Stochastic Analysis for Cylindrical Lévy processes

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Abstract

The purpose of this thesis is to lay down the theoretical foundations necessary for the successful application of cylindrical Lévy processes as models of random perturbations of infinite-dimensional systems. This is accomplished in three main steps.

First, we provide a comprehensive theory of stochastic integration with respect to cylindrical Lévy processes in Hilbert space. In fact, we go further than simply introducing the stochastic integral, and give a complete analytic characterisation of the largest set of predictable Hilbert-Schmidt operator-valued processes integrable with respect to a cylindrical Lévy process. We demonstrate the strength of the developed integration theory by establishing a stochastic dominated convergence result.

Second, we prove an Itô formula for Itô processes driven by cylindrical α-stable noise. It turns out that in the case of standard symmetric α-stable cylindrical Lévy processes, our integration theory simplifies significantly and it is possible to identify the largest space of predictable Hilbert-Schmidt operator-valued integrands with the collection of all predictable processes that have paths in the Bochner space L^α. As an application of our developed integration theory, we carry out an in-depth analysis of the jump structure of stochastic integral processes driven by standard symmetric α-stable cylindrical Lévy processes, which allows us to establish an Itô formula in this setting.

Finally, we consider stochastic evolution equations driven by α-stable noise and prove the existence of a mild solution, establish long-term regularity of the solutions via a Lyapunov functional approach, and prove an Itô formula for mild solutions to the evolution equations under consideration. The main tool for establishing these results is a Yoshida approximation of the solution, which we combine with the crucial observation that these approximations converge in the space $\mathcal{C}([0,T], L^p(\Omega, H))$ of p-th mean continuous Hilbert space-valued processes for any $p < \alpha$.

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Contents

1 Introduction

Any rigorous discussion of stochastic partial differential equations (SPDEs) must begin with the precise mathematical formulation of the driving noise. In an infinite-dimensional setting, this task is highly non-trivial, since even the most fundamental choice, the standard Brownian motion, fails to exist in the usual sense as a stochastic process with values in the underlying infinite-dimensional space. In fact, these are exactly the considerations that lead naturally to the theory of cylindrical processes and their application as models of random perturbations of infinite-dimensional systems.

In the literature, one can find several alternative definitions for cylindrical Lévy noise in an infinite-dimensional setting. The most popular approach is to introduce cylindrical Lévy noise in a diagonal manner, wherein the noise is expressed as an infinite sum of independent one-dimensional Lévy processes along an orthonormal basis, see Peszat and Zabczyk [60], or Priola and Zabczyk [63] for the case when the components of the orthogonal expansion are real-valued normalised symmetric α-stable processes. Another approach, see for example Brzeźniak and Zabczyk [9], defines a cylindrical Lévy process by subordinating a cylindrical Wiener process, which is then used as a theoretical foundation of their models of random perturbations. Alternatively, in Peszat and Zabczyk [59], the authors introduce the notion of a cylindrical Lévy process, which they call an impulsive cylindrical process, via Poission random measures.

While all these definitions are mathematically rigorous, they are somewhat dissatisfying for the following reasons. First, a common theme in all the aforementioned papers is the observation that a cylindrical Lévy process can be embedded into a larger space where it becomes a classical Lévy process. However, within the context of SPDEs, this point of view leads to the unnatural phenomenon that one needs to impose conditions on the larger auxiliary space, which a priori has no relation to the problem under consideration, see Brzeźniak and Zabczyk [9]. Second, if we introduce cylindrical Lévy noise using one of the above definitions, we impose great limitations on the behaviour of the random perturbation. Indeed, taking the approach in Peszat and Zabczyk [60] with diagonal noise as an example, the assumption that the one-dimensional components are independent imply that these processes can only jump along the basis vectors, since the probability that two different components jump at the same time is zero. Hence, there arose the natural question of whether it is possible to find a general framework for modeling Lévy-type noise in an infinite-dimensional setting, which allows for a unified treatment of a wide variety of different random perturbations.

To provide an answer to this question, the first systematic treatment of the concept of a cylindrical Lévy process was introduced by Applebaum and Riedle in their work [4]. Their goal was to provide a definition of a cylindrical Lévy process, that is both analytically rigorous and, at the same time, can be seen as a natural generalisation of finite-dimensional Lévy processes. This has been accomplished by building on the well-established theory of cylindrical measures, pioneered by the French school of mathematics in a series of papers, see Maurey [45], Schwartz [72] or Badrikian and Chevet [5], among many others. Yet, before this rather general object could be used within the context of SPDEs as a model of noise, it became unavoidable to overcome certain technical difficulties. SPDEs with additive noise, driven by specific examples of cylindrical Lévy processes were considered in Brzeźniak and Zabczyk [9], Priola and Zabczyk [63] and Peszat and Zabczyk [60]. However, in order to consider SPDEs with a more general noise structure than the simple additive case, it became necessary to develop a theory of stochastic integration with respect to general cylindrical Lévy processes.

The classical construction of the stochastic integral in finite-dimensional spaces is based on the semi-martingale decomposition; see Dellacherie and Meyer [15]. Another approach, beginning with the class of "good integrators" is introduced in Protter [64], or in Kurtz and Protter [39], in the infinite-dimensional setting. However, it is worth noting that even in Protter's work [64], the extension of the space of integrable processes from adapted, càglàd to general predictable processes relies heavily on the semimartingale decomposition. Alternatively, one might approach the problem of defining stochastic integrals from the point of view of random measures. For an in-depth discussion of this perspective, see Métivier and Pellaumail [48] or Rao [65]. In yet another approach, see Bichteler [7], the author successfully mimics the construction of the well-known Daniell integral from calculus to introduce the notion of the stochastic integral. The theory of stochastic integration presented by Kwapień and Woyczyński in their works [40] and [41] relies crucially on a decoupling inequality for tangent sequences. In their approach, they also go beyond the construction of the stochastic integral, and provide a complete characterisation of the largest space of predictable integrands as a randomized Musielak-Orlicz space. These ideas were later generalised to Hilbert-valued quasi-left continuous semimartingales in Nowak [57].

Considering stochastic integration with respect to cylindrical processes, an extensive literature is available in the special case when the integrator is a cylindrical Brownian motion, see for example van Neerven et. al. [77] for a recent extension to the general setting of UMD Banach spaces. The problem of stochastic integration with respect to cylindrical processes other than cylindrical Brownian motion has received significantly less attention. As a matter of fact, the only other class of cylindrical processes with respect to which a theory of stochastic integration has been developed is the collection of cylindrical martingales. This has been achieved through a Doléans measure approach by Métivier and Pellaumail in [47], via the construction of a family of reproducing kernel Hilbert spaces by Mikulevičius and Rozovskiĭ in [50] and [51], or alternatively by introducing a new type of quadratic variation for cylindrical continuous local martingales in UMD Banach spaces in Veraar and Yaroslavtsev [78]. This limitation of the literature to cylindrical martingales is due to the technical difficulty that in general, cylindrical semi-martingales do not enjoy a semi-martingale decomposition in a cylindrical sense, see Jakubowski and Riedle [30, Re. 2.2]. As a consequence, one cannot use the above mentioned classical approach to develop a theory of stochastic integration in this setting.

The problem of stochastic integration with respect to arbitrary cylindrical Lévy processes in Hilbert space has been successfully addressed by Jakubowski and Riedle in their work [30]. Here, the authors introduce the stochastic integral for càglàd integrands using tightness arguments in Skorokhod space and the crucial observation that tightness of the decoupled version of a collection of stochastic integrals implies tightness of the original integrals. While the integration theory developed by Jakubowski and Riedle in [30] made the first crucial step towards the possibility of considering SPDEs with a rather general noise structure, it also had certain shortcomings. Most notably, as observed in the paper by Brzeźniak et al. [8], solutions to SPDEs driven by cylindrical Lévy processes exhibit a rather interesting behaviour, wherein a solution can be so irregular that it does not have a càdlàg modification. Hence, to ensure that we have sufficiently powerful theoretical tools to deal with this situation, it became necessary to consider stochastic integrals with predictable integrands. Since the integration theory presented by Jakubowski and Riedle in [30] relies fundamentally on convergence arguments in Skorokhod space, the extension of the notion of stochastic integral to predictable integrands could not be achieved within this framework.

The first part of this thesis is devoted to filling this gap in the literature by providing a theory of stochastic integration for predictable integrands with respect to arbitrary cylindrical Lévy processes in Hilbert space. In fact, by building on the original ideas of Kwapień and Woyczyński, see for example [42], not only does it become possible to introduce stochastic integrals with predictable integrands, but we also give a complete characterisation of the largest space of predictable processes integrable with respect to an arbitrary cylindrical Lévy process. As a by-product of our construction, we obtain an explicit analytic condition for the integrability of a predictable process, which is expressible in terms of the cylindrical characteristics of the integrator. Structurally speaking, the collection of all integrable processes forms a generalised modular space, see Nakano [55], where the topology induced by the modular can be equivalently metrised as a Polish space. As a demonstration of the robustness of our approach, we provide a stochastic dominated convergence theorem in our setting, which allows for the interchange of the limit and stochastic integral.

As a special case of our general integration theory, we consider separately stochastic integrals with respect to standard symmetric α-stable cylindrical Lévy processes for $\alpha \in (0,2)$. Using the work of Kosmala and Riedle in [37], and utilising the well-known tail properties of stable distributions, see for example Linde [43], we show that in case the integrator is a standard symmetric α-stable cylindrical Lévy process, the largest space of predictable integrands coincides with the collection of predictable processes that almost surely have paths in the Bochner space L^α. This condition aligns perfectly with its real-valued analogue, see Rosinski and Woyczynski [69], where the authors obtain the same integrability condition for classical real-valued standard symmetric α-stable Lévy processes via a random time change.

Moving away from stochastic integration, we consider another cornerstone of stochastic analysis, the Itô formula. Historically speaking, the earliest form of the Itô formula dates back to the seminal work of Itô [26]. Since then, Itô's formula has been extended in several directions. First, instead of a standard Brownian motion, one might consider Itô processes driven by more general stochastic processes, like Lévy processes or general semimartingales, see for example Jacod and Shiryaev [27], Meyer [49] or Protter [64]. Second, in keeping with modern mathematical developments, it is possible to leave the one-dimensional setting behind and consider generalisations of Itô's formula to more abstract spaces, see Metivier [46] for a proof in Hilbert space, or Gyöngy and Krylov [22] for the Banach-valued setting.

In this thesis, we develop a (strong) Itô formula for processes of the form:

$$dX(t) = F(t)\,\mathrm{d}t + G(t)\,\mathrm{d}L(t) \quad \text{for } t \in [0,T], \tag{1.1}$$

where L is a standard symmetric α-stable cylindrical Lévy process for $\alpha \in (1,2)$, while $F\colon \Omega \times [0,T] \to H$ and $G\colon \Omega \times [0,T] \to L_2(U,H)$ are predictable processes satisfying the integrability condition

$$\int_0^T \|F(t)\| + \|G(t)\|_{L_2(U,H)}^{\alpha}\,\mathrm{d}t < \infty \quad \text{a.s.} \tag{1.2}$$

Whereas the process in (1.1) is a Hilbert-valued semimartingale, and a classical Itô formula is available in this setting, see Metivier [46, Th. 27.2], in order to successfully apply this formula, it is often necessary to identify the martingale and bounded variation parts. While this is usually accomplished through the semimartingale decomposition of the driving noise, since cylindrical Lévy processes do not have a semimartingale decomposition, obtaining a useful form of the Itô formula in this setting requires a different approach. To tackle this problem, we draw upon the theory of random measures and compensators, e.g. Jacod and Shiryaev [27], and carry out an analysis of the jump structure of stochastic integral processes driven by a standard symmetric α-stable cylindrical Lévy process, which leads us naturally to a particularly useful form of the Itô formula in this setting.

Armed with our Itô formula, we turn to stochastic evolution equations driven by a standard symmetric α-stable cylindrical Lévy process. In the literature, there are essentially two alternative approaches to this subject. The random field approach, which originates from the work of Walsh [79], and the semigroup approach, introduced by Da Prato and Zabyczyk in the monograph [13]. While in the random field approach, one can find numerous publications with SPDEs perturbed by α-stable noise, see for example Chong [10], Mytnik [53], or Mueller [52], only in the work of Kosmala and Riedle [37] do the authors take the semigroup approach. The scarcity of results in the latter case can be attributed to the serious obstacle that similarly to cylindrical Brownian motion, standard symmetric α-stable cylindrical Lévy processes exist only in the generalised sense as cylindrical processes.

In the final chapter of this thesis, we take the semigroup approach and utilise our developed integration theory and strong Itô formula to explore various aspects of solutions to stochastic evolution equations driven by a standard symmetric α-stable cylindrical Lévy process in a Hilbert space U for $\alpha \in (1,2)$. More precisely, we consider equations of the form:

$$\begin{aligned} \mathrm{d}X(t) &= \big(AX(t) + F(X(t))\big)\,\mathrm{d}t + G(X(t-))\,\mathrm{d}L(t) \qquad \text{for } t \in [0,T], \\ X(0) &= x_0, \end{aligned} \tag{1.3}$$

where A is a generator of a C_0-semigroup $(S(t))_{t\geq 0}$ in a Hilbert space H, x_0 is an $\mathcal{F}_0$-measurable H-valued random variable, $F\colon H \to H$ and $G\colon H \to L_2(U,H)$ are measurable mappings and $T > 0$. Our work in this direction is comprised of three main results.

First, we build on the publication of Kosmala and Riedle in [37], and prove the existence of a mild solution to (1.3). In fact, we significantly improve the existence result presented by Kosmala and Riedle in [37], by showing that it is sufficient to impose some rather natural Lipschitz and boundedness conditions on the coefficients F and G in (1.3). The main tool in our proof is a Yosida approximation for the solution, which is combined with the important observation that while solutions to the Yosida approximating equations are pathwise discontinuous, they naturally live in the space $\mathcal{C}([0,T], L^p(\Omega,H))$ of p-th mean continuous Hilbert space-valued processes for any $p < \alpha$. This observation allows us to use the well-known Arzela-Ascoli theorem to establish relative compactness of solutions to the Yosida approximating equations in the space $\mathcal{C}([0,T], L^p(\Omega, H))$, from which our existence result follows immediately.

Second, we follow the work of Ichikawa [25] and take the functional Lyapunov approach to investigate regularity properties of mild solutions to (1.3). More precisely, we provide explicit conditions on the coefficients F and G under which the mild solution to (1.3) is exponentially ultimately bounded in the p-th moment for every $p \in (0,1)$. The main ingredient of our proof is the strong Itô formula developed earlier in this thesis. However, since mild solutions to SPDEs are not semimartingales, our strong Itô formula cannot be applied directly. To circumvent this problem, we make use of the fact that Yosida approximations have strong solutions, which makes it possible to apply the developed strong Itô formula to these, and establish exponentially ultimate boundedness of the solutions to the Yosida approximations. Since Yosida approximations converge to the mild solution of (1.3) in $\mathcal{C}([0,T], L^p(\Omega, H))$, we then take the limit to obtain exponential ultimate boundedness of the mild solution.

Finally, we consider the problem of proving an Itô formula for mild solutions to (1.3). Since mild solutions to evolution equations are not semimartingales, the classical Itô formula for semimartingales is not applicable. Hence, it becomes necessary to develop a specific version of the Itô formula tailor-made for the evolution equation under consideration. In the case of Gaussian noise, this problem was considered by Ichikawa in [25], who provided one of the first examples of such a specialised Itô formula for mild solutions. For a more recent treatment of the Gaussian case, see also the work Da Prato et. al. [12]. For classical Lévy processes, an Itô formula for mild solutions was obtained by Alberverio et. al. in [2]. As a final result of this thesis, we provide an Itô formula for mild solutions to (1.3) by first applying our strong Itô formula to the Yosida approximations, and then carefully extending this result to mild solutions via a limiting argument.

2 Preliminaries

2.1 Generalities

Throughout this thesis, unless otherwise stated, capital letters G, H, U, V denote separable Hilbert spaces with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. We identify the dual of a Hilbert space by the space itself. The Borel σ-algebra of a Hilbert space G is denoted by $\mathfrak{B}(G)$ and we denote by $B_G(r)$ the open ball in G with centre 0 and radius r. In the special case when $r = 1$, we use the notation B_G to denote the open unit ball around the origin.

The Banach space of bounded linear operators from G to H will be denoted by $L(G, H)$ with the operator norm $\|\cdot\|_{G \to H}$, when $G = H$ we use the shorthand $L(G) := L(G, G)$. For each $T \in L(G, H)$, we denote by T^* the adjoint operator of T. The subspace $L_2(G, H) \subseteq L(G, H)$ of Hilbert-Schmidt operators is endowed with the norm $\|F\|^2_{L_2(G,H)} := \sum_{k=1}^{\infty} \|Fa_k\|^2$ for $F \in L_2(G, H)$, where $(a_k)_{k \in \mathbb{N}}$ is an orthonormal basis of G.

Let $(S, \mathcal{S}, \nu)$ be a complete finite measure space and H a Hilbert space. We denote by $L^0_\nu(S, H)$ the space of equivalence classes of $\mathcal{S}$-measurable functions $f\colon S \to H$, equipped with the topology of convergence in measure. In a similar manner, for each $p > 0$, we denote by $L^p_\nu(S, H)$ the equivalence classes of measurable functions with finite p-th moments. For $p \geq 1$, this is a Banach space when equipped with the usual norm $\|f\|^p_{L^p} := \int_S \|f(s)\|^p \, \nu(\mathrm{d}s)$, and for $0 < p < 1$ it is a metric space under the translation invariant metric $d(f, g) = \int_S \|f(s) - g(s)\|^p \, \nu(\mathrm{d}s)$. For ease of notation, we also use the notation $\|f\|_{L^p}$ to denote the metric $d(0, f)$ for $0 < p < 1$.

2.2 Infinitely divisible measures

The notion of infinitely divisible measures on a Hilbert space H can be defined in essentially the same way as in Euclidean space; see Parthasarathy [58]. Much like in finite dimensions, infinitely divisible distributions on a Hilbert space H are uniquely determined by a triplet (b, Q, λ), where $b \in H$, $Q\colon H \to H$ is nuclear and non-negative mapping, and λ is a σ-finite measure on $\mathfrak{B}(H)$ satisfying the conditions that $\lambda(\{0\}) = 0$ and $\int_H \left(\|h\|^2 \wedge 1 \right) \lambda(\mathrm{d}h) < \infty$. A measure λ on $\mathfrak{B}(H)$ satisfying these two properties is called a Lévy measure. Given any $\delta > 0$ and Lévy measure λ on $\mathfrak{B}(H)$, we say that δ is a continuity point of λ, or in short $\delta \in C(\lambda)$, if $\lambda(\{h \in H : \|h\| = \delta\}) = 0$. A sequence of infinitely divisible measures $\mu_n \overset{\mathcal{D}}{=} (b_n, Q_n, \lambda_n)$ with associated sequence $(T_n)_{n\in\mathbb{N}}$ of S-operators $T_n\colon H \to H$, defined by

$$\langle T_n h_1, h_2 \rangle = \langle Q_n h_1, h_2 \rangle + \int_{\|h_1\| \le 1} \langle h_1, u \rangle \langle h_2, u \rangle \, \lambda_n(\mathrm{d}u) \qquad \text{for all } h_1, h_2 \in H,$$

converges weakly to an infinitely divisible measure $\mu \overset{\mathcal{D}}{=} (b, Q, \lambda)$ if and only if the following conditions are satisfied:

$$(1)\ b = \lim_{\substack{\delta \downarrow 1 \\ \delta \in C(\lambda)}} \lim_{n\to\infty} \left(b_n + \int_{1 < \|h\| \le \delta} h \, \lambda_n(dh) \right); \tag{2.1}$$

$$(2)\ \lim_{\delta \downarrow 0} \limsup_{n\to\infty} \int_{\|h\| \le \delta} \langle h, u \rangle^2 \, \lambda_n(\mathrm{d}u) + (Q_n h, h) = (Qh, h) \text{ for all } h \in H; \tag{2.2}$$

$$(3)\ \lambda_n \to \lambda \text{ weakly outside of every closed neighbourhood of the origin;} \tag{2.3}$$

$$(4)\ (T_n)_{n\in\mathbb{N}} \text{ is compact in the space of nuclear operators.} \tag{2.4}$$

The necessity of these conditions can be found in [43, Pr. 5.7.4], while their sufficiency is an adaption of [58, Th. VI.5.5] to the case of a discontinuous truncation function.

In our preceding discussion of the characteristic triplet, we always assumed that the truncation function $f : H \to H$ is of the form $f(h) = h\mathbb{1}_{\bar{B}_H}$. However, when dealing with limit theorems, it is often preferable to use a continuous truncation function. In the following definition, we give a specific example of a continuous truncation function, which will play an important role in the rest of this work.

Definition 2.1. *Let $\theta\colon H \to H$ be defined by*

$$\theta(h) = \begin{cases} h & \text{if } \|h\| \leq 1; \\ \frac{h}{\|h\|} & \text{if } \|h\| > 1. \end{cases}$$

We will denote by (b^θ, Q, λ) the infinitely divisible characteristics expressed with respect to the truncation function θ. One of the advantages of using a continuous truncation function is highlighted by the following observation.

Remark 2.2. *Let $(\mathcal{I}, \|\cdot\|_0)$ denote the collection of H-valued infinitely divisible random variables endowed with a translation invariant metric $\|\cdot\|_0$ generating the topology of convergence in probability. Define the mapping*

$$g : \mathcal{I} \to H, \quad g(X) = b^\theta_X,$$

where b^θ_X denotes the first characteristic of X with respect to the truncation function θ. By [58, Th. VI.5.5], g is continuous and hence, by the topological characterisation of continuity, for all $\epsilon > 0$ there exists $\delta > 0$, depending only on ϵ and the metric $\|\cdot\|_0$, such that for all $X \in \mathcal{I}$ we have the implication:

$$\|X\|_0 < \delta \implies \left\| b^\theta_X \right\| < \epsilon.$$

2.3 Characteristics of Lévy processes

The notion of characteristics for general real-valued semimartingales has been defined in Jacod and Shiryaev [27]. In the case of quasi-left continuous real-valued semimartingales, an alternative construction was given in Kwapien and Woyczinsky [42], which was later generalised to the Hilbert-valued setting in Nowak [56].

In the special case when the quasi-left continuous semimartingale is a Lévy process L with values in a separable Hilbert space H, one might approach the problem of defining its characteristics in two different ways. On the one hand, one can define the characteristics of L through the Lévy-Khinchine formula. On the other hand, it is possible to use the arguments in Nowak [56] and define the characteristics as limits of certain increments of the process L over a suitably chosen sequence of partitions. As one might expect, these two approaches essentially lead to the same answer provided that we use the correct truncation function. An important relationship between the two definitions of characteristics is given in the following theorem, for which we first need to introduce the concept of a *nested normal sequence of partitions.*

Definition 2.3. *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions of the interval $[s,t]$ of the form*

$$\pi_n = \left\{ s = p_{0,n} < p_{1,n} < \ldots < p_{N(n),n} = t \right\}.$$

We say that $(\pi_n)_{n\in\mathbb{N}}$ is a nested normal sequence of partitions if

(1) $\pi_n \subseteq \pi_m$ *for all* $n \leq m$*;*

(2) $\lim_{n\to\infty} \max_{i\in\{1,\ldots,N(n)\}} |p_{i,n} - p_{i-1,n}| = 0.$

Theorem 2.4. *Let L be an H-valued Lévy process with characteristics (b^θ, Q, λ), and let $(\pi_n)_{n\in\mathbb{N}}$ be a nested normal sequence of partitions of $[s,t]$. If we put $d_{i,n} = L(p_{i,n}) - L(p_{i-1,n})$, then we have*

(1) $\lim_{n\to\infty} \sum_{\pi_n} E\left[\theta(d_{i,n})\right] = (t-s)b^\theta$*;*

(2) $\lim_{n\to\infty} \sum_{\pi_n} E\left[\|d_{i,n}\|^2 \wedge 1\right] = (t-s)\left(\int_H \left(\|h\|^2 \wedge 1\right) \lambda(\mathrm{d}h) + \mathrm{Tr}(Q)\right).$

Proof. For a proof, see [57, Le. 3.4], or the Appendix for an argument based purely on the theory of infinitely divisible measures. □

2.4 Cylindrical measures, random variables and processes

The first systematic treatment of the concept of a cylindrical Lévy process has been introduced in the work of Applebaum and Riedle [4]. Analogously to cylindrical Brownian motion, cylindrical Lévy processes exist only in the generalised sense of Gel'fand and Vilenkin [21] or Segal [73] as cylindrical processes. As a first step towards a rigorous definition of these generalised processes, we give a brief overview of the most crucial concepts from the theory of cylindrical measures and random variables. For a detailed account of these topics, see Schwartz [71] or Vakhania [76].

Let G be a Hilbert space and $S \subseteq G$. For each fixed $n \in \mathbb{N}$, elements $g_1, ..., g_n \in S$ and Borel set $A \in \mathfrak{B}(\mathbb{R}^n)$, we define

$$C(g_1, ..., g_n; A) := \{g \in G : (\langle g, g_1 \rangle, ..., \langle g, g_n \rangle) \in A\}.$$

Such sets are called *cylindrical sets* with respect to A. The collection of all these cylindrical sets is denoted by $\mathcal{Z}(G, S)$. In general, $\mathcal{Z}(G, S)$ forms an algebra of sets, however, in the special case when S is finite, it becomes a σ-algebra. We will use the shorthand $\mathcal{Z}(G)$ to denote $\mathcal{Z}(G, G)$.

Definition 2.5. *A set function $\mu : \mathcal{Z}(G) \to [0, \infty]$ is called a cylindrical measure on $\mathcal{Z}(G)$ if for each finite subset $S \subseteq G$, the restriction of μ to the σ-algebra $\mathcal{Z}(G, S)$ is a σ-additive measure.*

A cylindrical measure μ on $\mathcal{Z}(G)$ is said to be finite if $\mu(G) < \infty$, and is called a *cylindrical probability measure* if we further require that $\mu(G) = 1$.

Definition 2.6. *A cylindrical random variable X in G is a linear and continuous mapping $X\colon G \to L_P^0(\Omega, \mathbb{R})$.*

Analogously to the case of real-valued random variables, each cylindrical random variable X in G defines a cylindrical probability measure μ_X by

$$\mu_X\colon \mathcal{Z}(G) \to [0,1], \qquad \mu_X(Z) = P\big((Xg_1, \ldots, Xg_n) \in A\big),$$

for cylindrical sets $Z = C(g_1, ..., g_n; A)$. A cylindrical probability measure μ_X obtained from a cylindrical random variable X through the above definition is called the *cylindrical distribution* of X. One can define the characteristic function φ_X of a cylindrical random variable X by

$$\varphi_X\colon G \to \mathbb{C}, \qquad \varphi_X(g) = E\big[e^{iXg}\big].$$

Assume that H is another Hilbert space and let $T\colon G \to H$ be a continuous, linear operator. By defining

$$TX\colon H \to L_P^0(\Omega, \mathbb{R}), \qquad (TX)h = X(T^*h),$$

we obtain a cylindrical random variable on H. An important special case of this transformation is when T is a Hilbert-Schmidt operator and hence 0-Radonifying by [76, Th. VI.5.2]. Then, it follows from [76, Pr. VI.5.3] that the cylindrical random variable TX is induced by an H-valued random variable $Y\colon \Omega \to H$ in the sense that $(TX)h = \langle Y, h\rangle$ for all $h \in H$. This procedure of mapping cylindrical random variables to classical ones is called Radonification. The following result shows that the inducing random variable Y depends continuously on the Hilbert-Schmidt operator.

Lemma 2.7. *Let X be a cylindrical random variable and $(F_n)_{n\in\mathbb{N}}$ a sequence in $L_2(G,H)$ converging to F in $\|\cdot\|_{L_2(G,H)}$. Then $(F_nX)_{n\in\mathbb{N}}$ converges to FX in $L_P^0(\Omega,H)$.*

Proof. Let μ_X denote the cylindrical distribution of X. As the sequence $(F_n)_{n\in\mathbb{N}}$ is compact in $L_2(G,H)$, the collection of measures $\{\mu_X \circ F_n^{-1} : n \in \mathbb{N}\}$ is relatively compact in the space of probability measures on $\mathfrak{B}(H)$; see [30, Pr. 5.3]. Continuity of X implies for all $h \in H$ that

$$\lim_{n\to\infty} \langle F_nX, h\rangle = \lim_{n\to\infty} X(F_n^*h) = X(F^*h) = \langle FX, h\rangle \qquad \text{in } L_P^0(\Omega,\mathbb{R}).$$

Together with relative compactness, this implies that $(F_nX)_{n\in\mathbb{N}}$ converges to FX in $L_P^0(\Omega,H)$; see e.g. [28, Le. 2.4]. □

Having defined the notion of a cylindrical random variable, we can now introduce cylindrical processes in the following natural way.

Definition 2.8. *An indexed family $(X(t) : t \geq 0)$ of cylindrical random variables is called a cylindrical process.*

In this thesis, we will be interested in a special subclass of cylindrical processes, the so called cylindrical Lévy processes. Simply put, these are cylindrical processes satisfying that their projections onto finite-dimensional subspaces become Lévy processes in the classical sense. This is made precise in the definition below.

Definition 2.9. *A family $(L(t) : t \geq 0)$ of cylindrical random variables $L(t)\colon G \to L_P^0(\Omega,\mathbb{R})$ is called a cylindrical Lévy process if for each $n \in \mathbb{N}$ and $g_1, ..., g_n \in G$, the stochastic process*

$$\big((L(t)g_1, ..., L(t)g_n) : t \geq 0\big)$$

is a Lévy process in $\mathbb{R}^n$.

One of the cornerstones of the theory of classical Lévy processes is the Lévy-Khinchine formula, which provides an invaluable representation of the characteristic function of Lévy processes. As it turns out, it is possible to obtain an analogous representation for the characteristic function of cylindrical Lévy processes, provided one defines the correct cylindrical analogue of Lévy measures. This is accomplished in the following.

We denote by $\mathcal{Z}_*(G)$ the collection

$$\big\{\{g \in G : (\langle g, g_1\rangle, ..., \langle g, g_n\rangle) \in B\} : n \in \mathbb{N}, g_1, ..., g_n \in G, B \in \mathfrak{B}(\mathbb{R}^n \setminus \{0\})\big\}$$

of cylindrical sets, which forms an algebra of subsets of G. Let L be a cylindrical Lévy process in G. For fixed $g_1, ..., g_n \in G$, we denote by $\lambda_{g_1,...,g_n}$ the Lévy measure of the n-dimensional Lévy process $\big((L(t)g_1, ..., L(t)g_n) : t \geq 0\big)$ obtained via the projection of the cylindrical Lévy process L onto the coordinate functions $g_1, ..., g_n$.

Definition 2.10. *We define a function* $\lambda\colon \mathcal{Z}_*(G) \to [0, \infty]$ *by*

$$\lambda(C) := \lambda_{g_1,...,g_n}(B) \quad \text{for } C = \{g \in G : (\langle g, g_1\rangle, ..., \langle g, g_n\rangle) \in B\},$$

where $B \in \mathfrak{B}(\mathbb{R}^n \setminus \{0\})$*. The fact that* λ *is well-defined is shown in [4]. The set function* λ *obtained in this manner is called the cylindrical Lévy measure of* L*.*

Similarly to how classical Lévy processes are related to infinitely divisible measures, cylindrical Lévy processes are related to the class of infinitely divisible cylindrical measures. For an in-depth study of the theory of infinitely divisible cylindrical measures, see [66]. Building on the theoretical foundations laid down in [66], we know that for a cylindrical Lévy process L in G, the characteristic function of $L(t)$ for each $t \geq 0$ takes the form

$$\varphi_{L(t)} : G \to \mathbb{C}, \qquad \varphi_{L(t)}(g) = \exp\left(tS(g)\right),$$

where the mapping $S : G \to \mathbb{C}$ is called the *cylindrical symbol* of L, and satisfies

$$S(g) = ia(g) - \frac{1}{2}\langle Qg, g\rangle + \int_G \left(e^{i\langle g,h\rangle} - 1 - i\langle g, h\rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle g, h\rangle) \right) \lambda(\mathrm{d}h).$$

Here, $a : G \to \mathbb{R}$ is a continuous mapping with $a(0) = 0$, $Q : G \to G$ is a positive and symmetric operator, and λ is a cylindrical Lévy measure on G. We call the triplet (a, Q, λ) the *cylindrical characteristics* of L.

In the second half of this thesis, we restrict our attention to the important class of standard symmetric α-stable cylindrical Lévy processes in G for $\alpha \in (0, 2)$. These are cylindrical Lévy processes with characteristic function $\varphi_{L(t)}(g) = \exp(-t \|g\|^\alpha)$ for each $t \geq 0$ and $g \in G$. For each $n \in \mathbb{N}$ and $g_1, ..., g_n \in G$ we define the projection $\pi_{g_1,...,g_n} : G \to \mathbb{R}^n$ via the prescription

$$\pi_{g_1,...,g_n}(g) = (\langle g, g_1\rangle, ..., \langle g, g_n\rangle).$$

If $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of G, then it follows from [68, Le. 2.4] that the cylindrical Lévy measure λ of the standard symmetric α-stable cylindrical Lévy processes in G satisfies the spectral representation

$$\lambda \circ \pi^{-1}_{e_1,...,e_n}(B) = \frac{\alpha}{c_\alpha} \int_{S_{\mathbb{R}^n}} \nu_n(\mathrm{d}x) \int_0^\infty \mathbb{1}_B(rx) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r \quad \text{for } B \in \mathfrak{B}(\mathbb{R}^n), \tag{2.5}$$

where $S_{\mathbb{R}^n} := \{\beta \in \mathbb{R}^n : |\beta| = 1\}$, $c_\alpha > 0$ is a constant dependent only on α, and ν_n denotes a uniform distribution on the sphere $S_{\mathbb{R}^n}$.

3 Stochastic integration with respect to cylindrical Lévy processes

3.1 Construction of the modular space

Originally introduced by Nakano [54], modular spaces serve as natural generalizations of Banach spaces. While numerous different definitions appear in the literature, see [20] for an overview of these, in this work we will always use the following adaption of Nakano's definition of a generalized modular, see [55].

Definition 3.1. *Let V be a real vector space. A function $\Delta : V \to [0, \infty]$ is called a modular if*

(1) $\Delta(-v) = \Delta(v)$ *for all* $v \in V$*;*

(2) $\inf_{\alpha > 0} \Delta(\alpha v) = 0$ *for all* $v \in V$*;*

(3) $\Delta(\alpha v) \le \Delta(\beta v)$ *for all* $0 \le \alpha \le \beta$ *and* $v \in V$*;*

(4) *there exists a constant* $c > 0$ *such that*

$$\Delta(v + w) \le c\,(\Delta(v) + \Delta(w)) \quad \textit{for all } v, w \in V.$$

Remark 3.2. *A function satisfying Condition* (4) *of Definition 3.1 is said to be of moderate growth.*

It is known, see for example [76], that Hilbert-Schmidt operators between Hilbert spaces map cylindrical random variables to genuine random variables. As it turns out, a similar correspondence can be established between cylindrical and genuine Lévy processes.

Lemma 3.3. *Let $(L(t) : t \geq 0)$ be a cylindrical Lévy process in G with cylindrical characteristics (a, Q, λ), and let $F \in L_2(G, H)$ be a Hilbert-Schmidt operator. Then, there exists an H-valued Lévy process $(L^F(t) : t \geq 0)$ satisfying for all $t \in [0, T]$ and $h \in H$ that $\langle L^F(t), h\rangle = L(t)(F^*h)$. Moreover, L^F has characteristics $(b_F, FQF^*, \lambda \circ F^{-1})$, where for all $u \in H$*

$$\langle b_F, u\rangle = a(F^*u) + \int_H \langle h, u\rangle \left(\mathbb{1}_{B_H}(h) - \mathbb{1}_{B_\mathbb{R}}(\langle h, u\rangle)\right) (\lambda \circ F^{-1})(\mathrm{d}h).$$

Proof. Existence of the H-valued Lévy process L^F follows from [29, Th.A]. To derive the characteristics, first apply [67, Le. 5.4] to obtain the cylindrical characteristics of L^F, and then use [67, Le. 5.8] to convert the cylindrical characteristics into genuine characteristics. □

Remark 3.4. *Let $F_1, F_2 \in L_2(G, H)$. Then, it follows from linearity of the cylindrical random variables $L(t)$ that $L^{F_1+F_2}(t) = L^{F_1}(t) + L^{F_2}(t)$ for all $t \geq 0$.*

Remark 3.5. *Note that in the special case, when the truncation function is θ, see Definition 2.1, the first characteristic b_F^θ satisfies for all $u \in H$ that*

$$\langle b_F^\theta, u\rangle = a(F^*u) + \int_H \langle \theta(h), u\rangle - \langle h, u\rangle \mathbb{1}_{B_\mathbb{R}}(\langle h, u\rangle) \left(\lambda \circ F^{-1}\right)(\mathrm{d}h). \tag{3.1}$$

For the rest of this chapter, we fix a cylindrical Lévy process L with cylindrical characteristics (a, Q, λ). Our aim is to define a modular m_L on a suitable subspace of the vector space of all measurable, Hilbert-Schmidt operator-valued functions $\psi : [0, T] \to L_2(G, H)$, with m_L explicitly expressed in terms of the cylindrical characteristics of L. The following functions play a key role in the definition of our modular.

Definition 3.6. *We define the functions $k_L, l_L : L_2(G, H) \to \mathbb{R}$ by*

$$
\begin{aligned}
k_L(F) &= \int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ F^{-1}\right)(\mathrm{d}h) + \mathrm{Tr}(FQF^*); \\
l_L(F) &= \sup_{O \in L(H)^1} \left\| b^\theta_{OF} \right\|,
\end{aligned}
$$

where $L(H)^1$ denotes the collection of bounded linear operators $O : H \to H$ satisfying $\|O\|_{H \to H} \leq 1$, (a, Q, λ) are the cylindrical characteristics of L, and for each $O \in L(H)^1$ the expression b^θ_{OF} denotes the first characteristic of the Radonified Lévy process L^{OF} as defined in Equation 3.1 above.

We are now ready to give the definition of our modular m_L.

Definition 3.7. *For a measurable function $\psi : [0, T] \to L_2(G, H)$ we define*

$$
\begin{aligned}
m'_L(\psi) &:= \int_0^T k_L(\psi(t)) + l_L(\psi(t))\, \mathrm{d}t; \\
m''(\psi) &:= \int_0^T \left(\|\psi(t)\|^2_{L_2(G,H)} \wedge 1\right) \mathrm{d}t; \\
m_L(\psi) &:= m'_L(\psi) + m''(\psi).
\end{aligned}
$$

We denote by $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L} := \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}(G, H)$ the space of Lebesgue a.e. equivalence classes of measurable functions $\psi : [0, T] \to L_2(G, H)$ for which $m_L(\psi) < \infty$.

Remark 3.8. *The fact that the integrals in the above definition are well defined follows from Lemma 3.12 below.*

The rest of this chapter will be devoted to proving that $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ is a vector space and m_L is a modular on $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ in the sense of Definition 3.1. As a first step towards this direction, the next lemma provides us with an alternative representation of l_L. This will be heavily used in the sequel when we investigate various properties of the modular.

Lemma 3.9. *Let L be a cylindrical Lévy process in G with characteristics (a, Q, λ). For all $F \in L_2(G,H)$ and $O \in L(H)$ it holds that*

$$b^{\theta}_{OF} = Ob^{\theta}_F + \int_H \theta(Oh) - O\theta(h)\,(\lambda \circ F^{-1})(\mathrm{d}h).$$

Proof. The proof follows from a direct calculation using the characteristic function of the Lévy process $(L^{OF}_t)_{t\geq 0}$. To get the left hand side, we Radonify L by the composition OF and apply Lemma 3.3. To obtain the right hand side, first we use Lemma 3.3 to Radonify L by F, and then transform the genuine Lévy process $(L^F_t)_{t\geq 0}$ by O using [11, Th. 4.1] to get another Lévy process. □

Remark 3.10. *It follows from Lemma 3.9 that for each $F \in L_2(G,H)$ the expression $l_L(F)$ is finite. To see this, we first note that for all $h \in \bar{B}_H$ and $O \in L(H)^1$ we have $\|\theta(Oh) - O\theta(h)\| = 0$, and for all $h \in H$ and $O \in L(H)^1$ it holds that $\|\theta(Oh) - O\theta(h)\| \leq 2$. By combining these observations with Lemma 3.9 we obtain*

$$\begin{aligned}\sup_{O\in L(H)^1} \left\|b^{\theta}_{OF}\right\| &\leq \sup_{O\in L(H)^1} \left\|Ob^{\theta}_F\right\| + \sup_{O\in L(H)^1} \left\|\int_H \theta(Oh) - O\theta(h)\,(\lambda \circ F^{-1})(\mathrm{d}h)\right\| \\ &\leq \left\|b^{\theta}_F\right\| + 2(\lambda \circ F^{-1})(\bar{B}^c_H) < \infty.\end{aligned}$$

Before we could prove that our modular m_L is well-defined, we need to establish a relationship between weak convergence of infinitely divisible measures and convergence of the corresponding characteristics in the following sense:

Lemma 3.11. *Let $\mu_n \stackrel{\mathcal{D}}{=} (b^{\theta}_n, Q_n, \lambda_n)$ be a sequence of infinitely divisible measures on $\mathfrak{B}(H)$ converging weakly to $\mu \stackrel{\mathcal{D}}{=} (b^{\theta}, Q, \lambda)$. Then, the following conditions hold:*

(1) $\lim_{n\to\infty} \left(\int_H \left(\|h\|^2 \wedge 1\right)\, \lambda_n\,(\mathrm{d}h) + \mathrm{Tr}(Q_n)\right) = \int_H \left(\|h\|^2 \wedge 1\right)\, \lambda\,(\mathrm{d}h) + \mathrm{Tr}(Q)$;

(2) $\lim_{n\to\infty} \left\|b^{\theta}_n - b^{\theta}\right\| = 0$.

Proof. The fact that (2) holds follows directly from [58, Th. VI.5.5/(1)]. To prove (1), fix $\delta \in (0,1]$ such that $\delta \in C(\lambda)$. By [58, Th. VI.5.5/(2)] we have

$$\lim_{n\to\infty} \int_{\|h\|>\delta} \left(\|h\|^2 \wedge 1 \right) \lambda_n(\mathrm{d}h) = \int_{\|h\|>\delta} \left(\|h\|^2 \wedge 1 \right) \lambda(\mathrm{d}h).$$

Therefore, it remains only to deal with the limit of the integrals over $\bar{B}_H(\delta)$. Let $\epsilon > 0$ be fixed. It follows from properties of the Lebesgue integral that there exists a $\delta_1 \in (0, \delta]$ such that

$$\int_{\|h\|\le\delta_1} \|h\|^2 \, \lambda(\mathrm{d}h) < \frac{\epsilon}{12}. \tag{3.2}$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H. Since Q is a trace class operator, there exists $K_1 \in \mathbb{N}$ such that

$$\sum_{k=K_1+1}^{\infty} \langle Qe_k, e_k\rangle < \frac{\epsilon}{12}.$$

By compactness of the associated S-operators, see [58, Th. VI.5.5/(3)], there exists $K_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\sum_{k=K_2+1}^{\infty} \left(\int_{\|h\|\le\delta} \langle e_k, h\rangle^2 \, \lambda_n(\mathrm{d}h) + \langle Q_n e_k, e_k\rangle \right) < \frac{\epsilon}{4}. \tag{3.3}$$

Moreover, by another application of [58, Th. VI.5.5/(3)], there exists a $\delta_2 < \delta_1$ and $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$ and for all $k \le K_2$ we have that

$$\left| \int_{\|h\|\le\delta_2} \langle e_k, h\rangle^2 \, \lambda_n(\mathrm{d}h) + \langle Q_n e_k, e_k\rangle - \langle Qe_k, e_k\rangle \right| < \frac{\epsilon}{12K}, \tag{3.4}$$

where $K := \max\{K_1, K_2\}$. Furthermore, by [58, Th. VI.5.5/(2)] , there exists $N_2 \in \mathbb{N}$

such that for all $n \geq N_2$ we have that

$$\left| \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda_n(\mathrm{d}h) - \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda(\mathrm{d}h) \right| < \frac{\epsilon}{2}. \tag{3.5}$$

Rewriting the integral over $\bar{B}_H(\delta)$ as

$$\int_{\|h\| \leq \delta} \|h\|^2 \ \lambda(\mathrm{d}h) = \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda(\mathrm{d}h) + \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda(\mathrm{d}h),$$

we obtain that

$$\begin{aligned} &\left| \int_{\|h\| \leq \delta} \|h\|^2 \ \lambda(\mathrm{d}h) + \mathrm{Tr}(Q) - \int_{\|h\| \leq \delta} \|h\|^2 \ \lambda_n(\mathrm{d}h) - \mathrm{Tr}(Q_n) \right| \\ &\leq \left| \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda(\mathrm{d}h) - \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda_n(\mathrm{d}h) \right| \\ &\qquad + \left| \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda(\mathrm{d}h) + \mathrm{Tr}(Q) - \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda_n(\mathrm{d}h) - \mathrm{Tr}(Q_n) \right|. \end{aligned} \tag{3.6}$$

We define $N := \max\{N_1, N_2\}$. If $n \geq N$ then Equation (3.5) implies

$$\left| \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda_n(\mathrm{d}h) - \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \ \lambda(\mathrm{d}h) \right| < \frac{\epsilon}{2}. \tag{3.7}$$

Thus, it remains only to control the second term on the right hand side of Equation (3.6). By Parseval's identity, Equations (3.2)-(3.4) and a repeated application of the triangle inequality, we obtain for all $n \geq N$ that

$$\begin{aligned} &\left| \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda(\mathrm{d}h) + \mathrm{Tr}(Q) - \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda_n(\mathrm{d}h) - \mathrm{Tr}(Q_n) \right| \\ &\leq \left| \mathrm{Tr}(Q) - \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda_n(\mathrm{d}h) - \mathrm{Tr}(Q_n) \right| + \left| \int_{\|h\| \leq \delta_2} \|h\|^2 \ \lambda(\mathrm{d}h) \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{k=1}^{\infty} \left(\langle Qe_k, e_k \rangle - \int_{\|h\| \le \delta_2} \langle h, e_k \rangle^2 \, \lambda_n(\mathrm{d}h) - \langle Q_n e_k, e_k \rangle \right) \right| + \left| \int_{\|h\| \le \delta_2} \|h\|^2 \, \lambda(\mathrm{d}h) \right| \\
&= \left| \sum_{k=1}^{K} \left(\langle Qe_k, e_k \rangle - \int_{\|h\| \le \delta_2} \langle h, e_k \rangle^2 \, \lambda_n(\mathrm{d}h) - \langle Q_n e_k, e_k \rangle \right) \right| \\
&\quad + \left| \sum_{k=K+1}^{\infty} \left(\langle Qe_k, e_k \rangle - \int_{\|h\| \le \delta_2} \langle h, e_k \rangle^2 \, \lambda_n(\mathrm{d}h) - \langle Q_n e_k, e_k \rangle \right) \right| + \left| \int_{\|h\| \le \delta_2} \|h\|^2 \, \lambda(\mathrm{d}h) \right| \\
&< \frac{\epsilon}{2}. \qquad (3.8)
\end{aligned}
$$

Hence, if $n \geq N$ then Equations (3.7) and (3.8) together imply

$$
\left| \int_{\|h\| \le \delta} \|h\|^2 \, \lambda(\mathrm{d}h) + \mathrm{Tr}(Q) - \int_{\|h\| \le \delta} \|h\|^2 \, \lambda_n(\mathrm{d}h) - \mathrm{Tr}(Q_n) \right| < \epsilon.
$$

Since $\epsilon > 0$ was arbitrary, the result follows. □

Lemma 3.12. *Let $k_L, l_L : L_2(G,H) \to \mathbb{R}$ be as in Definition 3.6. Then we have:*

(1) *k_L is continuous;*

(2) *l_L is lower-semicontinuous and continuous at 0.*

Proof. Continuity of k_L follows immediately from Lemmata 2.7 and 3.11. To prove lower-semicontinuity of l_L, we fix $F \in L_2(G,H)$ and a sequence $(F_n)_{n\in\mathbb{N}} \subseteq L_2(G,H)$ satisfying that $\lim_{n\to\infty} \|F_n - F\|_{L_2(G,H)} = 0$. Let $\epsilon > 0$ be fixed. Since Remark 3.10 implies that for each $F \in L_2(G,H)$ the expression $\sup_{O\in L(H)^1} \left\|b^\theta_{OF}\right\|$ is finite, by the very definition of the supremum, there exists $O_\epsilon \in L(H)^1$ such that $\sup_{O\in L(H)^1} \left\|b^\theta_{OF}\right\| \leq \left\|b^\theta_{O_\epsilon F}\right\| + \epsilon$. Moreover, for this O_ϵ, we have by Lemma 2.7 and [58, Th. VI.5.5] that $\lim_{n\to\infty} \left\|b^\theta_{O\epsilon F_n}\right\| = \left\|b^\theta_{O\epsilon F}\right\|$. Since when the limit exists it is equivalent to the limit inferior, we obtain

$$\sup_{O\in L(H)^1}\left\|b^\theta_{OF}\right\| \le \left\|b^\theta_{O_\epsilon F}\right\| + \epsilon = \lim_{n\to\infty}\left\|b^\theta_{O_\epsilon F_n}\right\| + \epsilon$$
$$= \liminf_{n\to\infty}\left\|b^\theta_{O_\epsilon F_n}\right\| + \epsilon \le \liminf_{n\to\infty}\sup_{O\in L(H)^1}\left\|b^\theta_{OF_n}\right\| + \epsilon.$$

As $\epsilon > 0$ is arbitrary, the above shows that

$$l_L(F) \le \liminf_{n\to\infty} l_L(F_n),$$

which proves lower-semicontinuity of l_L. To show continuity of l_L at 0, note that by Lemma 2.7 and Remark 2.2, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|F\|_{L_2(G,H)} \le \delta \implies \left\|b^\theta_F\right\| \le \epsilon$. Since for all $O \in L(H)^1$ it holds that $\|OF\|_{L_2(G,H)} \le \|F\|_{L_2(G,H)}$, we have

$$\|F\|_{L_2(G,H)} \le \delta \implies \sup_{O\in L(H)^1}\left\|b^\theta_{OF}\right\| \le \epsilon.$$

This concludes the proof. □

In preparation for showing that m_L is of moderate growth, see Definition 3.1/(4), we prove the following technical lemmata.

Lemma 3.13. *Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of G and let $P_n\colon G \to G$ be the projection onto Span$\{e_1, ..., e_n\}$. Then, for all $F \in L_2(G,H)$ we have*

$$\lim_{n\to\infty}\|FP_n - F\|_{L_2(G,H)} = 0.$$

Proof. Since $P_n e_i = e_i$ for $i \le n$, and $P_n e_i = 0$ for $i > n$, we have

$$\|FP_n - F\|^2_{L_2(G,H)} = \sum_{i=1}^{\infty}\|(FP_n - F)e_i\|^2_H = \sum_{i=n+1}^{\infty}\|Fe_i\|^2_H.$$

As F is Hilbert-Schmidt, the sum $\sum_{i=1}^{\infty} \|Fe_i\|_H^2$ converges, which implies that

$$\lim_{n\to\infty} \|FP_n - F\|_{L_2(G,H)}^2 = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} \|Fe_i\|_H^2 = 0,$$

which concludes the proof of our claim. □

Lemma 3.14. *For all $F, F_1, F_2 \in L_2(G,H)$ we have*

(1) $k_L(F_1 + F_2) \le 2\,(k_L(F_1) + k_L(F_2))$;

(2) $\sup_{O \in L(H)^1} k_L(OF) \le k_L(F)$.

Proof. Let $P_n\colon G \to G$ denote the projections from Lemma 3.13. Using the inequality

$$(a+b)^2 \wedge 1 \le 2\left[(a^2 \wedge 1) + (b^2 \wedge 1)\right] \text{ for all } a, b \in \mathbb{R}, \tag{3.9}$$

we observe that for each $n \in \mathbb{N}$

$$\begin{aligned}
&\int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ ((F_1+F_2)P_n)^{-1}\right)(\mathrm{d}h) \\
=&\int_H \left(\|h\|^2 \wedge 1\right) \left((\lambda \circ P_n^{-1}) \circ (F_1+F_2)^{-1}\right)(\mathrm{d}h) \\
=&\int_G \left(\|(F_1+F_2)g\|^2 \wedge 1\right) \left(\lambda \circ P_n^{-1}\right)(\mathrm{d}g) \\
\le&2\left(\int_G \left(\|F_1 g\|^2 \wedge 1\right) \left(\lambda \circ P_n^{-1}\right)(\mathrm{d}g) + \int_G \left(\|F_2 g\|^2 \wedge 1\right) \left(\lambda \circ P_n^{-1}\right)(\mathrm{d}g)\right) \\
=&2\left(\int_H \left(\|h\|^2 \wedge 1\right) \left((\lambda \circ P_n^{-1}) \circ F_1^{-1}\right)(\mathrm{d}h) + \int_H \left(\|h\|^2 \wedge 1\right) \left((\lambda \circ P_n^{-1}) \circ F_2^{-1}\right)(\mathrm{d}h)\right) \\
=&2\left(\int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ (F_1 P_n)^{-1}\right)(\mathrm{d}h) + \int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ (F_2 P_n)^{-1}\right)(\mathrm{d}h)\right).
\end{aligned} \tag{3.10}$$

Moreover, by symmetry and positivity of Q, and basic properties of the trace operator

$$\begin{aligned}
&\operatorname{Tr}\left(((F_1+F_2)P_n)Q((F_1+F_2)P_n)^*\right)\\
&\quad= \operatorname{Tr}((F_1P_n)Q(F_1P_n)^*) + \operatorname{Tr}((F_1P_n)Q(F_2P_n)^*)\\
&\qquad\qquad + \operatorname{Tr}((F_2P_n)Q(F_1P_n)^*) + \operatorname{Tr}((F_2P_n)Q(F_2P_n)^*)\\
&\quad\le \left(\left\|F_1P_nQ^{1/2}\right\|_{L_2(G,H)} + \left\|F_2P_nQ^{1/2}\right\|_{L_2(G,H)}\right)^2\\
&\quad\le 2\left(\left\|F_1P_nQ^{1/2}\right\|^2_{L_2(G,H)} + \left\|F_2P_nQ^{1/2}\right\|^2_{L_2(G,H)}\right)\\
&\quad= 2\left(\operatorname{Tr}((F_1P_n)Q(F_1P_n)^*) + \operatorname{Tr}((F_2P_n)Q(F_2P_n)^*)\right).
\end{aligned} \tag{3.11}$$

By adding the Inequalities in (3.10) and (3.11) we get

$$k_L((F_1+F_2)P_n) \le 2\left(k_L(F_1P_n) + k_L(F_2P_n)\right).$$

By taking limits on both sides, and using continuity of k_L, see Lemma 3.12/(1), the first part of this Lemma is now proved.

To prove the second part, we fix $F \in L_2(G,H)$ and obtain for all $O \in L(H)^1$ and $n \in \mathbb{N}$ that

$$\begin{aligned}
\int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ (OFP_n)^{-1}\right)(\mathrm{d}h) &= \int_G \left(\|(OF)g\|^2 \wedge 1\right) \left(\lambda \circ P_n^{-1}\right)(\mathrm{d}g)\\
&\le \int_G \left(\|Fg\|^2 \wedge 1\right) \left(\lambda \circ P_n^{-1}\right)(\mathrm{d}g)\\
&= \int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ (FP_n)^{-1}\right)(\mathrm{d}h).
\end{aligned} \tag{3.12}$$

Moreover, using the relationship between the Hilbert-Schmidt norm and the trace operator, we obtain for all $O \in L(H)^1$ and $n \in \mathbb{N}$ that

$$\begin{aligned}\operatorname{Tr}((OFP_n)Q(OFP_n)^*) &= \operatorname{Tr}((OFP_nQ^{1/2})(OFP_nQ^{1/2})^*) \\ &= \left\| OFP_nQ^{1/2} \right\|^2_{L_2(G,H)} \\ &\leq \left\| FP_nQ^{1/2} \right\|^2_{L_2(G,H)} = \operatorname{Tr}((FP_n)Q(FP_n)^*).\end{aligned} \tag{3.13}$$

By adding Inequalities (3.12) and (3.13), and taking limits on both sides, the result now follows from Lemmata 3.11 and 3.13. □

Lemma 3.15. *For all $\psi_1, \psi_2 \in \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ we have*

$$m_L(\psi_1 + \psi_2) \leq 4\left(m_L\left(\psi_1\right) + m_L\left(\psi_2\right)\right).$$

Proof. Let $F_1, F_2 \in L_2(G,H)$ and $(\pi_n)_{n\in\mathbb{N}}$ be a nested normal sequence of partitions of the interval $[0,1]$. By Lemma 3.3 and the limit characterisation of Lévy characteristics in Theorem 2.4, we obtain

$$b^{\theta}_{F_1+F_2} = \lim_{n\to\infty} \sum_{i=1}^{N(n)} E\left[\theta\left(L^{F_1+F_2}(t_{i,n}) - L^{F_1+F_2}(t_{i-1,n})\right)\right].$$

By Remark 3.4, we can rewrite the sum as

$$\sum_{i=1}^{N(n)} E\left[\theta\left(\left(L^{F_1}(t_{i,n}) - L^{F_1}(t_{i-1,n})\right) + \left(L^{F_2}(t_{i,n}) - L^{F_2}(t_{i-1,n})\right)\right)\right].$$

In order to simplify the notation, for each $n \in \mathbb{N}$ and $i \in \{1, ..., N(n)\}$ we define

$$A_{i,n} := L^{F_1}(t_{i,n}) - L^{F_1}(t_{i-1,n}) \text{ and } B_{i,n} := L^{F_2}(t_{i,n}) - L^{F_2}(t_{i-1,n}).$$

Using this notation, an application of the triangle inequality yields

$$
\begin{aligned}
&\left\| b^{\theta}_{F_1+F_2} \right\| \\
&= \lim_{n\to\infty} \left\| \sum_{i=1}^{N(n)} E\left[\theta(A_{i,n}+B_{i,n})\right] \right\| \\
&= \lim_{n\to\infty} \left\| \sum_{i=1}^{N(n)} E\left[\theta(A_{i,n}+B_{i,n}) - \theta(A_{i,n}) - \theta(B_{i,n})\right] + \sum_{i=1}^{N(n)} E\left[\theta(A_{i,n}) + \theta(B_{i,n})\right] \right\| \\
&\leq \lim_{n\to\infty} \left(\left\| \sum_{i=1}^{N(n)} E\left[\theta(A_{i,n}+B_{i,n}) - \theta(A_{i,n}) - \theta(B_{i,n})\right] \right\| + \left\| \sum_{i=1}^{N(n)} E\left[\theta(A_{i,n}) + \theta(B_{i,n})\right] \right\| \right).
\end{aligned}
\tag{3.14}
$$

Applying the inequality

$$
\|\theta(h_1+h_2) - \theta(h_1) - \theta(h_2)\| \leq 2\left(\theta(\|h_1\|)^2 + \theta(\|h_2\|)^2\right) \quad \text{for all } h_1, h_2 \in H,
$$

we see that

$$
\begin{aligned}
&\left\| \sum_{i=1}^{N(n)} E\left[\theta(A_{i,n}+B_{i,n}) - \theta(A_{i,n}) - \theta(B_{i,n})\right] \right\| \\
&\qquad \leq \sum_{i=1}^{N(n)} E\left[\|\theta(A_{i,n}+B_{i,n}) - \theta(A_{i,n}) - \theta(B_{i,n})\|\right] \\
&\qquad \leq \sum_{i=1}^{N(n)} E\left[2\left(\theta(\|A_{i,n}\|)^2 + \theta(\|B_{i,n}\|)^2\right)\right] \\
&\qquad = 2\sum_{i=1}^{N(n)} E\left[\theta(\|A_{i,n}\|)^2\right] + 2\sum_{i=1}^{N(n)} E\left[\theta(\|B_{i,n}\|)^2\right].
\end{aligned}
\tag{3.15}
$$

Moreover, by the triangle inequality we have

$$\left\|\sum_{i=1}^{N(n)} E\left[\theta(A_{i,n})+\theta(B_{i,n})\right]\right\| \leq \left\|\sum_{i=1}^{N(n)} E\left[\theta(A_{i,n})\right]\right\| + \left\|\sum_{i=1}^{N(n)} E\left[\theta(B_{i,n})\right]\right\|. \tag{3.16}$$

Combining Equations (3.14)-(3.16), we get

$$\begin{aligned}\left\|b^{\theta}_{F_1+F_2}\right\| \leq \lim_{n\to\infty}\Bigg(2\sum_{i=1}^{N(n)} E\left[\theta(\|A_{i,n}\|)^2\right] + 2\sum_{i=1}^{N(n)} E\left[\theta(\|B_{i,n}\|)^2\right] \\ + \left\|\sum_{i=1}^{N(n)} E\left[\theta(A_{i,n})\right]\right\| + \left\|\sum_{i=1}^{N(n)} E\left[\theta(B_{i,n})\right]\right\|\Bigg).\end{aligned}$$

By taking the limit as $n \to \infty$ and using the limit characterisation of Lévy characteristics from Theorem 2.4, we obtain

$$\begin{aligned}\left\|b^{\theta}_{F_1+F_2}\right\| \leq\ & 2\left(\int_H \left(\|h\|^2 \wedge 1\right)\left(\lambda \circ F_1^{-1}\right)(\mathrm{d}h) + \mathrm{Tr}(F_1 Q F_1^*)\right) \\ &+\ 2\left(\int_H \left(\|h\|^2 \wedge 1\right)\left(\lambda \circ F_2^{-1}\right)(\mathrm{d}h) + \mathrm{Tr}(F_2 Q F_2^*)\right) \\ &+ \left\|b^{\theta}_{F_1}\right\| + \left\|b^{\theta}_{F_2}\right\| \\ =\ & 2\left(k_L(F_1)+k_L(F_2)\right) + \left\|b^{\theta}_{F_1}\right\| + \left\|b^{\theta}_{F_2}\right\|. \end{aligned} \tag{3.17}$$

Therefore, by Equation (3.17) and Lemma 3.14/(2), we get

$$l_L(F_1+F_2) := \sup_{O\in L(H)^1} \left\|b^{\theta}_{O(F_1+F_2)}\right\| \leq 2\left(k_L(F_1)+k_L(F_2)\right) + l_L(F_1) + l_L(F_2). \tag{3.18}$$

Combining Equation (3.18) with Lemma 3.14/(1) yields that

$$k_L(F_1+F_2) + l_L(F_1+F_2) \leq 4\left(k_L(F_1)+k_L(F_2)+l_L(F_1)+l_L(F_2)\right). \tag{3.19}$$

Hence, an application of Equation (3.19) and Inequality (3.9) implies for all measurable functions $\psi_1, \psi_2 \in \mathcal{M}^{\text{HS}}_{\text{det},L}$ that

$$\begin{aligned}
&m_L(\psi_1+\psi_2)\\
&= \int_0^T k_L\big(\psi_1(t)+\psi_2(t)\big) + l_L\big(\psi_1(t)+\psi_2(t)\big)\,\mathrm{d}t + \int_0^T \left(\|\psi_1(t)+\psi_2(t)\|^2_{L_2(G,H)} \wedge 1\right)\,\mathrm{d}t\\
&\leq 4\left(\int_0^T k_L(\psi_1(t) + l_L(\psi_1(t)\,\mathrm{d}t + \int_0^T k_L(\psi_2(t) + l_L(\psi_2(t)\,\mathrm{d}t\right)\\
&\qquad\qquad + 2\left(\int_0^T \left(\|\psi_1(t)\|^2_{L_2(G,H)} \wedge 1\right)\,\mathrm{d}t + \int_0^T \left(\|\psi_2(t)\|^2_{L_2(G,H)} \wedge 1\right)\,\mathrm{d}t\right)\\
&\leq 4\left(m_L(\psi_1) + m_L(\psi_2)\right).
\end{aligned}$$

This concludes the proof. □

Lemma 3.16. *For all $r > 0$ there exists $c_r > 0$ such that*

$$\sup_{\|F\|_{L_2(G,H)}\leq r} \left(k_L(F) + l_L(F)\right) \leq c_r.$$

Proof. By Lemma 3.12, $k_L + l_L$ is continuous at 0, from which it follows that there exists a $\delta > 0$ such that $\|F\|_{L_2(G,H)} \leq \delta$ implies $(k_L + l_L)(F) \leq 1$. Let $r > 0$ be fixed. If we choose $N_r \in \mathbb{N}$ to be large enough so that $\frac{r}{N_r} \leq \delta$, then by a repeated use of Equation (3.19), we obtain for some $c_r > 0$ that

$$\begin{aligned}
\sup_{\|F\|_{L_2(G,H)}\leq r} (k_L + l_L)(F) &= \sup_{\|F\|_{L_2(G,H)}\leq r} (k_L + l_L)\left(N_r \frac{F}{N_r}\right)\\
&\leq c_r \sup_{\|F\|_{L_2(G,H)}\leq r} (k_L + l_L)\left(\frac{F}{N_r}\right) \leq c_r,
\end{aligned}$$

which completes the proof. □

Remark 3.17. *In particular, Lemma 3.16 implies that for all bounded functions $\psi : [0,T] \to L_2(G,H)$ we have that $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$. Indeed, if $\sup_{t\in[0,T]} \|\psi(t)\|_{L_2(G,H)} \le r$ for some $r > 0$, then by Lemma 3.16 there exists $c_r > 0$ such that it holds that $\sup_{\|F\|_{L_2(G,H)}\le r} (k_L(F) + l_L(F)) \le c_r$. Hence we obtain*

$$m'_L(\psi) := \int_0^T \Big(k_L(\psi(t)) + l_L(\psi(t))\Big)\,\mathrm{d}t \le T\,c_r < \infty.$$

Since by the very definition of m'' we have that $m''(\psi) < \infty$, it follows that $m_L(\psi) < \infty$.

Having developed all the technical tools, we now present the main result of this section, which shows that $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ is a vector space and m_L is a modular on $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ in the sense of Definition 3.1.

Theorem 3.18. *$\mathcal{M}^{\mathrm{HS}}_{\det,L}$ is a linear space and m_L is a modular on $\mathcal{M}^{\mathrm{HS}}_{\det,L}$.*

Proof. Let $\psi_1, \psi_2 \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$. By Lemma 3.15, we have

$$m_L(\psi_1 + \psi_2) \le 4\,(m_L(\psi_1) + m_L(\psi_2)) < \infty,$$

which implies that $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ is closed under addition. A similar argument as in Lemma 3.16 shows that $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ is closed under multiplication by scalars, which completes the proof that $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ is a vector space. Hence, it remains only to show that m_L satisfies the conditions of Definition 3.1. It follows directly from the definition of m_L that $m_L(-\psi) = m_L(\psi)$ for all $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$. Condition (2) of Definition 3.1 is a consequence of Lemma 3.12. Condition (3) of Definition 3.1 follows from an argument similar to Lemma 3.14/(2) and the very definition of l_L. Finally, Condition (4) of Definition 3.1 is a direct consequence of Lemma 3.15. □

Remark 3.19. *We say that a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}^{\mathrm{HS}}_{\det,L}$ converges in the modular topology to $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$ if $\lim_{n\to\infty} m_L(\psi_n - \psi) = 0$. Since $m_L(\psi) = 0$ if and only* *if $\psi(t) = 0$ for Lebesgue almost all $t \in [0,T]$, we have that limits of sequences in the modular topology are Lebesgue a.e. uniquely determined. For this and further properties of the modular topology, see Section 2 of Nakano [55].*

Later on, we will be interested in the space $L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\det,L})$ of $\mathcal{M}^{\mathrm{HS}}_{\det,L}$-valued random elements. In order to make precise mathematical sense of $L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\det,L})$, we prove that the modular space $\mathcal{M}^{\mathrm{HS}}_{\det,L}$, considered as a topological space with the modular topology, is a Polish space.

Lemma 3.20. *The modular topology on $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ induced by m_L is complete.*

Proof. Let $(\psi_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}^{\mathrm{HS}}_{\det,L}$ be such that $\lim_{i,j\to\infty} m_L(\psi_i - \psi_j) = 0$. Then, for all $\epsilon \in (0,1)$ we have by Markov's inequality that

$$\begin{aligned}
&\lim_{i,j\to\infty} \mathrm{Leb}\Big(t \in [0,T] : \|\psi_i(t) - \psi_j(t)\|_{L_2(G,H)} > \epsilon\Big) \\
&\le \lim_{i,j\to\infty} \frac{1}{\epsilon^2} \int_0^T \Big(\|\psi_i(t) - \psi_j(t)\|^2_{L_2(G,H)} \wedge 1\Big)\,\mathrm{d}t \le \lim_{i,j\to\infty} \frac{1}{\epsilon^2} m_L(\psi_i - \psi_j) = 0,
\end{aligned}$$

which implies that the sequence $(\psi_i)_{i\in\mathbb{N}}$ is Cauchy in Lebesgue measure. Hence, there exists a subsequence $(\psi_{i_n})_{n\in\mathbb{N}}$ converging Lebesgue almost everywhere to a measurable function $\psi : [0,T] \to L_2(G,H)$.

Let $\epsilon > 0$ be fixed. By assumption, there exists $N \in \mathbb{N}$ such that for all $i,j \ge N$ we have $m_L(\psi_i - \psi_j) < \epsilon/2$. Since by Lemma 3.12, k_L is continuous and l_L is lower-semicontinuous, Fatou's lemma implies for all $i \ge N$ that

$$\begin{aligned}
m'_L(\psi_i - \psi) &= \int_0^T (k_L + l_L)(\psi_i(t) - \psi(t))\,\mathrm{d}t \\
&\le \int_0^T \liminf_{n\to\infty} (k_L + l_L)(\psi_i(t) - \psi_{i_n}(t))\,\mathrm{d}t \qquad (3.20) \\
&\le \liminf_{n\to\infty} \int_0^T (k_L + l_L)(\psi_i(t) - \psi_{i_n}(t))\,\mathrm{d}t \le \liminf_{n\to\infty} m_L(\psi_i - \psi_{i_n}) < \frac{\epsilon}{2}.
\end{aligned}$$

Since $(\psi_{i_n})_{n\in\mathbb{N}}$ converges Lebesgue a.e. to ψ, using the dominated convergence theorem we obtain

$$\begin{aligned} m''(\psi_i - \psi) &= \int_0^T \left(\|\psi_i(t) - \psi(t)\|^2_{L_2(G,H)} \wedge 1\right) \mathrm{d}t \\ &= \int_0^T \lim_{n\to\infty} \left(\|\psi_i(t) - \psi_{i_n}(t)\|^2_{L_2(G,H)} \wedge 1\right) \mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^T \left(\|\psi_i(t) - \psi_{i_n}(t)\|^2_{L_2(G,H)} \wedge 1\right) \mathrm{d}t \leq \lim_{n\to\infty} m_L(\psi_i - \psi_{i_n}) < \frac{\epsilon}{2}. \end{aligned} \tag{3.21}$$

Equations (3.20) and (3.21) together imply that $m_L(\psi_i - \psi) < \epsilon$ for all $i \geq N$. Since $\epsilon > 0$ can be chosen to be arbitrarily small, we conclude that $\lim_{i\to\infty} m_L(\psi_i - \psi) = 0$. Finally, to see that $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$, fix $i_0 \in \mathbb{N}$ such that $m_L(\psi_{i_0} - \psi) \leq 1$. Since $\psi_{i_0} \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$, we have that $m_L(\psi_{i_0}) < \infty$ and hence

$$m_L(\psi) \leq 4\left(m_L(\psi - \psi_{i_0}) + m_L(\psi_{i_0})\right) \leq 4\left(1 + m_L(\psi_{i_0})\right) < \infty.$$

Thus, we have that $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$, which concludes the proof. □

Remark 3.21. *Note that Lemma 3.20 explains the role of m'' in the modular m_L. In particular, m'' is needed to establish completeness of the modular topology by allowing the identification of a potential m_L-limit of an m_L-Cauchy sequence.*

Our next goal is to establish that step functions are dense in the modular space $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ under the modular topology induced by m_L. In particular, this will immediately yield that the modular space is separable.

Lemma 3.22. *The collection of Hilbert-Schmidt operator-valued step functions of the form*

$$\psi\colon [0,T] \to L_2(G,H), \qquad \psi(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_1 < \cdots < t_n = T$, $F_i \in L_2(G,H)$ *for each* $i \in \{0, ..., n-1\}$, *is dense in* $\mathcal{M}^{\mathrm{HS}}_{\det,L}$ *with the modular topology. Moreover, the modular topology is separable.*

Proof. First, it follows directly from Remark 3.17 that each step function of the above form is an element of the modular space $\mathcal{M}^{\mathrm{HS}}_{\det,L}$. To prove the claimed result, we first assume that $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$ is bounded, that is, there exists a constant $r > 0$ such that $\sup_{t\in[0,T]} \|\psi(t)\|_{L_2(G,H)} \leq r$. By [24, Le. 1.2.19], there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of step functions satisfying:

(1) $\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\psi_n(t)\|_{L_2(G,H)} \leq r$;

(2) $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ Lebesgue a.e.

Then, we have

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\psi_n(t) - \psi(t)\|_{L_2(G,H)} \leq 2r,$$

from which it follows by Lemma 3.16 that there exists a constant $c > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} (k_L + l_L)(\psi_n(t) - \psi(t)) \leq c. \tag{3.22}$$

Since by Lemma 3.12, k_L and l_L are continuous at 0, using Equation (3.22) to obtain a dominating function, and noting that $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ Lebesgue a.e. , Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^T (k_L + l_L)(\psi_n(t) - \psi(t))\, \mathrm{d}t = \int_0^T \lim_{n\to\infty} (k_L + l_L)(\psi_n(t) - \psi(t))\, \mathrm{d}t = 0.$$

Arguing similarly, since $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ Lebesgue a.e. , another application of Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} \int_0^T \left(\|\psi_n(t) - \psi(t)\|^2_{L_2(G,H)} \wedge 1 \right) \mathrm{d}t$$

$$= \int_0^T \lim_{n\to\infty} \left(\|\psi_n(t) - \psi(t)\|^2_{L_2(G,H)} \wedge 1 \right) \, \mathrm{d}t = 0,$$

which proves our claim for any bounded $\psi \in \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$.

In the case of a general $\psi \in \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$, we define a sequence of functions

$$\psi_n : [0,T] \to L_2(G,H), \qquad \psi_n(t) = \begin{cases} \psi(t) & \text{if } \|\psi(t)\|_{L_2(G,H)} \le n, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the very definition of ψ_n that for every $n \in \mathbb{N}$ and $t \in [0,T]$ we have

$$(k_L + l_L)(\psi_{n+1}(t) - \psi(t)) \le (k_L + l_L)(\psi_n(t) - \psi(t)) \le (k_L + l_L)(\psi(t)).$$

Since $m_L(\psi) < \infty$ we get

$$\int_0^T (k_L + l_L)(\psi_1(t) - \psi(t)) \, \mathrm{d}t \le \int_0^T (k_L + l_L)(\psi(t)) \, \mathrm{d}t \le m_L(\psi) < \infty.$$

Thus, by applying the monotone convergence theorem for the non-negative, pointwise decreasing to 0 sequence of functions $((k_L + l_L)(\psi_n - \psi))_{n\in\mathbb{N}}$, we obtain

$$\lim_{n\to\infty} \int_0^T (k_L + l_L)(\psi_n(t) - \psi(t)) \, \mathrm{d}t = 0. \tag{3.23}$$

Moreover, since $(\psi_n)_{n\in\mathbb{N}}$ converges pointwise to ψ, by Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^T \left(\|\psi_n(t) - \psi(t)\|^2_{L_2(G,H)} \wedge 1 \right) \, \mathrm{d}t = 0. \tag{3.24}$$

Hence, it follows from Equations (3.23) and (3.24) that $\lim_{n\to\infty} m_L(\psi_n - \psi) = 0$. By the first part of this lemma, for each $n \in \mathbb{N}$ there exists a sequence $(\psi_{n,i})_{i\in\mathbb{N}}$ of step functions converging to ψ_n in the modular m_L as $i \to \infty$. For each $n \in \mathbb{N}$ we can choose $i_n \in \mathbb{N}$ such that $m_L(\psi_n - \psi_{n,i_n}) < \frac{1}{n}$. Then, it follows from Lemma 3.15 that

$$\lim_{n\to\infty} m_L(\psi - \psi_{n,i_n}) \le \lim_{n\to\infty} 4\,(m_L(\psi - \psi_n) + m_L(\psi_n - \psi_{n,i_n})) = 0.$$

Since one might require that the approximating sequence of step functions are defined on rational partitions of the time domain and, by separability of $L_2(G,H)$, only take values in a countable dense subset of $L_2(G,H)$, separability of the modular topology follows. $\square$

Proposition 3.23. *The space $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$, considered as a topological space with the modular topology induced by m_L, is a Polish space.*

Proof. Since m_L is of moderate growth, it follows from [1] that the modular topology on $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ is metrizable. More precisely, by [1, Th. I], there exists a metric d_L on $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ satisfying for some $\alpha > 1$ and $c_1, c_2 > 0$ that

$$c_1\, m_L(\psi_1 - \psi_2) \le d_L(\psi_1,\psi_2)^\alpha \le c_2\, m_L(\psi_1 - \psi_2) \quad \text{for all } \psi_1, \psi_1 \in \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}. \tag{3.25}$$

Combining Equation (3.25) with Lemma 3.20, Theorem 3.18 and Lemma 3.22 we obtain that $(\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}, d_L)$ is a complete and separable metric linear space. Thus, [34, Cor. 2.6] implies that there exists a translation invariant metric ρ_L, equivalent to d_L, such that $(\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}, \rho_L)$ is a Polish space. $\square$

3.2 Characterisation of deterministic integrable processes

The definition of the stochastic integral for deterministic integrands with respect to a cylindrical Lévy process L depends heavily on two classes of step functions. We give in the following a precise definition of what is meant by a step function.

Definition 3.24.

(1) *An $L_2(G,H)$-valued step function is of the form*

$$\psi\colon [0,T] \to L_2(G,H), \qquad \psi(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i,t_{i+1}]}(t), \tag{3.26}$$

where $0 = t_1 < \cdots < t_n = T$, $F_i \in L_2(G,H)$ for each $i \in \{0,...,n-1\}$. The space of $L_2(G,H)$-valued step functions is denoted by $\mathcal{S}^{\mathrm{HS}}_{\mathrm{det}} := \mathcal{S}^{\mathrm{HS}}_{\mathrm{det}}(G,H)$.

(2) *An $L(H)$-valued step function is of the form*

$$\gamma\colon [0,T] \to L(H), \qquad \gamma(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i,t_{i+1}]}(t), \tag{3.27}$$

where $0 = t_1 < \cdots < t_n = T$ and $F_i \in L(H)$ for each $i \in \{0,...,n-1\}$. The space of $L(H)$-valued step functions with $\sup_{t\in[0,T]} \|\gamma(t)\|_{H\to H} \le 1$ is denoted by $\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}} := \mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}(H,H)$.

Let $L(t_{i+1}) - L(t_i)$ be an increment of the cylindrical Lévy process L and assume that $F_i \in L_2(G,H)$ for each $i \in \{1,...,n-1\}$. Since Hilbert-Schmidt operators are 0-Radonifying by [76, Th. VI.5.2], it follows from [76, Pr. VI.5.3] that there exist genuine random variables $F_i\big(L(t_{i+1}) - L(t_i)\big)\colon \Omega \to H$ for each $i \in \{1,...,n-1\}$ satisfying

$$(L(t_{i+1}) - L(t_i))(F_i^* h) = \langle F_i(L(t_{i+1}) - L(t_i)), h\rangle \quad P\text{-a.s. for all } h \in H.$$

We call the random variables $F_i\big(L(t_{i+1}) - L(t_i)\big)$ for each $i \in \{1,...,n-1\}$ Radonified increments. The stochastic integral is defined for any $\psi \in \mathcal{S}^{\mathrm{HS}}_{\mathrm{det}}$ with representation (3.26) as the sum of the Radonified increments

$$I(\psi) := \int_0^T \psi \,\mathrm{d}L = \sum_{i=1}^{n-1} F_i(L(t_{i+1}) - L(t_i)).$$

Thus, the integral $I(\psi) : \Omega \to H$ is a genuine H-valued random variable.

The following definition of the stochastic integral can be traced back to the theory of vector measures, and was adapted to the probabilistic setting in [75] by Urbanik and Woyczyński.

Definition 3.25. *A function* $\psi \colon [0,T] \to L_2(G,H)$ *is integrable if there exists a sequence* $(\psi_n)_{n\in\mathbb{N}}$ *of elements of* $\mathcal{S}_{\text{det}}^{\text{HS}}$ *satisfying*

(1) $(\psi_n)_{n\in\mathbb{N}}$ *converges to* ψ *Lebesgue a.e.;*

(2) $\displaystyle \lim_{m,n\to\infty} \sup_{\gamma\in\mathcal{S}_{\text{det}}^{1,\text{op}}} E\left[\left\|\int_0^T \gamma(\psi_m-\psi_n)\,\mathrm{d}L\right\| \wedge 1\right] = 0.$

In this case, the stochastic integral of the deterministic function ψ *is defined by*

$$I(\psi) := \int_0^T \psi\,\mathrm{d}L := \lim_{n\to\infty}\int_0^T \psi_n\,\mathrm{d}L \quad \text{in } L_P^0(\Omega,H).$$

The class of all deterministic L*-integrable Hilbert-Schmidt operator-valued functions is denoted by* $\mathcal{I}_{\text{det},L}^{\text{HS}} := \mathcal{I}_{\text{det},L}^{\text{HS}}(G,H)$.

Remark 3.26. *If Conditions (1) and (2) in Definition 3.25 are satisfied, then completeness of* $L_P^0(\Omega,H)$ *implies the existence of the limit. Furthermore, it follows that the integral process* $(\int_0^t \psi\,\mathrm{d}L)_{t\geq 0}$*, defined by* $\int_0^t \psi\,\mathrm{d}L := \int_0^T \mathbb{1}_{[0,t]}\psi\,\mathrm{d}L$ *has cádlág paths. To see this, note that for each* $m,n\in\mathbb{N}$ *the process* $(\int_0^t (\psi_m-\psi_n)\,\mathrm{d}L)_{t\geq 0}$ *has cádlág paths. By an extension of [42, Pr. 8.2.1] to* H*-valued processes and Condition (2) above, we obtain*

$$\lim_{m,n\to\infty} P\left(\sup_{0\leq t\leq T}\left\|\int_0^t (\psi_m-\psi_n)\,\mathrm{d}L\right\| > \epsilon\right) \leq 3\lim_{m,n\to\infty}\sup_{0\leq t\leq T} P\left(\left\|\int_0^t (\psi_m-\psi_n)\,\mathrm{d}L\right\| > \frac{\epsilon}{3}\right) = 0.$$

By passing on to a suitable subsequence if necessary, we obtain that there exists a subsequence $(\int_0^t \psi_{n_k} \, \mathrm{d}L)_{k\in\mathbb{N}}$ *that converges uniformly almost surely, which guarantees that the limiting process has cádlág paths.*

The following is the main result of this section identifying the largest space of L-integrable Hilbert-Schmidt operator-valued functions with the modular space $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$.

Theorem 3.27. *The space* $\mathcal{I}^{\mathrm{HS}}_{\mathrm{det},L}$ *of deterministic functions integrable with respect to the cylindrical Lévy process* L *in* G *coincides with the modular space* $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$.

The remainder of this section is devoted to proving the above theorem. As a first step, we prove a key Lemma, which shows that convergence of step functions in the modular topology is equivalent to convergence of the corresponding stochastic integrals in the following sense.

Lemma 3.28. *Let* L *be a cylindrical Lévy process in* G*, and* $(\psi_n)_{n\in\mathbb{N}}$ *a sequence in* $\mathcal{S}^{\mathrm{HS}}_{\mathrm{det}}$. *Then the following are equivalent:*

(a) $\lim_{n\to\infty} m_L(\psi_n) = 0$;

(b) $\lim_{n\to\infty} \sup_{\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} E\left[\left\|\int_0^T \gamma\psi_n \, dL\right\| \wedge 1\right] = 0$ *and* $\lim_{n\to\infty} m''(\psi_n) = 0$.

The proof of the implication (a) ⇒ (b) relies on two technical lemmata. The first of these gives a limit representation of the modular.

Lemma 3.29. *Let* $(L(t) : t \geq 0)$ *be a cylindrical Lévy process in* G *with cylindrical characteristics* (a, Q, λ) *and assume that* $\psi \in \mathcal{S}^{\mathrm{HS}}_{\mathrm{det}}$ *has the representation as in* (3.26). *If* $(\pi_k)_{k\in\mathbb{N}}$ *is a nested normal sequence of partitions of* $[0, T]$ *containing the jumps of* ψ *then we have*

$$\lim_{k\to\infty} \sum_{i=1}^{n-1} \sum_{\substack{p_{j,k}\in\pi_k \\ t_i<p_{j,k}\leq t_{i+1}}} E\left[\theta\left(L^{F_i}(p_{j,k}) - L^{F_i}(p_{j-1,k})\right)\right] = \int_0^T b^{\theta}_{\psi(t)} \, \mathrm{d}t,$$

and

$$\lim_{k\to\infty}\sum_{i=1}^{n-1}\sum_{\substack{p_{j,k}\in\pi_k\\ t_i<p_{j,k}\le t_{i+1}}} E\left[\left\|\theta\left(L^{F_i}(p_{j,k})-L^{F_i}(p_{j-1,k})\right)\right\|^2\right]$$
$$=\int_0^T\int_H\left(\|h\|^2\wedge 1\right)\left(\lambda\circ\psi(t)^{-1}\right)(\mathrm{d}h)\mathrm{d}t+\int_0^T \mathrm{Tr}(\psi(t)Q\psi(t)^*)\,\mathrm{d}t.$$

Proof. The proof is a direct application of Lemma 3.3 and the limit characterisation of Lévy characteristics in Theorem 2.4. □

The next result establishes the relationship between the limit characterization of the modular given in Lemma 3.29, and the size of the stochastic integral in $L^0_P(\Omega,H)$.

Lemma 3.30. *For all $\epsilon>0$ there exists $\delta>0$ such that if $(X_n)_{n\in\{1,\dots,N\}}$ is a sequence of independent H-valued random variables satisfying that*

$$\left\|\sum_{n=1}^N E[\theta(X_n)]\right\|<\delta \quad \text{and} \quad \sum_{n=1}^N E\left[\|\theta(X_n)\|^2\right]<\delta$$

then

$$E\left[\left\|\sum_{n=1}^N X_n\right\|\wedge 1\right]<\epsilon.$$

Proof. See [42, Pr. 8.1.1/(ii)]. □

Proof of (a) ⇒ (b) *in Lemma 3.28.* Let $\epsilon>0$ be fixed and choose $\delta>0$ so that the implication in Lemma 3.30 holds. By assumption, we have that $m_L(\psi_n)\to 0$, from which it follows that there exists an $N\in\mathbb{N}$ such that for all $n\ge N$ we have $m(\psi_n)<\delta$. To conclude the proof, it suffices to show that for all $n\ge N$ and $\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}$ we have

$$E\left[\left\|\int_0^T\gamma\psi_n\,dL\right\|\wedge 1\right]<\epsilon.$$

Let $n_0 \geq N$ and $\gamma_0 \in \mathcal{S}_{\text{det}}^{1,\text{op}}$ be fixed. We assume the representation

$$\gamma_0 \psi_{n_0}(t) = O_{0,n_0} F_{0,n_0} \mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{N(n_0)-1} O_{i,n_0} F_{i,n_0} \mathbb{1}_{(t_{i,n_0}, t_{i+1,n_0}]}(t),$$

where $0 = t_{0,n_0} < t_{1,n_0} < \ldots < t_{N(n_0),n_0} = T$, $O_{i,n_0} \in L(H)^1$ and $F_{i,n_0} \in L_2(G, H)$. Let $(\pi_k)_{k \in \mathbb{N}}$ be a nested normal sequence of partitions containing the jumps of $\gamma_0 \psi_{n_0}$. Since by Lemma 3.14 and the very definition of l_L we have $m_L(\gamma_0 \psi_{n_0}) \leq m_L(\psi_{n_0}) < \delta$, Lemma 3.29 guarantees that there exists a $K \in \mathbb{N}$ such that the partition π_K satisfies

$$\left\| \sum_{i=0}^{N(n_0)-1} \sum_{\substack{p_{j,K} \in \pi_K \\ t_{i,n_0} < p_{j,K} \leq t_{i+1,n_0}}} E\left[\theta\left(L^{O_{i,n_0} F_{i,n_0}}(p_{j,K}) - L^{O_{i,n_0} F_{i,n_0}}(p_{j-1,K})\right)\right] \right\| < \delta \qquad (3.28)$$

and

$$\sum_{i=0}^{N(n_0)-1} \sum_{\substack{p_{j,K} \in \pi_K \\ t_{i,n_0} < p_{j,K} \leq t_{i+1,n_0}}} E\left[\left\|\theta\left(L^{O_{i,n_0} F_{i,n_0}}(p_{j,K}) - L^{O_{i,n_0} F_{i,n_0}}(p_{j-1,K})\right)\right\|^2\right] < \delta. \qquad (3.29)$$

Since π_K contains the jumps of $\gamma_0 \psi_{n_0}$, by Lemma 3.30, the estimates (3.28) and (3.29) together imply

$$\begin{aligned}
E\left[\left\|\int_0^T \gamma_0 \psi_{n_0}\, dL\right\| \wedge 1\right] &= E\left[\left\|\sum_{i=0}^{N(n_0)-1} O_{i,n_0} F_{i,n_0} \left(L(t_{i+1,n_0}) - L(t_{i,n_0})\right)\right\| \wedge 1\right] \\
&= E\left[\left\|\sum_{i=0}^{N(n_0)-1} \left(L^{O_{i,n_0} F_{i,n_0}}(t_{i+1,n_0}) - L^{O_{i,n_0} F_{i,n_0}}(t_{i,n_0})\right)\right\| \wedge 1\right] \\
&< \epsilon.
\end{aligned}$$

This concludes the proof of the implication. □

In order to prove the reverse implication (b) $\Rightarrow$ (a) in Lemma 3.28, we need a number of preliminary results. The key technical tool used in our arguments will be the Kuratowski-Ryll Nardzewski measurable selection theorem, which we quote below.

Theorem 3.31. *Let $(X, \mathcal{S})$ be a measurable space, Y a Polish space, and denote by 2^Y the family of all subsets of Y. If a set-valued function $\Gamma : X \to 2^Y$ satisfies that $\{x \in X : \Gamma(x) \cap A \neq \emptyset\} \in \mathcal{S}$ for all open sets $A \subseteq Y$, then Γ admits an $\mathcal{S}/\mathfrak{B}(Y)$-measurable selector $\gamma : X \to Y$.*

Proof. See [38]. □

Remark 3.32. *To avoid issues with non-separability of the norm topology, we will always endow $L(H)^1$ with the strong topology, which turns $L(H)^1$ into a Polish space, see [32]. In particular, this implies that there exists a countable subset $\mathcal{D}^1 \subseteq L(H)^1$, which is dense in $L(H)^1$ under the strong topology.*

The following lemma provides an alternative form of $\int_0^T l_L(\psi(t))\,\mathrm{d}t$, by allowing us to move the supremum out of the integral.

Lemma 3.33. *If $\psi \in \mathcal{S}_{\mathrm{det}}^{\mathrm{HS}}$ then*

$$\int_0^T l_L(\psi(t))\,\mathrm{d}t := \int_0^T \sup_{O \in L(H)^1} \left\| b^\theta_{O\psi(t)} \right\| \mathrm{d}t = \sup_{\gamma \in \mathcal{S}_{\mathrm{det}}^{1,\mathrm{op}}} \left\| \int_0^T b^\theta_{\gamma\psi(t)}\,\mathrm{d}t \right\|,$$

where $\mathcal{S}_{\mathrm{det}}^{1,\mathrm{op}}$ was defined in Definition 3.24/(2).

Proof. Fix an element $e \in H$ such that $\|e\| = 1$. If $h \in H$ is linearly independent of e, we define $A_h := \mathrm{Span}\{e, h\}$. For each $h \in H$ we define the mapping $f : H \to L(H)^1$ by

$$f(h)(h') = \begin{cases} R_{A_h}(h'_{A_h}) + h'_{A_h^\perp} & \text{if } h \in H \setminus \mathrm{Span}\{e\} \\ \mathrm{sgn}(\lambda) h' & \text{if } h = \lambda e, \end{cases} \tag{3.30}$$

where h'_{A_h} and $h'_{A_h^\perp}$ denote the projections of h' onto the subspace A_h and its orthogonal complement $A_h^\perp$, respectively, and R_{A_h} denotes the rotation on the plane A_h around the origin by the angle $\angle(h,e)$, that is, by the angle which rotates the vector h into e. We claim that f satisfies the following properties:

(1) for each $h \in H$ the mapping $f(h) : H \to H$ is a linear isometry;

(2) for each $h \in H$ and $F \in L_2(G,H)$ we have $b^\theta_{f(h)F} = f(h)b^\theta_F$;

(3) for each $h \in H$ it holds that $\langle e, f(h)(h)\rangle = \|f(h)(h)\|$.

Proof of (1): We first assume that $h \in H \setminus \mathrm{Span}\{e\}$. Then, since for any $h' \in H$ we have that $h' = h'_{A_h} + h'_{A_h^\perp}$, and h'_{A_h} is orthogonal to $h'_{A_h^\perp}$, it follows that

$$\begin{aligned}\|h'\|^2 = \left\|h'_{A_h} + h'_{A_h^\perp}\right\|^2 &= \left\langle h'_{A_h} + h'_{A_h^\perp}, h'_{A_h} + h'_{A_h^\perp}\right\rangle \\ &= \langle h'_{A_h}, h'_{A_h}\rangle + \langle h'_{A_h^\perp}, h'_{A_h^\perp}\rangle = \left\|h'_{A_h}\right\|^2 + \left\|h'_{A_h^\perp}\right\|^2. \end{aligned} \tag{3.31}$$

Since rotations are isometries, we have $\left\|R_{A_h}(h'_{A_h})\right\| = \left\|h'_{A_h}\right\|$, moreover, it follows from the definition of the rotation R_{A_h} that $R_{A_h}(h'_{A_h})$ is orthogonal to $h'_{A_h^\perp}$. Using these observations, a similar argument as above yields for all $h \in H \setminus \mathrm{Span}\{e\}$ and $h' \in H$ that

$$\|f(h)(h')\|^2 = \left\|R_{A_h}(h'_{A_h}) + h'_{A_h^\perp}\right\|^2 = \|R_{A_h}(h'_{A_h})\|^2 + \left\|h'_{A_h^\perp}\right\|^2 = \|h'_{A_h}\|^2 + \left\|h'_{A_h^\perp}\right\|^2. \tag{3.32}$$

Hence, if $h \in H \setminus \mathrm{Span}\{e\}$ then by Equations (3.31) and (3.32) we have for all $h' \in H$ that $\|f(h)(h')\| = \|h'\|$. If, on the other hand, we have $h = \lambda e$ for some $\lambda \in \mathbb{R}$, then it follows from the very definition of f that $\|f(h)(h')\| = \|\mathrm{sgn}(\lambda)h'\| = \|h'\|$, which finishes the proof that for all $h \in H$ the mapping $f(h) : H \to H$ is an isometry. Linearity of $f(h)$ follows directly from the definition.

Proof of (2): By Lemma 3.9 and the fact that by Step (1), for each $h_0 \in H$ the mapping $f(h_0) : H \to H$ is a linear isometry, we obtain for all $F \in L_2(G,H)$ that

$$\begin{aligned} b^{\theta}_{f(h_0)F} &= f(h_0)b^{\theta}_{F} + \int_H \theta(f(h_0)h) - f(h_0)\theta(h)\,(\lambda \circ F^{-1})(\mathrm{d}h) \\ &= f(h_0)b^{\theta}_{F} + \int_H f(h_0)\theta(h) - f(h_0)\theta(h)\,(\lambda \circ F^{-1})(\mathrm{d}h) = f(h_0)b^{\theta}_{F}. \end{aligned}$$

Proof of (3): Assume first that $h \in H \setminus \mathrm{Span}\{e\}$. Then, we have that $h_{A_h} = h$ and $h_{A_h^{\perp}} = 0$. By combining these observations with the fact that by its very definition, R_h rotates the vector h into e, we get

$$\langle e, f(h)(h)\rangle = \langle e, R_h(h)\rangle = \langle e, \|h\|\, e\rangle = \|h\|\,\langle e, e\rangle = \|h\| = \|R_h(h)\| = \|f(h)(h)\|\,. \tag{3.33}$$

If, on the other hand, we assume that $h = \lambda e$ for some $\lambda \in \mathbb{R}$ then

$$\langle e, f(h)(h)\rangle = \langle e, \mathrm{sgn}(\lambda)\lambda e\rangle = |\lambda| = \|\mathrm{sgn}(\lambda)\lambda e\| = \|f(h)h\|\,. \tag{3.34}$$

Equations (3.33) and (3.34) together imply that for all $h \in H$ we have $\langle e, f(h)(h)\rangle = \|f(h)(h)\|$.

To finish the proof of this lemma, let $\epsilon > 0$ be fixed. We define a set-valued function $g : L_2(G,H) \to 2^{L(H)^1}$ by

$$g(F) = \left\{ O \in L(H)^1 : \sup_{Q \in L(H)^1} \left\| b^{\theta}_{QF} \right\| - \left\| b^{\theta}_{OF} \right\| < \frac{\epsilon}{T} \right\}.$$

In order to prove the existence of a measurable selector for g, by Theorem 3.31, it suffices to show that for all open sets $S \subseteq L(H)^1$ we have

$$\{F \in L_2(G,H) : g(F) \cap S \neq \emptyset\} \in \mathfrak{B}(L_2(G,H)).$$

We define the mapping $h : L_2(G, H) \times L(H)^1 \to \mathbb{R}$ by

$$h(F, O) = \sup_{Q \in L(H)^1} \left\| b^{\theta}_{QF} \right\| - \left\| b^{\theta}_{OF} \right\|.$$

It follows from lower-semicontinuity of l_L, see Lemma 3.12, that for each fixed $O \in L(H)^1$ the mapping $F \mapsto h(F, O)$ is lower-semicontinuous. Moreover, by [36, Th.3], Lemma 2.7 and [58, Th.VI.5.5], we have that for each fixed $F \in L_2(G, H)$ the mapping $O \mapsto h(F, O)$ is continuous with the strong topology. By using these observations and noting that S is an open set in the strong topology, we obtain that

$$\begin{aligned}\{F \in L_2(G, H) : g(F) \cap S \neq \emptyset\} &= \cup_{O \in S} \left\{F \in L_2(G, H) : h(F, O) < \frac{\epsilon}{T}\right\} \\ &= \cup_{O \in S \cap \mathcal{D}^1} \left\{F \in L_2(G, H) : h(F, O) < \frac{\epsilon}{T}\right\},\end{aligned}$$

where $\mathcal{D}^1$ denotes a countable dense subset of $L(H)^1$ with the strong topology, see Remark 3.32. Since for each fixed $O \in L(H)^1$, the mapping $F \mapsto h(F, O)$ is lower-semicontinuous and hence measurable, for each fixed $O \in S$ it holds that

$$\left\{F \in L_2(G, H) : h(F, O) < \frac{\epsilon}{T}\right\} \in \mathfrak{B}(L_2(G, H)).$$

Since $\mathcal{D}^1$ is countable, we obtain

$$\cup_{O \in S \cap \mathcal{D}^1} \left\{F \in L_2(G, H) : h(F, O) < \frac{\epsilon}{T}\right\} \in \mathfrak{B}(L_2(G, H)).$$

Hence, we can apply the Kuratowski-Ryll Nardzewski measurable selection theorem, to conclude that there exists a measurable selector function $i : L_2(G, H) \to L(H)^1$ satisfying for all $F \in L_2(G, H)$ that $i(F) \in g(F)$.

Finally, we define a function $\eta : [0,T] \to L(H)^1$ by

$$\eta = f(b^\theta_{i(\psi)\psi}) \circ i(\psi),$$

where η is measurable since it is the composition of measurable functions. Using properties $(1)-(3)$ of the mapping f, we obtain

$$\begin{aligned}
\left\| \int_0^T b^\theta_{\eta\psi} \,\mathrm{d}t \right\| &\geq \int_0^T \langle b^\theta_{\eta\psi}, e\rangle \,\mathrm{d}t \\
&= \int_0^T \langle f(b^\theta_{i(\psi)\psi}) b^\theta_{i(\psi)\psi}, e\rangle \,\mathrm{d}t \\
&= \int_0^T \left\| f(b^\theta_{i(\psi)\psi}) b^\theta_{i(\psi)\psi} \right\| \,\mathrm{d}t \\
&= \int_0^T \left\| b^\theta_{i(\psi)\psi} \right\| \,\mathrm{d}t \\
&\geq \int_0^T \left(\sup_{O \in L(H)^1} \left\| b^\theta_{O\psi(t)} \right\| - \frac{\epsilon}{T} \right) \mathrm{d}t \\
&= \int_0^T \sup_{O \in L(H)^1} \left\| b^\theta_{O\psi(t)} \right\| \,\mathrm{d}t - \epsilon. \qquad (3.35)
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, and by approximating η using processes from $\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}$ we conclude

$$\sup_{\gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} \left\| \int_0^T b^\theta_{\gamma\psi} \,\mathrm{d}t \right\| \geq \int_0^T \sup_{O \in L(H)^1} \left\| b^\theta_{O\psi(t)} \right\| \,\mathrm{d}t.$$

As the reverse inequality follows directly from basic properties of the Bochner integral, the proof is complete. □

Lemma 3.34. *If a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^{\mathrm{HS}}_{\mathrm{det}}$ satisfies*

$$\lim_{n\to\infty} \sup_{\gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} E\left[\left\| \int_0^T \gamma\psi_n \, dL \right\| \wedge 1 \right] = 0,$$

then

$$\lim_{n\to\infty} \sup_{\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} \left\| \int_0^T b^{\theta}_{\gamma\psi_n(t)}\, \mathrm{d}t \right\| = 0.$$

Proof. Assume, aiming for a contradiction, that this is not the case. Then, by passing on to a suitable subsequence if necessary, there exists an $\epsilon > 0$ and a sequence $(\gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}$ satisfying for all $n \in \mathbb{N}$ that

$$\left\| \int_0^T b^{\theta}_{\gamma_n\psi_n(t)}\, \mathrm{d}t \right\| > \epsilon. \tag{3.36}$$

On the other hand, the condition

$$\lim_{n\to\infty} \sup_{\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} E\left[\left\| \int_0^T \gamma\psi_n\, dL \right\| \wedge 1 \right] = 0$$

implies that the sequence $(I(\gamma_n\psi_n))_{n\in\mathbb{N}}$ of infinitely divisible random variables with corresponding sequence of first characteristics $\left(\int_0^T b^{\theta}_{\gamma_n\psi_n(t)}\, dt\right)_{n\in\mathbb{N}}$ converges to 0 in probability. By [58, Th.VI.5.5/(1)], we then have

$$\lim_{n\to\infty} \int_0^T b^{\theta}_{\gamma_n\psi_n(t)}\, dt = 0,$$

which contradicts Equation (3.36). Hence, the result follows. □

The product measure of two cylindrical measures is defined analogously to the case of Radon measures; see [71, Ch. II.2.2]. The following lemma provides an alternative representation of an integral with respect to the product measure of the cylindrical Lévy measure of L and the Lebesgue measure on a finite interval. To make sense out of this, the Lebesgue measure is considered as a cylindrical measure on $\mathfrak{B}(\mathbb{R})$.

Lemma 3.35. *Let L be a cylindrical Lévy process in G with cylindrical Lévy measure λ. Then we have for each $\psi \in \mathcal{S}_{\det}^{\mathrm{HS}}$ with $\psi(0) = 0$ that*

$$\int_0^T \int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ \psi(t)^{-1}\right) (\mathrm{d}h)\, \mathrm{d}t = \int_H \left(\|h\|^2 \wedge 1\right) \left((\lambda \otimes \mathrm{Leb}) \circ \kappa_\psi^{-1}\right) (\mathrm{d}h),$$

where $\kappa_\psi \colon G \times [0,T] \to H$ is defined by $\kappa_\psi(g,t) = \psi(t)g$.

Proof. First, we show that the result holds for $\psi = F\mathbb{1}_{(t_i,t_{i+1}]}$, where $F \in L_2(G,H)$ and $0 \le t_i < t_{i+1} \le T$. In this case, we see that for all $C \in \mathcal{Z}_*(H)$

$$\big(\lambda \otimes \mathrm{Leb}\big) \circ \kappa_\psi^{-1}(C) = \big(\lambda \otimes \mathrm{Leb}\big)\big(F^{-1}(C) \times (t_i, t_{i+1}]\big) = (t_i - t_{i+1})\big(\lambda \circ F^{-1}\big)(C). \tag{3.37}$$

Since the cylindrical measure on the right hand side of Equation (3.37) is the cylindrical Lévy measure of the Radonified increment $F(L(t_{i+1}) - L(t_i))$, it extends to a genuine Lévy measure on $\mathfrak{B}(H)$ for which we keep the notation $\lambda \circ F^{-1}$. Consequently, the cylindrical Lévy measure on the left hand side of Equation (3.37) extends to a genuine Lévy measure on $\mathfrak{B}(H)$, and the two extensions agree on $\mathfrak{B}(H)$. It follows that

$$\begin{aligned} &\int_H \left(\|h\|^2 \wedge 1\right)\left(\left(\lambda \otimes \mathrm{Leb}\right) \circ \kappa_\psi^{-1}\right) (\mathrm{d}h) \qquad (3.38)\\ &= \int_{t_i}^{t_{i+1}} \int_H \left(\|h\|^2 \wedge 1\right)\left(\lambda \circ F^{-1}\right) (\mathrm{d}h)\, \mathrm{d}t = \int_0^T \int_H \left(\|h\|^2 \wedge 1\right)\left(\lambda \circ \psi(t)^{-1}\right) (\mathrm{d}h)\, \mathrm{d}t. \end{aligned}$$

Let $\psi \in \mathcal{S}_{\det}^{\mathrm{HS}}$ be of the form as in (3.26) with $\psi(0) = 0$. For each $C \in \mathcal{Z}_*(H)$ we obtain

$$\kappa_\psi^{-1}(C) = \bigcup_{i=1}^{n-1} \Big\{(g,t) \in G \times [0,T] : F_i g \mathbb{1}_{(t_i,t_{i+1}]} \in C\Big\}.$$

Since the above is a finite union of disjoint cylindrical sets, it follows

$$\Big((\lambda \otimes \mathrm{Leb}) \circ \kappa_{\psi}^{-1}\Big)(C) = \sum_{i=1}^{n-1} \Big((\lambda \otimes \mathrm{Leb}) \circ \kappa_{F_i \mathbb{1}_{(t_i,t_{i+1}]}}^{-1}\Big)(C). \tag{3.39}$$

As the measure on the right side of Equation (3.39) extends to a genuine Lévy measure on $\mathfrak{B}(H)$ according to the first part of this proof, the measure on the left extends to a genuine Lévy measure on $\mathfrak{B}(H)$. It follows from Equation (3.38) that

$$\begin{aligned}
&\int_H \Big(\|h\|^2 \wedge 1\Big)\Big((\lambda \otimes \mathrm{Leb}) \circ \kappa_{\psi}^{-1}\Big)(\mathrm{d}h) \\
&\qquad = \sum_{i=1}^{n-1} \int_H \Big(\|h\|^2 \wedge 1\Big)\Big((\lambda \otimes \mathrm{Leb}) \circ \kappa_{F_i \mathbb{1}_{(t_i,t_{i+1}]}}^{-1}\Big)(\mathrm{d}h) \\
&\qquad = \sum_{i=1}^{n-1} \int_0^T \int_H \Big(\|h\|^2 \wedge 1\Big)\big(\lambda \circ (F_i \mathbb{1}_{(t_i,t_{i+1}]}(t))^{-1}\big)(\mathrm{d}h)\,\mathrm{d}t \\
&\qquad = \int_0^T \int_H \Big(\|h\|^2 \wedge 1\Big)\big(\lambda \circ \psi(t)^{-1}\big)(\mathrm{d}h)\,\mathrm{d}t,
\end{aligned}$$

which completes the proof. □

Proof of (b) $\Rightarrow$ (a) *in Lemma 3.28.* By assumption, we have that $\lim_{n\to\infty} m''(\psi_n) = 0$ and

$$\lim_{n\to\infty} \sup_{\gamma \in \mathcal{S}_{\mathrm{det}}^{1,\mathrm{op}}} E\left[\left\|\int_0^T \gamma \psi_n \, dL\right\| \wedge 1\right] = 0. \tag{3.40}$$

Since for each $n \in \mathbb{N}$, ψ_n has a representation of the form

$$\psi_n(t) = F_0^n \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{N(n)-1} F_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t),$$

where $0 = t_1^n < ... < t_{N(n)}^n = T$, and $F_i^n \in L_2(G,H)$ for each $i \in \{0,...,N(n)-1\}$, the integral $I(\psi_n)$ takes the form

$$I(\psi_n) = \sum_{i=1}^{N(n)-1} F_i^n \left(L(t_{i+1}^n) - L(t_i^n)\right).$$

Since the integral $I(\psi_n)$ is the sum of independent infinitely divisible random variables, $I(\psi_n)$ is also infinitely divisible and has characteristics

$$\left(\sum_{i=1}^{N(n)-1} (t_{i+1}^n - t_i^n) b^\theta_{F_i^n}, \sum_{i=1}^{N(n)-1} (t_{i+1}^n - t_i^n) F_i^n Q (F_i^n)^*, \sum_{i=1}^{N(n)-1} (t_{i+1}^n - t_i^n) \left(\lambda \circ (F_i^n)^{-1}\right) \right).$$

Since Equation (3.40) implies that $\lim_{n\to\infty} I(\psi_n) = 0$ in $L_P^0(\Omega, H)$, and we may assume that $\psi_n(0) = 0$ since it plays no role in the definition of the stochastic integral, we conclude from Lemmata 3.11 and 3.35 that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T k_L(\psi(t))\, dt \\
&= \lim_{n\to\infty} \int_0^T \int_H \left(\|h\|^2 \wedge 1\right) \left(\lambda \circ \psi_n(t)^{-1}\right)(\mathrm{d}h)\, \mathrm{d}t + \int_0^T \mathrm{Tr}\left(\psi_n(t) Q \psi_n(t)^*\right)\, \mathrm{d}t \\
&= \lim_{n\to\infty} \int_H \left(\|h\|^2 \wedge 1\right) \left((\lambda \otimes \mathrm{Leb}) \circ \kappa_{\psi_n}^{-1}\right)(\mathrm{d}h) + \mathrm{Tr}\left(\int_0^T \left(\psi_n(t) Q \psi_n(t)^*\right)\, \mathrm{d}t\right) = 0.
\end{aligned} \tag{3.41}$$

Furthermore, it follows from Equation (3.40) by Lemmata 3.33 and 3.34 that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^T l_L(\psi_n(t)) dt &= \lim_{n\to\infty} \int_0^T \sup_{O \in L(H)^1} \left\| b^\theta_{O\psi_n(t)} \right\| \mathrm{d}t \\
&= \lim_{n\to\infty} \sup_{\gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} \left\| \int_0^T b^\theta_{\gamma\psi_n(t)}\, \mathrm{d}t \right\| = 0.
\end{aligned} \tag{3.42}$$

Equations (3.41) and (3.42) together imply that $\lim_{n\to\infty} m'_L(\psi_n) = 0$. Since by assumption, we have $\lim_{n\to\infty} m''(\psi_n) = 0$, we obtain that $\lim_{n\to\infty} m_L(\psi_n) = 0$. □

We are now ready to present the proof of the main result of this section characterising the largest space of deterministic Hilbert-Schmidt operator-valued functions which are integrable with respect to a cylindrical Lévy process L in Hilbert space.

Proof of Theorem 3.27. If $\psi \in \mathcal{I}^{\mathrm{HS}}_{\det,L}$ then by the very definition of integrability, see Definition 3.25, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^{\mathrm{HS}}_{\det}$ such that $\psi_n \to \psi$ Lebesgue a.e. and $\sup_{\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\det}} E[\|I(\gamma(\psi_n-\psi_m))\| \wedge 1] \to 0$. By Lemma 3.28, this implies that $m_L(\psi_n - \psi_m) \to 0$. Completeness of the modular space $\mathcal{M}^{\mathrm{HS}}_{\det,L}$, see Lemma 3.20, and the fact that $\psi_n \to \psi$ Lebesgue a.e. allows us to conclude that $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$.

Conversely, if $\psi \in \mathcal{M}^{\mathrm{HS}}_{\det,L}$, then Lemma 3.22 implies that there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}^{\mathrm{HS}}_{\det}$ such that $\psi_n \to \psi$ Lebesgue a.e. and $m_L(\psi_n - \psi) \to 0$. It follows that $m_L(\psi_n - \psi_m) \to 0$, which implies $\sup_{\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\det}} E[\|I(\gamma(\psi_n-\psi_m))\| \wedge 1] \to 0$ by Lemma 3.28 and establishes that $\psi \in \mathcal{I}^{\mathrm{HS}}_{\det,L}$. □

3.3 Predictable integrands

For the remainder of this section, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. As in the case of deterministic integrands, we begin by introducing two classes of functions on which our definition of the stochastic integral depend.

Definition 3.36.

(1) *An $L_2(G,H)$-valued predictable step process $\Psi\colon \Omega\times[0,T] \to L_2(G,H)$ is of the form*

$$\Psi(\omega,t) = \left(\sum_{k=1}^{N(0)} F_{0,k}\mathbb{1}_{A_{0,k}}(\omega)\right)\mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1}\left(\sum_{k=1}^{N(i)} F_{i,k}\mathbb{1}_{A_{i,k}}(\omega)\right)\mathbb{1}_{(t_i,t_{i+1}]}(t), \tag{3.43}$$

where $0 = t_1 < \cdots < t_n = T$, $A_{0,k} \in \mathcal{F}_0$ and $F_{0,k} \in L_2(G,H)$ for all $k =$

$1,\dots,N(0)$, $A_{i,k}\in\mathcal{F}_{t_i}$ and $F_{i,k}\in L_2(G,H)$ for all $i=1,\dots,n-1$ and $k=1,\dots,N(i)$. The space of all $L_2(G,H)$-valued predictable step processes is denoted by $\mathcal{S}_{\mathrm{prd}}^{\mathrm{HS}}:=\mathcal{S}_{\mathrm{prd}}^{\mathrm{HS}}(G,H)$.

(2) *An $L(H)$-valued predictable step process $\Gamma\colon\Omega\times[0,T]\to L(H,H)$ is of the form*

$$\Gamma(\omega,t)=\left(\sum_{k=1}^{N(0)}O_{0,k}\mathbb{1}_{A_{0,k}}(\omega)\right)\mathbb{1}_{\{0\}}(t)+\sum_{i=1}^{n-1}\left(\sum_{k=1}^{N(i)}O_{i,k}\mathbb{1}_{A_{i,k}}(\omega)\right)\mathbb{1}_{(t_i,t_{i+1}]}(t), \tag{3.44}$$

where $0=t_1<\cdots<t_n=T$, $A_{0,k}\in\mathcal{F}_0$ and $O_{0,k}\in L(H)$ for all $k=1,\dots,N(0)$, $A_{i,k}\in\mathcal{F}_{t_i}$ and $O_{i,k}\in L(H)$ for all $i=1,\dots,n-1$ and $k=1,\dots,N(i)$. The space of all $L(H)$-valued predictable step processes with

$$\sup_{(\omega,t)\in\Omega\times[0,T]}\|\Gamma(\omega,t)\|_{H\to H}\le 1$$

is denoted by $\mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}}:=\mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}}(H,H)$.

Let $\Psi\in\mathcal{S}_{\mathrm{prd}}^{\mathrm{HS}}$ be of the form (3.43). Since Hilbert-Schmidt operators are 0-Radonifying by [76, Th. VI.5.2], it follows from [76, Pr. VI.5.3] that there exists an H-valued random variable $F_{i,k}(L(t_{i+1})-L(t_i))\colon\Omega\to H$ for each $i=1,\dots,n-1$ and $k=1,\dots,N(i)$, satisfying

$$\big(L(t_{i+1})-L(t_i)\big)(F_{i,k}^*h)=\langle F_{i,k}(L(t_{i+1})-L(t_i)),h\rangle\quad P\text{-a.s. for all } h\in H.$$

In this case, the stochastic integral of Ψ is defined by

$$I(\Psi):=\int_0^T\Psi(t)\,\mathrm{d}L(t):=\sum_{i=1}^{n-1}\sum_{k=1}^{N(i)}\mathbb{1}_{A_{i,k}}F_{i,k}(L(t_{i+1})-L(t_i)).$$

Thus, the integral $I(\Psi) : \Omega \to H$ is a genuine H-valued random variable.

For the purposes of this section, it is convenient to introduce the measure space $\big(\Omega \times [0,T], \mathcal{P}, P_T\big)$, where $\mathcal{P}$ denotes the predictable σ-algebra and the measure P_T is defined by $P_T := P \otimes \mathrm{Leb}|_{[0,T]}$.

Definition 3.37. *We say that a predictable process Ψ is L-integrable if there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of processes in $\mathcal{S}_{\mathrm{prd}}^{\mathrm{HS}}$ such that*

(1) $(\Psi_n)_{n\in\mathbb{N}}$ *converges P_T-a.e. to Ψ;*

(2) $\displaystyle \lim_{m,n\to\infty} \sup_{\Gamma\in\mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}}} E\left[\left\| \int_0^T \Gamma(\Psi_m - \Psi_n)\,\mathrm{d}L \right\| \wedge 1\right] = 0.$

In this case, the stochastic integral of Ψ is defined by

$$I(\Psi) := \int_0^T \Psi \,\mathrm{d}L = \lim_{n\to\infty} \int_0^T \Psi_n \,\mathrm{d}L \quad \textit{in } L_P^0(\Omega, H).$$

The class of all L-integrable $L_2(G,H)$-valued predictable processes will be denoted by $\mathcal{I}_{\mathrm{prd},L}^{\mathrm{HS}} := \mathcal{I}_{\mathrm{prd},L}^{\mathrm{HS}}(G,H)$. As usual, for $t \in [0,T]$, we define $\int_0^t \Psi\,\mathrm{d}L := \int_0^T \mathbb{1}_{\Omega\times(0,t]}\Psi\,\mathrm{d}L$.

Remark 3.38. *An extension of [41, Le. 2.3] to H-valued random variables shows that Condition (2) of Definition 3.37 implies*

$$\lim_{m,n\to\infty} E\left[\sup_{t\in[0,T]} \left\| \int_0^t \Psi_m - \Psi_n \,\mathrm{d}L \right\| \wedge 1\right] = 0.$$

Hence, the notion of convergence introduced in Definition 3.37 is stronger than ucp convergence. In particular, this immediately gives that for each $\Psi \in \mathcal{I}_{\mathrm{prd},L}^{\mathrm{HS}}$ the process $\left(\int_0^t \Psi\,\mathrm{d}L\right)_{t\in[0,T]}$ has cádlág paths. For details, see the end of Remark 3.26.

Let $L_P^0(\Omega, (\mathcal{M}_{\det,L}^{\mathrm{HS}}, \rho_L))$ denote the collection of all random variables $\Psi : \Omega \to \mathcal{M}_{\det,L}^{\mathrm{HS}}$ taking values in the Polish space $(\mathcal{M}_{\det,L}^{\mathrm{HS}}, \rho_L)$, see Proposition 3.23. We endow the space $L^0_P(\Omega, (\mathcal{M}^{\mathrm{HS}}_{\det,L}, \rho_L))$ with the translation invariant metric

$$|||\Psi_1 - \Psi_2|||_L := E\left[\rho_L(\Psi_1 - \Psi_2) \wedge 1\right] \quad \text{for } \Psi_1, \Psi_2 \in L^0_P(\Omega, (\mathcal{M}^{\mathrm{HS}}_{\det,L}, \rho_L)).$$

Recall, see for example [18, Th. 9.2.3] or [31, Le. 3.6], that $\left(L^0_P(\Omega, (\mathcal{M}^{\mathrm{HS}}_{\det,L}, \rho_L)), |||\cdot|||_L\right)$ is a complete metric linear space. For ease of notation we will often use the shorthand $L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\det,L}) := \left(L^0_P(\Omega, (\mathcal{M}^{\mathrm{HS}}_{\det,L}, \rho_L)), |||\cdot|||_L\right)$.

Lemma 3.39. *Let Ψ be a predictable stochastic process in $L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\det,L})$. Then there exists a sequence $(\Psi_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ converging to Ψ both in the metric $|||\cdot|||_L$ and P_T-a.e.*

Proof. If Ψ is bounded, then $\Psi \in L^\infty_{P_T}\big(\Omega \times [0,T], L_2(G,H)\big)$. Since the algebra of sets

$$\mathcal{A}' = \left\{(s,t] \times B : s < t, B \in \mathcal{F}_s\right\} \cup \left\{\{0\} \times B : B \in \mathcal{F}_0\right\}$$

generates $\mathcal{P}$, we conclude from [24, Le. 1.2.19] and [24, Re. 1.2.20] that there exists a sequence $(\Psi_k)_{k\in\mathbb{N}}$ of uniformly bounded processes in $\mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ such that $\Psi_k \to \Psi$ P_T-a.e. Thus, there exists a set $N \in \mathcal{P}$ such that $P_T(N) = 0$ and $\big(\Psi_k(\omega,t) - \Psi(\omega,t)\big) \to 0$ for all $(\omega,t) \in N^c$. Fubini's theorem implies that

$$P_T(N) = P \otimes \mathrm{Leb}|_{[0,T]}(N) = \int_\Omega \mathrm{Leb}|_{[0,T]}(N_\omega)\, P(\mathrm{d}\omega) = 0,$$

where for each fixed $\omega \in \Omega$ we define

$$N_\omega := \left\{t \in [0,T] \colon \big(\Psi_k(\omega,t) - \Psi(\omega,t)\big)_{m\in\mathbb{N}} \text{ does not converge to } 0\right\}.$$

The above implies that $\mathrm{Leb}|_{[0,T]}(N_\omega) = 0$ for almost all $\omega \in \Omega$, that is, there exists an

$\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ we have

$$\mathrm{Leb}|_{[0,T]}\Big(t \in [0,T] \colon \big(\Psi_k(\omega,t) - \Psi(\omega,t)\big)_{m\in\mathbb{N}} \text{ does not converge to } 0\Big) = 0.$$

Because $(\Psi_k)_{k\in\mathbb{N}}$ is uniformly bounded and Ψ is bounded, we can conclude from Lebesgue's dominated convergence theorem that $\lim_{n\to\infty} m_L\left(\Psi_k(\omega,\cdot) - \Psi(\omega,\cdot)\right) = 0$ for each $\omega \in \Omega_0$. Since m_L and ρ_L generate the same topology on $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$, we also have $\rho_L(\Psi_k(\omega,\cdot) - \Psi(\omega,\cdot)) \to 0$ as $k \to \infty$ for each $\omega \in \Omega_0$. Another application of Lebesgue's dominated convergence theorem yields

$$\lim_{k\to\infty} |||\Psi_k - \Psi|||_L = \lim_{k\to\infty} \int_\Omega \Big(\rho_L(\Psi_k(\omega,\cdot) - \Psi(\omega,\cdot)) \wedge 1\Big)\,\mathrm{d}P = 0,$$

which shows the claim if Ψ is bounded. In the case of a general Ψ, we define

$$\Psi_n : \Omega \times [0,T] \to L_2(G,H), \qquad \Psi_n(\omega,t) = \begin{cases} \Psi(\omega,t) & \text{if } \|\Psi(\omega,t)\|_{L_2(G,H)} \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\lim_{n\to\infty} |||\Psi - \Psi_n|||_L = 0$. The first part of the proof shows that for each $n \in \mathbb{N}$ there exists a sequence $(\Psi_{n,k})_{k\in\mathbb{N}} \subseteq \mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ converging to Ψ_n as $k \to \infty$ in $|||\cdot|||_L$ and P_T-a.e. For each $n \in \mathbb{N}$ choose $k_n \in \mathbb{N}$ such that $|||(\Psi_n - \Psi_{n,k_n})|||_L < \frac{1}{n}$. It follows that

$$\lim_{n\to\infty} |||(\Psi - \Psi_{n,k_n})|||_L \le \lim_{n\to\infty} \Big(|||(\Psi - \Psi_n)|||_L + |||(\Psi_n - \Psi_{n,k_n})|||_L \Big) = 0,$$

which completes the proof, since by passing on to a suitable subsequence, we also have convergence P_T-a.e. □

3.4 Construction of the decoupled tangent sequence

The technique of constructing decoupled tangent sequences is a powerful tool to obtain strong results on a sequence of possibly dependent random variables. In this section, we briefly recall the fundamental definition, see e.g. Kwapień and Woyczyński [42] or de la Peña and Giné [61], and construct the decoupled tangent sequence in our setting which will enable us to identify the largest space of predictable integrands in the next section.

Remark 3.40. *We repeatedly use the fact in the following that given a random variable X on $(\Omega, \mathcal{F}, P)$ and another probability space $(\Omega', \mathcal{F}', P')$, the random variable X can always be considered as a random variable on the product space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P')$ by defining*

$$X(\omega, \omega') = X(\omega) \quad \text{for all } (\omega, \omega') \in \Omega \times \Omega'.$$

In this case, if X is real-valued and P-integrable we have $E_P[X] = E_{P \otimes P'}[X]$.

In the next definition, we follow closely Chapter 4.3 of [42].

Definition 3.41. *Let $\big(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n \in \mathbb{N}}\big)$ be a filtered probability space and $(X_n)_{n \in \mathbb{N}}$ an $(\mathcal{F}_n)$-adapted sequence of H-valued random variables. If $\big(\Omega', \mathcal{F}', P', ({\mathcal{F}'}_n)_{n \in \mathbb{N}}\big)$ is another filtered probability space, then a sequence $(Y_n)_{n \in \mathbb{N}}$ of H-valued random variables defined on $\big(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P', (\mathcal{F}_n \otimes {\mathcal{F}'}_n)_{n \in \mathbb{N}}\big)$ is said to be a decoupled tangent sequence to $(X_n)_{n \in \mathbb{N}}$ if*

(1) *for each $\omega \in \Omega$, we have that $(Y_n(\omega, \cdot))_{n \in \mathbb{N}}$ is a sequence of independent random variables on $(\Omega', \mathcal{F}', P')$;*

(2) *the sequences $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ satisfy for each $n \in \mathbb{N}$ that*

$$\mathcal{L}(X_n | \mathcal{F}_{n-1} \otimes {\mathcal{F}'}_{n-1}) = \mathcal{L}(Y_n | \mathcal{F}_{n-1} \otimes {\mathcal{F}'}_{n-1}) \quad P \otimes P' - a.s.$$

Remark 3.42. *The importance of decoupled tangent sequences within the framework of stochastic integration lies in the existence of a collection of inequalities, frequently called decoupling inequalities, which relate convergence of an adapted sequence of random variables to convergence of their decoupled tangent sequence. More precisely, by [42, Pr. 5.7.1.(ii)], there exists a constant $c_1 > 0$ such that for all finite adapted sequences $(X_n)_{n=1,...,N}$ of H-valued random variables with corresponding decoupled tangent sequence $(Y_n)_{n=1,...,N}$ it holds that*

$$E_P\left[\left\|\sum_{n=1}^N X_n\right\| \wedge 1\right] \le c_1 E_{P\otimes P}\left[\left\|\sum_{n=1}^N Y_n\right\| \wedge 1\right].$$

Moreover, by [42, Pr. 5.7.2], there exists $c_2 > 0$ such that the following "recoupling" inequality also holds

$$E_{P\otimes P}\left[\left\|\sum_{n=1}^N Y_n\right\| \wedge 1\right] \le c_2 \sup_{\epsilon_n \in\{\pm 1\}} E_P\left[\left\|\sum_{n=1}^N \epsilon_n X_n\right\| \wedge 1\right].$$

The main tool for establishing the stochastic integral in the next section is a cylindrical Lévy process $\widetilde{L}$ on an enlarged probability space, whose Radonified increments are decoupled to the Radonified increments of the original cylindrical Lévy process. This cylindrical Lévy process $\widetilde{L}$ is explicitly constructed in the following result.

Proposition 3.43. *Let L be a cylindrical Lévy process in G, $0 = t_0 \le ... \le t_N = T$ be a partition of $[0,T]$ and for each $n = 1, ..., N$ we define $\Theta_n := \sum_{k=1}^{M(n)} F_{n,k}\mathbb{1}_{A_{n,k}}$, where $F_{n,k} \in L_2(G,H)$, $A_{n,k} \in \mathcal{F}_{t_{n-1}}$ for all $k = 1, ..., M(n)$. By defining cylindrical random variables*

$$\widetilde{L}(t)\colon G \to L^0_{P\otimes P}(\Omega\times\Omega;\mathbb{R}), \qquad \Big(\widetilde{L}(t)g\Big)(\omega,\omega') = \Big(L(t)g\Big)(\omega'),$$

it follows that $(\widetilde{L}(t) : t \ge 0)$ is a cylindrical Lévy process on G and the sequence of its

Radonified increments

$$\Big(\Theta_n\big(\widetilde{L}(t_n)-\widetilde{L}(t_{n-1})\big)\Big)_{n\in\{1,\dots,N\}}$$

defined on $\big(\Omega\times\Omega,\mathcal{F}\otimes\mathcal{F},P\otimes P,(\mathcal{F}_{t_n}\otimes\mathcal{F}_{t_n})_{n\in\{0,\dots,N\}}\big)$ *is a decoupled tangent sequence to the sequence of Radonified increments*

$$\Big(\Theta_n\big(L(t_n)-L(t_{n-1})\big)\Big)_{n\in\{1,\dots,N\}}$$

defined on $\big(\Omega,\mathcal{F},P,(\mathcal{F}_{t_n})_{n\in\{0,\dots,N\}}\big)$.

Proof. In order to make it easier to follow this proof, we define $\Omega'=\Omega$, $\mathcal{F}'=\mathcal{F}$, $P'=P$ and $\mathcal{F}'_{t_n}=\mathcal{F}_{t_n}$ for all $n\in\{0,...,N\}$ and instead of denoting the filtered product space by

$$\Big(\Omega\times\Omega,\mathcal{F}\otimes\mathcal{F},P\otimes P,(\mathcal{F}_{t_n}\otimes\mathcal{F}_{t_n})_{n\in\{0,\dots,N\}}\Big),$$

we write

$$\Big(\Omega\times\Omega',\mathcal{F}\otimes\mathcal{F}',P\otimes P',(\mathcal{F}_{t_n}\otimes\mathcal{F}'_{t_n})_{n\in\{0,\dots,N\}}\Big).$$

The fact that for each $t\geq 0$ the mapping $\widetilde{L}(t)\colon G\to L^0_{P\otimes P'}(\Omega\times\Omega',\mathbb{R})$ is continuous follows directly from the definition of $\widetilde{L}$ and Remark 3.40. Thus $\widetilde{L}$ is a cylindrical stochastic process. To prove that it is in fact a cylindrical Lévy process, let us fix $n\in\mathbb{N}$ and $g_1,\dots,g_n\in G$ and consider the n-dimensional processes Y and Z defined by $Y(t)=(\widetilde{L}(t)g_1,\dots,\widetilde{L}(t)g_n)$ and $Z(t)=(L(t)g_1,\dots,L(t)g_n)$. It is enough to show that for any $m\in\mathbb{N}$ and times $0\leq t_0<\cdots<t_m\leq T$ the random variables $Y(t_m)-Y(t_{m-1}),\dots,Y(t_1)-Y(t_0)$ and $Z(t_m)-Z(t_{m-1}),\dots,Z(t_1)-Z(t_0)$ have the same distribution. Here we only prove that for any $0\leq s<t\leq T$ the random variables $Y(t)-Y(s)$ and $Z(t)-Z(s)$ have the same distribution. The general case follows

analogously. To see this, let $A \in \mathfrak{B}(\mathbb{R}^n)$ be arbitrary. The very definition of $\widetilde{L}$ shows

$$\begin{aligned}
&(P \otimes P')\,(Y(t) - Y(s) \in A) \\
&\quad = (P \otimes P')\,\Big((\widetilde{L}(t)g_1 - \widetilde{L}(s)g_1, ..., \widetilde{L}(t)g_n - \widetilde{L}(s)g_n) \in A\Big) \\
&\quad = (P \otimes P')\,(\Omega \times \{(L(t)g_1 - L(s)g_1, ..., L(t)g_n - L(s)g_n) \in A\}) \\
&\quad = P'\,((L(t)g_1 - L(s)g_1, ..., L(t)g_n - L(s)g_n) \in A) \\
&\quad = P\,(Z(t) - Z(s) \in A)\,.
\end{aligned}$$

To show that the Radonified increments of $\tilde{L}$ satisfy Condition (1) of Definition 3.41, fix some $\omega \in \Omega$. Then $\Theta_n(\omega)$ is a (deterministic) Hilbert-Schmidt operator and $(\widetilde{L}(t)(\omega, \cdot) : t \geq 0)$ is a cylindrical Lévy process in G. Thus, for a fixed $\omega \in \Omega$ and $n \in \{1, ..., N\}$, the Radonified increment $\Theta_n(\omega)(\widetilde{L}(t_n)(\omega, \cdot) - \widetilde{L}(t_{n-1})(\omega, \cdot))$ is an $\mathcal{F}'_{t_n}$-measurable H-valued random variable on $(\Omega', \mathcal{F}', P')$ independent of $\mathcal{F}'_{t_{n-1}}$. It follows for each $\omega \in \Omega$ that

$$\Big(\Theta_n(\omega)(\widetilde{L}(t_n)(\omega, \cdot) - \widetilde{L}(t_{n-1})(\omega, \cdot))\Big)_{n \in \{1, ..., N\}}$$

is a sequence of independent random variables.

For establishing Condition (2) of Definition 3.41, we define for each $n \in \{1, ..., N\}$ the H-valued random variables

$$\begin{aligned}
X_n &:= \Theta_n\big(L(t_n) - L(t_{n-1})\big) := \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k}} F_{n,k}\big(L(t_n) - L(t_{n-1})\big), \\
Y_n &:= \Theta_n\big(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})\big) := \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k}} F_{n,k}\big(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})\big),
\end{aligned}$$

where $F_{n,k}\big(L(t_n) - L(t_{n-1})\big)$ and $F_{n,k}\big(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})\big)$ refer to the Radonified increments, and by taking another representation of Θ_n if necessary, we may assume that for each $n \in \mathbb{N}$ the representation of Θ_n satisfies that $A_{n,k} \cap A_{n,l} = \emptyset$ for $k \neq l$ and

$\cup_{k=1}^{M(n)} A_{n,k} = \Omega$. Choose regular versions of the conditional distributions

$$\begin{aligned}
&(P \otimes P')_{X_n} \colon \mathfrak{B}(H) \times (\Omega \times \Omega') \to [0,1], \\
&\qquad (P \otimes P')_{X_n}\big(B, (\omega, \omega')\big) = (P \otimes P')(X_n \in B | \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}})(\omega, \omega'), \\
&(P \otimes P')_{Y_n} \colon \mathfrak{B}(H) \times (\Omega \times \Omega') \to [0,1], \\
&\qquad (P \otimes P')_{Y_n}\big(B, (\omega, \omega')\big) = (P \otimes P')(Y_n \in B | \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}})(\omega, \omega').
\end{aligned}$$

Since $\widetilde{L}(t)$ is a cylindrical Lévy process, and for each $n \in \mathbb{N}$ we have $A_{n,k} \cap A_{n,l} = \emptyset$ for $k \neq l$ and $\cup_{k=1}^{M(n)} A_{n,k} = \Omega$, we obtain for all $h \in H$ and $n \in \mathbb{N}$ that

$$\begin{aligned}
&E_{P \otimes P'}\Big[e^{i\langle Y_n, h\rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}}\Big] \\
&\quad = E_{P \otimes P'}\left[e^{i\left\langle \left(\sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'} F_{n,k}\right)(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})), h\right\rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}}\right] \\
&\quad = \sum_{k=1}^{M(n)} E_{P \otimes P'}\Big[\mathbb{1}_{A_{n,k} \times \Omega'}\, e^{i\langle F_{n,k}(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})), h\rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}}\Big] \\
&\quad = \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'}\, E_{P \otimes P'}\Big[e^{i\langle F_{n,k}(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})), h\rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}}\Big] \\
&\quad = \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'}\, E_{P \otimes P'}\Big[e^{i\langle F_{n,k}(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})), h\rangle}\Big] \\
&\quad = \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'}\, E_{P'}\Big[e^{i\langle F_{n,k}(L(t_n) - L(t_{n-1})), h\rangle}\Big] \\
&\quad = \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'}\, e^{(t_n - t_{n-1}) S(F_{n,k}^* h)} \\
&\quad = e^{(t_n - t_{n-1}) S(\Theta_n^* h)} \quad P \otimes P' - \text{a.s.}, \qquad (3.45)
\end{aligned}$$

where S denotes the cylindrical Lévy symbol of L. In the same way we obtain

$$E_{P\otimes P'}\left[e^{i\langle X_n,h\rangle}\middle|\mathcal{F}_{t_{n-1}}\otimes\mathcal{F}'_{t_{n-1}}\right]=e^{(t_n-t_{n-1})S(\Theta_n^*h)}\quad P\otimes P'-\text{a.s.}\tag{3.46}$$

It follows from (3.45) and (3.46) by calculating the conditional expectation from the conditional probability, see e.g. [31, Th. 6.4], that for $P\otimes P'$ a.a. $(\omega,\omega')\in\Omega\times\Omega'$ and for all $u\in H$ we have

$$\begin{aligned}\varphi_{(P\otimes P')_{X_n}(\cdot,(\omega,\omega'))}(u)&=\int_H e^{i\langle h,u\rangle}\,(P\otimes P')_{X_n}\big(\mathrm{d}h,(\omega,\omega')\big)\\&=E_{P\otimes P'}\left[e^{i\langle X_n,u\rangle}\middle|\mathcal{F}_{t_{n-1}}\otimes\mathcal{F}'_{t_{n-1}}\right](\omega,\omega')\\&=E_{P\otimes P'}\left[e^{i\langle Y_n,u\rangle}\middle|\mathcal{F}_{t_{n-1}}\otimes\mathcal{F}'_{t_{n-1}}\right](\omega,\omega')\\&=\int_H e^{i\langle h,u\rangle}\,(P\otimes P')_{X_n}\big(\mathrm{d}h,(\omega,\omega')\big)=\varphi_{(P\otimes P')_{Y_n}(\cdot,(\omega,\omega'))}(u).\end{aligned}$$

Since characteristic functions uniquely determine distributions on $\mathfrak{B}(H)$, we obtain

$$(P\otimes P')_{X_n}(\cdot,(\omega,\omega'))=(P\otimes P')_{Y_n}(\cdot,(\omega,\omega'))\quad P\otimes P'-\text{a.s.},$$

establishing Condition (2) of Definition 3.41. □

3.5 Characterisation of random integrable processes

The following is the main result of this chapter characterising the largest space of predictable integrands which are stochastically integrable with respect to a cylindrical Lévy process L in Hilbert space.

Theorem 3.44. *The space $\mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$ of predicable Hilbert-Schmidt operator-valued processes integrable with respect to a cylindrical Lévy process L in G coincides with the class of predictable processes in $L^0_P(\Omega,\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L})$.*

As in the case of deterministic integrands, the above characterisation of the space of L-integrable predictable processes strongly relies on the equivalent notion of convergences in two spaces.

Lemma 3.45. *Let L be a cylindrical Lévy process in G, and $(\Psi_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$. Then the following are equivalent:*

(a) $\lim_{n\to\infty} |||\Psi_n|||_L = 0$;

(b) $\lim_{n\to\infty} \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E\left[\left\|\int_0^T \Gamma\Psi_n \,\mathrm{d}L\right\| \wedge 1\right] = 0$ *and* $\lim_{n\to\infty} E\left[m''(\Psi_n)\wedge 1\right] = 0.$

Proof. To prove (a) ⇒ (b), let $\epsilon > 0$ be fixed. Lemma 3.28 and the fact that m_L and ρ_L generate the same topology on $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ enables us to choose $\delta > 0$ such that for every $\psi \in \mathcal{S}^{\mathrm{HS}}_{\mathrm{det}}$ we have the implication:

$$\rho_L(\psi) \le \delta \;\Rightarrow\; \sup_{\gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{det}}} P\left(\left\|\int_0^T \gamma\psi \,\mathrm{d}L\right\| > \epsilon\right) \le \epsilon. \tag{3.47}$$

Since $\lim_{n\to\infty} |||\Psi_n|||_L = 0$, there exists $n_0\in\mathbb{N}$ such that the set

$$A_n := \{\omega\in\Omega : \rho_L(\Psi_n(\omega)) \le \delta\}$$

satisfies $P(A_n) \ge 1-\epsilon$ for all $n \ge n_0$. By recalling the definition of $\widetilde{L}$ and $(\Omega', \mathcal{F}', P')$ from Proposition 3.43, implication (3.47) implies for all $\omega \in A_n$ and $n \ge n_0$ that

$$\sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} P'\left(\omega'\in\Omega'\colon \left\|\left(\int_0^T \Gamma(\omega)\Psi_n(\omega)\,\mathrm{d}\widetilde{L}(\omega,\cdot)\right)(\omega')\right\| > \epsilon\right) \le \epsilon.$$

Since $P(A_n) \ge 1-\epsilon$ for all $n \ge n_0$, we obtain

$$P\left(\omega\in\Omega\colon \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} P'\left(\omega'\in\Omega'\colon \left\|\left(\int_0^T \Gamma(\omega)\Psi_n(\omega)\;\mathrm{d}\widetilde{L}(\omega,\cdot)\right)(\omega')\right\| > \epsilon\right) \le \epsilon\right)$$

$$\geq P(A_n) \geq 1 - \epsilon.$$

Fubini's theorem implies for all $n \geq n_0$ and $\Gamma \in \mathcal{S}_{\rm prd}^{1,\rm op}$ that

$$\begin{aligned}
&(P \otimes P')\left((\omega, \omega') \in \Omega \times \Omega' \colon \left\| \left(\int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right)(\omega, \omega') \right\| > \epsilon \right) \\
&= \int_\Omega P' \left(\omega' \in \Omega' : \left\| \left(\int_0^T \Gamma(\omega) \Psi_n(\omega) \, \mathrm{d}\widetilde{L}(\omega, \cdot) \right)(\omega') \right\| > \epsilon \right) P(\mathrm{d}\omega) \leq \epsilon + \epsilon(1 - \epsilon).
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\lim_{n \to \infty} \sup_{\Gamma \in \mathcal{S}_{\rm prd}^{1,\rm op}} E_{P \otimes P'} \left[\left\| \int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right\| \wedge 1 \right] = 0. \tag{3.48}$$

By the ideal property of $L_2(G, H)$, for each $n \in \mathbb{N}$ and $\Gamma \in \mathcal{S}_{\rm prd}^{1,\rm op}$ the integrand $\Gamma \Psi_n$ lies in $\mathcal{S}_{\rm prd}^{\rm HS}$ and has a representation of the from

$$\Gamma \Psi_n = \Gamma_0^n F_0^n \mathbb{1}_{\{0\}} + \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}, \tag{3.49}$$

where $0 = t_1^n \leq \cdots < t_{N(n)}^n = T$, and $\Gamma_i^n F_i^n$ is an $\mathcal{F}_{t_i^n}$-measurable $L_2(G, H)$-valued random variable taking only finitely many values for each $i = 0, \ldots, N(n) - 1$. Proposition 3.43 guarantees for each $n \in \mathbb{N}$ that the sequence of Radonified increments

$$\left(\Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n)) \right)_{i=1,\ldots,N_n-1}$$

has the decoupled tangent sequence

$$\left(\Gamma_i^n F_i^n (\widetilde{L}(t_{i+1}^n) - \widetilde{L}(t_i^n)) \right)_{i=1,\ldots,N_n-1}.$$

We conclude from the decoupling inequality [42, Pr. 5.7.1.(ii)] that there exists a constant $c > 0$ such that, for all $n \in \mathbb{N}$ and $\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}$, we have

$$\begin{aligned} E_{P\otimes P'}\left[\left\|\int_0^T \Gamma\Psi_n \,\mathrm{d}L\right\| \wedge 1\right] &= E_{P\otimes P'}\left[\left\|\sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n))\right\| \wedge 1\right] \\ &\le c E_{P\otimes P'}\left[\left\|\sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n (\widetilde{L}(t_{i+1}^n) - \widetilde{L}(t_i^n))\right\| \wedge 1\right] \\ &= c E_{P\otimes P'}\left[\left\|\int_0^T \Gamma\Psi_n \,\mathrm{d}\widetilde{L}\right\| \wedge 1\right]. \end{aligned}$$

We conclude from Remark 3.40 and (3.48) that

$$\lim_{n\to\infty} \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E_P\left[\left\|\int_0^T \Gamma\Psi_n \,\mathrm{d}L\right\| \wedge 1\right] = \lim_{n\to\infty} \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E_{P\otimes P'}\left[\left\|\int_0^T \Gamma\Psi_n \,\mathrm{d}L\right\| \wedge 1\right] = 0.$$

Seeing that m_L and ρ_L generate the same topology on $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$, our assumption that $\lim_{n\to\infty} |||\Psi_n|||_L = 0$ implies $\lim_{n\to\infty} E[m''(\Psi_n) \wedge 1] = 0$, which immediately gives (b).

For establishing (b) $\Rightarrow$ (a), given any $\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}$ we may assume that $\Gamma\Psi_n$ has a representation of the form (3.49). We conclude from [42, Pr. 5.7.2] that there exists a constant $c > 0$ such that for all $\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}$ we have

$$\begin{aligned} E_{P\otimes P'}\left[\left\|\int_0^T \Gamma\Psi_n \,\mathrm{d}\widetilde{L}\right\| \wedge 1\right] &= E_{P\otimes P'}\left[\left\|\sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n (\widetilde{L}(t_{i+1}^n) - \widetilde{L}(t_i^n))\right\| \wedge 1\right] \\ &\le c \max_{\epsilon_i\in\{\pm 1\}} E_{P\otimes P'}\left[\left\|\sum_{i=1}^{N(n)-1} \epsilon_i\Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n))\right\| \wedge 1\right] \\ &= c \max_{\epsilon_i\in\{\pm 1\}} E_{P}\left[\left\|\sum_{i=1}^{N(n)-1} \epsilon_i\Gamma_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n))\right\| \wedge 1\right] \\ &\le c \sup_{\Theta\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E_{P}\left[\left\|\sum_{i=1}^{N(n)-1} \Theta_i^n F_i^n (L(t_{i+1}^n) - L(t_i^n))\right\| \wedge 1\right] \end{aligned}$$

$$= c \sup_{\Theta \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E_P\left[\left\|\int_0^T \Theta \Psi_n \,\mathrm{d}L\right\| \wedge 1\right]. \tag{3.50}$$

By choosing $\Gamma = \mathrm{Id}\mathbb{1}_{\Omega \times (0,T]}$, the hypothesis on $(\Psi_n)_{n\in\mathbb{N}}$ implies

$$\lim_{n\to\infty} E_{P\otimes P'}\left[\left\|\int_0^T \Psi_n \,\mathrm{d}\widetilde{L}\right\| \wedge 1\right] = 0.$$

It follows that for every subsequence $(\Psi_{n_m})_{m\in\mathbb{N}}$ of $(\Psi_n)_{n\in\mathbb{N}}$, there exists a further subsequence $(\Psi_{n_{m_j}})_{j\in\mathbb{N}}$ and a set $N \subseteq \Omega \times \Omega'$ with $(P \otimes P')(N) = 0$ satisfying

$$\lim_{j\to\infty}\left(\int_0^T \Psi_{n_{m_j}} \,\mathrm{d}\widetilde{L}\right)(\omega,\omega') = 0 \quad \text{for each } (\omega,\omega') \in N^c.$$

Define the section of the set N for each $\omega \in \Omega$ by

$$N_\omega = \left\{\omega' \in \Omega' \colon \lim_{j\to\infty}\left(\int_0^T \Psi_{n_{m_j}}(\omega) \,\mathrm{d}\widetilde{L}(\omega,\cdot)\right)(\omega') \neq 0\right\},$$

where we note that since $\Psi_{n_{m_j}}$ are predictable step processes, it holds that

$$\left(\int_0^T \Psi_{n_{m_j}} \,\mathrm{d}\widetilde{L}\right)(\omega,\cdot) = \int_0^T \Psi_{n_{m_j}}(\omega) \,\mathrm{d}\widetilde{L}(\omega,\cdot) \qquad \text{for all } \omega \in \Omega.$$

Fubini's theorem implies $0 = (P \otimes P')(N) = \int_\Omega P'(N_\omega)\mathrm{d}P(\omega)$, from which it follows that there exists $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that $P'(N_\omega) = 0$ for all $\omega \in \Omega_1$. In other words, for each fixed $\omega \in \Omega_1$, the sequence of random variables

$$\left(\int_0^T \Psi_{n_{m_j}}(\omega) \,\mathrm{d}\widetilde{L}(\omega,\cdot)\right)_{j\in\mathbb{N}}$$

converges P'-a.s. to 0 as H-valued random variables on $(\Omega', \mathcal{F}', P')$. Since for each fixed

$\omega \in \Omega_1$, the above sequence is infinitely divisible and has characteristics

$$\left(\int_0^T b^\theta_{\Psi_{n_{m_j}}(\omega,t)}\,\mathrm{d}t,\ \int_0^T \Psi_{n_{m_j}}(\omega,t) Q \Psi^*_{n_{m_j}}(\omega,t)\,\mathrm{d}t,\ (\lambda\otimes \mathrm{Leb})\circ \kappa^{-1}_{\Psi_{n_{m_j}}(\omega)} \right),$$

by Lemmata 3.11, 3.35 and the fact that for each $\omega \in \Omega$ the cylindrical Lévy process $\widetilde{L}(\omega,\cdot)$ has the same cylindrical characteristics as L, we obtain for all $\omega \in \Omega_1$ that

$$\lim_{j\to\infty} k_L(\Psi_{n_{m_j}}(\omega)) = \lim_{j\to\infty} k_{\widetilde{L}(\omega)}(\Psi_{n_{m_j}}(\omega)) = 0.$$

As $P(\Omega_1) = 1$, the above argument proves that for all $\epsilon > 0$ we have

$$\lim_{n\to\infty} P\left(\omega \in \Omega : \int_0^T k_L(\Psi_n(\omega,t))\,\mathrm{d}t > \epsilon\right) = 0. \tag{3.51}$$

To finish the proof, it remains to show that for all $\epsilon > 0$ we have

$$\lim_{n\to\infty} P\left(\omega \in \Omega : \int_0^T l_L(\Psi_n(\omega,t))\,\mathrm{d}t > \epsilon\right) = 0. \tag{3.52}$$

It follows from Equation (3.50) that the sequence $(\Psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ satisfies

$$\lim_{n\to\infty} \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E_{P\otimes P'}\left[\left\| \int_0^T \Gamma\Psi_n\,\mathrm{d}\widetilde{L} \right\| \wedge 1 \right] = 0. \tag{3.53}$$

Let $\epsilon \in (0,1)$ be fixed. Since stochastic integrals with deterministic integrands with respect to L are infinitely divisible, Remark 2.2 implies that there exists $\delta \in (0,\epsilon)$ such that for all $\psi \in \mathcal{M}^{\mathrm{HS}}_{\mathrm{det}}$ we have the implication

$$P\left(\left\| \int_0^T \psi\,\mathrm{d}L \right\| > \sqrt{\delta} \right) < \sqrt{\delta} \implies \left\| \int_0^T b^\theta_{\psi(t)}\,\mathrm{d}t \right\| < \epsilon. \tag{3.54}$$

By Equation (3.53), there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sup_{\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} (P \otimes P') \left(\left\| \int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right\| > \delta \right) < \delta. \tag{3.55}$$

Chebyshev's inequality, Fubini's theorem and Equation (3.55) imply for all $n \geq N$ and $\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}$ that

$$\begin{aligned} P\left(P'\left(\left\| \int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right\| > \delta \right) < \sqrt{\delta} \right) &= 1 - P\left(P'\left(\left\| \int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right\| > \delta \right) \geq \sqrt{\delta} \right) \\ &\geq 1 - \frac{1}{\sqrt{\delta}} \int_\Omega P'\left(\left\| \int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right\| > \delta \right) \mathrm{d}P \\ &= 1 - \frac{1}{\sqrt{\delta}} (P \otimes P') \left(\left\| \int_0^T \Gamma \Psi_n \, \mathrm{d}\widetilde{L} \right\| > \delta \right) \\ &\geq 1 - \sqrt{\delta}. \end{aligned} \tag{3.56}$$

In light of Equations (3.54) and (3.56), we have for all $n \geq N$ and $\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}$ that

$$P\left(\left\| \int_0^T b^\theta_{\Gamma(\omega)\Psi_n(\omega)} \, \mathrm{d}t \right\| < \epsilon \right) \geq 1 - \sqrt{\delta},$$

or equivalently, for all $n \geq N$ we have

$$\sup_{\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} P\left(\left\| \int_0^T b^\theta_{\Gamma(\omega)\Psi_n(\omega)} \, \mathrm{d}t \right\| \geq \epsilon \right) \leq \sqrt{\delta}.$$

The above inequality, combined with an approximation argument using functions in $\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}$ shows that for any predictable $L(H)^1$-valued process Λ and $n \geq N$ it holds that

$$P\left(\left\| \int_0^T b^\theta_{\Lambda(\omega)\Psi_n(\omega)} \, \mathrm{d}t \right\| \geq \epsilon \right) \leq \sqrt{\delta}. \tag{3.57}$$

For each fixed $n \geq N$, define a process $H_n : \Omega \times [0,T] \to L(H)^1$ by

$$H_n(\omega) = f(b^\theta_{i(\Psi_n(\omega))\Psi_n(\omega)}) \circ i(\Psi_n(\omega)),$$

with i and f as in Lemma 3.33. Then, H_n is predictable and, by the same argument as in Equation (3.35), it satisfies for each $\omega \in \Omega$ that

$$\int_0^T \sup_{O \in L(H)^1} \left\| b^\theta_{O\Psi_n(\omega,t)} \right\| \, \mathrm{d}t \leq \left\| \int_0^T b^\theta_{H_n(\omega,t)\Psi_n(\omega,t)} \, \mathrm{d}t \right\| + \epsilon. \tag{3.58}$$

By Equation (3.58), and replacing Λ by H_n in Equation (3.57), for all $n \geq \mathbb{N}$ we obtain

$$P\left(\int_0^T \sup_{O \in L(H)^1} \left\| b^\theta_{O\Psi_n(\omega,t)} \right\| \, \mathrm{d}t \geq 2\epsilon \right) \leq P\left(\left\| \int_0^T b^\theta_{H_n(\omega,t)\Psi_n(\omega,t)} \, \mathrm{d}t \right\| \geq \epsilon \right) \leq \sqrt{\delta}.$$

Since we have that $\delta < \epsilon$, this finishes the proof of the claim in Equation (3.52). Finally, by Equations (3.51), (3.52), and the assumption that $\lim_{n\to\infty} E\left[m''(\Psi_n) \wedge 1\right] = 0$, we obtain that $\lim_{n\to\infty} E\left[m_L(\Psi_n) \wedge 1\right] = 0$. This completes the proof, since m_L and ρ_L generate the same topology. □

Remark 3.46. *In light of Lemma 3.45, it follows by the same argument as in Remark 3.38 that conditions* (a) *and* (b) *of Lemma 3.45 both imply*

$$\lim_{n\to\infty} E\left[\sup_{t\in[0,T]} \left\| \int_0^t \Psi_n \, \mathrm{d}L \right\| \wedge 1 \right] = 0.$$

Proof of Theorem 3.44. If $\Psi \in \mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$ then Definition 3.37 guarantees the existence of a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ converging P_T-a.e. to Ψ and satisfying

$$\lim_{m,n\to\infty} \sup_{\Gamma \in \mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E\left[\left\| \int_0^T \Gamma(\Psi_m - \Psi_n) \, \mathrm{d}L \right\| \wedge 1 \right] = 0.$$

Lemma 3.45 implies that $\lim_{m,n\to\infty} \left\lVert\!\left\lvert \Psi_m - \Psi_n \right\rvert\!\right\rVert_L = 0$. Completeness of the metric space $(L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}), \left\lVert\!\left\lvert\cdot\right\rvert\!\right\rVert_L)$ and the fact that $(\Psi_n)_{n\in\mathbb{N}}$ converges P_T-a.e. to Ψ together yield that the sequence $(\Psi_n)_{n\in\mathbb{N}}$ has a limit in $L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L})$ and that this limit necessarily coincides with Ψ. Thus $\Psi \in L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L})$.

To establish the reverse inclusion, let Ψ be a predictable process in the space $L^0_P(\Omega, \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L})$. Lemma 3.39 guarantees that there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ converging to Ψ in $\left\lVert\!\left\lvert\cdot\right\rvert\!\right\rVert_L$ and P_T-a.e. Then, $(\Psi_m - \Psi_n)$ converges to 0 both in $\left\lVert\!\left\lvert\cdot\right\rvert\!\right\rVert_L$ and P_T-a.e. as $m, n \to \infty$. This implies by Lemma 3.45 that

$$\lim_{m,n\to\infty} \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E\left[\left\lVert \int_0^T \Gamma(\Psi_m - \Psi_n)\,\mathrm{d}L \right\rVert \wedge 1\right] = 0.$$

Thus Ψ satisfies the conditions of Definition 3.37, which means that $\Psi \in \mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$. □

Lemma 3.45 is crucial to characterise the space of integrable predictable processes in Theorem 3.44, as it describes convergence of predictable step processes in the space of integrands in terms of convergence in the randomised modular space. Having identified the space of integrable predictable processes, we can extend Lemma 3.45 to the whole space of integrable predictable processes.

Corollary 3.47. *Let L be a cylindrical Lévy process in G, and $(\Psi_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$. Then the following are equivalent:*

(a) $\lim\limits_{n\to\infty} \left\lVert\!\left\lvert \Psi_n \right\rvert\!\right\rVert_L = 0$;

(b) $\lim\limits_{n\to\infty} \sup\limits_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E\left[\left\lVert \int_0^T \Gamma\Psi_n\,\mathrm{d}L \right\rVert \wedge 1\right] = 0$ *and* $\lim\limits_{n\to\infty} E\left[m''(\Psi_n) \wedge 1\right] = 0.$

Proof. To establish the implication (a) ⇒ (b), first note that it follows from the definition of $\left\lVert\!\left\lvert\cdot\right\rvert\!\right\rVert_L$ and the fact that ρ_L generate the same topology as m_L that $m''(\Psi_n) \to 0$ in probability. Let $\epsilon > 0$ be fixed. Lemma 3.45 implies that there exists a $\delta(\epsilon) > 0$ such that we have for all $\Psi \in \mathcal{S}_{\text{prd}}^{\text{HS}}$ the implication:

$$|||\Psi|||_L < \delta(\epsilon) \quad \Rightarrow \quad \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi \, \mathrm{d}L\right\| \wedge 1\right] < \epsilon. \tag{3.59}$$

Since $\lim_{n\to\infty} |||\Psi_n|||_L = 0$, there exists an $n_0 \in \mathbb{N}$ such that $|||\Psi_n|||_L < \frac{\delta(\epsilon)}{2}$ for all $n \geq n_0$. By Theorem 3.44 we have that $(\Psi_n)_{n\in\mathbb{N}} \subseteq L_P^0(\Omega, \mathcal{M}_{\det,L}^{\text{HS}})$, hence Lemma 3.39 guarantees for each $n \in \mathbb{N}$ the existence of a sequence $(\Psi_n^m)_{m\in\mathbb{N}} \subseteq \mathcal{S}_{\text{prd}}^{\text{HS}}$ converging to Ψ_n in $|||\cdot|||_L$ and P_T-a.e. Consequently, we can find $m_0(n,\epsilon) \in \mathbb{N}$ for each $n \in \mathbb{N}$ such that for all $m \geq m_0(n,\epsilon)$ we have $|||\Psi_n^m - \Psi_n|||_L < \frac{\delta(\epsilon)}{2}$. We obtain for each $n \geq n_0$ and $m \geq m_0(n,\epsilon)$ that

$$|||\Psi_n^m|||_L \leq |||\Psi_n^m - \Psi_n|||_L + |||\Psi_n|||_L < \delta(\epsilon),$$

which implies by (3.59) that

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi_n^m \, \mathrm{d}L\right\| \wedge 1\right] < \epsilon. \tag{3.60}$$

Thus, if we fix an $n \geq n_0$ and recall that the integral of Ψ_n is defined to be the limit in probability of the integrals of Ψ_n^m as $m \to \infty$, we obtain from Equation (3.60) that

$$\begin{aligned} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi_n \, \mathrm{d}L\right\| \wedge 1\right] &= \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} \lim_{m\to\infty} E\left[\left\|\int_0^T \Gamma\Psi_n^m \, \mathrm{d}L\right\| \wedge 1\right] \\ &\leq \lim_{m\to\infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi_n^m \, \mathrm{d}L\right\| \wedge 1\right] < \epsilon. \end{aligned}$$

To establish the reverse implication (b) $\Rightarrow$ (a), let $\epsilon > 0$ be fixed. Lemma 3.45

implies that there exists a $\delta(\epsilon) > 0$ such that we have for all $\Psi \in \mathcal{S}_{\text{prd}}^{\text{HS}}$ the implication:

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi \, \mathrm{d}L\right\| \wedge 1\right] + E\left[m''(\Psi) \wedge 1\right] < \delta(\epsilon) \quad \Rightarrow \quad \left\|\!\left|\Psi\right|\!\right\|_L < \tfrac{\epsilon}{2}. \tag{3.61}$$

By assumption, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi_n \, \mathrm{d}L\right\| \wedge 1\right] + E\left[m''(\Psi_n) \wedge 1\right] < \tfrac{\delta(\epsilon)}{4}. \tag{3.62}$$

As $(\Psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{I}_{\text{prd},L}^{\text{HS}}$, it follows from Theorem 3.44 and Lemma 3.39 that for each $n \in \mathbb{N}$ there exists a sequence $(\Psi_n^m)_{m\in\mathbb{N}}$ of elements of $\mathcal{S}_{\text{prd}}^{\text{HS}}$ converging to Ψ_n in $\left\|\!\left|\cdot\right|\!\right\|_L$ and P_T-a.e. Consequently, we can find $m_0(n,\epsilon) \in \mathbb{N}$ for each $n \in \mathbb{N}$, such that for all $m \geq m_0(n,\epsilon)$ we have

$$\left\|\!\left|\Psi_n^m - \Psi_n\right|\!\right\|_L < \epsilon/2. \tag{3.63}$$

Since for each $n \in \mathbb{N}$ we have that $\lim_{m\to\infty} \left\|\!\left|\Psi_n^m - \Psi_n\right|\!\right\|_L = 0$, the first part of this Corollary and the reverse triangle inequality shows that for each $n \in \mathbb{N}$ there exists an $m_1(n,\epsilon) \in \mathbb{N}$ such that for all $m \geq m_1(n,\epsilon)$ we have

$$\left|\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi_n \, \mathrm{d}L\right\| \wedge 1\right] - \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E\left[\left\|\int_0^T \Gamma\Psi_n^m \, \mathrm{d}L\right\| \wedge 1\right]\right| < \tfrac{\delta(\epsilon)}{4}. \tag{3.64}$$

Moreover, since for each $n \in \mathbb{N}$ we have that $\Psi_n^m \to \Psi_n$ P_T-a.e. as $m \to \infty$, there exists $m_2(n,\epsilon) \in \mathbb{N}$ such that for all $m \geq m_2(n,\epsilon)$ it holds that

$$\left|E\left[m''(\Psi_n) \wedge 1\right] - E\left[m''(\Psi_n^m) \wedge 1\right]\right| < \frac{\delta(\epsilon)}{4}. \tag{3.65}$$

By combining Equations (3.62),(3.64) and (3.65), we obtain for all $n \geq n_0$ and $m \geq$

$\max\{m_0(n,\epsilon), m_1(n,\epsilon), m_2(n,\epsilon)\}$ that

$$\sup_{\Gamma \in \mathcal{S}^{1,\text{op}}_{\text{prd}}} E\left[\left\|\int_0^T \Gamma \Psi_n^m \,\mathrm{d}L\right\| \wedge 1\right] + E\left[m''(\Psi_n^m) \wedge 1\right] < \delta(\epsilon),$$

which implies by (3.61) and (3.63) that

$$|||\Psi_n|||_L \le |||\Psi_n - \Psi_n^m|||_L + |||\Psi_n^m|||_L < \epsilon.$$

As $\epsilon > 0$ was arbitrary, this concludes the proof. □

Remark 3.48. *Using Remark 3.46, a similar argument as in implication* (a) $\Rightarrow$ (b) *of Corollary 3.47 can be used to show that conditions* (a) *and* (b) *of Lemma 3.47 both imply*

$$\lim_{n\to\infty} E\left[\sup_{t\in[0,T]} \left\|\int_0^t \Psi_n \,\mathrm{d}L\right\| \wedge 1\right] = 0.$$

Having introduced the notion of the stochastic integral, we now show that stochastic integral processes, obtained by fixing an integrand and varying the upper limit of the stochastic integral, are in fact semimartingales.

Theorem 3.49. *If* $\Psi \in \mathcal{I}^{\text{HS}}_{\text{prd},L}$*, then the integral process* $(I(\Psi)(t) : t \in [0,T])$ *defined by*

$$I(\Psi)(t) := \int_0^T \mathbb{1}_{[0,t]}(s)\Psi(s)\,L(\mathrm{d}s) \qquad \textit{for } t \in [0,T],$$

is a semimartingale.

Proof. Let $\Gamma \in \mathcal{S}^{1,\text{op}}_{\text{prd}}$ be of the form

$$\Gamma(t) = \Gamma_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{N-1} \Gamma_k \mathbb{1}_{(s_k, s_{k+1}]}(t),$$

where $0 = s_1 < ... < s_N = T$ are deterministic times, $\Gamma_0 : \Omega \to L(H)^1$ is $\mathcal{F}_0$-measurable, and each $\Gamma_k : \Omega \to L(H)^1$ is an $\mathcal{F}_{s_k}$-measurable random variable taking only finitely many values for $k = 1, ..., N-1$. Then we define the stochastic integral

$$\int_0^T \Gamma \, \mathrm{d}I(\Psi) := \sum_{k=1}^{N-1} \Gamma_k \Big(I(\Psi)(s_{k+1}) - I(\Psi)(s_k) \Big).$$

To prove the claim, by [29, Th. 2.1], it suffices to show that the set

$$\left\{ \int_0^T \Gamma \, \mathrm{d}I(\Psi) : \Gamma \in \mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}} \right\}$$

is bounded in probability. Suppose, aiming for a contradiction, that it is not the case. Then there exists an $\epsilon > 0$ and a sequence $(\Gamma_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}}$ satisfying for all $n \in \mathbb{N}$ that

$$P\left(\left\| \int_0^T \Gamma_n \, \mathrm{d}I(\Psi) \right\| > n \right) \geq \epsilon. \tag{3.66}$$

For each $\Psi \in \mathcal{S}_{\mathrm{prd}}^{\mathrm{HS}}$ and $\Gamma \in \mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}}$, the very definitions of stochastic integrals show

$$\int_0^T \Gamma \, \mathrm{d}I(\Psi) = \int_0^T \Gamma\Psi \, \mathrm{d}L.$$

This equality can be generalised to arbitrary $\Psi \in \mathcal{I}_{\mathrm{prd},L}^{\mathrm{HS}}$ and $\Gamma \in \mathcal{S}_{\mathrm{prd}}^{1,\mathrm{op}}$ by a standard approximation argument. Using this to rewrite Equation (3.66), we obtain for all $n \in \mathbb{N}$ that

$$\epsilon \leq P\left(\left\| \int_0^T \Gamma_n \, \mathrm{d}I(\Psi) \right\| > n \right) = P\left(\left\| \int_0^T \frac{1}{n}\Gamma_n \Psi \, \mathrm{d}L \right\| > 1 \right). \tag{3.67}$$

On the other hand, since $\left|\left|\left| \frac{1}{n}\Gamma_n\Psi \right|\right|\right|_L \to 0$ as $n \to \infty$, Corollary 3.47 implies

$$\lim_{n\to\infty} E\left[\left\| \int_0^T \frac{1}{n}\Gamma_n \Psi \, \mathrm{d}L \right\| \wedge 1 \right] = 0,$$

which contradicts (3.67) because of the equivalent characterisation of the topology in $L^0_P(\Omega, H)$. $\square$

We finish this section with a stochastic dominated convergence theorem.

Theorem 3.50. *Let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence of processes in $\mathcal{I}^{\rm HS}_{{\rm prd},L}$ such that*

(1) *$(\Psi_n)_{n\in\mathbb{N}}$ converges P_T-a.e. to an $L_2(G,H)$-valued predictable process Ψ;*

(2) *there exists a process $\Upsilon \in \mathcal{I}^{\rm HS}_{{\rm prd},L}$ satisfying for all $n \in \mathbb{N}$ that*

$$(k_L + l_L)(\Psi_n(\omega,t)) \le (k_L + l_L)(\Upsilon(\omega,t)) \quad \text{for } P_T\text{-a.a. } (\omega,t) \in \Omega \times [0,T].$$

Then it follows that $\Psi \in \mathcal{I}^{\rm HS}_{{\rm prd},L}$ and

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \left\| \int_0^t \Psi_n \, {\rm d}L - \int_0^t \Psi \, {\rm d}L \right\| > \epsilon \right) = 0 \qquad \text{for all } \epsilon > 0.$$

Proof. By assumption, there exists a set $N \subseteq \Omega \times [0,T]$ with $P_T(N) = 0$ such that $\lim_{n\to\infty} \Psi_n(\omega,t) = \Psi(\omega,t)$ and $(k_L + l_L)(\Psi_n(\omega,t)) \le (k_L + l_L)(\Upsilon(\omega,t))$ for all $(\omega,t) \in N^c$ and $n \in \mathbb{N}$. Fubini's theorem yields that

$$0 = P_T(N) = \int_\Omega {\rm Leb}|_{[0,T]}(N_\omega)\, P({\rm d}\omega),$$

where

$$N_\omega := \Big\{ t \in [0,T] : \lim_{n\to\infty} \Psi_n(\omega,t) \neq \Psi(\omega,t)\} \text{ or } (k_L + l_L)(\Psi_n(\omega,t)) > (k_L + l_L)(\Upsilon(\omega,t)) \Big\}.$$

It follows that there exists an $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that ${\rm Leb}|_{[0,T]}(N_\omega) = 0$ for all $\omega \in \Omega_1$. Consequently, for each $\omega \in \Omega_1$ we have $(k_L + l_L)(\Psi_n(\omega,t)) \le (k_L + l_L)(\Upsilon(\omega,t))$

and $\lim_{n\to\infty} \Psi_n(\omega, t) = \Psi(\omega, t)$ for Lebesgue almost every $t \in [0, T]$. Theorem 3.44 guarantees that there exists $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) = 1$ such that $m_L(\Upsilon(\omega, \cdot)) < \infty$ for all $\omega \in \Omega_2$. Continuity of k_L and l_L at 0, see Lemma 3.12, and the classical version of Lebesgue's dominated convergence theorem implies that for all $\omega \in \Omega_1 \cap \Omega_2$ we have

$$\begin{aligned}
&\lim_{m,n\to\infty} m_L(\Psi_m(\omega, \cdot) - \Psi_n(\omega, \cdot)) \\
&\quad = \lim_{m,n\to\infty} \Bigg(\int_0^T k_L(\Psi_m(\omega, t) - \Psi_n(\omega, t)) + l_L(\Psi_m(\omega, t) - \Psi_n(\omega, t))\, \mathrm{d}t \\
&\qquad\qquad + \int_0^T \|\Psi_m(\omega, t) - \Psi_n(\omega, t)\|^2_{L_2(G,H)} \wedge 1\, \mathrm{d}t \Bigg) = 0.
\end{aligned}$$

Hence, for each $\omega \in \Omega_1 \cap \Omega_2$ the sequence $(\Psi_n)_{n\in\mathbb{N}}$ is Cauchy in the modular topology, which by Lemma 3.20 and the fact that $\Psi_n(\omega) \to \Psi(\omega)$ for Lebesgue a.a. $t \in [0, T]$ allows us to conclude that $\Psi(\omega) \in \mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$. Since $P(\Omega_1 \cap \Omega_2) = 1$, Theorem 3.44 shows $\Psi \in \mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$.

Another application of Lebesgue's dominated convergence theorem establishes that $\lim_{n\to\infty} |||\Psi_n - \Psi|||_L = 0$. Corollary 3.47 implies

$$\lim_{n\to\infty} \sup_{\Gamma\in\mathcal{S}^{1,\mathrm{op}}_{\mathrm{prd}}} E\left[\left\| \int_0^T \Gamma(\Psi_n - \Psi)\, \mathrm{d}L \right\| \wedge 1 \right] = 0,$$

which, by Remark 3.48, implies ucp convergence. This allows us to conclude that the sequence $(I(\Psi_n))_{n\in\mathbb{N}}$ of processes converges in probability on compact time intervals to the process $I(\Psi)$. Hence we have

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \left\| \int_0^t \Psi_n\, \mathrm{d}L - \int_0^t \Psi\, \mathrm{d}L \right\| > \epsilon \right) = 0 \qquad \text{for all } \epsilon > 0,$$

which completes the proof. □

4 Integral processes driven by standard symmetric α-stable cylindrical Lévy processes

4.1 Integration theory for α-stable cylindrical Lévy processes

From this point on, instead of working with general cylindrical Lévy processes, we restrict our attention to the subclass of standard symmetric α-stable cylindrical Lévy processes for $\alpha \in (0, 2)$. Recall that these are cylindrical Lévy processes with characteristic function $\varphi_{L(t)}(g) = \exp(-t \|g\|^\alpha)$ for each $t \geq 0$ and $g \in G$.

In this section, we revisit the problem of stochastic integration and show that in the special case when the integrator is a standard symmetric α-stable cylindrical Lévy process, our integration theory simplifies significantly and the abstract modular space $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ can be identified with the Bochner space $L^\alpha_{\mathrm{Leb}}([0,T], L_2(G, H))$. As we shall see, this simplification is possible due to the special tail properties of stable distributions and the fact that standard symmetric α-stable cylindrical Lévy processes have cylindrical characteristics $(0, 0, \lambda)$, which implies that the modular m_L takes a much simpler form than in the general case.

In order to identify the modular space $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L}$ with $L^\alpha_{\mathrm{Leb}}([0,T], L_2(G, H))$, it suffices to obtain both upper and lower bounds on the modular m_L in terms of $\|\cdot\|_{L^\alpha}$. This is accomplished below in two technical lemmata.

Lemma 4.1. *Let L be the canonical α-stable cylindrical Lévy process in G with cylindrical Lévy measure λ. Then there exists a constant $c_\alpha > 0$ such that*

$$\int_0^T \|\psi(t)\|^\alpha_{L_2(G,H)} \,\mathrm{d}t \leq c_\alpha \int_0^T \int_H \left(\|h\|^2 \wedge 1 \right) (\lambda \circ \psi(t)^{-1})(\mathrm{d}h) \,\mathrm{d}t$$

for all measurable functions $\psi : [0, T] \to L_2(G, H)$.

Proof. Let F be an operator in $L_2(G, H)$. The spectral theorem for compact operators, see e.g. [17, Th. 4.1], guarantees that F has a decomposition of the form

$$F = \sum_{j=1}^{\infty} \gamma_j \langle a_j, \cdot \rangle b_j, \tag{4.1}$$

where $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ are orthonormal bases of G and H, respectively, and $(\gamma_j)_{j\in\mathbb{N}}$ is a sequence in $\mathbb{R}$. Let $\pi_n \colon H \to H$ be the projection onto $\mathrm{Span}\{b_1, ..., b_n\}$. We conclude from the spectral representation (2.5) of the stable measure $\lambda \circ \pi^{-1}_{a_1,\dots,a_n}$ for each $n \in \mathbb{N}$ that

$$\begin{aligned}
(\lambda \circ F^{-1} \circ \pi_n^{-1})(\bar{B}_H^c) &= \lambda\Big(\Big\{g \in G : \sum_{j=1}^{n} \gamma_j^2 \langle a_j, g\rangle^2 > 1\Big\}\Big) \\
&= \lambda \circ \pi^{-1}_{a_1,\dots,a_n}\Big(\Big\{x \in \mathbb{R}^n : \sum_{j=1}^{n} \gamma_j^2 x_j^2 > 1\Big\}\Big) \\
&= \frac{\alpha}{c_\alpha} \int_{S_{\mathbb{R}^n}} \int_0^\infty \mathbb{1}_{\{y\in\mathbb{R}^n : \sum_{j=1}^n \gamma_j^2 y_j^2 > 1\}}(rx) \frac{1}{r^{1+\alpha}} \,\mathrm{d}r\, \nu_n(\mathrm{d}x) \\
&= \frac{1}{c_\alpha} \int_{S_{\mathbb{R}^n}} \Big(\sum_{j=1}^{n} \gamma_j^2 x_j^2\Big)^{\alpha/2} \nu_n\,(\mathrm{d}x),
\end{aligned}$$

where ν_n is a uniform distribution on the sphere $S_{\mathbb{R}^n}$ not necessarily of unit mass. By defining $c_n := \sum_{j=1}^n \gamma_j^2$ and applying Jensen's inequality to the concave function $\beta \mapsto \beta^{\alpha/2}$ with respect to the discrete probability measure $\{c_n^{-1}\gamma_1^2, ..., c_n^{-1}\gamma_n^2\}$, we obtain

$$\frac{1}{c_\alpha} \int_{S_{\mathbb{R}^n}} \Big(\sum_{j=1}^{n} \gamma_j^2 x_j^2\Big)^{\alpha/2} \nu_n(\mathrm{d}x) \ge \frac{c_n^{\alpha/2}}{c_\alpha} \int_{S_{\mathbb{R}^n}} \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} |x_j|^\alpha \,\nu_n(\mathrm{d}x).$$

Letting $\nu_n^1 = \frac{1}{\nu_n(S_{\mathbb{R}^n})}\nu_n$, Lemma 2.4 and A2 in [68] imply

$$\frac{c_n^{\alpha/2}}{c_\alpha} \int_{S_{\mathbb{R}^n}} \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} |x_j|^\alpha \,\nu_n(\mathrm{d}x) = \frac{c_n^{\alpha/2}}{c_\alpha} \nu_n(S_{\mathbb{R}^n}) \sum_{j=1}^{n} \frac{\gamma_j^2}{c_n} \int_{S_{\mathbb{R}^n}} |x_j|^\alpha \nu_n^1(\mathrm{d}x)$$

$$
\begin{aligned}
&= \frac{c_n^{\alpha/2}}{c_\alpha} \nu_n(S_{\mathbb{R}^n}) \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n+\alpha}{2})} \sum_{j=1}^n \frac{\gamma_j^2}{c_n} \\
&= \frac{1}{c_\alpha}\Big(\sum_{j=1}^n \gamma_j^2\Big)^{\alpha/2} \\
&= \frac{1}{c_\alpha}\Big(\sum_{j=1}^n \|Fa_j\|^2\Big)^{\alpha/2},
\end{aligned}
$$

where the last step follows from the spectral representation (4.1). Thus, for all $n \in \mathbb{N}$ it holds that

$$
\frac{1}{c_\alpha}\Big(\sum_{j=1}^n \|Fa_j\|^2\Big)^{\alpha/2} \le (\lambda \circ F^{-1} \circ \pi_n^{-1})(\bar{B}_H^c). \tag{4.2}
$$

Since $\pi_n F \to F$ in $L_2(G,H)$, Lemma 2.7 implies that $\big((\pi_n \circ F)(L(1))\big)_{n\in\mathbb{N}}$ converges in probability to the random variable $F(L(1))$. Condition (2.3) yields

$$
\lim_{n\to\infty} (\lambda \circ F^{-1} \circ \pi_n^{-1})(\bar{B}_H^c) = (\lambda \circ F^{-1})(\bar{B}_H^c).
$$

By taking limits as $n \to \infty$ on both sides in (4.2), we obtain

$$
\frac{1}{c_\alpha} \|F\|_{L_2(G,H)}^\alpha \le (\lambda \circ F^{-1})(\bar{B}_H^c).
$$

It follows for any measurable function $\psi : [0,T] \to L_2(G,H)$ that

$$
\begin{aligned}
\int_0^T \|\psi(t)\|_{L_2(G,H)}^\alpha \,\mathrm{d}t &\le c_\alpha \int_0^T \int_{\bar{B}_H^c} (\lambda \circ \psi(t)^{-1})\,(\mathrm{d}h)\,\mathrm{d}t \\
&\le c_\alpha \int_0^T \int_H \big(\|h\|^2 \wedge 1\big)(\lambda \circ \psi(t)^{-1})\,(\mathrm{d}h)\,\mathrm{d}t,
\end{aligned}
$$

which completes the proof. □

The next lemma is a direct consequence of the inequality

$$\left(\lambda \circ F^{-1}\right)(B_H^c) \leq c \, \|F\|_{L_2(G,H)}^{\alpha}, \quad F \in L_2(G,H) \tag{4.3}$$

for some constant $c > 0$ depending only on α, where λ denotes the cylindrical Lévy measure of a standard symmetric α-stable cylindrical Lévy process, see [37, Le. 1].

Lemma 4.2. *Let L be a standard symmetric α-stable cylindrical Lévy process with cylindrical Lévy measure λ. Then, for all $\alpha \in (0,2)$ and $m \in \mathbb{N}$ there exists $a_\alpha^m < \infty$ depending only on α and m such that*

$$\int_{\|h\| \leq 1/m} ||h||^2 \, (\lambda \circ F^{-1})(dh) \leq a_\alpha^m \, ||F||_{L_2(G,H)}^{\alpha} \tag{4.4}$$

for all $F \in L_2(G,H)$, where $\lim_{m\to\infty} a_\alpha^m = 0$. Moreover, if $\alpha \in (1,2)$ then for all $m \in \mathbb{N}$ there exists $d_\alpha^m < \infty$ depending only on α and m such that

$$\int_{\|h\| \leq 1/m} ||h||^2 \, (\lambda \circ F^{-1})(dh) + \int_{\|h\| > m} ||h|| \, (\lambda \circ F^{-1})(dh) \leq d_\alpha^m \, ||F||_{L_2(G,H)}^{\alpha} \tag{4.5}$$

for all $F \in L_2(G,H)$, where $\lim_{m\to\infty} d_\alpha^m = 0$.

Proof. Step 1: We begin by proving that there exists an a_α^m such that

$$\int_{\|h\| \leq 1/m} ||h||^2 \, (\lambda \circ F^{-1})(dh) \leq a_\alpha^m \, ||F||_{L_2(G,H)}^{\alpha} \tag{4.6}$$

for some $a_\alpha^m < \infty$. For each $m, n \in \mathbb{N}$, we define

$$B_{i,m,n} := \left\{ h \in H : \|h\| \leq \frac{i}{m2^n} \right\} \quad \text{for } i = 1, ..., 2^n.$$

We approximate the function $\|\cdot\|^2 : H \to \mathbb{R}$ on $B_H(\frac{1}{m})$ by

$$f_n^m : H \to \mathbb{R}, \quad f_n^m(h) = \sum_{i=1}^{2^n-1} \left(\frac{i}{m2^n}\right)^2 \mathbb{1}_{B^c_{i,m,n} \setminus B^c_{i+1,m,n}}(h).$$

Since $\lambda \circ F^{-1}$ is a genuine α-stable measure in H, by [43, Thm. 6.2.7] we have

$$(\lambda \circ F^{-1})(B_H^c(r)) = r^{-\alpha}(\lambda \circ F^{-1})(B_H^c) \tag{4.7}$$

for $r > 0$, from which we obtain

$$\begin{aligned}
&\int_{\|h\|\le 1/m} f_n^m(h)(\lambda \circ F^{-1})(dh) = \\
&\qquad = \frac{1}{m^2} \sum_{i=1}^{2^n-1} \left(\frac{i}{2^n}\right)^2 \left((\lambda \circ F^{-1}) \left(B^c_{i,m,n}\right) - (\lambda \circ F^{-1}) \left(B^c_{i+1,m,n}\right) \right) \\
&\qquad = \frac{1}{m^2} (\lambda \circ F^{-1})(B_H^c) \sum_{i=1}^{2^n-1} \left(\frac{i}{2^n}\right)^2 \left(\left(\frac{i}{m2^n}\right)^{-\alpha} - \left(\frac{i+1}{m2^n}\right)^{-\alpha} \right) \\
&\qquad = m^{\alpha-2} (\lambda \circ F^{-1})(B_H^c) \int_0^1 \left(\sum_{i=1}^{2^n-1} - \left(\frac{i}{2^n}\right)^2 \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(r) \right) \mathrm{d}(r^{-\alpha}) \\
&\qquad = m^{\alpha-2} (\lambda \circ F^{-1})(B_H^c) \int_0^1 \left(\sum_{i=1}^{2^n-1} \left(\frac{i}{2^n}\right)^2 \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(r) \right) \alpha r^{-(\alpha+1)} \,\mathrm{d}r
\end{aligned}$$

for $n \in \mathbb{N}$. Hence, by the Monotone Convergence Theorem we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\|h\|\le 1/m} f_n^m(h)(\lambda \circ F^{-1})(dh) \\
&\qquad = m^{\alpha-2}\alpha(\lambda \circ F^{-1})(B_H^c) \lim_{n\to\infty} \int_0^1 \left(\sum_{i=1}^{2^n-1} \left(\frac{i}{2^n}\right)^2 \mathbb{1}_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(r) \right) r^{-(\alpha+1)} \,\mathrm{d}r \\
&\qquad = m^{\alpha-2}\alpha(\lambda \circ F^{-1})(B_H^c) \int_0^1 r^2 r^{-(\alpha+1)} \,\mathrm{d}r \\
&\qquad = m^{\alpha-2}\alpha(\lambda \circ F^{-1})(B_H^c) \int_0^1 r^{1-\alpha} \,\mathrm{d}r
\end{aligned}$$

$$= m^{\alpha-2}\frac{\alpha}{2-\alpha}(\lambda\circ F^{-1})(B_H^c). \tag{4.8}$$

At the same time, since for all $h \in B_H(\frac{1}{m})$ we have $\lim_{n\to\infty} f_n^m(h) = \|h\|^2$ and $f_n^m(h) \leq \|h\|^2$ for all $m, n \in \mathbb{N}$, we can use the Monotone Convergence Theorem to conclude that

$$\lim_{n\to\infty}\int_{\|h\|\leq 1/m} f_n^m(h)(\lambda\circ F^{-1})(dh) = \int_{\|h\|\leq 1/m}\|h\|^2(\lambda\circ F^{-1})(dh). \tag{4.9}$$

Combining Equations (4.8) and (4.9) and applying estimate (4.3), we get

$$\int_{\|h\|\leq 1/m}\|h\|^2(\lambda\circ F^{-1})(\mathrm{d}h) = m^{\alpha-2}\frac{\alpha}{2-\alpha}(\lambda\circ F^{-1})(B_H^c) \leq m^{\alpha-2}\frac{\alpha}{2-\alpha}c\,\|F\|^{\alpha}_{L_2(G,H)}.$$

Hence, we arrive at the estimate (4.6), with $a_\alpha^m = m^{\alpha-2}\frac{\alpha}{2-\alpha}c$. Since $\alpha < 2$, it follows that $\lim_{m\to\infty} a_\alpha^m = 0$.

Step 2: We claim that for all $\alpha \in (1,2)$ and $m \in \mathbb{N}$ there exists $b_\alpha^m < \infty$, depending only on α and m, such that

$$\int_{\|h\|>m}||h||\,(\lambda\circ F^{-1})(dh) \leq b_\alpha^m\,||F||^{\alpha}_{L_2(G,H)}$$

for all $F \in L_2(G,H)$, where for each $\alpha \in (1,2)$ we have $\lim_{m\to\infty} b_\alpha^m = 0$. The proof is analogous to *Step 1.* □

Theorem 4.3. *The space $\mathcal{I}^{\mathrm{HS}}_{\mathrm{det},L}$ of deterministic functions integrable with respect to the standard symmetric α-stable cylindrical Lévy process in G for $\alpha \in (0,2)$ coincides with $L^{\alpha}_{\mathrm{Leb}}\big([0,T], L_2(G,H)\big)$.*

Proof. Combining Lemma 4.1, Lemma 4.2 and Inequality (4.3), we get that there exists constants $c_\alpha, d_\alpha > 0$, depending only on α, such that for all measurable functions

$\psi : [0,T] \to L_2(G,H)$ we have

$$\begin{aligned}\frac{1}{c_\alpha}\int_0^T \|\psi(t)\|^\alpha_{L_2(G,H)}\,\mathrm{d}t &\le \int_0^T\int_H \big(\|h\|^2\wedge 1\big)\,(\lambda\circ\psi(t)^{-1})(\mathrm{d}h)\,\mathrm{d}t \\ &\le d_\alpha\int_0^T \|\psi(t)\|^\alpha_{L_2(G,H)}\,\mathrm{d}t. \qquad (4.10)\end{aligned}$$

Since L has cylindrical characteristics $(0,0,\lambda)$, for each measurable function $\psi : [0,T] \to L_2(G,H)$, the modular m_L takes the form

$$m_L(\psi) = \int_0^T\int_H \big(\|h\|^2\wedge 1\big)\,(\lambda\circ\psi(t)^{-1})(\mathrm{d}h)\,\mathrm{d}t + \int_0^T \Big(\|\psi(t)\|^2_{\mathrm{HS}}\wedge 1\Big)\,\mathrm{d}t.$$

Hence, it follows from Equation (4.10) that $m_L(\psi)$ is finite if and only if $\|\psi\|_{L^\alpha}$ is finite, which implies that $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L} = L^\alpha_{\mathrm{Leb}}\big([0,T],L_2(G,H)\big)$. Since by Theorem 3.27, we have that $\mathcal{M}^{\mathrm{HS}}_{\mathrm{det},L} = \mathcal{I}^{\mathrm{HS}}_{\mathrm{det},L}$, the result follows. □

In light of the above theorem, we can now complete the picture, and characterise the largest space of predictable integrands which are stochastically integrable with respect to a standard symmetric α-stable cylindrical Lévy process. This result might be viewed as the natural extension of the integrability condition obtained for real-valued standard symmetric α-stable Lévy processes by Rosinski and Woyczynski in [69, Th. 4.1].

Theorem 4.4. *The space $\mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$ of predicable Hilbert-Schmidt operator-valued processes integrable with respect to a standard symmetric α-stable cylindrical Lévy process in G for $\alpha\in(0,2)$ coincides with predictable processes in $L^0_P\big(\Omega, L^\alpha_{\mathrm{Leb}}\big([0,T],L_2(G,H)\big)\big)$.*

Proof. The proof is a direct consequence of Theorems 3.44 and 4.3. □

While a stochastic dominated convergence theorem for general cylindrical Lévy processes was proved in Theorem 3.50, due to the complicated nature of the general modular, in practice, it might be rather difficult to apply this result. However, as one might expect, in the case of a standard symmetric α-stable cylindrical Lévy process, the conditions become much simpler.

Theorem 4.5. *Let L be a standard symmetric α-stable cylindrical Lévy process and $(\Psi_n)_{n\in\mathbb{N}}$ a sequence of processes in $\mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$ such that*

(1) *$(\Psi_n)_{n\in\mathbb{N}}$ converges P_T-a.e. to an $L_2(G,H)$-valued predictable process Ψ;*

(2) *there exists a process $\Upsilon \in \mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$ satisfying for all $n\in\mathbb{N}$ that*

$$\|\Psi_n(\omega,t)\|_{L_2(G,H)} \le \|\Upsilon(\omega,t)\|_{L_2(G,H)} \quad \text{for } P_T\text{-a.a. } (\omega,t)\in\Omega\times[0,T].$$

Then it follows that $\Psi\in\mathcal{I}^{\mathrm{HS}}_{\mathrm{prd},L}$ and

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]}\left\|\int_0^t \Psi_n\,\mathrm{d}L - \int_0^t \Psi\,\mathrm{d}L\right\| > \epsilon\right) = 0 \qquad \text{for all } \epsilon>0.$$

Proof. Since Equation (4.10) implies that $m_L(\cdot)$ and $\|\cdot\|_{L^\alpha}$ generate the same topology when L is a standard symmetric α-stable cylindrical Lévy process, we can apply Theorem 3.50 to conclude the proof. □

Later, we will need the following stochastic Fubini theorem, which provides conditions under which it is possible to interchange stochastic and Lebsegue integrals.

Theorem 4.6 (Stochastic Fubini Theorem)**.** *Let L be a standard symmetric α-stable cylindrical Lévy process for $\alpha\in(1,2)$. If $\Psi\colon\Omega\times[0,T]^2\to L_2(G,H)$ is measurable, $\Psi(t,\cdot)$ is predictable for every $t\in[0,T]$, and $\int_0^T\int_0^T \|\Psi(t,s)\|^\alpha_{L_2(G,H)}\,\mathrm{d}t\,\mathrm{d}s<\infty$ a.s. then it follows:*

(a) *$\Psi(t,\cdot)$ is stochastically integrable for every $t\in[0,T]$ and $\int_0^T \Psi(\cdot,s)\mathrm{d}L(s)$ is a.s. Bochner integrable;*

(b) $\Psi(\cdot, s)$ *is a.s. Bochner integrable for every* $s \in [0,T]$ *and* $\int_0^T \Psi(t,\cdot)\,\mathrm{d}t$ *is stochastically integrable;*

(c) $\displaystyle\int_0^T \left(\int_0^T \Psi(t,s)\,\mathrm{d}t\right) \mathrm{d}L(s) = \int_0^T \left(\int_0^T \Psi(t,s)\,\mathrm{d}L(s)\right) \mathrm{d}t$ *a.s.*

Proof. The proof is similar as in finite dimensions; see [82]. □

Finally, we recall an important moment estimate for stochastic integrals with respect to standard symmetric α-stable cylindrical Lévy processes, which will be heavily used in the sequel.

Lemma 4.7. *For every* $0 < p < \alpha$ *and stochastically integrable predictable process* Ψ *we have*

$$E\left[\sup_{t\in[0,T]} \left|\left|\int_0^t \Psi\,\mathrm{d}L\right|\right|^p\right] \le e_{p,\alpha}\left(E\left[\int_0^T ||\Psi(t)||^\alpha_{\mathcal{L}_2(U,H)}\,\mathrm{d}t\right]\right)^{p/\alpha}, \tag{4.11}$$

where L *is a standard symmetric* α*-stable cylindrical Lévy process and* $e_{p,\alpha} = \frac{\alpha}{\alpha-p} e_{2,\alpha}^{p/\alpha}$ *for some* $e_{2,\alpha} \in (0,\infty)$ *that depends only on* α*.*

Proof. A straightforward extension of [37, Cor. 3] to predictable integrands. □

4.2 Random measures and compensators

In this section, we briefly recall some results on random measures and their compensators from [27, Ch. II].

Definition 4.8 (Random measure)**.** *A family* $\mu = \{\mu(\omega; \mathrm{d}t, \mathrm{d}h) : \omega \in \Omega\}$ *is called a random measure on* $[0,T]\times H$ *if* $\mu(\omega)$ *is a measure on* $\mathfrak{B}([0,T])\otimes\mathfrak{B}(H)$ *for each* $\omega \in \Omega$*. It is said to be an integer-valued random measure if moreover, we have:*

(i) $\mu(\{t\}\times H) \le 1$ *for all* $t \in [0,T]$ P*-a.s.;*

(ii) μ *takes values in* $\mathbb{N}\cup\{\infty\}$ P*-a.s.*

We denote by $\tilde{\mathcal{P}}$ (resp. $\tilde{\mathcal{O}}$) the *predictable* (resp. *optional*) σ-algebra on $\Omega\times[0,T]\times H$ and call a function $W:\Omega\times[0,T]\times H\mapsto\mathbb{R}$ *predictable* (resp. *optional*) if it is $\tilde{\mathcal{P}}$ (resp. $\tilde{\mathcal{O}}$) measurable.

If μ is a random measure and W is optional we define

$$\left(\int_0^t\int_H W(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h)\right)(\omega)$$
$$:=\begin{cases}\int_0^t\int_H W(\omega,s,h)\,\mu(\omega)(\mathrm{d}s,\mathrm{d}h), & \text{if } \int_0^t\int_H |W(\omega,s,h)|\,\mu(\omega)(\mathrm{d}s,\mathrm{d}h)<\infty,\\ \infty, & \text{otherwise.}\end{cases}$$

A random measure μ is called *predictable* (resp. *optional*) if $(\int_0^t\int_H W(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h) : t\in[0,T])$ is predictable (resp. optional) for every predictable (resp. optional) function W. An optional random measure μ is called $\tilde{\mathcal{P}}$-σ-finite if there exists a sequence $(A_n)_{n\in\mathbb{N}}\subset\tilde{\mathcal{P}}$ with $\bigcup_{n=1}^\infty A_n=\Omega\times[0,T]\times H$, such that $E\left[\int_0^T\int_H \mathbb{1}_{A_n}(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h)\right]<\infty$ for each $n\in\mathbb{N}$.

For each $\tilde{\mathcal{P}}$-σ-finite, optional random measure μ on $[0,T]\times H$ there exists a predictable random measure ν on $[0,T]\times H$ such that

$$E\left[\int_0^t\int_H W(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h)\right]=E\left[\int_0^t\int_H W(s,h)\,\nu(\mathrm{d}s,\mathrm{d}h)\right] \tag{4.12}$$

for all $t\in[0,T]$, and any non-negative predictable function W. The measure ν is determined uniquely up to a set of probability zero by (4.12) and is called the *compensator* of μ; see [27, th. II.1.8].

If Y is an H-valued, adapted càdlàg process then the integer-valued random measure μ^Y characterised by

$$\mu^Y((0,t]\times B)=\sum_{0\le s\le t}\mathbb{1}_B(\Delta Y(s)),\quad t\in(0,T], B\in\mathfrak{B}(H), 0\notin B,$$

where $\Delta Y(s) := Y(s) - \lim_{h \searrow 0+} Y(s-h)$ for $s \in [0,T]$, is an optional and $\tilde{\mathcal{P}}$-σ-finite random measure on $[0,T] \times H$. Thus, its compensator exists, which we denote by ν^Y.

Example 4.9. *Let L be a genuine H-valued Lévy process with Lévy measure λ. Then the compensator ν^L of the jump measure μ^L is given as the extension of $\mu^L((s,t] \times B) = (t-s)\lambda(B)$, $0 \le s < t \le T$, $B \in \mathfrak{B}(H)$ to $\mathfrak{B}([0,T]) \otimes \mathfrak{B}(H)$.*

In the sequel, we will make use of another characterisation of compensators of jump-measures. We denote by $\mathcal{C}^+(H)$ the class of non-negative, continuous functions $k : H \to \mathbb{R}$ bounded on H and vanishing inside a neighbourhood of 0.

Proposition 4.10. *The compensator ν^Y of the jump-measure μ^Y of an H-valued càdlàg semimartingale Y is characterised by being predictable and satisfying either of the following:*

(i) *The process*

$$\left(\int_0^t \int_H k(h)\, \mu^Y(\mathrm{d}s, \mathrm{d}h) - \int_0^t \int_H k(h)\, \nu^Y(\mathrm{d}s, \mathrm{d}h) :\ t \in [0,T] \right)$$

is a local martingale for every $k \in \mathcal{C}^+(H)$.

(ii) *If W is predictable and the process*

$$\left(\int_0^t \int_H W(s,h)\, \mu^Y(\mathrm{d}s, \mathrm{d}h) :\ t \in [0,T] \right) \tag{4.13}$$

is locally integrable, then so is

$$\left(\int_0^t \int_H W(s,h)\, \nu^Y(\mathrm{d}s, \mathrm{d}h) :\ t \in [0,T] \right)$$

and

$$\left(\int_0^t \int_H W(s,h)\,\mu^Y(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H W(s,h)\,\nu^Y(\mathrm{d}s,\mathrm{d}h) :\ t \in [0,T]\right)$$

is a local martingale.

Proof. The equivalence between (i) and (ii) follows by the same argument as in the proof of [27, Th. II.2.21.]. The fact that (ii) is an equivalent definition of the compensator is proved in [27, Th. II.1.8.]. □

Proposition 4.10 justifies the following standard notation: if W is predictable and (4.13) is locally integrable, we define the following local martingale

$$\int_0^t \int_H W(s,h)\,(\mu^Y - \nu^Y)(\mathrm{d}s,\mathrm{d}h) := \int_0^t \int_H W(s,h)\,\mu^Y(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H W(s,h)\,\nu^Y(\mathrm{d}s,\mathrm{d}h)$$

for each $t \in [0,T]$.

4.3 Predictable compensator of the integral process

In what follows, we will be interested in stochastic integral processes driven by standard symmetric α-stable cylindrical Lévy processes L in a Hilbert space U. More precisely, we fix a predictable stochastic process $G \in L^0_P\big(\Omega, L^\alpha_{\mathrm{Leb}}\big([0,T], L_2(U,H)\big)\big)$, and consider the integral process

$$\left(\int_0^t G\, dL\right)_{t\in[0,T]}.$$

By Theorem 3.49, we know that these processes are semimartingales. However, as we shall see below, when L is a standard symmetric α-stable cylindrical Lévy process for some $\alpha \in (1,2)$, these integral processes become local martingales.

Lemma 4.11. *If G is a predictable stochastic process, stochastically integrable with respect to a standard symmetric α-stable cylindrical Lévy process L in U for some $\alpha \in (1,2)$, then the integral process $\int_0^\cdot G \, \mathrm{d}L$ is a local martingale.*

Proof. Define the predictable stopping times $\tau_n = \inf \left\{ t > 0 : \int_0^t ||G(s)||^\alpha_{L_2(U,H)} \, \mathrm{d}s > n \right\}$ for $n \in \mathbb{N}$. It follows from Proposition 4.22(ii) and Lemma 1.3 in [13] that for each $n \in \mathbb{N}$ there exists a sequence of predictable step processes $(G_{n,k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} E\left[\int_0^T \left|\left|G(s)\mathbb{1}_{[0,\tau_n]}(s) - G_{n,k}(s)\right|\right|^\alpha_{L_2(U,H)} \, \mathrm{d}s\right] = 0. \tag{4.14}$$

Since inequality (4.11) guarantees for each $k, n \in \mathbb{N}$ that

$$E\left[\sup_{0\le t\le T} \left|\left|\int_0^t G_{n,k} \, \mathrm{d}L\right|\right|\right] \le e_{1,\alpha} \left(E\left[\int_0^T ||G_{n,k}(s)||^\alpha_{L_2(U,H)} \, \mathrm{d}s\right]\right)^{1/\alpha} < \infty,$$

the same arguments as in [64, Th. I:51] show that the processes $\int_0^\cdot G_{n,k} \, \mathrm{d}L$ are martingales. By Inequality (4.11) and Equation (4.14), we have that $\int_0^\cdot G\mathbb{1}_{[0,\tau_n]} \, \mathrm{d}L$ is a limit of martingales in $L^1(\Omega, H)$, and thus a martingale. Since standard arguments, see e.g. [64, Th. I.12], establish

$$\left(\int_0^\cdot G \, \mathrm{d}L\right)^{\tau_n} = \int_0^\cdot G\mathbb{1}_{[0,\tau_n]} \, \mathrm{d}L \quad \text{a.s.}, \tag{4.15}$$

for the stopped integral process, the proof is completed. □

For a standard symmetric α-stable cylindrical Lévy process L for some $\alpha \in (1,2)$ and a stochastically integrable predictable process G, we define the integral process $X = \int_0^\cdot G \mathrm{d}L$ and

$$\nu\left((0,t] \times B\right) := \int_0^t \left(\lambda \circ G(s)^{-1}\right)(B) \, \mathrm{d}s \qquad \text{for each } t \in (0,T], B \in \mathfrak{B}(H) \text{ with } 0 \notin \bar{B}. \tag{4.16}$$

The main result of this section is that ν extends to a random measure on $[0,T] \times H$ and that the extension is the predictable compensator of the jump measure of X. We will derive this result in a number of Lemmata.

Lemma 4.12. *The set function ν defined in (4.16) is well defined and extends to a predictable random measure on $[0,T] \times H$. This extension is unique among the class of σ-finite random measures on $[0,T] \times H$ that assign 0 mass to the origin.*

Proof. Step 1: We show that for all open sets $B \subseteq H$ with $0 \notin \bar{B}$ the process

$$f \colon \Omega \times [0,T] \to \mathbb{R}, \quad f(\omega,t) = \left(\lambda \circ G(\omega,t)^{-1}\right)(B)$$

is predictable. Since the set B is assumed to be open and bounded away from 0, it follows from Lemma 2.7 and the Portmanteau Theorem that the function $h \colon L_2(U,H) \to \mathbb{R}$ defined by $h(F) = (\lambda \circ F^{-1})(B)$ is lower semicontinuous, and hence measurable. Moreover, as the stochastic process $G \colon \Omega \times [0,T] \to L_2(U,H)$ is predictable, it follows that the composition $f = h \circ G$ is predictable.

Step 2: We show that f is predictable for all $B \in \mathfrak{B}(H \setminus \{0\})$, which will immediately imply that (4.16) is almost surely well defined and predictable as it is then just an integral of a non-negative predictable process. We define

$$\mathcal{D} = \left\{ B \in \mathfrak{B}(H \setminus \{0\}) : \lambda \circ G(\cdot,\cdot)^{-1}(B) \text{ is predictable} \right\},$$

and claim that $\mathcal{D}$ is a λ-system. Continuity of measures implies that $H \setminus \{0\} \in \mathcal{D}$ since, for all $t \in (0,T]$ and $\omega \in \Omega$, we have

$$\left(\lambda \circ G(\omega,t)^{-1}\right)(H \setminus \{0\}) = \lim_{n\to\infty} \left(\lambda \circ G(\omega,t)^{-1}\right)\left(\bar{B}_H(1/n)^c\right),$$

where the right hand side is the limit of processes that are predictable by Step 1. If

$B \in \mathcal{D}$ then $B^c \in \mathcal{D}$ since

$$\left(\lambda \circ G(\omega,t)^{-1}\right)(B^c) = \left(\lambda \circ G(\omega,t)^{-1}\right)(H \setminus \{0\}) - \left(\lambda \circ G(\omega,t)^{-1}\right)(B).$$

The collection $\mathcal{D}$ is closed under union of increasing sequences, which follows as above from continuity of measures and predictability of the pointwise limit. This concludes the proof of the claim that $\mathcal{D}$ is a λ-system.

We define the π-system.

$$\mathcal{I} = \{B \in \mathfrak{B}(H \setminus \{0\}) : B \text{ is open}\}.$$

The family $\mathcal{I}$ is contained in $\mathcal{D}$, since for each $B \in \mathcal{I}$ we have

$$\left(\lambda \circ G(\omega,t)^{-1}\right)(B) = \lim_{n\to\infty} \left(\lambda \circ G(\omega,t)^{-1}\right)\left(B \cap \bar{B}_H(1/n)^c\right),$$

and the right-hand side is predictable by Step 1. The Dynkin π-λ theorem for sets, see e.g. [31, Th. 1.1] implies $\sigma(\mathcal{I}) \subseteq \mathcal{D}$, and thus $\mathcal{D} = \mathfrak{B}(H \setminus \{0\})$.

Step 3: Let $\omega \in \Omega$ be such that $\int_0^T \|G(\omega,s)\|^\alpha_{L_2(U,H)}\,\mathrm{d}s < \infty$. Equation (4.16) defines the set function $\nu(\omega)$ on the semi-ring

$$\mathcal{S} = \{(0,t] \times B : t \in [0,T] \text{ and } B \in \mathfrak{B}(H \setminus \{0\})\}.$$

The set function $\nu(\omega)$ is σ-additive by its very definition and σ-finite, since for $n \in \mathbb{N}$ we have by (4.3) and (4.7) that

$$\begin{aligned}\nu(\omega)\left((0,T] \times \overline{B}_H^c(1/n)\right) &= \int_0^T \left(\lambda \circ G(\omega,s)^{-1}\right)\left(\overline{B}_H^c(1/n)\right)\,\mathrm{d}s \\ &= n^\alpha \int_0^T \left(\lambda \circ G(\omega,s)^{-1}\right)\left(\overline{B}_H^c\right)\,\mathrm{d}s\end{aligned}$$

$$\leq n^{\alpha}\, c \int_0^T \|G(\omega,s)\|_{L_2(U,H)}^{\alpha}\, \mathrm{d}s < \infty.$$

Carathéodory's extension theorem, see e.g. [31, Th. 2.5], implies that the set function $\nu(\omega)$ extends uniquely to a measure on $\mathfrak{B}([0,T]) \otimes \mathfrak{B}(H \setminus \{0\})$ which we also denote by $\nu(\omega)$.

Step 4: It remains to show that ν is predictable. Applying the monotone class theorem as above shows that the process $\int_0^{\cdot} \left(\lambda \circ G(s)^{-1}\right)(B)\, \mathrm{d}s$ is predictable for each $B \in \mathfrak{B}(H \setminus \{0\})$. Since

$$\int_0^{\cdot} \mathbb{1}_{(s,t]}(u) \mathbb{1}_A \left(\lambda \circ G(u)^{-1}\right)(B)\, \mathrm{d}u = \int_0^{\cdot} \left(\lambda \circ \left(\mathbb{1}_{(s,t]}(u) \mathbb{1}_A G(u)\right)^{-1}\right)(B)\, \mathrm{d}u,$$

it follows that the process $\int_0^{\cdot} \int_H W(u,h)\, \nu(\mathrm{d}u, \mathrm{d}h)(\cdot)$ is predictable for all functions $W = \mathbb{1}_{(s,t]} \mathbb{1}_A \mathbb{1}_B$ with $0 < s < t \leq T$, $A \in \mathcal{F}_s$ and $B \in \mathfrak{B}(H \setminus \{0\})$. An application of the functional monotone class theorem (follows e.g. from [81, Th. 3.14]) extends this result to all predictable processes W on $\Omega \times [0,T] \times H$, which shows predictability of the random measure ν on $[0,T] \times H \setminus \{0\}$. Defining $\nu((s,t] \times \{0\}) := 0$ for any $0 \leq s < t \leq T$ extends ν to a predictable random measure on $[0,T] \times H$. □

To show that the random measure ν characterised by (4.16) is the compensator of the jump-measure μ^X of the integral process X, we first consider the case when the integrand is a predictable step process.

Lemma 4.13. *Suppose that G is a predictable step process in $\mathcal{S}_{\mathrm{prd}}^{\mathrm{HS}}$. Then the random measure ν obtained in Lemma 4.12 is the predictable compensator of μ^X.*

Proof. Since Lemma 4.12 guarantees that ν is predictable, it remains to show (4.12), which by the functional monotone class theorem reduces to proving

$$E\left[\mathbb{1}_A \sum_{s<u\leq t} \mathbb{1}_B(\Delta X(u))\right] = E\left[\mathbb{1}_A \int_s^t \left(\lambda \circ G(u)^{-1}\right)(B)\,\mathrm{d}u\right]$$

for any $0 < s < t \leq T$, $A \in \mathcal{F}_s$ and $B \in \mathfrak{B}(H)$ with $0 \notin \bar{B}$. Let G be of the form

$$G = F_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{N} F_i \mathbb{1}_{(t_{i-1},t_i]}, \tag{4.17}$$

where $N \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_N = T$ and F_i is an $\mathcal{F}_{t_{i-1}}$-measurable and $L_2(U,H)$-valued random variable taking finitely many values. Moreover, without the loss of generality, we may assume that the points of the time partition contain s and t; otherwise these can be added without changing G. Then X takes the form

$$X(t) := \int_0^t G\,\mathrm{d}L := \sum_{i=1}^{N} F_i\big(L(t_i \wedge t) - L(t_{i-1} \wedge t)\big), \quad t \in [0,T], \tag{4.18}$$

and it follows that

$$\begin{aligned}\mathbb{1}_A \sum_{s<u\leq t} \mathbb{1}_B(\Delta X(u)) &= \mathbb{1}_A \sum_{i=1}^{N} \sum_{s\leq t_{i-1}<u\leq t_i\leq t} \mathbb{1}_B\Big(\Delta\big(G(t_i)L(u)\big)\Big) \\ &= \mathbb{1}_A \sum_{i=1}^{N} \sum_{s\leq t_{i-1}<u\leq t_i\leq t} \mathbb{1}_B\Big(\Delta\big(F_i L(u)\big)\Big).\end{aligned}$$

For each $i \in \{1,\ldots,N\}$, the random variable F_i is of the form $F_i = \sum_{j=1}^{m_i} \mathbb{1}_{A_{i,j}}\varphi_{i,j}$ for some pairwise disjoint sets $A_{i,j} \in \mathcal{F}_{t_{i-1}}$ and $\varphi_{i,j} \in L_2(U,H)$ for $j \in \{1,\ldots,m_i\}$. Since $0 \notin \bar{B}$, we have

$$E\left[\mathbb{1}_A \sum_{t_{i-1}<u\leq t_i} \mathbb{1}_B\Big(\Delta\big(F_i L(u)\big)\Big)\right] = \sum_{j=1}^{m_i} E\left[\mathbb{1}_{A\cap A_{i,j}} \sum_{t_{i-1}<u\leq t_i} \mathbb{1}_B\Big(\Delta\big(\varphi_{i,j} L(u)\big)\Big)\right]$$

$$= \sum_{j=1}^{m_i} (t_i - t_{i-1}) E \left[\mathbb{1}_{A \cap A_{i,j}} \left(\lambda \circ \varphi_{i,j}^{-1} \right) (B) \right]$$

$$= (t_i - t_{i-1}) E \left[\mathbb{1}_A \left(\lambda \circ F_i^{-1} \right) (B) \right],$$

because $A \cap A_{i,j} \in \mathcal{F}_{t_{i-1}}$ and the compensator of the jump measure of the Lévy process $\varphi_{i,j} L$ in H is given by $\left(\lambda \circ \varphi_{i,j}^{-1}\right) \mathrm{d}h \, \mathrm{d}t$ since its Lévy measure is $\left(\lambda \circ \varphi_{i,j}^{-1}\right)$, see Example 4.9. □

Before we show that the result of Lemma 4.13 can be extended to general integrands, we need to prove some technical Lemmata. Recall the class of functions $\mathcal{C}^+(H)$ used in Proposition 4.10 (and defined just before) to determine the compensator.

Lemma 4.14. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of càdlàg functions $f_n \colon [0,T] \to H$ converging uniformly to $f \colon [0,T] \to H$. Then we have for any $k \in \mathcal{C}^+(H)$ that*

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \sum_{0 \le s \le t} k(\Delta f_n(s)) - \sum_{0 \le s \le t} k(\Delta f(s)) \right| = 0. \tag{4.19}$$

Proof. Both sums in (4.19) are finite by the càdlàg property of f, f_n and since k vanishes inside a neighbourhood of 0. The assumed uniform convergence implies

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|\Delta f_n(t) - \Delta f(t)\| = 0. \tag{4.20}$$

Denoting $\mathrm{supp}(k) := \{h \in H : k(h) \neq 0\}$ and $\delta := \frac{1}{2}\mathrm{dist}(0, \mathrm{supp}(k))$, we obtain that $\mathrm{supp}(k)_\delta := \{h \in H, \mathrm{dist}(h, \mathrm{supp}(k)) < \delta\}$, is bounded away from zero, i.e. $0 \notin \overline{\mathrm{supp}(k)_\delta}$. It follows that the set $D := \{t \in [0,T] : \Delta f(t) \in \mathrm{supp}(k)_\delta\}$ is finite, which together with continuity of k and (4.20) implies

$$\lim_{n \to \infty} \sup_{t \in D} |k(\Delta f_n(t)) - k(\Delta f(t))| = 0. \tag{4.21}$$

Since (4.20) guarantees that there exists $n_0 \in \mathbb{N}$ such that we have $\Delta f_n(t) \notin \operatorname{supp}(k)$ for all $n \geq n_0$ and $t \in [0,T] \setminus D$, we conclude from (4.21) for $n \geq n_0$ that

$$\begin{aligned}
\sup_{t\in[0,T]} &\left| \sum_{0\leq s\leq t} k(\Delta f_n(s)) - \sum_{0\leq s\leq t} k(\Delta f(s)) \right| \\
&= \sup_{t\in[0,T]} \left| \sum_{s\in D\cap[0,t]} k(\Delta f_n(s)) - \sum_{s\in D\cap[0,t]} k(\Delta f(s)) \right| \\
&\leq |D| \sup_{t\in D} |k(\Delta f_n(t)) - k(\Delta f(t))| \to 0, \quad n \to \infty.
\end{aligned}$$

The proof is complete. □

Lemma 4.15. *Let $g_n, g \in L^\alpha_{\text{Leb}}([0,T], L_2(U,H))$, $n \in \mathbb{N}$, be such that g_n converges to g in $L^\alpha_{\text{Leb}}([0,T], L_2(U,H))$ and pointwise for almost every $s \in [0,T]$. Then we obtain for each $k \in \mathcal{C}^+(H)$ that*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \int_H k(h) \left(\lambda \circ g_n(s)^{-1}\right) \mathrm{d}h \, \mathrm{d}s - \int_0^t \int_H k(h) \left(\lambda \circ g(s)^{-1}\right) \mathrm{d}h \, \mathrm{d}s \right| = 0.$$

Proof. Let $k \in \mathcal{C}^+(H)$ be fixed. By Lemma 2.7, we have for Lebesgue almost all $s \in [0,T]$ that

$$\lim_{n\to\infty} \int_H k(h) \left(\lambda \circ g_n(s)^{-1}\right) \mathrm{d}h = \int_H k(h) \left(\lambda \circ g(s)^{-1}\right) \mathrm{d}h.$$

Since k is bounded and vanishes in a neighbourhood of 0, we conclude from Inequality (4.3) and Equation 4.7 that

$$\int_H k(h) \left(\lambda \circ g_n(s)^{-1}\right) \mathrm{d}h \leq c_{k,\alpha} \left|\left|g_n(s)\right|\right|^\alpha_{L_2(U,H)},$$

for a constant $c_{k,\alpha}$ independent of $s \in [0,T]$ and $n \in \mathbb{N}$. Since it follows from our assumptions that for each $t \in [0,T]$ we have

$$\lim_{n\to\infty} \int_0^t ||g_n(s)||^\alpha_{L_2(U,H)} \,\mathrm{d}s = \int_0^t ||g(s)||^\alpha_{L_2(U,H)} \,\mathrm{d}s,$$

the generalised dominated convergence theorem, see e.g. [70, Th. 4.19], implies

$$\lim_{n\to\infty} \int_0^t \int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \,\mathrm{d}h\,\mathrm{d}s = \int_0^t \int_H k(h) \, \left(\lambda \circ g(s)^{-1}\right) \,\mathrm{d}h\,\mathrm{d}s.$$

As the functions

$$t \mapsto \int_0^t \int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \,\mathrm{d}h\,\mathrm{d}s$$

are continuous monotone and converge pointwise to a continuous limit on $[0,T]$, the convergence is uniform by [62, p. 81/127] (or deuxième théorème de Dini). □

Now we can prove the main result of this section.

Theorem 4.16. *Let L be a standard symmetric α-stable cylindrical Lévy process L for some $\alpha \in (1,2)$, and G a stochastically integrable predictable process. Then the predictable compensator ν^X of the jump measure μ^X of $X := \int_0^\cdot G\,\mathrm{d}L$ is characterised by (4.16).*

Proof. In light of Proposition 4.10, it suffices to show that the process M^k defined by

$$\left(M^k(t) := \int_0^t \int_H k(h)\,\mu^X(\mathrm{d}s, \mathrm{d}h) - \int_0^t \int_H k(h) \, \left(\lambda \circ G(s)^{-1}\right) \,\mathrm{d}h\,\mathrm{d}s,\, t \in [0,T]\right),$$

is a local martingale for any $k \in \mathcal{C}^+(H)$. By Theorem 3.44, we can use Lemma 3.39 to show that there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of predictable step processes in $\mathcal{S}^{\mathrm{HS}}_{\mathrm{prd}}$ converging in probability in $L^\alpha_{\mathrm{Leb}}([0,T], L_2(U,H))$ and P_T − a.e. to G. By taking a suitable subsequence, we may assume that $(G_n)_{n\in\mathbb{N}}$ converges almost surely in

$L^{\alpha}_{\text{Leb}}([0,T], L_2(U,H))$. Moreover, it follows from Corollary 3.47 and Remark 3.48 that by taking another subsequence if necessary, we can assume that the sequence of integral processes $(\int_0^{\cdot} G_n \mathrm{d}L)_{n\in\mathbb{N}}$ converges uniformly almost surely to the integral process $\int_0^{\cdot} G \mathrm{d}L$. Letting $X_n := \int_0^{\cdot} G_n \mathrm{d}L$ and denoting the jump-measure of X_n by μ^{X_n}, we define for each $k \in \mathcal{C}^+(H)$ and $n \in \mathbb{N}$ a process M_n^k by

$$\left(M_n^k(t) := \int_0^t \int_H k(h)\, \mu^{X_n}(\mathrm{d}s, \mathrm{d}h) - \int_0^t \int_H k(h) \, \left(\lambda \circ G_n(s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s, \; t \in [0,T] \right).$$

Proposition 4.10 and Lemma 4.13 imply that M_n^k is a local martingale for all $n \in \mathbb{N}$. Since for each $n \in \mathbb{N}$ and $t \in [0,T]$ we have that $\mu^{X_n}(\{t\} \times H) \leq 1$ almost surely, it follows that for each $n \in \mathbb{N}$ and $t \in [0,T]$ we have

$$\left| \Delta \left(\int_0^t \int_H k(h)\, \mu^{X_n}(\mathrm{d}h, \mathrm{d}s) \right) \right| \leq \|k\|_\infty \quad \text{a.s.},$$

which shows that $\left|\Delta M_n^k(t)\right| \leq \|k\|_\infty$ a.s. for all $n \in \mathbb{N}$ and $t \in [0,T]$.

Almost sure uniform convergence of X_n and Lemma 4.14 guarantee that there exists an $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that, for all $\omega \in \Omega_1$, we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \int_H k(h)\, \mu^{X_n(\omega)}(\mathrm{d}h, \mathrm{d}s) - \int_0^t \int_H k(h)\, \mu^{X(\omega)}(\mathrm{d}h, \mathrm{d}s) \right| = 0. \tag{4.22}$$

In the same way, convergence of G_n both in $L^{\alpha}_{\text{Leb}}([0,T], L_2(U,H))$ a.s. and $P_T -$ a.e. allows us to conclude by Lemma 4.15 that there exists an $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) = 1$ such that, for all $\omega \in \Omega_2$, we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \int_H k(h) \, \left(\lambda \circ G_n(\omega, s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s - \int_0^t \int_H k(h) \, \left(\lambda \circ G(\omega, s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s \right| = 0. \tag{4.23}$$

Equations (4.22) and (4.23) show that M_n^k converges uniformly to M^k almost surely.

As the jumps of M_n^k are a.s. uniformly bounded by $\|k\|_\infty$, we conclude from [27, Co. IX.1.19] that M^k is a local martingale and the proof is complete. $\square$

4.4 Quadratic variation of the integral process

The covariation of two real-valued càdlàg semimartingales V_1 and V_2 starting from zero is the process $[V_1, V_2]$ defined by

$$[V_1, V_2](t) := V_1(t)V_2(t) - \int_0^t V_1(s-)\,\mathrm{d}V_2(s) - \int_0^t V_2(s-)\,\mathrm{d}V_1(s), \quad t \in [0,T].$$

When $V := V_1 = V_2$, we call the process $[V] := [V,V]$ the quadratic variation of V. The continuous part of $[V]$ is defined by

$$[V]^c(t) = [V](t) - \sum_{0\le s\le t} (\Delta V(s))^2 \quad \text{for each } t \in [0,T]. \tag{4.24}$$

If $[V]^c = 0$ we say that V is purely discontinuous; see e.g. [64, Se. II.6].

The concept of quadratic variation is generalised for a càdlàg semimartingale Z with values in the separable Hilbert space H in [46, Sec. 26]. Let $(f_i)_{i\in\mathbb{N}}$ denote an orthonormal basis of H. There exists a unique stochastic process $[[Z]]$ with values in the Hilbert-Schmidt tensor product of H satisfying

$$\langle [[Z]], f_i \otimes f_j \rangle = [Z_i, Z_j] \quad \text{for all } i,j \in \mathbb{N},$$

where $\otimes$ denotes the tensor product and $Z_i(t) = \langle Z(t), f_i\rangle$ for $t \in [0,T]$ are the projection processes of Z; see [46, Se. 21.2] for brief introduction. The process $[[Z]]$ does not depend on the choice of the orthonormal basis $(f_i)_{i\in\mathbb{N}}$. The process $[[Z]]$ is called the tensor quadratic variation of Z and its continuous part $[[Z]]^c$ is defined by

$$\begin{aligned}&\langle [[Z]]^c(t), f_i \otimes f_j\rangle \\ &\qquad = \langle [[Z]](t), f_i \otimes f_j\rangle - \sum_{0\le s\le t} \Delta\left(Z^i(s)Z^j(s)\right) \text{ for all } t\in[0,T],\ i,j\in\mathbb{N}.\end{aligned}$$

We say that Z is purely discontinuous if $[[Z]]^c{=}0$.

Proposition 4.17. *Let L be a standard symmetric α-stable cylindrical Lévy process for some $\alpha\in(1,2)$, and G a stochastically integrable predictable process with values in $L_2(U,H)$. Then the integral process $X := \int_0^\cdot G\,\mathrm{d}L$ is purely discontinuous.*

Proof. We proceed in three steps.

Step 1: Assume $H=\mathbb{R}$ and $U=\mathbb{R}^d$ for some $d\in\mathbb{N}$. In this case, L is a U-valued standard symmetric α-stable Lévy process, and therefore purely discontinuous; see e.g. [64, p. 71]. Pure discontinuity is preserved also for the integral process; see e.g. [27, Se. IX.5.5a] or [64, Th. II.29].

Step 2: Assume $H=\mathbb{R}$, but without any further restrictions on U. In that case, by the identification $U\simeq L_2(U,\mathbb{R})$, the integrand G is a U-valued process satisfying

$$\int_0^T \|G(t)\|^\alpha\,\mathrm{d}t<\infty \quad \text{a.s.} \tag{4.25}$$

Fix an orthonormal basis $(f_k)_{k\in\mathbb{N}}$ in U and define for each $n\in\mathbb{N}$ the projection

$$\pi_n\colon U\to U, \qquad \pi_n(u)=\sum_{k=1}^n \langle u, f_k\rangle f_k.$$

Since the projection π_n is a Hilbert-Schmidt operator, there exists a U-valued Lévy process L_n with the property $\langle L_n, u\rangle = L(\pi_n^* u)$ for all $u\in U$. We define the approximations

$$X_n := \int_0^{\cdot} G \mathrm{d}L_n, \quad n \in \mathbb{N}.$$

Since L_n attains values in a finite-dimensional subspace and is a symmetric α-stable process by [68, Le. 2.4], it follows that X_n is purely discontinuous by *Step 1.*

Let M be a real-valued, continuous martingale and define for $k \in \mathbb{N}$ the stopping times

$$\tau_k = \inf\left\{t > 0 : \int_0^t \|G(s)\|^\alpha \,\mathrm{d}s \geq k\right\} \wedge \inf\left\{t > 0 : |M(t)| \geq k\right\} \wedge T.$$

It follows that $\tau_k \to T$ as $k \to \infty$ by (4.25). Since X_n is purely discontinuous, it follows from [27, Le. I.4.14] that $(X_n M)^{\tau_k}$ is a local martingale for each k, $n \in \mathbb{N}$. Since applying Inequality (4.11) and Equation (4.15) shows

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} |(X_n M)^{\tau_k}(t)|\right] &= E\left[\sup_{0\leq t\leq T} |M(t)^{\tau_k}| \left|\int_0^t \mathbb{1}_{[0,\tau_k]} G \,\mathrm{d}L_n\right|\right] \\ &\leq k\, e_{1,\alpha} E\left[\int_0^{\tau_k} \|G(s)\|^\alpha \,\mathrm{d}s\right] \leq e_{1,\alpha}\, k^2 < \infty, \end{aligned}$$

we obtain that $(X_n M)^{\tau_k}$ is a martingale by [64, Th. I:51].

Noting that $\int_0^{\cdot} G \,\mathrm{d}L_n = \int_0^{\cdot} G\pi_n \,\mathrm{d}L$, Inequality (4.11) and Equation (4.15) establish for each $t \geq 0$ that

$$\begin{aligned} \lim_{n\to\infty} E\left[|(X_n M - XM)^{\tau_k}(t)|\right] &\leq \lim_{n\to\infty} kE\left[\left|\int_0^t \mathbb{1}_{[0,\tau_k]} G(\pi_n - I) \,\mathrm{d}L\right|\right] \\ &\leq \lim_{n\to\infty} e_{1,\alpha} k \left(E\left[\int_0^T \|G(s)(\pi_n - I)\|^\alpha_{\mathrm{L}_2(\mathrm{U},\mathrm{H})} \,\mathrm{d}s\right]\right)^{1/\alpha} \\ &= 0. \end{aligned}$$

It follows that the process $(XM)^{\tau_k}$ as a limit of martingales is itself a martingale. Since X is a local martingale according to Lemma 4.11 and M is an arbitrary real-valued continuous martingale, it follows from [27, Le. I.4.14] that X is purely discontinuous.

Step 3: For the general case, we fix an orthonormal basis $(f_i)_{i\in\mathbb{N}}$ in H and choose any $i, j \in \mathbb{N}$. Since $\langle X(t), f_i\rangle = \int_0^t G^* f_i \,\mathrm{d}L := \int_0^t \langle G^* f_i, \cdot\rangle \,\mathrm{d}L$ for every $t \geq 0$, the polarisation formula for real-valued covariation shows

$$\begin{aligned}\langle [[X]], e_i \otimes e_j\rangle &= \left[\int_0^\cdot G^* e_i \,\mathrm{d}L, \int_0^\cdot G^* e_j \,\mathrm{d}L\right] \\ &= \frac{1}{2}\left(\left[\int_0^\cdot G^*(e_i + e_j)\,\mathrm{d}L\right] - \left[\int_0^\cdot G^* e_i \,\mathrm{d}L\right] - \left[\int_0^\cdot G^* e_j \,\mathrm{d}L\right]\right).\end{aligned}$$

Linearity of the integral and binomial formula enable us to conclude

$$\begin{aligned}&\sum_{0\leq s\leq t} \Delta\langle X(s), e_i\rangle\langle X(s), e_j\rangle \\ &= \sum_{0\leq s\leq t} \Delta\left(\int_0^s G^* e_i \,\mathrm{d}L\right)\left(\int_0^s G^* e_j \,\mathrm{d}L\right) \\ &= \sum_{0\leq s\leq t} \frac{1}{2}\left(\Delta\left(\int_0^s G^*(e_i + e_j)\,\mathrm{d}L\right)^2 - \Delta\left(\int_0^s G^* e_i \,\mathrm{d}L\right)^2 - \Delta\left(\int_0^s G^* e_j \,\mathrm{d}L\right)^2\right).\end{aligned}$$

The very definition (4.24) of the continuous part leads us to

$$\begin{aligned}\langle [[X]]^c, e_i \otimes e_j\rangle &= \langle [[X]], e_i \otimes e_j\rangle - \sum_{0\leq s\leq t} \Delta\langle X(s), e_i\rangle\langle X(s), e_j\rangle \\ &= \frac{1}{2}\left(\left[\int_0^\cdot G^*(e_i + e_j)\,\mathrm{d}L\right]^c - \left[\int_0^\cdot G^* e_i \,\mathrm{d}L\right]^c - \left[\int_0^\cdot G^* e_j \,\mathrm{d}L\right]^c\right).\end{aligned}$$

Since *Step 2* guarantees that the processes $\int_0^\cdot G^*(e_i + e_j)\,\mathrm{d}L$, $\int_0^\cdot G^* e_i \,\mathrm{d}L$ and $\int_0^\cdot G^* e_i \,\mathrm{d}L$ are purely discontinuous, it follows that $\langle [[X]]^c, e_i \otimes e_j\rangle = 0$ for all $i, j \in \mathbb{N}$ which completes the proof. □

4.5 Strong Itô formula

In this section, we establish an Itô formula for processes that are given by a differential driven by a standard symmetric α-stable cylindrical Lévy process L for $\alpha \in (1,2)$ and are of the form

$$dX(t) = F(t)\,\mathrm{d}t + G(t)\,\mathrm{d}L(t) \quad \text{for } t \in [0,T], \tag{4.26}$$

where $F\colon \Omega \times [0,T] \to H$, $G\colon \Omega \times [0,T] \to L_2(U,H)$ are predictable and satisfy

$$\int_0^T \|F(t)\| + \|G(t)\|_{L_2(U,H)}^\alpha \,\mathrm{d}t < \infty \quad \text{a.s.} \tag{4.27}$$

We denote by $\mathcal{C}_b^2(H)$ the space of continuous functions $f\colon H \to \mathbb{R}$ having bounded first and second Fréchet derivatives, which are denoted by Df and $D^2 f$, respectively.

Theorem 4.18. *Let X be a stochastic process of the form (4.26). It follows for each $f \in \mathcal{C}_b^2(H)$ and $t \in [0,T]$ that*

$$\begin{aligned} &f(X(t)) \\ &\quad = f(X(0)) + \int_0^t \langle Df(X(s)), F(s)\rangle \,\mathrm{d}s + \int_0^t \langle G(s)^* Df(X(s-)), \cdot \rangle \,\mathrm{d}L(s) + M_f(t) \\ &\qquad + \int_0^t \int_H \Big(f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle \Big)\, \nu^X(\mathrm{d}h, \mathrm{d}s), \end{aligned}$$

where $M_f := \big(M_f(t) : t \in [0,T]\big)$ is a local martingale defined by

$$M_f(t) := \int_0^t \int_H \big(f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle \big)\, (\mu^X - \nu^X)(\mathrm{d}h, \mathrm{d}s),$$

and we have

$$\nu^X(\mathrm{d}h, \mathrm{d}s) = \big(\lambda \circ G(s)^{-1}\big)(\mathrm{d}h)\,\mathrm{d}s.$$

Lemma 4.19. *Let λ be the cylindrical Lévy measure of a standard symmetric α-stable cylindrical Lévy process for $\alpha \in (1,2)$. Then we have for each $f \in \mathcal{C}_b^2(H)$, $h \in H$, and $F \in L_2(U,H)$ that*

$$\int_H \left| f(h+g) - f(h) - \langle Df(h), g\rangle \right| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\ \leq d_\alpha^1 \left(2 \left|\left|Df\right|\right|_\infty + \frac{1}{2} \left|\left|D^2 f\right|\right|_\infty \right) \|F\|_{L_2(U,H)}^\alpha ,$$

where d_α^1 is a constant depending only on α as defined in Inequality (4.5).

Proof. Taylor's remainder theorem in the integral form, see [3, Th. VII.5.8], and Inequality (4.5) imply

$$\begin{aligned}
&\int_{\overline{B}_H} \left| f(h+g) - f(h) - \langle Df(h), g\rangle \right| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\
&\quad = \int_{\overline{B}_H} \left| \int_0^1 \langle D^2 f(h+\theta g) g, g\rangle (1-\theta)\, \mathrm{d}\theta \right| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\
&\quad \leq \left|\left|D^2 f\right|\right|_\infty \int_{\overline{B}_H} \left(\int_0^1 \|g\|^2 (1-\theta)\, \mathrm{d}\theta \right) \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\
&\quad = \frac{1}{2} \left\|D^2 f\right\|_\infty \int_{\overline{B}_H} \|g\|^2 \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\
&\quad \leq d_\alpha^1 \frac{1}{2} \left\|D^2 f\right\|_\infty \|F\|_{L_2(U,H)}^\alpha .
\end{aligned} \tag{4.28}$$

Similarly, Taylor's remainder theorem in the integral form and Inequality (4.5) show

$$\begin{aligned}
\int_{\overline{B}_H^c} \left| f(h+g) - f(h) \right| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) &= \int_{\overline{B}_H^c} \left| \int_0^1 \langle Df(h+\theta g), g\rangle d\theta \right| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\
&\leq \left|\left|Df\right|\right|_\infty \int_{\overline{B}_H^c} \left(\int_0^1 \left|\left|g\right|\right|\, \mathrm{d}\theta \right) \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \\
&\leq d_\alpha^1 \left|\left|Df\right|\right|_\infty \left|\left|F\right|\right|_{L_2(U,H)}^\alpha .
\end{aligned} \tag{4.29}$$

Another application of Inequality (4.5) shows

$$\int_{\overline{B}_H^c} |\langle Df(h), g\rangle| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g) \le ||Df||_\infty \int_{\overline{B}_H^c} ||g|| \left(\lambda \circ F^{-1}\right)(\mathrm{d}g)$$
$$\le d_\alpha^1 \, ||Df||_\infty \, ||F||_{L_2(U,H)}^\alpha . \tag{4.30}$$

Combining Inequalities (4.28) to (4.30) completes the proof. □

Proof of Theorem 4.18. The stochastic process X given by (4.26) is purely discontinuous as it is the sum of a finite-variation process and a purely discontinuous process according to Proposition 4.17. The Itô formula in [46, Th. 27.2] takes for all $t \in [0, T]$ the form

$$\mathrm{d}f(X(t)) = \langle Df(X(t-)), \cdot\rangle \, \mathrm{d}X(t) + \int_H \left(f(X(t-)+h) - f(X(t-)) - \langle Df(X(t-)), h\rangle\right) \mu^X(\mathrm{d}h, \mathrm{d}t). \tag{4.31}$$

One can show by approximating with simple integrands that

$$\langle Df(X(t-)), \cdot\rangle \, \mathrm{d}X(t) = \langle Df(X(t-)), F(t)\rangle \, \mathrm{d}t + \langle G(t)^* Df(X(t-)), \cdot\rangle \, \mathrm{d}L(t),$$

where both integrals are well-defined since (4.27) guarantees

$$\int_0^T |\langle Df(X(t-)), F(t)\rangle| + \|\langle G(t)^* Df(X(t-)), \cdot\rangle\|_{L_2(U,\mathbb{R})}^\alpha \, \mathrm{d}t$$
$$\le \|Df\|_\infty \int_0^T \|F(t)\| \, \mathrm{d}t + \|Df\|_\infty^\alpha \int_0^T \|G(t)\|_{L_2(U,H)}^\alpha \, \mathrm{d}t < \infty \quad \text{a.s.}$$

The definition of the compensator ν^X and Lemma 4.19 imply

$$
\begin{aligned}
&E\left[\int_0^T \int_H \left|f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle\right| \mu^X(\mathrm{d}s, \mathrm{d}h)\right] \\
&\quad = E\left[\int_0^T \int_H \left|f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle\right| \nu^X(\mathrm{d}s, \mathrm{d}h)\right] \\
&\quad \le d_\alpha^1 \left(2\left|\left|Df\right|\right|_\infty + \frac{1}{2}\left|\left|D^2 f\right|\right|_\infty\right) E\left[\int_0^T \|G(s)\|_{L_2(U,H)}^\alpha \, \mathrm{d}s\right]. \qquad (4.32)
\end{aligned}
$$

The stopping times $\tau_n := \inf\left\{t > 0 : \int_0^t ||G(s)||_{L_2(U,H)}^\alpha \, \mathrm{d}s \ge n\right\} \wedge T$ satisfy $\tau_n \to T$ as $n \to \infty$ by (4.27). Since Inequality (4.32) guarantees for all $n \in \mathbb{N}$ that

$$
E\left[\int_0^{T\wedge\tau_n} \int_H |f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle| \mu^X(\mathrm{d}s, \mathrm{d}h)\right] < \infty,
$$

Proposition 4.10 shows that M_f is a local martingale. This concludes the proof, since the claimed formula is just a different form of (4.31). □

5 Stochastic evolution equations driven by standard symmetric α-stable cylindrical Lévy processes

5.1 Mild solutions to stochastic evolution equations

We recall that U and H are separable Hilbert spaces with norms $\|\cdot\|$ and L is a standard symmetric α-stable cylindrical $(\mathcal{F}_t)$-Lévy process in U with $\alpha \in (1,2)$. In this section we consider the mild solution of the stochastic evolution equation:

$$\begin{aligned} \mathrm{d}X(t) &= \big(AX(t) + F(X(t))\big)\,\mathrm{d}t + G(X(t-))\,\mathrm{d}L(t) \qquad \text{for } t \in [0,T], \\ X(0) &= x_0, \end{aligned} \tag{5.1}$$

where A is a generator of a C_0-semigroup $(S(t))_{t\geq 0}$ in H, x_0 is an $\mathcal{F}_0$-measurable H-valued random variable, $F\colon H \to H$ and $G\colon H \to L_2(U,H)$ are measurable mappings and $T > 0$.

Definition 5.1. *An H-valued predictable process X is a mild solution to (5.1) if*

$$X(t) = S(t)x_0 + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)G(X(s-))\mathrm{d}L(s)$$

for every $t \in [0,T]$.

We work under the following assumptions:

(A1) The C_0-semigroup $(S(t))_{t\geq 0}$ is compact, analytic and a semigroup of contractions, and 0 is an element of the resolvent set of A.

(A2) The mapping F is Lipschitz and bounded, i.e. there exists $K_F \in (0,\infty)$ such that

$$\|F(h_1) - F(h_2)\| \leq K_F \|h_1 - h_2\|, \quad \|F(h)\| \leq K_F \tag{5.2}$$

for every $h_1, h_2, h \in H$.

(A3) The mapping G is Lipschitz and bounded, i.e. there exists $K_G \in (0,\infty)$ such that

$$\|G(h_1) - G(h_2)\|_{L_2(U,H)} \leq K_G \|h_1 - h_2\|, \quad \|G(h)\|_{L_2(U,H)} \leq K_G \tag{5.3}$$

for every $h_1, h_2, h \in H$.

(A4) The initial condition x_0 has finite p-th moment for every $p < \alpha$.

Remark 5.2. *We shall use the notation $D^\delta := Dom((-A)^\delta)$ for the domain of the fractional generator $(-A)^\delta$ for $\delta \in [0,1]$, and equip D^δ with the norm $\|h\|_\delta := \left\|(-A)^\delta h\right\|$. It follows from Assumption (A1) that the embedding of Hilbert spaces $D^\delta \hookrightarrow D^\gamma$ is dense and compact for every $0 \leq \gamma < \delta \leq 1$, cf. [6, Cor. 3.8.2].*

Remark 5.3. *Assumption (A1) implies, cf. [35, p. 289], that for every $\delta \geq 0$ there exists $c_\delta \in (0,\infty)$ depending only on δ such that*

$$\|S(t)\|_{L(H,D^\delta)} \leq c_\delta t^{-\delta} \qquad \text{for every } t > 0. \tag{5.4}$$

Remark 5.4. *By considering the cases $\|h_1 - h_2\| \leq 1$ and $\|h_1 - h_2\| > 1$ separately, we conclude from Assumptions (A2) and (A3) that there exist $K_F, K_G \in (0,\infty)$ such that for any $\beta \in (0,1)$ we have*

$$\|F(h_1) - F(h_2)\| \leq K_F \|h_1 - h_2\|^\beta, \quad \|F(h)\| \leq K_F, \tag{5.5}$$

and

$$\|G(h_1) - G(h_2)\|_{L_2(U,H)} \leq K_G \|h_1 - h_2\|^\beta, \quad \|G(h)\|_{L_2(U,H)} \leq K_G, \tag{5.6}$$

for every $h_1, h_2, h \in H$.

The first main theorem of this chapter is the following existence result, which also includes properties on the path regularity of the solution.

Theorem 5.5. *Under the assumptions (A1)-(A4), there exists a mild solution X to (5.1). The mild solution X is an element of $\mathcal{C}([0,T], L^p(\Omega,H))$ for every $p < \alpha$ and has càdlàg paths in H.*

We will obtain the solution to (5.1) by using Yosida approximations. For this purpose, we define $R_n = n\,(n\mathrm{I} - A)^{-1}$ for $n \in \mathbb{N}$ and denote by X_n the mild solution to

$$
\begin{aligned}
\mathrm{d}X_n(t) &= \big(AX_n(t) + R_nF(X_n(t))\big)\,\mathrm{d}t + R_nG(X_n(t-))\,\mathrm{d}L(t), \\
X_n(0) &= R_nx_0.
\end{aligned} \tag{5.7}
$$

Existence of the mild solution X_n to Equation (5.7) with cádlág paths is guaranteed by [37, Th. 12].

Remark 5.6. *We recall that under Assumption (A1) we have for all $\delta \in [0,1]$ that*

$$
\|R_n\|_{L(D^\delta)} \le 1, \quad n \in \mathbb{N}.
$$

This follows from the fact, that if an operator commutes with A then it commutes with A^γ, see e.g. [23, Pr. 3.1.1], which enables us to conclude for every $n \in \mathbb{N}$ that

$$
\|R_n\|_{L(D^\gamma)} = \sup_{\|(-A)^\gamma h\| \le 1} \left\|n(nI - A)^{-1}(-A)^\gamma h\right\| \le \sup_{\|h\| \le 1} \left\|n(nI - A)^{-1}h\right\| = \|R_n\|_{L(H)}\,.
$$

Since $(S(t))_{t\ge 0}$ is a contraction semigroup, [19, Th. 3.5] guarantees $\|R_n\|_{L(H)} \le 1$ for all $n \in \mathbb{N}$.

The solution to (5.1) will be constructed as a limit of X_n in $\mathcal{C}([0,T], L^p(\Omega,H))$ for an arbitrary but fixed $p < \alpha$. In the first three Lemmata, we establish relative compactness of the Yosida approximation $\{X_n : n \in \mathbb{N}\}$ in the space $\mathcal{C}([0,T], L^p(\Omega,H))$.

Lemma 5.7. *The set $\{X_n(t) : n \in \mathbb{N}\}$ is tight in H for every $t \in [0,T]$.*

Proof. The case $t = 0$ follows immediately from the strong convergence of R_n. For the case $t \in (0,T]$ we first prove that for every $1 \leq q < \alpha$, and $0 \leq \delta < 1/\alpha$ we have

$$\sup_{n\in\mathbb{N}} E\left[\|X_n(t)\|_\delta^q\right] < \infty. \tag{5.8}$$

Applying Hölder's inequality and inequality (4.11) shows for every $n \in \mathbb{N}$ that

$$\begin{aligned}
&E\left[\|X_n(t)\|_\delta^q\right] \\
&\quad = E\left[\left\|S(t)R_n x_0 + \int_0^t S(t-s)R_n F(X_n(s))\mathrm{d}s + \int_0^t S(t-s)R_n G(X_n(s-))\mathrm{d}L\right\|_\delta^p\right] \\
&\quad \leq 3^{q-1}\Bigg(E\left[\|S(t)R_n x_0\|_\delta^q\right] + t^{q-1}E\left[\int_0^t \|S(t-s)R_n F(X_n(s))\|_\delta^q \,\mathrm{d}s\right] \\
&\qquad\qquad + e_{q,\alpha}\left(E\left[\int_0^t \|S(t-s)R_n G(X_n(s))\|_{L_2(U,D^\delta)}^\alpha \,\mathrm{d}s\right]\right)^{\frac{q}{\alpha}}\Bigg).
\end{aligned}$$

Commutativity of S and R_n, Remark 5.3 and Remark 5.6 verify

$$E\left[\|S(t)R_n x_0\|_\delta^q\right] \leq c_\delta^q t^{-q\delta} \sup_{n\in\mathbb{N}} \|R_n\|_{L(D^\delta)}^q E\left[\|x_0\|^q\right] < \infty.$$

Assumption (A2) on boundedness of F together with Remark 5.3 and Remark 5.6 yield

$$E\left[\int_0^t \|S(t-s)R_n F(X_n(s))\|_\delta^q \,\mathrm{d}s\right] \leq c_\delta^q \frac{t^{1-q\delta}}{1-q\delta} \sup_{n\in\mathbb{N}} \|R_n\|_{L(D^\delta)}^q K_F^q < \infty.$$

Similarly, Assumption (A3) on boundedness of G implies

$$\begin{aligned}
&\left(E\left[\int_0^t \|S(t-s)R_n G(X_n(s))\|_{L_2(U,D^\gamma)}^\alpha \,\mathrm{d}s\right]\right)^{\frac{q}{\alpha}} \\
&\qquad\qquad \leq c_\delta^q \left(\frac{t^{1-\alpha\delta}}{1-\alpha\delta}\right)^{\frac{q}{\alpha}} \sup_{n\in\mathbb{N}} \|R_n\|_{L(D^\delta)}^q K_G^q < \infty.
\end{aligned}$$

Combining the above estimates establishes (5.8), which in turn gives the statement of the Lemma. Indeed, choose any $\delta \in (0, 1/\alpha)$ and use Markov's inequality and (5.8) for $q = 1$ to obtain for each $N > 0$ that

$$\sup_{n\in\mathbb{N}} P\left(\|X_n(t)\|_\delta > N\right) \leq \frac{c}{N}$$

for some constant $c \in (0, \infty)$. Since the embedding $D^\delta \hookrightarrow H$ is compact according to Remark 5.2, we obtain tightness of $\{X_n(t) : n \in \mathbb{N}\}$ by Prokhorov's theorem. $\square$

The following technical result will turn out to be useful in the sequel.

Lemma 5.8. *Let V be a separable Hilbert space with the norm $\|\cdot\|_V$ and let $A_m \in L(V)$ be a sequence of operators converging strongly to 0. If $(B_n)_{n\in\mathbb{N}}$ is a tight sequence of uniformly bounded V-valued random variables, then it follows for all $p > 0$ that*

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} E\left[\|A_m B_n\|_V^p\right] = 0.$$

Proof. Let $\epsilon > 0$ be fixed. Our assumptions guarantee that there exists a constant $c > 0$ such that $\sup_{n,m\in\mathbb{N}} \|A_m B_n\|_V^p \leq c$ a.s. and a compact set $K_\epsilon \subseteq V$ satisfying $P(B_n \notin K_\epsilon) < \frac{\epsilon}{c}$ for every $n \in \mathbb{N}$. Since any continuous mapping converging to zero converges uniformly on compacts, there exists $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$ we have

$$\sup_{n\in\mathbb{N}} \int_{\{B_n\in K_\epsilon\}} \|A_m B_n(\omega)\|_V^p \; P(\mathrm{d}\omega) < \epsilon.$$

It follows for all $n \in \mathbb{N}$ and $m \geq m_1$ that

$$E\left[\|A_m B_n\|_V^p\right] = \int_{\{B_n\in K_\epsilon\}} \|A_m B_n(\omega)\|_V^p \; P(\mathrm{d}\omega) + \int_{\{B_n\notin K_\epsilon\}} \|A_m B_n(\omega)\|_V^p \; P(\mathrm{d}\omega) \leq 2\epsilon,$$

which completes the proof. $\square$

Lemma 5.9. *The sequence* $\{X_n(t) : n \in \mathbb{N}\}$ *is relatively compact in* $L^0_P(\Omega, H)$ *for every* $t \in [0,T]$.

Proof. First, we impose the additional assumption that

$$T 12^2 e_{2,\alpha} \left(K_F^\alpha + K_G^\alpha\right) < 1, \tag{5.9}$$

where K_F, K_G come from (5.5), (5.6) and $e_{2,\alpha}$ is defined just below (4.11). For $1 < p < \alpha$ and $m, n \in \mathbb{N}$ we estimate the p-th moment of the difference $X_m(t) - X_n(t)$ by

$$\begin{aligned}
&E\left[\|X_m(t) - X_n(t)\|^p\right] \\
&\quad \leq 3^{p-1}\Bigg(E\left[\|S(t)(R_m - R_n)x_0\|^p\right] \\
&\qquad + E\left[\left\|\int_0^t S(t-s)(R_m F(X_m(s)) - R_n F(X_n(s)))\mathrm{d}s\right\|^p\right] \\
&\qquad + E\left[\left\|\int_0^t S(t-s)(R_m G(X_m(s-)) - R_n G(X_n(s-)))\mathrm{d}L(s)\right\|^p\right]\Bigg) \\
&\quad \leq 6^{p-1}\Bigg(E\left[\|S(t)(R_m - R_n)x_0\|^p\right] \\
&\qquad + E\left[\left\|\int_0^t S(t-s)(R_m - R_n)F(X_m(s))\mathrm{d}s\right\|^p\right] \\
&\qquad + E\left[\left\|\int_0^t S(t-s)R_n(F(X_m(s)) - F(X_n(s)))\mathrm{d}s\right\|^p\right] \\
&\qquad + E\left[\left\|\int_0^t S(t-s)(R_m - R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right] \\
&\qquad + E\left[\left\|\int_0^t S(t-s)R_n(G(X_m(s-)) - G(X_n(s-)))\mathrm{d}L(s)\right\|^p\right]\Bigg).
\end{aligned} \tag{5.10}$$

Furthermore, using Hölder's inequality, Inequality (4.11), Remark 5.6, (A1) and the estimates (5.5) and (5.6) with $\beta = p/\alpha$, we obtain

$$\begin{aligned} &E\left[\left\|\int_0^t S(t-s)R_n(F(X_m(s))-F(X_n(s)))\mathrm{d}s\right\|^p\ \right] \\ &\quad\le T^{p-\frac{p}{\alpha}}\left(E\left[\int_0^t \|S(t-s)R_n(F(X_m(s))-F(X_n(s)))\|^\alpha\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} \\ &\quad\le T^{p-\frac{p}{\alpha}}K_F^p\left(E\left[\int_0^t \|X_m(s)-X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}}, \end{aligned}$$

and

$$\begin{aligned} &E\left[\left\|\int_0^t S(t-s)R_n(G(X_m(s-))-G(X_n(s-)))\mathrm{d}L(s)\right\|^p\ \right] \\ &\quad\le e_{p,\alpha}\left(E\left[\int_0^t \|S(t-s)R_n(G(X_m(s))-G(X_n(s)))\|^\alpha\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} \\ &\quad\le e_{p,\alpha}K_G^p\left(E\left[\int_0^t \|X_m(s)-X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}}, \end{aligned}$$

which together with (5.10) yield

$$\begin{aligned} &E\left[\|X_m(t)-X_n(t)\|^p\right] \\ &\quad\le 6^{p-1}\Bigg(E\left[\|S(t)(R_m-R_n)x_0\|^p\right]+E\left[\left\|\int_0^t S(t-s)(R_m-R_n)F(X_m(s))\mathrm{d}s\right\|^p\right] \\ &\qquad\qquad + T^{p-\frac{p}{\alpha}}K_F^p\left(E\left[\int_0^t \|X_m(s)-X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} \\ &\qquad\qquad + E\left[\left\|\int_0^t S(t-s)(R_m-R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right] \\ &\qquad\qquad + e_{p,\alpha}K_G^p\left(E\left[\int_0^t \|X_m(s)-X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}}\Bigg). \end{aligned} \tag{5.11}$$

If we define

$$u_{n,m,p}(t) := \left(E\left[\|X_m(t)-X_n(t)\|^p\right]\right)^{\frac{\alpha}{p}}$$

$$
\begin{aligned}
u^0_{n,m,p} := 5^{\frac{\alpha}{p}-1}6^{\frac{\alpha}{p}(p-1)}\Bigg(& \sup_{t\in[0,T]} \|S(t)\|^{\alpha}_{L(H)} \left(E\left[\|(R_m-R_n)x_0\|^p\right]\right)^{\frac{\alpha}{p}} \\
& + \left(\sup_{t\in[0,T]} E\left[\left\|\int_0^t S(t-s)(R_m-R_n)F(X_m(s))\mathrm{d}s\right\|^p\right]\right)^{\frac{\alpha}{p}} \\
& + \left(\sup_{t\in[0,T]} E\left[\left\|\int_0^t S(t-s)(R_m-R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right]\right)^{\frac{\alpha}{p}}\Bigg) \\
w_p := 5^{\frac{\alpha}{p}-1}6^{\frac{\alpha}{p}(p-1)} & \left(T^{\alpha-1}K^{\alpha}_F + e^{\frac{\alpha}{p}}_{p,\alpha}K^{\alpha}_G\right)
\end{aligned}
$$

for $t \in [0,T]$, then after raising both sides of (5.11) to the power of α/p and simple algebraic steps we obtain

$$
u_{n,m,p}(t) \le u^0_{n,m,p} + w_p \int_0^t \left(u_{n,m,p}(s)\right)^{\frac{p}{\alpha}} \mathrm{d}s,
$$

which in turn by Gronwall's inequality in [80, Th. 2] gives

$$
\begin{aligned}
& u_{n,m,p}(t) \\
& \quad \le 2^{\frac{\alpha}{\alpha-p}-1}\left(u^0_{n,m,p} + \left(\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}}\right) \le 2^{\frac{\alpha}{\alpha-p}}u^0_{n,m,p} + 2^{\frac{\alpha}{\alpha-p}}\left(\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}}.
\end{aligned} \tag{5.12}
$$

If we show that

$$
\lim_{p\to\alpha-} 2^{\frac{\alpha}{\alpha-p}}\left(\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}} = \lim_{p\to\alpha-}\left(2\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}} = 0, \tag{5.13}
$$

and for any $1 < p < \alpha$

$$
\lim_{m,n\to\infty} u^0_{n,m,p} = 0, \tag{5.14}
$$

then by (5.12), for each $\epsilon \in (0,1)$ we can find $p^* \in (1,\alpha)$ such that

$$2^{\frac{\alpha}{\alpha-p^*}}\left(\frac{\alpha-p^*}{\alpha}tw_{p^*}\right)^{\frac{\alpha}{\alpha-p^*}} < \frac{\epsilon^{\alpha+1}}{2},$$

and an $N \in \mathbb{N}$ such that for all $m,n \geq N$

$$2^{\frac{\alpha}{\alpha-p^*}} u^0_{n,m,p^*} \leq \frac{\epsilon^{\alpha+1}}{2}.$$

Thus, for any $m,n \geq N$ we obtain using Markov's inequality that

$$\begin{aligned} P\left(\|X_m(t)-X_n(t)\| \geq \epsilon\right) &\leq \frac{1}{\epsilon^{p^*}} E\left[\|X_m(t)-X_n(t)\|^{p^*}\right] \\ &= \frac{1}{\epsilon^{p^*}}\left(u_{n,m,p^*}(t)\right)^{\frac{p^*}{\alpha}} \\ &\leq \epsilon^{\frac{p^*}{\alpha}(\alpha+1)-p^*} = \epsilon^{\frac{p^*}{\alpha}} \leq \sqrt{\epsilon}, \end{aligned}$$

which concludes the proof under the additional assumption (5.9).

Argument for (5.13)*:* Recall that $1 < p$ and $e_{p,\alpha} = \frac{\alpha}{\alpha-p} e_{2,\alpha}^{p/\alpha}$ for some $e_{2,\alpha} \in (0,\infty)$ independent of p. Thus, if p is sufficiently close to α we have $T^{\alpha-1} \leq e_{p,\alpha}^{\frac{\alpha}{p}}$ and estimate

$$\begin{aligned} \left(2\frac{\alpha-p}{p}tw_p\right)^{\frac{\alpha}{\alpha-p}} &= \left(2\frac{\alpha-p}{\alpha}t5^{\frac{\alpha-p}{p}}6^{\frac{\alpha}{p}(p-1)}\left(T^{\alpha-1}K_F^{\alpha} + e_{p,\alpha}^{\frac{\alpha}{p}}K_G^{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}} \\ &\leq 5^{\frac{\alpha}{p}}\left(2\frac{\alpha-p}{\alpha}t6^2\left(\frac{\alpha-p}{\alpha}\right)^{-\frac{\alpha}{p}} e_{2,\alpha}\left(K_F^{\alpha}+K_G^{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}} \\ &\leq 5^{\alpha}\left(\frac{\alpha-p}{\alpha}\right)^{-\frac{\alpha}{p}}\left(t12^2 e_{2,\alpha}\left(K_F^{\alpha}+K_G^{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}} \\ &\leq 5^{\alpha}\left(\frac{\alpha-p}{\alpha}\right)^{-2}\left(T12^2 e_{2,\alpha}\left(K_F^{\alpha}+K_G^{\alpha}\right)\right)^{\frac{\alpha}{\alpha-p}}. \end{aligned}$$

Thus, (5.13) follows from the limit

$$\lim_{y\to\infty} y^2 \left(T12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)\right)^y = \lim_{y\to\infty} \frac{y^2}{\left(T12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)\right)^{-y}} = 0$$

by using L'Hospital's rule and the assumption in Equation (5.9).

Argument for (5.14)*:* Let $p \in (1,\alpha)$ be fixed. By strong convergence of R_n and Lebesgue's dominated convergence theorem we have

$$\lim_{m,n\to\infty} E\left[\|(R_m - R_n)x_0\|^p\right] = 0. \tag{5.15}$$

Moreover, by Hölder's inequality, strong convergence of R_n, Lemma 5.8 and Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\lim_{m,n\to\infty}&\left(\sup_{t\in[0,T]} E\left[\left\|\int_0^t S(t-s)(R_m-R_n)F(X_m(s))\mathrm{d}s\right\|^p\right]\right)\\ \le& T^{p-1}\sup_{t\in[0,T]}\|S(t)\|_{L(H)}^p \lim_{m,n\to\infty}\left(E\left[\int_0^T \|(R_m-R_n)F(X_m(s))\|^p\,\mathrm{d}s\right]\right) = 0,\end{aligned} \tag{5.16}$$

where the assumptions of Lemma 5.8 are satisfied by boundedness of F, see (5.2), and tightness of $\{F(X_n(s)), m\in\mathbb{N}\}$ implied by Lemma 5.7 and continuity of F. Similarly, using inequality (4.11) and (5.6), we obtain

$$\begin{aligned}\lim_{m,n\to\infty}&\left(\sup_{t\in[0,T]} E\left[\left\|\int_0^t S(t-s)(R_m-R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right]\right)\\ \le& e_{p,\alpha}\sup_{t\in[0,T]}\|S(t)\|_{L(H)}^p \lim_{m,n\to\infty}\left(E\left[\int_0^T \|(R_m-R_n)G(X_m(s))\|^\alpha\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} = 0.\end{aligned} \tag{5.17}$$

To prove the general case without the assumption (5.9), we proceed by induction.

Fix a time

$$T_0 \in \left(0, \frac{1}{12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)}\right). \tag{5.18}$$

If $t \in [0, T_0]$, relative compactness of $\{X_n(t),\, n \in \mathbb{N}\}$ in $L_P^0(\Omega, H)$ follows from (5.18) by the previous arguments. Assume that the collection $\{X_n(t),\, n \in \mathbb{N}\}$ is relatively compact in $L_P^0(\Omega, H)$ for all $t \in ((k-1)T_0, kT_0]$. In order to finish the proof, we need to show that the result then also holds for all $t \in (kT_0, (k+1)T_0]$. To see this, we fix $t \in (kT_0, (k+1)T_0]$ and write

$$\begin{aligned} X_m(t) - X_n(t) = S(t - kT_0) \Bigg(& S(kT_0)(R_m - R_n)x_0 \\ & + \int_0^{kT_0} S(kT_0 - s)(R_m F(X_m(s) - R_n F(X_n(s)))\,\mathrm{d}s \\ & + \int_0^{kT_0} S(kT_0 - s)(R_m G(X_m(s-) - R_n G(X_n(s-)))\,\mathrm{d}L(s) \Bigg) \\ & + \int_{kT_0}^t S(t-s)(R_m F(X_m(s) - R_n F(X_n(s)))\,\mathrm{d}s \\ & + \int_{kT_0}^t S(t-s)(R_m G(X_m(s-) - R_n G(X_n(s-)))\,\mathrm{d}L(s) \\ = S(t - kT_0)&(X_m(kT_0) - X_n(kT_0)) \\ & + \int_{kT_0}^t S(t-s)(R_m F(X_m(s) - R_n F(X_n(s)))\,\mathrm{d}s \\ & + \int_{kT_0}^t S(t-s)(R_m G(X_m(s-) - R_n G(X_n(s-)))\,\mathrm{d}L(s). \end{aligned}$$

Since our inductive hypothesis implies that $\{X_n(kT_0),\, n \in \mathbb{N}\}$ is relatively compact in $L_P^0(\Omega, H)$, it follows that the collection $\{S(t - T_0)X_n(kT_0),\, n \in \mathbb{N}\}$ is also relatively compact in $L_P^0(\Omega, H)$. Since $t - kT_0 < T_0$, it follows from Equation (5.18) that we have $(t - kT_0)\, 12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right) < 1$, and we can use the same argument as in the first part of the proof to obtain relative compactness of $(X_n(t))_{n\in\mathbb{N}}$ in $L^0_P(\Omega, H)$ for each $t \in (kT_0, (k+1)T_0]$. By mathematical induction, we can now cover intervals of arbitrary length. This concludes the proof of the general case. $\square$

We now step from relative compactness of $\{X_n(t) : n \in \mathbb{N}\}$ in $L^0_P(\Omega, H)$ for fixed time t to relative compactness of the processes $\{X_n : n \in \mathbb{N}\}$ in the space $\mathcal{C}([0,T], L^p(\Omega, H))$ for $0 < p < \alpha$ using the Arzelà–Ascoli Theorem.

Lemma 5.10. *The collection $\{X_n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}([0,T], L^p(\Omega, H))$ for any $0 < p < \alpha$.*

Proof. We consider the case $1 < p < \alpha$ as the case $p \leq 1$ follows from the fact that relative compactness in $\mathcal{C}([0,T], L^p(\Omega, H))$ implies relative compactness in $\mathcal{C}([0,T], L^{p'}(\Omega, H))$ for $p > p'$. In light of the Arzelà–Ascoli Theorem, cf. e.g. [33, Th. 7.17]), it suffices to show that

(a) $\{X_n(t) : n \in \mathbb{N}\} \subset L^p(\Omega, H)$ is relatively compact for each $t \in [0, T]$;

(b) $\{X_n : n \in \mathbb{N}\} \subset \mathcal{C}([0,T], L^p(\Omega, H))$ is equicontinuous.

The claim in (a) follows from [16, Cor. 3.3] by Lemmata 5.7, 5.9 and the fact that Equation (5.8) with $\delta = 0$ and any $q \in (p, \alpha)$ implies via the Vallee-Poussin Theorem [14, Th. II.22] that the collection $\{X_n(t) : n \in \mathbb{N}\}$ is p-uniformly integrable and bounded in $L^p(\Omega, H)$. Hence, it remains only to prove (b). To that end, we take $t \in [0, T)$ and $h \in (0, T - t]$, and estimate

$$\begin{aligned}
&\|X_n(t+h) - X_n(t)\|^p \\
&\leq 5^{p-1}\Bigg(\|(S(h) - I)\, S(t) R_n x_0\|^p + \left\| \int_t^{t+h} S(t+h-s) R_n F(X_n(s))\, \mathrm{d}s \right\|^p \\
&\qquad + \left\| \int_t^{t+h} S(t+h-s) R_n G(X_n(s-))\, \mathrm{d}L(s) \right\|^p
\end{aligned}$$

$$
+ \left\| \int_0^t (S(h) - I)\, S(t-s) R_n F(X_n(s))\, \mathrm{d}s \right\|^p
$$

$$
+ \left\| \int_0^t (S(h) - I)\, S(t-s) R_n G(X_n(s-))\, \mathrm{d}L(s) \right\|^p \Bigg). \tag{5.19}
$$

Commutativity of R_n and S and contractivity of S implies

$$
E\left[\|(S(h) - I)\, S(t) R_n x_0\|^p\right] \le \sup_{n\in\mathbb{N}} \|R_n\|_{L(H)}^p\, E\left[\|(S(h) - I)\, x_0\|^p\right]. \tag{5.20}
$$

Boundedness of F in Assumption (A2) and contractivity of S imply

$$
E\left[\left\| \int_t^{t+h} S(t+h-s) R_n F(X_n(s))\, \mathrm{d}s \right\|^p\right] \le h^p \sup_{n\in\mathbb{N}} \|R_n\|_{L(H)}^p\, K_F^p. \tag{5.21}
$$

We conclude from Inequality (4.11) by using boundedness of G in Assumption (A3) and contractivity of S that

$$
\begin{aligned}
&E\left[\left\| \int_t^{t+h} S(t+h-s) R_n G(X_n(s-))\, \mathrm{d}L(s) \right\|^p\right] \qquad (5.22)\\
&\quad \le e_{p,\alpha} \left(E\left[\int_t^{t+h} \|S(t+h-s) G(X_n(s))\|_{L_2(U,H)}^\alpha\, \mathrm{d}s \right] \right)^{p/\alpha} \\
&\quad \le e_{p,\alpha} \sup_{n\in\mathbb{N}} \|R_n\|_{L(H)}^p\, K_G^p h^{p/\alpha}.
\end{aligned}
$$

It follows from Lemma 5.7 that $\{X_n(s) : n \in \mathbb{N}\}$ is tight in H for every $s \in [0,t]$. Lemma 5.8 implies

$$
\lim_{h\searrow 0} \sup_{n\in\mathbb{N}} E\left[\|(S(h) - I)\, S(t-s) R_n F(X_n(s))\|^p\right] = 0.
$$

Lebesgue's dominated convergence theorem shows

$$
\lim_{h\searrow 0} \int_0^t \sup_{n\in\mathbb{N}} E\left[\|(S(h) - I)\, S(t-s) R_n F(X_n(s))\|^p\right] \mathrm{d}s = 0. \tag{5.23}
$$

In the same way, after applying Inequality (4.11), we obtain from Lemma 5.8

$$\lim_{h\searrow 0}\sup_{n\in\mathbb{N}} E\left[\left\|\int_0^t \left(S(h)-I\right)S(t-s)R_nG(X_n(s-))\,\mathrm{d}L(s)\right\|^p\right]=0. \tag{5.24}$$

Applying (5.20) - (5.24) to Inequality (5.19) shows uniform continuity from the right. Similar arguments establish uniform continuity from the left, which proves (b), and thus completes the proof. □

Proof of Theorem 5.5. Let $p\in(0,\alpha)$ be fixed and fix $\beta\in(0,1)$ such that $p=\alpha\beta$, where β is the Hölder exponent from (5.6). Lemma 5.10 guarantees that there exists a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}\sup_{t\in[0,T]} E\left[\|X_{n_k}(t)-Z(t)\|^p\right]=0 \tag{5.25}$$

for some $Z\in\mathcal{C}([0,T],L^p(\Omega,H))$. The proof will be complete if we show that Z is a mild solution to (5.1). We conclude for each $k\in\mathbb{N}$ and $t\in[0,T]$ from Lipschitz continuity of F and Hölder continuity of G in (5.2) and (5.6) and contractivity of S by applying Hölder's inequality and Inequality (4.11) that

$$\begin{aligned}
&E\left[\left\|Z(t)-S(t)x_0-\int_0^t S(t-s)F(Z(s))\,\mathrm{d}s-\int_0^t S(t-s)G(Z(s-))\,\mathrm{d}L(s)\right\|^p\right]\\
&\quad\le (1\wedge 4^{p-1})\Bigg(E\left[\|S(t)(R_{n_k}-I)x_0\|^p\right]+E\left[\|Z(t)-X_{n_k}(t)\|^p\right]\\
&\qquad\qquad+E\left[\left\|\int_0^t S(t-s)\big(F(Z(s))-F(X_{n_k}(s))\big)\,\mathrm{d}s\right\|^p\right]\\
&\qquad\qquad+E\left[\left\|\int_0^t S(t-s)\big(G(Z(s-))-G(X_{n_k}(s-))\big)\,\mathrm{d}L(s)\right\|^p\right]\Bigg)\\
&\quad\le (1\wedge 4^{p-1})\Bigg(E\left[\|S(t)(R_{n_k}-I)x_0\|^p\right]+E\left[\|Z(t)-X_{n_k}(t)\|^p\right]
\end{aligned}$$

$$
\begin{aligned}
&+ T^{p-1} E\left[\int_0^t \left\|S(t-s)\big(F(Z(s)) - F(X_{n_k}(s))\big)\right\|^p \,\mathrm{d}s\right] \\
&+ e_{p,\alpha}\left(E\left[\int_0^t \left\|S(t-s)\big(G(Z(s)) - G(X_{n_k}(s))\big)\right\|_{L_2(U,H)}^\alpha \,\mathrm{d}s\right]\right)^{p/\alpha}\Bigg) \\
\leq (1 \wedge 4^{p-1})\Bigg(& E\left[\|S(t)(R_{n_k} - I)x_0\|^p\right] + E\left[\|Z(t) - X_{n_k}(t)\|^p\right] \\
&+ T^{p-1} K_F^p \sup_{t\in[0,T]} \|S(t)\|_{L(H)}^p \, E\left[\int_0^t \|Z(s) - X_{n_k}(s)\|^p \,\mathrm{d}s\right] \\
&+ e_{p,\alpha} K_G^p \sup_{t\in[0,T]} \|S(t)\|_{L(H)}^p \left(E\left[\int_0^t \|Z(s) - X_{n_k}(s)\|_{L_2(U,H)}^{\alpha\beta} \,\mathrm{d}s\right]\right)^{p/\alpha}\Bigg) \\
\leq (1 \wedge 4^{p-1})\Bigg(& E\left[\|S(t)(R_{n_k} - I)x_0\|^p\right] + \big(1 + T^p K_F^p\big) \sup_{t\in[0,T]} E\left[\|Z(t) - X_{n_k}(t)\|^p\right] \\
&+ e_{p,\alpha} K_G^p T^{p/\alpha} \sup_{t\in[0,T]} \left(E\left[\|Z(t) - X_{n_k}(t)\|^p\right]\right)^\beta\Bigg)
\end{aligned}
$$

As the last line tends to 0 as $k \to \infty$ by (5.25) and strong convergence of R_{n_k} to I, it follows that Z is a mild solution to (5.1).

It remains only to establish that Z has càdlàg paths, which will follow immediately from the following corollary as well as the observation that X_n has càdlàg paths. □

At the end of this section, we present a stronger convergence result for Yosida approximations that not only completes the proof of Theorem 5.5 but also turns out to be useful in applications as will be seen in the following sections.

Corollary 5.11. *For all $0 < p < \alpha$ there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ of the Yosida approximations, which converges to a solution to (5.1) both in $\mathcal{C}([0,T], L^p(\Omega, H))$ and uniformly on $[0,T]$ almost surely.*

Proof. Lemma 5.10 enables us to choose a subsequence $(X_n)_{n\in\mathbb{N}}$ of the Yosida approximations which converges to the mild solution X in $\mathcal{C}([0,T], L^p(\Omega, H))$. To prove almost sure convergence, we fix an arbitrary $r > 0$ and estimate

$$
\begin{aligned}
P &\left(\sup_{t\in[0,T]} \|X(t) - X_n(t)\| > r \right) \\
&\le P\left(\sup_{t\in[0,T]} \|S(t)(I - R_n)x_0\| > \frac{r}{3} \right) \\
&\quad + P\left(\sup_{t\in[0,T]} \left\| \int_0^t S(t-s)\big(F(X(s)) - R_nF(X_n(s))\big)\,\mathrm{d}s \right\| > \frac{r}{3} \right) \\
&\quad + P\left(\sup_{t\in[0,T]} \left\| \int_0^t S(t-s)\big(G(X(s-)) - R_nG(X_n(s-))\big)\,\mathrm{d}L(s) \right\| > \frac{r}{3} \right). \qquad (5.26)
\end{aligned}
$$

For the following arguments, we define $m := \sup_{t\in[0,T]} \|S(t)\|_{L(H)}$. As $I - R_n$ converges to zero strongly as $n \to \infty$ we obtain

$$
P\left(\sup_{t\in[0,T]} \|S(t)(I - R_n)x_0\| > \frac{r}{3} \right) \le P\left(m\,\|(I - R_n)x_0\| > \frac{r}{3} \right) \to 0.
$$

For estimating the second term in (5.26), we apply Markov's inequality and Lipschitz continuity of F in (A2) to obtain

$$
\begin{aligned}
P &\left(\sup_{t\in[0,T]} \left\| \int_0^t S(t-s)\big(F(X(s)) - R_nF(X_n(s))\big)\,\mathrm{d}s \right\| > \frac{r}{3} \right) \\
&\le P\left(m \int_0^T \|F(X(s)) - R_nF(X_n(s))\|\,\mathrm{d}s > \frac{r}{3} \right) \\
&\le P\left(\int_0^T \|(I - R_n)F(X(s))\|\,\mathrm{d}s > \frac{r}{6m} \right) \\
&\qquad\qquad + P\left(\int_0^T \|R_n(F(X(s)) - F(X_n(s)))\|\,\mathrm{d}s > \frac{r}{6m} \right) \\
&\le \frac{6m}{r} E\left[\int_0^T \|(I - R_n)F(X(s))\|\,\mathrm{d}s \right] \\
&\qquad\qquad + \frac{6m}{r} E\left[\int_0^T \|R_n(F(X(s)) - F(X_n(s)))\|\,\mathrm{d}s \right]
\end{aligned}
$$

$$\leq \frac{6m}{r} E\left[\int_0^T \|(I-R_n)F(X(s))\| \,\mathrm{d}s\right] \\ + \frac{6m}{r}\left(\sup_{n\in\mathbb{N}} \|R_n\|_{L(H)}\right) TK_F \sup_{t\in[0,T]} E\left[\|X(t)-X_n(t)\|\right].$$

We conclude from the last inequality by Lebesgue's dominated convergence theorem and convergence of X_n to X in $\mathcal{C}([0,T], L^1(\Omega, H))$ that

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \left\|\int_0^t S(t-s)(F(X(s)) - R_nF(X_n(s)))\,\mathrm{d}s\right\| > \frac{r}{3}\right) = 0.$$

To estimate the last term in (5.26), we apply the dilation theorem for contraction semigroups, see [74, Th. I.8.1]): there exists a C_0-group $(\hat{S}(t))_{t\in\mathbb{R}}$ of unitary operators $\hat{S}(t)$ on a larger Hilbert space $\hat{H}$ in which H is continuously embedded satisfying $S(t) = \pi\hat{S}(t)i$ for all $t \geq 0$, where π is the projection from $\hat{H}$ to H and i is the continuous embedding of H into $\hat{H}$. Thus, if we denote $m = \sup_{t\in[0,T]} \left\|\pi\hat{S}(t)\right\|_{L(\hat{H},H)}$, $k = \sup_{s\in[-T,0]} \left\|\hat{S}(s)i\right\|_{L(H,\hat{H})}$, we may estimate using Markov's inequality, Inequality (4.11) and Hölder continuity of G in (5.6)

$$\begin{aligned}
&P\left(\sup_{t\in[0,T]} \left\|\int_0^t S(t-s)\left(G(X(s-)) - R_nG(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{3}\right) \\
&\leq P\left(\sup_{t\in[0,T]} \left\|\int_0^t S(t-s)\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\| > \frac{r}{6}\right) \\
&\qquad + P\left(\sup_{t\in[0,T]} \left\|\int_0^t S(t-s)R_n\left(G(X(s-)) - G(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{6}\right) \\
&= P\left(\sup_{t\in[0,T]} \left\|\int_0^t \pi\hat{S}(t-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\| > \frac{r}{6}\right) \\
&\qquad + P\left(\sup_{t\in[0,T]} \left\|\int_0^t \pi\hat{S}(t-s)iR_n\left(G(X(s-)) - G(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{6}\right)
\end{aligned}$$

$$
\begin{aligned}
&= P\left(\sup_{t\in[0,T]}\left\|\pi\hat{S}(t)\int_0^t \hat{S}(-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\| > \frac{r}{6}\right) \\
&\qquad + P\left(\sup_{t\in[0,T]}\left\|\pi\hat{S}(t)\int_0^t \hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{6}\right) \\
&\le P\left(\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\|_{\hat{H}} > \frac{r}{6m}\right) \\
&\qquad + P\left(\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\|_{\hat{H}} > \frac{r}{6m}\right) \\
&\le \frac{6m}{r}E\left[\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\|_{\hat{H}}\right] \\
&\qquad + \frac{6m}{r}E\left[\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\|_{\hat{H}}\right] \\
&\le e_{1,\alpha}\frac{6m}{r}\left(E\left[\int_0^T\left\|\hat{S}(-s)i\left(I-R_n\right)G(X(s-))\right\|^{\alpha}_{L_2(U,\hat{H})}\mathrm{d}s\right]\right)^{1/\alpha} \\
&\qquad + e_{1,\alpha}\frac{6m}{r}\left(E\left[\int_0^T\left\|\hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\right\|^{\alpha}_{L_2(U,\hat{H})}\mathrm{d}s\right]\right)^{1/\alpha} \\
&\le e_{1,\alpha}\frac{6m}{r}k\left(E\left[\int_0^T\left\|\left(I-R_n\right)G(X(s-))\right\|^{\alpha}_{L_2(U,\hat{H})}\mathrm{d}s\right]\right)^{1/\alpha} \\
&\qquad + e_{1,\alpha}\frac{6m}{r}kK_GT^{1/\alpha}\left(\sup_{t\in[0,T]}E\left[\left\|(X(t))-X_n(t)\right\|^{\alpha\beta}\right]\right)^{1/\alpha},
\end{aligned}
$$

for $\beta \in (0,1)$. We conclude from the last inequality by Lebesgue's dominated convergence, strong convergence of R_n to I, boundedness G in (5.6) and convergence of X_n to X in $\mathcal{C}([0,T], L^{\alpha\beta}(\Omega, H))$ that

$$
\lim_{n\to\infty} P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)(G(X(s-))-R_nG(X_n(s-)))\,\mathrm{d}L(s)\right\| > \frac{r}{3}\right) = 0,
$$

We have shown that all the terms on the right hand side of (5.26) converge to zero as n tends to infinity which gives uniform convergence of X_n to X in probability on

$[0,T]$. This concludes the proof, since uniform convergence in probability implies the existence of a desired subsequence. $\square$

5.2 Moment boundedness of evolution equations

In this section, we investigate stability properties of the solution for the stochastic evolution equation (5.1) by applying the strong Itô formula derived in Theorem 4.18. More precisely, we shall provide conditions on the coefficients such that the mild solution X is ultimately exponentially bounded in the p-th moment, that is, there exist constants $m_1, m_2, m_3 > 0$ such that

$$E\left[\|X(t)\|^p\right] \leq m_1 e^{-tm_2} E\left[\|x_0\|^p\right] + m_3 \quad \text{for all } t \geq 0.$$

Recall that $\mathcal{C}_b^2(H)$ denotes the space of continuous real-valued functions defined on H with bounded first and second Fréchet derivatives. In what follows, our goal is to derive a Lyapunov-type criterion using the following operator on $\mathcal{C}_b^2(H)$:

$$\begin{aligned}\mathcal{L}f(h) &= \langle Df(h), Ah + F(h)\rangle \\ &\quad + \int_H \big(f(h+g) - f(h) - \langle Df(h), g\rangle\big)\left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g), \quad h \in D^1 \end{aligned} \tag{5.27}$$

for $f \in \mathcal{C}_b^2(H)$. Note that the right hand side of (5.27) is well defined by Lemma 4.19. We can now state the main result of this section, the following general moment boundedness criterion.

Theorem 5.12. *Let $p \in (0,1)$ be fixed and V be a function in $\mathcal{C}_b^2(H)$ satisfying for some constants $\beta_1, \beta_2, \beta_3, k_1, k_3 > 0$ the inequalities*

$$\beta_1 \|h\|^p - k_1 \leq V(h) \leq \beta_2 \|h\|^p \quad \textit{for all } h \in H, \tag{5.28}$$

$$\mathcal{L}V(h) \leq -\beta_3 V(h) + k_3 \quad \textit{for all } h \in D^1. \tag{5.29}$$

Then the solution X to (5.1) is exponentially ultimately bounded in the p-th moment:

$$E\left[||X(t)||^p\right] \le \frac{\beta_2}{\beta_1} e^{-\beta_3 t} E\left[\|x_0\|^p\right] + \frac{1}{\beta_1}\left(k_1 + \frac{k_3}{\beta_3}\right).$$

Before we prove Theorem 5.12 we demonstrate its application by deriving conditions for moment boundedness in terms of the coefficients of Equation (5.1).

Corollary 5.13. *Suppose that there exists $\epsilon > 0$ such that*

$$\langle Ah + F(h), h\rangle \le -\epsilon\, ||h||^2 \quad \textit{for all } h \in D^1,$$

then the solution to (5.1) is exponentially ultimately bounded in the p-th moment for every $p \in (0,1)$.

Proof. Fix $p \in (0,1)$ and let ζ be a function in $\mathcal{C}^2([0,\infty))$ satisfying $\zeta(x) = x^{p/2}$ for $x \ge 1$ and $\zeta(x) \le 1$ for $x < 1$. By defining $V(h) = \zeta(\|h\|^2)$ for all $h \in H$, we obtain $V \in \mathcal{C}_b^2(H)$ and

$$V(h) = ||h||^p \quad \text{for all } h \in \overline{B}_H^c \quad \text{and} \quad 0 \le V(h) \le 1 \quad \text{for all } h \in \overline{B}_H.$$

It follows that (5.28) holds with $\beta_1 = \beta_2 = k_1 = 1$. We show that (5.29) also holds. By the definition of V, it follows for each $h \in D^1 \cap \overline{B}_H^c$ that

$$\langle DV(h), Ah + F(h)\rangle = p\,||h||^{p-2}\langle h, Ah + F(h)\rangle \le -\epsilon p\,||h||^p = -\epsilon p V(h).$$

For $h \in D^1 \cap \overline{B}_H$, one obtains by boundedness of F in Assumption (A2) that

$$\langle DV(h), Ah + F(h)\rangle \le ||DV||_\infty \left(||A||_{L(D^1)} + K_F\right).$$

Since Lemma 4.19 together with boundedness of G in Assumption (A3) implies for each

$h \in H$ that

$$\int_H \left(V(h+g) - V(h) - \langle DV(h), g\rangle\right) \left(\lambda \circ G^{-1}(h)\right)(\mathrm{d}g)$$
$$\leq d_\alpha^1 \left(2\,||DV||_\infty + \frac{1}{2}\,\left|\left|D^2V\right|\right|_\infty\right) K_G^\alpha,$$

we have verified Condition (5.29). Hence, an application of Theorem 5.12 completes the proof. □

In the remaining of this section, we prove Theorem 5.12 using the Yosida approximations established in the previous sections. For this purpose, let X_n denote the mild solution to the approximating equations (5.7) for each $n \in \mathbb{N}$. We may assume due to Corollary 5.11, by passing to a subsequence if necessary, that X_n converges to the solution X of (5.1) uniformly almost surely on $[0, T]$. In what follows, we will routinely pass on to a subsequence without changing the indices.

Proposition 5.14. *The mild solution X_n of (5.7) is a strong solution attaining values in D^1, that is, for each $t \in [0, T]$, it satisfies*

$$X_n(t) = R_n x_0 + \int_0^t \left(AX_n(s) + R_nF(X_n(s))\right) \mathrm{d}s + \int_0^t R_nG(X_n(s-))\,\mathrm{d}L(s).$$

Proof. Our argument will follow closely the proof of [2, Th. 2]. As mild solution, X_n satisfies

$$X_n(t) = S(t)R_n x_0 + \int_0^t S(t-s)R_nF(X_n(s))\,\mathrm{d}s + \int_0^t S(t-s)R_nG(X_n(s-))\,\mathrm{d}L(s). \tag{5.30}$$

The process X_n is $(\mathcal{F}_t)$-measurable with càdlàg paths and attains values in D^1. First, we obtain from (5.30) by interchanging integrals and $A \in L(D^1)$ for $t \geq 0$ that

$$AX_n(t) = AS(t)R_n x_0 + \int_0^t AS(t-s))R_n F(X_n(s))\,\mathrm{d}s + \int_0^t AS(t-s)R_n G(X_n(s-))\,\mathrm{d}L(s). \quad (5.31)$$

Each term on the right hand side of (5.31) is almost surely Bochner integrable. Indeed, integrability of the first term is immediate from the uniform boundedness principle. For the second term, boundedness of F in Condition (A2) and commutativity of S and R_n implies

$$\int_0^t \int_0^s \|AS(s-r)R_n F(X_n(r))\|_1 \,\mathrm{d}r\,\mathrm{d}s \leq \|A\|_{L(D^1)} \|R_n\|_{L(H,D^1)} \int_0^t \int_0^s \|S(s-r)F(X_n(s))\| \,\mathrm{d}r\,\mathrm{d}s < \infty \quad \text{a.s.}$$

Almost sure Bochner integrability of the stochastic integral in (5.31) follows from boundedness of G in Assumption (A3) and commutativity of S and R_n via the estimate

$$E\left[\int_0^t \int_0^s \|AS(s-r)R_n G(X_n(r))\|^\alpha_{L_2(U,D^1)} \,\mathrm{d}r\,\mathrm{d}s\right] \leq \|A\|^\alpha_{L(D^1)} \|R_n\|^\alpha_{L(H,D^1)} E\left[\int_0^t \int_0^s \|S(s-r)G(X_n(r))\|^\alpha_{L_2(U,H)} \,\mathrm{d}r\,\mathrm{d}s\right] < \infty.$$

Integrating both sides of (5.31) results in the equality

$$\int_0^t AX_n(s)\,\mathrm{d}s = \int_0^t AS(s)R_n x_0\,\mathrm{d}s + \int_0^t \int_0^s AS(s-r)R_n F(X_n(r))\,\mathrm{d}r\,\mathrm{d}s + \int_0^t \int_0^s AS(s-r)R_n G(X_n(r-))\,\mathrm{d}L(r)\,\mathrm{d}s.$$

Applying Fubini's theorems, see Theorem 4.6 for the stochastic version, and the equality $\int_0^t AS(s)R_n h \,\mathrm{d}s = S(t)R_n h - R_n h$ for all $h \in H$ enable us to conclude

$$\begin{aligned}
\int_0^t AX_n(s)\,\mathrm{d}s &= \int_0^t AS(s)R_n x_0 \,\mathrm{d}s + \int_0^t \int_r^t AS(s-r)R_n F(X_n(r))\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^t \int_r^t AS(s-r)R_n G(X_n(r-))\,\mathrm{d}s\,\mathrm{d}L(r) \\
&= S(t)R_n x_0 - R_n x_0 + \int_0^t S(t-r)R_n F(X_n(r))\,\mathrm{d}r - \int_0^t R_n F(X_n(r))\,\mathrm{d}r \\
&\quad + \int_0^t S(t-r)R_n G(X_n(r-))\,\mathrm{d}L(r) - \int_0^t R_n G(X_n(r-))\,\mathrm{d}L(r) \\
&= X_n(t) - R_n x_0 - \int_0^t R_n F(X_n(r))\,\mathrm{d}r - \int_0^t R_n G(X_n(r-))\,\mathrm{d}L(r),
\end{aligned}$$

which verifies X_n as a strong solution to (5.1). □

We denote by $\mathcal{L}_n$ the usual generator associated with the Yosida approximations X_n, $n \in \mathbb{N}$, defined for $f \in \mathcal{C}_b^2(H)$ and $h \in D^1$ by

$$\begin{aligned}
\mathcal{L}_n f(h) &= \langle Df(h), Ah + R_n F(h)\rangle \\
&\quad + \int_H \big(f(h+R_n g) - f(h) - \langle Df(h), R_n g\rangle\big)\,\big(\lambda \circ G(h)^{-1}\big)(\mathrm{d}g). \qquad (5.32)
\end{aligned}$$

The right hand side of (5.32) is well defined by Lemma 4.19. Recall that the counterpart to $\mathcal{L}_n$ for the mild solution X denoted by $\mathcal{L}$ was introduced in (5.27). The generators $\mathcal{L}_n$ and $\mathcal{L}$ are related by the following crucial convergence result.

Lemma 5.15. *Let $(X_n)_{n\in\mathbb{N}}$ be solutions of (5.7) which a.s. converges uniformly to the solution of (5.1). It follows for each $f \in \mathcal{C}_b^2(H)$ that*

$$\lim_{n\to\infty} E\left[\int_0^T \Big|\mathcal{L}_n f(X_n(s)) - \mathcal{L} f(X_n(s))\Big|\,\mathrm{d}s\right] = 0.$$

Proof. We obtain for each $h \in D^1$ that

$$
\begin{aligned}
|\mathcal{L}f(h) - \mathcal{L}_n f(h)| &\leq ||Df||_\infty \, ||(I - R_n)\, F(h)|| \\
&+ \int_{\overline{B}_H} |f(h+g) - f(h + R_n g) - \langle Df(h), (I - R_n)\, g\rangle| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g) \\
&+ \int_{\overline{B}_H^c} |f(h+g) - f(h + R_n g)| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g) \\
&+ \int_{\overline{B}_H^c} |\langle Df(h), (I - R_n)\, g\rangle| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g).
\end{aligned} \tag{5.33}
$$

Taylor's remainder theorem in the integral form implies

$$
\begin{aligned}
&\int_{\overline{B}_H} |f(h+g) - f(h + R_n g) - \langle Df(h), (I - R_n)\, g\rangle| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g) \\
&\leq \int_{\overline{B}_H} \int_0^1 |\langle D^2 f(h + \theta\,(I - R_n)\, g)\,(I - R_n)\, g, (I - R_n)\, g\rangle (1-\theta)\,|\mathrm{d}\theta \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g) \\
&\leq \frac{1}{2} \left|\left|D^2 f\right|\right|_\infty \int_{\overline{B}_H} ||(I - R_n)\, g||^2 \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g).
\end{aligned}
$$

In the same way, we obtain

$$
\begin{aligned}
&\int_{\overline{B}_H^c} |f(h+g) - f(h + R_n g)| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g) \\
&\qquad\qquad \leq ||Df||_\infty \int_{\overline{B}_H^c} ||(I - R_n)\, g|| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g),
\end{aligned}
$$

and also

$$
\begin{aligned}
&\int_{\overline{B}_H^c} |\langle Df(h), (I - R_n)\, g\rangle| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g) \\
&\qquad\qquad \leq ||Df||_\infty \int_{\overline{B}_H^c} ||(I - R_n)\, g|| \left(\lambda \circ G(h)^{-1}\right)(\mathrm{d}g).
\end{aligned}
$$

Applying the last three estimates to (5.33) and taking expectation on both sides, it follows from Inequality (4.5) and for each $n \in \mathbb{N}$ that

$$\begin{aligned}
&E\left[\int_0^T |\mathcal{L}_n f(X_n(s)) - \mathcal{L} f(X_n(s))| \,\mathrm{d}s\right] \\
&\quad \le ||Df||_\infty \, E\left[\int_0^T ||(I - R_n)\, F(X_n(s))|| \,\mathrm{d}s\right] \\
&\qquad + cE\left[\int_0^T ||(I - R_n)\, G(X_n(s))||^\alpha_{L_2(U,H)} \,\mathrm{d}s\right],
\end{aligned}$$

where $c := d^1_\alpha \left(2\,||Df||_\infty + \frac{1}{2}\left|\left|D^2 f\right|\right|_\infty\right)$.

To complete the proof, it remains to show that both

$$\lim_{n\to\infty} E\left[\int_0^T ||(I - R_n)\, F(X_n(s))|| \,\mathrm{d}s\right] = 0, \tag{5.34}$$

$$\lim_{n\to\infty} E\left[\int_0^T ||(I - R_n)\, G(X_n(s))||^\alpha_{L_2(U,H)} \,\mathrm{d}s\right] = 0. \tag{5.35}$$

Let $t \in [0,T]$ be arbitrary but fixed, and recall that we chose $X_n(t)$ almost surely convergent and thus $\{X_m(t)(\omega) : m \in \mathbb{N}\} \subset H$ is compact for almost all $\omega \in \Omega$. Strong convergence of $I - R_n$ to zero, see [19, Le. 3.4], continuity of F and G and the fact that continuous mapping converging pointwise to a continuous mapping converge uniformly over compacts together imply for each $t \in [0,T]$ that, almost surely, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} ||(I - R_n)\, F(X_n(t))|| \le \lim_{n\to\infty} \sup_{m\in\mathbb{N}} ||(I - R_n)\, F(X_m(t))|| = 0, \\
&\lim_{n\to\infty} ||(I - R_n)\, G(X_n(t))||^\alpha_{L_2(U,H)} \le \lim_{n\to\infty} \sup_{m\in\mathbb{N}} ||(I - R_n)\, G(X_m(t))||^\alpha_{L_2(U,H)} = 0.
\end{aligned}$$

Since the boundedness conditions in (A2) and (A3) guarantee

$$\begin{aligned}
||(I - R_n)\, F(X_n(t))|| &\le \left(\sup_{n\in\mathbb{N}} ||I - R_n||_{L(H)}\right) K_F \quad \text{a.s.}, \\
||(I - R_n)\, G(X_n(t))||^\alpha_{L_2(U,H)} &\le \left(\sup_{n\in\mathbb{N}} ||I - R_n||^\alpha_{L(H)}\right) K_G^\alpha \quad \text{a.s.}
\end{aligned}$$

an application of Lebesgue's dominated convergence theorem verifies (5.34) and (5.35), which completes the proof. □

Proof of Theorem 5.12. Let $(X_n)_{n\in\mathbb{N}}$ be the solutions of (5.7). Because of Corollary 5.11, we can assume that $(X_n)_{n\in\mathbb{N}}$ converges uniformly to the solution of (5.1) a.s. Proposition 5.14 enables us to apply our strong Itô formula in Theorem 4.18 to X_n, which results in

$$
\begin{aligned}
&V(X_n(t)) \\
&\quad = V(X_n(0)) + \int_0^t \mathcal{L}_n V(X_n(s))\mathrm{d}s + \int_0^t \langle G(X_n(s-))^* R_n^* DV(X_n(s-)), \cdot\rangle \mathrm{d}L(s) \\
&\qquad + \int_0^t \int_H V(X_n(s-)+h) - V(X_n(s-)) - \langle DV(X_n(s-)), h\rangle (\mu^{X_n} - \nu^{X_n})(\mathrm{d}s, \mathrm{d}h)
\end{aligned}
\tag{5.36}
$$

almost surely for all $t \geq 0$. Applying the product formula to the real-valued semi-martingale $V(X_n(\cdot))$ and the function $t \mapsto e^{\beta_3 t}$ and taking expectations on both sides of (5.36) shows

$$
e^{\beta_3 t} E\left[V(X_n(t))\right] = E\left[V(X_n(0))\right] + E\left[\int_0^t e^{\beta_3 s}\left(\beta_3 V(X_n(s)) + \mathcal{L}_n V(X_n(s))\right)\mathrm{d}s\right].
\tag{5.37}
$$

Here, we used the fact that the last two integrals in (5.36) define martingales, and thus have expectation zero. This follows from the observation that they are local martingales according to Lemma 4.11 and Theorem 4.18 and are uniformly bounded in mean, see [64, Th I.51]. The latter is guaranteed by the boundedness of G in (A3), since

$$
\begin{aligned}
&E\left[\int_0^t ||\langle G(X_n(s))^* R_n^* DV(X_n(s)), \cdot\rangle||^\alpha_{L_2(U,\mathbb{R})}\, \mathrm{d}s\right] \\
&\qquad = E\left[\int_0^t ||G(X_n(s))^* R_n^* DV(X_n(s))||^\alpha\, \mathrm{d}s\right] \leq ||R_n||^\alpha_{L(H)}\, ||DV||^\alpha_\infty\, T K_G^\alpha < \infty,
\end{aligned}
$$

and similarly, by using Lemma 4.19,

$$E\left[\int_0^t \int_H \left|V(X_n(s-)+h) - V(X_n(s-)) - \langle DV(X_n(s-),h\rangle\right| \nu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right]$$
$$\leq d_\alpha^1 \left(2\,||DV||_\infty + \frac{1}{2}\,||D^2V||_\infty\right) ||R_n||_{L(H)}^\alpha\, E\left[\int_0^t ||G(X_n(s))||_{L_2(U,H)}^\alpha \,\mathrm{d}s\right] < \infty.$$

The first term on the right hand side in (5.37) is finite since

$$E\left[V(X_n(0))\right] \leq \beta_2 \left\|R_n\right\|_{L(H)}^p E\left[\|x_0\|^p\right] < \infty.$$

The same holds for the second term, which can be shown using the same arguments as in the proof of Lemma 5.15. By applying Inequality (5.29) to (5.37), we conclude

$$\begin{aligned}
&e^{\beta_3 t} E\left[\left[V(X_n(t))\right]\right.\\
&\leq E\left[V(X_n(0))\right] + E\left[\int_0^t e^{\beta_3 s}\Big(- \mathcal{L}V(X_n(s)) + \mathcal{L}_n V(X_n(s)) + k_3\Big)\mathrm{d}s\right]\\
&\leq E\left[V(X_n(0))\right] + e^{\beta_3 T} E\left[\int_0^t \Big|\mathcal{L}_n V(X_n(s)) - \mathcal{L}V(X_n(s))\Big|\mathrm{d}s\right] + \frac{k_3}{\beta_3}\left(e^{\beta_3 t} - 1\right).
\end{aligned}$$

Lemma 5.15 together with Fatou's lemma implies

$$E\left[V(X(t))\right] \leq \liminf_{n\to\infty} E\left[V(X_n(t))\right] \leq e^{-\beta_3 t} E\left[V(x_0)\right] + \frac{k_3}{\beta_3}.$$

Applying Assumption (5.28) completes the proof. □

5.3 Mild Itô formula

In this section, we prove an Itô formula for mild solutions of Equation (5.1) and mappings $f \in \mathcal{C}_b^2(H)$ such that the second derivative $D^2 f$ is not only continuous but satisfies

$$\lim_{n\to\infty} \|g_n - g\| = 0 \implies \lim_{n\to\infty} \sup_{h\in\overline{B}_H} \left\| D^2 f(g_n + h) - D^2 f(g+h) \right\|_{L(H)} = 0. \tag{5.38}$$

The subspace of all these functions is denoted by $\mathcal{C}_{b,u}^2(H)$.

Theorem 5.16 (Itô formula for mild solutions)**.** *A mild solution X of (5.1) satisfies for each $f \in \mathcal{C}_{b,u}^2(H)$ and $t \geq 0$ that*

$$\begin{aligned} f(X(t)) = f(x_0) &+ \int_0^t \langle G(X(s-)^* Df(X(s-))), \cdot\rangle \,\mathrm{d}L(s) \\ &+ \int_0^t \int_H \big(f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle\big) \left(\mu^X - \nu^X\right)(\mathrm{d}s, \mathrm{d}h) \\ &+ \lim_{n\to\infty} \left(\int_0^t \langle Df(X_n(s)), AX_n(s)\rangle \,\mathrm{d}s \right) + \int_0^t \langle Df(X(s)), F(X(s))\rangle \,\mathrm{d}s \\ &+ \int_0^t \int_H \big(f(X(s)+h) - f(X(s)) - \langle Df(X(s)), h\rangle\big) \left(\lambda \circ G(X(s))^{-1}\right)(\mathrm{d}h)\,\mathrm{d}s, \end{aligned} \tag{5.39}$$

where the limit is taken in $L_P^0(\Omega, \mathbb{R})$.

Remark 5.17. *Note that while X may not be a semimartingale, the compensated measure $\mu^X - \nu^X$ in (5.39) still exists as X is both adapted and càdlàg; see [27, Chap. II].*

Remark 5.18. *Unlike in similar situation with the driving noise being Gaussian, see e.g. [44], we do not identify the limit in (5.39) as then the imposed assumptions on f are very restrictive. In applications (see e.g. [2]), it is usually enough to establish some* *bound on the limit*

$$\lim_{n\to\infty}\left(\int_0^t \langle Df(X_n(s)), AX_n(s)\rangle\,\mathrm{d}s\right),$$

which leads to natural assumptions on the generator A.

We divide the proof of the above theorem into a few technical lemmata. To simplify the notation, we introduce the function $T_f\colon H\times H\to\mathbb{R}$ for $f\in\mathcal{C}^2_{b,u}(H)$ defined by

$$T_f(g,h)=f(g+h)-f(g)-\langle Df(g),h\rangle,\quad g,h\in H.$$

Lemma 5.19. *Let λ be the cylindrical Lévy measure of a standard symmetric α-stable cylindrical Lévy process L for $\alpha\in(1,2)$. It follows for every $f\in\mathcal{C}^2_b(H)$, $\varphi\in L_2(U,H)$ and $g,h\in H$ that*

$$\begin{aligned}&\int_H |T_f(g,b)-T_f(h,b)|\left(\lambda\circ\varphi^{-1}\right)(\mathrm{d}b)\\ &\quad\le 2d_\alpha^1\,\|\varphi\|_{L_2(U,H)}^\alpha\left(\sup_{b\in\overline{B}_H}\left\|D^2f(g+b)-D^2f(h+b)\right\|_{L(H)}+\left\|D^2f\right\|_\infty\|g-h\|\right).\end{aligned}$$

Proof. Taylor's remainder theorem in the integral form and Inequality (4.5) imply

$$\begin{aligned}&\int_{\overline{B}_H}|T_f(g,b)-T_f(h,b)|\left(\lambda\circ\varphi^{-1}\right)(\mathrm{d}b)\\ &\qquad=\int_{\overline{B}_H}\left|\int_0^1\langle(D^2f(g+\theta b)-D^2f(h+\theta b))b,b\rangle(1-\theta)\,\mathrm{d}\theta\right|\left(\lambda\circ\varphi^{-1}\right)(\mathrm{d}b)\\ &\qquad\le\frac{1}{2}\sup_{b\in\overline{B}_H}\left\|D^2f(g+b)-D^2f(h+b)\right\|_{L(H)}\int_{\overline{B}_H}\|b\|^2\left(\lambda\circ\varphi^{-1}\right)(\mathrm{d}b)\\ &\qquad\le\frac{1}{2}d_\alpha^1\left(\sup_{b\in\overline{B}_H}\left\|D^2f(g+b)-D^2f(h+b)\right\|_{L(H)}\right)\|\varphi\|_{L_2(U,H)}^\alpha.\end{aligned}\tag{5.40}$$

A similar argument yields

$$
\begin{aligned}
&\int_{\overline{B}_H^c} |T_f(g,b) - T_f(h,b)| \left(\lambda \circ \varphi^{-1}\right)(\mathrm{d}b) \\
&\qquad \le \int_{\overline{B}_H^c} \left| \int_0^1 \langle Df(g+\theta b) - Df(h+\theta b), b\rangle \, \mathrm{d}\theta \right| \left(\lambda \circ \varphi^{-1}\right)(\mathrm{d}b) \\
&\qquad\qquad\qquad + \int_{\overline{B}_H^c} |\langle Df(g) - Df(h), b\rangle| \left(\lambda \circ \varphi^{-1}\right)(\mathrm{d}b) \\
&\qquad \le \left(\sup_{b\in H} \|Df(g+b) - Df(h+b)\| + \|Df(g) - Df(h)\| \right) \int_{\overline{B}_H^c} \|b\| \left(\lambda \circ \varphi^{-1}\right)(\mathrm{d}b) \\
&\qquad \le 2d_\alpha^1 \left\| D^2 f \right\|_\infty \|g-h\| \, \|\varphi\|_{L_2(U,H)}^\alpha . \qquad (5.41)
\end{aligned}
$$

Combining Inequalities (5.40) and (5.41) completes the proof. □

Lemma 5.20. *Let $(X_n)_{n\in\mathbb{N}}$ denote a subsequence of the solutions to the Yosida approximating equations (5.7), satisfying that $(X_n)_{n\in\mathbb{N}}$ converges to the mild solution X to (5.1) both in $\mathcal{C}([0,T], L^p(\Omega,H))$ and uniformly on $[0,T]$ almost surely. Then it follows for any $f \in \mathcal{C}_{b,u}^2(H)$ and $t \in [0,T]$ that*

$$
\lim_{n\to\infty} \int_0^t \int_H T_f(X_n(s-),h)\, \mu^{X_n}(\mathrm{d}s,\mathrm{d}h) = \int_0^t \int_H T_f(X(s-),h)\, \mu^X(\mathrm{d}s,\mathrm{d}h) \;\; \textit{in } L_P^0(\Omega,\mathbb{R}).
$$

Proof. First, the existence of a subsequence $(X_n)_{n\in\mathbb{N}}$ with the claimed properties is guaranteed by Corollary 5.11. By recalling the form of X_n in Proposition 5.14, it follows from Theorem 4.16 that for each $n \in \mathbb{N}$ we have

$$
\begin{aligned}
&E\left[\left| \int_0^t \int_H \left(T_f(X_n(s-),h) - T_f(X(s-),h)\right) \mu^{X_n}(\mathrm{d}s,\mathrm{d}h) \right| \right] \\
&\qquad \le E\left[\int_0^t \int_H \left| T_f(X_n(s-),h) - T_f(X(s-),h) \right| \mu^{X_n}(\mathrm{d}s,\mathrm{d}h) \right] \\
&\qquad = E\left[\int_0^t \int_H \left| T_f(X_n(s),h) - T_f(X(s),h) \right| \nu^{X_n}(\mathrm{d}s,\mathrm{d}h) \right]
\end{aligned}
$$

$$= E\left[\int_0^t \int_H \left|T_f(X_n(s),h) - T_f(X(s),h)\right| \left(\lambda \circ (R_n G(X_n(s-)))^{-1}\right)(\mathrm{d}h)\,\mathrm{d}s\right]. \tag{5.42}$$

Since Remark 5.6 guarantees $c := 2d_\alpha^1 K_G^\alpha \sup_{n\in\mathbb{N}} \|R_n\|_{L(H)}^\alpha < \infty$, we obtain from Lemma 5.19 for P_T-a.a. $(\omega, s) \in \Omega \times [0,T]$ that

$$\begin{aligned}
&\int_H |T_f(X_n(s)(\omega),h) - T_f(X(s)(\omega),h)| \left(\lambda \circ (R_n G(X_n(s-)(\omega)))^{-1}\right)(\mathrm{d}h) \\
&\le c\Bigg(\sup_{b\in\overline{B}_H} \left\|D^2 f(X_n(s)(\omega)+b) - D^2 f(X(s)(\omega)+b)\right\|_{L(H)} \\
&\qquad\qquad + \left\|D^2 f\right\|_\infty \|f(X_n(s)(\omega)) - f(X(s)(\omega))\|\Bigg).
\end{aligned}$$

Since f satisfies (5.38), Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} E\left[\left|\int_0^t \int_H \big(T_f(X_n(s-),h) - T_f(X(s-),h)\big)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right|\right] = 0. \tag{5.43}$$

For the next step, fix $\epsilon, \epsilon' > 0$, and bound for any $m, n \in \mathbb{N}$ the expression

$$\begin{aligned}
&P\left(\left|\int_0^t \int_H T_f(X(s-),h)\left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^X(\mathrm{d}s,\mathrm{d}h)\right)\right| > \epsilon\right) \\
&\le P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \\
&\quad + P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \\
&\quad + P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)^c} T_f(X(s-),h)\,\left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^X(\mathrm{d}s,\mathrm{d}h)\right)\right| > \frac{\epsilon}{3}\right).
\end{aligned} \tag{5.44}$$

Since Taylor's remainder theorem in the integral form implies that for all $h \in H$ we have $|T_f(X(s-),h)| \le \frac{1}{2}\left\|D^2 f\right\|_\infty \|h\|^2$, we obtain by applying Theorem 4.16 and Inequality

(4.5) that

$$
\begin{aligned}
&E\left[\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right|\right] \\
&\quad\le \frac{1}{2}\left\|D^2 f\right\|_\infty E\left[\int_0^t \int_{\overline{B}_H(1/m)} \|h\|^2 \left(\lambda\circ (R_n G(X_n(s)))^{-1}\right)(\mathrm{d}h)\mathrm{d}s\right] \\
&\quad\le d_\alpha^m K_G^\alpha \frac{1}{2}\left\|D^2 f\right\|_\infty T,
\end{aligned}
$$

where in the last step we used that by Remark 5.6 we have $\|R_n\|_{L(H)} \le 1$ for all $n \in \mathbb{N}$. Since the last line is independent of $n \in \mathbb{N}$ and $d_\alpha^m \to 0$ as $m \to \infty$ according to Inequality (4.5), Markov's inequality implies that there exists $m_1 \in \mathbb{N}$ such that for all $m \ge m_1$ and all $n \in \mathbb{N}$

$$
P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \le \epsilon'. \tag{5.45}
$$

Exactly the same arguments establish that for all $m \ge m_1$

$$
P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \le \epsilon'. \tag{5.46}
$$

To bound the last term in (5.44), first note that for each $m, n \in \mathbb{N}$ we have that

$$
\begin{aligned}
&\int_0^t \int_{\overline{B}_H(1/m)^c} T_f(X(s-),h)\,\left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^{X}(\mathrm{d}s,\mathrm{d}h)\right) \\
&= \sum_{0\le s\le t} T_f(X(s-),\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s)) \\
&\qquad - \sum_{0\le s\le t} T_f(X(s-),\Delta X(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)) \\
&= \sum_{0\le s\le t} T_f(X(s-),\Delta X_n(s))\left(\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s)) - \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s))\right)
\end{aligned}
$$

$$+\sum_{0\le s\le t}\big(T_f(X(s-),\Delta X_n(s))-T_f(X(s-),\Delta X(s))\big)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)). \quad (5.47)$$

For estimating the first term in the last line, we use the equality $\mathbb{1}_A(x)-\mathbb{1}_A(y)=\mathbb{1}_A(x)\mathbb{1}_{A^c}(y)-\mathbb{1}_{A^c}(x)\mathbb{1}_A(y)$. Applying this identity to the first term on the right hand side of Equation (5.47), we conclude from Taylor's remainder theorem in the integral form that

$$\begin{aligned}
&\left|\sum_{0\le s\le t} T_f\left(X(s-),\Delta X_n(s)\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s))\right| \\
&\le \sum_{0\le s\le t}\big|T_f(X(s-),\Delta X_n(s))\big|\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \\
&\le \frac{1}{2}\left\|D^2 f\right\|_\infty \sum_{0\le s\le t}\|\Delta X_n(s)\|^2\,\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \\
&\le c_f\sum_{0\le s\le t}\Big(\|\Delta X(s)\|^2+\|\Delta X_n(s)-\Delta X(s)\|^2\Big)\,\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)),
\end{aligned} \quad (5.48)$$

where we used the notation $c_f := \left\|D^2 f\right\|_\infty$. Applying Theorem 4.16 and using the boundedness assumption on G in (A3) allows us to conclude that

$$\begin{aligned}
E\left[\sum_{0\le s\le t}\|\Delta X(s)\|^2\,\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s))\right] &= E\left[\int_0^t\int_{\overline{B}_H(1/m)}\|h\|^2\,\mu^X(\mathrm{d}s,\mathrm{d}h)\right] \\
&= E\left[\int_0^t\int_{\overline{B}_H(1/m)}\|h\|^2\,\left(\lambda\circ G(X(s-))^{-1}\right)(\mathrm{d}h,\mathrm{d}s)\right]\le d_\alpha^m T K_G^\alpha.
\end{aligned}$$

Since $d_\alpha^m\to 0$ as $m\to\infty$, Markov's inequality implies that there exists $m_2\in\mathbb{N}$ with $m_2\ge m_1$ such that for all $m\ge m_2$ and all $n\in\mathbb{N}$

$$P\left(\sum_{0\le s\le t}\|\Delta X(s)\|^2\,\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s))\ge\frac{\epsilon}{24c_f}\right)\le\frac{\epsilon'}{8}. \quad (5.49)$$

In the remaining part of the proof fix $m \geq m_2 \geq m_1$ such that (5.49) is satisfied. Since $(X_n)_{n\in\mathbb{N}}$ converges to X uniformly almost surely, there exists $n_1 \in \mathbb{N}$ such that the set $A_n := \{\sup_{s\in[0,t]} \|\Delta X_n(s) - \Delta X(s)\| \leq \frac{1}{2m}\}$ satisfies $P(A_n^c) \leq \frac{\epsilon'}{16}$ for all $n \geq n_1$. Hence, if $n \geq n_1$ and $\omega \in A_n$ then by the reverse triangle inequality it holds that if $\|\Delta X_n(\omega, s)\| \geq \frac{1}{m}$ then

$$\|\Delta X(\omega, s)\| \geq \|\Delta X_n(\omega, s)\| - \|\Delta X_n(\omega, s) - \Delta X(\omega, s)\| \geq \frac{1}{2m}.$$

Consequently, we obtain for all $n \geq n_1$ that

$$\begin{aligned} &P\left(\sum_{0\leq s\leq t} \|\Delta X_n(s) - \Delta X(s)\|^2 \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \geq \frac{\epsilon}{24c_f}\right) \\ &\qquad \leq P\left(\sum_{0\leq s\leq t} \|\Delta X_n(s) - \Delta X(s)\|^2 \mathbb{1}_{\overline{B}_H(1/2m)^c}(\Delta X(s)) \geq \frac{\epsilon}{24c_f}\right) + \frac{\epsilon'}{16}. \end{aligned}$$

Since X has only finitely many jumps in $\overline{B}_H(1/2m)^c$ on $[0,t]$ and $\Delta X_n(s)$ converges to $\Delta X(s)$ for all $s \in [0,t]$, there exists n_2 such that for all $n \geq n_2$

$$P\left(\sum_{0\leq s\leq t} \|\Delta X_n(s) - \Delta X(s)\|^2 \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \geq \frac{\epsilon}{24c_f}\right) \leq \frac{\epsilon'}{8}.$$

Applying this together with (5.49) to Inequality (5.48) proves that for $m \geq m_2$ there exists n_2 such that for all $n \geq n_2$

$$P\left(\sum_{0\leq s\leq t} T_f\left(X(s-), \Delta X_n(s)\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \geq \frac{\epsilon}{12}\right) \leq \frac{\epsilon'}{4}. \tag{5.50}$$

As ΔX_n converges to ΔX uniformly on $[0,T]$ almost surely we obtain that for almost all $\omega \in \Omega$ we have

$$\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X_n(s)(\omega))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)(\omega)) = 0$$

if n is large enough. Therefore

$$\lim_{n\to\infty}\left(\sum_{0\le s\le t} T_f(X(s-),\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s))\right) = 0 \quad \text{a.s.}$$

and thus we obtain that there exists n_3 such that for all $n \ge n_3$ we have

$$P\left(\sum_{0\le s\le t} T_f(X(s-),\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)) \ge \frac{\epsilon}{12}\right) \le \frac{\epsilon'}{4}.$$

Combining this with (5.50) shows that for $m \ge m_2$ and $n \ge \max\{n_2, n_3\}$ we have

$$P\left(\sum_{0\le s\le t} T_f(X(s-),\Delta X_n(s))\left(\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s)) - \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s))\right) \ge \frac{\epsilon}{6}\right) \le \frac{\epsilon'}{2}. \tag{5.51}$$

Since X has only finitely many jumps in $\overline{B}_H(1/m)^c$ on $[0,t]$ and $\Delta X_n(s)$ converges to $\Delta X(s)$ for all $s \in [0,t]$, there exits n_4 such that for all $n \ge n_4$

$$P\left(\sum_{0\le s\le t} \big(T_f(X(s-),\Delta X_n(s)) - T_f(X(s-),\Delta X(s))\big)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)) \ge \frac{\epsilon}{6}\right) \le \frac{\epsilon'}{2}.$$

Applying this together with (5.51) to (5.47) shows

$$P\left(\int_0^t \int_{\overline{B}_H(1/m)^c} T_f(X(s-),h)\,\big(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^X(\mathrm{d}s,\mathrm{d}h)\big| \ge \frac{\epsilon}{3}\right) \le \epsilon'. \tag{5.52}$$

By applying Equations (5.45),(5.46) and (5.52) to (5.44), the proof is now complete. □

Proof of Theorem 5.16. Let $(X_n)_{n\in\mathbb{N}}$ be the solutions to (5.7). According to Corollary 5.11, we can assume that $(X_n)_{n\in\mathbb{N}}$ converges both in $\mathcal{C}([0,T],L^p(\Omega,H))$ and uniformly on $[0,T]$ almost surely to the mild solution X, which has càdlàg paths. Since X_n is a strong solution to (5.7) according to Proposition 5.14, the Itô formula in Theorem 4.18 implies for all $t \geq 0$ and $n \in \mathbb{N}$ that

$$\begin{aligned} f(X_n(t)) = f(R_n x_0) &+ \int_0^t \langle G(X_n(s-))^* R_n^* Df(X_n(s-)), \cdot\rangle \,\mathrm{d}L(s) \\ &+ \int_0^t \langle Df(X_n(s)), AX_n(s)\rangle \,\mathrm{d}s + \int_0^t \langle Df(X_n(s)), R_n F(X_n(s))\rangle \,\mathrm{d}s \\ &+ \int_0^t \int_H \Big(f(X_n(s-)+h) - f(X_n(s-)) - \langle Df(X_n(s-)), h\rangle \Big)\, \mu^{X_n}(\mathrm{d}s, \mathrm{d}h). \end{aligned} \tag{5.53}$$

Continuity of f shows that $f(X_n(t)) \to f(X(t))$ and $f(R_n x_0) \to f(x_0)$ a.s. Inequality (4.11) implies for the first integral in (5.53) that

$$\begin{aligned} &E\left[\left\| \int_0^t \langle G(X_n(s-))^* R_n^* Df(X_n(s-)), \cdot\rangle \,\mathrm{d}L(s) - \int_0^t \langle G(X(s-))^* Df(X(s-)), \cdot\rangle \,\mathrm{d}L(s) \right\|\right] \\ &\quad \leq e_{1,\alpha} \left(E\left[\int_0^t \| G(X_n(s))^* R_n^* Df(X_n(s)) - G(X(s))^* Df(X(s)) \|_{L_2(U,\mathbb{R})}^\alpha \,\mathrm{d}s \right]\right)^{1/\alpha}, \end{aligned}$$

which tends to zero by a similar argument as in the proof of Lemma 5.15. It follows in $L_P^1(\Omega,\mathbb{R})$ that

$$\lim_{n\to\infty} \int_0^t \langle G(X_n(s-))^* R_n^* Df(X_n(s-)), \cdot\rangle \,\mathrm{d}L(s) = \int_0^t \langle G(X(s-))^* Df(X(s-)), \cdot\rangle \,\mathrm{d}L(s).$$

Lemma 5.20 implies in $L_P^0(\Omega,\mathbb{R})$ that

$$\lim_{n\to\infty} \int_0^t \int_H \big(f(X_n(s-)+h) - f(X_n(s-)) - \langle Df(X_n(s-)), h\rangle \big)\, \mu^{X_n}(\mathrm{d}s, \mathrm{d}h)$$

$$= \int_0^t \int_H \big(f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle \big)\, \mu^X(\mathrm{d}s, \mathrm{d}h).$$

Almost sure uniform convergence of $(X_n)_{n\in\mathbb{N}}$ and Lebesgue's dominated convergence theorem yields that

$$\lim_{n\to\infty} \int_0^t \langle Df(X_n(s)), R_n F(X_n(s))\rangle \,\mathrm{d}s = \int_0^t \langle Df(X(s)), F(X(s))\rangle \,\mathrm{d}s \quad \text{a.s.}$$

As all terms in (5.53) converge in $L^0_P(\Omega, \mathbb{R})$, it follows that the remaining term

$$\int_0^t \langle Df(X_n(s)), AX_n(s)\,\rangle \mathrm{d}s$$

also converges in $L^0_P(\Omega, \mathbb{R})$, which completes the proof. □

6 Appendix

Proof of Theorem 2.4. We prove the lemma for the case when $s = 0$ and $t = 1$, the general case follows similarly. By [56, Th. 3.9], both limits exist and are independent of the partition $(\pi_n)_{n\in\mathbb{N}}$.[1] Thus, we may choose

$$\pi_n := \left\{0 < \frac{1}{2^n} < ... < (2^n - 1)\frac{1}{2^n} < 1\right\}.$$

Using the notation $\mu \stackrel{\mathcal{D}}{=} (b^\theta, Q, \lambda)$ to denote the cylindrical distribution of $L(1)$, since the partitions are evenly spaced, it follows from [43, Th. 5.7.3/(iii)] that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{\pi_n} E\left[\theta(d_i^n)\right] \\
&= \lim_{n\to\infty} \int_H \theta(h) \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) \\
&= \lim_{\substack{\delta\downarrow 1\\ \delta\in C(\lambda)}} \lim_{n\to\infty} \left(\int_{\|h\|>\delta} \theta(h) \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) + \int_{\|h\|\leq\delta} \theta(h) \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h)\right) \\
&= \lim_{\substack{\delta\downarrow 1\\ \delta\in C(\lambda)}} \lim_{n\to\infty} \int_{\|h\|>\delta} \theta(h) \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) + b^{h\mathbb{1}_{\bar{B}_H}} \\
&= \lim_{\substack{\delta\downarrow 1\\ \delta\in C(\lambda)}} \int_{\|h\|>\delta} \theta(h)\, \lambda(\mathrm{d}h) + b^{h\mathbb{1}_{\bar{B}_H}} \\
&= \int_{\|h\|>1} \theta(h)\, \lambda(\mathrm{d}h) + b^{h\mathbb{1}_{\bar{B}_H}} \\
&= \int_H \left(\theta(h) - h\mathbb{1}_{\bar{B}_H}\right) \lambda(\mathrm{d}h) + b^{h\mathbb{1}_{\bar{B}_H}} = b^\theta.
\end{aligned}$$

[1]For this thesis, it would be enough to prove this lemma under the assumption of equidistant partitions, since all step functions could be defined over rational time partitions. Hence, there is no need to rely on any external results here.

To prove the second assertion, recall that by [43, Pr. 5.7.2]

$$e\left(2^n \mu^{*\frac{1}{2^n}}\right) \Rightarrow \mu \tag{6.1}$$

where $e\left(2^n \mu^{*\frac{1}{2^n}}\right)$ denotes the exponent measure of the finite measure $\left(2^n \mu^{*\frac{1}{2^n}}\right)$. If we express the characteristics of the exponent measure in terms of the truncation function θ, we get

$$e\left(2^n \mu^{*\frac{1}{2^n}}\right) \stackrel{\mathcal{D}}{=} \left(\int_H \theta(h)\ \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h), 0, \left(2^n \mu^{*\frac{1}{2^n}}\right)\right) \quad \text{and} \quad \mu \stackrel{\mathcal{D}}{=} (b^\theta, Q, \lambda). \tag{6.2}$$

From this, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{\pi_n} E\left[\|d_i^n\|^2 \wedge 1\right] \\
=&\lim_{n\to\infty} \int_H \left(\|h\|^2 \wedge 1\right)\ \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) \\
=&\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} \left(\int_{\|h\|>\epsilon} \left(\|h\|^2 \wedge 1\right)\ \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) + \int_{\|h\|\le\epsilon} \left(\|h\|^2 \wedge 1\right)\ \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h)\right).
\end{aligned} \tag{6.3}$$

Splitting (6.3) into the sum of two limits, we first see that

$$\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} \int_{\|h\|>\epsilon} \left(\|h\|^2 \wedge 1\right)\ \left(2^n \mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) = \int_H \left(\|h\|^2 \wedge 1\right)\ \lambda(\mathrm{d}h).$$

It remains only to identify the second limit in (6.3). Without the loss of generality, we may fix an orthonormal basis $(e_i)_{i\in\mathbb{N}} \subseteq H$, and assume that $\epsilon \le 1$. Then, it follows from the Parseval identity, [43, Th. 5.7.3/(ii)] and compactness of the sequence of S-operators corresponding to the sequence of exponential measures $\left(e\left(2^n \mu^{*\frac{1}{2^n}}\right)\right)_{n\in\mathbb{N}}$ that for all

$\delta > 0$ there exists an $N_{\delta,1} \in \mathbb{N}$ such that

$$\sup_{\epsilon\in(0,1]}\sup_{n\in\mathbb{N}}\sum_{i=N_{\delta,1}}^{\infty}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) < \frac{\delta}{2}.$$

Moreover, since $\mathrm{Tr}(Q) < \infty$, for all $\delta > 0$ there exists $N_{\delta,2} \in \mathbb{N}$ such that

$$\sum_{i=N_{\delta,2}}^{\infty}\langle Qe_i,e_i\rangle < \frac{\delta}{2}.$$

Let $\delta > 0$ be fixed. Then, if we define $N_\delta = \max\{N_{\delta,1}, N_{\delta,2}\}$ then we see that

$$\begin{aligned}
&\left|\int_{\|h\|\leq\epsilon}\left(\|h\|^2\wedge 1\right)\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \mathrm{Tr}(Q)\right| \\
\leq&\left|\int_{\|h\|\leq\epsilon}\left(\|h\|^2\wedge 1\right)\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \sum_{i=1}^{N_\delta-1}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h)\right| \\
&\quad+\left|\sum_{i=1}^{N_\delta-1}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \mathrm{Tr}(Q)\right| \\
\leq&\left|\int_{\|h\|\leq\epsilon}\left(\|h\|^2\wedge 1\right)\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \sum_{i=1}^{N_\delta-1}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h)\right| \\
&\quad+\left|\sum_{i=1}^{N_\delta-1}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \sum_{i=1}^{N_\delta-1}\langle Qe_i,e_i\rangle\right| + \left|\sum_{i=1}^{N_\delta-1}\langle Qe_i,e_i\rangle - \mathrm{Tr}(Q)\right| \\
<&\,\delta + \left|\sum_{i=1}^{N_\delta-1}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \sum_{i=1}^{N_\delta-1}\langle Qe_i,e_i\rangle\right|.
\end{aligned}$$

Taking limits on both sides of the inequality above, it follows from Equations (6.1) and (6.2) by [58, VI.Th 5.5]/(3) that

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\left|\sum_{i=1}^{N_\delta-1}\int_{\|h\|\leq\epsilon}\langle h,e_i\rangle^2\left(2^n\mu^{*\frac{1}{2^n}}\right)(\mathrm{d}h) - \sum_{i=1}^{N_\delta-1}\langle Qe_i,e_i\rangle\right| = 0,$$

which immediately implies

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \left| \int_{\|h\| \leq \epsilon} \left(\|h\|^2 \wedge 1 \right) \left(2^n \mu^{*\frac{1}{2^n}} \right) (\mathrm{d}h) - \mathrm{Tr}(Q) \right| \leq \delta.$$

Since $\delta > 0$ is arbitrary, this concludes the proof. $\square$

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