in large part, in [AHL23] (see page 8).

3.1 Classical Spheres, Part I: Spheres over $\mathbb{R}$ and $\mathbb{C}$

3.1.1 Symmetric Spheres, $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$

The isotropy representation here is the standard representation of $\mathrm{SO}(n-1)$ on $\mathbb{R}^{n-1}$, which is irreducible, hence the only invariant metrics correspond to negative multiples of the Killing form (equivalently, positive multiples of B_0). We remark that any such metric is naturally reductive. The embedding $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$ may be realized as the the lower right hand $(n-1)\times(n-1)$ block, and we choose the reductive complement $\mathfrak{m} = \mathfrak{so}(n-1)^{\perp}$ with respect to the Killing form. Explicitly,

$$\begin{aligned}\mathfrak{so}(n) &= \mathrm{span}_{\mathbb{R}}\{E^{(n)}_{i,j}\}_{1\leq i<j\leq n},\\ \mathfrak{so}(n-1) &= \mathrm{span}_{\mathbb{R}}\{E^{(n)}_{i,j}\}_{2\leq i<j\leq n},\end{aligned}$$

and

$$\mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{E^{(n)}_{1,j}\}_{2\leq j\leq n}.$$

One sees immediately from the main proposition in [Wan89] that these standard round spheres are not very interesting from the viewpoint of homogeneous spin geometry:

Theorem 3.1.1. *The spheres $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ do not admit a non-trivial invariant spinor for any choice of invariant metric.*

Remark 3.1.2. Since any invariant metric on $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ is normal homogeneous, and in particular naturally reductive, the Ambrose-Singer connection always has totally skew-symmetric torsion, $T^{\mathrm{AS}} \in \mathcal{T}_{\mathrm{skew}}$ (in fact, $T^{\mathrm{AS}} = 0$ since the space is symmetric). This also applies to the other two realizations of the sphere with irreducible isotropy representation, $S^6 = \mathrm{G}_2/\mathrm{SU}(3)$ and $S^7 = \mathrm{Spin}(7)/\mathrm{G}_2$.

3.1.2 Hermitian Spheres, $S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1)$

The isotropy representation splits into one copy of the trivial representation $\mathbb{R}$ and one copy of $\mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1}$, leading to a 2-parameter family of invariant metrics. Note, however, that the Killing form is no longer non-degenerate so more care must be taken when choosing a reductive complement. The embedding $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n)$ may be realized as the lower right hand $(n-1)\times(n-1)$ block, leading to the realization of Lie algebras given by

$$\mathfrak{u}(n) = \mathrm{span}_{\mathbb{R}}\{E^{(n)}_{j,k}, iF^{(n)}_{p,q}\}_{\substack{1\le j<k\le n,\\ p,q=1,\dots n}}$$
$$\mathfrak{u}(n-1) = \mathrm{span}_{\mathbb{R}}\{E^{(n)}_{j,k}, iF^{(n)}_{p,q}\}_{\substack{2\le j<k\le n,\\ p,q=2,\dots n}}$$

and one verifies that

$$\mathfrak{m} := \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{1,1}, E^{(n)}_{1,j+1}, iF^{(n)}_{1,j+1}\}_{1\le j\le n-1}$$

is a reductive complement. The two irreducible isotropy submodules are given by

$$\mathfrak{m}_1 := \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{1,1}\}, \quad \mathfrak{m}_2 := \mathrm{span}_{\mathbb{R}}\{E^{(n)}_{1,j+1}, iF^{(n)}_{1,j+1}\}_{1\le j\le n-1},$$

and the 2-parameter family of invariant metrics is given by

$$g_{a,b} := aB_0|_{\mathfrak{m}_1\times\mathfrak{m}_1} + bB_0|_{\mathfrak{m}_2\times\mathfrak{m}_2}, \quad a,b>0.$$

These spheres are the complex analog of the previous case and, as such, one may deduce a similar result about the space of invariant spinors from [Wan89] by noting that $\mathfrak{m}_2 \simeq \mathbb{C}^{n-1}$ is isomorphic to the standard representation of $\mathrm{U}(n-1)$ and that the spinor modules in dimensions $2n-2$ and $2n-1$ are naturally identified. Here we give an alternative elementary proof of this result:

Theorem 3.1.3. *The spheres $S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1)$ do not admit a non-trivial invariant* *spinor for any choice of invariant metric.*

Proof. The basis $e_1, \dots, e_{2n-1}$ for $\mathfrak{m}$ given by

$$e_1 := \frac{1}{\sqrt{a}} iF^{(n)}_{1,1}, \quad e_{2j} := \frac{1}{\sqrt{2b}} E^{(n)}_{1,j+1}, \quad e_{2j+1} := \frac{1}{\sqrt{2b}} iF^{(n)}_{1,j+1},$$

for $j = 1, \dots, n-1$ is orthonormal with respect to $g_{a,b}$, and the isotropy algebra is spanned by the operators

$$\begin{aligned} \mathrm{ad}(E^{(n)}_{j,k}) &= e_{2j-2} \wedge e_{2k-2} + e_{2j-1} \wedge e_{2k-1}, \\ \mathrm{ad}(iF^{(n)}_{p,q}) &= e_{2p-2} \wedge e_{2q-1} + e_{2q-2} \wedge e_{2p-1} \quad (p \neq q), \\ \mathrm{ad}(iF^{(n)}_{p,p}) &= e_{2p-2} \wedge e_{2p-1}. \end{aligned}$$

where $2 \leq j < k \leq n$ and $p, q = 2, \dots, n$. In particular the lifts of the operators $\mathrm{ad}(iF^{(n+1)}_{p,p})$ act on the spinor bundle via Clifford multiplication by $\frac{1}{2} e_{2p-2} \cdot e_{2p-1}$, and the result then follows by noting that if $\psi \in \Sigma_{\mathrm{inv}}$ then

$$0 = ||e_{2p-2} \cdot e_{2p-1} \cdot \psi||^2 = \langle e_{2p-2} \cdot e_{2p-1} \cdot \psi, e_{2p-2} \cdot e_{2p-1} \cdot \psi \rangle = \langle \psi, \psi \rangle = ||\psi||^2.$$

□

Next, we calculate the Ambrose-Singer torsion and determine its type:

Proposition 3.1.4. *For any $a, b > 0$ the sphere $(S^{2n-1} = \mathrm{U}(n)/\mathrm{U}(n-1), g_{a,b})$ has Ambrose-Singer torsion of type $\mathcal{T}_{\mathrm{skew}} \oplus \mathcal{T}_{\mathrm{CT}}$, given by*

$$\begin{aligned} T^{\mathrm{AS}}(e_1, e_{2j}) &= \frac{1}{\sqrt{a}} e_{2j+1}, \quad T^{\mathrm{AS}}(e_1, e_{2j+1}) = \frac{-1}{\sqrt{a}} e_{2j}, \\ T^{\mathrm{AS}}(e_{2j}, e_{2l}) &= T^{\mathrm{AS}}(e_{2j+1}, e_{2l+1}) = 0, \quad T^{\mathrm{AS}}(e_{2j}, e_{2l+1}) = \frac{\delta_{j,l} \sqrt{a}}{b} e_1, \end{aligned}$$

for all $j, l = 1, \dots, n-1$. The projection of T^{AS} onto $\mathcal{T}_{\mathrm{skew}}$ is

$$T^{\mathrm{AS}}_{\mathrm{skew}} := \left(\frac{a + 2b}{3b\sqrt{a}} \right) \sum_{j=1}^{n-1} e_1 \wedge e_{2j} \wedge e_{2j+1},$$

with $T^{\mathrm{AS}} = T^{\mathrm{AS}}_{\mathrm{skew}}$ if and only if $a = b$ (i.e. $g_{a,b}$ is a multiple of the Killing form).

Proof. Straightforward calculation of the commutator relations, and subsequent application of

(2.13), (2.19)-(2.21) and the isomorphism (2.17). □

3.1.3 Special Hermitian Spheres, $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$

The isotropy group $\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n)$ may be realized as the lower right hand $(n-1)\times(n-1)$ block. We take the reductive complement $\mathfrak{m} := \mathfrak{su}(n-1)^{\perp}$, where the orthogonal complement is taken with respect to B_0. At the level of Lie algebras,

$$\mathfrak{su}(n) = \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{p,q}, E^{(n)}_{p,q}, i\big(\sum_{l=2}^{n} F^{(n)}_{l,l} - (n-1)F^{(n)}_{1,1}\big), i(F^{(n)}_{r,r} - F^{(n)}_{r+1,r+1})\}_{\substack{1\le p<q\le n,\\ r=2,\dots,n}}$$

$$\mathfrak{su}(n-1) = \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{p,q}, E^{(n)}_{p,q}, i(F^{(n)}_{r,r} - F^{(n)}_{r+1,r+1})\}_{\substack{2\le p<q\le n,\\ r=2,\dots,n}}$$

and

$$\mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{i\big(\sum_{l=2}^{n} F^{(n)}_{l,l} - (n-1)F^{(n)}_{1,1}\big), iF^{(n)}_{1,p}, E^{(n)}_{1,p}\}_{p=2,\dots,n}.$$

The isotropy representation splits into one copy of the trivial representation and one copy of the standard representation, $\mathfrak{m} \simeq \mathfrak{m}_1 \oplus \mathfrak{m}_2$, leading to the 2-parameter family of invariant metrics

$$g_{a,b} := aB_0|_{\mathfrak{m}_1} + bB_0|_{\mathfrak{m}_2}, \quad a,b>0.$$

A $g_{a,b}$-orthonormal basis of $\mathfrak{m}$ is given by $\{e_i\}_{i=1}^{2n-1}$, where

$$e_1 := \frac{1}{\sqrt{an(n-1)}}\big(\sum_{l=2}^{n} iF^{(n)}_{l,l} - (n-1)iF^{(n)}_{1,1}\big), \quad e_{2p} := \frac{1}{\sqrt{2b}}E^{(n)}_{1,p+1}, \quad e_{2p+1} := \frac{i}{\sqrt{2b}}F^{(n)}_{1,p+1},$$

for $p = 1,\dots,n-1$, and the two isotropy summands are given explicitly in terms of this basis as

$$\mathfrak{m}_1 = \mathrm{span}_{\mathbb{R}}\{e_1\}, \quad \mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{e_2,\dots,e_{2n-1}\}.$$

The complexified algebra $\mathfrak{su}(n)^{\mathbb{C}}$ has a Cartan subalgebra spanned by

$$\tau_k := \frac{1}{\sqrt{k(k+1)}}\big(\sum_{p=2}^{k+1} F^{(n)}_{p,p} - kF^{(n)}_{k+2,k+2}\big), \qquad k=1,\dots,n-2,$$

$$\tau_{n-1} := i\sqrt{a}e_1,$$

and the elements $\tau_1, \dots, \tau_{n-2}$ span a Cartan subalgebra for the complexified isotropy algebra $\mathfrak{su}(n-1)^{\mathbb{C}}$. A straightforward calculation then gives,

Proposition 3.1.5. *The above Cartan subalgebra of $\mathfrak{su}(n-1)^{\mathbb{C}}$ acts on $\mathfrak{m}^{\mathbb{C}}$ via*

$$\mathrm{ad}(\tau_k)e_{2p} = \begin{cases} \frac{-i}{\sqrt{k(k+1)}} e_{2p+1} & \text{if } p \leq k, \\ \frac{ik}{\sqrt{k(k+1)}} e_{2p+1} & \text{if } p = k+1, \\ 0 & \text{if } p \geq k+2, \end{cases} \qquad \mathrm{ad}(\tau_k)e_{2p+1} = \begin{cases} \frac{i}{\sqrt{k(k+1)}} e_{2p} & \text{if } p \leq k, \\ \frac{-ik}{\sqrt{k(k+1)}} e_{2p} & \text{if } p = k+1, \\ 0 & \text{if } p \geq k+2, \end{cases}$$

and $\mathrm{ad}(\tau_k)e_1 = 0$ *for* $k = 1, \dots, n-2$.

Corollary 3.1.6. *The isotropy representation maps the above Cartan subalgebra of $\mathfrak{su}(n-1)^{\mathbb{C}}$ into $\mathfrak{so}(\mathfrak{m}^{\mathbb{C}}, g^{\mathbb{C}}_{a,b}) \cong \mathfrak{so}(2n-1, \mathbb{C})$ as the operators*

$$\tau_k \mapsto \mathrm{ad}(\tau_k)|_{\mathfrak{m}^{\mathbb{C}}} = \frac{-i}{\sqrt{k(k+1)}} \left(\sum_{p=1}^{k} e_{2p} \wedge e_{2p+1} - k e_{2k+2} \wedge e_{2k+3} \right), \tag{3.1}$$

for $k = 1, \dots, n-2$.

Theorem 3.1.7. *Using the above orthonormal basis and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors on $(S^{2n-1} = \frac{\mathrm{SU}(n)}{\mathrm{SU}(n-1)}, g_{a,b})$ for any $a, b > 0$ is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{\psi_+ := 1, \ \psi_- := y_1 \wedge y_2 \wedge \cdots \wedge y_{n-1}\}.$$

Proof. Considering the spin lifts of the operators in (3.1), one notes that $\widetilde{\mathrm{ad}(\tau_k)}|_{\mathfrak{m}^{\mathbb{C}}} \cdot \psi = 0$ for all $k = 1, \dots, n-2$ if and only if $e_{2p} \cdot e_{2p+1} \cdot \psi = e_{2p+2} \cdot e_{2p+3} \cdot \psi$ for all $p = 1, \dots, n-2$. Note that this condition is necessarily satisfied if $\psi \in \Sigma_{\mathrm{inv}}$. Using the Clifford multiplication formulas (2.4), one finds

$$\begin{aligned} e_{2p} \cdot e_{2p+1} \cdot \psi &= i(x_p \lrcorner + y_p \wedge)(y_p \wedge - x_p \lrcorner)\psi = \cdots = i[\psi - 2y_p \wedge (x_p \lrcorner \psi)], \\ e_{2p+2} \cdot e_{2p+3} \cdot \psi &= i(x_{p+1} \lrcorner + y_{p+1} \wedge)(y_{p+1} \wedge - x_{p+1} \lrcorner)\psi = \cdots = i[\psi - 2y_{p+1} \wedge (x_{p+1} \lrcorner \psi)], \end{aligned}$$

and hence $\Sigma_{\mathrm{inv}} \subseteq \mathrm{span}_{\mathbb{C}}\{1, y_1 \wedge y_2 \wedge \cdots \wedge y_{n-1}\}$. Thus it suffices to show that there are two linearly independent invariant spinors. Since the isotropy representation decomposes as the sum of one copy of the trivial representation and one non-trivial module, the number of invariant spinors is independent of the choice of $a, b > 0$. In particular we consider the round metric, corresponding to the parameters $a = \frac{n-1}{n}$, $b = \frac{1}{2}$, together with its usual SU(n)-invariant Sasakian structure (see [DGP18] for a more detailed description). Denoting by $(\varphi, \xi := e_1, \eta := \xi^\flat)$ the Sasakian structure tensors, we recall from Theorem 2.3.8 (using slightly different notation) that the spaces

$$E_\pm := \{\psi \in \Gamma(\Sigma M) \colon (\pm 2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot \psi = 0 \quad \forall X \in TM\}$$

satisfy $\dim(E_+ + E_-) = 2$, and hence it suffices to show that they have a basis consisting of invariant spinors. One also remarks from [DGP18] that φ is an invariant tensor (in fact, using their setup one finds the explicit algebraic description $\varphi = \frac{n-1}{n}\operatorname{ad}(\xi)$). Let $\phi \in \Gamma(E_+)$, so that

$$(2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot \phi = 0 \quad \text{for all } X \in TM.$$

Since φ and ξ are both invariant tensors, it suffices to consider this defining equation at the origin (i.e. for $X \in \mathfrak{m}$). By performing a similar type of calculation as in the proof of [Kat00, Prop. 7.1], it follows that for any $g_0 \in \mathrm{SU}(n)$ we have

$$\begin{aligned}((2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot (g_0\phi))(g) &= (2\varphi(X) \cdot (g_o\phi) + \xi \cdot X \cdot (g_0\phi) - X \cdot \xi \cdot (g_0\phi))(g) \\ &= ((2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot \phi)(g_0^{-1}g) \\ &= 0,\end{aligned}$$

where we have slightly abused notation to denote a spinor and the corresponding SU($n-1$)-equivariant map $\mathrm{SU}(n) \to \Sigma$ by the same symbol. One argues similarly for $\phi \in \Gamma(E_-)$. This shows that the spaces $E_\pm$ are representations of SU(n). But $\dim(E_\pm) \leq 2$ (see Table 2.1), and thus they must be trivial representations for $n \geq 3$, proving the result in these cases. For $n = 2$ the isotropy group is trivial $\mathrm{SU}(1) = \{e\}$, so every spinor is invariant. The spinor module in this dimension is 2-dimensional, so in particular there are two linearly independent invariant spinors. □

Remark 3.1.8. The fact that the space of invariant spinors is 2-dimensional also follows as a consequence of the main proposition in [Wan89] together with the natural identification of the spinor modules in dimensions $2n-2$ and $2n-1$, by noting that the isotropy representation acts trivially on $\mathbb{R}e_1$.

Remark 3.1.9. A priori, choosing a different orthonormal basis for $\mathfrak{m}$ can lead to different expressions for the invariant spinors, since the identification from Chapter 2.1 of spinors with (algebraic) exterior forms is very much basis dependent. This runs counter to the natural expectation that the invariants here should be spanned by 1 and the anti-holomorphic volume form, however, by choosing a well-suited orthonormal basis for $\mathfrak{m}$ one can avoid this problem. Indeed, our chosen $g_{a.b}$-orthonormal basis $\{e_i\}$ is *adapted* to the invariant almost complex structure $\varphi := \sqrt{\frac{a(n-1)}{n}}\,\mathrm{ad}(e_1)$ on $\mathfrak{m}_2 = (\mathbb{R}e_1)^\perp$ in the sense that $\varphi(e_{2p}) = e_{2p+1}$ for $p = 1, \dots, n-1$. This gives $\mathfrak{m}_2$ the structure of a complex representation (which is isomorphic to the standard representation of $\mathrm{SU}(n-1)$ on $\mathbb{C}^{n-1}$), and complexifying the full isotropy representation therefore gives:

$$\mathfrak{m}^{\mathbb{C}} = (\mathfrak{m}_1 \oplus \mathfrak{m}_2)^{\mathbb{C}} \simeq (\mathbb{R} \oplus \mathfrak{m}_2)^{\mathbb{C}} \simeq \mathbb{C} \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_2^*.$$

In particular, this shows that the image of the isotropy representation lies inside $\mathfrak{gl}(L) \subseteq \mathfrak{so}(\mathfrak{m}^{\mathbb{C}})$ (see [AHL23, Section 4.1.2] for the details of this inclusion), which by [AHL23, Prop. 4.5] then implies that $\Sigma \simeq \Lambda^{0,\bullet}\mathfrak{m}$ as complex representations. It follows that the spinors $\psi_+ = 1$ and $\psi_- = y_1 \wedge \cdots \wedge y_{n-1}$ are unaffected by orthonormal changes of adapted basis, since they are unaffected when viewed as anti-holomorphic forms. More generally, this argument shows that it is possible to choose expressions for the spinors in a consistent way whenever G/H admits an invariant orthogonal almost complex structure or an invariant almost contact metric structure, corresponding to the cases $G = \mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\,\mathrm{U}(1)$, and G_2 in this chapter.

Next, we calculate the Ambrose-Singer torsion and determine its type:

Proposition 3.1.10. *For any $a, b > 0$ the sphere $(S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1), g_{a,b})$ has Ambrose-Singer torsion of type $\mathcal{T}_{\mathrm{skew}} \oplus \mathcal{T}_{\mathrm{CT}}$, given by*

$$T^{\mathrm{AS}}(e_1, e_{2p}) = -\sqrt{\frac{n}{a(n-1)}}e_{2p+1}, \quad T^{\mathrm{AS}}(e_1, e_{2p+1}) = \sqrt{\frac{n}{a(n-1)}}e_{2p},$$
$$T^{\mathrm{AS}}(e_{2p}, e_{2q}) = T^{\mathrm{AS}}(e_{2p+1}, e_{2q+1}) = 0, \quad T^{\mathrm{AS}}(e_{2p}, e_{2q+1}) = \frac{-\delta_{p,q}\sqrt{an}}{b\sqrt{n-1}}e_1,$$

for all $p, q = 1, \dots, n-1$. The projection of T^{AS} onto $\mathcal{T}_{\mathrm{skew}}$ is

$$T^{\mathrm{AS}}_{\mathrm{skew}} := -\frac{(a+2b)\sqrt{n}}{3b\sqrt{a(n-1)}}\sum_{p=1}^{n-1} e_1 \wedge e_{2p} \wedge e_{2p+1},$$

with $T^{\mathrm{AS}} = T^{\mathrm{AS}}_{\mathrm{skew}}$ if and only if $a = b$.

In order to differentiate the invariant spinors $\psi_\pm$ from Theorem 3.1.7, we note that the Nomizu map corresponding to $\nabla^{g_{a,b}}$ may be described in terms of the Lie bracket on $\mathfrak{su}(n)$ as follows:

Lemma 3.1.11. *The Nomizu map for the Levi-Civita connection is given by*

$$\Lambda^{g_{a,b}}(x_1)x_2 = 0, \qquad \Lambda^{g_{a,b}}(x)y = (1-\frac{a}{2b})[x,y]_{\mathfrak{m}},$$
$$\Lambda^{g_{a,b}}(y)x = \frac{a}{2b}[y,x]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b}}(y_1)y_2 = \frac{1}{2}[y_1,y_2]_{\mathfrak{m}},$$

for $x, x_1, x_2 \in \mathfrak{m}_1$, $y, y_1, y_2 \in \mathfrak{m}_2$, where $[\ ,\ ]_{\mathfrak{m}}$ denotes the orthogonal projection of the Lie bracket onto $\mathfrak{m} \subseteq \mathfrak{su}(n)$.

These formulas may be proved by directly checking that $\Lambda^{g_{a,b}}$ is skew-symmetric with respect to $g_{a,b}$ and satisfies $\Lambda^{g_{a,b}}(v)w - \Lambda^{g_{a,b}}(w)v - [v,w]_{\mathfrak{m}} = 0$ for all $v, w \in \mathfrak{m}$. Combining the preceding proposition and lemma gives:

Corollary 3.1.12. *The Nomizu map of the Levi-Civita connection on $(S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1), g_{a,b})$ is given in terms of the orthonormal basis $e_1, \dots, e_{2n-1}$ by*

$$\Lambda^{g_{a,b}}(e_1) = (1-\frac{a}{2b})\sqrt{\frac{n}{a(n-1)}}\sum_{l=1}^{n-1} e_{2l}\wedge e_{2l+1}, \quad \Lambda^{g_{a,b}}(e_{2p}) = -\frac{1}{2b}\sqrt{\frac{an}{n-1}}e_1\wedge e_{2p+1},$$
$$\Lambda^{g_{a,b}}(e_{2p+1}) = \frac{1}{2b}\sqrt{\frac{an}{n-1}}e_1\wedge e_{2p},$$

for $p = 1, \dots, n-1$.

Lifting these to the spin bundle and applying them to $\psi_\pm$ gives:

Theorem 3.1.13. *The invariant spinors $\psi_\pm$ are generalized Killing spinors, i.e. $\nabla^{g_{a,b}}_X \psi_\pm = A_\pm(X)\cdot\psi_\pm$, for the endomorphisms*

$$A_+ := \lambda_1 \operatorname{Id}|_{\mathfrak{m}_1} + \lambda_2 \operatorname{Id}|_{\mathfrak{m}_2}, \quad A_- := (-1)^n A_+,$$

where $\lambda_1 := \frac{(2b-a)\sqrt{n(n-1)}}{4b\sqrt{a}}$, $\lambda_2 := \frac{\sqrt{an}}{4b\sqrt{n-1}}$.

Proof. The proof proceeds by direct calculation. As an example, we show that the desired equation holds for ψ_+ in the direction of $X = e_1$. Using the preceding corollary, we differentiate at the origin $o = eH$:

$$\widetilde{\Lambda}^{g_{a,b}}(e_1)\cdot\psi_+ = \frac{1}{2}(1-\frac{a}{2b})\sqrt{\frac{n}{a(n-1)}}\sum_{l=1}^{n-1} e_{2l}\cdot e_{2l+1}\cdot\psi_+$$
$$= \frac{2b-a}{4b}\sqrt{\frac{n}{a(n-1)}}\sum_{l=1}^{n-1} i(x_l\lrcorner + y_l\wedge)(y_l\wedge - x_l\lrcorner)1$$

$$= \frac{2b-a}{4b}\sqrt{\frac{n}{a(n-1)}}\sum_{l=1}^{n-1} i = \left(\frac{(2b-a)\sqrt{n(n-1)}}{4b\sqrt{a}}\right) i = \lambda_1 e_1 \cdot \psi_+.$$

□

Corollary 3.1.14. *The spinors $\psi_\pm$ are Killing spinors if and only if $a = \frac{2b(n-1)}{n}$, leading to $\lambda_1 = \lambda_2 = \frac{1}{2\sqrt{2b}}$. The round metric corresponds to the parameters $a = \frac{n-1}{n}$, $b = \frac{1}{2}$ (cf. [DGP18]), in which case we recover the usual Sasakian Killing spinors for the constants $\frac{1}{2}$, $\frac{-1}{2}$ (or $\frac{1}{2}$, $\frac{1}{2}$, depending on n).*

Generalizing the usual Sasakian structure, we have:

Proposition 3.1.15. *The sphere $(S^{2n-1} = \frac{\mathrm{SU}(n)}{\mathrm{SU}(n-1)}, g_{a,b})$ admits:*

(i) a compatible invariant normal almost contact metric structure for all $a, b > 0$.

(ii) a compatible invariant α-contact structure if and only if $\alpha = \frac{\sqrt{an}}{2b\sqrt{n-1}}$.

(iii) a compatible invariant α-K-contact structure if and only if $\alpha = \frac{\sqrt{an}}{2b\sqrt{n-1}}$.

In particular there exists a compatible invariant α-Sasakian structure if and only if $\alpha = \frac{\sqrt{an}}{2b\sqrt{n-1}}$.

Proof. In order for the structure to be invariant, the only choices for the Reeb vector field are $\xi = \pm e_1$. We note that the 2-form $\Phi := g_{a,b}(\cdot, \varphi(\cdot))$ is invariant if and only if

$$\Phi \in (\Lambda^2 \mathfrak{m}_2)^{\mathrm{SU}(n)} \simeq \mathrm{span}_{\mathbb{R}}\{\mathrm{ad}\,\xi|_{\mathfrak{m}_2}\},$$

and the metric compatibility condition $g_{a,b}(\varphi(X), \varphi(Y)) = g_{a,b}(X, Y) - g_{a,b}(\xi, X) g_{a,b}(\xi, Y)$ is satisfied if and only if

$$\varphi = \sqrt{\frac{a(n-1)}{n}}\,\mathrm{ad}\,\xi.$$

A tedious but straightforward Lie algebra computation then shows that the Nijenhuis tensor vanishes for any values of a, b, and the structure is α-contact ($d\eta = 2\alpha\Phi$) and α-K-contact ($\nabla^g_X \xi = -\alpha\varphi(X)$) if and only if $\alpha = \frac{\sqrt{an}}{2b\sqrt{n-1}}$. □

Remark 3.1.16. More generally, for the parameters $a = \frac{-(n-1)\epsilon}{n}$, $b = \frac{1}{2}$ one has the Berger metrics g_ϵ (see e.g. [DGP18]), with $\epsilon = -1$ corresponding to the round metric. We would like to determine the spinorial equations satisfied by the invariant spinors $\psi_\pm$ with respect to the invariant connections constructed in [DGP18]. In order to deal only with the Riemannian case, we will require $\epsilon < 0$. Let us focus on dimensions not equal to 5, 7 ($n \neq 3, 4$), in which case there is a 1-parameter family of invariant connections with skew torsion,

$$\nabla^s = \nabla^{g_\epsilon} - \epsilon s \ \Phi \wedge \eta, \quad s \in \mathbb{R}, \tag{3.2}$$

with torsion $T^s = -2\epsilon s \ \Phi \wedge \eta$, where Φ is the invariant 2-form defined in Section 2.2 of [DGP18] and η is the metric dual of $\xi := e_1$.

Generalizing Theorem 3.1.13, we have:

Proposition 3.1.17. *For $n \neq 3, 4$ the invariant spinors $\psi_\pm$ on $(S^{2n-1} = \frac{\mathrm{SU}(n)}{\mathrm{SU}(n-1)}, g_\epsilon)$ satisfy the generalized Killing equation with torsion,*

$$\nabla^s_X \psi_+ = A^s_+(X) \cdot \psi_+, \quad \nabla^s_X \psi_- = A^s_-(X) \cdot \psi_-,$$

for the endomorphisms

$$A^s_+ := A_+ - \frac{\epsilon s(n-1)}{2} \mathrm{Id}|_{\mathfrak{m}_1} + \frac{\epsilon s}{2} \mathrm{Id}|_{\mathfrak{m}_2},$$
$$A^s_- := A_- - \frac{(-1)^n \epsilon s(n-1)}{2} \mathrm{Id}|_{\mathfrak{m}_1} + \frac{(-1)^n \epsilon s}{2} \mathrm{Id}|_{\mathfrak{m}_2}.$$

Proof. Suppose that $a = \frac{-(n-1)\epsilon}{n}$, $b = \frac{1}{2}$. With respect to our chosen orthonormal basis $\{e_i\}_{i=1}^{2n-1}$, the invariant 2-form Φ takes the form

$$\Phi = -\sum_{p=1}^{n-1} e_{2p} \wedge e_{2p+1} = -\sum_{p=1}^{n-1} e_{2p} \wedge \varphi(e_{2p}),$$

where $\varphi = \frac{(n-1)\sqrt{-\epsilon}}{n} \mathrm{ad}(e_1)$. One easily calculates,

$$\Phi \cdot \psi_+ = (n-1)\xi \cdot \psi_+, \quad \Phi \cdot \psi_- = (-1)^n (n-1)\xi \cdot \psi_-, \quad \xi \cdot e_{2p} \cdot \psi_+ = e_{2p+1} \cdot \psi_+,$$
$$\xi \cdot e_{2p} \cdot \psi_- = (-1)^n e_{2p+1} \cdot \psi_-, \quad \xi \cdot e_{2p+1} \cdot \psi_+ = -e_{2p} \cdot \psi_+, \quad \xi \cdot e_{2p+1} \cdot \psi_- = (-1)^{n+1} e_{2p} \cdot \psi_-.$$

We now consider all possible cases:

1. If $Z = \xi$ then $Z \lrcorner T^s = -2\epsilon s \ \Phi$, and we have

$$\nabla^s_\xi \psi_+ = \nabla^{g_\epsilon}_\xi \psi_+ - \frac{1}{2}\epsilon s \Phi \cdot \psi_+ = A_+(\xi) \cdot \psi_+ - \frac{1}{2}\epsilon s(n-1)\xi \cdot \psi_+,$$
$$\nabla^s_\xi \psi_- = \nabla^{g_\epsilon}_\xi \psi_- - \frac{1}{2}\epsilon s \Phi \cdot \psi_- = A_-(\xi) \cdot \psi_- - \frac{1}{2}\epsilon s(n-1)(-1)^n \xi \cdot \psi_-.$$

2. If $Z = e_{2p}$ or $Z = e_{2p+1}$ then $Z \lrcorner T^s = 2\epsilon s\ \varphi(Z) \wedge \eta = -2\epsilon s\ \eta \wedge \varphi(Z)$, and we have

$$\begin{aligned}
\nabla^s_{e_{2p}} \psi_+ &= \nabla^{g_\epsilon}_{e_{2p}} \psi_+ - \frac{1}{2}\epsilon s \xi \cdot e_{2p+1} \cdot \psi_+ = A_+(e_{2p}) \cdot \psi_+ + \frac{1}{2}\epsilon s e_{2p} \cdot \psi_+, \\
\nabla^s_{e_{2p}} \psi_- &= \nabla^{g_\epsilon}_{e_{2p}} \psi_- - \frac{1}{2}\epsilon s \xi \cdot e_{2p+1} \cdot \psi_- = A_-(e_{2p}) \cdot \psi_- + \frac{1}{2}\epsilon s (-1)^n e_{2p} \cdot \psi_-, \\
\nabla^s_{e_{2p+1}} \psi_+ &= \nabla^{g_\epsilon}_{e_{2p+1}} \psi_+ + \frac{1}{2}\epsilon s \xi \cdot e_{2p} \cdot \psi_+ = A_+(e_{2p+1}) \cdot \psi_+ + \frac{1}{2}\epsilon s e_{2p+1} \cdot \psi_+, \\
\nabla^s_{e_{2p+1}} \psi_- &= \nabla^{g_\epsilon}_{e_{2p+1}} \psi_- + \frac{1}{2}\epsilon s \xi \cdot e_{2p} \cdot \psi_- = A_-(e_{2p+1}) \cdot \psi_- + \frac{1}{2}\epsilon s (-1)^n e_{2p+1} \cdot \psi_-.
\end{aligned}$$

□

Remark 3.1.18. For $n = 3, 4$ the families of invariant metric connections with skew torsion are larger, and depend on certain special tensors available in these dimensions [DGP18]. We omit these cases here in the interest of brevity.

3.2 Classical Spheres, Part II: Spheres over $\mathbb{H}$

In this section we consider the quaternionic spheres $S^{4n-1} = \frac{\mathrm{Sp}(n)\cdot K}{\mathrm{Sp}(n-1)\cdot K}$ where $K = \{1\}$, U(1), or Sp(1). Under an appropriate identification of the reductive complements in the three cases, the isotropy representations may be viewed as extensions of the standard representation of $\mathrm{Sp}(n-1)$ on $\mathbb{R}^{4n-4}$ to the group $\mathrm{Sp}(n-1) \cdot K$. In particular, this allows us to easily deduce the invariant spinors for the latter two cases from those for $K = \{1\}$. In each case we find an explicit basis for the space of invariant spinors and discuss the relevant geometric structures at play.

3.2.1 Standard Quaternionic Spheres, $S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$

This is the case corresponding to $K = \{1\}$. The isotropy representation splits into three copies of the trivial representation and one copy of the standard representation of $\mathrm{Sp}(n-1)$ on $\mathbb{R}^{4n-4}$,

$$\mathfrak{m} \simeq \bigoplus_{i=1}^{4} \mathfrak{m}_i, \qquad \text{where } \mathfrak{m}_i \simeq \mathbb{R}\ (i = 1, 2, 3),\ \mathfrak{m}_4 \simeq \mathbb{R}^{4n-4}.$$

Up to isometry, there is a 4-parameter family of invariant metrics (see [Zil82]), and these are given by rescaling B_0 separately on the isotropy components:

$$g_{\vec{a}} := a_1 B_0|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + a_2 B_0|_{\mathfrak{m}_2 \times \mathfrak{m}_2} + a_3 B_0|_{\mathfrak{m}_3 \times \mathfrak{m}_3} + a_4 B_0|_{\mathfrak{m}_4 \times \mathfrak{m}_4}, \qquad a_1, a_2, a_3, a_4 > 0. \tag{3.3}$$

We now give an explicit description of $\mathfrak{m}$, together with a $g_{\vec{a}}$-orthonormal basis, which will be needed for subsequent calculations. The embedding $\mathrm{Sp}(n-1) \hookrightarrow \mathrm{Sp}(n)$ may be realized as the lower right hand $(n-1) \times (n-1)$ block, and we take $\mathfrak{m} := \mathfrak{sp}(n-1)^{\perp}$ with respect to the Killing form, $\kappa_{\mathfrak{sp}(n)} := -4(n+1)B_0$. We then have, at the level of Lie algebras,

$$\mathfrak{sp}(n) = \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{p,p}, jF^{(n)}_{p,p}, kF^{(n)}_{p,p}, iF^{(n)}_{r,s}, jF^{(n)}_{r,s}, kF^{(n)}_{r,s}, E^{(n)}_{r,s}\}_{\substack{p=1,\dots,n,\\ 1\le r<s\le n}}$$
$$\mathfrak{sp}(n-1) = \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{p,p}, jF^{(n)}_{p,p}, kF^{(n)}_{p,p}, iF^{(n)}_{r,s}, jF^{(n)}_{r,s}, kF^{(n)}_{r,s}, E^{(n)}_{r,s}\}_{\substack{p=2,\dots,n,\\ 2\le r<s\le n}}$$

and

$$\mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{iF^{(n)}_{1,1}, jF^{(n)}_{1,1}, kF^{(n)}_{1,1}, iF^{(n)}_{1,p}, jF^{(n)}_{1,p}, kF^{(n)}_{1,p}, E^{(n)}_{1,p}\}_{p=2,\dots,n}. \tag{3.4}$$

A $g_{\vec{a}}$-orthonormal basis is then given by

$$e_1 := \frac{1}{\sqrt{a_1}}\, iF^{(n)}_{1,1}, \quad e_2 := \frac{-1}{\sqrt{a_2}}\, kF^{(n)}_{1,1}, \quad e_3 := \frac{1}{\sqrt{a_3}}\, jF^{(n)}_{1,1}, \quad e_{4p} := \frac{1}{\sqrt{2a_4}}\, jF^{(n)}_{1,p+1}, \tag{3.5}$$
$$e_{4p+1} := \frac{1}{\sqrt{2a_4}}\, kF^{(n)}_{1,p+1}, \quad e_{4p+2} := \frac{1}{\sqrt{2a_4}}\, iF^{(n)}_{1,p+1}, \quad e_{4p+3} := \frac{1}{\sqrt{2a_4}}\, E^{(n)}_{1,p+1}, \tag{3.6}$$

for $p = 1, \dots, n-1$, and the isotropy summands are

$$\mathfrak{m}_1 = \mathbb{R}e_1, \quad \mathfrak{m}_2 = \mathbb{R}e_2, \quad \mathfrak{m}_3 = \mathbb{R}e_3, \quad \mathfrak{m}_4 = \mathrm{span}_{\mathbb{R}}\{e_4, \dots, e_{4n-1}\}.$$

We define the *vertical* and *horizontal* spaces by $\mathcal{V} := \bigoplus_{i=1}^{3} \mathfrak{m}_i$ and $\mathcal{H} := \mathfrak{m}_4$ respectively; these will be relevant to our discussion of the 3-(α,δ)-Sasaki subfamily of metrics appearing later in the section. From [AHL23, Eqn. (37)], we have:

Theorem 3.2.1. *Using the above $g_{\vec{a}}$-orthonormal basis and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors on $(S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1), g_{\vec{a}})$ for any $a_1, a_2, a_3, a_4 > 0$ is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{\omega^j,\ y_1 \wedge \omega^j\}_{j=0}^{n-1},$$

where $\omega := \sum_{i=1}^{n-1} y_{2i} \wedge y_{2i+1}$.

Remark 3.2.2. We note that the choice of invariant metric in the preceding theorem is immaterial, since the isotropy representation acts trivially in the vertical directions. Thus, by choosing the 3-Sasakian metric, the result also follows from Theorem 4.3.10. The fact that the space of invariant spinors is $2n$-dimensional may also be deduced from the main proposition in [Wan89] by noting that the spinor module Σ_{4n-1} in dimension $4n-1$ is the tensor product of $\mathbb{C}^2$ with the spinor module Σ_{4n-4} in dimension $4n-4$, and the isotropy representation acts trivially on the span of e_1, e_2, e_3. Explicitly, the proposition in [Wan89] gives n linearly independent $\mathrm{Sp}(n-1)$-stabilized spinors in Σ_{4n-4}, and one then takes the tensor products of these with a basis of $\mathbb{C}^2$ to obtain $2n$ invariant spinors in Σ_{4n-1}. Our approach has the added benefit of providing an explicit description of the spinors, and allowing us to treat the cases $G = \mathrm{Sp}(n)$, $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$, and $\mathrm{Sp}(n)\,\mathrm{U}(1)$ in a unified way.

Before discussing the 3-(α, δ)-Sasaki case in more detail, we first calculate the Ambrose-Singer torsion in the general case and determine its type:

Proposition 3.2.3. *For any $a_1, a_2, a_3, a_4 > 0$ the sphere $(S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1), g_{\vec{a}})$ has Ambrose-Singer torsion of type $\mathcal{T}_{\text{skew}} \oplus \mathcal{T}_{\text{CT}}$, given by*

$$\begin{aligned}
T^{\text{AS}}(e_1, e_2) &= \frac{-2\sqrt{a_3}}{\sqrt{a_1 a_2}} e_3, \quad T^{\text{AS}}(e_1, e_3) = \frac{2\sqrt{a_2}}{\sqrt{a_1 a_3}} e_2, \quad T^{\text{AS}}(e_2, e_3) = \frac{-2\sqrt{a_1}}{\sqrt{a_2 a_3}} e_1, \\
T^{\text{AS}}(e_1, -)|_{\mathfrak{m}_4} &= \frac{1}{\sqrt{a_1}} \Phi_1|_{\mathfrak{m}_4}, \quad T^{\text{AS}}(e_2, -)|_{\mathfrak{m}_4} = \frac{1}{\sqrt{a_2}} \Phi_2|_{\mathfrak{m}_4}, \quad T^{\text{AS}}(e_3, -)|_{\mathfrak{m}_4} = \frac{1}{\sqrt{a_3}} \Phi_3|_{\mathfrak{m}_4}, \\
T^{\text{AS}}(e_{4p}, e_{4q}) &= T^{\text{AS}}(e_{4p+1}, e_{4q+1}) = T^{\text{AS}}(e_{4p+2}, e_{4q+2}) = T^{\text{AS}}(e_{4p+3}, e_{4q+3}) = 0 \\
T^{\text{AS}}(e_{4p}, e_{4q+1}) &= \frac{-\delta_{p,q}\sqrt{a_1}}{a_4} e_1, \quad T^{\text{AS}}(e_{4p}, e_{4q+2}) = \frac{-\delta_{p,q}\sqrt{a_2}}{a_4} e_2, \quad T^{\text{AS}}(e_{4p}, e_{4q+3}) = \frac{-\delta_{p,q}\sqrt{a_3}}{a_4} e_3, \\
T^{\text{AS}}(e_{4p+1}, e_{4q+2}) &= \frac{-\delta_{p,q}\sqrt{a_3}}{a_4} e_3, \quad T^{\text{AS}}(e_{4p+1}, e_{4q+3}) = \frac{\delta_{p,q}\sqrt{a_2}}{a_4} e_2, \quad T^{\text{AS}}(e_{4p+2}, e_{4q+3}) = \frac{-\delta_{p,q}\sqrt{a_1}}{a_4} e_1,
\end{aligned}$$

for $p, q = 1, \dots, n-1$, where Φ_1, Φ_2, Φ_3 are defined formally as in (3.12)-(3.14). The projection of T^{AS} onto $\mathcal{T}_{\text{skew}}$ is

$$T^{\text{AS}}_{\text{skew}} = -\frac{2}{3}\left(\frac{a_1 + a_2 + a_3}{\sqrt{a_1 a_2 a_3}}\right) e_1 \wedge e_2 \wedge e_3 + \frac{1}{3}\sum_{i=1}^{3}\left(\frac{a_i + 2a_4}{a_4\sqrt{a_i}}\right) e_i \wedge \Phi_i|_{\mathfrak{m}_4}, \tag{3.7}$$

with $T^{\text{AS}} = T^{\text{AS}}_{\text{skew}}$ if and only if $a_1 = a_2 = a_3 = a_4$.

3.2.1.1 Spinors on 3-(α, δ)-Sasaki spheres

Among the metrics (3.3), we consider in this subsection the distinguished subfamily of 3-(α, δ)-Sasaki metrics $g_{\alpha,\delta}$. Following the notation and setup of Theorem 2.5.7, and noting that the Killing form on $\mathfrak{sp}(n)$ is $\kappa_{\mathfrak{sp}(n)} := -4(n+1)B_0$, we define the 3-$(\alpha, \delta)$-Sasaki structure tensors

$$\xi_1 := i\delta F_{1,1}, \quad \xi_2 := -k\delta F_{1,1}, \quad \xi_3 := j\delta F_{1,1}, \tag{3.8}$$

$$g_{\alpha,\delta} := \frac{1}{\delta^2} B_0|_{\mathcal{V}\times\mathcal{V}} + \frac{1}{2\alpha\delta} B_0|_{\mathcal{H}\times\mathcal{H}}, \quad \varphi_p := \frac{1}{2\delta}\,\mathrm{ad}(\xi_p)|_{\mathcal{V}} + \frac{1}{\delta}\,\mathrm{ad}(\xi_p)|_{\mathcal{H}}, \tag{3.9}$$

for $p = 1, 2, 3$. Noting that $g_{\alpha,\delta}$ is obtained from $g_{\vec{a}}$ by setting $a_1 = a_2 = a_3 = \frac{1}{\delta^2}$, $a_4 = \frac{1}{2\alpha\delta}$, we inherit from (3.5)-(3.6) the $g_{\alpha,\delta}$-orthonormal basis

$$e_r := \xi_r, \quad e_{4p} := j\sqrt{\alpha\delta} F^{(n)}_{1,p+1}, \quad e_{4p+1} := k\sqrt{\alpha\delta} F^{(n)}_{1,p+1}, \tag{3.10}$$

$$e_{4p+2} := i\sqrt{\alpha\delta} F^{(n)}_{1,p+1}, \quad e_{4p+3} := \sqrt{\alpha\delta} E^{(n)}_{1,p+1}, \tag{3.11}$$

for $r = 1, 2, 3$ and $p = 1, \ldots, n-1$. The fundamental 2-forms $\Phi_r(X, Y) := g(X, \varphi_r(Y))$ are given in terms of this basis by

$$\Phi_1 = -\xi_2 \wedge \xi_3 - \sum_{p=1}^{n-1} (e_{4p} \wedge e_{4p+1} + e_{4p+2} \wedge e_{4p+3}), \tag{3.12}$$

$$\Phi_2 = \xi_1 \wedge \xi_3 - \sum_{p=1}^{n-1} (e_{4p} \wedge e_{4p+2} - e_{4p+1} \wedge e_{4p+3}), \tag{3.13}$$

$$\Phi_3 = -\xi_1 \wedge \xi_2 - \sum_{p=1}^{n-1} (e_{4p} \wedge e_{4p+3} + e_{4p+1} \wedge e_{4p+2}). \tag{3.14}$$

Remark 3.2.4. It is worth noting that the spinors ω^j, $y_1 \wedge \omega^j$ appearing in Theorem 3.2.1 have an interpretation in terms of the 3-(α, δ)-Sasaki structure tensors. Indeed, using the spin representation described in Chapter 2.1 one has

$$y_1 = \frac{1}{\sqrt{2}}(\xi_2 + i\xi_3), \quad \omega = -\frac{1}{2}(\Phi_2|_{\mathcal{H}} + i\Phi_3|_{\mathcal{H}}).$$

Finally, before discussing the situation in dimension 7, we recall the existence of the second Einstein metric on a 3-Sasakian manifold:

Remark 3.2.5. It was shown in [BGM94] that a 3-Sasakian manifold admits (uniquely up to homothety) a second Einstein metric of positive scalar curvature, which differs from the 3-Sasakian metric by a rescaling along the fibres of the canonical fibration. In dimension 7 it is known from [FKMS97] that this scaling factor is $\frac{1}{5}$, and, more generally, it was shown in [AD20, Prop. 2.3.3] that a 3-(α, δ)-Sasaki manifold of dimension $4n - 1$ is Riemannian Einstein if and only if $\delta = \alpha$ or $\delta = (2n + 1)\alpha$; in particular, comparing with (3.9) easily recovers the factor of $\frac{1}{5}$ in the 7-dimensional 3-Sasakian case. It was furthermore shown in [FKMS97] that the second Einstein metric admits a proper nearly parallel G_2-structure (equivalently, a unique Killing spinor up to scaling), and we shall see in the following example that this spinor turns out to be the canonical spinor of the 3-(α, δ)-Sasaki structure.

Example 3.2.6. In this example we consider in more detail the 3-(α, δ)-Sasaki 7-sphere, $(S^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1), g_{\alpha,\delta})$, and compare the spinors from Theorem 3.2.1 with those described previously in [AF10, AD20]. At the Lie algebra level, we decompose $\mathfrak{sp}(2) = \mathfrak{sp}(1) \oplus_{\perp_{\kappa_{\mathfrak{sp}(2)}}} \mathfrak{m}$, where

$$\begin{aligned}
\mathfrak{sp}(1) &= \mathrm{span}_{\mathbb{R}} \left\{ iF^{(2)}_{2,2}, jF^{(2)}_{2,2}, kF^{(2)}_{2,2} \right\}, \\
\mathcal{V} &= \mathrm{span}_{\mathbb{R}} \left\{ \xi_1 := i\delta F^{(2)}_{1,1},\ \xi_2 := -k\delta F^{(2)}_{1,1},\ \xi_3 := j\delta F^{(2)}_{1,1} \right\}, \\
\mathcal{H} &= \mathrm{span}_{\mathbb{R}} \left\{ e_4 := j\sqrt{\alpha\delta} F^{(2)}_{1,2},\ e_5 := k\sqrt{\alpha\delta} F^{(2)}_{1,2},\ e_6 := i\sqrt{\alpha\delta} F^{(2)}_{1,2},\ e_7 := \sqrt{\alpha\delta} E^{(2)}_{1,2} \right\}, \\
\mathfrak{m} &:= \mathcal{V} \oplus \mathcal{H},
\end{aligned}$$

and orthogonality is with respect to the Killing form $\kappa_{\mathfrak{sp}(2)} = -12B_0$ on $\mathfrak{sp}(2)$. By Theorem 2.5.7, the 3-(α, δ)-Sasaki structure is given by the tensors $g_{\alpha,\delta}$, ξ_p, φ_p, $(p = 1, 2, 3)$ described above. The above basis for $\mathfrak{m}$ is $g_{\alpha,\delta}$-orthonormal, and adapted to the 3-(α, δ)-Sasaki structure in the sense of Definition 2.3.6, i.e. the fundamental 2-forms are given by

$$\Phi_1 = -(\xi_{2,3} + e_{4,5} + e_{6,7}), \quad \Phi_2 = -(\xi_{3,1} + e_{4,6} - e_{5,7}), \quad \Phi_3 = -(\xi_{1,2} + e_{4,7} + e_{5,6}).$$

Using the spin representation described in Chapter 2.1, it follows from Theorem 3.2.1 that the space of invariant spinors is

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{1, \omega, y_1, y_1 \wedge \omega\},$$

where $\omega := y_2 \wedge y_3$.

Let us illustrate in detail the process of finding these invariant spinors by hand. To begin, one finds that the isotropy operators are given by

$$\mathrm{ad}(iF^{(2)}_{2,2})|_{\mathfrak{m}} = e_{4,5} - e_{6,7}, \quad \mathrm{ad}(jF^{(2)}_{2,2})|_{\mathfrak{m}} = -e_{4,7} + e_{5,6}, \quad \mathrm{ad}(kF^{(2)}_{2,2})|_{\mathfrak{m}} = -e_{4,6} - e_{5,7}. \tag{3.15}$$

Now, applying the first operator in (3.15) to aritrary $\eta \in \Sigma = \Lambda^\bullet L'$ gives

$$\widetilde{\mathrm{ad}(iF^{(2)}_{2,2})} \cdot \eta = \frac{1}{2}(e_4 \cdot e_5 - e_6 \cdot e_7) \cdot \eta$$

$$= \frac{1}{2}\left[i(x_2\lrcorner + y_2\wedge)\ (y_2\wedge -x_2\lrcorner)\eta - i(x_3\lrcorner + y_3\wedge)\ (y_3\wedge -x_3\lrcorner)\eta\right],$$

and hence

$$\operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot 1 = 0,\quad \operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot y_1 = 0,\quad \operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot y_2 = -iy_2,$$
$$\operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot y_3 = iy_3,\quad \operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot (y_1\wedge y_2) = -iy_1\wedge y_2,\quad \operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot (y_2\wedge y_3) = 0,$$
$$\operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot (y_1\wedge y_3) = iy_1\wedge y_3,\quad \operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\cdot (y_1\wedge y_2\wedge y_3) = 0.$$

The kernel of this operator is therefore given by

$$\ker\operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}} = \operatorname{span}_{\mathbb{C}}\{1,\ y_2\wedge y_3,\ y_1,\ y_1\wedge y_2\wedge y_3\}.$$

Continuing similarly for the other two operators in (3.15) and taking the intersection of the three kernels gives

$$\Sigma_{\text{inv}} = \left(\ker\operatorname{ad}(\widetilde{iF^{(2)}_{2,2}})|_{\mathfrak{m}}\right)\cap\left(\ker\operatorname{ad}(\widetilde{jF^{(2)}_{2,2}})|_{\mathfrak{m}}\right)\cap\left(\ker\operatorname{ad}(\widetilde{kF^{(2)}_{2,2}})|_{\mathfrak{m}}\right) = \operatorname{span}_{\mathbb{C}}\{1,\ y_2\wedge y_3,\ y_1,\ y_1\wedge y_2\wedge y_3\}.$$

Remark 3.2.7. The canonical spinor ψ_0 and three auxiliary spinors $\psi_r := \xi_r\cdot\psi_0$ $(r = 1,2,3)$ described in Theorem 4.5.2 of [AD20] are given in terms of the above basis of Σ_{inv} by

$$\psi_0 = \frac{1}{\sqrt{2}}(\omega + iy_1),\quad \psi_1 = \frac{1}{\sqrt{2}}(i\omega + y_1),\quad \psi_2 = \frac{1}{\sqrt{2}}(-1 + iy_1\wedge\omega),\quad \psi_3 = \frac{1}{\sqrt{2}}(-i + y_1\wedge\omega).$$

Let us now differentiate the spinors ψ_i, $i = 0,1,2,3$ from Remark 3.2.7 and compare to the spinorial equations in [AD20, Thm. 4.5.2]. Using the expression for the Nomizu map from Theorem 2.5.7, we calculate

$$\Lambda^{g_{\alpha,\delta}}(\xi_1) = \delta\xi_{2,3} + \delta(1-\frac{\alpha}{\delta})(e_{4,5}+e_{6,7}),\quad \Lambda^{g_{\alpha,\delta}}(\xi_2) = \delta\xi_{3,1} + \delta(1-\frac{\alpha}{\delta})(e_{4,6}-e_{5,7}),$$
$$\Lambda^{g_{\alpha,\delta}}(\xi_3) = \delta\xi_{1,2} + \delta(1-\frac{\alpha}{\delta})(e_{4,7}+e_{5,6}),\quad \Lambda^{g_{\alpha,\delta}}(e_4) = \alpha(-\xi_1\wedge e_5 - \xi_2\wedge e_6 - \xi_3\wedge e_7),$$
$$\Lambda^{g_{\alpha,\delta}}(e_5) = \alpha(\xi_1\wedge e_4 + \xi_2\wedge e_7 - \xi_3\wedge e_6),\quad \Lambda^{g_{\alpha,\delta}}(e_6) = \alpha(-\xi_1\wedge e_7 + \xi_2\wedge e_4 + \xi_3\wedge e_5),$$
$$\Lambda^{g_{\alpha,\delta}}(e_7) = \alpha(\xi_1\wedge e_6 - \xi_2\wedge e_5 + \xi_3\wedge e_4).$$

Lifting these to the spin bundle and calculating in the spin representation, as we did above for the isotropy operators, then gives the desired generalized Killing equations in [AD20, Thm.

4.5.2]:

$$\nabla^{g_{\alpha,\delta}}_X \psi_0 = \begin{cases} -\frac{3\alpha}{2} X \cdot \psi_0 & X \in \mathcal{H}, \\ \frac{2\alpha-\delta}{2} X \cdot \psi_0 & X \in \mathcal{V}, \end{cases} \qquad \nabla^{g_{\alpha,\delta}}_X \psi_i = \begin{cases} \frac{2\alpha-\delta}{2} \xi_i \cdot \psi_i & X = \xi_i, \\ \frac{3\delta-2\alpha}{2} \xi_j \cdot \psi_i & X = \xi_j \ (j \neq i), \\ \frac{\alpha}{2} X \cdot \psi_i & X \in \mathcal{H}, \end{cases} \tag{3.16}$$

for $i = 1, 2, 3$. For example, one calculates

$$\begin{aligned} \widetilde{\Lambda^{g_{\alpha,\delta}}(\xi_1)} \cdot \psi_0 &= \frac{\delta}{2} \left[i(x_1\lrcorner + y_1\wedge)(y_1 \wedge -x_1\lrcorner) \ \frac{1}{\sqrt{2}}(\omega + iy_1) \right] \\ &\quad + \frac{(\delta-\alpha)}{2} \left[i(x_2\lrcorner + y_2\wedge)(y_2 \wedge -x_2\lrcorner) \ \frac{1}{\sqrt{2}}(\omega + iy_1) \right] \\ &\quad + \frac{(\delta-\alpha)}{2} \left[i(x_3\lrcorner + y_3\wedge)(y_3 \wedge -x_3\lrcorner) \ \frac{1}{\sqrt{2}}(\omega + iy_1) \right] \\ &= \frac{\delta}{2} \left[i(x_1\lrcorner + y_1\wedge) \ \frac{1}{\sqrt{2}}(-i + y_1 \wedge \omega) \right] \\ &\quad + \frac{(\delta-\alpha)}{2} \left[i(x_2\lrcorner + y_2\wedge) \ \frac{1}{\sqrt{2}}(iy_2 \wedge y_1 - y_3) + i(x_3\lrcorner + y_3\wedge) \ \frac{1}{\sqrt{2}}(iy_3 \wedge y_1 + y_2) \right] \\ &= \frac{i\delta}{2\sqrt{2}} \left[-iy_1 + \omega \right] + \frac{i(\delta-\alpha)}{2\sqrt{2}} \left[(iy_1 - y_2 \wedge y_3) + (iy_1 + y_3 \wedge y_2) \right] \\ &= \frac{(2\alpha-\delta)}{2} \ \frac{1}{\sqrt{2}}(y_1 + i\omega) = \frac{(2\alpha-\delta)}{2}\psi_1 = \frac{(2\alpha-\delta)}{2}\xi_1 \cdot \psi_0. \end{aligned}$$

Later, we will show in Proposition 6.3.5 that the second equation in (3.16) (for the auxiliary spinors ψ_r, $r = 1, 2, 3$) is equivalent to the deformed Killing equation (5.1) in dimension 7. We conclude the 7-dimensional example by observing that substituting the parameters for the second Einstein metric, $g_2 := g_{\alpha,\delta}|_{\delta=5\alpha}$ (see Remark 3.2.5), into (3.16) gives

$$\nabla^{g_2}_X \psi_0 = -\frac{3\alpha}{2} X \cdot \psi_0, \quad \nabla^{g_2}_X \psi_i = \begin{cases} -\frac{3\alpha}{2} \xi_i \cdot \psi_i & X = \xi_i, \\ \frac{13\alpha}{2} \xi_j \cdot \psi_i & X = \xi_j \ (j \neq i), \\ \frac{\alpha}{2} X \cdot \psi_i & X \in \mathcal{H}. \end{cases}$$

In particular, this shows that ψ_0 is the Killing spinor determining the proper nearly parallel G_2-structure described in [FKMS97].

Finally, before discussing the general invariant metrics (3.3) in more detail, we compare the Ambrose-Singer connection to the canonical connection of the 3-(α,δ)-Sasaki structure introduced in [AD20]:

Corollary 3.2.8. *The canonical connection of the 3-(α,δ)-Sasaki space ($S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-$*

1), $g_{\alpha,\delta}$) coincides with the Ambrose-Singer connection if and only if the 3-(α,δ)-Sasaki structure is parallel ($\delta = 2\alpha$).

Proof. We have seen that $g_{\alpha,\delta}$ is obtained from $g_{\vec{a}}$ by setting $a_1 = a_2 = a_3 = \frac{1}{\delta^2}$ and $a_4 = \frac{1}{2\alpha\delta}$. Recalling that the canonical connection has skew torsion (Theorem 4.4.1 in [AD20]), if the two connections are assumed to coincide then Proposition 3.2.3 implies $a_1 = a_2 = a_3 = a_4$, hence $\delta = 2\alpha$. Conversely, if $\delta = 2\alpha$ then $a_1 = a_2 = a_3 = a_4 = \frac{1}{4\alpha^2}$, and Proposition 3.2.3 implies that the Ambrose-Singer connection has skew torsion given by

$$T^{\mathrm{AS}} = -4\alpha\ e_1 \wedge e_2 \wedge e_3 + 2\alpha \sum_{i=1}^{3} e_i \wedge \Phi_i|_{\mathfrak{m}_4}.$$

The result then follows by comparing this to Theorem 4.4.1 in [AD20]. □

3.2.1.2 General Invariant Metrics on $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)$

We now leave the 3-(α,δ)-Sasaki setting and return to the general invariant metrics (3.3). In order to differentiate the invariant spinors from Theorem 3.2.1, it is helpful to compare $g_{\vec{a}}$ with the round (3-Sasakian) metric, $g' := g_{\alpha,\delta}|_{\alpha=\delta=1}$; they are related by

$$g_{\vec{a}} = b_1 g'|_{\mathfrak{m}_1\times\mathfrak{m}_1} + b_2 g'|_{\mathfrak{m}_2\times\mathfrak{m}_2} + b_3 g'|_{\mathfrak{m}_3\times\mathfrak{m}_3} + b_4 g'|_{\mathfrak{m}_4\times\mathfrak{m}_4},$$

where $b_i := a_i$ $(i = 1,2,3)$ and $b_4 := 2a_4$. We denote by $\{\overline{e}_i\}$ the $g_{\vec{a}}$-orthonormal basis defined in (3.5)-(3.6), and by $\{e_i\}$ the g'-orthonormal basis defined by setting $\alpha = \delta = 1$ in (3.10)-(3.11). By adapting the proof of Proposition 2.33 in [BHM+15], we obtain:

Lemma 3.2.9. *The Levi-Civita connection 1-forms $\overline{\omega}_{i,j} := g_{\vec{a}}(\nabla^{g_{\vec{a}}}\overline{e}_i, \overline{e}_j)$ and $\omega'_{i,j} := g'(\nabla^{g'}e_i, e_j)$ are related by*

$$\overline{\omega}_{i,j}(\overline{e}_k) = \frac{1}{2}\left(\Theta^p_{q,r} + \Theta^q_{p,r}\right)\omega'_{i,j}(e_k) + \frac{1}{2}\left(\Theta^q_{p,r} - \Theta^r_{p,q}\right)\omega'_{j,k}(e_i) + \frac{1}{2}\left(\Theta^r_{p,q} - \Theta^p_{q,r}\right)\omega'_{i,k}(e_j)$$

for $e_i \in \mathfrak{m}_p,\ e_j \in \mathfrak{m}_q,\ e_k \in \mathfrak{m}_r$, where $\Theta^l_{m,n} := \sqrt{\frac{b_l}{b_m b_n}}$.

Proof. Let $e_i, e_j, e_k, \Theta^l_{m,n}$ be as in the statement of the lemma. Using the Koszul formula and the fact that $\nabla^{g'}$ is torsion-free, we calculate

$$\overline{\omega}_{i,j}(\overline{e}_k) = g_{\vec{a}}(\nabla^{g_{\vec{a}}}_{\overline{e}_k}\overline{e}_i, \overline{e}_j) = \frac{1}{2}\left[-g_{\vec{a}}([\overline{e}_i,\overline{e}_k]_{\mathfrak{m}}, \overline{e}_j) - g_{\vec{a}}([\overline{e}_k,\overline{e}_j]_{\mathfrak{m}}, \overline{e}_i) - g_{\vec{a}}([\overline{e}_i,\overline{e}_j]_{\mathfrak{m}}, \overline{e}_k)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[-\Theta^q_{p,r} g'([e_i,e_k]_{\mathfrak{m}},e_j) - \Theta^p_{q,r} g'([e_k,e_j]_{\mathfrak{m}},e_i) - \Theta^r_{p,q} g'([e_i,e_j]_{\mathfrak{m}},e_k)\right] \\
&= -\frac{1}{2}\Theta^q_{p,r} g'(\Lambda^{g'}(e_i)e_k - \Lambda^{g'}(e_k)e_i, e_j) - \frac{1}{2}\Theta^p_{q,r} g'(\Lambda^{g'}(e_k)e_j - \Lambda^{g'}(e_j)e_k, e_i) \\
&\quad - \frac{1}{2}\Theta^r_{p,q} g'(\Lambda^{g'}(e_i)e_j - \Lambda^{g'}(e_j)e_i, e_k),
\end{aligned}
$$

and the result then follows from the fact that $\nabla^{g'}$ is metric (for g'). □

In dimension 7, this comparison with the round metric allows us to easily find new examples of generalized Killing spinors:

Proposition 3.2.10. *The spinors ψ_i, $i = 0,1,2,3$ on the 7-sphere $(S^7 = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)}, g_{\vec{a}})$, defined as in Remark 3.2.7, are generalized Killing spinors for the endomorphisms*

$$
A_i = \lambda_{i,1}\,\mathrm{Id}|_{\mathfrak{m}_1} + \lambda_{i,2}\,\mathrm{Id}|_{\mathfrak{m}_2} + \lambda_{i,3}\,\mathrm{Id}|_{\mathfrak{m}_3} + \lambda_{i,4}\,\mathrm{Id}|_{\mathfrak{m}_4}, \quad i = 0,1,2,3,
$$

with eigenvalues

$$
\lambda_{0,p} = \begin{cases} \frac{1}{2}(-\Theta^p_{p+1,p+2} + \Theta^{p+1}_{p,p+2} + \Theta^{p+2}_{p,p+1}) - (\Theta^4_{p,4} - \Theta^p_{4,4}) & p = 1,2,3, \\ -\frac{1}{2}(\Theta^1_{4,4} + \Theta^2_{4,4} + \Theta^3_{4,4}) & p = 4, \end{cases}
$$

$$
\lambda_{k,p} = \begin{cases} \frac{1}{2}(-\Theta^p_{p+1,p+2} + \Theta^{p+1}_{p,p+2} + \Theta^{p+2}_{p,p+1}) - (\Theta^4_{p,4} - \Theta^p_{4,4}) & k = p \text{ and } p = 1,2,3, \\ \frac{1}{2}(-\Theta^p_{p+1,p+2} + \Theta^{p+1}_{p,p+2} + \Theta^{p+2}_{p,p+1}) + (\Theta^4_{p,4} - \Theta^p_{4,4}) & k \neq p \text{ and } p = 1,2,3, \\ \frac{1}{2}(-\Theta^k_{4,4} + \Theta^{k+1}_{4,4} + \Theta^{k+2}_{4,4}) & p = 4, \end{cases}
$$

where $k = 1,2,3$, and the indices $k, k+1, k+2, p, p+1, p+2$ on the right hand side are taken modulo 3.

Proof. Using the preceding lemma together with the explicit formulas for the Nomizu map of the round 7-sphere (set $\alpha = \delta = 1$ in Example 3.2.6), we obtain:

$$
\begin{aligned}
&\Lambda^{g_{\vec{a}}}(\bar{e}_p) = (\Theta^p_{p+1,p+2} - \Theta^{p+1}_{p,p+2} - \Theta^{p+2}_{p,p+1})\ \overline{\Phi}_p|_{\mathcal{V}} - (\Theta^4_{p,4} - \Theta^p_{4,4})\ \overline{\Phi}_p|_{\mathcal{H}}, \quad p = 1,2,3, \\
&\Lambda^{g_{\vec{a}}}(\bar{e}_4) = -\Theta^1_{4,4}\bar{e}_{1,5} - \Theta^2_{4,4}\bar{e}_{2,6} - \Theta^3_{4,4}\bar{e}_{3,7}, \quad \Lambda^{g_{\vec{a}}}(\bar{e}_5) = \Theta^1_{4,4}\bar{e}_{1,4} + \Theta^2_{4,4}\bar{e}_{2,7} - \Theta^3_{4,4}\bar{e}_{3,6}, \\
&\Lambda^{g_{\vec{a}}}(\bar{e}_6) = -\Theta^1_{4,4}\bar{e}_{1,7} + \Theta^2_{4,4}\bar{e}_{2,4} + \Theta^3_{4,4}\bar{e}_{3,5}, \quad \Lambda^{g_{\vec{a}}}(\bar{e}_7) = \Theta^1_{4,4}\bar{e}_{1,6} - \Theta^2_{4,4}\bar{e}_{2,5} + \Theta^3_{4,4}\bar{e}_{3,4},
\end{aligned}
$$

where the indices $p, p+1, p+2$ are taken modulo 3, and $\overline{\Phi}_p$ are the forms defined by replacing each e_i with $\bar{e}_i$ (and replacing each ξ_i with $\bar{e}_i$, $i = 1,2,3$) in (3.12)-(3.14). The result then follows by lifting these operators and calculating the Clifford products with ψ_i, $i = 0, 1, 2, 3$ in the spin representation. □

Remark 3.2.11. By choosing the metric parameters a_1, a_2, a_3, a_4 in (3.3) appropriately, the endomorphisms A_i from the preceding proposition can be arranged to have 4 distinct eigenvalues, providing, to the author's knowledge, the first example of generalized Killing spinors whose endomorphism has four distinct eigenvalues (see [AF10, AD20] for examples of generalized Killing spinors with two or three distinct eigenvalues). We also note that, in the case of the 3-(α, δ)-Sasaki metric ($b_1 = b_2 = b_3 = \frac{1}{\delta^2}$, $b_4 = \frac{1}{\alpha\delta}$), we have

$$\Theta^p_{q,r} = \Theta^4_{p,4} = |\delta|, \qquad \Theta^p_{4,4} = |\alpha| \qquad \text{for } p, q, r \in \{1, 2, 3\}.$$

Since S^{4n-1} is compact we have, by convention, $\alpha\delta > 0$ (cf. Theorem 2.5.7), and thus α and δ have the same sign. If $\alpha, \delta > 0$, then the generalized Killing equations in Proposition 3.2.10 immediately recover the known equations (3.16). If $\alpha, \delta < 0$, then we recover the equations (3.16) up to a factor of -1, corresponding to the fact that replacing α, δ with $-\alpha, -\delta$ in the orthonormal basis (3.8), (3.10)-(3.11) gives a basis with the opposite orientation.

Somewhat surprisingly, performing a similar deformation of the 3-Sasakian Killing spinors in dimensions larger than 7 is not guaranteed to produce generalized Killing spinors, as the following proposition shows:

Proposition 3.2.12. *Let $\psi = \mu_1 1 + \mu_2 y_1 \wedge \omega^{n-1} \in E_1^-$ ($\mu_1, \mu_2 \in \mathbb{C}$) be an invariant Killing spinor for the round metric on $S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$. If $n > 2$, then the spinor ψ on $(S^{4n-1} = \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}, g_{\vec{a}})$ defined by the same formula is a generalized Killing spinor if and only if $b_2 = b_3 = b_4$. If $b_2 = b_3 = b_4$, then ψ is a generalized Killing spinor for the endomorphism*

$$A = \frac{1}{2}\left[(1-2n)\Theta^1_{2,2} + 2n\Theta^2_{1,2}\right] \mathrm{Id}|_{\mathfrak{m}_1} + \frac{1}{2}\Theta^1_{2,2}\,\mathrm{Id}|_{\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4}$$

with at most two distinct eigenvalues.

Proof. Using Lemma 3.2.9, the Nomizu map for the Levi-Civita connection of $g_{\vec{a}}$ takes the same form as in the proof of the preceding proposition, with $\overline{e}_4$ replaced with $\overline{e}_{4p}$, $\overline{e}_5$ replaced with $\overline{e}_{4p+1}$, and so on. Using the spin representation described in Chapter 2.1, one sees that Clifford multiplication by $\overline{\Phi}_2|_{\mathcal{H}}$ and $\overline{\Phi}_3|_{\mathcal{H}}$ (resp. $\overline{\Phi}_2|_{\mathcal{V}}$ and $\overline{\Phi}_3|_{\mathcal{V}}$) changes the degree of the spinors 1 and $y_1 \wedge \omega^{n-1}$ by two (resp. one). On the other hand, Clifford multiplication by a vector changes the degree by at most one. Thus, by comparing the degrees of $\widetilde{\Lambda^{g_{\vec{a}}}}(\overline{e}_2) \cdot \psi$ and $\widetilde{\Lambda^{g_{\vec{a}}}}(\overline{e}_3) \cdot \psi$ with elements of $\mathfrak{m} \cdot \psi$, we see that if ψ is a generalized Killing spinor and $n > 2$ then $\Theta^4_{2,4} - \Theta^2_{4,4} = 0 = \Theta^4_{3,4} - \Theta^3_{4,4}$. Simplifying these equations gives $b_2 = b_3 = b_4$, as desired. Conversely, if $b_2 = b_3 = b_4$ then lifting the Nomizu operators and calculating the Clifford product with ψ in the spin representation gives the result. □

The preceding proposition shows that attempting to produce generalized Killing spinors with a certain number of distinct eigenvalues by rescaling the isotropy components of metrics carrying Killing spinors is not a straightforward process. Indeed, if one starts with an arbitrary invariant Killing spinor for the round metric on $S^{4n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$, which, as we shall prove in Chapter 4.4, may be written as a linear combination of $\psi_k := \omega^{k+1} - i(k+1)y_1 \wedge \omega^k$ $(-1 \leq k \leq n-1)$, the resulting system of algebraic equations determining precisely which linear combination of the ψ_k's is needed is difficult to solve. It remains to be understood why this deformation technique works in some situations but not others, and whether it can be used to produce other interesting examples of generalized Killing spinors.

3.2.2 S^3-Quaternionic Spheres, $S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{Sp}(1)}{\mathrm{Sp}(n-1)\,\mathrm{Sp}(1)}$

This is the case corresponding to $K = \mathrm{Sp}(1)$. We begin by discussing the general case, then pass to the 7-dimensional setting, where the invariant spinor is related to the exceptional G_2-geometry available in this dimension. Using [AHL23, Eqn. (24)], we have at the level of Lie algebras

$$\mathfrak{sp}(n) \oplus \mathfrak{sp}(1) = \mathrm{span}_{\mathbb{R}}\{(iF^{(n)}_{p,q}, 0), (jF^{(n)}_{p,q}, 0), (kF^{(n)}_{p,q}, 0), (E^{(n)}_{r,s}, 0), (0, i), (0, j), (0, k)\}_{\substack{1 \leq p \leq q \leq n,\\ 1 \leq r < s \leq n}}$$

$$\mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1) = \mathrm{span}_{\mathbb{R}}\{(iF^{(n)}_{p,q}, 0), (jF^{(n)}_{p,q}, 0), (kF^{(n)}_{p,q}, 0), (E^{(n)}_{r,s}, 0), (iF^{(n)}_{1,1}, i), (jF^{(n)}_{1,1}, j), (kF^{(n)}_{1,1}, k)\}_{\substack{2 \leq p \leq q \leq n,\\ 2 \leq r < s \leq n}}$$

and for a reductive complement we take the orthogonal complement $\mathfrak{m} := (\mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1))^{\perp}$ with respect to the Killing form κ on $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$,

$$\kappa((A, z), (A', z')) := -4(n+1)B_0(A, A') + 8\Re(zz'). \tag{3.17}$$

The isotropy representation decomposes into two inequivalent irreducible summands, $\mathfrak{m} \simeq \mathfrak{m}_1 \oplus \mathfrak{m}_2$, leading to the 2-parameter family of invariant metrics,

$$g_{a,b} := -a\kappa|_{\mathfrak{m}_1 \times \mathfrak{m}_1} - b\kappa|_{\mathfrak{m}_2 \times \mathfrak{m}_2}, \quad a, b > 0.$$

A $g_{a,b}$-orthonormal basis for $\mathfrak{m}$ is given by

$$\xi_1 := \frac{1}{\Omega}\left(iF^{(n)}_{1,1}, -\left(\frac{n+1}{2}\right)i\right), \quad \xi_2 := \frac{1}{\Omega}\left(-kF^{(n)}_{1,1}, \left(\frac{n+1}{2}\right)k\right), \quad \xi_3 := \frac{1}{\Omega}\left(jF^{(n)}_{1,1}, -\left(\frac{n+1}{2}\right)j\right),$$
$$e_{4p} := \frac{1}{2\sqrt{2b(n+1)}}(jF^{(n)}_{1,p+1}, 0), \quad e_{4p+1} := \frac{1}{2\sqrt{2b(n+1)}}(kF^{(n)}_{1,p+1}, 0),$$
$$e_{4p+2} := \frac{1}{2\sqrt{2b(n+1)}}(iF^{(n)}_{1,p+1}, 0), \quad e_{4p+3} := \frac{1}{2\sqrt{2b(n+1)}}(E^{(n)}_{1,p+1}, 0)$$

for $p = 1, \ldots, n-1$, where $\Omega := \sqrt{2a(n+1)(n+3)}$. In terms of this basis, the two isotropy summands are

$$\mathfrak{m}_1 := \mathrm{span}_{\mathbb{R}}\{\xi_1, \xi_2, \xi_3\}, \quad \mathfrak{m}_2 := \mathrm{span}_{\mathbb{R}}\{e_4, \ldots, e_{4n-1}\}.$$

From [AHL23, Prop. 4.7] we obtain:

Theorem 3.2.13. *Using the above orthonormal basis and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors on* $(S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{Sp}(1)}{\mathrm{Sp}(n-1)\,\mathrm{Sp}(1)}, g_{a,b})$ *for any* $a, b > 0$ *is trivial unless* $n = 2$*, in which case* $\dim_{\mathbb{C}} \Sigma_{\mathrm{inv}} = 1$*. In this case, the* 1*-dimensional* Σ_{inv} *is contained in the span of* y_1 *and* $\omega := \sum_{i=1}^{n-1} y_{2i} \wedge y_{2i+1}$.

Proof. The result follows directly from [AHL23, Prop. 4.7] by noting that $\omega = \sum_{i=1}^{n-1} y_{2i} \wedge y_{2i+1}$ is the symplectic form stabilized by $\mathfrak{sp}(2n-2, \mathbb{C})$. □

The 1-dimensional space of invariant spinors obtained in dimension 7 is explicitly constructed in Example 3.2.15, which appears immediately after the following proposition describing the Ambrose-Singer torsion in the general case:

Proposition 3.2.14. *For any* $a, b > 0$ *the sphere* $(S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{Sp}(1)}{\mathrm{Sp}(n-1)\,\mathrm{Sp}(1)}, g_{a,b})$ *has Ambrose-Singer torsion of type* $\mathcal{T}_{\mathrm{skew}} \oplus \mathcal{T}_{\mathrm{CT}}$*, given by*

$$T^{\mathrm{AS}}(\xi_1, \xi_2) = \frac{(n-1)}{\Omega}\xi_3, \quad T^{\mathrm{AS}}(\xi_1, \xi_3) = -\frac{(n-1)}{\Omega}\xi_2, \quad T^{\mathrm{AS}}(\xi_2, \xi_3) = \frac{(n-1)}{\Omega}\xi_1,$$
$$T^{\mathrm{AS}}(\xi_1, -)|_{\mathfrak{m}_2} = \frac{1}{\Omega}\Phi_1|_{\mathfrak{m}_2}, \quad T^{\mathrm{AS}}(\xi_2, -)|_{\mathfrak{m}_2} = \frac{1}{\Omega}\Phi_2|_{\mathfrak{m}_2}, \quad T^{\mathrm{AS}}(\xi_3, -)|_{\mathfrak{m}_2} = \frac{1}{\Omega}\Phi_3|_{\mathfrak{m}_2},$$
$$T^{\mathrm{AS}}(e_{4p}, e_{4q}) = T^{\mathrm{AS}}(e_{4p+1}, e_{4q+1}) = T^{\mathrm{AS}}(e_{4p+2}, e_{4q+2}) = T^{\mathrm{AS}}(e_{4p+3}, e_{4q+3}) = 0$$
$$T^{\mathrm{AS}}(e_{4p}, e_{4q+1}) = \frac{-a\delta_{p,q}}{b\Omega}\xi_1, \quad T^{\mathrm{AS}}(e_{4p}, e_{4q+2}) = \frac{-a\delta_{p,q}}{b\Omega}\xi_2, \quad T^{\mathrm{AS}}(e_{4p}, e_{4q+3}) = \frac{-a\delta_{p,q}}{b\Omega}\xi_3,$$
$$T^{\mathrm{AS}}(e_{4p+1}, e_{4q+2}) = \frac{-a\delta_{p,q}}{b\Omega}\xi_3, \quad T^{\mathrm{AS}}(e_{4p+1}, e_{4q+3}) = \frac{a\delta_{p,q}}{b\Omega}\xi_2, \quad T^{\mathrm{AS}}(e_{4p+2}, e_{4q+3}) = \frac{-a\delta_{p,q}}{b\Omega}\xi_1,$$

for $p, q = 1, \ldots, n-1$*, where* Φ_1, Φ_2, Φ_3 *are defined formally as in (3.12)-(3.14). The projection* *of T^{AS} onto $\mathcal{T}_{\mathrm{skew}}$ is*

$$T^{\mathrm{AS}}_{\mathrm{skew}} = \left(\frac{n-1}{\Omega}\right)\xi_1\wedge\xi_2\wedge\xi_3 + \frac{1}{3}\left(\frac{2}{\Omega}+\frac{a}{b\Omega}\right)\sum_{i=1}^{3}\xi_i\wedge\Phi_i|_{\mathfrak{m}_2},$$

with $T^{\mathrm{AS}} = T^{\mathrm{AS}}_{\mathrm{skew}}$ if and only if $a = b$.

The remainder of the section is devoted to discussion of the 7-dimensional example, $S^7 = \frac{Sp(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}$. We shall explicitly determine the invariant spinor in this dimension, and discuss how it fits into the larger picture of the well-known correspondence between spinors and G_2-structures in dimension 7.

Example 3.2.15. Following the setup outlined above, the isotropy algebra is

$$\mathfrak{sp}(1)\oplus\mathfrak{sp}(1) = \left\{\left(iF^{(2)}_{2,2},0\right),\left(jF^{(2)}_{2,2},0\right),\left(kF^{(2)}_{2,2},0\right),\left(iF^{(2)}_{1,1},i\right),\left(jF^{(2)}_{1,1},j\right),\left(kF^{(2)}_{1,1},k\right)\right\},$$

and the two isotropy summands $\mathfrak{m}_1$, $\mathfrak{m}_2$ are given by

$$\begin{aligned}\mathfrak{m}_1 &= \mathrm{span}_{\mathbb{R}}\left\{\frac{1}{\sqrt{30a}}\left(iF^{(2)}_{1,1},\frac{-3i}{2}\right),\frac{1}{\sqrt{30a}}\left(-kF^{(2)}_{1,1},\frac{3k}{2}\right),\frac{1}{\sqrt{30a}}\left(jF^{(2)}_{1,1},\frac{-3j}{2}\right)\right\}\\ &=: \{\xi_1,\xi_2,\xi_3\},\\ \mathfrak{m}_2 &= \mathrm{span}_{\mathbb{R}}\left\{\frac{1}{\sqrt{24b}}\left(jF^{(2)}_{1,2},0\right),\frac{1}{\sqrt{24b}}\left(kF^{(2)}_{1,2},0\right),\frac{1}{\sqrt{24b}}\left(iF^{(2)}_{1,2},0\right),\frac{1}{\sqrt{24b}}\left(E^{(2)}_{1,2},0\right)\right\}\\ &=: \{e_4,e_5,e_6,e_7\}.\end{aligned}$$

The basis $\{\xi_1,\xi_2,\xi_3,e_4,e_5,e_6,e_7\}$ for $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2$ is orthonormal with respect to the invariant metric $g_{a,b}$ described above, and we shall also denote $e_i := \xi_i$ $(i=1,2,3)$ in certain places. Fixing the associated Clifford algebra representation as in Chapter 2.1, and letting $\omega := y_2\wedge y_3$, we have,

Theorem 3.2.16. *For any $a,b>0$, the space of invariant spinors on $(S^7 = \frac{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}, g_{a,b})$ is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{\psi_0 = \frac{1}{\sqrt{2}}(\omega + iy_1)\}.$$

Proof. Considering [AHL23, Prop. 4.7, Cor. 4.8] and the spinors ψ_i, $i=0,1,2,3$ from Remark 3.2.7, the space of invariant spinors is the subspace of $\mathrm{span}_{\mathbb{C}}\{\psi_0,\psi_1,\psi_2,\psi_3\}$ annihilated by the spin lifts of the three additional isotropy operators

$$\mathrm{ad}(iF^{(2)}_{1,1}, i)|_{\mathfrak{m}} = 2\xi_2 \wedge \xi_3 + e_4 \wedge e_5 + e_6 \wedge e_7, \quad \mathrm{ad}(jF^{(2)}_{1,1}, j)|_{\mathfrak{m}} = 2\xi_1 \wedge \xi_2 + e_4 \wedge e_7 + e_5 \wedge e_6,$$
$$\mathrm{ad}(kF^{(2)}_{1,1}, k)|_{\mathfrak{m}} = 2\xi_1 \wedge \xi_3 - e_4 \wedge e_6 + e_5 \wedge e_7.$$

A calculation similar to Example 3.2.6 then gives the result. □

In order to differentiate the spinor ψ_0, we remark the commutator relations

$$[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V} \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)), \qquad [\mathcal{V}, \mathcal{H}] \subseteq \mathcal{H}, \qquad [\mathcal{H}, \mathcal{H}] \subseteq \mathcal{V} \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)),$$

and one then finds that the Nomizu map for the Levi-Civita connection is given by

$$\Lambda^{g_{a,b}}(V)W = \begin{cases} \frac{1}{2}[V, W]_{\mathfrak{m}} & V, W \in \mathcal{V}, \\ (1 - \frac{a}{2b})[V, W]_{\mathfrak{m}} & V \in \mathcal{V}, W \in \mathcal{H}, \\ \frac{a}{2b}[V, W]_{\mathfrak{m}} & V \in \mathcal{H}, W \in \mathcal{V}, \\ \frac{1}{2}[V, W]_{\mathfrak{m}} & V, W \in \mathcal{H}. \end{cases}$$

In terms of our chosen basis, for $\mathfrak{m}_1$ this takes the form

$$\Lambda^{g_{a,b}}(\xi_1) = \frac{1}{\sqrt{30a}}(-\frac{1}{2}\xi_2 \wedge \xi_3 + (1 - \frac{a}{2b})e_4 \wedge e_5 + (1 - \frac{a}{2b})e_6 \wedge e_7),$$
$$\Lambda^{g_{a,b}}(\xi_2) = \frac{1}{\sqrt{30a}}(\frac{1}{2}\xi_1 \wedge \xi_3 + (1 - \frac{a}{2b})e_4 \wedge e_6 - (1 - \frac{a}{2b})e_5 \wedge e_7),$$
$$\Lambda^{g_{a,b}}(\xi_3) = \frac{1}{\sqrt{30a}}(-\frac{1}{2}\xi_1 \wedge \xi_2 + (1 - \frac{a}{2b})e_4 \wedge e_7 + (1 - \frac{a}{2b})e_5 \wedge e_6),$$

and likewise for $\mathfrak{m}_2$,

$$\Lambda^{g_{a,b}}(e_4) = \frac{\sqrt{a}}{2b\sqrt{30}}(-\xi_1 \wedge e_5 - \xi_2 \wedge e_6 - \xi_3 \wedge e_7), \quad \Lambda^{g_{a,b}}(e_5) = \frac{\sqrt{a}}{2b\sqrt{30}}(\xi_1 \wedge e_4 + \xi_2 \wedge e_7 - \xi_3 \wedge e_6),$$
$$\Lambda^{g_{a,b}}(e_6) = \frac{\sqrt{a}}{2b\sqrt{30}}(-\xi_1 \wedge e_7 + \xi_2 \wedge e_4 + \xi_3 \wedge e_5), \quad \Lambda^{g_{a,b}}(e_7) = \frac{\sqrt{a}}{2b\sqrt{30}}(\xi_1 \wedge e_6 - \xi_2 \wedge e_5 + \xi_3 \wedge e_4).$$

Applying the spin lifts of these operators to ψ_0 gives:

Proposition 3.2.17. *The spinor ψ_0 is a generalized Killing spinor, $\nabla^{g_{a,b}}_X \psi_0 = A(X) \cdot \psi_0$, for the endomorphism*

$$A = \frac{(2a - 5b)}{4b\sqrt{30a}} \mathrm{Id}|_{\mathfrak{m}_1} - \frac{\sqrt{3a}}{4b\sqrt{10}} \mathrm{Id}|_{\mathfrak{m}_2},$$

and it is a Riemannian Killing spinor if and only $a = b$ ($\iff$ $g_{a,b}$ is a multiple of the second Einstein metric).

As in Remark 3.2.5, we briefly discuss the second Einstein metric in this case:

Remark 3.2.18. Here, the 3-Sasakian metric (the round metric) is given by $g_{a,b}|_{a=\frac{5}{24},b=\frac{1}{24}}$ and the second Einstein metric is given by rescaling by $\frac{1}{5}$ on the vertical component, yielding the normal homogeneous metric $g_{a,b}|_{a=b=\frac{1}{24}}$. From Theorem 3.2.16 and Proposition 3.2.17, we see that the 1-dimensional space of invariant spinors in this example is spanned by the Killing spinor determining the proper nearly parallel G_2-structure on $(S^7 = \frac{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}, g_{a,b}|_{a=b})$.

We now recall (see e.g. [FKMS97, ACFH15]) the well-known fact that a unit length spinor ψ in dimension 7 induces a G_2-structure, via the 3-form

$$\omega_\psi(X, Y, Z) := \langle X \cdot Y \cdot Z \cdot \psi, \psi \rangle. \tag{3.18}$$

In particular, by comparing Proposition 3.2.17 with [Agr06, Table 6] and [ACFH15, Lemma 4.5], one sees that the G_2-structure on $(S^7 = \frac{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}, g_{a,b})$ induced by ψ_0 is cocalibrated for all $a, b > 0$ and nearly parallel when $a = b$.

Proposition 3.2.19. *The G_2-form induced by ψ_0 is given with respect to our chosen orthonormal basis by*

$$\omega_{\psi_0} = -e_{123} + e_{145} + e_{167} + e_{246} - e_{257} + e_{347} + e_{356} = -\eta_1 \wedge \eta_2 \wedge \eta_3 - \sum_{i=1}^{3} \eta_i \wedge \Phi_i|_{\mathcal{H}}$$

and is invariant.

Proposition 4.4 in [ACFH15] says that the intrinsic torsion of this G_2-structure is given by $\Gamma = -\frac{2}{3} A \lrcorner \omega_{\psi_0}$, where the contraction of an endomorphism into a 3-form means the 3-form composed with the endormorphism in the first argument: $(A \lrcorner \omega_{\psi_0})(X, Y, Z) := \omega_{\psi_0}(A(X), Y, Z)$. Considering the G_2-connection $\nabla^n := \nabla^{g_{a,b}} - \Gamma$, one easily verifies $\nabla^n \psi_0 = 0$, as expected. We also find:

Proposition 3.2.20. *The torsion T^n of the G_2-connection ∇^n is a 3-form if and only if $a = b$. When $a = b$,*

$$T^n = \frac{-1}{\sqrt{30a}}\, \omega_{\psi_0}$$

is given by a multiple of the G_2-form, and thus in particular is invariant and ∇^n-parallel.

Proof. From $\nabla^n = \nabla^{g_{a,b}} - \Gamma$ we see that the difference tensor between ∇^n and $\nabla^{g_{a,b}}$ is $-\Gamma = \frac{2}{3}A \lrcorner \omega_{\psi_0}$, and thus the torsion is a 3-form if and only if $(X, Y, Z) \mapsto \frac{2}{3}\omega_{\psi_0}(A(X), Y, Z)$ is totally skew-symmetric. Clearly this happens precisely when A is a multiple of the identity, which occurs if and only if $a = b$. When $a = b$ we have $A = \frac{-\sqrt{3}}{4\sqrt{10a}}\,\mathrm{Id}$ and thus the torsion of ∇^n is given by

$$T^n = -2\Gamma = -2\left(-\frac{2}{3}A \lrcorner \omega_{\psi_0}\right) = \frac{-1}{\sqrt{30a}}\omega_{\psi_0}.$$

□

Remark 3.2.21. From [FI02] it is known that the characteristic connection ∇^c (which exists in this example since the G_2-structure is cocalibrated) is unique, so when $a = b$ it coincides with ∇^n. When $a \neq b$ we need to find a different way to describe ∇^c. Using Theorem 4.8 in [FI02], and taking into account that our G_2-structure is cocalibrated and differs from the reference G_2-form in [FI02] by an orientation-reversing change of basis, the torsion 3-form of ∇^c is given by

$$T^c = -\frac{1}{6}\langle d\omega_{\psi_0}, *\omega_{\psi_0}\rangle \omega_{\psi_0} + *d\omega_{\psi_0}. \tag{3.19}$$

In order to compute T^c using (3.19), we first prove the following lemma:

Lemma 3.2.22. *The Hodge dual and exterior derivative of ω_{ψ_0} are given by*

(i) $*\omega_{\psi_0} = -e_{4567} + e_{2367} + e_{2345} + e_{1357} - e_{1346} + e_{1256} + e_{1247}$,

(ii) $d\omega_{\psi_0} = \frac{a+5b}{b\sqrt{30a}}(*\omega_{\psi_0}) + \frac{-5a+5b}{b\sqrt{30a}}e_{4567}$.

Proof. The first claim is straightforward. The proof of the second claim proceeds by a lengthy yet straightforward computation using the above expression of the Nomizu map $\Lambda^{g_{a,b}}$ together with the identity $d = \sum_{i=1}^{7} e_i \wedge \nabla^{g_{a,b}}_{e_i}$. □

Recalling (e.g. from [Agr06, Table 6]) that a G_2-structure ω is nearly parallel if and only if $d\omega$ is a non-zero scalar multiple of $*\omega$, the preceding lemma also gives an alternate proof of the fact that the G_2-structure induced by ψ_0 is nearly parallel if and only if $a = b$.

Corollary 3.2.23. *The characteristic connection ∇^c is given by*

$$\nabla^c = \nabla^{g_{a,b}} - \frac{1}{2}\left(\frac{a}{b\sqrt{30a}}\omega_{\psi_0} + \frac{5a-5b}{b\sqrt{30a}}e_{123}\right),$$

and the defining 3-form ω_ψ and defining spinor ψ_0 are both ∇^c-parallel.

Proof. Using (3.19) and the preceding lemma, we calculate

$$\begin{aligned}
T^c &= -\frac{1}{6}\langle \frac{a+5b}{b\sqrt{30a}}(*\omega_{\psi_0}) + \frac{-5a+5b}{b\sqrt{30a}}e_{4567}, *\omega_{\psi_0}\rangle\omega_{\psi_0} + *\left(\frac{a+5b}{b\sqrt{30a}}(*\omega_{\psi_0}) + \frac{-5a+5b}{b\sqrt{30a}}e_{4567}\right)\\
&= \left(\frac{-7(a+5b)}{6b\sqrt{30a}} + \frac{-5a+5b}{6b\sqrt{30a}}\right)\omega_{\psi_0} + \left(\frac{a+5b}{b\sqrt{30a}}\omega_{\psi_0} + \frac{-5a+5b}{b\sqrt{30a}}e_{123}\right)\\
&= -\frac{a}{b\sqrt{30a}}\omega_{\psi_0} - \frac{5a-5b}{b\sqrt{30a}}e_{123}.
\end{aligned}$$

The fact that $\nabla^c\omega = 0 = \nabla^c\psi_0$ follows since ∇^c is a G_2-connection. □

A natural question is whether there are other invariant differential forms. To that end, one finds:

Proposition 3.2.24. *Up to taking Hodge duals, the invariant differential forms on $(S^7 = \frac{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}, g_{a,b})$ are given in Table 3.1.*

Proof. By hand, or using the LiE computer algebra package ([LCL88]), one finds that the dimensions of the spaces of invariant 1-, 2-, and 3-forms on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ are 3, 6, and 10 respectively. Bases for these spaces are then given by $\{\eta_i\}_{i=1}^3$, $\{\Phi_i|_\mathcal{V}, \Phi_i|_\mathcal{H}\}_{i=1,2,3}$, and $\{\eta_1\wedge\eta_2\wedge\eta_3\}\cup\{\eta_i\wedge\Phi_j|_\mathcal{H}\}_{i,j=1,2,3}$ respectively. The result then follows by checking which elements in these spans are invariant under the additional three isotropy operators described in the proof of Theorem 3.2.16. □

k	$\dim\Lambda^k_{\mathrm{inv}}$	Basis for Λ^k_{inv}
0	1	1
1	0	0
2	0	0
3	2	ω_{ψ_0}, $\xi_{1,2,3}$

Table 3.1: Invariant Differential Forms on $(S^7 = \frac{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}, g_{a,b})$

Noting from the preceding proposition that there are no invariant 1-forms, one obtains:

Corollary 3.2.25. *The space $(S^7 = \frac{\mathrm{Sp}(2)\,\mathrm{Sp}(1)}{\mathrm{Sp}(1)\,\mathrm{Sp}(1)}, g_{a,b})$ does not admit an invariant Einstein-Sasakian or 3-Sasakian structure for any values of $a, b > 0$.*

3.2.3 S^1-Quaternionic Spheres, $S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{U}(1)}{\mathrm{Sp}(n-1)\,\mathrm{U}(1)}$

This is the case corresponding to $K = \mathrm{U}(1)$. Viewing $\mathrm{U}(1)$ as a subgroup of $\mathrm{Sp}(1)$ via $e^{i\theta} \mapsto \cos\theta + i\sin\theta + 0j + 0k$, and from [AHL23, Eqn. (24)], we have at the level of Lie algebras:

$$\mathfrak{sp}(n) \oplus \mathfrak{u}(1) = \mathrm{span}_{\mathbb{R}}\{(iF_{p,q}^{(n)}, 0), (jF_{p,q}^{(n)}, 0), (kF_{p,q}^{(n)}, 0), (E_{r,s}^{(n)}, 0), (0, i)\}_{\substack{1\le p\le q\le n,\\ 1\le r<s\le n}}$$
$$\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1) = \mathrm{span}_{\mathbb{R}}\{(iF_{p,q}^{(n)}, 0), (jF_{p,q}^{(n)}, 0), (kF_{p,q}^{(n)}, 0), (E_{r,s}^{(n)}, 0), (iF_{1,1}^{(n)}, i)\}_{\substack{2\le p\le q\le n.\\ 2\le r<s\le n}}$$

Note that the Killing form of $\mathfrak{sp}(n) \oplus \mathfrak{u}(1)$ fails to be non-degenerate, so in order to choose a reductive complement we instead take the orthogonal complement with respect to the restriction of the inner product κ from (3.17) to the subalgebra $\mathfrak{sp}(n) \oplus \mathfrak{u}(1) \subset \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$,

$$\mathfrak{m} := (\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1))^{\perp_\kappa}.$$

The isotropy representation splits into one copy of the trivial representation and two non-isomorphic irreducible representations:

$$\mathfrak{m} \simeq \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

where $\dim_{\mathbb{R}} \mathfrak{m}_1 = 1$, $\dim_{\mathbb{R}} \mathfrak{m}_2 = 2$, and $\dim_{\mathbb{R}} \mathfrak{m}_3 = 4(n-1)$. This gives a 3-parameter family of invariant metrics,

$$g_{a,b,c} := -a\kappa|_{\mathfrak{m}_1\times\mathfrak{m}_1} - b\kappa|_{\mathfrak{m}_2\times\mathfrak{m}_2} - c\kappa|_{\mathfrak{m}_3\times\mathfrak{m}_3}, \quad a, b, c > 0.$$

In particular, by manually checking the sectional curvatures, one finds that the round metric is given by the parameters

$$a = \frac{n+3}{8(n+1)}, \quad b = \frac{1}{4(n+1)}, \quad c = \frac{1}{8(n+1)}.$$

For general $a, b, c > 0$, a $g_{a,b,c}$-orthonormal basis for $\mathfrak{m}$ is given by

$$\xi_1 := \frac{1}{\Omega}\left(iF_{1,1}, -\left(\frac{n+1}{2}\right)i\right), \quad \xi_2 := \frac{1}{2\sqrt{b(n+1)}}\left(-kF_{1,1}, 0\right), \quad \xi_3 := \frac{1}{2\sqrt{b(n+1)}}\left(jF_{1,1}, 0\right),$$
$$e_{4p} := \frac{1}{2\sqrt{2c(n+1)}}(jF_{1,p+1}, 0), \quad e_{4p+1} := \frac{1}{2\sqrt{2c(n+1)}}(kF_{1,p+1}, 0),$$
$$e_{4p+2} := \frac{1}{2\sqrt{2c(n+1)}}(iF_{1,p+1}, 0), \quad e_{4p+3} := \frac{1}{2\sqrt{2c(n+1)}}(E_{1,p+1}, 0),$$

for $1 \leq p \leq n-1$, where $\Omega := \sqrt{2a(n+1)(n+3)}$, and, in terms of this basis, the isotropy summands are

$$\mathfrak{m}_1 = \mathrm{span}_{\mathbb{R}}\{\xi_1\}, \quad \mathfrak{m}_2 = \mathrm{span}_{\mathbb{R}}\{\xi_2, \xi_3\}, \quad \mathfrak{m}_3 = \mathrm{span}_{\mathbb{R}}\{e_{4p}, e_{4p+1}, e_{4p+2}, e_{4p+3}\}_{p=1}^{n-1}.$$

From [AHL23, Eqn. (41)] we obtain:

Theorem 3.2.26. *Using the above orthonormal basis and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors on* $(S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{U}(1)}{\mathrm{Sp}(n-1)\,\mathrm{U}(1)}, g_{a,b,c})$ *for any* $a, b, c > 0$ *is trivial unless* n *is even. If* n *is even,*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{\omega^{n/2},\ y_1 \wedge \omega^{(n-2)/2}\},$$

where $\omega := \sum_{i=1}^{n-1} y_{2i} \wedge y_{2i+1}$.

Proof. This follows directly from [AHL23, Eqn. (41)]. Alternatively, one can argue exactly as in the proof of Theorem 3.2.16, using only the spin lift of the operator

$$\mathrm{ad}(iF_{1,1}^{(n)}, i)|_{\mathfrak{m}} = \xi_2 \wedge \xi_3 - \Phi_1. \tag{3.20}$$

□

In order to differentiate these spinors, we calculate:

Lemma 3.2.27. *The Nomizu map for the Levi-Civita connection of* $g_{a,b,c}$ *is given by*

$$\begin{aligned}
&\Lambda^{g_{a,b,c}}(x_1)x_2 = \frac{1}{2}[x_1, x_2]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b,c}}(x)y = (1 - \frac{a}{2b})[x, y]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b,c}}(y)x = \frac{a}{2b}[y, x]_{\mathfrak{m}},\\
&\Lambda^{g_{a,b,c}}(y_1)y_2 = \frac{1}{2}[y_1, y_2]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b,c}}(x)z = (1 - \frac{a}{2c})[x, z]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b,c}}(z)x = \frac{a}{2c}[z, x]_{\mathfrak{m}},\\
&\Lambda^{g_{a,b,c}}(z_1)z_2 = \frac{1}{2}[z_1, z_2]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b,c}}(y)z = (1 - \frac{b}{2c})[y, z]_{\mathfrak{m}}, \quad \Lambda^{g_{a,b,c}}(z)y = \frac{b}{2c}[z, y]_{\mathfrak{m}},
\end{aligned}$$

for all $x, x_1, x_2 \in \mathfrak{m}_1$, $y, y_1, y_2 \in \mathfrak{m}_2$, $z, z_1, z_2 \in \mathfrak{m}_3$.

Proof. This is a straightforward calculation using the fact that κ is ad-invariant and the commutator relations

$$\begin{aligned}
&[\mathfrak{m}_1, \mathfrak{m}_1] = 0, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathfrak{m}_2, \quad [\mathfrak{m}_1, \mathfrak{m}_3] \subseteq \mathfrak{m}_3, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subseteq \mathfrak{m}_1 \oplus (\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1)),\\
&[\mathfrak{m}_2, \mathfrak{m}_3] \subseteq \mathfrak{m}_3, \quad [\mathfrak{m}_3, \mathfrak{m}_3] \subseteq \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus (\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1)).
\end{aligned}$$

□

Remark 3.2.28. In terms of our chosen basis, the operators in the preceding lemma take the form

$$\Lambda^{g_{a,b,c}}(\xi_1) = \frac{2(1-\frac{a}{2b})}{\Omega}\ \xi_2 \wedge \xi_3 + \frac{(1-\frac{a}{2c})}{\Omega}\sum_{p=1}^{n-1}(e_{4p}\wedge e_{4p+1} + e_{4p+2}\wedge e_{4p+3}),$$
$$\Lambda^{g_{a,b,c}}(\xi_2) = \frac{a}{b\Omega}\ \xi_3 \wedge \xi_1 + \frac{(1-\frac{b}{2c})}{2\sqrt{b(n+1)}}\sum_{p=1}^{n-1}(e_{4p}\wedge e_{4p+2} - e_{4p+1}\wedge e_{4p+3}),$$
$$\Lambda^{g_{a,b,c}}(\xi_3) = \frac{a}{b\Omega}\ \xi_1 \wedge \xi_2 + \frac{(1-\frac{b}{2c})}{2\sqrt{b(n+1)}}\sum_{p=1}^{n-1}(e_{4p}\wedge e_{4p+3} + e_{4p+1}\wedge e_{4p+2}),$$

in the vertical directions and, in the horizontal directions,

$$\Lambda^{g_{a,b,c}}(e_{4p}) = -\frac{a}{2c\Omega}\ \xi_1 \wedge e_{4p+1} - \frac{\sqrt{b}}{4c\sqrt{n+1}}(\xi_2\wedge e_{4p+2} + \xi_3\wedge e_{4p+3}),$$
$$\Lambda^{g_{a,b,c}}(e_{4p+1}) = \frac{a}{2c\Omega}\ \xi_1 \wedge e_{4p} + \frac{\sqrt{b}}{4c\sqrt{n+1}}(\xi_2\wedge e_{4p+3} - \xi_3\wedge e_{4p+2}),$$
$$\Lambda^{g_{a,b,c}}(e_{4p+2}) = -\frac{a}{2c\Omega}\ \xi_1 \wedge e_{4p+3} + \frac{\sqrt{b}}{4c\sqrt{n+1}}(\xi_2\wedge e_{4p} + \xi_3\wedge e_{4p+1}),$$
$$\Lambda^{g_{a,b,c}}(e_{4p+3}) = \frac{a}{2c\Omega}\ \xi_1 \wedge e_{4p+2} - \frac{\sqrt{b}}{4c\sqrt{n+1}}(\xi_2\wedge e_{4p+1} - \xi_3\wedge e_{4p}).$$

In contrast to the previous section, here we do have an invariant vector field, namely ξ_1, leading to the possibility of invariant almost contact structures. The following proposition gives necessary and sufficient conditions for the existence of various types of these structures:

Proposition 3.2.29. *The sphere* $(S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{U}(1)}{\mathrm{Sp}(n-1)\,\mathrm{U}(1)}, g_{a,b,c})$ *admits:*

(i) a compatible invariant normal almost contact metric structure for all $a, b, c > 0$.

(ii) a compatible invariant α*-contact structure if and only if* $\frac{a}{b\Omega} = \frac{a}{2c\Omega} = \alpha$.

(iii) a compatible invariant α*-K-contact structure if and only if* $\frac{a}{b\Omega} = \frac{a}{2c\Omega} = \alpha$.

In particular there exists a compatible invariant α*-Sasakian structure if and only if* $\frac{a}{b\Omega} = \frac{a}{2c\Omega} = \alpha$.

Proof. Following a similar argument as in the proof of Proposition 3.1.15, the only choices for the Reeb vector field are $\xi := \pm\xi_1$. We claim that the 2-form $\Phi := g_{a,b,c}(\cdot, \varphi(\cdot))$ is invariant if and only if

$$\Phi \in (\Lambda^2 \mathfrak{m}_2)^{\mathrm{Sp}(n-1)\,\mathrm{U}(1)} \oplus (\Lambda^2 \mathfrak{m}_3)^{\mathrm{Sp}(n-1)\,\mathrm{U}(1)} \simeq \mathrm{span}_{\mathbb{R}}\{\mathrm{ad}\,\xi|_{\mathfrak{m}_2}\} \oplus \mathrm{span}_{\mathbb{R}}\{\mathrm{ad}\,\xi|_{\mathfrak{m}_3}\}. \qquad (3.21)$$

The proof of the claim consists of two parts: First, by noting that $\mathfrak{m}_1$, $\mathfrak{m}_2, \mathfrak{m}_3$ are irreducible and have pairwise distinct dimensions, we see that any invariant 2-form has trivial $\mathfrak{m}_1 \otimes \mathfrak{m}_2$, $\mathfrak{m}_1 \otimes \mathfrak{m}_3$, and $\mathfrak{m}_2 \otimes \mathfrak{m}_3$ components. Second, we note that $\Lambda^2 \mathfrak{m}_1 = 0$ (for dimension reasons) and that each $\Lambda^2 \mathfrak{m}_i$ $(i = 2, 3)$ has a real 3-dimensional space of $\mathrm{Sp}(n-1)$-invariant 2-forms, corresponding to the quaternionic structure

$$\mathcal{I}_i := \Phi_1|_{\mathfrak{m}_i}, \quad \mathcal{J}_i := \Phi_2|_{\mathfrak{m}_i}, \quad \mathcal{K}_i := \Phi_3|_{\mathfrak{m}_i}, \qquad i = 2, 3.$$

Imposing the additional condition of U(1)-invariance is equivalent to also requiring $\mathcal{I}$-complex linearity, giving $(\Lambda^2 \mathfrak{m}_i)^{\mathrm{Sp}(n-1)\,\mathrm{U}(1)} \simeq \mathrm{span}_{\mathbb{R}}\{\mathcal{I}_i\} = \mathrm{span}_{\mathbb{R}}\{\mathrm{ad}\,\xi|_{\mathfrak{m}_i}\}$ for $i = 2, 3$.

Returning to the main proof, (3.21) is equivalent to $\varphi = \lambda_1\,\mathrm{ad}\,\xi|_{\mathfrak{m}_2} \oplus \lambda_2\,\mathrm{ad}\,\xi|_{\mathfrak{m}_3}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$, and the metric compatibility condition $g_{a,b,c}(\varphi(X), \varphi(Y)) = g_{a,b,c}(X, Y) - g_{a,b,c}(\xi, X) g_{a,b,c}(\xi, Y)$ necessitates $\lambda_1 = \Omega/2$, $\lambda_2 = \Omega$, i.e.

$$\varphi = \frac{\Omega}{2}\,\mathrm{ad}\,\xi|_{\mathfrak{m}_2} \oplus \Omega\,\mathrm{ad}\,\xi|_{\mathfrak{m}_3}.$$

One then calculates that the Nijenhuis tensor vanishes for any values of a, b, c, and the structure is α-contact ($d\eta = 2\alpha\Phi$) and α-K-contact ($\nabla^g_X \xi = -\alpha\varphi(X)$) if and only if $\frac{a}{b\Omega} = \frac{a}{2c\Omega} = \alpha$. □

Solving the equations in the preceding proposition for a, b, c, we immediately obtain:

Corollary 3.2.30. *For each fixed $\alpha > 0$, the invariant α-Sasakian structures occur in a 1-parameter family:*

$$a = 8\lambda^2\alpha^2(n+1)(n+3), \quad b = 2\lambda, \quad c = \lambda, \qquad (\lambda > 0). \qquad (3.22)$$

In particular, the round metric occurs for the parameter $\lambda = \frac{1}{8(n+1)}$.

In contrast to the case $G = \mathrm{Sp}(n)$, we shall see in the following remark that the relationship between invariant Einstein-Sasakian structures and invariant Killing spinors is more complicated for $G = \mathrm{Sp}(n)\,\mathrm{U}(1)$, stemming from the fact that this group has a non-trivial 2-dimensional complex representation.

Remark 3.2.31. We remark that the above 2-form Φ takes the form (3.12). Therefore, if an invariant Einstein-Sasakian structure exists then Theorem 2.3.8 and Proposition 4.3.11 imply that there are two linearly independent Killing spinors inside $E_1^- = \mathrm{span}_{\mathbb{C}}\{1,\ y_1 \wedge \omega^{n-1}\}$. If $n = 1$ then $\dim_{\mathbb{C}} \Sigma = 2$, and it follows that $\Sigma = E_1^-$ is spanned by two linearly independent (non-invariant) Killing spinors. In fact, for all $n \geq 1$, one sees from Theorem 3.2.26 that the intersection of E_1^- with the space of invariant spinors is trivial, so the Killing spinors spanning E_1^- are not invariant. This contrasts with the behaviour in the cases $G = \mathrm{Sp}(n)$, $\mathrm{SU}(n)$, where the spaces $E_i^\pm$ had bases of invariant spinors. Indeed, in each of the three cases $G = \mathrm{Sp}(n)\,\mathrm{U}(1)$, $\mathrm{Sp}(n)$, $\mathrm{SU}(n+1)$ it is easy to see using Kath's arguments from [Kat00, Prop. 7.1, Thm. 7.1] that there is a G-representation on $E_i^\pm$ which must be trivial in the latter two cases for dimensional reasons. However, unlike the other two groups, $G = \mathrm{Sp}(n)\,\mathrm{U}(1)$ has a nontrivial 2-dimensional representation, which is what allows the Killing spinors spanning E_1^- to be non-invariant in this case. This behaviour is somewhat surprising; one would intuitively expect the Killing spinors associated to an invariant Einstein-Sasakian structure to also be invariant.

Theorem 3.2.32. *The sphere* $(S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{U}(1)}{\mathrm{Sp}(n-1)\,\mathrm{U}(1)}, g_{a,b,c})$ *admits an invariant generalized Killing spinor if and only if $n = 2$. If $n = 2$ there exists a pair ψ_0, ψ_1 of linearly independent invariant generalized Killing spinors,*

$$\nabla_X^{g_{a,b,c}} \psi_i = A_i(X) \cdot \psi_i, \quad i = 0, 1,$$

for the endomorphisms

$$A_0 := \frac{a}{2\Omega}(\frac{1}{c} - \frac{1}{b}) \,\mathrm{Id}|_{\mathfrak{m}_1} + \left(\frac{a}{2b\Omega} - \frac{(1 - \frac{b}{2c})}{2\sqrt{3}\sqrt{b}} \right) \mathrm{Id}|_{\mathfrak{m}_2} + \left(-\frac{a}{4c\Omega} - \frac{\sqrt{b}}{4c\sqrt{3}} \right) \mathrm{Id}|_{\mathfrak{m}_3},$$
$$A_1 := \frac{a}{2\Omega}(\frac{1}{c} - \frac{1}{b}) \,\mathrm{Id}|_{\mathfrak{m}_1} + \left(\frac{a}{2b\Omega} + \frac{(1 - \frac{b}{2c})}{2\sqrt{3}\sqrt{b}} \right) \mathrm{Id}|_{\mathfrak{m}_2} + \left(-\frac{a}{4c\Omega} + \frac{\sqrt{b}}{4c\sqrt{3}} \right) \mathrm{Id}|_{\mathfrak{m}_3}$$

with at most three distinct eigenvalues.

Proof. Suppose there exists an invariant generalized Killing spinor $\nabla_X^{g_{a,b,c}} \psi = A(X) \cdot \psi$ and, using Theorem 3.2.26, write $\psi = \lambda_1 \omega^{\frac{n}{2}} + \lambda_2 y_1 \wedge \omega^{\frac{n-2}{2}}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Let us examine the spin lift of the operator $\Lambda^{g_{a,b,c}}(e_{4p})$. One calculates

$$\frac{1}{2}\xi_1 \cdot e_{4p+1} \cdot \psi = \frac{i}{2} \left[\lambda_1 \left(\frac{n}{2} y_{2p+1} \wedge \omega^{\frac{n-2}{2}} - y_{2p} \wedge \omega^{\frac{n}{2}} \right) + \lambda_2 \left(\frac{n-2}{2} y_1 \wedge y_{2p+1} \wedge \omega^{\frac{n-4}{2}} - y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}} \right) \right],$$
$$\frac{1}{2}\xi_2 \cdot e_{4p+2} \cdot \psi = \frac{1}{2} \left[\lambda_1 \left(\frac{n}{2} y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}} - y_1 \wedge y_{2p+1} \wedge \omega^{\frac{n}{2}} \right) + \lambda_2 \left(y_{2p+1} \wedge \omega^{\frac{n-2}{2}} - \frac{n-2}{2} y_{2p} \wedge \omega^{\frac{n-4}{2}} \right) \right],$$

$$\frac{1}{2}\xi_3 \cdot e_{4p+3} \cdot \psi = \frac{1}{2}\left[\lambda_1\left(\frac{n}{2}y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}} + y_1 \wedge y_{2p+1} \wedge \omega^{\frac{n}{2}}\right) + \lambda_2\left(y_{2p+1} \wedge \omega^{\frac{n-2}{2}} + \frac{n-2}{2}y_{2p} \wedge \omega^{\frac{n-4}{2}}\right)\right],$$

and the explicit formula from Remark 3.2.28 then gives

$$\begin{aligned}\widetilde{\Lambda^{g_{a,b,c}}}(e_{4p}) \cdot \psi = &-\left(\frac{ian\lambda_1}{8c\Omega} + \frac{\sqrt{b}\lambda_2}{4c\sqrt{n+1}}\right) y_{2p+1} \wedge \omega^{\frac{n-2}{2}} + \left(\frac{ia\lambda_2}{4c\Omega} + \frac{n\lambda_1}{2}\right) y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}} \\ &+ \left(\frac{ia\lambda_1}{4c\Omega}\right) y_{2p} \wedge \omega^{\frac{n}{2}} - \left(\frac{ia(n-2)\lambda_2}{8c\Omega}\right) y_1 \wedge y_{2p+1} \wedge \omega^{\frac{n-4}{2}}\end{aligned}$$

In order for the right hand side of this expression to be equal to the Clifford product of a (real) tangent vector with ψ, one sees that such a vector must be of the form $se_{4p} + te_{4p+1}$ for some $s, t \in \mathbb{R}$. Comparing with

$$\begin{aligned}(se_{4p} + te_{4p+1}) \cdot \psi = &\left(\frac{ins\lambda_1}{2} - \frac{nt\lambda_1}{2}\right) y_{2p+1} \wedge \omega^{\frac{n-2}{2}} + (is\lambda_1 + t\lambda_1) y_{2p} \wedge \omega^{\frac{n}{2}} \\ &+ \left(-\frac{i(n-2)s\lambda_2}{2} + \frac{(n-2)t\lambda_2}{2}\right) y_1 \wedge y_{2p+1} \wedge \omega^{\frac{n-4}{2}} \\ &+ (-is\lambda_2 - t\lambda_2) y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}}\end{aligned}$$

gives the necessary conditions

$$\begin{aligned}\left(\frac{ia\lambda_1}{4c\Omega}\right) y_{2p} \wedge \omega^{\frac{n}{2}} &= (is\lambda_1 + t\lambda_1) y_{2p} \wedge \omega^{\frac{n}{2}}, \\ \left(\frac{ia\lambda_2}{4c\Omega} + \frac{n\lambda_1}{2}\right) y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}} &= -(is\lambda_1 + t\lambda_1) y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}}.\end{aligned}$$

If $n \neq 2$ then $y_{2p} \wedge \omega^{\frac{n}{2}}$ and $y_1 \wedge y_{2p} \wedge \omega^{\frac{n-2}{2}}$ are non-zero (the latter is non-zero independent of n), and we have

$$\left(\frac{ia\lambda_1}{4c\Omega}\right) = -\left(\frac{ia\lambda_2}{4c\Omega} + \frac{n\lambda_1}{2}\right).$$

It follows by taking the real part of both sides that $\lambda_1 = \lambda_2 = 0$, i.e. $\psi \equiv 0$ is trivial. For $n = 2$, lifting the operators in Remark 3.2.28 and applying them to the spinors

$$\psi_0 := \frac{1}{\sqrt{2}}(\omega + iy_1), \quad \psi_1 := \xi_1 \cdot \psi_0 = \frac{1}{\sqrt{2}}(i\omega + y_1)$$

gives the result. □

Next, we calculate the Ambrose-Singer torsion and determine its type:

Proposition 3.2.33. *For any $a, b, c > 0$ the sphere $(S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{U}(1)}{\mathrm{Sp}(n-1)\,\mathrm{U}(1)}, g_{a,b,c})$ has Ambrose-*

Singer torsion of type $\mathcal{T}_{\text{skew}} \oplus \mathcal{T}_{\text{CT}}$, given by

$$T^{\text{AS}}(\xi_1,\xi_2) = -\frac{2}{\Omega}\xi_3, \quad T^{\text{AS}}(\xi_1,\xi_3) = \frac{2}{\Omega}\xi_2, \quad T^{\text{AS}}(\xi_2,\xi_3) = -\frac{2a}{b\Omega}\xi_1,$$
$$T^{\text{AS}}(\xi_1,-)|_{\mathfrak{m}_3} = \frac{1}{\Omega}\Phi_1|_{\mathfrak{m}_3}, \quad T^{\text{AS}}(\xi_2,-)|_{\mathfrak{m}_3} = \frac{1}{2\sqrt{b(n+1)}}\Phi_2|_{\mathfrak{m}_3}, \quad T^{\text{AS}}(\xi_3,-)|_{\mathfrak{m}_3} = \frac{1}{2\sqrt{b(n+1)}}\Phi_3|_{\mathfrak{m}_3},$$
$$T^{\text{AS}}(e_{4p},e_{4q}) = T^{\text{AS}}(e_{4p+1},e_{4q+1}) = T^{\text{AS}}(e_{4p+2},e_{4q+2}) = T^{\text{AS}}(e_{4p+3},e_{4q+3}) = 0$$
$$T^{\text{AS}}(e_{4p},e_{4q+1}) = -\frac{a\delta_{p,q}}{c\Omega}\xi_1, \quad T^{\text{AS}}(e_{4p},e_{4q+2}) = -\frac{\sqrt{b}\delta_{p,q}}{2c\sqrt{n+1}}\xi_2, \quad T^{\text{AS}}(e_{4p},e_{4q+3}) = -\frac{\sqrt{b}\delta_{p,q}}{2c\sqrt{n+1}}\xi_3,$$
$$T^{\text{AS}}(e_{4p+1},e_{4q+2}) = -\frac{\sqrt{b}\delta_{p,q}}{2c\sqrt{n+1}}\xi_3, \quad T^{\text{AS}}(e_{4p+1},e_{4q+3}) = \frac{\sqrt{b}\delta_{p,q}}{2c\sqrt{n+1}}\xi_2, \quad T^{\text{AS}}(e_{4p+2},e_{4q+3}) = -\frac{a\delta_{p,q}}{c\Omega}\xi_1,$$

for $p,q = 1,\dots,n-1$, where Φ_1,Φ_2,Φ_3 are defined formally as in (3.12)-(3.14). The projection of T^{AS} onto $\mathcal{T}_{\text{skew}}$ is

$$T^{\text{AS}}_{\text{skew}} = -\left(\frac{2a+4b}{3b\Omega}\right)\xi_{1,2,3} + \left(\frac{a+2c}{3c\Omega}\right)\xi_1\wedge\Phi_1|_{\mathfrak{m}_3} + \left(\frac{b+2c}{6c\sqrt{b(n+1)}}\right)\sum_{i=2}^{3}\xi_i\wedge\Phi_i|_{\mathfrak{m}_3},$$

with $T^{\text{AS}} = T^{\text{AS}}_{\text{skew}}$ if and only if $a=b=c$.

Before studying the situation in dimension 7 in more detail, we conclude our discussion of the general case with a description of the invariant differential forms of degree $k \leq 3$:

Proposition 3.2.34. *The invariant differential forms of degree less than or equal to 3 on $(S^{4n-1} = \frac{\text{Sp}(n)\,\text{U}(1)}{\text{Sp}(n-1)\,\text{U}(1)}, g_{a,b,c})$ are given in Table 3.2.*

Proof. The result for degrees $k = 0,1$ is clear, and for $k = 2$ it follows from the proof of Proposition 3.2.29. For $k = 3$ one proceeds as in the proof of Proposition 3.2.24 (a description of the $\text{Sp}(n-1)$-invariant forms in general dimension can be found in Corollary 4.3.7, for example), using only the additional operator (3.20). □

k	$\dim\Lambda^k_{\text{inv}}$	Basis for Λ^k_{inv}
0	1	1
1	1	ξ_1
2	2	$\Phi_1|_{\mathfrak{m}_2}$, $\Phi_1|_{\mathfrak{m}_3}$
3	4	$\xi_{1,2,3}$, $\xi_1\wedge\Phi_1|_{\mathfrak{m}_3}$, $(\xi_2\wedge\Phi_2|_{\mathfrak{m}_3}+\xi_3\wedge\Phi_3|_{\mathfrak{m}_3})$, $(\xi_2\wedge\Phi_3|_{\mathfrak{m}_3}-\xi_3\wedge\Phi_2|_{\mathfrak{m}_3})$

Table 3.2: Invariant Forms of Low Degree on $(S^{4n-1} = \frac{\text{Sp}(n)\,\text{U}(1)}{\text{Sp}(n-1)\,\text{U}(1)}, g_{a,b,c})$

In what follows, we examine more closely the generalized Killing spinors in dimension 7 and compare with the known results for the round (3-Sasakian) metric and the second Einstein metric (see [FKMS97, AF10, MS14b]).

Remark 3.2.35. In the case of an invariant Sasakian metric, substituting (3.22) into the endomorphisms from Theorem 3.2.32 gives

$$A_0 = \frac{1}{2}\,\mathrm{Id}|_{\mathfrak{m}_1\oplus\mathfrak{m}_2} + \frac{-\sqrt{6\lambda}-1}{2\sqrt{6\lambda}}\,\mathrm{Id}|_{\mathfrak{m}_3}, \quad A_1 = \frac{1}{2}\,\mathrm{Id}|_{\mathfrak{m}_1\oplus\mathfrak{m}_2} + \frac{-\sqrt{6\lambda}+1}{2\sqrt{6\lambda}}\,\mathrm{Id}|_{\mathfrak{m}_3}.$$

It is clear that A_0 is never a multiple of the identity map, and A_1 is a multiple of the identity if and only if $\lambda = 1/24$, which occurs precisely when $g_{a,b,c}$ is the round metric (see Corollary 3.2.30). When $\lambda = 1/24$, we have

$$A_0 = \frac{1}{2}\,\mathrm{Id}|_{\mathfrak{m}_1\oplus\mathfrak{m}_2} - \frac{3}{2}\,\mathrm{Id}|_{\mathfrak{m}_3}, \quad A_1 = \frac{1}{2}\,\mathrm{Id}\,.$$

In this case there are three linearly independent Killing spinors (the invariant Killing spinor ψ_1 and two non-invariant linearly independent Killing spinors inside E_1^-), and the result of [FK90] then implies the existence of a 3-Sasakian structure, as expected. Starting with the generalized Killing spinor ψ_0, the existence of a 3-Sasakian structure also follows from [MS14b, Thm. 4.10], which notes moreover that ψ_0 corresponds to the canonical spinor described in [AF10] for a general 3-Sasakian manifold of dimension 7 (and we then see that $\psi_1 = \xi_i \cdot \psi_0$ corresponds to one of the auxiliary spinors described therein).

Finally, as in Remark 3.2.5, we comment on the second Einstein metric in this example:

Remark 3.2.36. Here, the second Einstein metric is given by $g_{a,b,c}|_{a=\frac{1}{24},b=\frac{1}{60},c=\frac{1}{24}}$, and it is clear that this metric is not part of the family (3.22). Substituting these values of a,b,c into the endomorphisms from Theorem 3.2.32 gives

$$A_0 = -\frac{3}{2\sqrt{5}}\,\mathrm{Id}, \qquad A_1 = -\frac{3}{2\sqrt{5}}\,\mathrm{Id}|_{\mathfrak{m}_1} + \frac{13}{2\sqrt{5}}\,\mathrm{Id}|_{\mathfrak{m}_2} + \frac{1}{2\sqrt{5}}\,\mathrm{Id}|_{\mathfrak{m}_3},$$

so ψ_0 is the Killing spinor determining the proper nearly parallel G_2-structure and ψ_1 is a generalized Killing spinor with 3 distinct eigenvalues.

3.3 Exceptional Spheres

3.3.1 $S^6 = \mathrm{G}_2 / \mathrm{SU}(3)$

The isotropy representation here is irreducible, so the only invariant metrics are obtained from multiples of the Killing form. For convenience we choose the invariant inner product $g = B_0$ on

$\mathfrak{g}_2$, and consider the reductive complement $\mathfrak{m} := (\mathfrak{su}(3))^{\perp}$ with respect to this inner product. Following the notation of Proposition A.3, we have

$$\mathfrak{g}_2 = \mathrm{span}_{\mathbb{R}}\{\nu_1, \dots, \nu_{14}\}, \quad \mathfrak{su}(3) = \mathrm{span}_{\mathbb{R}}\{\nu_1, \dots \nu_8\},$$

and for $\mathfrak{m}$ we take the g-orthonormal basis given by

$$\mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{e_1, \dots, e_6\}, \quad \text{where } e_i := \begin{cases} \nu_{8+i} & i = 1, 2, 4, 6, \\ -\nu_{8+i} & i = 3, 5. \end{cases}$$

The motivation for this choice of basis relates to the existence of an invariant nearly Kähler structure; it is well-known that $S^6 = \mathrm{G}_2 / \mathrm{SU}(3)$ admits a unique invariant nearly Kähler structure up to sign [EL51]. Indeed, it appears as one of the four cases in Butruille's classification of homogeneous nearly Kähler 6-manifolds [But05, But10]. Inhomogeneous examples were later constructed in [CV15, FH17], and a nice overview of the topic of nearly Kähler 6-manifolds can be found in [ABF18]. For $(S^6 = \mathrm{G}_2 / \mathrm{SU}(3), g)$, a direct calculation (by hand or using computer algebra software) shows that the 1-dimensional space of invariant 2-forms is spanned by

$$J = e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6.$$

In particular, the invariant almost complex structures are $\pm J$ and the given basis $\{e_i\}$ is *adapted* to J in the sense that $J(e_1) = e_2$, $J(e_3) = e_4$, $J(e_5) = e_6$. This property allows us to give an easy proof of the following theorem:

Theorem 3.3.1. *Using the above orthonormal basis and the corresponding description of the spinor module from Remark 2.1.2, the space of invariant spinors on* $(S^6 = \mathrm{G}_2 / \mathrm{SU}(3), g)$ *is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{1, y_1 \wedge y_2 \wedge y_3\},$$

and the spinors $\psi_{\pm} := 1 \pm y_1 \wedge y_2 \wedge y_3$ *are Riemannian Killing spinors,*

$$\nabla^g_X \psi_{\pm} = \pm \frac{1}{2\sqrt{3}} X \cdot \psi_{\pm}.$$

Proof. Since the orthonormal basis $\{e_i\}$ is adapted to the invariant almost complex structure J, it follows from Remark 3.1.9 that $\Sigma \simeq \Lambda^{0,\bullet}\mathfrak{m}$ as complex representations of $\mathrm{SU}(3)$, and the space of invariants is therefore spanned by 1 and the complex volume form $y_1 \wedge y_2 \wedge y_3$ (see e.g. [FH91, Prop. F.10]). $\square$

Remark 3.3.2. The preceding theorem can also be proved directly, by performing a similar calculation as in Example 3.2.6, using the operators

$$\begin{aligned}
&\mathrm{ad}(\nu_1)|_{\mathfrak{m}} = \frac{1}{2}(e_{1,2} - e_{3,4}), \quad \mathrm{ad}(\nu_2)|_{\mathfrak{m}} = \frac{1}{2}(e_{3,5} + e_{4,6}), \quad \mathrm{ad}(\nu_3)|_{\mathfrak{m}} = \frac{1}{2}(-e_{3,6} + e_{4,5}),\\
&\mathrm{ad}(\nu_4)|_{\mathfrak{m}} = \frac{1}{2}(e_{1,6} - e_{2,5}), \quad \mathrm{ad}(\nu_5)|_{\mathfrak{m}} = \frac{1}{2}(-e_{1,5} - e_{2,6}), \quad \mathrm{ad}(\nu_6)|_{\mathfrak{m}} = \frac{1}{2}(-e_{1,4} + e_{2,3}),\\
&\mathrm{ad}(\nu_7)|_{\mathfrak{m}} = \frac{1}{2}(e_{1,3} + e_{2,4}), \quad \mathrm{ad}(\nu_8)|_{\mathfrak{m}} = \frac{1}{2\sqrt{3}}(-e_{1,2} - e_{3,4} + 2e_{5,6}).
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda^g(e_1) = \frac{1}{2\sqrt{3}}(e_{3,6} + e_{4,5}), \quad \Lambda^g(e_2) = \frac{1}{2\sqrt{3}}(e_{3,5} - e_{4,6}), \quad \Lambda^g(e_3) = \frac{1}{2\sqrt{3}}(-e_{1,6} - e_{2,5}),\\
&\Lambda^g(e_4) = \frac{1}{2\sqrt{3}}(-e_{1,5} + e_{2,6}), \quad \Lambda^g(e_5) = \frac{1}{2\sqrt{3}}(e_{1,4} + e_{2,3}), \quad \Lambda^g(e_6) = \frac{1}{2\sqrt{3}}(e_{1,3} - e_{2,4}).
\end{aligned}$$

Remark 3.3.3. It is noted in [AHL23, Remark 5.3] that the spinors $\psi_\pm$ induce the nearly Kähler structure J via

$$J(X) \cdot (\psi_\pm)_0 = iX \cdot (\psi_\pm)_0 \quad \text{for all } X \in TM,$$

where $(\psi_\pm)_0$ denotes the projection onto the even half-spinor module $\Sigma_0 \subseteq \Sigma$ (the fact that this relation uniquely defines an almost complex structure is a consequence of the injectivity of Clifford multiplication by real tangent vectors). This is a special case of the well-known relationship between Killing spinors and nearly Kähler structures in dimension 6, see e.g. [Gru90, BFGK91, ACFH15].

Next, we calculate the Ambrose-Singer torsion and determine its type:

Proposition 3.3.4. *The sphere* $(S^6 = \mathrm{G}_2 / \mathrm{SU}(3), g)$ *has Ambrose-Singer torsion of type* $\mathcal{T}_{\mathrm{skew}}$ *given by*

$$T^{\mathrm{AS}} = \frac{1}{\sqrt{3}}(-e_{1,3,6} - e_{1,4,5} - e_{2,3,5} + e_{2,4,6}).$$

3.3.2 $S^7 = \mathrm{Spin}(7)/\,\mathrm{G}_2$

The isotropy representation is again irreducible, so we use the invariant metric induced by the inner product $g = B_0$ on $\mathfrak{spin}(7)$, and choose the reductive complement $\mathfrak{m} := (\mathfrak{g}_2)^\perp$ with respect to this inner product. Following the notation of Proposition A.4, we have

$$\mathfrak{spin}(7) = \{\nu_1, \ldots, \nu_{14}, \nu'_{15}, \ldots, \nu'_{21}\}, \qquad \mathfrak{g}_2 = \{\nu_1, \ldots \nu_{14}\}, \qquad \mathfrak{m} = \{e_1, \ldots, e_7\},$$

where $e_i := \nu'_{14+i}$. A straightforward calculation in the spin representation gives:

Theorem 3.3.5. *(cf. [Wan89, §3 Case 4]). Using the above orthonormal basis and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors on $(S^7 = \mathrm{Spin}(7)/\,\mathrm{G}_2, g)$ is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{-1 + y_1 \wedge y_2 \wedge y_3\},$$

and the spinor $\psi := -1 + y_1 \wedge y_2 \wedge y_3$ is a Riemannian Killing spinor,

$$\nabla^g_X \psi = \frac{\sqrt{3}}{4\sqrt{2}} X \cdot \psi.$$

Proof. This follows from the same type of calculation as in Example 3.2.6, using the operators

$$\begin{aligned}
\mathrm{ad}(\nu_1)|_{\mathfrak{m}} &= \frac{1}{2}(e_{2,3} - e_{6,7}), \quad \mathrm{ad}(\nu_2)|_{\mathfrak{m}} = \frac{1}{2}(-e_{2,4} - e_{3,5}), \quad \mathrm{ad}(\nu_3)|_{\mathfrak{m}} = \frac{1}{2}(-e_{2,5} + e_{3,4}),\\
\mathrm{ad}(\nu_4)|_{\mathfrak{m}} &= \frac{1}{2}(e_{4,7} - e_{5,6}), \quad \mathrm{ad}(\nu_5)|_{\mathfrak{m}} = \frac{1}{2}(-e_{4,6} - e_{5,7}), \quad \mathrm{ad}(\nu_6)|_{\mathfrak{m}} = \frac{1}{2}(e_{2,7} - e_{3,6}),\\
\mathrm{ad}(\nu_7)|_{\mathfrak{m}} &= \frac{1}{2}(-e_{2,6} - e_{3,7}), \quad \mathrm{ad}(\nu_8)|_{\mathfrak{m}} = \frac{1}{2\sqrt{3}}(e_{2,3} - 2e_{4,5} + e_{6,7}),\\
\mathrm{ad}(\nu_9)|_{\mathfrak{m}} &= \frac{1}{2\sqrt{3}}(2e_{1,7} - e_{2,5} - e_{3,4}), \quad \mathrm{ad}(\nu_{10})|_{\mathfrak{m}} = \frac{1}{2\sqrt{3}}(2e_{1,6} + e_{2,4} - e_{3,5}),\\
\mathrm{ad}(\nu_{11})|_{\mathfrak{m}} &= \frac{1}{2\sqrt{3}}(-2e_{1,3} + e_{4,7} + e_{5,6}), \quad \mathrm{ad}(\nu_{12})|_{\mathfrak{m}} = \frac{1}{2\sqrt{3}}(2e_{1,2} + e_{4,6} - e_{5,7}),\\
\mathrm{ad}(\nu_{13})|_{\mathfrak{m}} &= \frac{1}{2\sqrt{3}}(2e_{1,5} + e_{2,7} + e_{3,6}), \quad \mathrm{ad}(\nu_{14})|_{\mathfrak{m}} = \frac{1}{2\sqrt{3}}(-2e_{1,4} + e_{2,6} - e_{3,7}),\\
\Lambda^g(e_1) &= \frac{1}{2\sqrt{6}}(e_{2,3} + e_{4,5} + e_{6,7}), \quad \Lambda^g(e_2) = \frac{1}{2\sqrt{6}}(-e_{1,3} - e_{4,7} - e_{5,6}),\\
\Lambda^g(e_3) &= \frac{1}{2\sqrt{6}}(e_{1,2} - e_{4,6} + e_{5,7}), \quad \Lambda^g(e_4) = \frac{1}{2\sqrt{6}}(-e_{1,5} + e_{2,7} + e_{3,6}),\\
\Lambda^g(e_5) &= \frac{1}{2\sqrt{6}}(e_{1,4} + e_{2,6} - e_{3,7}), \quad \Lambda^g(e_6) = \frac{1}{2\sqrt{6}}(-e_{1,7} - e_{2,5} - e_{3,4}),\\
\Lambda^g(e_7) &= \frac{1}{2\sqrt{6}}(e_{1,6} - e_{2,4} + e_{3,5}).
\end{aligned}$$

□

Next, we calculate the Ambrose-Singer torsion and determine its type:

Proposition 3.3.6. *The sphere* $(S^7 = \mathrm{Spin}(7)/\,\mathrm{G}_2, g)$ *has Ambrose-Singer torsion of type* $\mathcal{T}_{\mathrm{skew}}$ *given by*

$$T^{\mathrm{AS}} = \frac{1}{\sqrt{6}}(-e_{1,2,3} - e_{1,4,5} - e_{1,6,7} + e_{2,4,7} + e_{2,5,6} + e_{3,4,6} - e_{3,5,7}).$$

3.3.3 $S^{15} = \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$

The isotropy representation in this case splits into two non-equivalent modules (one copy of the spin representation and one copy of the standard representation), hence there is a two-dimensional family of invariant metrics. Indeed, include $\mathfrak{spin}(7) \subseteq \mathfrak{spin}(9)$ as in Proposition A.4 and set $\mathfrak{m} := (\mathfrak{spin}(7))^{\perp}$, where orthogonality is taken with respect to the Killing form on $\mathfrak{spin}(9)$. Explicitly, in the notation of Proposition A.4,

$$\begin{aligned}
\mathfrak{spin}(9) &= \mathrm{span}_{\mathbb{R}}\{\iota(\nu_1), \dots, \iota(\nu_{14}), \iota(\nu'_{15}), \dots, \iota(\nu'_{21}), \nu'_{22}, \dots, \nu'_{36}\},\\
\mathfrak{spin}(7) &= \mathrm{span}_{\mathbb{R}}\{\nu'_{22}, \dots, \nu'_{36}\},
\end{aligned}$$

and

$$\mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{\widehat{e}_1, \dots, \widehat{e}_{15}\}, \quad \text{where } \widehat{e}_i := \nu'_{21+i}.$$

The two irreducible isotropy summands are given by

$$\mathfrak{m}_F := \mathrm{span}_{\mathbb{R}}\{\widehat{e}_1, \dots, \widehat{e}_7\}, \quad \mathfrak{m}_B := \mathrm{span}_{\mathbb{R}}\{\widehat{e}_8, \dots, \widehat{e}_{15}\},$$

corresponding to the tangent spaces of the fiber and base respectively of the octonionic Hopf fibration,

$$S^7 = \frac{\mathrm{Spin}(8)}{\mathrm{Spin}(7)} \hookrightarrow S^{15} = \frac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} \to S^8 = \frac{\mathrm{Spin}(9)}{\mathrm{Spin}(8)}. \tag{3.23}$$

The two-dimensional family of invariant metrics is parameterized by

$$g_{a,b} := aB_0|_{\mathfrak{m}_F \times \mathfrak{m}_F} + bB_0|_{\mathfrak{m}_B \times \mathfrak{m}_B}, \quad a, b > 0,$$

and a $g_{a,b}$-orthonormal basis of $\mathfrak{m}$ is given by $\{e_i\}_{i=1}^{15}$, where

$$e_i := \begin{cases} \frac{1}{\sqrt{a}}\widehat{e}_i & \text{if } i = 1, \dots, 7, \\ \frac{1}{\sqrt{b}}\widehat{e}_i & \text{if } i = 8, \dots, 15. \end{cases} \tag{3.24}$$

By manually checking the sectional curvatures, one finds that the round metric corresponds to the parameters $a = \frac{1}{2}$, $b = \frac{1}{8}$ and we recall (see e.g. [Zil82]) that the round metric is the only member of the family $g_{a,b}$ which can be isometric to one of the metrics for $G = \mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, or $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ considered in previous sections. In the following proposition we give a complete description of the invariant differential forms:

Proposition 3.3.7. *Up to taking Hodge duals, the invariant differential forms on* $(S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), g_{a,b})$ *are given in Table 3.3, where* $\omega \in \Lambda^3\mathfrak{m}$, $\Psi \in \Lambda^4\mathfrak{m}$ *and their exterior derivatives are defined by the formulas in Appendix B, and* $\mathrm{pr}_{i,j}$ *denotes the projection onto* $\Lambda^i\mathfrak{m}_F \otimes \Lambda^j\mathfrak{m}_B$.

k	$\dim \Lambda^k_{\mathrm{inv}}$	Basis for Λ^k_{inv}
0	1	1
1	0	0
2	0	0
3	1	ω
4	2	$d\omega, \Psi$
5	1	$d\Psi$
6	0	0
7	4	$\mathrm{pr}_{1,6}(\omega \wedge d\omega),\ \mathrm{pr}_{3,4}(\omega \wedge d\omega),\ *(\omega \wedge d\Psi),\ *(\Psi \wedge \Psi)$

Table 3.3: Invariant Differential Forms on $(S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), g_{a,b})$

In particular, this shows that the invariant differential forms on $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$ are generated by ω and Ψ and their derivatives, and in what follows we shall examine more closely the geometric and spinorial features of these forms.

Remark 3.3.8. The 4-form Ψ is purely horizontal, i.e. $\Psi \in \Lambda^4\mathfrak{m}_B$, and the horizontal component of $d\omega$ is a multiple of Ψ. One finds that Ψ is *not* invariant for the larger group $\mathrm{Spin}(8)$, hence it does not descend to an invariant 4-form on the base space $S^8 = \mathrm{Spin}(9)/\mathrm{Spin}(8)$ of the octonionic Hopf fibration (3.23).

In the following theorem we describe the space of invariant spinors:

Theorem 3.3.9. *Using the above $g_{a,b}$-orthonormal basis and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors on* $(S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), g_{a,b})$ *is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{\psi\},$$

where

$$\psi := \frac{1}{2\sqrt{2}}(-iy_{1,5} + y_{1,2,3} + y_{2,5,7} - y_{3,5,6} + iy_{1,2,4,7} - iy_{1,3,4,6} - iy_{4,5,6,7} + y_{2,3,4,6,7}),$$

and we use the notation $y_{i_1,\dots,i_p} := y_{i_1} \wedge \dots \wedge y_{i_p}$*. It satisfies the spinorial equation*

$$\nabla^g_{e_i}\psi = \begin{cases} \left(\frac{a-2b}{40b\sqrt{2a}}\right)\omega \cdot e_i \cdot \psi & \text{if } i = 1,\dots,7, \\ \left(\frac{a+2b}{16b\sqrt{2}(a-4b)}\right)\omega \cdot e_i \cdot \psi + \left(\frac{\sqrt{a}}{16(a-4b)}\right) d\omega \cdot e_i \cdot \psi & \text{if } i = 8,\dots,15. \end{cases} \tag{3.25}$$

Proof. We follow the same procedure as in Example 3.2.6. To differentiate ψ, we note that the Nomizu map of the Levi-Civita connection is given by

$$\Lambda^g(X)Y = \begin{cases} \frac{1}{2}[X,Y]_{\mathfrak{m}} & \text{if } X,Y \in \mathfrak{m}_F, \\ (1-\frac{a}{2b})[X,Y]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m}_F,\, Y \in \mathfrak{m}_B, \\ \frac{1}{2b}[X,Y]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m}_B,\, Y \in \mathfrak{m}_F, \\ \frac{1}{2}[X,Y]_{\mathfrak{m}} & \text{if } X,Y \in \mathfrak{m}_B. \end{cases}$$

□

The fact that the spinorial equation (3.25) depends only on the invariant 3-form ω suggests that there is an intrinsic relationship between ψ and ω. Indeed, the spinor ψ determines ω via the *squaring construction*:

$$\omega(X,Y,Z) = -2\langle (X \wedge Y \wedge Z)\cdot\psi, \psi\rangle \quad \text{for all } X,Y,Z \in TM,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Hermitian inner product on the spinor bundle. Conversely, (3.25) shows that ω determines ψ up to first order. In general, for each integer $k \geq 0$ the squaring construction determines an invariant k-form $\omega_{(k)}$ via

$$\omega_{(k)}(X_1,\dots,X_k) := \Re\langle (X_1 \wedge \dots \wedge X_k)\cdot\psi, \psi\rangle \quad \text{for all } X_1,\dots X_k \in TM, \tag{3.26}$$

and one can ask whether ψ is related to other invariant forms from Table 3.3 by this construction.

Proposition 3.3.10. *Up to taking Hodge duals, the differential forms obtained from the invariant spinor* ψ *via the squaring construction are given in Table 3.4.*

Proof. Note that invariance of ψ implies that each $\omega_{(n)}$ is invariant too, and thus Proposition 3.3.7 greatly limits the possibilities for $\omega_{(n)}$. One immediately sees that $\omega_{(1)} = \omega_{(2)} = \omega_{(6)} = 0$, and the case $k = 3$ is given above. Examining next the case of 4-forms, we consider the projection

$$\mathrm{pr}_{2,2}(d\omega) = d\omega + \frac{3\sqrt{a}}{b\sqrt{2}}\Psi$$

of the invariant 4-form $d\omega$ onto $\Lambda^2\mathfrak{m}_F \otimes \Lambda^2\mathfrak{m}_B$, and it is easy to see that $\{\Psi,\ \mathrm{pr}_{2,2}(d\omega)\}$ is a basis for the space of invariant 4-forms. Writing $\omega_{(4)} = \lambda_1\Psi + \lambda_2\mathrm{pr}_{2,2}(d\omega)$, a straightforward calculation in the spin representation shows that

$$\omega_{(4)}(e_8, e_9, e_{10}, e_{11}) = 0, \qquad \omega_{(4)}(e_1, e_2, e_8, e_{11}) = -\frac{1}{2},$$

hence $\lambda_1 = 0$ and $\lambda_2 = -\frac{\sqrt{a}}{2\sqrt{2}}$ (by comparing with the formulas in Appendix B) Similarly, for 5-forms, one calculates

$$\omega_{(5)}(e_1, e_8, e_{10}, e_{12}, e_{15}) = 0 \neq \sqrt{\frac{2}{a}} = d\Psi(e_1, e_8, e_{10}, e_{12}, e_{15}),$$

and we conclude from Table 3.3 that $\omega_{(5)} = 0$. In degree 7, we write

$$\omega_{(7)} = \lambda_1\mathrm{pr}_{1,6}(\omega \wedge d\omega) + \lambda_2\mathrm{pr}_{3,4}(\omega \wedge d\omega) + \lambda_3(*(\omega \wedge d\Psi)) + \lambda_4(*(\Psi \wedge \Psi))$$

as a linear combination of the 7-forms from Table 3.3. A straightforward calculation gives

$$\begin{aligned} \omega_{(7)}(e_1, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}) &= \frac{1}{2}, \qquad \omega_{(7)}(e_1, e_2, e_3, e_8, e_9, e_{10}, e_{11}) = 1 \\ \omega_{(7)}(e_3, e_4, e_5, e_6, e_7, e_{13}, e_{14}) &= \frac{1}{2}, \qquad \omega_{(7)}(e_1, e_2, e_3, e_4, e_5, e_6, e_7) = -1, \end{aligned}$$

and it then follows from the formulas in Appendix B that

$$\lambda_1 = -\frac{b\sqrt{2}}{18\sqrt{a}}, \quad \lambda_2 = \frac{\sqrt{a}}{6\sqrt{2}}, \quad \lambda_3 = -\frac{\sqrt{a}}{8\sqrt{2}}, \quad \lambda_4 = -\frac{1}{14}.$$

For higher degrees, we note that the squaring construction is compatible with the Hodge star in the following sense: For any multi-index $I = \{i_1, \ldots, i_k\}$, define a multi-vector $e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$, and recall that $*e_I = \mathrm{sign}(I \cup \widehat{I})e_{\widehat{I}}$, where $\widehat{I} = \{1, 2, \ldots, 15\} \setminus I$ (with union and complement taken as ordered sets). Since $e_I \wedge (e_{\widehat{I}}) = \mathrm{sign}(I \cup \widehat{I})e_1 \wedge e_2 \wedge \cdots \wedge e_{15}$ acts on the spin representation by $\mathrm{sign}(I \cup \widehat{I})\,\mathrm{Id}$, we then have

$$\omega_{(k)}(e_I) = \Re\langle e_I \cdot \psi, \psi\rangle = \mathrm{sign}(I \cup \widehat{I})\ \Re\langle e_I \cdot e_I \cdot e_{\widehat{I}} \cdot \psi, \psi\rangle = (-1)^{\frac{k(k+1)}{2}}\mathrm{sign}(I \cup \widehat{I})\ \Re\langle e_{\widehat{I}} \cdot \psi, \psi\rangle$$

$$= (-1)^{\frac{k(k+1)}{2}} \omega_{(15-k)}(*e_I),$$

and it follows that $\omega_{(15-k)} = (-1)^{\frac{k(k+1)}{2}} * \omega_{(k)}$. In particular, the forms $\omega_{(k)}$ with $k \geq 8$ are determined by those in Table 3.4. □

k	$\omega_{(n)}$
0	1
1	0
2	0
3	ω
4	$-\frac{\sqrt{a}}{2\sqrt{2}}\mathrm{pr}_{2,2}(d\omega)$
5	0
6	0
7	$-\frac{b\sqrt{2}}{18\sqrt{a}}\mathrm{pr}_{1,6}(\omega \wedge d\omega) + \frac{\sqrt{a}}{6\sqrt{2}}\mathrm{pr}_{3,4}(\omega \wedge d\omega) - \frac{\sqrt{a}}{8\sqrt{2}} * (\omega \wedge d\Psi) - \frac{1}{14} * (\Psi \wedge \Psi)$

Table 3.4: Forms on $(S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), g_{a,b})$ obtained from ψ via the squaring construction

Finally, we calculate the Ambrose-Singer torsion and determine its type:

Proposition 3.3.11. *For any $a, b > 0$ the sphere $(S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7), g_{a,b})$ has Ambrose-Singer torsion of type $\mathcal{T}_{\mathrm{skew}} \oplus \mathcal{T}_{\mathrm{CT}}$, given by*

$$T^{\mathrm{AS}}(e_i, -) = \frac{1}{2\sqrt{2a}} e_i \lrcorner \omega \quad (i = 1, \ldots, 7), \quad T^{\mathrm{AS}}(e_i, -) = \omega_i' \quad (i = 8, \ldots, 15),$$

where ω_i' are the $(0, 2)$-tensors determined by

$$\omega_i'(e_j, e_k) = \begin{cases} \frac{1}{2\sqrt{2a}}\omega(e_i, e_j, e_k) & \text{if } j < k, \\ \frac{\sqrt{a}}{2b\sqrt{2}}\omega(e_i, e_j, e_k) & \text{if } j > k, \\ 0 & \text{if } j = k. \end{cases}$$

The projection of T^{AS} onto $\mathcal{T}_{\mathrm{skew}}$ is

$$T^{\mathrm{AS}}_{\mathrm{skew}} = \frac{(a + 2b)}{6b\sqrt{2a}}\omega,$$

with $T^{\mathrm{AS}} = T^{\mathrm{AS}}_{\mathrm{skew}}$ if and only if $a = b$.

3.4 Generalized Killing Spinors on Round Spheres

In this section we revisit our findings for the round metric in each case, and compare with the known results of Moroianu and Semmelmann [MS14a, MS14b]. In Section 4 of [MS14b] it is shown that, for the round sphere S^n ($n \geq 3$), if there exists a generalized Killing spinor with exactly two eigenvalues, then those eigenvalues are equal to $\frac{1}{2}$, $-\frac{3}{2}$ (up to a change of orientation) and they occur with multiplicity $m_{\frac{1}{2}}$ and $m_{-\frac{3}{2}} = m_{\frac{1}{2}} + 1$ respectively. Furthermore, they show that if such a spinor exists, then $n = 3$ or 7, giving the multiplicities $(m_{\frac{1}{2}}, m_{-\frac{3}{2}}) = (1, 2)$ or $(3, 4)$. In what follows we examine each of the 3- and 7-dimensional cases from our classification and determine whether, when equipped with the round metric, they admit invariant generalized Killing spinors with exactly two eigenvalues.

(I). $G = \mathrm{SO}(4)$, $\mathrm{SO}(8)$, $\mathrm{U}(2)$, or $\mathrm{U}(4)$. By Theorems 3.1.1 and 3.1.3 there are no invariant spinors to consider in these cases.

(II). $G = \mathrm{SU}(2) \cong \mathrm{Sp}(1)$. By Theorem 3.1.7 and Corollary 3.1.14, the round metric $g_{a,b}|_{a=b=\frac{1}{2}}$ admits a pair of invariant Killing spinors for the constant $\frac{1}{2}$, but no invariant generalized Killing spinors.

(III). $G = \mathrm{Sp}(1)\,\mathrm{Sp}(1)$ or $\mathrm{Sp}(1)\,\mathrm{U}(1)$. By Theorems 3.2.13 and 3.2.26 there are no invariant spinors to consider in these cases.

(IV). $G = \mathrm{Sp}(2)$. Considering Example 3.2.6 with the round metric $g_{\alpha,\delta}|_{\alpha=\delta=1}$, one recovers the canonical spinor described in [AF10, AD20], consistent also with [MS14b, Thm. 4.10].

(V). $G = \mathrm{Sp}(2)\,\mathrm{Sp}(1)$. The round metric in this case is given by $g_{a,b}|_{a=\frac{5}{24},\ b=\frac{1}{24}}$. Substituting these values of a and b into the endomorphism A from Proposition 3.2.17 shows that the spinor ψ_0 from Theorem 3.2.16 is an invariant generalized Killing spinor with two distinct eigenvalues, and the associated endomorphism is consistent with [MS14b, Thm. 4.10]; this theorem also implies the existence of a 3-Sasakian structure with ψ_0 as the canonical spinor, however this structure cannot be invariant as a consequence of Corollary 3.2.25.

(VI). $G = \mathrm{Sp}(2)\,\mathrm{U}(1)$. The round metric in this case is given by $g_{a,b,c}|_{a=\frac{5}{24},\ b=\frac{1}{12},\ c=\frac{1}{24}}$, and by Corollary 3.2.30, there is a compatible invariant Sasakian structure. By Remark 3.2.35, the spinor ψ_0 from Theorem 3.2.32 is an invariant generalized Killing spinor with two distinct eigenvalues, and the associated endomorphism is consistent with [MS14b, Thm. 4.10]; as in the previous case this theorem also implies the existence of a non-invariant 3-Sasakian structure with ψ_0 as the canonical spinor (non-invariance in this case follows from the fact that the space of invariant vectors is 1-dimensional).

(VII). $G = \mathrm{Spin}(7)$. In this case, for any invariant metric, the 1-dimensional space of invariant spinors consists of Killing spinors.

4

Homogeneous (3-)Sasakian Structures from the Spinorial Viewpoint

This chapter contains work which has appeared in the preprint [Hof22].

4.1 The (3-)Sasakian Structures Induced by Killing Spinors

In this section we expain how to construct Sasakian and 3-Sasakian structures from Riemannian Killing spinors, generalizing the constructions of Friedrich and Kath in dimensions 5 and 7 [FK88, FK89, FK90]. By considering the rank two subbundles $E_i^- \subseteq \Sigma M$ defined in [FK90] (recalled in Theorem 2.3.8), we also show that all Sasakian and 3-Sasakian structures on connected, simply-connected Einstein-Sasakian or 3-Sasakian manifolds arise from this construction. In this chapter we shall only consider real Killing spinors for the Killing numbers $\lambda = \pm\frac{1}{2}$.

Generalizing to arbitrary dimension the 1-form and dual vector field considered in [FK88] (also those considered in [FK90, Section 5], [BFGK91, Chapter 4.4], and, by letting the two spinors in Definition 4.1.1 be multiples of each other, the 1-form considered in [FK89, Section 4]), we make the following definition:

Definition 4.1.1. Let (M, g) be a Riemannian spin manifold. Given a pair of spinors ψ_1, ψ_2, the *associated* 1-*form* η_{ψ_1,ψ_2} and its metric dual ξ_{ψ_1,ψ_2} (the *associated vector field*) are defined by

$$\eta_{\psi_1,\psi_2}(X) := \Re\langle\psi_1, X \cdot \psi_2\rangle, \qquad \xi_{\psi_1,\psi_2} := \eta^\sharp_{\psi_1,\psi_2}, \qquad \text{for all } X \in TM, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Hermitian metric on the spinor bundle and $\Re$ is the real part. We also define the *associated endomorphism field* φ_{ψ_1,ψ_2} by

$$\varphi_{\psi_1,\psi_2}(X) := -\frac{1}{2}(X \lrcorner d\eta_{\psi_1,\psi_2})^\sharp, \qquad \text{for all } X \in TM. \tag{4.2}$$

Remark 4.1.2. Observe that the operator $\Re$ in (4.1) is unnecessary when $\psi_1 \in \mathbb{R}i\psi_2$, since skew-symmetry of the Clifford multiplication with respect to the Hermitian product $\langle\ ,\ \rangle$ ensures that $\langle i\psi, X \cdot \psi\rangle$ is purely real. Thus, by setting $\psi_1 = i\psi_2$ in (4.1), our definition recovers the 1-form and vector field considered in [FK89, Section 4] and [BFGK91, Chapter 1.5]. We also note that it is possible to choose non-vanishing spinors ψ_1, ψ_2 such that the associated vector field ξ_{ψ_1,ψ_2} is identically zero. This is necessarily the case when $\psi_1 = \psi_2$, for example.

As we will frequently encounter the tensors (4.1), (4.2) and their derivatives, we summarize here some relevant well-known facts:

Lemma 4.1.3. *Let (M, g) be a Riemannian spin manifold with spinor bundle ΣM, and denote by $\langle\ ,\ \rangle$ the usual Hermitian scalar product on the fibers of ΣM.*

(i) Differentiation of the Hermitian product $\langle\ ,\ \rangle$ commutes with $\Re$, i.e.

$$X(\Re\langle\varphi, \psi\rangle) = \Re(X\langle\varphi, \psi\rangle)$$

for all $X \in TM$ and all spinors $\varphi, \psi \in \Gamma(\Sigma M)$.

(ii) For any spinor $\psi \in \Gamma(\Sigma M)$ and any vector fields $X, Y \in TM$ we have

$$\Re\langle X \cdot \psi, Y \cdot \psi\rangle = g(X, Y)\ ||\psi||^2.$$

(iii) If ψ is a Riemannian Killing spinor then the length function $||\psi||$ is constant on each connected component of M.

Proof. These can be found elsewhere throughout the literature; in the interest of comprehensiveness we recall the proofs here.

(i) One easily calculates:

$$\begin{aligned} X(\Re\langle\varphi, \psi\rangle) &= \frac{1}{2}X(\langle\varphi, \psi\rangle + \overline{\langle\varphi, \psi\rangle}) = \frac{1}{2}X(\langle\varphi, \psi\rangle + \langle\psi, \varphi\rangle) \\ &= \frac{1}{2}\left(\langle\nabla^g_X\varphi, \psi\rangle + \langle\varphi, \nabla^g_X\psi\rangle + \langle\nabla^g_X\psi, \varphi\rangle + \langle\psi, \nabla^g_X\varphi\rangle\right) \end{aligned}$$

$$= \frac{1}{2}\left(\langle\nabla^g_X\varphi,\psi\rangle + \overline{\langle\nabla^g_X\varphi,\psi\rangle} + \langle\varphi,\nabla^g_X\psi\rangle + \overline{\langle\varphi,\nabla^g_X\psi\rangle}\right)$$
$$= \Re\langle\nabla^g_X\varphi,\psi\rangle + \Re\langle\varphi,\nabla^g_X\psi\rangle = \Re(X\langle\varphi,\psi\rangle).$$

(ii) Using the skew-symmetry of the Hermitian product with respect to Clifford multiplication, we calculate:

$$\begin{aligned}\Re\langle X\cdot\psi, Y\cdot\psi\rangle &= -\Re\langle\psi, X\cdot Y\cdot\psi\rangle = \Re\langle\psi, Y\cdot X\cdot\psi\rangle + \Re\langle\psi, 2g(X,Y)\psi\rangle\\ &= \Re\overline{\langle Y\cdot X\cdot\psi,\psi\rangle} + 2g(X,Y)\ ||\psi||^2 = \Re\langle Y\cdot X\cdot\psi,\psi\rangle + 2g(X,Y)\ ||\psi||^2\\ &= -\Re\langle X\cdot\psi, Y\cdot\psi\rangle + 2g(X,Y)\ ||\psi||^2,\end{aligned}$$

and the result follows.

(iii) Let ψ be a Riemannian Killing spinor for the Killing number λ. Differentiating the square of the norm with respect to any $X \in TM$ and using skew-symmetry of the Hermitian product with respect to Clifford multiplication gives:

$$X||\psi||^2 = X\langle\psi,\psi\rangle = \langle\nabla^g_X\psi,\psi\rangle + \langle\psi,\nabla^g_X\psi\rangle = \lambda\langle X\cdot\psi,\psi\rangle + \lambda\langle\psi, X\cdot\psi\rangle = 0.$$

□

Since one of the defining conditions of a Sasakian structure involves the exterior derivative of the Reeb 1-form, we also need an identity relating $d\eta_{\psi_1,\psi_2}$ to the spinors ψ_1,ψ_2. Generalizing the identity calculated by Friedrich and Kath in the proof of [FK89, Thm. 2], we have:

Lemma 4.1.4. *If (M,g) is a Riemannian spin manifold carring a pair of Killing spinors ψ_1,ψ_2 (not necessarily linearly independent) for the same Killing number λ, then the exterior derivative of the 1-form η_{ψ_1,ψ_2} is given by*

$$d\eta_{\psi_1,\psi_2}(X,Y) = 2\lambda\Re\langle\psi_1,(Y\cdot X - X\cdot Y)\cdot\psi_2\rangle \quad \textit{for all } X,Y \in TM. \tag{4.3}$$

Remark 4.1.5. Recalling the squaring construction (3.26), we have $\eta_{\psi_1,\psi_2} = -\omega_{(1)}$ by definition, and it follows from the preceding lemma that $d\eta_{\psi_1,\psi_2} = 4\lambda\omega_{(2)}$.

We now describe the relationship between the length of the vector field ξ_{ψ_1,ψ_2} and the kernel of $d\eta_{\psi_1,\psi_2}$:

Lemma 4.1.6. *If ψ_1, ψ_2 are Killing spinors (not necessarily linearly independent) for the same Killing number λ, then the length function $\ell := ||\xi_{\psi_1,\psi_2}||$ is locally constant if and only if $\xi_{\psi_1,\psi_2} \lrcorner d\eta_{\psi_1,\psi_2} = 0$.*

Proof. One easily sees from the calculation on [BFGK91, p. 30] that ξ_{ψ_1,ψ_2} is a Killing vector field (actually they prove this only for the case $\psi_1 = i\psi_2$, but the same argument works in general by using Lemma 4.1.3(i)). Therefore we have

$$\frac{1}{2} d\eta_{\psi_1,\psi_2}(X,Y) = g(\nabla^g_X \xi_{\psi_1,\psi_2}, Y) \quad \text{for all } X,Y \in TM \tag{4.4}$$

(see e.g. [Bla76, p. 64], noting the slightly different conventions), and the result then follows from the calculation

$$X(\ell^2) = Xg(\xi_{\psi_1,\psi_2}, \xi_{\psi_1,\psi_2}) = 2g(\nabla_X \xi_{\psi_1,\psi_2}, \xi_{\psi_1,\psi_2}) = d\eta_{\psi_1,\psi_2}(X, \xi_{\psi_1,\psi_2}).$$

□

For a Sasakian structure (g, ξ, η, φ) one always has $\xi \lrcorner d\eta = 0$, so we see from the preceding lemma that it is necessary for ℓ to be locally constant (and non-zero) in order for the tensors $\frac{1}{\ell}\xi_{\psi_1,\psi_2}, \frac{1}{\ell}\eta_{\psi_1,\psi_2}, \frac{1}{\ell}\varphi_{\psi_1,\psi_2}$ to constitute a Sasakian structure. We now arrive at the first main result of this section, which shows that for Killing spinors this condition is also sufficient:

Theorem 4.1.7. *Suppose that (M, g) is a Riemannian spin manifold carrying a pair ψ_1, ψ_2 of Killing spinors (not necessarily linearly independent) for the same Killing number $\lambda \in \{\frac{1}{2}, \frac{-1}{2}\}$, and suppose furthermore that $\ell := ||\xi_{\psi_1,\psi_2}||$ is locally constant and non-zero. Then, the tensors*

$$\xi := \frac{1}{\ell}\xi_{\psi_1,\psi_2}, \quad \eta := \frac{1}{\ell}\eta_{\psi_1,\psi_2}, \quad \varphi := \frac{1}{\ell}\varphi_{\psi_1,\psi_2}$$

determine a Sasakian structure on (M, g).

Proof. By [BG99, Prop. 2.1.2], it suffices to show that ξ is a unit length Killing vector field and the $(1,1)$-tensor $\alpha(X) := -\nabla^g_X \xi$ satisfies

$$(\nabla^g_X \alpha)Y = g(X,Y)\xi - \eta(Y)X. \tag{4.5}$$

It is clear that $||\xi|| = 1$ by construction, and we have already seen in the proof of Lemma 4.1.6 that ξ_{ψ_1,ψ_2} is a Killing vector field (as ℓ is constant on each connected component of M we then have that $\xi = \frac{1}{\ell}\xi_{\psi_1,\psi_2}$ is a Killing vector field as well). The identity (4.5) has a very elegant proof in terms of conformal Killing forms, i.e. differential forms $\Theta \in \Lambda^k T^*M$ satisfying

$$\nabla^g_X \Theta = \frac{1}{k+1} X \lrcorner d\Theta - \frac{1}{n-k+1} X^* \wedge \delta\Theta, \tag{4.6}$$

where $\delta\colon \Lambda^k T^*M \to \Lambda^{k-1}T^*M$ denotes the codifferential and X^* is the 1-form dual to X via the Riemannian metric[1]. Indeed, from Remark 4.1.5 we have $d\eta = \frac{1}{\ell} d\eta_{\psi_1,\psi_2} = \frac{4\lambda}{\ell}\omega_{(2)}$, thus by [Sem03, Prop. 2.2] $d\eta$ is a conformal Killing form (using the fact that ψ_1, ψ_2 are Killing spinors, hence twistor spinors–see e.g. [BFGK91]). The existence of the Killing spinors ψ_1, ψ_2 for the Killing number $\lambda \in \{\frac{1}{2}, \frac{-1}{2}\}$ implies that (M^n, g) is Einstein with scalar curvature $4n(n-1)\lambda^2 = n(n-1)$ (see [Fri80, BFGK91]), and the result then follows by [Sem03, Prop. 3.4]. □

In fact, we shall prove in the next theorem that, *any* Einstein-Sasakian structure on a simply-connected manifold arises from this construction. To that end, we recall from Theorem 2.3.8 (using slightly different notation) the bundles

$$E_\pm := \{\psi \in \Sigma M \colon (\pm 2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot \psi = 0 \quad \forall X \in TM\}.$$

For our purposes here, it is important to recall that $\operatorname{rank}(E_-) \geq 1$ and that E_- has a basis consisting of Killing spinors for the constant $\frac{1}{2}$. Using these bundles, we prove:

Theorem 4.1.8. *If $(M^{2n-1}, g, \xi, \eta, \varphi)$ is a simply-connected Einstein-Sasakian manifold, then the Sasakian structure (ξ, η, φ) arises from the preceding construction.*

Proof. Let $\psi \in \Gamma(E_-)$ be a (non-trivial) Killing spinor for the Killing number $\lambda = \frac{1}{2}$, and assume without loss of generality that $||\psi|| = 1$ (if not, by Lemma 4.1.3(iii), we may rescale it to unit length using locally constant functions). Defining $\psi' := -\xi \cdot \psi$, we note that $\psi' \in \Gamma(E_-)$, since ξ anti-commutes in the Clifford algebra with the operators $(-2\varphi(X) + \xi \cdot X - X \cdot \xi)$ defining the bundle E_-. Furthermore, ψ' is a Killing spinor due to the calculation

$$\begin{aligned}\nabla^g_X \psi' &= -\nabla^g_X(\xi \cdot \psi) = -(\nabla^g_X \xi) \cdot \psi - \xi \cdot \nabla^g_X \psi = \varphi(X) \cdot \psi - \frac{1}{2}\xi \cdot X \cdot \psi \\ &= \frac{1}{2}(\xi \cdot X - X \cdot \xi) \cdot \psi - \frac{1}{2}\xi \cdot X \cdot \psi = -\frac{1}{2} X \cdot \xi \cdot \psi = \frac{1}{2} X \cdot \psi',\end{aligned}$$

where we have used the identity $\varphi = -\nabla^g \xi$ and the condition defining E_-. Using Lemma

[1] The author would like to thank Prof. Uwe Semmelmann for pointing out this argument.

4.1.3(ii), we calculate

$$\eta_{\psi,\psi'}(X) = \Re\langle\psi, X\cdot\psi'\rangle = -\Re\langle\psi, X\cdot\xi\cdot\psi\rangle = g(X,\xi)\ ||\psi||^2 = \eta(X)$$

for any $X \in TM$, hence $\eta_{\psi,\psi'} = \eta$ (and also $\xi_{\psi,\psi'} = \xi$). As $\ell = ||\xi_{\psi_1,\psi_2}|| = ||\xi|| = 1$, it follows from Theorem 4.1.7 that the pair ψ, ψ' induces a Sasakian structure on (M, g), and, since $\xi_{\psi_1,\psi_2} = \xi$, the two Sasakian structures must coincide (because $\varphi_{\psi,\psi'} = -\nabla^g\xi_{\psi,\psi'} = -\nabla^g\xi = \varphi$). The proof for the case $\psi \in \Gamma(E_+)$, $\lambda = -\frac{1}{2}$ is similar. □

As corollaries, we obtain analogous construction and uniqueness results for 3-Sasakian manifolds:

Theorem 4.1.9. *Suppose (M, g) is a Riemannian spin manifold carrying Killing spinors $\psi_1, \psi_2, \psi_3, \psi_4$ (not necessarily linearly independent) for the same Killing number $\lambda \in \{\frac{1}{2}, \frac{-1}{2}\}$. If ξ_{ψ_1,ψ_2} and ξ_{ψ_3,ψ_4} are orthogonal vector fields with locally constant non-zero length, then the two Sasakian structures induced by Theorem 4.1.7 determine a 3-Sasakian structure on (M, g).*

Proof. This follows from Theorem 4.1.7 and the fact that two Sasakian structures with orthogonal Reeb vector fields uniquely determine a 3-Sasakian structure (see e.g. [FK90, p. 556] for the construction of the structure tensors of the third Sasakian structure in terms of those of the other two). □

To prove a uniqueness result for 3-Sasakian manifolds analogous to Theorem 4.1.8, we consider the three bundles E_i^- from Theorem 2.3.8. Indeed, performing the argument from the proof of Theorem 4.1.8 for each of the three Sasakian structures individually yields:

Theorem 4.1.10. *If $(M^{4n-1}, g, \xi_i, \eta_i, \varphi_i)$ is a simply-connected 3-Sasakian manifold, then the 3-Sasakian structure arises from the preceding construction.*

Let us briefly compare our results with those of Friedrich and Kath in the 5- and 7-dimensional setting. These can be found in [FK88], [FK89, Section 4], [FK90, Sections 5, 6], and also appeared subsequently in the book chapters [BFGK91, Chapters 4.3, 4.4]. In dimension 5, their construction uses the fact that the spin representation of Spin(5) acts transitively on the unit sphere in the spinor module (which no longer holds in dimension > 9 [MS43]). Given a non-zero spinor ψ, this allows them to arrange a particular choice of frame in which a unique unit length solution $\xi \in TM^5$ to the equation $\xi\cdot\psi = i\psi$ is readily apparent. For a non-zero Killing spinor ψ, and under an appropriate normalization of the scalar curvature, they then show that this vector field ξ determines a Sasakian structure. Similarly, given an orthonormal pair of Killing spinors ψ_1, ψ_2 in dimension 7, Friedrich and Kath use the orthogonal decomposition $\Sigma M^7 = \mathbb{C}\psi_1 \oplus_\perp (TM^7 \cdot \psi_1)$ to find a unique unit length vector field ξ satisfying $\xi \cdot \psi_1 = \psi_2$, which goes on to become the Reeb vector field of a Sasakian structure. This decomposition of the spinor bundle occurs as a coincidence in dimension 7, and fails in higher dimensions since the dimension of the spinor module grows much faster than the dimension of the manifold. In both cases, Friedrich and Kath note that their vector field ξ is dual to the 1-form defined essentially by (4.1), but there is no mention of the fact that this 1-form can be taken as the starting point to perform such a construction in arbitrary dimension, as we have done here.

Aside from the preceding comments about dimension, we also note that our results are slightly different in spirit: Friedrich and Kath prove that a Killing spinor in dimension 5 (resp. two Killing spinors in dimension 7) defines a specific unit length vector field which is in fact the Reeb vector field of a Sasakian structure. On the other hand, our Theorem 4.1.7 requires the assumption that the vector field ξ_{ψ_1,ψ_2} induced by a pair of Killing spinors has locally constant positive length; this is then shown in Theorem 4.1.8 to be a reasonable assumption in the sense that simply-connected Einstein-Sasakian manifolds always carry spinors ψ_1, ψ_2 such that $||\xi_{\psi_1,\psi_2}|| = 1$, so no cases are lost by imposing this. The similarities and differences are much the same when comparing [FK90, Section 6] to our Theorem 4.1.9.

4.2 Invariance of Spinors and their Associated (3-)Sasakian Structures

Given the relationship described above it is natural to ask whether, on a homogeneous manifold, invariance of a (3-)Sasakian structure implies invariance of the associated unit spinors and vice versa. One already sees from [AHL23, Remark 4.43] that the homogeneous sphere $S^{4n-1} = \frac{\mathrm{Sp}(n)\,\mathrm{U}(1)}{\mathrm{Sp}(n-1)\,\mathrm{U}(1)}$ equipped with the round metric (which is invariant and carries an invariant Einstein-Sasakian structure) admits non-invariant Killing spinors $\psi \in \Gamma(E_1^-)$. However, it turns out that if one applies Theorem 4.1.7 to a pair of invariant Killing spinors then the resulting Sasakian structure must also be invariant, as we prove in this section. This suggests that an invariant spinor is a more fundamental geometric object than an invariant (3-)Sasakian structure, capturing more of the homogeneity data of the space. To begin, we have the following lemma:

Lemma 4.2.1. *For any $X \in \mathbb{R}^k$ and $\theta \in \Lambda^p\mathbb{R}^k$, the identity*

$$\theta \cdot X - X \cdot \theta = ((-1)^p + 1)\ X \lrcorner \theta + ((-1)^p - 1)\ X \wedge \theta \tag{4.7}$$

holds in the Clifford algebra.

Proof. The Clifford algebra identities

$$X \cdot \theta = (X \wedge \theta) - (X \lrcorner \theta), \qquad \theta \cdot X = (-1)^p [X \wedge \theta + X \lrcorner \theta]$$

appear as Equations (1.4) in Chapter 1.2 of [BFGK91], and the result follows by subtracting them. □

Considering a 2-form $T = \sum_{i<j} T_{ij} e_i \wedge e_j \in \Lambda^2 \mathbb{R}^k \cong \mathfrak{so}(k)$ and its spin lift $\widetilde{T} = \frac{1}{2} \sum_{i<j} T_{ij} e_i \cdot e_j$, we immediately obtain the corollary:

Corollary 4.2.2. *Let $T \in \mathfrak{so}(k)$ be a skew-symmetric linear transformation and $\widetilde{T} \in \mathfrak{spin}(k)$ its spin lift under the Lie algebra isomorphism $\mathfrak{so}(k) \cong \mathfrak{spin}(k)$. Then, for any $X \in \mathbb{R}^k$, the identity*

$$\widetilde{T} \cdot X - X \cdot \widetilde{T} = T(X) \tag{4.8}$$

holds in the Clifford algebra.

These formulas also easily generalize for the commutator of a 2-form with a form of arbitrary degree:

Lemma 4.2.3. *Let $T \in \mathfrak{so}(k)$ be a skew-symmetric linear transformation and $\widetilde{T} \in \mathfrak{spin}(k)$ its spin lift under the Lie algebra isomorphism $\mathfrak{so}(k) \cong \mathfrak{spin}(k)$. Then, for any $\theta \in \Lambda^\bullet \mathbb{R}^k$, the identity*

$$\widetilde{T} \cdot \theta - \theta \cdot \widetilde{T} = T(\theta), \tag{4.9}$$

holds in the Clifford algebra, where $T(\theta)$ refers to the standard action of $\mathfrak{so}(k)$ on $\Lambda^\bullet \mathbb{R}^k$.

Proof. It suffices to prove the result for $T = e_i \wedge e_j$ and $\theta = e_{l_1} \wedge \cdots \wedge e_{l_p}$. We calculate:

$$\begin{aligned}
\widetilde{T} \cdot \theta - \theta \cdot \widetilde{T} &= \frac{1}{2} \left(e_i \cdot e_j \cdot e_{l_1} \cdot \dots \cdot e_{l_p} - e_{l_1} \cdot \dots \cdot e_{l_p} \cdot e_i \cdot e_j \right) \\
&= \begin{cases} 0 & i, j \in \{l_1, \dots, l_p\} \text{ or } i, j \notin \{l_1, \dots, l_p\}, \\ e_i \cdot e_j \cdot \theta & \text{otherwise} \end{cases} \\
&= T(\theta).
\end{aligned}$$

□

Finally, we show that the Clifford product of an invariant vector or differential form with an invariant spinor is again invariant.

Lemma 4.2.4. *If $(M = G/H, g)$ is a homogeneous spin manifold carrying an invariant spinor ψ, then $\theta \cdot \psi \in \Sigma$ is invariant for any invariant form $\theta \in \Lambda^k_{\mathrm{inv}}\mathfrak{m}$, $k \geq 0$.*

Proof. For any isotropy operator $h \in \mathfrak{h} \subseteq \mathfrak{so}(\mathfrak{m})$, it follows from (4.9) that

$$\widetilde{h} \cdot \theta \cdot \psi - \theta \cdot \widetilde{h} \cdot \psi = h(\theta) \cdot \psi.$$

Invariance of ψ and θ gives $\widetilde{h} \cdot \psi = 0 = h(\theta)$, hence $\widetilde{h} \cdot \theta \cdot \psi = 0$ as desired. □

With these lemmas, it is easy to prove that invariant Killing spinors induce invariant (3-)Sasakian structures via the construction from Chapter 4.1:

Theorem 4.2.5. *If $(M = G/H, g)$ is a Riemannian homogeneous spin manifold carrying a pair ψ_1, ψ_2 of invariant spinors, then the associated tensors ξ_{ψ_1,ψ_2}, η_{ψ_1,ψ_2}, and φ_{ψ_1,ψ_2} are also invariant. In particular, if ψ_1, ψ_2 are invariant Killing spinors for the same Killing number $\lambda \in \{\frac{1}{2}, \frac{-1}{2}\}$, and ξ_{ψ_1,ψ_2} has locally constant positive length $\ell > 0$, then the induced Sasakian structure $(\frac{1}{\ell}\xi_{\psi_1,\psi_2}, \frac{1}{\ell}\eta_{\psi_1,\psi_2}, \frac{1}{\ell}\varphi_{\psi_1,\psi_2})$ is invariant.*

Proof. We show that the tensors $\xi_{\psi_1,\psi_2}, \eta_{\psi_1,\psi_2}, \varphi_{\psi_1,\psi_2}$ are invariant. Using (4.8) and invariance of g, $\langle\ ,\ \rangle$, ψ_1, and ψ_2, we calculate:

$$\begin{aligned} g([h, \xi_{\psi_1,\psi_2}], X) &= -g(\xi_{\psi_1,\psi_2}, [h, X]) = -\Re\langle \psi_1, [h, X] \cdot \psi_2 \rangle \\ &= -\Re\langle \psi_1, (\widetilde{\mathrm{ad}(h)|_{\mathfrak{m}}} \cdot X - X \cdot \widetilde{\mathrm{ad}(h)|_{\mathfrak{m}}}) \cdot \psi_2 \rangle = -\Re\langle \psi_1, \widetilde{\mathrm{ad}(h)|_{\mathfrak{m}}} \cdot X \cdot \psi_2 \rangle \\ &= \Re\langle \widetilde{\mathrm{ad}(h)|_{\mathfrak{m}}} \cdot \psi_1, X \cdot \psi_2 \rangle = 0 \end{aligned}$$

for all $h \in \mathfrak{h}$, $X \in \mathfrak{m}$, hence ξ_{ψ_1,ψ_2} and $\eta_{\psi_1,\psi_2} = \xi^\flat_{\psi_1,\psi_2}$ are invariant. Invariance of φ_{ψ_1,ψ_2} then follows from (4.2), completing the proof. □

By the same argument, one also obtains the analogous result in the 3-Sasakian setting:

Theorem 4.2.6. *If $(M = G/H, g)$ is a Riemannian homogeneous spin manifold carrying invariant Killing spinors $\psi_1, \psi_2, \psi_3, \psi_4$ for the same Killing number $\lambda \in \{\frac{1}{2}, \frac{-1}{2}\}$, and such that ξ_{ψ_1,ψ_2} and ξ_{ψ_3,ψ_4} are orthogonal and have locally constant positive length, then the induced 3-Sasakian structure is invariant.*

4.3 Invariant Differential Forms and Spinors

Expanding upon the work [DOP20], in this section we describe the invariant φ_1-(anti-)holomorphic differential forms on homogeneous 3-Sasakian spaces. We also describe the invariant spinors carried by these spaces, and the relationship between the forms and spinors. We would like to emphasize that this approach exploits the exterior form viewpoint of the spin representation, which greatly simplifies calculations and allows one to easily prove results for spaces of arbitrary dimension.

Remark 4.3.1. Before discussing the invariant forms and spinors, we comment briefly about the homogeneous 3-Sasakian space $\mathbb{RP}^{4n-1} \cong \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)\times\mathbb{Z}_2}$. The isotropy group $\mathrm{Sp}(n-1)\times\mathbb{Z}_2$ is not connected, leading to non-uniqueness of lifts of the isotropy representation (see Figure 2.1), and, consequently, non-uniqueness of homogeneous spin structures. Moreover, non-connectedness of the isotropy group precludes us from the usual strategy of finding invariants at the level of Lie algebras (since the exponential map in this case is not surjective). To avoid the intricacies of this special situation, we exclude $\mathbb{RP}^{4n-1}$ from consideration in this chapter by requiring that our homogeneous 3-Sasakian spaces be simply-connected; it is the only non simply-connected space in Theorem 2.4.1.

In order to prove the first major result of this section, we will make use of the First Fundamental Theorems of Invariant Theory for the classical complex simple Lie groups, which can be found e.g. in [FH91, Sch08]; we will use the formulations presented in [Sch08] as these are more suited to our purposes. We will also need the description of the exterior powers of the standard representation of $\mathrm{SO}(n,\mathbb{C})$ as highest weight modules (see e.g. [GW09, Chapter 5.5.2]). We summarize these results in the following three theorems:

Theorem 4.3.2. *(Based on the First Fundamental Theorems in [Sch08, Section 5]). Let* $\mathrm{SO}(n,\mathbb{C})$ *act on* $\mathbb{C}^n$ *by its standard representation and, if* $n = 2l$ *is even, let* $\mathrm{Sp}(2l,\mathbb{C})$ *also act by its standard representation. Denote by* $e_1,\dots,e_n$ *(resp.* $e_1^*,\dots,e_n^*$*) the standard basis for* $\mathbb{C}^n$ *(resp. the images of the standard basis vectors under the isomorphism* $\mathbb{C}^n \simeq (\mathbb{C}^n)^*$ *given by the non-degenerate bilinear form defining the group), and let*

$$\mathcal{T} := T(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$$

denote the algebra of tensors on $\mathbb{C}^n \oplus (\mathbb{C}^n)^*$*, with the natural algebra multiplication given by concatenation of tensors. The subalgebra of invariant tensors for the two groups are described* *up to mutations (i.e. permutations of the tensor factors) as follows:*

(i) FFT for $\mathrm{SO}(n,\mathbb{C})$: *The space $\mathcal{T}^{\mathrm{SO}(n,\mathbb{C})}$ of invariant tensors is the $\mathbb{C}$-span of all mutations of tensor products of flips of the tensors*

$$\det := \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(n)}, \qquad I := \sum_{i=1}^{n} e_i\otimes e_i^*,$$

where a flip means applying the isomorphism $\mathbb{C}^n \simeq (\mathbb{C}^n)^*$, $e_i \mapsto e_i^*$ or its inverse to one of the tensor factors.

(ii) FFT for $\mathrm{Sp}(2l,\mathbb{C})$: *The space $\mathcal{T}^{\mathrm{Sp}(2l,\mathbb{C})}$ of invariant tensors is the $\mathbb{C}$-span of all mutations of tensor products of p, p^*, I, where*

$$p := \sum_{i=1}^{l} (e_i\otimes e_{l+i} - e_{l+i}\otimes e_i), \qquad I := \sum_{i=1}^{n} e_i\otimes e_i^*.$$

Remark 4.3.3. Note that our $\mathcal{T}$ is defined slightly differently than the tensor algebra considered in [Sch08]; we don't require all the covariant and contravariant factors to be collected, i.e. we don't require elements of $\mathcal{T}$ to lie in $T(\mathbb{C}^n)\otimes T((\mathbb{C}^n)^*)$. To compensate for this, our notion of mutations includes *all* permutations of the tensor factors, not just those that permute covariant factors and contravariant factors amongst themselves (as was the definition in [Sch08]). It is easy to see that our mutations intertwine the group action, and that invariant elements of $\mathcal{T}$ are generated by invariant elements of $T(\mathbb{C}^n)\otimes T((\mathbb{C}^n)^*)$ by taking linear combinations of mutations.

Theorem 4.3.4. *(Based on [GW09, Thm. 5.5.11]). Denote by $\omega_1,\dots,\omega_{n-1}$ the fundamental weights of $\mathrm{SL}(n,\mathbb{C})$, and $\Lambda^r\mathbb{C}^n$ the r^{th} exterior power of the standard representation. The representation $\Lambda^r\mathbb{C}^n$ is irreducible for all $r=1,\dots,n$, with highest weight ω_r for $r=1,\dots,n-1$.*

Theorem 4.3.5. *(Based on [GW09, Thm. 5.5.13]). For $n=2l$ or $2l+1$, let $\omega_1,\dots\omega_l$ denote the fundamental weights of $\mathrm{SO}(n,\mathbb{C})$, and $\Lambda^r\mathbb{C}^n$ the r^{th} exterior power of the standard representation.*

(i) *For $n=2l+1\geq 3$: The representation $\Lambda^r\mathbb{C}^n$ is irreducible for all $r=1,\dots,n$, with highest weight ω_r for $r=1,\dots,l-1$ and $2\omega_l$ for $r=l$.*

(ii) *For $n=2l\geq 4$: The representation $\Lambda^r\mathbb{C}^n$ is irreducible for $r=1,\dots,l-1$, with highest weight ω_r for $r=1,\dots,l-2$ and $\omega_{l-1}+\omega_l$ for $r=l-1$. The representation $\Lambda^l\mathbb{C}^n$ splits as the direct sum of two irreducible representations with highest weights $2\omega_{l-1}$ and $2\omega_l$.*

We begin with an easy result regarding the algebra of $\mathrm{Sp}(n)$-invariant horizontal complex tensors for the classical case of the 3-Sasakian round sphere:

Proposition 4.3.6. *The algebra $T_{\mathrm{inv}}(\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}})$ of invariant horizontal complex tensors on the 3-Sasakian sphere $(S^{4n-1} = \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}, g_{\mathrm{round}})$ is generated, up to mutations, by its degree 0 and 2 elements.*

Proof. The isotropy algebra is $\mathfrak{h} = \mathfrak{sp}(n-1)$, and complexifying the isotropy representation gives $\mathfrak{m}^{\mathbb{C}} \simeq 3\mathbb{C} \oplus 2\mathbb{C}^{2n-2}$, where $\mathbb{C}^{2n-2}$ denotes the standard representation of $\mathfrak{h}^{\mathbb{C}} = \mathfrak{sp}(2n-2, \mathbb{C})$. Using the fact that $\mathbb{C}^{2n-2}$ is self-dual as an $\mathfrak{h}^{\mathbb{C}}$-representation, the space of horizontal complex tensors is

$$T(\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}}) \simeq T(2\mathbb{C}^{2n-2}) \simeq T(\mathbb{C}^{2n-2} \oplus (\mathbb{C}^{2n-2})^*) \simeq \mathcal{T}, \tag{4.10}$$

and the result follows from Theorem 4.3.2(ii).

□

As a consequence, we are able to deduce a description of the $\mathrm{Sp}(n)$-invariant forms on the round sphere:

Corollary 4.3.7. *For any $k \geq 0$, the space $\Lambda^k_{\mathrm{inv}}\mathfrak{m}$ of invariant k-forms on the 3-Sasakian sphere $(S^{4n-1} = \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}, g_{\mathrm{round}})$ is spanned by wedge products of invariant 1- and 2- forms. Explicitly, the invariant algebra $\Lambda^{\bullet}_{\mathrm{inv}}\mathfrak{m}$ is spanned by elements of the form*

$$\tau_{\epsilon_1,\epsilon_2,\epsilon_3,a_1,a_2,a_3} := \eta_1^{\epsilon_1} \wedge \eta_2^{\epsilon_2} \wedge \eta_3^{\epsilon_3} \wedge (\Phi_1|_{\mathcal{H}})^{a_1} \wedge (\Phi_2|_{\mathcal{H}})^{a_2} \wedge (\Phi_3|_{\mathcal{H}})^{a_3},$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0,1\}$, $a_1, a_2, a_3 \in \mathbb{Z}_{\geq 0}$.

Proof. Any degree 2 horizontal tensor decomposes uniquely into symmetric and skew-symmetric parts according to

$$\mathfrak{m}_{\mathcal{H}} \otimes \mathfrak{m}_{\mathcal{H}} \simeq S^2(\mathfrak{m}_{\mathcal{H}}) \oplus \Lambda^2(\mathfrak{m}_{\mathcal{H}}),$$

and this decomposition holds as H-representations. By Proposition 4.3.6, we then have that $T_{\mathrm{inv}}(\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}})$ is generated, up to mutations, by its degree 0 elements together with its symmetric and skew-symmetric degree 2 elements:

$$T_{\mathrm{inv}}(\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}})/\mathrm{mut.} \simeq \bigoplus_{k\geq 0} (\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}} \otimes \mathfrak{m}_{\mathcal{H}}^{\mathbb{C}})^{\otimes k}_{\mathrm{inv}} \simeq \bigoplus_{k\geq 0} \left(S^2_{\mathrm{inv}}(\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}}) \oplus \Lambda^2_{\mathrm{inv}}(\mathfrak{m}_{\mathcal{H}}^{\mathbb{C}})\right)^{\otimes k}.$$

But any tensor containing a factor from $S^2_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$, or any mutation of such a tensor, is symmetric in at least two positions and therefore lies in the kernel of the projection onto the skew-symmetric tensors. In particular, this shows that

$$\Lambda^{\bullet}_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) \simeq \bigoplus_{k\geq 0} \Lambda^k \left(\Lambda^2_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})\right)$$

(see Case 3(i) in the proof of Proposition 4.3.8 below for more details about the required properties of the skew-symmetrization map), so the algebra of complex horizontal differential forms is generated by its (degree 0 and) degree 2 elements. For $U = \mathbb{C}^{2n-2}$ the standard representation of $\mathfrak{h}^{\mathbb{C}} = \mathfrak{sp}(2n-2, \mathbb{C})$, it is known that $\dim_{\mathbb{C}} \Lambda^2_{\text{inv}} U = 1$ (see [DOP20, p. 834]), hence

$$\dim_{\mathbb{C}} \Lambda^2_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) = \dim_{\mathbb{C}} \Lambda^2_{\text{inv}}(U \oplus U) = 2 \dim_{\mathbb{C}} \Lambda^2_{\text{inv}} U + \dim_{\mathbb{C}}(U \otimes U)^{H^{\mathbb{C}}} = 2 \cdot 1 + 1 = 3$$

(since U is a self-dual $H^{\mathbb{C}}$-representation), and it follows that $\Lambda^2_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$ is spanned by $\Phi_1|_{\mathcal{H}}, \Phi_2|_{\mathcal{H}}$, and $\Phi_3|_{\mathcal{H}}$ (viewed as complex forms). In particular, together with the constant function 1, these generate the complex algebra $\Lambda^{\bullet}_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$ and also the real subalgebra $\Lambda^{\bullet}_{\text{inv}}(\mathfrak{m}_{\mathcal{H}})$. The result then follows by noting that the isotropy representation acts trivially in the vertical directions. □

In fact, we expect that Corollary 4.3.7 should hold for all homogeneous 3-Sasakian spaces (with an additional generator $\Phi_0 \in \Lambda^2_{\text{inv}}(\mathfrak{m}_{\mathcal{H}})$ for the case $G = \mathrm{SU}(n+1)$), however the arguments for the remaining cases become much harder (due to reducibility of $\mathfrak{m}_{\mathcal{H}}$ in the $\mathrm{SU}(n+1)$ case, the appearance of extra tensors in certain dimensions for the $G = \mathrm{SO}(n+3)$ case, and the lack of invariant theoretic tools for the exceptional cases). It seems likely that other methods would be needed to prove such a result in general. For this reason we now prove a somewhat weaker result, but one which can be shown for all the cases and which will nonetheless be sufficient for the purpose of finding the spaces of invariant spinors:

Proposition 4.3.8. *If $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is a simply-connected homogeneous 3-Sasakian space, then the algebras $\Lambda^{\bullet,0}_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$, $\Lambda^{0,\bullet}_{\text{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$ of invariant horizontal φ_1-(anti-)holomorphic forms are generated by their degree 0 and 2 elements.*

Proof. Employing the basis for $\mathfrak{sp}(1) = \mathrm{span}_{\mathbb{R}}\{\xi_1, \xi_2, \xi_3\}$ given in [DOP20, Eqn. (17)], the almost complex structure $\varphi_1|_{\mathcal{H}} = \mathrm{ad}(\xi_1)$ acts on $(\mathfrak{m}_{\mathcal{H}})^{\mathbb{C}} = (\mathfrak{g}_1)^{\mathbb{C}}$ via $i\,\mathrm{Id}$ on $(1,0) \otimes U$ and $-i\,\mathrm{Id}$ on $(0,1) \otimes U$, hence the φ_1-holomorphic (resp. φ_1-anti-holomorphic) horizontal cotangent bundles are given by $\Lambda^{1,0}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) = (1,0)\otimes U \simeq U$ (resp. $\Lambda^{0,1}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) = (0,1)\otimes U \simeq U$). Therefore we have $\Lambda^k U \simeq \Lambda^{k,0}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) \simeq \Lambda^{0,k}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$ for all $k \geq 0$, so it suffices to consider the invariant exterior forms on U. We use this approach to treat the cases for G individually.

Case 1: $G = \mathrm{Sp}(n)$. Here the complexified isotropy group is $\mathrm{Sp}(n-1)^{\mathbb{C}} \cong \mathrm{Sp}(2n-2,\mathbb{C})$ and we have $U = \mathbb{C}^{2n-2}$ (the standard representation). It then follows from Theorem 4.3.2(ii) that $\Lambda^k_{\mathrm{inv}} U \cong \Lambda^k_{\mathrm{inv}} \mathbb{C}^{2n-2}$ is generated by alternating tensor powers (i.e. wedge powers) of the 2-form $p \in \Lambda^2\mathbb{C}^{2n-2}$ stabilized by $\mathrm{Sp}(2n-2,\mathbb{C})$.

Case 2: $G = \mathrm{SU}(n+1)$. The isotropy group in this case is $H = S(\mathrm{U}(n-1)\times \mathrm{U}(1))$, and we consider separately the cases $n > 2$ and $n = 2$. When $n > 2$ we have $U = \mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^*$, with the action of $\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{sl}(n-1,\mathbb{C}) \oplus \mathfrak{u}(1)^{\mathbb{C}}$ on U via the standard (resp. dual of the standard) representation of $\mathfrak{sl}(n,\mathbb{C})$ on $\mathbb{C}^{n-1}$ (resp. $(\mathbb{C}^{n-1})^*$) and the action of $1 \in \mathfrak{u}(1)^{\mathbb{C}} \cong \mathbb{C}$ given by

$$1\cdot v = \left(1 + \frac{n-1}{2}\right) v, \qquad 1\cdot v' = -\left(1+\frac{n-1}{2}\right) v' \tag{4.11}$$

for all $v \in \mathbb{C}^{n-1}$, $v' \in (\mathbb{C}^{n-1})^*$ (see [DOP20, Section 4.5]). We then have

$$\Lambda^k U \simeq \Lambda^k(\mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^*) \simeq \bigoplus_{p+q=k} (\Lambda^p\mathbb{C}^{n-1}) \otimes (\Lambda^q(\mathbb{C}^{n-1})^*), \tag{4.12}$$

and examining the action of $\mathfrak{u}(1)^{\mathbb{C}}$ in (4.11) shows that an element in one of the summands on the right hand side of (4.12) is $\mathfrak{u}(1)^{\mathbb{C}}$-invariant if and only if it has the same number of $\mathbb{C}^{n-1}$ and $(\mathbb{C}^{n-1})^*$ factors. By Theorem 4.3.4, the $\mathrm{SL}(n-1,\mathbb{C})$-modules $\Lambda^p\mathbb{C}^{n-1}$ and $\Lambda^q\mathbb{C}^{n-1}$ are irreducible and non-isomorphic unless $p = q$, and it then follows from (4.12) that

$$\dim_{\mathbb{C}} \Lambda^k_{\mathrm{inv}}(U) = \begin{cases} 1 & \text{if } k \text{ is even and } k \leq \dim_{\mathbb{C}} U, \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

In particular one checks in a basis that $\omega_{1,0} := (\Phi_2|_{\mathcal{H}} - i\Phi_3|_{\mathcal{H}})$ (resp. $\omega_{0,1} := (\Phi_2|_{\mathcal{H}} + i\Phi_3|_{\mathcal{H}})$) is an element of $\Lambda^{2,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}})$ (resp. $\Lambda^{0,2}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}})$), and that the top power of $\omega_{1,0}$ (resp. $\omega_{0,1}$) is a φ_1-holomorphic (resp. φ_1-anti-holomorphic) volume form. It follows that lower powers are non-zero, hence they span the 1-dimensional spaces of invariant φ_1-(anti-)holomorphic forms in the relevant dimensions. The argument for $n = 2$ is similar, except one only needs to consider the action of $\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{u}(1)^{\mathbb{C}}$ via (4.11).

Case 3: $G = \mathrm{SO}(n+3)$. The isotropy group in this case is $H = \mathrm{SO}(n-1)\times \mathrm{Sp}(1)$, and it is shown in [DOP20, Section 4.3] that $U = \mathbb{C}^{n-1}\times\mathbb{C}^{n-1}$ with the following action of $H^{\mathbb{C}}$: identify each element $(a,b) \in U$ with $[a|b] \in \mathrm{Mat}_{(n-1)\times 2}(\mathbb{C})$ (i.e. the matrix with first column $a \in \mathbb{C}^{n-1}$ and second column $b \in \mathbb{C}^{n-1}$), and let $\mathrm{SO}(n-1,\mathbb{C})$ act by left multiplication on the columns of $[a|b]$ and $\mathrm{Sp}(1)^{\mathbb{C}} \cong \mathrm{Sp}(2,\mathbb{C})$ by right multiplication on the rows. This gives decompositions $U \simeq \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ as $\mathrm{SO}(n-1,\mathbb{C})$-modules and $U \simeq (\mathbb{C}^2)^{\oplus(n-1)}$ as $\mathrm{Sp}(2,\mathbb{C})$-modules (direct sums of the standard representation in both cases). Letting $e_1, \dots e_{n-1}$ (resp. $e_1^*, \dots, e_{n-1}^*$) be the standard basis vectors for the first (resp. second) copy of $\mathbb{C}^{n-1}$ in the $\mathrm{SO}(n-1,\mathbb{C})$-decomposition, the i^{th} copy of $\mathbb{C}^2$ in the $\mathrm{Sp}(2,\mathbb{C})$-decomposition has basis e_i, e_i^*. In particular, on the i^{th} copy of $\mathbb{C}^2$ (which we shall henceforth denote by $(\mathbb{C}^2)_i$) the symplectic form stabilized by $\mathrm{Sp}(2,\mathbb{C})$ is given by

$$p_i := e_i \otimes e_i^* - e_i^* \otimes e_i. \tag{4.14}$$

For each pair $i,j \in \{1,\dots,n-1\}$, the copies $(\mathbb{C}^2)_i$ and $(\mathbb{C}^2)_j$ are isomorphic as $\mathrm{Sp}(2,\mathbb{C})$-modules via $f_{i,j}\colon e_i \mapsto e_j, e_i^* \mapsto e_j^*$, and the natural extension of this map to tensors satisfies $f_{i,j}(p_i) = p_j$. By Theorem 4.3.2(ii), the $\mathrm{Sp}(2,\mathbb{C})$-invariant tensors in $T(\mathbb{C}^2)$ are spanned by mutations of tensor powers of the symplectic form defining the group. The tensor algebra of $U = \bigoplus_{i=1}^{n-1} (\mathbb{C}^2)_i$ is the direct sum of all possible tensor products of the spaces $(\mathbb{C}^2)_i$, $i = 1, \dots n-1$, hence the $\mathrm{Sp}(2,\mathbb{C})$-invariant tensors are spanned by mutations of tensor products of the symplectic forms p_i, $i = 1, \dots, n-1$. We now consider separately the two subcases $\Lambda^k_{\text{inv}} U$ with $k = n-1$ and $k \neq n-1$. It suffices to show that (4.13) holds in both subcases, as the result will then follow by the same argument as in Case 2.

(i) $k = n-1$: The symplectic forms p_i have degree 2, hence $(U^{\otimes(n-1)})_{\text{inv}}$ can only be non-trivial if $n-1$ is even. We assume for the rest of the subcase that $n-1 = 2l$. Multilinearly expanding a tensor product of the form $p_{i_1} \otimes \cdots \otimes p_{i_l}$, or any mutation thereof, one sees that the vectors $e_{i_s}, e_{i_s}^*$, $s = 1, \dots l$ appear in each term. Similarly, each term of the $\mathrm{SO}(n-1,\mathbb{C})$-invariant tensor $I = \sum_{i=1}^{n-1} e_i \otimes e_i^*$ from Theorem 4.3.2(i) contains a pair of vectors of the form e_i, e_i^*. The flips of I are precisely

$$I = \sum_{i=1}^{n-1} e_i \otimes e_i^*, \quad I_1 := \sum_{i=1}^{n-1} e_i^* \otimes e_i, \quad I_2 := \sum_{i=1}^{n-1} e_i \otimes e_i, \quad I_3 := \sum_{i=1}^{n-1} e_i^* \otimes e_i^*, \tag{4.15}$$

hence any mutation of an l-fold tensor product of flips of I has the property that, when fully expanded, each term contains a pair of vectors of the form e_i, e_i^* or e_i, e_i or e_i^*, e_i^* for some $i = 1, \dots, n-1$. On the other hand, the $\mathrm{SO}(n-1,\mathbb{C})$-invariant tensor $\det = \sum_{\sigma \in S_{n-1}} \mathrm{sign}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n-1)}$ from Theorem 4.3.2(i) (and any mutation and/or flip thereof) has the property that, for each $i = 1, \dots, n-1$, each term contains either exactly one copy of e_i or exactly one copy of e_i^*. Said differently, linear combinations of mutations of l-fold tensor products of p_i, I, I_1, I_2, I_3 have repeated indices in each term, whereas linear combinations mutations of flips of det do not have any terms with repeated indices. Comparing Theorem 4.3.2(i) with the above observation that the $\mathrm{Sp}(2,\mathbb{C})$-invariant tensors are spanned by mutations of tensor products of the p_i, we then have that $(U^{\otimes(n-1)})_{\mathrm{inv}}$ is contained in the span of all mutations of (l-fold) tensor products of flips of I. Consider the skew-symmetrization map $\mathrm{Alt}\colon T(U) \to \Lambda^\bullet U$ given by

$$u_{i_1} \otimes \cdots \otimes u_{i_k} \mapsto \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) u_{\sigma(i_1)} \otimes \cdots \otimes u_{\sigma(i_k)}.$$

Indeed, any invariant exterior form, viewed as a skew-symmetric tensor, is mapped to a positive multiple of itself under Alt, so it suffices to consider the image of $(U^{\otimes 2l})_{\mathrm{inv}}$ under Alt. Noting that the skew-symmetrizations of a tensor and any mutation of that tensor agree up to sign, it follows that $\Lambda^{2l}_{\mathrm{inv}} U$ is contained in the span of all images under Alt of l-fold tensor products of flips of I. Furthermore, since for any tensors α, β the exterior form $\mathrm{Alt}(\alpha \otimes \beta)$ agrees up to positive scaling with $\mathrm{Alt}(\alpha) \wedge \mathrm{Alt}(\beta)$, we have

$$\Lambda^{2l}_{\mathrm{inv}} U \subseteq \Lambda^l \mathcal{S}, \qquad \text{where } \mathcal{S} := \mathrm{span}_{\mathbb{C}}\{\mathrm{Alt}(I), \mathrm{Alt}(I_1), \mathrm{Alt}(I_2), \mathrm{Alt}(I_3)\}.$$

One easily checks that $\mathrm{Alt}(I) = I - I_1 = -\mathrm{Alt}(I_1)$, $\mathrm{Alt}(I_2) = \mathrm{Alt}(I_3) = 0$, and we note furthermore that the tensor

$$\mathcal{I} := \mathrm{Alt}(I) = \sum_{i=1}^{n-1} (e_i \otimes e_i^* - e_i^* \otimes e_i) = \sum_{i=1}^{n-1} p_i$$

is $\mathrm{Sp}(2,\mathbb{C})$-invariant. Thus $\mathcal{S} = \mathbb{C}\mathcal{I}$ is the trivial $H^{\mathbb{C}}$-representation, and it follows that $\Lambda^{2l}_{\mathrm{inv}} U = \Lambda^l \mathcal{S} = \mathbb{C}(\mathcal{I} \wedge \cdots \wedge \mathcal{I})$ (l times). In particular we have shown that $\dim_{\mathbb{C}} \Lambda^{n-1}_{\mathrm{inv}} U$ is equal to 1 if $n-1$ is even and 0 if $n-1$ is odd.

(ii) $k \neq n-1$: Similarly to (4.12), taking the k^{th} exterior power of the $\mathrm{SO}(n-1,\mathbb{C})$-decomposition $U \simeq \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$ gives

$$\Lambda^k U \simeq \Lambda^k(\mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}) \simeq \bigoplus_{p+q=k} (\Lambda^p \mathbb{C}^{n-1}) \otimes (\Lambda^q \mathbb{C}^{n-1}) \tag{4.16}$$

as $\mathrm{SO}(n-1,\mathbb{C})$-modules. If $n-1$ is odd, then by Theorem 4.3.5(i) the (self-dual) $\mathrm{SO}(n-$

$1, \mathbb{C})$-modules $\Lambda^p\mathbb{C}^{n-1}, \Lambda^q\mathbb{C}^{n-1}$ in (4.16) are irreducible for all p, q and non-isomorphic unless $p = q$ or $p = n-1-q$ (the latter isomorphism being the Hodge star operator). The case $p = n-1-q$ is excluded by the assumption $k \neq n-1$, so by self-duality we have that (4.13) holds. If $n-1$ is even, then by Theorem 4.3.5(ii) the (self-dual) $\mathrm{SO}(n-1, \mathbb{C})$-modules $\Lambda^p\mathbb{C}^{n-1}, \Lambda^q\mathbb{C}^{n-1}$ in (4.16) are irreducible for $p, q \neq \frac{n-1}{2}$ and non-isomorphic unless $p = q$ or $p = n-1-q$. The assumption $k \neq n-1$ then ensures that (4.13) holds for all $k \neq n-1$.

Case 4: The Five Exceptional Spaces. These are the five spaces from Theorem 2.4.1 with G an exceptional Lie group. Following [DOP20], we denote the corresponding 3-Sasakian data by

$$\mathfrak{g}^s = \mathfrak{g}_0^s, \oplus \mathfrak{g}_1^s, \qquad (\mathfrak{g}_1^s)^{\mathbb{C}} \cong \mathbb{C}^2 \otimes U^s, \qquad s = 1, 2, 3, 4, 5,$$

and we recall that the $(\mathfrak{h}^s)^{\mathbb{C}}$-modules U^s have been described in terms of highest weight modules on [DOP20, p. 841]. This information is summarized in Table 4.1.

	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
G^s	G_2	F_4	E_6	E_7	E_8
H^s	$\mathrm{Sp}(1)$	$\mathrm{Sp}(3)$	$\mathrm{SU}(6)$	$\mathrm{Spin}(12)$	E_7
$(\mathfrak{h}^s)^{\mathbb{C}}$	$A_1 = \mathfrak{sp}(2, \mathbb{C})$	$C_3 = \mathfrak{sp}(6, \mathbb{C})$	$A_5 = \mathfrak{sl}(6, \mathbb{C})$	$D_6 = \mathfrak{so}(12, \mathbb{C})$	$\mathrm{E}_7 = \mathfrak{e}_7^{\mathbb{C}}$
U^s	$V(3)$	$V(\lambda_3)$	$V(\lambda_3)$	$V(\lambda_5)$	$V(\lambda_7)$

Table 4.1: The Exceptional Homogeneous 3-Sasakian Spaces

Using the LiE computer algebra package ([LCL88]), one checks that (4.13) holds for each $s = 1, 2, 3, 4, 5$, and the result in this case then follows by the same argument as in Case 2. □

As a consequence, we immediately obtain a description of the invariant φ_1-(anti-)holomorphic forms on the full tangent bundle:

Theorem 4.3.9. *If $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is a simply-connected homogeneus 3-Sasakian space, then the invariant φ_1-(anti-)holomorphic forms are given by*

$$\Lambda^{\bullet,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}) = \mathrm{span}_{\mathbb{C}}\{\omega_{1,0}^k,\ y_{1,0} \wedge \omega_{1,0}^k\}_{k=0}^{n-1}, \qquad \Lambda^{0,\bullet}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}) = \mathrm{span}_{\mathbb{C}}\{\omega_{0,1}^k,\ y_{0,1} \wedge \omega_{0,1}^k\}_{k=0}^{n-1},$$

where

$$y_{1,0} := (\xi_2 - i\xi_3), \quad y_{0,1} := \overline{y_{1,0}}, \qquad \omega_{1,0} := (\Phi_2|_{\mathcal{H}} - i\Phi_3|_{\mathcal{H}}), \quad \omega_{0,1} := \overline{\omega_{1,0}}.$$

Proof. Since the isotropy group acts trivially in the vertical directions, we have

$$\Lambda^{\bullet,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}) = \Lambda^{\bullet,0}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{V}}) \otimes \Lambda^{\bullet,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) \cong [1 \otimes \Lambda^{\bullet,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})] \oplus [y_{1,0} \otimes \Lambda^{\bullet,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})],$$
$$\Lambda^{0,\bullet}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}) = \Lambda^{0,\bullet}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{V}}) \otimes \Lambda^{0,\bullet}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}}) \cong [1 \otimes \Lambda^{0,\bullet}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})] \oplus [y_{0,1} \otimes \Lambda^{0,\bullet}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})],$$

and the result then follows from an analogous argument as in Case 2 in the proof of the preceding proposition, where it was noted that $1, \omega_{1,0}$ (resp. $1, \omega_{0,1}$) are generators of $\Lambda^{\bullet,0}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$ (resp. $\Lambda^{0,\bullet}_{\mathrm{inv}}(\mathfrak{m}^{\mathbb{C}}_{\mathcal{H}})$). □

Using Theorem 4.3.9, we are now ready to prove the main result of the section:

Theorem 4.3.10. *Let $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ be a simply-connected homogeneous 3-Sasakian manifold of dimension $4n-1$. For any adapted orthonormal basis of the reductive complement, and the corresponding description of the spinor module from Chapter 2.1, the space of invariant spinors is given by*

$$\Sigma_{\mathrm{inv}} = \mathrm{span}_{\mathbb{C}}\{\omega^k,\ y_1 \wedge \omega^k\}_{k=0}^{n-1}, \tag{4.17}$$

where $\omega := \sum_{p=1}^{n-1} y_{2p} \wedge y_{2p+1}$.

Proof. It is well-known that the 3-Sasakian structure on M gives a reduction of the structure group of the tangent bundle to $\mathrm{Sp}(n-1) \subset \mathrm{SO}(4n-1)$ (see [Kuo70, Thm. 5]). Furthermore, since the 3-Sasakian structure on $M = G/H$ is assumed to be homogeneous, we have that the image of the isotropy representation ι is contained in this reduction:

$$\iota(H) \subseteq \mathrm{Sp}(n-1) \subseteq \mathrm{SO}(4n-1).$$

The lifted action of $\mathrm{Sp}(n-1)$ on $\Sigma = \Lambda^\bullet L'$ is given explicitly in [AHL23, Prop. 4.5] (see also the discussion in [AHL23, Section 4.1.4]): since operators in the symplectic group (and hence operators in $\iota(H)$) are traceless, it is simply the usual action via exterior powers of the dual of the standard representation. Thus $\Sigma M \cong \Lambda^{0,\bullet}(T^*_{\mathbb{C}}M)$ as homogeneous bundles, and the result follows from Theorem 4.3.9 by noting that $y_1 = \frac{1}{\sqrt{2}} y_{0,1}$ and $\omega = -\frac{1}{2}\omega_{0,1}$. □

One immediate consequence of this characterization of the invariant spinors is a simple description of the bundle E_1^- from Theorem 2.3.8:

Proposition 4.3.11. *If $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is a simply-connected homogeneous 3-Sasakian* *manifold, with an adapted basis for the reductive complement and the associated spinor module as above, then*

$$E_1^- = \mathrm{span}_{\mathbb{C}}\{1,\ y_1 \wedge \omega^{n-1}\}.$$

Proof. As noted in the proof of [FK90, Thm. 1], the defining condition of E_1^- implies that $e_j \cdot \varphi_1(e_j) \cdot \psi = \xi_1 \cdot \psi$ for all $\psi \in \Gamma(E_1^-)$ and all $j = 2, \dots 4n-1$. Indeed, it is not hard to see that the two conditions are equivalent, so it suffices to show that

$$e_{2p} \cdot e_{2p+1} \cdot 1 = \xi_1 \cdot 1 \quad \text{and} \quad e_{2p} \cdot e_{2p+1} \cdot (y_1 \wedge \omega^{n-1}) = \xi_1 \cdot (y_1 \wedge \omega^{n-1})$$

for all $p = 1, \dots, 2n-1$. Using the formulas (2.4), we calculate

$$e_{2p} \cdot e_{2p+1} = i(x_p\lrcorner\ +\ y_p\wedge) \circ (y_p \wedge\ -\ x_p\lrcorner) = i\left[x_p\lrcorner \circ y_p \wedge - y_p \wedge \circ x_p\lrcorner\right],$$

and hence

$$\begin{aligned} e_{2p} \cdot e_{2p+1} \cdot 1 &= i\left[x_p\lrcorner(y_p \wedge 1) - y_p \wedge (x_p\lrcorner 1)\right] = i = \xi_1 \cdot 1, \\ e_{2p} \cdot e_{2p+1} \cdot (y_1 \wedge \omega^{n-1}) &= i\left[x_p\lrcorner(y_p \wedge y_1 \wedge \omega^{n-1}) - y_p \wedge (x_p\lrcorner(y_1 \wedge \omega^{n-1}))\right] \\ &= -iy_1 \wedge \omega^{n-1} = \xi_1 \cdot (y_1 \wedge \omega^{n-1}) \end{aligned}$$

for all $p = 1, \dots, 2n-1$. The penultimate equality follows by considering separately the cases $p = 1$, $p \neq 1$ and using the fact that ω^{n-1} is a multiple of $y_2 \wedge y_3 \wedge \cdots \wedge y_{2n-1}$, hence $y_p \wedge (x_p\lrcorner\omega^{n-1}) = \omega^{n-1}$ for $p \neq 1$. □

Remark 4.3.12. Note that the preceding proposition is an immediate consequence of Proposition 4.4.4 in the next section, where it is proved using a different method. We have also included it here as an illustrative example of how to perform spinorial calculations using the exterior forms realization of the spin representation.

In the proof of Theorem 4.3.10 we relied on the identification $\Sigma \cong \Lambda^{0,\bullet}_{\mathbb{C}}(M)$ between invariant φ_1-anti-holomorphic forms and invariant spinors. A natural question then arises as to the relationship between invariant real differential forms and invariant spinors. The next few results are devoted to the exploration of this topic. From this point forward we fix, without further mention, the Clifford algebra representation associated to an adapted basis, so that the invariant spinors take the form (4.17).

Lemma 4.3.13. *Let $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ be a simply-connected homogeneous 3-Sasakian manifold of dimension $4n-1$. Then for any integer $k \geq 0$ we have*

$$\Phi_0 \cdot \omega^k = 0, \tag{4.18}$$

$$(\Phi_1|_{\mathcal{H}}) \cdot \omega^k = 2i(2k - n + 1)\omega^k, \tag{4.19}$$

$$(\Phi_2|_{\mathcal{H}}) \cdot \omega^k = 2(\omega^{k+1} - k(n-1)\omega^{k-1}), \tag{4.20}$$

$$(\Phi_3|_{\mathcal{H}}) \cdot \omega^k = -2i(\omega^{k+1} + k(n-1)\omega^{k-1}), \tag{4.21}$$

where $\omega := \sum_{p=1}^{n-1} y_{2p} \wedge y_{2p+1}$ and we use the convention $\omega^0 = 1$.

Proof. These identities follow from a straightforward calculation in the spin representation. First we consider $\Phi_0 = \sum_{p=1}^{n-1}(e_{4p} \wedge e_{4p+1} - e_{4p+2} \wedge e_{4p+3})$. For $k \geq 0$, we calculate:

$$\begin{aligned}
\Phi_0 \cdot \omega^k &= \sum_{p=1}^{n-1}(e_{4p} \wedge e_{4p+1} - e_{4p+2} \wedge e_{4p+3}) \cdot \omega^k \\
&= i\sum_{p=1}^{n-1}[(x_{2p}\lrcorner + y_{2p}\wedge)(y_{2p} \wedge -x_{2p}\lrcorner) - (x_{2p+1}\lrcorner + y_{2p+1}\wedge)(y_{2p+1} \wedge -x_{2p+1}\lrcorner)]\omega^k \\
&= i\sum_{p=1}^{n-1}[x_{2p}\lrcorner(y_{2p} \wedge \omega^k) - y_{2p} \wedge (x_{2p}\lrcorner\omega^k) - x_{2p+1}\lrcorner(y_{2p+1} \wedge \omega^k) + y_{2p+1} \wedge (x_{2p+1}\lrcorner\omega^k)] \\
&= i\sum_{p=1}^{n-1}[\omega^k - 2ky_{2p} \wedge y_{2p+1} \wedge \omega^{k-1} - \omega^k + 2ky_{2p+1} \wedge (-y_{2p})] \\
&= 0.
\end{aligned}$$

The calculations for $(\Phi_i|_{\mathcal{H}}) \cdot \omega^k$, $i = 1, 2, 3$ are analogous, and we include them below for the sake of completeness. For $k \geq 0$, we calculate:

$$\begin{aligned}
(\Phi_1|_{\mathcal{H}}) \cdot \omega^k &= -\sum_{p=1}^{n-1}(e_{4p} \wedge e_{4p+1} + e_{4p+2} \wedge e_{4p+3}) \cdot \omega^k \\
&= -i\sum_{p=1}^{n-1}[(x_{2p}\lrcorner + y_{2p}\wedge)(y_{2p} \wedge -x_{2p}\lrcorner) + (x_{2p+1}\lrcorner + y_{2p+1}\wedge)(y_{2p+1} \wedge -\, x_{2p+1}\lrcorner)]\omega^k \\
&= -i\sum_{p=1}^{n-1}(-ky_{2p} \wedge y_{2p+1} \wedge \omega^{k-1} + \omega^k - ky_{2p+1} \wedge (-y_{2p}) \wedge \omega^{k-1} \\
&\qquad\qquad - ky_{2p} \wedge y_{2p+1} \wedge \omega^{k-1} + \omega^k - ky_{2p} \wedge y_{2p+1} \wedge \omega^{k-1}) \\
&= i(4k\omega^k - 2(n-1)\omega^k) = 2i(2k - n + 1)\omega^k,
\end{aligned}$$

$$
\begin{aligned}
(\Phi_2|_{\mathcal{H}}) \cdot \omega^k &= -\sum_{p=1}^{n-1}(e_{4p} \cdot e_{4p+2} - e_{4p+1} \cdot e_{4p+3}) \cdot \omega^k \\
&= \sum_{p=1}^{n-1}[(x_{2p}\lrcorner + y_{2p}\wedge)(x_{2p+1}\lrcorner + y_{2p+1}\wedge) + (y_{2p}\wedge - x_{2p}\lrcorner)(y_{2p+1}\wedge - x_{2p+1}\lrcorner)]\omega^k \\
&= \left(\sum_{p,q=1}^{n-1} 2y_{2p} \wedge y_{2p+1} \wedge y_{2q} \wedge y_{2q+1}\right) \wedge \left(\sum_{q=1}^{n-1} y_{2q} \wedge y_{2q+1}\right)^{k-1} \\
&\qquad + k\left(\sum_{p=1}^{n-1}(-2)\right) \wedge \left(\sum_{q=1}^{n-1} y_{2q} \wedge y_{2q+1}\right)^{k-1} \\
&= 2(\omega^{k+1} - k(n-1)\omega^{k-1}),
\end{aligned}
$$

$$
\begin{aligned}
(\Phi_3|_{\mathcal{H}}) \cdot \omega^k &= -\sum_{p=1}^{n-1}(e_{4p} \cdot e_{4p+3} + e_{4p+1} \cdot e_{4p+2}) \cdot \omega^k \\
&= -i\sum_{p=1}^{n-1}[(x_{2p}\lrcorner + y_{2p}\wedge)(y_{2p+1}\wedge - x_{2p+1}\lrcorner) + (y_{2p}\wedge - x_{2p}\lrcorner)(x_{2p+1}\lrcorner + y_{2p+1}\wedge)]\omega^k \\
&= -i\left(\sum_{p,q=1}^{n-1} 2y_{2p} \wedge y_{2p+1} \wedge y_{2q} \wedge y_{2q+1}\right) \wedge \left(\sum_{q=1}^{n-1} y_{2q} \wedge y_{2q+1}\right)^{k-1} \\
&\qquad - ik\left(\sum_{p=1}^{n-1} 2\right) \wedge \left(\sum_{q=1}^{n-1} y_{2q} \wedge y_{2q+1}\right)^{k-1} \\
&= -2i(\omega^{k+1} + k(n-1)\omega^{k-1}).
\end{aligned}
$$

□

We immediately deduce:

Corollary 4.3.14. *For $i \in \{0, 1, 2, 3\}$, let S_i denote the complex span of the spinors $(\Phi_i|_{\mathcal{H}})^k \cdot 1$ with $k = 1, \ldots, 2n-1$. We have:*

$$S_0 = \{0\}, \quad S_1 = \mathrm{span}_{\mathbb{C}}\{1\}, \quad S_2 = S_3 = \mathrm{span}_{\mathbb{C}}\{\omega^k\}_{k=0}^{n-1}.$$

Proof. The cases S_0 and S_1 are clear from the preceding lemma. From (4.20) we note that the Clifford product of the form $\frac{1}{2}\Phi_2|_{\mathcal{H}}$ with the spinor ω^k is a monic degree $(k+1)$ polynomial in ω, hence $\frac{1}{2^k}(\Phi_2|_{\mathcal{H}})^k \cdot 1$ is a monic degree k polynomial in ω. It is straightforward to see (by induction) that

$$\mathrm{span}_{\mathbb{C}}\{\tfrac{1}{2^k}(\Phi_2|_{\mathcal{H}})^k \cdot 1\}_{k=0}^{k_0} = \mathrm{span}_{\mathbb{C}}\{\omega^k\}_{k=0}^{k_0}$$

for any $k_0 \in \{1, \dots, n-1\}$, and the result for S_2 follows. The proof for S_3 is analogous. $\square$

This also gives a nice description of the invariant spinors in terms of the invariant real differential forms:

Theorem 4.3.15. *The space Σ_{inv} of invariant spinors on a simply-connected homogeneous 3-Sasakian manifold is spanned by Clifford products of invariant differential forms with the invariant spinor $1 \in \Sigma_{\mathrm{inv}}$.*

Proof. In light of Theorem 4.3.10, it suffices to show that spinors of the form ω^k and $y_1 \wedge \omega^k$ can be obtained as linear combinations of Clifford products of invariant differential forms with $1 \in \Sigma_{\mathrm{inv}}$. This follows from Corollary 4.3.14 by noting that $\omega^k \in S_2$ and $y_1 \wedge \omega^k \in \xi_2 \cdot S_2$. $\square$

Remark 4.3.16. We would like to point out that the results of this section so far also hold in the more general setting of compact simply-connected homogeneous 3-(α, δ)-Sasaki spaces; The reason for this is that the generalized 3-Sasakian data used to define homogeneous 3-(α, δ)-Sasaki structures coincides, in the case of a compact space, with the notion of 3-Sasakian data (compare Theorems 2.4.2 and 2.5.7). In particular, the isotropy representation of a family of compact homogeneous 3-(α, δ)-Sasaki spaces parameterized by $\alpha, \delta > 0$ is isomorphic to the isotropy representation of the corresponding homogeneous 3-Sasakian space obtained by setting $\alpha = \delta = 1$. The next section discusses Killing spinors on homogeneous 3-Sasakian spaces, which do not carry over to the corresponding 3-(α, δ)-Sasaki spaces. Rather, certain deformations of Killing spinors in the 3-(α, δ)-Sasaki setting are investigated in Chapter 5.

4.4 The Space of Riemannian Killing Spinors

We conclude the chapter with an explicit description of the Riemannian Killing spinors on a homogeneous 3-Sasakian space:

Theorem 4.4.1. *Let $(M^{4n-1} = G/H, g, \xi_i, \eta_i, \varphi_i)$ be a simply-connected homogeneous 3-Sasakian manifold, and fix a description of the spinor module relative to an adapted basis as in the previous section. If $n \geq 2$, then the space of invariant Killing spinors has a basis given by*

$$\psi_k := \omega^{k+1} - i(k+1) y_1 \wedge \omega^k, \qquad -1 \leq k \leq n-1,$$

where we use the conventions $\omega^{-1} = 0$ and $\omega^0 = 1$. If $n = 1$ then the space of invariant Killing spinors has a basis given by 1, y_1. Furthermore, if $(M, g) \ncong (S^{4n-1}, g_{\mathrm{round}})$ then any Killing *spinor is invariant.*

Proof. Let Λ, $\Lambda^g\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ denote the Nomizu maps of the canonical and Levi-Civita connections respectively. First we consider the horizontal directions $X\in\mathcal{H}$. From (2.8) we have $\Lambda(X)=0$ for all $X\in\mathcal{H}$, and thus any invariant Killing spinor ψ satisfies the algebraic equation

$$0=\widetilde{\Lambda}(X)\cdot\psi=\widetilde{\Lambda^g}(X)\cdot\psi+\frac{1}{4}(X\lrcorner T)\cdot\psi=\frac{1}{2}X\cdot\psi+\frac{1}{2}\sum_{i=1}^{3}\xi_i\cdot\varphi_i(X)\cdot\psi \tag{4.22}$$

for all $X\in\mathcal{H}$. Assume first that $n\geq 2$. Calculating in an adapted basis $\{e_{4p},e_{4p+1},e_{4p+2},e_{4p+3}\}$, we find:

$$\begin{aligned}\frac{1}{2}e_{4p}\cdot\omega^k+\frac{1}{2}\sum_{i=1}^{3}\xi_i\cdot\varphi_i(e_{4p})\cdot\omega^k&=\frac{1}{2}\left(e_{4p}\cdot\omega^k+\xi_1\cdot e_{4p+1}\cdot\omega^k+\xi_2\cdot e_{4p+2}\cdot\omega^k+\xi_3\cdot e_{4p+3}\cdot\omega^k\right)\\&=\frac{1}{2}\left(2ix_{2p}\lrcorner\omega^k+\xi_2\cdot e_{4p+2}\cdot\omega^k+\xi_3\cdot e_{4p+3}\cdot\omega^k\right)\\&=ix_{2p}\lrcorner\omega^k+\frac{1}{2}\left(-y_1\wedge(y_{2p+1}\wedge+x_{2p+1}\lrcorner)\omega^k+y_1\wedge(y_{2p+1}\wedge-x_{2p+1}\lrcorner)\omega^k\right)\\&=ix_{2p}\lrcorner\omega^k-y_1\wedge(x_{2p+1}\lrcorner\omega^k)\end{aligned}$$

and similarly,

$$\frac{1}{2}e_{4p}\cdot(y_1\wedge\omega^k)+\frac{1}{2}\sum_{i=1}^{3}\xi_i\cdot\varphi_i(e_{4p})\cdot(y_1\wedge\omega^k)=iy_{2p}\wedge(y_1\wedge\omega^k)+y_{2p+1}\wedge\omega^k.$$

Writing

$$\psi:=\sum_{k=0}^{n-1}\lambda_k\omega^k+\sum_{k=0}^{n-1}\lambda'_k(y_1\wedge\omega^k)$$

in terms of the basis from Theorem 4.3.10, we have

$$\frac{1}{2}e_{4p}\cdot\psi+\frac{1}{2}\sum_{i=1}^{3}\xi_i\cdot\varphi_i(e_{4p})\cdot\psi=\sum_{k=0}^{n-1}\lambda_k[ix_{2p}\lrcorner\omega^k-y_1\wedge(x_{2p+1}\lrcorner\omega^k)]+\sum_{k=0}^{n-1}\lambda'_k[y_{2p+1}\wedge\omega^k+iy_{2p}\wedge(y_1\wedge\omega^k)]. \tag{4.23}$$

For the $k=l$ index of the summations on the right hand side of (4.23), the degrees of the four terms are $2l-1$, $2l$, $2l+1$ and $2l+2$ respectively. Considering separately the even and odd degree parts of (4.23), we are seeking solutions of

$$\sum_{k=0}^{n-1}[i\lambda_k x_{2p}\lrcorner\omega^k+\lambda'_k y_{2p+1}\wedge\omega^k]=0=\sum_{k=0}^{n-1}[-\lambda_k y_1\wedge(x_{2p+1}\lrcorner\omega^k)+i\lambda'_k y_{2p}\wedge(y_1\wedge\omega^k)],$$

or equivalently, solutions of the linear system of equations

$$\lambda_k' = -i(k+1)\lambda_{k+1}, \quad -1 \leq k \leq n-1.$$

This gives $n+1$ linearly independent spinors

$$\psi_k := \omega^{k+1} - i(k+1)y_1 \wedge \omega^k, \quad -1 \leq k \leq n-1,$$

and a straightforward calculation of the other horizontal derivatives (by substituting $X = e_{4p+1}, e_{4p+2}$, and e_{4p+3} into (4.22)) shows that these spinors satisfy the Killing equation in the horizontal directions. In the vertical directions, one sees from Theorem 2.5.7 that the Nomizu map for the Levi-Civita connection satisfies $\Lambda^g(\xi_i) = \xi_j \wedge \xi_k$ for any even permutation (i,j,k) of $(1,2,3)$. Considering the spin lifts $\widetilde{\Lambda^g}(\xi_i) = \frac{1}{2}\xi_j \cdot \xi_k$, another straightforward calculation in the spin representation shows that any spinor of the form ω^k or $y_1 \wedge \omega^k$ satisfies the Killing equation in the vertical directions, and we conclude that the ψ_k are Killing spinors in the case $n \geq 2$. Assuming now that $n = 1$, the horizontal distribution is trivial, and consequently Equation (4.22) does not apply. In this dimension the spinor bundle has complex dimension equal to $n + 1 = 1 + 1 = 2$, hence it is spanned by 1, y_1, and the above argument for the vertical directions shows that these are Killing spinors. The dimension of the space of Killing spinors on a 3-Sasakian manifold $(M^{4n-1}, g) \not\cong (S^{4n-1}, g_{\mathrm{round}})$ is equal to $n+1$ (see [Bär93]), and the result follows. □

Remark 4.4.2. The final assertion of Theorem 4.4.1–that any Killing spinor on a homogeneous 3-Sasakian space $(M^{4n-1} = G/H, g) \not\cong (S^{4n-1}, g_{\mathrm{round}})$ is invariant–was previously proved using a different method in [Kat00, Thm. 7.1]: Kath showed that G has a representation on the space of Killing spinors, and then deduced that any Killing spinor is invariant (equivalently, this representation is trivial) by comparing in each case the dimension of the space of Killing spinors with the dimension of the smallest non-trivial representation of G.

Remark 4.4.3. Let us comment briefly on the case of the round sphere. Using Bär's correspondence between Killing spinors on a manifold and parallel spinors on its cone ([Bär93]), it is easy to see that the spinor bundle of the round sphere is parallelized by Killing spinors; the cone over the round sphere is the Euclidean space of one dimension higher, which has trivial holonomy and therefore a parallelization of its spinor bundle by parallel spinors. In particular, for the 3-Sasakian round sphere $(S^{4n-1} = \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}, g, \xi_i, \eta_i, \varphi_i)$, the $\mathrm{Sp}(n)$-invariant Killing spinors ψ_k

from Theorem 4.4.1 fail to span the whole spinor bundle when $n > 1$, so there exist non-invariant Killing spinors in this case. An explicit construction of the Killing spinors on the round sphere in stereographic coordinates can be found on [BFGK91, p. 37].

We conclude the chapter by exploring which of the invariant Killing spinors from Theorem 4.4.1 recover the invariant 3-Sasakian structure on $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ via the construction in Chapter 4.1. By Theorem 4.1.10 and the proof of Theorem 4.1.8, we see that in order to recover the Sasakian structure $(\xi_i, \eta_i, \varphi_i)$ it suffices to find a Killing spinor Ψ such that $\Psi' := -\xi_i \cdot \Psi$ is also a Killing spinor. Thus we consider the subbundles $\mathcal{E}_i$ of the spinor bundle spanned (over the space $C^\infty(M)$ of smooth real-valued functions) by invariant spinors with this property:

$$\mathcal{E}_i := \operatorname{span}_{C^\infty(M)}\{\Psi \in \kappa_{\text{inv}}(M,g)\colon\ \Psi' := -\xi_i \cdot \Psi \in \kappa_{\text{inv}}(M,g)\}, \quad i = 1,2,3,$$

where $\kappa_{\text{inv}}(M,g) := \operatorname{span}_{\mathbb{C}}\{\psi_k\}_{k=-1}^{n-1}$ denotes the space of invariant Killing spinors. In fact, it turns out that these subbundles coincide with the E_i^- in the homogeneous setting, as the following proposition shows:

Proposition 4.4.4. *If $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is a simply-connected homogeneous 3-Sasakian space, then $\mathcal{E}_i = E_i^-$ for $i = 1,2,3$. Furthermore, each $\mathcal{E}_i$ has a basis $\Psi_{\mathcal{E}_i,0}, \Psi_{\mathcal{E}_i,1}$ given by*

$$\Psi_{\mathcal{E}_1,0} := 1, \quad \Psi_{\mathcal{E}_1,1} := y_1 \wedge \omega^{n-1}, \quad \Psi_{\mathcal{E}_2,0} := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k}{(2k+1)!}\psi_{2k},$$
$$\Psi_{\mathcal{E}_2,1} := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{(2k)!}\psi_{2k-1}, \quad \Psi_{\mathcal{E}_3,0} := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!}\psi_{2k}, \quad \Psi_{\mathcal{E}_3,1} := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2k)!}\psi_{2k-1}.$$

Proof. We consider first the case $(M^{4n-1}, g) \not\cong (S^{4n-1}, g_{\text{round}})$. By Theorem 4.4.1 any Killing spinor is invariant, thus it follows from the proof of Theorem 4.1.8 that $E_i^- \subseteq \mathcal{E}_i$. Therefore, in order to show that $E_i^- = \mathcal{E}_i$ it suffices to show that $\operatorname{rank}(\mathcal{E}_i) \leq 2$. To find elements $\Psi \in \mathcal{E}_i$, we write $\Psi = \sum_{k=-1}^{n-1} \lambda_k \psi_k$ in terms of our basis of invariant Killing spinors and seek to determine for which values of $\lambda_{-1}, \dots, \lambda_{n-1}$ there exist $\Theta_{-1}, \dots, \Theta_{n-1} \in \mathbb{C}$ satisfying

$$\xi_i \cdot \Psi = \sum_{k=-1}^{n-1} \Theta_k \psi_k. \tag{4.24}$$

To show that $\operatorname{rank}(\mathcal{E}_i) \leq 2$ for $i = 1,2,3$, we treat the subcases $i = 1,2,3$ individually:

(i) $\underline{i=1}$: First, we note that

$$\xi_1 \cdot \psi_k = i\omega^{k+1} - (k+1)y_1 \wedge \omega^k.$$

Substituting this into (4.24), we are looking for solutions of

$$\sum_{k=-1}^{n-1} \lambda_k[i\omega^{k+1} - (k+1)y_1 \wedge \omega^k] = \sum_{k=-1}^{n-1} \Theta_k[\omega^{k+1} - i(k+1)y_1 \wedge \omega^k],$$

or equivalently, solutions of the linear equations

$$i\lambda_{-1} = \Theta_{-1}, \quad \lambda_{n-1} = i\Theta_{n-1}, \quad i\lambda_k = \Theta_k, \quad \lambda_k = i\Theta_k, \quad \text{for } k = 0, \dots, n-2.$$

The solutions of this system of equations necessarily have $\lambda_k = \Theta_k = 0$ for all $k \neq -1, n-1$, hence $\mathcal{E}_1$ contained in the span of 1 and $y_1 \wedge \omega^{n-1}$.

(ii) $\underline{i=2}$: Proceeding similarly as in the previous subcase, we first note that

$$\xi_2 \cdot \psi_k = iy_1 \wedge \omega^{k+1} + (k+1)\omega^k.$$

Substituting this into (4.24) gives

$$\sum_{k=-1}^{n-1} \lambda_k[iy_1 \wedge \omega^{k+1} + (k+1)\omega^k] = \sum_{k=-1}^{n-1} \Theta_k[\omega^{k+1} - i(k+1)y_1 \wedge \omega^k],$$

or equivalently, the linear system

$$(k+1)\lambda_k = \Theta_{k-1}, \quad \lambda_{k-1} = -(k+1)\Theta_k, \quad \text{for } k = 0, \dots, n-1.$$

These equations give rise to the recursive relation $\lambda_k = \frac{1}{k+1}\Theta_{k-1} = \frac{-1}{k(k+1)}\lambda_{k-2}$, whose space of solutions lies within the span of the spinors $\Psi_{\mathcal{E}_2,0}$, $\Psi_{\mathcal{E}_2,1}$.

(iii) $\underline{i=3}$: Similarly to the previous two subcases, we first note that

$$\xi_3 \cdot \psi_k = y_1 \wedge \omega^{k+1} + i(k+1)\omega^k.$$

Substituting this into (4.24) gives

$$\sum_{k=-1}^{n-1} \lambda_k[y_1 \wedge \omega^{k+1} + i(k+1)\omega^k] = \sum_{k=-1}^{n-1} \Theta_k[\omega^{k+1} - i(k+1)y_1 \wedge \omega^k],$$

which is equivalent to the linear system

$$i(k+1)\lambda_k = \Theta_{k-1}, \quad \lambda_{k-1} = -i(k+1)\Theta_k, \quad \text{for } k = 0, \ldots, n-1.$$

These give the recursive relation $\lambda_k = \frac{-i}{k+1}\Theta_{k-1} = \frac{1}{k(k+1)}\lambda_{k-2}$, whose space of solutions lies within the span of $\Psi_{\mathcal{E}_3,0}$, $\Psi_{\mathcal{E}_3,1}$.

Thus, for $(M^{4n-1}, g) \ncong (S^{4n-1}, g_{\text{round}})$ we have shown that $\mathcal{E}_i = E_i^-$ for $i = 1, 2, 3$, and that $\{\Psi_{\mathcal{E}_i,0}, \Psi_{\mathcal{E}_i,1}\}$ is a basis (over $C^\infty(M)$) for this vector bundle. But the defining equation for E_i^- is an algebraic equation depending only on the choice of adapted basis and the associated Clifford multiplication, hence the result also holds for $(M^{4n-1}, g) \cong (S^{4n-1}, g_{\text{round}})$. □

As noted above, it immediately follows that these spinors recover the homogeneous 3-Sasakian structure via the construction in Chapter 4.1, giving a full picture of this construction in the homogeneous 3-Sasakian setting:

Theorem 4.4.5. *If $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is a simply-connected homogeneous 3-Sasakian space then, for each $i \in \{1, 2, 3\}$, the Sasakian structure $(\xi_i, \eta_i, \varphi_i)$ arises from the spinors $\Psi_i := \Psi_{\mathcal{E}_i,0}$ and $\Psi_i' := -\xi_i \cdot \Psi_{\mathcal{E}_i,0}$ (or $\Psi_i := \Psi_{\mathcal{E}_i,1}$ and $\Psi_i' := -\xi_i \cdot \Psi_{\mathcal{E}_i,1}$) via the construction in Chapter 4.1.*

The values of the spinors $\Psi_{\mathcal{E}_i,0}$ and $\Psi_{\mathcal{E}_i,1}$, in terms of the basis of invariant spinors from Theorem 4.3.10, are tabulated for a few low dimensions in Table 4.2.

$\dim(M)$	$\Psi_{\mathcal{E}_1,0}$	$\Psi_{\mathcal{E}_2,0}$	$\Psi_{\mathcal{E}_3,0}$
7	1	$\omega - iy_1$	$\omega - iy_1$
11	1	$\omega - iy_1 + \frac{1}{2}iy_1 \wedge \omega^2$	$\omega - iy_1 - \frac{1}{2}y_1 \wedge \omega^2$
15	1	$\omega - iy_1 + \frac{1}{2}iy_1 \wedge \omega^2 - \frac{1}{6}\omega^3$	$\omega - iy_1 - \frac{1}{2}y_1 \wedge \omega^2 + \frac{1}{6}\omega^2$
$\dim(M)$	$\Psi_{\mathcal{E}_1,1}$	$\Psi_{\mathcal{E}_2,1}$	$\Psi_{\mathcal{E}_3,1}$
7	$y_1 \wedge \omega$	$1 + iy_1 \wedge \omega$	$1 - iy_1 \wedge \omega$
11	$y_1 \wedge \omega^2$	$1 + iy_1 \wedge \omega - \frac{1}{2}\omega^2$	$1 - iy_1 \wedge \omega + \frac{1}{2}\omega^2$
15	$y_1 \wedge \omega^3$	$1 + iy_1 \wedge \omega - \frac{1}{2}\omega^2 - \frac{1}{6}iy_1 \wedge \omega^3$	$1 - iy_1 \wedge \omega + \frac{1}{2}\omega^2 - \frac{1}{6}iy_1 \wedge \omega^3$

Table 4.2: The Spinors $\Psi_{\mathcal{E}_i,0}$ and $\Psi_{\mathcal{E}_i,1}$ in Low Dimensions

5

Deformed Killing Spinors on 3-(α, δ)-Sasaki Manifolds

This chapter contains joint work with Prof. Dr. habil. Ilka Agricola (see page 8).

Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-(α, δ)-Sasaki manifold with Levi-Civita connection ∇^g and canonical connection ∇. We recall from Theorem 2.3.8 Friedrich and Kath's rank two subbundles of the spinor bundle:

$$E_i^- := \{\psi \in \Gamma(\Sigma M) : (-2\varphi_i(X) + \xi_i \cdot X - X \cdot \xi_i) \cdot \psi = 0 \quad \forall X \in TM\}, \qquad i = 1, 2, 3.$$

These were shown in [FK90] to have bases consisting of Killing spinors in the 3-Sasakian setting, and will also be interesting in the 3-(α, δ)-Sasaki setting, where we will show that they carry spinors satisfying the *deformed Killing equation* (5.1). To prove this, we shall adapt the method of Friedrich and Kath in [FK90]. While their overall strategy still works in this new setting, the curvature calculations become somewhat more complicated than in the 3-Sasakian case. We will mitigate this by using the canonical connection for some arguments, in order to take advantage of the identities calculated in [ADS23]. The main result of this chapter is the following theorem:

Theorem 5.0.1. *On a 3-(α, δ)-Sasaki manifold $(M, g, \xi_i, \eta_i, \varphi_i)$, the bundle $E := E_1^- + E_2^- + E_3^-$ has a basis of spinors satisfying*

$$\nabla^g_X \psi = \frac{\alpha}{2} X \cdot \psi + \frac{\alpha - \delta}{2} \sum_{p=1}^{3} \eta_p(X) \Phi_p \cdot \psi \qquad \textit{for all } X \in TM. \tag{5.1}$$

The rest of the chapter is devoted to the proof of Theorem 5.0.1.

5.1 The Modified Spinorial Connection

In this section we introduce the *modified connection* $\widehat{\nabla}$ on the spinor bundle and show that it preserves the bundles E_i^-, $i = 1, 2, 3$. To start, we recall that the Levi-Civita derivatives of the 3-(α, δ)-Sasaki structure tensors are given by:

Proposition 5.1.1. *([AD20, Prop. 2.3.2, Cor. 2.3.1]). If $(M, g, \xi_i, \eta_i, \varphi_i)$ is a 3-(α, δ)-Sasaki manifold then*

$$\begin{aligned}(\nabla^g_Y \varphi_i)X &= \alpha[g(X, Y)\xi_i - \eta_i(X)Y] - 2(\alpha - \delta)[\eta_k(Y)\varphi_j(X) - \eta_j(Y)\varphi_k(X)] \qquad (5.2)\\ &\quad + (\alpha - \delta)[\eta_j(Y)\eta_j(X) + \eta_k(Y)\eta_k(X)]\xi_i\\ &\quad - (\alpha - \delta)\eta_i(X)[\eta_j(Y)\xi_j + \eta_k(Y)\xi_k],\\ \nabla^g_Y \xi_i &= -\alpha\varphi_i(Y) - (\alpha - \delta)[\eta_k(Y)\xi_j - \eta_j(Y)\xi_k], \qquad (5.3)\end{aligned}$$

for any even permutation (i, j, k) of $(1, 2, 3)$.

We now prove two technical lemmas that will be necessary for subsequent calculations:

Lemma 5.1.2. *Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-(α, δ)-Sasaki manifold. If $\psi \in \Gamma(E_i^-)$ and (i, j, k) is an even permutation of $(1, 2, 3)$, then*

$$(-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \Phi_j \cdot \psi = [8\varphi_k(X) - 2\xi_k X + 2X\xi_k - 4\eta_i(X)\xi_j + 4\eta_j(X)\xi_i] \cdot \psi, \qquad (5.4)$$

$$(-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \Phi_k \cdot \psi = [-8\varphi_j(X) + 2\xi_j X - 2X\xi_j - 4\eta_i(X)\xi_k + 4\eta_k(X)\xi_i] \cdot \psi, \qquad (5.5)$$

for all $X \in \mathfrak{X}(M)$.

Proof. We calculate in the Clifford algebra,

$$\begin{aligned}-2\varphi_i(X) \cdot \Phi_j &= 2(2\varphi_i(X) \lrcorner \Phi_j - \Phi_j \cdot \varphi_i(X)) = 2(-2\varphi_j(\varphi_i(X)) - \Phi_j \cdot \varphi_i(X))\\ &= -4\eta_i(X)\xi_j + 4\varphi_k(X) - 2\Phi_j \cdot \varphi_i(X),\\ \xi_i \cdot X \cdot \Phi_j &= \xi_i \cdot (-2X \lrcorner \Phi_j + \Phi_j \cdot X) = 2\xi_i \cdot \varphi_j(X) + \xi_i \cdot \Phi_j \cdot X\\ &= 2\xi_i \cdot \varphi_j(X) + (-2\xi_i \lrcorner \Phi_j + \Phi_j \cdot \xi_i) \cdot X\\ &= 2\xi_i \cdot \varphi_j(X) - 2\xi_k \cdot X + \Phi_j \cdot \xi_i \cdot X,\\ -X \cdot \xi_i \cdot \Phi_j &= -X \cdot (-2\xi_i \lrcorner \Phi_j + \Phi_j \cdot \xi_i) = 2X \cdot \xi_k + (2X \lrcorner \Phi_j - \Phi_j \cdot X) \cdot \xi_i\\ &= 2X \cdot \xi_k - 2\varphi_j(X) \cdot \xi_i - \Phi_j \cdot X \cdot \xi_i.\end{aligned}$$

Adding these equations and using the defining relation for E_i^- gives

$$\begin{aligned}(-2\varphi_i(X)+\xi_iX-X\xi_i)\cdot\Phi_j\cdot\psi &= [4\varphi_k(X)-2\xi_kX+2X\xi_k]\cdot\psi+[-4\eta_i(X)\xi_j+2\xi_i\cdot\varphi_j(X)-2\varphi_j(X)\cdot\xi_i]\cdot\psi\\ &= [4\varphi_k(X)-2\xi_kX+2X\xi_k]\cdot\psi+[-4\eta_i(X)\xi_j+4\varphi_i(\varphi_j(X))]\cdot\psi\\ &= [4\varphi_k(X)-2\xi_kX+2X\xi_k]\cdot\psi+[-4\eta_i(X)\xi_j+4\varphi_k(X)+4\eta_j(X)\xi_i]\cdot\psi\\ &= [8\varphi_k(X)-2\xi_kX+2X\xi_k-4\eta_i(X)\xi_j+4\eta_j(X)\xi_i]\cdot\psi.\end{aligned}$$

The second equation, (5.5), is proved analogously. □

The second technical lemma is an identity which arises from Friedrich and Kath's calculations for the 3-Sasakian case, and also holds for 3-(α,δ)-Sasaki manifolds:

Lemma 5.1.3. *(Based on the proof of [FK90, Thm. 1]). If $(M,g,\xi_i,\eta_i,\varphi_i)$ is a 3-(α,δ)-Sasaki manifold and $\psi\in\Gamma(E_i^-)$, then*

$$[-2g(X,Y)\xi_i+2\eta_i(X)Y-\varphi_i(Y)X+X\varphi_i(Y)]\cdot\psi+(-2\varphi_i(X)+\xi_iX-X\xi_i)\cdot\left(\frac{1}{2}Y\cdot\psi\right)=0. \qquad (5.6)$$

Proof. This follows by the same calculation as on p.547 of [FK90] (note that their calculation has a small typo on the second last line; the term "$+XY\xi$" should instead say "$\pm XY\xi$"). □

Using the preceding lemmas, we now show:

Proposition 5.1.4. *If $(M,g,\xi_i,\eta_i,\varphi_i)$ is a 3-(α,δ)-Sasaki manifold then the modified spinorial connection*

$$\widehat{\nabla}_Y\psi:=\nabla^g_Y\psi-\frac{\alpha}{2}Y\cdot\psi-\frac{\alpha-\delta}{2}\sum_{p=1}^{3}\eta_p(Y)\Phi_p\cdot\psi,\qquad Y\in TM,\ \psi\in\Sigma M$$

preserves the bundles E_i^-, $i=1,2,3$.

Proof. Let $\psi\in\Gamma(E_i^-)$, and take (i,j,k) an even permutation of $(1,2,3)$. Differentiating the defining equation

$$(-2\varphi_i(X)+\xi_iX-X\xi_i)\psi=0$$

with respect to Y gives

$$\begin{aligned}0 &= [-2(\nabla^g_Y\varphi_i)X-2\varphi_i(\nabla^g_YX)+(\nabla^g_Y\xi_i)X+\xi_i(\nabla^g_YX)-(\nabla^g_YX)\xi_i-X(\nabla^g_Y\xi_i)]\cdot\psi\\ &\quad+(-2\varphi_i(X)+\xi_iX-X\xi_i)\cdot\nabla^g_Y\psi\\ &= [-2(\nabla^g_Y\varphi_i)X+(\nabla^g_Y\xi_i)X-X(\nabla^g_Y\xi_i)]\cdot\psi \qquad (5.7)\end{aligned}$$

$$+ (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \left(\frac{\alpha}{2} Y \cdot \psi + \frac{\alpha - \delta}{2} \sum_{p=1}^{3} \eta_p(Y) \Phi_p \cdot \psi \right)$$
$$+ (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \widehat{\nabla}_Y \psi.$$

(i) If $Y \in \mathcal{H}$ then (5.7) simplifies, using Proposition 5.1.1, to

$$0 = [-2\alpha(g(X,Y)\xi_i - \eta_i(X)Y) - \alpha\varphi_i(Y)X + \alpha X\varphi_i(Y)] \cdot \psi + (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \left(\frac{\alpha}{2} Y \cdot \psi \right)$$
$$+ (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \widehat{\nabla}_Y \psi,$$

and the fact that the first two terms on the right hand side of the above equation sum to zero follows immediately from (5.6).

(ii) If $Y = \xi_i$ then Proposition 5.1.1 reduces (5.7) to

$$0 = (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \left(\frac{\alpha}{2} \xi_i \cdot \psi + \frac{\alpha - \delta}{2} \Phi_i \cdot \psi \right) + (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \widehat{\nabla}_{\xi_i} \psi,$$

whose first term vanishes due to the fact that Φ_i acts on E_i^- as a multiple of ξ_i, and ξ_i anti-commutes with the operator defining E_i^-.

(iii) If $Y = \xi_j$ then (5.7) reduces to

$$0 = -2[\alpha(\eta_j(X)\xi_i - \eta_i(X)\xi_j) + 2(\alpha - \delta)\varphi_k(X) + (\alpha - \delta)\eta_j(X)\xi_i - (\alpha - \delta)\eta_i(X)\xi_j] \cdot \psi$$
$$+ [(-\alpha\xi_k + (\alpha - \delta)\xi_k)X - X(-\alpha\xi_k + (\alpha - \delta)\xi_k)] \cdot \psi$$
$$+ (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \left(\frac{\alpha}{2} \xi_j \cdot \psi + \frac{\alpha - \delta}{2} \Phi_j \cdot \psi \right)$$
$$+ (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \widehat{\nabla}_{\xi_j} \psi$$
$$= (\alpha - \delta)[-4\varphi_k(X) - 2\eta_j(X)\xi_i + 2\eta_i(X)\xi_j + \xi_k X - X\xi_k + \frac{1}{2}(-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \Phi_j] \cdot \psi$$
$$+ (-2\varphi_i(X) + \xi_i X - X\xi_i) \cdot \widehat{\nabla}_{\xi_j} \psi$$

(for the second equality we have again used (5.6) to eliminate some of the terms). The vanishing of the first term then follows immediately by substituting (5.4).

(iv) For the case $Y = \xi_k$, one performs an analogous calculation using (5.5).

□

5.2 Curvature and Torsion Identities for the Canonical Connection

In this section we derive some curvature and torsion identities which will be useful later. We denote by ∇^g, ∇, $\widehat{\nabla}$ the Levi-Civita, canonical, and modified connections, and by R^g, R, $\widehat{R}$ their respective curvature operators. For the purposes of our calculations, it will be convenient to work in an adapted frame (in the sense of Definition 2.3.6), and to define the constant $\beta := 2(\delta - 2\alpha)$.

We begin by proving a curvature identity for the canonical connection:

Proposition 5.2.1. *If $(M, g, \xi_i, \eta_i, \varphi_i)$ is a 3-(α, δ)-Sasaki manifold and $e_1, \ldots, e_{4n-1}$ an adapted orthonormal (local) frame, then*

$$\sum_{s=1}^{4n-1} R(X, Y, e_s, \varphi_i(e_s)) = \begin{cases} 4n\alpha\beta\Phi_i(X,Y) & X, Y \in \mathcal{H}, \\ 0 & X \in \mathcal{V}, Y \in \mathcal{H} \text{ or } X \in \mathcal{H}, Y \in \mathcal{V}, \\ 8n\alpha\beta\Phi_i(X,Y) & X, Y \in \mathcal{V}. \end{cases}$$

Proof. We consider the cases one at a time:

(i) Suppose first that $X, Y \in \mathcal{H}$. For horizontal $e_s \in \mathcal{H}$ and any even permutation (i, j, k) of $(1, 2, 3)$, it follows from [ADS23, Eqn. (2.6)] that

$$\begin{aligned} 2\alpha\beta\Phi_i(X,Y) &= R(X, Y, e_s, \varphi_i(e_s)) + R(X, Y, \varphi_j(e_s), \varphi_k(e_s)) \\ &= R(X, Y, e_s, \varphi_i(e_s)) + R(X, Y, \varphi_j(e_s), \varphi_i(\varphi_j(e_s))), \end{aligned}$$

and taking the sum over all $e_s \in \mathcal{H}$ then gives

$$2\sum_{s=4}^{4n-1} R(X, Y, e_s, \varphi_i(e_s)) = 2(4n-4)\alpha\beta\Phi_i(X,Y) \tag{5.8}$$

(since $\varphi_j(e_s)$ runs through the list $\pm e_4, \ldots, \pm e_{4n-1}$ as s runs through $4, \ldots, 4n-1$). In the vertical directions, it follows from [ADS23, Eqns. (1.2), (2.5)] that $R(X, Y, \xi_j, \xi_k) = 2\alpha\beta\Phi_i(X,Y)$, and hence

$$\sum_{s=1}^{3} R(X, Y, e_s, \varphi_i(e_s)) = 4\alpha\beta\Phi_i(X,Y). \tag{5.9}$$

Combining (5.8) and (5.9) then gives the result in this case:

$$\sum_{i=1}^{4n-1} R(X,Y,e_s,\varphi_i(e_s)) = [(4n-4)\alpha\beta + 4\alpha\beta]\Phi_i(X,Y) = 4n\alpha\beta\Phi_i(X,Y).$$

(ii) For the mixed cases, the first paragraph of [ADS23, Section 2.2] shows that $R(X,Y)$ is the zero operator when $X \in \mathcal{H}, Y \in \mathcal{V}$ or $X \in \mathcal{V}$, $Y \in \mathcal{H}$.

(iii) Suppose now that $X, Y \in \mathcal{V}$, and without loss of generality write $X = \xi_p$, $Y = \xi_q$ for (p,q,r) an even permutation of $(1,2,3)$. Letting $e_s \in \mathcal{H}$, it follows from [ADS23, Eqns. (2.4), (2.5)] respectively that

$$R(\xi_p,\xi_q,\xi_j,\xi_k) = -4\alpha\beta(\delta_{p,j}\delta_{q,k} - \delta_{p,k}\delta_{q,j}) = -4\alpha\beta(\eta_p \wedge \eta_q)(\xi_j,\xi_k) = 4\alpha\beta\Phi_r(\xi_j,\xi_k),$$
$$R(\xi_p,\xi_q,e_s,\varphi_i e_s) = 2\alpha\beta\Phi_r(e_s,\varphi_i e_s),$$

and combining these gives

$$\begin{aligned}\sum_{s=1}^{4n-1} R(X,Y,e_s,\varphi_i e_s) &= \sum_{s=1}^{4n-1} R(\xi_p,\xi_q,e_s,\varphi_i e_s) = 2[4\alpha\beta\Phi_r(\xi_j,\xi_k)] + 2\alpha\beta\sum_{s=4}^{4n-1}\Phi_r(e_s,\varphi_i e_s)\\ &= 8\alpha\beta\Phi_r(\xi_j,\xi_k) - 2\alpha\beta(4n-4)\delta_{i,r} = -8\alpha\beta\delta_{i,r} - 8(n-1)\alpha\beta\delta_{i,r}\\ &= -8n\alpha\beta\delta_{i,r} = 8n\alpha\beta\Phi_i(\xi_p,\xi_q) = 8n\alpha\beta\Phi_i(X,Y),\end{aligned}$$

where we have calculated using an even permutation (i,j,k) of $(1,2,3)$.

□

In order to re-translate the preceding curvature identity back in terms of the Levi-Civita connection, it is necessary to prove several identities involving the canonical torsion:

Proposition 5.2.2. *If $(M,g,\xi_i,\eta_i,\varphi_i)$ is a 3-(α,δ)-Sasaki manifold and $e_1,\dots,e_{4n-1}$ an adapted orthonormal (local) frame, then*

$$\sum_{s=1}^{4n-1} g(T(X,Y),T(e_s,\varphi_i(e_s))) = \begin{cases} \{-16(n-1)\alpha^2 + 8\alpha(\delta-4\alpha)\}\Phi_i(X,Y) & X,Y \in \mathcal{H},\\ 0 & X \in \mathcal{H}, Y \in \mathcal{V},\\ 0 & X \in \mathcal{V}, Y \in \mathcal{H},\\ 8(\delta-4\alpha)(2(n+1)\alpha-\delta)\Phi_i(X,Y) & X,Y \in \mathcal{V},\end{cases}$$

and

$$\sum_{s=1}^{4n-1} dT(X,Y,e_s,\varphi_i e_s) = \begin{cases} \{-16\alpha^2(2n-1)+8\alpha\beta\}\Phi_i(X,Y) & X,Y\in\mathcal{H}, \\ 0 & X\in\mathcal{H}, Y\in\mathcal{V}, \\ 0 & X\in\mathcal{V}, Y\in\mathcal{H}, \\ 32(n-1)\alpha(\delta-2\alpha)\Phi_i(X,Y) & X,Y\in\mathcal{V}. \end{cases}$$

Proof. From [ADS23, Eqns. (1.10), (1.11), (1.12)], the canonical torsion and its exterior derivative satisfy

$$T(\xi_j,\xi_k) = 2(\delta-4\alpha)\xi_i \tag{5.10}$$

$$T(X,Y) = 2\alpha\sum_{p=1}^{3}[\eta_p(Y)\varphi_p(X)-\eta_p(X)\varphi_p(Y)+\Phi_p(X,Y)\xi_p] - 2(\alpha-\delta)\mathfrak{S}_{i,j,k}\eta_{ij}(X,Y)\xi_k, \tag{5.11}$$

$$dT = 4\alpha^2\sum_{p=1}^{3}\Phi_p|_{\mathcal{H}}\wedge\Phi_p|_{\mathcal{H}} + 8\alpha(\delta-2\alpha)\mathfrak{S}_{i,j,k}\Phi_i|_{\mathcal{H}}\wedge\eta_{jk}. \tag{5.12}$$

We treat the cases one at a time:

(i) Suppose first that $X,Y\in\mathcal{H}$. If $e_s\in\mathcal{H}$, then substituting (5.10) and (5.11) and gives

$$g(T(X,Y),T(\xi_j,\xi_k)) = g\left(2\alpha\sum_{p=1}^{3}\Phi_p(X,Y)\xi_p,\ 2(\delta-4\alpha)\xi_i\right) = 4\alpha(\delta-4\alpha)\Phi_i(X,Y),$$

$$g(T(X,Y),T(e_s,\varphi_i e_s)) = g\left(2\alpha\sum_{p=1}^{3}\Phi_p(X,Y)\xi_p,\ 2\alpha\sum_{l=1}^{3}\Phi_l(e_s,\varphi_i e_s)\xi_l\right) = -4\alpha^2\Phi_i(X,Y),$$

and hence

$$\begin{aligned}\sum_{s=1}^{4n-1} g(T(X,Y),T(e_s,\varphi_i e_s)) &= (4n-4)[-4\alpha^2\Phi_i(X,Y)] + 2[4\alpha(\delta-4\alpha)\Phi_i(X,Y)] \\ &= \{-16(n-1)\alpha^2 + 8\alpha(\delta-4\alpha)\}\Phi_i(X,Y).\end{aligned}$$

The formula

$$\sum_{s=1}^{4n-1} dT(X,Y,e_s,\varphi_i e_s) = \{-16\alpha^2(2n-1)+8\alpha\beta\}\Phi_i(X,Y)$$

appears as [ADS23, Eqn. (3.4)] (note that we are working in dimension $4n-1$ rather than $4n+3$).

(ii) Suppose that $X \in \mathcal{H}$, $Y \in \mathcal{V}$ or $X \in \mathcal{V}$, $Y \in \mathcal{H}$. If $e_s \in \mathcal{H}$, then substituting (5.10) and (5.11) gives

$$\begin{aligned} g(T(X,Y),T(\xi_j,\xi_k)) &= g\left(2\alpha\sum_{p=1}^{3}[\eta_p(Y)\varphi_p X - \eta_p(X)\varphi_p Y],\ 2(\delta-4\alpha)\xi_i\right) \\ &= 4\alpha(\delta-4\alpha)\sum_{p=1}^{3}[\eta_p(Y)g(\varphi_p X,\xi_i) - \eta_p(X)g(\varphi_p Y,\xi_i)] = 0 \end{aligned}$$

and

$$\begin{aligned} g(T(X,Y),T(e_s,\varphi_i e_s)) &= g\left(2\alpha\sum_{p=1}^{3}[\eta_p(Y)\varphi_p X - \eta_p(X)\varphi_p Y],\ 2\alpha\sum_{l=1}^{3}\Phi_l(e_s,\varphi_i e_s)\xi_l\right) \\ &= -4\alpha^2\sum_{p=1}^{3}[\eta_p(Y)g(\varphi_p X,\xi_i) - \eta_p(X)g(\varphi_p Y,\xi_i)] = 0. \end{aligned}$$

The formula (5.12) immediately implies that $dT(X,Y,e_s,\varphi_i e_s) = dT(X,Y,\xi_j,\xi_k) = 0$. Both of the desired formulas then follow by taking sums.

(iii) Suppose that $X, Y \in \mathcal{V}$. If $e_s \in \mathcal{H}$, then substituting (5.10) and (5.11) gives

$$\begin{aligned} g(T(X,Y),T(\xi_j,\xi_k)) &= 4\alpha(\delta-4\alpha)\sum_{p=1}^{3}[\eta_p(Y)g(\varphi_p X,\xi_i) - \eta_p(X)g(\varphi_p Y,\xi_i)] \\ &\quad + 4\alpha(\delta-4\alpha)\Phi_i(X,Y) - 4(\alpha-\delta)(\delta-4\alpha)\eta_{jk}(X,Y) \\ &= 4\alpha(\delta-4\alpha)[2\Phi_i(X,Y)] + 4\alpha(\delta-4\alpha)\Phi_i(X,Y) \\ &\quad + 4(\alpha-\delta)(\delta-4\alpha)\Phi_i(X,Y) \\ &= -4(\delta-4\alpha)^2\Phi_i(X,Y) \end{aligned}$$

and

$$\begin{aligned} g(T(X,Y),T(e_s,\varphi_i e_s)) &= g(T(X,Y),-2\alpha\xi_i) \\ &= -4\alpha^2\sum_{p=1}^{3}[\eta_p(Y)g(\varphi_p X,\xi_i) - \eta_p(X)g(\varphi_p Y,\xi_i)] \\ &\quad - 4\alpha^2\Phi_i(X,Y) + 4\alpha(\alpha-\delta)\eta_{jk}(X,Y) \\ &= -4\alpha^2[2\Phi_i(X,Y)] - 4\alpha^2\Phi_i(X,Y) - 4\alpha(\alpha-\delta)\Phi_i(X,Y) \\ &= [-16\alpha^2 + 4\alpha\delta]\Phi_i(X,Y), \end{aligned}$$

and it follows that

$$\sum_{s=1}^{4n-1} g(T(X,Y),T(e_s,\varphi_i e_s)) = (4n-4)[-16\alpha^2+4\alpha\delta]\Phi_i(X,Y)+2[-4(\delta-4\alpha)^2\Phi_i(X,Y)]$$
$$= 8(\delta-4\alpha)(2(n+1)\alpha-\delta)\Phi_i(X,Y).$$

On the other hand, if $e_s \in \mathcal{H}$, then (5.12) gives

$$dT(X,Y,e_s,\varphi_i e_s) = 8\alpha(\delta-2\alpha)\mathfrak{S}_{p,q,r}\Phi_p|_{\mathcal{H}}(e_s,\varphi_i e_s)\eta_{qr}(X,Y) = 8\alpha(\delta-2\alpha)\Phi_i(X,Y),$$
$$dT(X,Y,\xi_j,\xi_k) = 0,$$

and hence

$$\sum_{s=1}^{4n-1} dT(X,Y,e_s,\varphi_i e_s) = (4n-4)[8\alpha(\delta-2\alpha)\Phi_i(X,Y)]+2[0] = 32(n-1)\alpha(\delta-2\alpha)\Phi_i(X,Y).$$

□

5.3 Projection Identities and Flatness of the Modified Connection

In this section we show that the restriction of the modified connection $\widehat{\nabla}$ to $E := E_1^- + E_2^- + E_3^-$ (the non-direct sum) is flat.

Proposition 5.3.1. *If $(M,g,\xi_i,\eta_i,\varphi_i)$ is a 3-(α,δ)-Sasaki manifold and $\psi \in \Gamma(E_i^-)$, then the orthogonal projection onto E_i^- of the spinorial curvature $R^g(\cdot,\cdot)\psi$ associated to the Levi-Civita connection is given by*

$$\mathrm{pr}_{E_i^-} R^g(X,Y)\psi = \begin{cases} \{-2(n-1)\alpha(\alpha-\delta)+\frac{1}{2}\delta^2\}\Phi_i(X,Y)\ \xi_i\cdot\psi & X,Y\in\mathcal{V}, \\ \{(2n-1)\alpha\delta-(2n-\frac{3}{2})\alpha^2\}\Phi_i(X,Y)\ \xi_i\cdot\psi & X,Y\in\mathcal{H}, \\ 0 & X\in\mathcal{H},Y\in\mathcal{V} \text{ or } X\in\mathcal{V},Y\in\mathcal{H}. \end{cases}$$

Proof. From [ADS23, Section 1.2], the difference between R and R^g is expressed by

$$R^g(X,Y,Z,V) = R(X,Y,Z,V) - \frac{1}{4}g(T(X,Y),T(Z,V)) - \frac{1}{8}dT(X,Y,Z,V). \tag{5.13}$$

Letting $e_1,\dots,e_{4n-1}$ be an adapted frame, we recall from the proof of [FK90, Thm. 1] that if $\psi \in \Gamma(E_i^-)$ then $e_p\cdot e_q\cdot\psi$ is orthogonal to E_i^- unless $e_q = e_p$ or $\pm\varphi_i e_p$. Additionally, the defining relation for E_i^- implies that $e_s \cdot \varphi_i e_s \cdot \psi = \xi_i \cdot \psi$ (for $e_s \neq \xi_i$). Using (5.13), we have:

$$\begin{aligned}\mathrm{pr}_{E_i^-} R^g(X,Y)\psi &= \frac{1}{4}\sum_{s=1}^{4n-1} R^g(X,Y,e_s,\varphi_i(e_s))\; e_s \cdot \varphi_i(e_s) \cdot \psi = \frac{1}{4}\sum_{s=1}^{4n-1} R^g(X,Y,e_s,\varphi_i(e_s))\; \xi_i \cdot \psi \\ &= \frac{1}{4}\sum_{s=1}^{4n-1}\Big(R(X,Y,e_s,\varphi_i(e_s)) - \frac{1}{4}g(T(X,Y),T(e_s,\varphi_i(e_s))) - \frac{1}{8}dT(X,Y,e_s,\varphi_i(e_s))\Big)\,\xi_i \cdot \psi.\end{aligned}$$

The result then follows by substituting the expressions from Propositions 5.2.1 and 5.2.2. □

In order to prove that $\widehat{\nabla}$ is flat on E_i^-, we first compute the orthogonal projections of various quantities onto E_i^-:

Lemma 5.3.2. *Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-(α,δ)-Sasaki manifold. For any $\psi \in \Gamma(E_i^-)$ and any even permutation (p,q,r) of $(1,2,3)$, we have*

(i) $\mathrm{pr}_{E_i^-}(\Phi_p \cdot \psi) = -\delta_{i,p}(2n-1)\xi_i \cdot \psi,$

(ii) $\mathrm{pr}_{E_i^-}(\Phi_p \cdot \Phi_q \cdot \psi - \Phi_q \cdot \Phi_p \cdot \psi) = 2\delta_{i,r}(4n-3)\xi_i \cdot \psi,$

(iii) $\mathrm{pr}_{E_i^-}((\nabla^g_{\xi_p}\Phi_q)\cdot\psi - (\nabla^g_{\xi_q}\Phi_p)\cdot\psi) = \delta_{i,r}[-2\delta + 8(n-1)(\alpha-\delta)]\xi_i \cdot \psi.$

Proof. Letting $\psi \in \Gamma(E_i^-)$, we prove the three identities one at a time.

(i) This follows by writing $\Phi_p = -\frac{1}{2}\sum_{s=1}^{4n-1} e_s \wedge \varphi_p(e_s)$ in an adapted frame and using

$$\mathrm{proj}_{E_i^-}(e_s \cdot \varphi_p(e_s) \cdot \psi) = \delta_{i,p}\;\xi_i \cdot \psi$$

(see the proof of [FK90, Thm. 1]).

(ii) We use the relation $V \lrcorner \Phi_q = -\frac{1}{2}(V \cdot \Phi_q - \Phi_q \cdot V)$, which may be deduced by subtracting Equations (1.4) in Chapter 1.2 of [BFGK91]. Considering first the horizontal part of Φ_p, we calculate

$$\begin{aligned}\Phi_p|_{\mathcal{H}} \cdot \Phi_q \cdot \psi &= -\frac{1}{2}\sum_{s=4}^{4n-1} e_s \cdot \varphi_p(e_s) \cdot \Phi_q \cdot \psi = -\frac{1}{2}\sum_{s=4}^{4n-1} e_s \cdot [-2(\varphi_p(e_s)\lrcorner\Phi_q) + \Phi_q \cdot \varphi_p(e_s)] \cdot \psi \\ &= -\frac{1}{2}\sum_{s=4}^{4n-1}[-2e_s \cdot \varphi_r(e_s) + e_s \cdot \Phi_q \cdot \varphi_p(e_s)] \cdot \psi \\ &= -\frac{1}{2}\sum_{s=4}^{4n-1}[-2e_s \cdot \varphi_r(e_s) + (-2e_s\lrcorner\Phi_q + \Phi_q \cdot e_s) \cdot \varphi_p(e_s)] \cdot \psi \\ &= -2\Phi_r|_{\mathcal{H}} \cdot \psi + \Phi_q \cdot \Phi_p|_{\mathcal{H}} \cdot \psi + \sum_{s=4}^{4n-1} \varphi_p(e_s) \cdot \varphi_q(e_s) \cdot \psi,\end{aligned}$$

and similarly for the vertical part,

$$\begin{aligned}\Phi_p|_{\mathcal{V}} \cdot \Phi_q \cdot \psi &= -\xi_q \cdot \xi_r \cdot \Phi_q \cdot \psi = -\xi_q \cdot (-2\xi_r \lrcorner \Phi_q + \Phi_q \cdot \xi_r) \cdot \psi = -\xi_q \cdot (2\xi_p + \Phi_q \cdot \xi_r) \cdot \psi \\ &= -2\xi_q \cdot \xi_p \cdot \psi + (2\xi_q \lrcorner \Phi_q - \Phi_q \cdot \xi_q \cdot \xi_r) \cdot \psi \\ &= -2\Phi_r|_{\mathcal{V}} \cdot \psi + \Phi_q \cdot \Phi_p|_{\mathcal{V}} \cdot \psi,\end{aligned}$$

Adding the above two equations, we deduce:

$$(\Phi_p \cdot \Phi_q - \Phi_q \cdot \Phi_p) \cdot \psi = -2\Phi_r \cdot \psi + \sum_{s=4}^{4n-1} \varphi_p(e_s) \cdot \varphi_q(e_s) \cdot \psi,$$

and projecting onto E_i^- using part (i) of this lemma gives the result.

(iii) From Proposition 5.1.1 we calculate

$$\begin{aligned}(\nabla^g_{\xi_p} \varphi_q - \nabla^g_{\xi_q} \varphi_p) \cdot \psi &= 2(2\alpha - \delta)\eta_p \otimes \xi_q - 2(2\alpha - \delta)\eta_q \otimes \xi_p - 4(\alpha - \delta)\varphi_r \\ &= 2(2\alpha - \delta)\varphi_r|_{\mathcal{V}} - 4(\alpha - \delta)\varphi_r\end{aligned}$$

The result then follows by lowering indices and projecting onto E_i^- using part (i) of this lemma.

□

The final step in the proof of Theorem 5.0.1 is the following proposition:

Proposition 5.3.3. *The restriction of the connection* $\widehat{\nabla}$ *to* $E := E_1^- + E_2^- + E_3^-$ *is flat, i.e.* $\widehat{R}(\cdot,\cdot)\psi \equiv 0$ *for all* $\psi \in \Gamma(E)$.

Proof. Suppose that $\psi \in \Gamma(E_i^-)$. Using the definition

$$\widehat{\nabla}_X \psi = \nabla^g_X \psi - \frac{\alpha}{2} X \cdot \psi - \frac{\alpha - \delta}{2} \sum_{p=1}^{3} \eta_p(X)\Phi_p \cdot \psi,$$

we calculate, for any vector fields X, Y,

$$\begin{aligned}\widehat{\nabla}_X \widehat{\nabla}_Y \psi &= \widehat{\nabla}_X [\nabla^g_Y \psi - \frac{\alpha}{2} Y \cdot \psi - \frac{\alpha - \delta}{2} \sum_{p=1}^{3} \eta_p(Y)\Phi_p \cdot \psi] \\ &= \nabla^g_X \nabla^g_Y \psi - \frac{\alpha}{2} X \cdot \nabla^g_Y \psi - \frac{\alpha - \delta}{2} \sum_{p=1}^{3} [\eta_p(X)\Phi_p \cdot \nabla^g_Y \psi] - \frac{\alpha}{2} \nabla^g_X (Y \cdot \psi) + \frac{\alpha^2}{4} X \cdot Y \cdot \psi\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha(\alpha-\delta)}{4}\sum_{p=1}^{3}[\eta_p(X)\Phi_p\cdot Y\cdot\psi]-\frac{\alpha-\delta}{2}\sum_{p=1}^{3}\nabla^g_X[\eta_p(Y)\cdot\Phi_p\cdot\psi]\\
&+\frac{\alpha(\alpha-\delta)}{4}\sum_{p=1}^{3}[\eta_p(Y)X\cdot\Phi_p\cdot\psi]+\frac{(\alpha-\delta)^2}{4}\sum_{p,q=1}^{3}[\eta_p(Y)\eta_q(X)\Phi_q\cdot\Phi_p\cdot\psi]\\
=&\ \nabla^g_X\nabla^g_Y\psi-\frac{\alpha}{2}X\cdot\nabla^g_Y\psi-\frac{\alpha-\delta}{2}\sum_{p=1}^{3}[\eta_p(X)\Phi_p\cdot\nabla^g_Y\psi]-\frac{\alpha}{2}[(\nabla^g_XY)\cdot\psi+Y\cdot\nabla^g_X\psi]\\
&+\frac{\alpha^2}{4}X\cdot Y\cdot\psi+\frac{\alpha(\alpha-\delta)}{4}\sum_{p=1}^{3}[\eta_p(X)\Phi_p\cdot Y\cdot\psi]\\
&-\frac{\alpha-\delta}{2}\sum_{p=1}^{3}\{(\alpha\Phi_p(X,Y)+(\alpha-\delta)\eta_{p+1,p+2}(X,Y)+\eta_p(\nabla^g_XY))\Phi_p\cdot\psi+\eta_p(Y)(\nabla^g_X\Phi_p)\cdot\psi\\
&+\eta_p(Y)\Phi_p\cdot\nabla^g_X\psi\}+\frac{\alpha(\alpha-\delta)}{4}\sum_{p=1}^{3}[\eta_p(Y)X\cdot\Phi_p\cdot\psi]+\frac{(\alpha-\delta)^2}{4}\sum_{p,q=1}^{3}[\eta_p(Y)\eta_q(X)\Phi_q\cdot\Phi_p\cdot\psi],
\end{aligned}
$$

and hence

$$
\begin{aligned}
\widehat{R}(X,Y)\psi&=\widehat{\nabla}_X\widehat{\nabla}_Y\psi-\widehat{\nabla}_Y\widehat{\nabla}_X\psi-\widehat{\nabla}_{[X,Y]}\psi\\
&=R^g(X,Y)\psi+\frac{\alpha^2}{4}(X\cdot Y-Y\cdot X)\cdot\psi+\frac{\alpha(\alpha-\delta)}{2}\sum_{p=1}^{3}[\eta_p(X)(Y\lrcorner\Phi_p)-\eta_p(Y)(X\lrcorner\Phi_p)]\cdot\psi\\
&-(\alpha-\delta)\sum_{p=1}^{3}[\alpha\Phi_p(X,Y)+(\alpha-\delta)\eta_{p+1,p+2}(X,Y)]\Phi_p\cdot\psi\\
&-\frac{\alpha-\delta}{2}\sum_{p=1}^{3}[\eta_p(Y)(\nabla^g_X\Phi_p)-\eta_p(X)(\nabla^g_Y\Phi_p)]\cdot\psi\\
&+\frac{(\alpha-\delta)^2}{4}\sum_{p,q=1}^{3}[\eta_p(Y)\eta_q(X)-\eta_p(X)\eta_q(Y)]\cdot\Phi_q\cdot\Phi_p\cdot\psi,
\end{aligned}
\tag{5.14}
$$

where the indices $p, p+1, p+2$ are taken modulo 3. The result then follows by considering the various cases of X, Y being in $\mathcal{H}, \mathcal{V}$ and projecting onto E_i^-, using the formulas from Proposition 5.3.1 and Lemma 5.3.2. □

Finally, we calculate the action of the (Riemannian) Dirac operator on deformed Killing spinors:

Remark 5.3.4. If $\psi \in \Gamma(E)$ is a spinor with $\widehat{\nabla}\psi = 0$ then the Dirac operator acts on it via

$$
\begin{aligned}
D\psi&=\sum_{s=1}^{4n-1}e_s\cdot\nabla^g_{e_s}\psi=\sum_{s=1}^{4n-1}e_s\cdot\left(\frac{\alpha}{2}e_s\cdot\psi+\frac{\alpha-\delta}{2}\sum_{p=1}^{3}\eta_p(e_s)\Phi_p\cdot\psi\right)\\
&=-\frac{(4n-1)\alpha}{2}\psi+\frac{\alpha-\delta}{2}\sum_{p=1}^{3}\xi_p\cdot\Phi_p\cdot\psi.
\end{aligned}
$$

6 Spinorial Duality for Riemannian Homogeneous Spaces Fibering Over a Symmetric Base

This chapter contains joint work with Prof. Dr. habil. Ilka Agricola (see page 8).

6.1 Duality of Extended Symmetric Data

Let us begin by defining certain Lie algebraic data generalizing the 3-Sasakian data recalled in Theorem 2.4.2 (and the generalized 3-Sasakian data recalled in Theorem 2.5.7).

Definition 6.1.1. *Extended symmetric data* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, g)$ consists of a triple of real Lie algebras with $\mathfrak{h}, \mathfrak{k} \subset \mathfrak{g}$, together with an inner product g on $\mathfrak{g}/\mathfrak{h}$, such that the following properties hold:

(i) The Lie algebra $\mathfrak{g}$ is semi-simple;

(ii) There is a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{k}$;

(iii) Under the natural identification $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{k} \oplus \mathfrak{g}_1$, the inner product g takes the form

$$g = \lambda_0 \kappa_{\mathfrak{g}}|_{\mathfrak{k}\times\mathfrak{k}} + \lambda_1 \kappa_{\mathfrak{g}}|_{\mathfrak{g}_1\times\mathfrak{g}_1}, \qquad \lambda_0, \lambda_1 \in \mathbb{R} \setminus \{0\}, \tag{6.1}$$

where $\kappa_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$.

The idea behind the preceding definition is that the Lie algebras $(\mathfrak{g}, \mathfrak{g}_0)$ constitute a Riemannian symmetric pair, thus the pair $(\mathfrak{g}, \mathfrak{h})$ can be viewed as the Lie algebraic data of a homogeneous space fibering over a symmetric base. Indeed, the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with the fact that $\mathfrak{k}$ and $\mathfrak{h}$ commute, gives the following commutator relations:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{h}, \mathfrak{k}] = 0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{h} \oplus \mathfrak{k}. \tag{6.2}$$

In particular one sees that $(\mathfrak{g}, \mathfrak{g}_0)$ satisfy the conditions of a Riemannian symmetric pair, as desired. The cases of most interest for us, 3-Sasakian data and generalized 3-Sasakian data, correspond to extended symmetric data with $\mathfrak{k} = \mathfrak{sp}(1)$ satisfying the additional condition Theorem 2.4.2(iii) (or equivalently, Theorem 2.5.7(ii)).

Remark 6.1.2. Associated to extended symmetric data $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, g)$ is the vector space

$$\mathfrak{m} := \mathfrak{k} \oplus \mathfrak{g}_1$$

serving as a reductive complement to $\mathfrak{h} \subseteq \mathfrak{g}$. Conversely, the Lie algebra $\mathfrak{k}$ may be recovered from $\mathfrak{m}$ via $\mathfrak{k} = \mathfrak{m} \cap \mathfrak{g}_0$. For this reason we shall also refer to $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, g)$ as extended symmetric data. We shall also denote by $\mathfrak{m}_i := \mathfrak{m} \cap \mathfrak{g}_i$, $i = 0, 1$ the components of $\mathfrak{m}$ with respect to the $\mathbb{Z}_2$-grading on $\mathfrak{g}$.

With the preceding remark in mind, we are ready to define our notion of duality at the Lie algebra level:

Definition 6.1.3. Given extended symmetric data $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, g)$, we define (for the same $\mathfrak{k}$) the dual extended symmetric data $(\mathfrak{g}', \mathfrak{h}, \mathfrak{m}', g')$ by setting

$$\mathfrak{g}' := \mathfrak{g}_0 \oplus i\mathfrak{g}_1 \subset \mathfrak{g}^{\mathbb{C}},$$
$$\mathfrak{m}' := \mathfrak{k} \oplus i\mathfrak{g}_1 \subset \mathfrak{g}',$$

and taking g' to be the real inner product induced on $\mathfrak{m}'$ by extending g sesquilinearly to $\mathfrak{g}^{\mathbb{C}}$ and restricting to the real form $\mathfrak{g}' \subset \mathfrak{g}^{\mathbb{C}}$.

In the preceding definition, the extension of g to the complexification is done sesquilinearly to ensure that g' is positive-definite (analogously to the duality for symmetric spaces). Let us briefly compare Definition 6.1.3 with Kath's notion of duality in [Kat00]. Apart from the fact that Kath's duality is between Riemannian and pseudo-Riemannian spaces, we note that it also depends on a Lie algebra involution T, whose 1-eigenspace (resp. (-1)-eigenspace) indicates which tangent directions on the compact side should correspond to tangent directions on the non-compact side with positive norm squared (resp. negative norm squared). We do not make use of such an involution, as the decomposition $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{g}_1$ automatically keeps track of which directions are to be modified. Rather, our duality construction is obtained from the duality between compact and non-compact symmetric spaces, by dualizing the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$.

Definition 6.1.4. Let K be a connected Lie group with Lie algebra $\mathfrak{k}$. An *extended symmetric space* (relative to K) is a connected reductive homogeneous space $(M := G/H, g)$ with Lie algebra decomposition as in Definition 6.1.1. Letting G' be the connected subgroup of $G^{\mathbb{C}}$ corresponding to the real Lie subalgebra $\mathfrak{g}'$, we define the dual of (M, g) to be $(M' := G'/H, g')$.

In order to investigate the spinorial properties of the dual, we must first describe the special orthogonal group $\mathrm{SO}(\mathfrak{m}', g')$ of the Riemannian metric g'. We define

$$\mathfrak{so}(\mathfrak{m})_0 := \{A \in \mathfrak{so}(\mathfrak{m}) : A(\mathfrak{k}) \subseteq \mathfrak{k} \text{ and } A(\mathfrak{m}_1) \subseteq \mathfrak{m}_1\},$$
$$\mathfrak{so}(\mathfrak{m})_1 := \{B \in \mathfrak{so}(\mathfrak{m}) : B(\mathfrak{k}) \subseteq \mathfrak{m}_1 \text{ and } B(\mathfrak{m}_1) \subseteq \mathfrak{k}\},$$

and the non-standard Lie bracket $[[\cdot,\cdot]]$ on $\mathfrak{so}(\mathfrak{m})_0 \oplus i\mathfrak{so}(\mathfrak{m})_1$ given by

$$[[A_1, A_2]] := [A_1, A_2]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}}, \quad [[A, iB]] := i[A, B]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}}, \quad [[iB_1, iB_2]] := [B_1, B_2]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}},$$

where $[\ ,\ ]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}}$ denotes the usual commutator in $\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}$. It is clear that the bracket $[[\cdot,\cdot]]$ is constructed so that $\mathfrak{so}(\mathfrak{m})_0 \oplus i\mathfrak{so}(\mathfrak{m})_1$ has the same commutators as $\mathfrak{so}(\mathfrak{m}) = \mathfrak{so}(\mathfrak{m})_0 \oplus \mathfrak{so}(\mathfrak{m})_1$. The following two lemmas and the subsequent proposition are analogous to [Kat00, Props. 6.1, 3.1, 4.1]:

Lemma 6.1.5. *Let* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, g)$ *be extended symmetric data, and* $(\mathfrak{g}', \mathfrak{h}, \mathfrak{m}', g')$ *the dual data. The map* $\tau\colon \mathfrak{so}(\mathfrak{m})_0 \oplus i\mathfrak{so}(\mathfrak{m})_1 \to \mathfrak{so}(\mathfrak{m}', g')$ *given by*

$$\tau(A)(x) = A(x), \quad \tau(A)(iy) = iA(y), \quad \tau(iB)(x) = iB(x), \quad \tau(iB)(iy) = B(y),$$

for all $A \in \mathfrak{so}(\mathfrak{m})_0$, $B \in \mathfrak{so}(\mathfrak{m})_1$, $x \in \mathfrak{k}$, $y \in \mathfrak{m}_1$ *is an isomorphism of Lie algebras.*

Proof. Let $x_1, x_2 \in \mathfrak{k}$, $y_1, y_2 \in \mathfrak{m}_1$, $A \in \mathfrak{so}(\mathfrak{m})_0$, and $B \in \mathfrak{so}(\mathfrak{m})_1$. Using the definitions of $\mathfrak{so}(\mathfrak{m})_0$, $\mathfrak{so}(\mathfrak{m})_1$, and g', we calculate

$$\begin{aligned}
g'(\tau(A+iB)x_1, x_2) + g'(x_1, \tau(A+iB)x_2) &= g'(Ax_1 + iBx_1, x_2) + g'(x_1, Ax_2 + iBx_2) \\
&= g(Ax_1, x_2) + g(x_1, Ax_2) = 0, \\
g'(\tau(A+iB)x_1, iy_1) + g'(x_1, \tau(A+iB)(iy_1)) &= g'(Ax_1 + iBx_1, iy_1) + g'(x_1, iAy_1 + By_1) \\
&= g(Bx_1, y_1) + g(x_1, By_1) = 0, \\
g'(\tau(A+iB)(iy_1), iy_2) + g'(iy_1, \tau(A+iB)(iy_2)) &= g'(iAy_1 + By_1, iy_2) + g'(iy_1, iAy_2 + By_2) \\
&= g(Ay_1, y_2) + g(y_1, Ay_2) = 0,
\end{aligned}$$

hence $\tau(A+iB) \in \mathfrak{so}(\mathfrak{m}', g')$. The map τ is a linear isomorphism, so it remains only to check that it is a Lie algebra homomorphism. One the one hand, we calculate

$$\tau[[A+iB, C+iD]] = \tau([A,C]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + [B,D]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + i[B,C]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + i[A,D]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}}),$$

and on the other hand,

$$\begin{aligned}
[\tau(A+iB), \tau(C+iD)]_{\mathfrak{so}(\mathfrak{m}',g')}(x) &= (\tau(A+iB)\circ\tau(C+iD))(x) - (\tau(C+iD)\circ\tau(A+iB))(x) \\
&= \tau(A+iB)(Cx+iDx) - \tau(C+iD)(Ax+iBx) \\
&= ACx + iADx + iBCx + BDx - (CAx + iCBx + iDAx + DBx) \\
&= \tau([A,C]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + i[A,D]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + i[B,C]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + [B,D]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}})(x)
\end{aligned}$$

in the $\mathfrak{k}$ directions, and

$$\begin{aligned}
[\tau(A+iB), \tau(C+iD)]_{\mathfrak{so}(\mathfrak{m}',g')}(iy) &= (\tau(A+iB)\circ\tau(C+iD))(iy) - (\tau(C+iD)\circ\tau(A+iB))(iy) \\
&= \tau(A+iB)(iCy+Dy) - \tau(C+iD)(iAy+By) \\
&= iACy + ADy + BCy + iBDy - (iCAy + CBy + DAy + iDBy) \\
&= \tau([A,C]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + i[A,D]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + i[B,C]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}} + [B,D]_{\mathfrak{so}(\mathfrak{m})^{\mathbb{C}}})(iy)
\end{aligned}$$

in the $\mathfrak{m}_1$ directions, completing the proof. □

Lemma 6.1.6. *Let $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, g)$ be extended symmetric data. If $x \in \mathfrak{k}$ and $y \in \mathfrak{m}_1$ then*

$$\mathrm{ad}(x) \in \mathfrak{so}(\mathfrak{m})_0, \quad \mathrm{proj}_{\mathfrak{m}} \circ \mathrm{ad}(y) \in \mathfrak{so}(\mathfrak{m})_1, \quad \Lambda^g(x) \in \mathfrak{so}(\mathfrak{m})_0, \quad \Lambda^g(y) \in \mathfrak{so}(\mathfrak{m})_1.$$

Proof. The first two items follow from the commutator relations (6.2). For the second two items, we use the first two together with the standard implicit formula for the Nomizu map (2.15). Indeed, letting $x, x_1, x_2, x_3 \in \mathfrak{k}$ and $y, y_1, y_2, y_3 \in \mathfrak{m}_1$ and using (2.16), (6.1), and (6.2), we calculate

$$\begin{aligned}
2g(U(x_1,x_2),y) &= g([y,x_1],x_2) + g(x_1,[y,x_2]) = 0+0 = 0, \\
2g(U(x_1,y),x_2) &= g([x_2,x_1],y) + g(x_1,[x_2,y]) = 0+0 = 0, \\
2g(U(y,x_1),x_2) &= g([x_2,y],x_1) + g(y,[x_2,x_1]) = 0+0 = 0, \\
2g(U(y_1,y_2),y_3) &= g(\mathrm{proj}_{\mathfrak{m}}[y_3,y_1],y_2) + g(y_1,\mathrm{proj}_{\mathfrak{m}}[y_3,y_2]) = 0+0 = 0.
\end{aligned}$$

Thus we have proved $(x\lrcorner U) \in \mathfrak{so}(\mathfrak{m})_0$ and $(y\lrcorner U) \in \mathfrak{so}(\mathfrak{m})_1$, and the result follows. □

Proposition 6.1.7. *Let (M, g) be an extended symmetric space, and (M', g') its dual. In terms of the identification τ, the Levi-Civita connection of (M', g') has Nomizu map given by*

$$\Lambda^{g'}(iy) = -\tau(i\Lambda^g(y)), \quad \Lambda^{g'}(x_1)x_2 = \tau(\Lambda^g(x_1))x_2, \quad \Lambda^{g'}(x)(iy) = -\tau(\Lambda^g(x))(iy) + 2i[x, y].$$

for $x \in \mathfrak{k}$, $y \in \mathfrak{m}_1$.

Proof. In order to show that the above expression for $\Lambda^{g'}$ induces a metric connection, it suffices to check that $\Lambda^{g'}(x)$ is skew-symmetric with respect to g'. Using (6.1), (6.2), Lemma 6.1.6, and the fact that the image of τ lies in $\mathfrak{so}(\mathfrak{m}', g')$, we calculate:

$$\begin{aligned}
&g'(\Lambda^{g'}(x_1)x_2, x_3) + g'(x_2, \Lambda^{g'}(x_1)x_3) = g'(\tau(\Lambda^g(x_1))x_2, x_3) + g'(x_2, \tau(\Lambda^g(x_1))x_3) = 0,\\
&g'(\Lambda^{g'}(x_1)x_2, iy) + g'(x_2, \Lambda^{g'}(x_1)(iy)) = g'(\tau(\Lambda^g(x_1))x_2, iy) + g'(x_2, -\tau(\Lambda^g(x_1))(iy) + 2i[x_1, y]) = 0,\\
&g'(\Lambda^{g'}(x)(iy_1), iy_2) + g'(iy_1, \Lambda^{g'}(x)(iy_2)) = g'(-\tau(\Lambda^g(x))(iy_1) + 2i[x, y_1], iy_2)\\
&\qquad + g'(iy_1, -\tau(\Lambda^g(x))(iy_2) + 2i[x, y_2]) = 2\lambda_1\kappa_{\mathfrak{g}}([x, y_1], y_2) + 2\lambda_1\kappa_{\mathfrak{g}}(y_1, [x, y_2]) = 0.
\end{aligned}$$

To see that the given expression for $\Lambda^{g'}$ is torsion-free, we use the fact that Λ^g is torsion-free to calculate:

$$\begin{aligned}
\Lambda^{g'}(x_1)x_2 - \Lambda^{g'}(x_2)x_1 - \mathrm{proj}_{\mathfrak{m}'}[x_1, x_2] &= \tau(\Lambda^g(x_1))x_2 - \tau(\Lambda^g(x_2))x_1 - [x_1, x_2]\\
&= \Lambda^g(x_1)x_2 - \Lambda^g(x_2)x_1 - [x_1, x_2] = 0,\\
\Lambda^{g'}(x)(iy) - \Lambda^{g'}(iy)(x) - \mathrm{proj}_{\mathfrak{m}'}[x, iy] &= -\tau(\Lambda^g(x))(iy) + 2i[x, y] + \tau(i\Lambda^g(y))(x) - [x, iy]\\
&= -i\Lambda^g(x)y + i[x, y] + i\Lambda^g(y)x = 0,\\
\Lambda^{g'}(iy_1)(iy_2) - \Lambda^{g'}(iy_2)(iy_1) - \mathrm{proj}_{\mathfrak{m}'}[iy_1, iy_2] &= -\tau(i\Lambda^g(y_1))(iy_2) + \tau(i\Lambda^g(y_2))(iy_1) + \mathrm{proj}_{\mathfrak{m}}[y_1, y_2]\\
&= -\Lambda^g(y_1)y_2 + \Lambda^g(y_2)y_1 + \mathrm{proj}_{\mathfrak{m}}[y_1, y_2] = 0.
\end{aligned}$$

The result then follows from the fact that the Levi-Civita connection is the unique torsion-free metric connection. □

Finally we turn our attention to the spinorial properties of the dual pairs. Inspired by [Kat00, Prop. 7.2], we have:

Proposition 6.1.8. *If $(M = G/H, g)$ and $(M' = G'/H, g')$ are a dual pair of extended symmetric spaces, then M admits a homogeneous spin structure if and only if M' admits one.*

Proof. Both directions of the 'if and only if' statement are identical, so we prove only the forward direction here. Supposing that $(M = G/H, g)$ admits a homogeneous spin structure, it follows from [DKL22, Prop. 1.3] that there is a lift $\widetilde{\mathrm{Ad}}\colon H \to \mathrm{Spin}(\mathfrak{m})$ of the isotropy representation $\mathrm{Ad}\colon H \to \mathrm{SO}(\mathfrak{m})$, and it suffices to show that $\mathrm{Ad}'\colon H \to \mathrm{SO}(\mathfrak{m}')$ lifts to a map $\widetilde{\mathrm{Ad}'}\colon H \to \mathrm{Spin}(\mathfrak{m}')$. If $\{v_1, \ldots, v_k, w_1, \ldots, w_l\}$ is a g-orthonormal basis for $\mathfrak{m}$ such that $\{v_t\}_{t=1}^k$ is a basis for $\mathfrak{k}$ and $\{w_t\}_{t=1}^l$ is a basis for $\mathfrak{m}_1$, then $\{v_1, \ldots, v_k, iw_1, \ldots, iw_l\}$ is a g'-orthonormal basis for $\mathfrak{m}'$. The identification $v_t \mapsto v_t$ and $w_t \mapsto iw_t$ is an H-equivariant isometry $\sigma\colon (\mathfrak{m}, g) \to (\mathfrak{m}', g')$, and we have $\mathrm{Ad}'(h) = \sigma \circ \mathrm{Ad}(h) \circ \sigma^{-1}$ for all $h \in H$. Noting that the isometry σ naturally extends to an isomorphism $\sigma\colon \mathrm{Spin}(\mathfrak{m}) \to \mathrm{Spin}(\mathfrak{m}')$ (by viewing these inside the respective Clifford algebras), we claim that the desired lift is given by $\widetilde{\mathrm{Ad}'}(h) := \sigma(\widetilde{\mathrm{Ad}}(h))$ for all $h \in H$. Denoting by λ (resp. λ') the covering map $\mathrm{Spin}(\mathfrak{m}) \to \mathrm{SO}(\mathfrak{m})$ (resp. $\mathrm{Spin}(\mathfrak{m}') \to \mathrm{SO}(\mathfrak{m}')$), this follows from the calculation

$$\begin{aligned}\lambda'(\widetilde{\mathrm{Ad}'}(h))(v) &= \lambda'(\sigma(\widetilde{\mathrm{Ad}}(h)))(v) = \sigma(\widetilde{\mathrm{Ad}}(h)) \cdot v \cdot \sigma(\widetilde{\mathrm{Ad}}(h))^{-1} = \sigma(\widetilde{\mathrm{Ad}}(h) \cdot \sigma^{-1}(v) \cdot \widetilde{\mathrm{Ad}}(h)^{-1}) \\ &= \sigma\big(\lambda(\widetilde{\mathrm{Ad}}(h))(\sigma^{-1}(v))\big) = \sigma\big(\mathrm{Ad}(h)(\sigma^{-1}(v))\big) = \mathrm{Ad}'(h)(v)\end{aligned}$$

for all $v \in \mathfrak{m}'$. □

6.2 Homogeneous 3-(α, δ)-Sasaki Dual Pairs

From this point forward we restrict attention to the case of generalized 3-Sasakian data, corresponding to extended symmetric data with $\mathfrak{k} = \mathfrak{sp}(1)$ satisfying the additional condition (ii) in Theorem 2.5.7.

Definition 6.2.1. The dual of a 3-(α, δ)-Sasaki homogeneous space $(M = G/H, g, \xi_i, \eta_i, \varphi_i)$ is the 3-(α', δ')-Sasaki space obtained by applying the duality construction to the corresponding generalized 3-Sasakian data, where $\alpha' := \alpha$ and $\delta' := -\delta$.

Remark 6.2.2. For a 3-(α, δ)-Sasaki homogeneous space $(M = G/H, g)$ fibering over a Wolf space, it is easy to check that Proposition 6.1.7 is compatible with the explicit expression for the Nomizu map of the Levi-Civita connection recalled in Theorem 2.5.7:

$$\begin{aligned}\Lambda^{g'}(x_1)(x_2) &= \tau(\Lambda^g(x_1))x_2 = \Lambda^g(x_1)x_2 = \frac{1}{2}[x_1, x_2], \\ \Lambda^{g'}(x)(iy) &= -\tau(\Lambda^g(x))(iy) + 2i[x, y] = -i\Lambda^g(x)y + 2i[x, y] \\ &= -i(1 - \frac{\alpha}{\delta})[x, y] + 2i[x, y] = (1 - \frac{\alpha'}{\delta'})[x, iy],\end{aligned}$$

$$\Lambda^{g'}(iy)(x) = -\tau(i\Lambda^g(y))x = -i\Lambda^g(y)x = -i\frac{\alpha}{\delta}[y,x] = \frac{\alpha'}{\delta'}[iy,x],$$
$$\Lambda^{g'}(iy_1)(iy_2) = -\tau(i\Lambda^g(y_1))(iy_2) = -\Lambda^g(y_1)(y_2) = -\frac{1}{2}\mathrm{proj}_{\mathfrak{sp}(1)}[y_1,y_2] = \frac{1}{2}\mathrm{proj}_{\mathfrak{sp}(1)}[iy_1,iy_2].$$

We can also find a relationship between the canonical connections of a 3-(α,δ)-Sasaki dual pair:

Proposition 6.2.3. *If (M,g) and (M',g') are a dual pair of homogeneous 3-(α,δ)-Sasaki spaces then the canonical connections $\Lambda\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ and $\Lambda'\colon \mathfrak{m}'\times\mathfrak{m}'\to\mathfrak{m}'$ are related by*

$$\Lambda'(V)W = \begin{cases} \tau(\Lambda(V))W - \frac{4\alpha'}{\delta'}[V,W] & V\in\mathfrak{sp}(1),\\ 0 & V\in i\mathfrak{m}_1.\end{cases}$$

Proof. Noting that M and M' fiber over Wolf spaces, Proposition 2.5.8 gives

$$\Lambda(V)W = \begin{cases} \frac{\delta-2\alpha}{\delta}[V,W] & V\in\mathfrak{sp}(1),\\ 0 & V\in \mathfrak{m}_1,\end{cases} \qquad \Lambda'(V)W = \begin{cases} \frac{\delta'-2\alpha'}{\delta'}[V,W] & V\in\mathfrak{sp}(1),\\ 0 & V\in i\mathfrak{m}_1.\end{cases}$$

The result then follows by calculating, for $V\in\mathfrak{sp}(1)$:

$$\frac{\delta'-2\alpha'}{\delta'}[V,W] = \frac{\delta+2\alpha}{\delta}[V,W] = \frac{\delta-2\alpha}{\delta}[V,W] + \frac{4\alpha}{\delta}[V,W] = \tau(\Lambda(V))W - \frac{4\alpha'}{\delta'}[V,W]$$

□

Inspired by Theorem 7.2 in [Kat00], we have:

Theorem 6.2.4. *Let (M,g) and (M',g') be a dual pair of homogeneous 3-(α,δ)-Sasaki spaces of dimension $4n-1$, and identify the spinor modules $\Sigma\cong\Sigma'\cong\Lambda^\bullet\mathbb{C}^{2n-1}$. If $\psi\colon G\to\Sigma$, $\psi\equiv u$ is a constant H-equivariant map whose corresponding spinor satisfies the deformed Killing equation*

$$\nabla^g_X\psi = \frac{\alpha}{2}X\cdot\psi + \frac{\alpha-\delta}{2}\sum_{i=1}^{3}\eta_i(X)\Phi_i\cdot\psi, \tag{6.3}$$

then the corresponding constant map $\psi'\colon G'\to\Sigma'$, $\psi'\equiv u$ is also H-equivariant and induces a spinor satisfying

$$\nabla^{g'}_X\psi' = \frac{\alpha'}{2}X\cdot\psi' + \frac{\alpha'-\delta'}{2}\sum_{i=1}^{3}\eta'_i(X)\Phi'_i\cdot\psi', \tag{6.4}$$

where $\alpha' := \alpha$, $\delta' := -\delta$. Similarly, if $\psi\equiv u$ is a parallel spinor for the canonical connection

$(\nabla_X\psi = 0)$ *then* $\psi' \equiv u$ *satisfies*

$$\nabla'_X\psi' = \begin{cases} 2\alpha'\Phi'_i\cdot\psi' - 2\alpha'\xi'_j\cdot\xi'_k\cdot\psi' & X = \xi'_i \in \mathcal{V}', \\ 0 & X \in \mathcal{H}', \end{cases} \tag{6.5}$$

for any even permutation (i,j,k) *of* $(1,2,3)$.

Proof. We would like to define an isometry $\theta\colon (\mathfrak{m}, g) \to (\mathfrak{m}', g')$ in order to compare the two spaces. The obvious choice is the identification $x \mapsto x$, $y \mapsto iy$ used in the proof of Proposition 6.1.8, however this leads to a problem with the orientations of the two spaces. Indeed, using the notation of [ADS21, Thm. 3.1.1], the Reeb vector fields of the dual pair are related by

$$\xi'_i = \delta'\sigma_i = -\delta\sigma_i = -\xi_i,$$

so an orthonormal frame $\{\xi_1, \xi_2, \xi_3, e_4, \ldots, e_{4n-4}\}$ inside the standard orientation for (M, g) would be identified with the frame $\{-\xi'_1 - \xi'_2, -\xi'_3, ie_4, \ldots ie_{4n-4}\}$, which is *not* oriented in the standard way for the dual 3-(α', δ')-Sasaki space (M', g'). Thus we choose instead the isometry

$$\theta := -\operatorname{Id}|_{\mathfrak{sp}(1)} \oplus i\operatorname{Id}|_{\mathfrak{m}_1},$$

which identifies ξ_i with ξ'_i. Then, under identification of the spinor modules as above, each tangent vector $V \in \mathfrak{m}$ and its image $\theta(V) \in \mathfrak{m}'$ act as the same operator by Clifford multiplication (likewise for the extension of θ to tensors and differential forms). We recall from Theorem 2.5.7 that

$$\varphi'_i = \frac{1}{2\delta'}\operatorname{ad}(\xi'_i)|_{\mathfrak{sp}(1)} + \frac{1}{\delta'}\operatorname{ad}(\xi'_i)|_{i\mathfrak{m}_1} = \frac{1}{2\delta}\operatorname{ad}(\xi_i)|_{\mathfrak{sp}(1)} + \frac{1}{\delta}\operatorname{ad}(\xi_i)|_{i\mathfrak{m}_1},$$

and therefore $\Phi'_i = \theta(\Phi_i)$. In the horizontal directions, it follows from Lemma 6.1.6 and Proposition 6.1.7 above, together with the explicit formula for the Nomizu map recalled in Theorem 2.5.7, that

$$\begin{aligned}\theta^{-1}(\Lambda^{g'}(iy)) = \theta^{-1}(-\tau(i\Lambda^{g}(y))) &= \theta^{-1}(-\frac{\alpha}{\delta}\operatorname{ad}(iy)|_{\mathfrak{sp}(1)} + \frac{1}{2}\operatorname{proj}_{\mathfrak{sp}(1)}\operatorname{ad}(iy)|_{i\mathfrak{m}_1}) \\ &= \frac{\alpha}{\delta}\operatorname{ad}(y)|_{\mathfrak{sp}(1)} + \frac{1}{2}\operatorname{proj}_{\mathfrak{sp}(1)}\operatorname{ad}(y)|_{\mathfrak{m}_1} \\ &= \Lambda^{g}(y).\end{aligned}$$

Thus if $\psi \equiv u$ is an invariant spinor satisfying (6.3), then we have

$$\widetilde{\Lambda}^{g'}(iy) \cdot u = \theta(\widetilde{\Lambda}^{g}(y)) \cdot u = \theta(\frac{\alpha}{2}y) \cdot u = \frac{\alpha'}{2}(iy) \cdot u,$$

where $\widetilde{\Lambda^g}$, $\widetilde{\Lambda^{g'}}$ denote the spin lifts of Λ^g, $\Lambda^{g'}$. The situation in the vertical directions is somewhat more complicated. Arguing as above, we have

$$\begin{aligned}
\theta^{-1}(\Lambda^{g'}(\xi_i)) &= \theta^{-1}(\tau(\Lambda^g(\xi_i))|_{\mathfrak{sp}(1)} - \tau(\Lambda^g(\xi_i))|_{i\mathfrak{m}_1} + 2\,\mathrm{ad}(\xi_i)|_{i\mathfrak{m}_1}) \\
&= \Lambda^g(\xi_i)|_{\mathfrak{sp}(1)} - \Lambda^g(\xi_i)|_{\mathfrak{m}_1} + 2\,\mathrm{ad}(\xi_i)|_{\mathfrak{m}_1} = \Lambda^g(\xi_i)|_{\mathfrak{sp}(1)} - \Lambda^g(\xi_i) + \Lambda^g(\xi_i)|_{\mathfrak{sp}(1)} + 2\,\mathrm{ad}(\xi_i)|_{\mathfrak{m}_1} \\
&= 2\Lambda^g(\xi_i)|_{\mathfrak{sp}(1)} - \Lambda^g(\xi_i) + 2\,\mathrm{ad}(\xi_i)|_{\mathfrak{m}_1} = \mathrm{ad}(\xi_i)|_{\mathfrak{sp}(1)} - \Lambda^g(\xi_i) + 2\,\mathrm{ad}(\xi_i)|_{\mathfrak{m}_1} \\
&= -2\delta\Phi_i - \Lambda^g(\xi_i)
\end{aligned}$$

(where for the final equality we have identified the endomorphism field $\mathrm{ad}(\xi_i)|_{\mathfrak{sp}(1)}$ with the 2-form $-2\delta\Phi_i|_{\mathcal{V}}$ using the metric g). Taking the spin lift then gives

$$\theta^{-1}(\widetilde{\Lambda}^{g'}(\xi_i)) = -\delta\Phi_i - \widetilde{\Lambda}^g(\xi_i),$$

or equivalently,

$$\theta^{-1}(\widetilde{\Lambda}^{g'}(\xi_i')) = -\delta'\Phi_i + \widetilde{\Lambda}^g(\xi_i),$$

and hence

$$\begin{aligned}
\widetilde{\Lambda^{g'}}(\xi_i') \cdot u &= \theta(-\delta'\Phi_i + \widetilde{\Lambda}^g(\xi_i)) \cdot u = -\delta'\Phi_i' \cdot u + \theta(\widetilde{\Lambda}^g(\xi_i)) \cdot u \\
&= -\delta'\Phi_i' \cdot u + \theta(\frac{\alpha}{2}\xi_i + \frac{\alpha-\delta}{2}\Phi_i) \cdot u = -\delta'\Phi_i' \cdot u + \frac{\alpha'}{2}\xi_i' + \frac{\alpha'+\delta'}{2}\Phi_i' \cdot u \\
&= \frac{\alpha'}{2}\xi_i' \cdot u + \frac{\alpha'-\delta'}{2}\Phi_i' \cdot u.
\end{aligned}$$

It follows that

$$\widetilde{\Lambda}^{g'}(V) \cdot u = \frac{\alpha'}{2}V \cdot u + \frac{\alpha'-\delta'}{2}\sum_{i=1}^{3}\eta_i'(V)\Phi_i' \cdot u, \qquad \text{for all } V \in \mathfrak{m}',$$

proving the first part of the theorem. Assume now that $\psi \equiv u$ is parallel for the canonical connection. It is immediately apparent from Proposition 6.2.3 that $\widetilde{\Lambda'}(iy) \cdot u = 0$ for all $iy \in i\mathfrak{m}_1$, and in the vertical directions

$$\theta^{-1}(\Lambda'(\xi_i')) = \theta^{-1}\left(\tau(\Lambda(\xi_i')) - \frac{4\alpha'}{\delta'}\,\mathrm{ad}(\xi_i')\right) = \Lambda(\xi_i') - \frac{4\alpha'}{\delta'}\,\mathrm{ad}(\xi_i')$$

$$= -\Lambda(\xi_i) - 4\alpha'(\varphi_i' + \varphi_i'|_{\mathfrak{sp}(1)}) = -\Lambda(\xi_i) + 4\alpha'(\Phi_i' + \Phi_i'|_{\mathcal{V}})$$

(where for the final equality we have identified skew-symmetric endomorphisms with differential forms using the metric g'). Applying the spin lift of this operator to $\psi \equiv u$ gives

$$\widetilde{\Lambda'}(\xi_i') \cdot u = 0 + 2\alpha' \Phi_i' \cdot u - 2\alpha' \xi_j' \cdot \xi_k' \cdot u,$$

as desired. □

6.3 The Special Case of Dimension 7

Let us examine more closely the situation in dimension 7. To start, let $(M^7, g, \varphi_i, \xi_i, \eta_i)$ be *any* 7-dimensional 3-(α, δ)-Sasaki manifold (not necessarily homogeneous). It is shown in [AD20, Thm. 4.5.1] that the canonical connection ∇ arises as the characteristic connection of the G_2-structure

$$\omega := \eta_1 \wedge \eta_2 \wedge \eta_3 + \sum_{i=1}^{3} \eta_i \wedge \Phi_i|_{\mathcal{H}} \tag{6.6}$$

defined naturally in terms of the 3-(α, δ)-Sasaki structure tensors. One then obtains a ∇-parallel spinor ψ_0 via the relationship described in [ACFH15] between unit spinors and G_2-structures in dimension 7. The spinor ψ_0 is called the *canonical spinor*, and can equivalently be realized, as in [AD20, Def. 4.5.1], as the unique spinor (up to sign) such that

$$\nabla \psi_0 = 0, \qquad \omega \cdot \psi_0 = -7\psi_0, \qquad |\psi_0| = 1.$$

The three *auxiliary spinors* are then defined via the Clifford products with the Reeb vector fields,

$$\psi_1 := \xi_1 \cdot \psi_0, \qquad \psi_2 := \xi_2 \cdot \psi_0, \qquad \psi_3 := \xi_3 \cdot \psi_0.$$

Theorem 6.3.1. *([AD20, Thm. 4.5.2]). The canonical and auxiliary spinors are Riemannian* *generalized Killing spinors,*

$$\nabla^g_X\psi_0 = \begin{cases} \frac{2\alpha-\delta}{2}X\cdot\psi_0 & X\in\mathcal{V}, \\ -\frac{3\alpha}{2}X\cdot\psi_0 & X\in\mathcal{H}, \end{cases} \qquad \nabla^g_X\psi_i = \begin{cases} \frac{2\alpha-\delta}{2}\xi_i\cdot\psi_i & X=\xi_i, \\ \frac{3\delta-2\alpha}{2}\xi_j\cdot\psi_i & X=\xi_j\ (j\neq i), \\ \frac{\alpha}{2}X\cdot\psi_i & X\in\mathcal{H}, \end{cases} \tag{6.7}$$

for $i,j=1,2,3$.

We refer the reader to Chapter 3.2.1.1 for a detailed discussion of the homogeneous example $S^{4n-1}=\mathrm{Sp}(n)/\mathrm{Sp}(n-1)$, including explicit calculations of the invariant spinors, and formulas for the canonical and auxiliary spinors in dimension 7.

Theorem 6.3.2. *Suppose that $(M=G/H,g)$ is a compact simply-connected 7-dimensional homogeneous 3-(α,δ)-Sasaki space, and $(M'=G'/H,g')$ its non-compact dual. Under the identification of spinor bundles $\Sigma\cong\Sigma'$ as in Theorem 6.2.4, the canonical and auxiliary spinors ψ_i, $i=0,1,2,3$ on (M,g) are given by constant H-equivariant maps $G\to\Sigma$, and the corresponding spinors $\psi_i'\in\Sigma'$, $i=0,1,2,3$ are the canonical and auxiliary spinors of the dual 3-(α',δ')-Sasaki space (M',g'). In particular they are Riemannian generalized Killing spinors, satisfying:*

$$\nabla^{g'}_X\psi_0' = \begin{cases} \frac{2\alpha'-\delta'}{2}X\cdot\psi_0' & X\in\mathcal{V}', \\ -\frac{3\alpha'}{2}X\cdot\psi_0' & X\in\mathcal{H}', \end{cases} \qquad \nabla^{g'}_X\psi_i' = \begin{cases} \frac{2\alpha'-\delta'}{2}\xi_i'\cdot\psi_i' & X=\xi_i', \\ \frac{3\delta'-2\alpha'}{2}\xi_j'\cdot\psi_i' & X=\xi_j'\ (j\neq i), \\ \frac{\alpha'}{2}X\cdot\psi_i' & X\in\mathcal{H}', \end{cases} \tag{6.8}$$

for $i,j=1,2,3$, where $\mathcal{V}'$, $\mathcal{H}'$ denote the vertical and horizontal bundles respectively of M'.

Proof. In dimension 7, one easily checks in a concrete realization of the spin representation that the space E from Theorem 5.0.1 is 3-dimensional, and given by

$$E=\mathrm{span}_{C^\infty(M)}\{\psi_1,\psi_2,\psi_3\}.$$

Since M is compact (and hence $\alpha\delta>0$), the generalized 3-Sasakian data determining it also determines a compact 7-dimensional 3-Sasakian homogeneous space. From the classification of homogeneous 3-Sasakian spaces (see Theorem 2.4.1) it follows that M is either

$$S^7=\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)} \quad\text{or}\quad \frac{\mathrm{SU}(3)}{S(\mathrm{U}(1)\times\mathrm{U}(1))},$$

(the case $\mathbb{RP}^7$ is excluded by the assumption that M is simply-connected) so in particular $G = \mathrm{Sp}(2)$ or $\mathrm{SU}(3)$. To see that ψ_0 is G-invariant, we note that it is determined up to sign as a unit length element of the 1-dimensional (-7)-eigenspace for the action of ω on Σ; then, using G-invariance of ω, we calculate at the origin

$$(\omega \cdot (g_0\psi_0))(g) = (\omega \cdot \psi_0)(g_0^{-1}g) = -7\psi_0(g_0^{-1}g) = -7(g_0\psi_0)(g), \qquad \text{for all } g_0, g \in G.$$

This shows that the 1-dimensional space $\mathbb{C}\psi_0$ is a G-subrepresentation of Σ, and this subrepresentation must be trivial since $G = \mathrm{Sp}(2)$ or $\mathrm{SU}(3)$ (i.e. ψ_0 corresponds to a constant map $G \to \Sigma$). The auxiliary spinors $\psi_i = \xi_i \cdot \psi_0$, $i = 1, 2, 3$ are then invariant as well by Lemma 4.2.4. Next, we observe that the canonical G_2-form ω' of (M', g'), defined using the 3-(α', δ')-Sasaki structure tensors analogously to (6.6), coincides with $\theta(\omega)$, and hence its (-7)-eigenspace inside Σ' coincides with the span of ψ_0. The auxiliary spinors of (M', g') therefore coincide with ψ_i, $i = 1, 2, 3$, and the spinorial equations (6.8) follow from Theorem 6.3.1 applied to M'. □

Similarly, for the canonical connection we find:

Proposition 6.3.3. *Let $(M = G/H, g)$ and (M', g') be a dual pair of homogeneous 7-dimensional 3-(α, δ)-Sasaki spaces, and identify the spinor modules as in Theorem 6.2.4. Then an invariant spinor $\psi \equiv u$ on M is ∇-parallel if and only if the correponding spinor $\psi' \equiv u$ on (M', g') is ∇'-parallel.*

Proof. By symmetry of the 'if and only if' statement, it suffices to prove either implication. If $\psi \equiv u$ is a ∇-parallel spinor on M then Theorem 6.2.4 implies that the dual spinor $\psi' \equiv u$ satisfies (6.5). Note that ψ is stabilized by $\mathrm{G}_2 = \mathrm{stab}_{\mathrm{SO}(7)}(\omega)$, since the canonical connection is a G_2-connection (see [AD20, Remark 4.4.4]). The action of G_2 on the spinor module Σ has only one fixed spinor up to scaling (see part (d) of the main proposition in [Wan89]), so we conclude that ψ is a multiple of the canonical spinor ψ_0. The result then follows by substituting (6.9) into (6.5). □

Finally, for any 7-dimensional 3-(α, δ)-Sasaki manifold (not necessarily homogeneous) we compare the spinorial equation satisfied by the auxiliary spinors (the second equation in (6.7)) to the deformed Killing equation (5.1). A straightforward calculation in the spin representation gives:

Lemma 6.3.4. *If $(M^7, g, \xi_i, \eta_i, \varphi_i)$ is a 7-dimensional 3-(α, δ)-Sasaki manifold, the canonical* *and auxiliary spinors satisfy*

$$\Phi_i \cdot \psi_0 = \psi_i, \quad \Phi_i \cdot \psi_i = \xi_i \cdot \psi_i, \quad \Phi_i \cdot \psi_j = -3\xi_i \cdot \psi_j,$$

for all $i, j = 1, 2, 3$ *with* $i \neq j$. *Furthermore, for any even permutation* (i, j, k) *of* $(1, 2, 3)$*, the canonical spinor satisfies*

$$(\Phi_i - \xi_j \cdot \xi_k) \cdot \psi_0 = 0. \tag{6.9}$$

The preceding lemma immediately gives:

Proposition 6.3.5. *If* $(M^7, g, \xi_i, \eta_i, \varphi_i)$ *is a 7-dimensional 3-*(α, δ)*-Sasaki manifold, then a spinor* $\psi \in \Gamma(E)$ *satisfies the deformed Killing equation (5.1) if and only if it satisfies the second equation in (6.7).*

Proof. Using Lemma 6.3.4, we calculate

$$\frac{\alpha}{2} X \cdot \psi_i + \frac{\alpha - \delta}{2} \sum_{p=1}^{3} \eta_p(X) \Phi_p \cdot \psi_i = \frac{\alpha}{2} X \cdot \psi_i + \frac{\alpha - \delta}{2} \left(\eta_i(X) \xi_i - 3\eta_{i+1}(X) \xi_{i+1} - 3\eta_{i+2}(X) \xi_{i+2} \right) \cdot \psi_i,$$

where the indices $i, i+1, i+2$ are taken modulo 3. The result then follows by substituting the various cases for X from (6.7). □

$\mathcal{A}$

An Explicit Construction of the Lie Algebras $\mathfrak{su}(3)$, $\mathfrak{g}_2$, $\mathfrak{spin}(7)$, and $\mathfrak{spin}(9)$

The calculations involved in finding the invariant spinors for the exceptional spheres in Table 1.1 can be greatly simplified by constructing in a unified manner the Lie algebra inclusions $\mathfrak{su}(3) \subset \mathfrak{g}_2 \subset \mathfrak{spin}(7) \subset \mathfrak{spin}(9)$. This appendix is devoted to the exposition of this construction, parts of which may be found in Chaper 4.4 of [BFGK91]:

Lemma A.1. *(Based on Lemma 15 in Chapter 4.4 of [BFGK91]). The Lie algebra $\mathfrak{g}_2$ (resp. $\mathfrak{su}(3)$) may be realized as the stabilizer of one (resp. two) spinors in the real spin representation $\Sigma_7^{\mathbb{R}} := \mathbb{R}^8$ of $\mathfrak{spin}(7)$. Explicitly, if $\epsilon_1, \ldots \epsilon_7$ is the standard basis of $\mathbb{R}^7$ and $\phi_1, \ldots, \phi_8$ is the standard basis of the real spinor module $\Sigma_7^{\mathbb{R}} = \mathbb{R}^8$, then $\mathfrak{g}_2$ and $\mathfrak{su}(3)$ are realized inside $\mathfrak{spin}(7) = \mathrm{span}_{\mathbb{R}}\{\epsilon_i \epsilon_j\}$ via*

$$\mathfrak{g}_2 \cong stab_{\mathfrak{spin}(7)}\{\phi_1\} \cong \left\{ \sum_{1 \le i < j \le 7} \omega_{i,j}\epsilon_i\epsilon_j : \begin{array}{r} \omega_{1,2} + \omega_{3,4} + \omega_{5,6} = 0, \\ -\omega_{1,3} + \omega_{2,4} - \omega_{6,7} = 0,\ -\omega_{1,4} - \omega_{2,3} - \omega_{5,7} = 0, \\ -\omega_{1,6} - \omega_{2,5} + \omega_{3,7} = 0,\ \omega_{1,5} - \omega_{2,6} - \omega_{4,7} = 0, \\ \omega_{1,7} + \omega_{3,6} + \omega_{4,5} = 0,\ \omega_{2,7} + \omega_{3,5} - \omega_{4,6} = 0, \end{array} \right\}$$

$$\mathfrak{su}(3) \cong stab_{\mathfrak{spin}(7)}\{\phi_1, \phi_2\} = \left\{ \sum_{1 \le i < j \le 7} \omega_{i,j}\epsilon_i\epsilon_j : \begin{array}{r} \omega_{1,2} + \omega_{3,4} + \omega_{5,6} = 0, \\ \omega_{1,3} = \omega_{2,4},\ \omega_{1,4} + \omega_{2,3} = 0,\ \omega_{1,5} = \omega_{2,6}, \\ \omega_{1,6} + \omega_{2,5} = 0,\ \omega_{3,5} = \omega_{4,6},\ \omega_{3,6} + \omega_{4,5} = 0, \\ \omega_{1,7} = \omega_{2,7} = \cdots = \omega_{6,7} = 0. \end{array} \right\}$$

Remark A.2. In order to find bases for these Lie algebras we use the explicit realization of the spin representation obtained from the following matrices:

$$\rho(\epsilon_1) := E^{(8)}_{1,8} + E^{(8)}_{2,7} - E^{(8)}_{3,6} - E^{(8)}_{4,5}, \quad \rho(\epsilon_2) := -E^{(8)}_{1,7} + E^{(8)}_{2,8} + E^{(8)}_{3,5} - E^{(8)}_{4,6},$$
$$\rho(\epsilon_3) := -E^{(8)}_{1,6} + E^{(8)}_{2,5} - E^{(8)}_{3,8} + E^{(8)}_{4,7}, \quad \rho(\epsilon_4) := -E^{(8)}_{1,5} - E^{(8)}_{2,6} - E^{(8)}_{3,7} - E^{(8)}_{4,8},$$
$$\rho(\epsilon_5) := -E^{(8)}_{1,3} - E^{(8)}_{2,4} + E^{(8)}_{5,7} + E^{(8)}_{6,8}, \quad \rho(\epsilon_6) := E^{(8)}_{1,4} - E^{(8)}_{2,3} - E^{(8)}_{5,8} + E^{(8)}_{6,7},$$
$$\rho(\epsilon_7) := E^{(8)}_{1,2} - E^{(8)}_{3,4} - E^{(8)}_{5,6} + E^{(8)}_{7,8}$$

(see Chapter 4.4 in [BFGK91]). By substituting these into the equations of the preceding lemma and subsequently orthogonalizing with respect to B_0, one obtains:

Proposition A.3. *A B_0-orthonormal basis for $\mathfrak{g}_2$ given by*

$$\nu_1 := \frac{1}{4}(\rho(\epsilon_1)\rho(\epsilon_2) - \rho(\epsilon_5)\rho(\epsilon_6)), \quad \nu_2 := \frac{1}{4}(\rho(\epsilon_3)\rho(\epsilon_5) + \rho(\epsilon_4)\rho(\epsilon_6)),$$
$$\nu_3 := \frac{1}{4}(\rho(\epsilon_3)\rho(\epsilon_6) - \rho(\epsilon_4)\rho(\epsilon_5)), \quad \nu_4 := \frac{1}{4}(\rho(\epsilon_1)\rho(\epsilon_3) + \rho(\epsilon_2)\rho(\epsilon_4)),$$
$$\nu_5 := \frac{1}{4}(\rho(\epsilon_1)\rho(\epsilon_4) - \rho(\epsilon_2)\rho(\epsilon_3)), \quad \nu_6 := \frac{1}{4}(\rho(\epsilon_1)\rho(\epsilon_5) + \rho(\epsilon_2)\rho(\epsilon_6)),$$
$$\nu_7 := \frac{1}{4}(\rho(\epsilon_1)\rho(\epsilon_6) - \rho(\epsilon_2)\rho(\epsilon_5)), \quad \nu_8 := -\frac{\rho(\epsilon_1)\rho(\epsilon_2) - 2\rho(\epsilon_3)\rho(\epsilon_4) + \rho(\epsilon_5)\rho(\epsilon_6)}{4\sqrt{3}},$$
$$\nu_9 := \frac{2\rho(\epsilon_1)\rho(\epsilon_7) - \rho(\epsilon_3)\rho(\epsilon_6) - \rho(\epsilon_4)\rho(\epsilon_5)}{4\sqrt{3}}, \quad \nu_{10} := \frac{2\rho(\epsilon_2)\rho(\epsilon_7) - \rho(\epsilon_3)\rho(\epsilon_5) + \rho(\epsilon_4)\rho(\epsilon_6)}{4\sqrt{3}},$$
$$\nu_{11} := \frac{\rho(\epsilon_1)\rho(\epsilon_3) - \rho(\epsilon_2)\rho(\epsilon_4) - 2\rho(\epsilon_6)\rho(\epsilon_7)}{4\sqrt{3}}, \quad \nu_{12} := \frac{\rho(\epsilon_1)\rho(\epsilon_4) + \rho(\epsilon_2)\rho(\epsilon_3) - 2\rho(\epsilon_5)\rho(\epsilon_7)}{4\sqrt{3}},$$
$$\nu_{13} := \frac{\rho(\epsilon_1)\rho(\epsilon_5) - \rho(\epsilon_2)\rho(\epsilon_6) + 2\rho(\epsilon_4)\rho(\epsilon_7)}{4\sqrt{3}}, \quad \nu_{14} := \frac{\rho(\epsilon_1)\rho(\epsilon_6) + \rho(\epsilon_2)\rho(\epsilon_5) + 2\rho(\epsilon_3)\rho(\epsilon_7)}{4\sqrt{3}},$$

with the first 8 elements $\nu_1, \ldots \nu_8$ forming a B_0-orthonormal basis for the subalgebra $\mathfrak{su}(3)$.

We now wish to extend this to B_0-orthonormal bases of $\mathfrak{spin}(7)$ and $\mathfrak{spin}(9)$. Denoting by $\iota\colon \mathrm{Mat}_8(\mathbb{R}) \hookrightarrow \mathrm{Mat}_9(\mathbb{R})$ the embedding as the lower right hand 8×8 block, one has:

Proposition A.4. *The basis $\{\nu_1, \ldots, \nu_{14}\}$ extends to a B_0-orthonormal basis of $\mathfrak{spin}(7)$ given by $\{\nu_1, \ldots, \nu_{14}, \nu'_{15}, \ldots, \nu'_{21}\}$ and a B_0-orthonormal basis of $\mathfrak{spin}(9)$ given by $\{\iota(\nu_1), \ldots, \iota(\nu_{14}), \iota(\nu'_{15}), \ldots, \iota(\nu'_{21}), \nu'_{22}, \ldots, \nu'_{36}\}$, where*

$$\nu'_{15} := \frac{\rho(\epsilon_1)\rho(\epsilon_2) + \rho(\epsilon_3)\rho(\epsilon_4) + \rho(\epsilon_5)\rho(\epsilon_6)}{2\sqrt{6}}, \quad \nu'_{16} := \frac{\rho(\epsilon_1)\rho(\epsilon_3) - \rho(\epsilon_2)\rho(\epsilon_4) + \rho(\epsilon_6)\rho(\epsilon_7)}{2\sqrt{6}},$$
$$\nu'_{17} := \frac{\rho(\epsilon_1)\rho(\epsilon_4) + \rho(\epsilon_2)\rho(\epsilon_3) + \rho(\epsilon_5)\rho(\epsilon_7)}{2\sqrt{6}}, \quad \nu'_{18} := \frac{-\rho(\epsilon_1)\rho(\epsilon_5) + \rho(\epsilon_2)\rho(\epsilon_6) + \rho(\epsilon_4)\rho(\epsilon_7)}{2\sqrt{6}},$$
$$\nu'_{19} := -\frac{\rho(\epsilon_1)\rho(\epsilon_6) + \rho(\epsilon_2)\rho(\epsilon_5) - \rho(\epsilon_3)\rho(\epsilon_7)}{2\sqrt{6}}, \quad \nu'_{20} := \frac{\rho(\epsilon_1)\rho(\epsilon_7) + \rho(\epsilon_3)\rho(\epsilon_6) + \rho(\epsilon_4)\rho(\epsilon_5)}{2\sqrt{6}},$$

$$\nu'_{21} := -\frac{\rho(\epsilon_2)\rho(\epsilon_7) + \rho(\epsilon_3)\rho(\epsilon_5) - \rho(\epsilon_4)\rho(\epsilon_6)}{2\sqrt{6}},$$

and

$$\nu'_{22} := \sqrt{2}\left(E^{(9)}_{2,3} - \sqrt{\frac{3}{2}}\ \iota(\nu'_{15})\right), \quad \nu'_{23} := \sqrt{2}\left(E^{(9)}_{2,4} + \sqrt{\frac{3}{2}}\ \iota(\nu'_{16})\right), \quad \nu'_{24} := \sqrt{2}\left(E^{(9)}_{2,5} + \sqrt{\frac{3}{2}}\ \iota(\nu'_{17})\right),$$
$$\nu'_{25} := \sqrt{2}\left(E^{(9)}_{2,6} - \sqrt{\frac{3}{2}}\ \iota(\nu'_{19})\right), \quad \nu'_{26} := \sqrt{2}\left(E^{(9)}_{2,7} + \sqrt{\frac{3}{2}}\ \iota(\nu'_{18})\right), \quad \nu'_{27} := \sqrt{2}\left(E^{(9)}_{2,8} - \sqrt{\frac{3}{2}}\ \iota(\nu'_{20})\right),$$
$$\nu'_{28} := \sqrt{2}\left(E^{(9)}_{2,9} + \sqrt{\frac{3}{2}}\ \iota(\nu'_{21})\right), \quad \nu'_{28+i} := \frac{1}{\sqrt{2}}E^{(9)}_{1,1+i},$$

for all $i = 1, \dots, 8$.

B

Invariant Differential Forms on $S^{15} = \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$

Here we give explicit formulas, in terms of the basis (3.24), for the differential forms on $S^{15} = \mathrm{Spin}(9)/\,\mathrm{Spin}(7)$ discussed in Chapter 3.3.3:

$$\begin{aligned}
\omega := {} & -e_{1,8,9} + e_{1,10,11} + e_{1,12,13} - e_{1,14,15} - e_{2,8,10} - e_{2,9,11} + e_{2,12,14} + e_{2,13,15} \\
& - e_{3,8,11} + e_{3,9,10} + e_{3,12,15} - e_{3,13,14} - e_{4,8,12} - e_{4,9,13} - e_{4,10,14} - e_{4,11,15} \\
& - e_{5,8,13} + e_{5,9,12} - e_{5,10,15} + e_{5,11,14} - e_{6,8,14} + e_{6,9,15} + e_{6,10,12} - e_{6,11,13} \\
& - e_{7,8,15} - e_{7,9,14} + e_{7,10,13} + e_{7,11,12}, \\
\Psi := {} & e_{8,9,10,11} + e_{8,9,12,13} - e_{8,9,14,15} + e_{8,10,12,14} + e_{8,10,13,15} + e_{8,11,12,15} \\
& - e_{8,11,13,14} - e_{9,10,12,15} + e_{9,10,13,14} + e_{9,11,12,14} + e_{9,11,13,15} \\
& - e_{10,11,12,13} + e_{10,11,14,15} + e_{12,13,14,15}, \\
\sqrt{\frac{a}{2}}\, d\omega = {} & e_{1,2,8,11} - e_{1,2,9,10} + e_{1,2,12,15} - e_{1,2,13,14} - e_{1,3,8,10} - e_{1,3,9,11} - e_{1,3,12,14} - e_{1,3,13,15} \\
& + e_{1,4,8,13} - e_{1,4,9,12} - e_{1,4,10,15} + e_{1,4,11,14} - e_{1,5,8,12} - e_{1,5,9,13} + e_{1,5,10,14} + e_{1,5,11,15} \\
& - e_{1,6,8,15} - e_{1,6,9,14} - e_{1,6,10,13} - e_{1,6,11,12} + e_{1,7,8,14} - e_{1,7,9,15} + e_{1,7,10,12} - e_{1,7,11,13} \\
& + e_{2,3,8,9} - e_{2,3,10,11} + e_{2,3,12,13} - e_{2,3,14,15} + e_{2,4,8,14} + e_{2,4,9,15} - e_{2,4,10,12} - e_{2,4,11,13} \\
& + e_{2,5,8,15} - e_{2,5,9,14} - e_{2,5,10,13} + e_{2,5,11,12} - e_{2,6,8,12} + e_{2,6,9,13} - e_{2,6,10,14} + e_{2,6,11,15} \\
& - e_{2,7,8,13} - e_{2,7,9,12} - e_{2,7,10,15} - e_{2,7,11,14} + e_{3,4,8,15} - e_{3,4,9,14} + e_{3,4,10,13} - e_{3,4,11,12} \\
& - e_{3,5,8,14} - e_{3,5,9,15} - e_{3,5,10,12} - e_{3,5,11,13} + e_{3,6,8,13} + e_{3,6,9,12} - e_{3,6,10,15} - e_{3,6,11,14} \\
& - e_{3,7,8,12} + e_{3,7,9,13} + e_{3,7,10,14} - e_{3,7,11,15} + e_{4,5,8,9} + e_{4,5,10,11} - e_{4,5,12,13} - e_{4,5,14,15} \\
& + e_{4,6,8,10} - e_{4,6,9,11} - e_{4,6,12,14} + e_{4,6,13,15} + e_{4,7,8,11} + e_{4,7,9,10} - e_{4,7,12,15} - e_{4,7,13,14} \\
& - e_{5,6,8,11} - e_{5,6,9,10} - e_{5,6,12,15} - e_{5,6,13,14} + e_{5,7,8,10} - e_{5,7,9,11} + e_{5,7,12,14} - e_{5,7,13,15} \\
& - e_{6,7,8,9} - e_{6,7,10,11} - e_{6,7,12,13} - e_{6,7,14,15} - \frac{3a e_{8,9,10,11}}{2b} - \frac{3a e_{8,9,12,13}}{2b} + \frac{3a e_{8,9,14,15}}{2b}
\end{aligned}$$

$$- \frac{3ae_{8,10,12,14}}{2b} - \frac{3ae_{8,10,13,15}}{2b} - \frac{3ae_{8,11,12,15}}{2b} + \frac{3ae_{8,11,13,14}}{2b} + \frac{3ae_{9,10,12,15}}{2b}$$
$$- \frac{3ae_{9,10,13,14}}{2b} - \frac{3ae_{9,11,12,14}}{2b} - \frac{3ae_{9,11,13,15}}{2b} + \frac{3ae_{10,11,12,13}}{2b} - \frac{3ae_{10,11,14,15}}{2b}$$
$$- \frac{3ae_{12,13,14,15}}{2b}$$

$$\sqrt{\frac{a}{2}}\, d\Psi = e_{1,8,10,12,15} - e_{1,8,10,13,14} - e_{1,8,11,12,14} - e_{1,8,11,13,15} + e_{1,9,10,12,14} + e_{1,9,10,13,15}$$
$$+ e_{1,9,11,12,15} - e_{1,9,11,13,14} - e_{2,8,9,12,15} + e_{2,8,9,13,14} + e_{2,8,11,12,13} - e_{2,8,11,14,15}$$
$$- e_{2,9,10,12,13} + e_{2,9,10,14,15} + e_{2,10,11,12,15} - e_{2,10,11,13,14} + e_{3,8,9,12,14} + e_{3,8,9,13,15}$$
$$- e_{3,8,10,12,13} + e_{3,8,10,14,15} - e_{3,9,11,12,13} + e_{3,9,11,14,15} - e_{3,10,11,12,14} - e_{3,10,11,13,15}$$
$$+ e_{4,8,9,10,15} - e_{4,8,9,11,14} + e_{4,8,10,11,13} - e_{4,8,13,14,15} - e_{4,9,10,11,12} + e_{4,9,12,14,15}$$
$$- e_{4,10,12,13,15} + e_{4,11,12,13,14} - e_{5,8,9,10,14} - e_{5,8,9,11,15} - e_{5,8,10,11,12} + e_{5,8,12,14,15}$$
$$- e_{5,9,10,11,13} + e_{5,9,13,14,15} + e_{5,10,12,13,14} + e_{5,11,12,13,15} + e_{6,8,9,10,13} + e_{6,8,9,11,12}$$
$$- e_{6,8,10,11,15} - e_{6,8,12,13,15} - e_{6,9,10,11,14} - e_{6,9,12,13,14} + e_{6,10,13,14,15} + e_{6,11,12,14,15}$$
$$- e_{7,8,9,10,12} + e_{7,8,9,11,13} + e_{7,8,10,11,14} + e_{7,8,12,13,14} - e_{7,9,10,11,15} - e_{7,9,12,13,15}$$
$$- e_{7,10,12,14,15} + e_{7,11,13,14,15}.$$

We also record in Table B.1 the *isotropy types* of the forms $\omega, \Psi, d\omega, d\Psi$ from Chapter 3.3.3, i.e. the number of factors from each isotropy component. The isotropy types of all other invariant forms in Table 3.3 may be easily deduced from these.

Form	Isotropy Type
ω	$\mathfrak{m}_F \otimes \Lambda^2 \mathfrak{m}_B$
Ψ	$\Lambda^4 \mathfrak{m}_B$
$d\omega$	$(\Lambda^2 \mathfrak{m}_F \otimes \Lambda^2 \mathfrak{m}_B) \oplus (\Lambda^4 \mathfrak{m}_B)$
$d\Psi$	$\mathfrak{m}_F \otimes \Lambda^4 \mathfrak{m}_B$

Table B.1: Isotropy Types of Invariant Forms on $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$

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