New techniques for integrable spin chains and their application to gauge theories

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NEW TECHNIQUES FOR INTEGRABLE SPIN CHAINS AND THEIR APPLICATION TO GAUGE THEORIES

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Abstract

In this thesis we study integrable systems known as spin chains and their applications to the study of the AdS/CFT duality, and in particular to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in four dimensions.

First, we introduce the necessary tools for the study of integrable periodic spin chains, which are based on algebraic and functional relations. From these tools, we derive in detail a technique that can be used to compute all the observables in these spin chains, known as Functional Separation of Variables. Then, we generalise our methods and results to a class of integrable spin chains with more general boundary conditions, known as open integrable spin chains.

In the second part, we study a cusped Maldacena-Wilson line in $\mathcal{N} = 4$ SYM with insertions of scalar fields at the cusp, in a simplifying limit called the ladders limit. We derive a rigorous duality between this observable and an open integrable spin chain, the open Fishchain. We solve the Baxter TQ relation for the spin chain to obtain the exact spectrum of scaling dimensions of this observable involving cusped Maldacena-Wilson line.

The open Fishchain and the application of Functional Separation of Variables to it form a very promising road for the study of the three-point functions of non-local operators in $\mathcal{N} = 4$ SYM via integrability.

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Introduction

This thesis is dedicated to the description of new techniques developed for the study of integrable systems, and in particular for the study of spin chains appearing in the special type of quantum field theories known as gauge theories.

Integrability is a property of some dynamical systems, known as *integrable systems*. The characteristic defining integrable systems is that they admit solutions for their dynamics that can be obtained through algebraic methods. Integrable systems describe real physical phenomena: thus, in spite of their solvability, their dynamics can still be quite complicated.

Integrable systems can be either classical or quantum, and the precise definition of integrability in these two cases is different. The main subject of this thesis are quantum integrable models, and their applications to the study of Quantum Field Theories. However, we will also briefly discuss classical integrability.

Quantum Field Theories (QFTs) describe a wide variety of real-world phenomena, with the most famous example being undoubtedly the Standard Model: computing observables in QFTs allows us to make predictions about the behaviour of the universe at microscopic scales. Unfortunately, performing such computations is an extremely hard task: in many cases, only perturbative methods can be used, and these are only applicable when the interactions in the QFT are weak.

Given that integrability lets us obtain solutions for theories, finding integrable structures in a QFT can drastically improve our ability to comprehend its dynamics, in particular in the strongly coupled regime.

Until the end of last century, integrability was not found in any 4-dimensional QFT. This fact changed dramatically thanks to the seminal papers [1, 2], where high energy hadron scattering in QCD was linked to a quantum integrable model, a spin chain. A few years later, the authors of [3] found that the perturbative corrections to some observables in maximally supersymmetric Yang-Mills theory ($\mathcal{N} = \mathbf{4}$ **SYM**) also correspond to the energy levels of a spin chain.

$\mathcal{N} = \mathbf{4}$ SYM is a 4-dimensional, supersymmetric, non-abelian QFT possessing conformal symmetry and $SU(N)$ gauge symmetry, which plays a pivotal role in modern Theoretical Physics. The observables of interest in $\mathcal{N} = 4$ SYM are its conformal data, i.e. the

scaling dimensions, or spectrum, Δ of local and non-local operators and their three-point structure constants. If we are able to compute them, the theory is said to be *solved*, as any other correlation function is fixed in terms of the conformal data by conformal symmetry. Solving $\mathcal{N} = 4$ SYM would provide invaluable insights into the properties and behavior of gauge theories, including the Standard Model, and a new way to probe quantum gravity, since $\mathcal{N} = 4$ SYM constitutes the Conformal Field Theory (CFT) side of the AdS_5/CFT_4 duality [4].

The results of [3] sparked a renewed interest in the study of $\mathcal{N} = 4$ SYM, exploiting the **integrable spin chains** appearing in it, see [5] for a review. These spin chains are often quite special, and many known methods for quantum integrability, such as the Bethe Ansatz, are not always adept for their study[1]. The existence of these technical challenges served as a catalyst for the development of new techniques for the analysis of quantum integrable models.

One extremely successful example of this story is the Quantum Spectral Curve (QSC), a method based on integrability that has been developed to solve the spectral problem for $\mathcal{N} = 4$ SYM in the large N (planar) limit[2]. The QSC lets us compute the scaling dimension of every local operator with extremely high numerical precision at any coupling [14]. Computing the three-point structure constants has proven to be a harder problem. Many new techniques have been developed in order to tackle it [15–22], but to this day it remains unsolved.

While local operators in $\mathcal{N} = 4$ SYM have received a lot of attention, the application of integrability to QFTs is not limited to them. For example, integrable spin chains also describe non-local operators in $\mathcal{N} = 4$ SYM - such as supersymmetric Wilson-Maldacena loops [23]. They also have been found in other high-dimensional QFTs, with one example being the Fishnet theory [24], an important toy model for holographic dualities and integrability.

These results underline the power of integrability - it lets us compute many observables in CFTs, to a level of precision never seen before. In particular, we hope that integrability, and the techniques that come from its study, can lead us to obtain the first complete solution of a non-abelian gauge theory in the planar limit.

As we have already mentioned, the calculation of three-point structure constants is the missing step to do in the road to the solution of $\mathcal{N} = 4$ SYM via integrability. We can identify two open questions, which are critical for making progress in this research program:

- Can we rigorously build a *non-perturbative* duality between a spin chain model and $\mathcal{N} = 4$ SYM?
- Once we have established such duality, can we build a general method that lets us compute all observables in this integrable spin chain, including those that are dual to

[1] This is especially true if we want to do higher loops or non-perturbative calculations.
[2] The Quantum Spectral Curve has also been extended to other theories, see for example [6–13].

the three-point structure constants of $\mathcal{N} = 4$ SYM?

The main goal of this thesis is to expose some recent progress towards the solution of these two problems. We aim to do so in a self-contained manner, assuming no prior knowledge of integrability and spin chains.

A promising method to solve the second problem is **Functional Separation of Variables** (FSoV). This framework can be applied to a wide range of spin chains, and in particular can already be used to compute some three-point functions in $\mathcal{N} = 4$ SYM [21, 25]. We will explore recent progress in this approach, and prove that using FSoV we can compute a complete set of observables for a large class of periodic integrable spin chains, even in those cases found in $\mathcal{N} = 4$ SYM. These are given in terms of the same building blocks of the QSC, the Baxter Q-functions, a fact that makes the applicability of this method to CFTs much more natural. From these observables, it is in principle possible to compute any physical quantity in the spin chain. Furthermore, we expand the range of applicability of FSoV to spin chains with *open boundary conditions*, which are dual to non-local operators in CFTs.

To make progress with the first problem, we develop the **open Fishchain**. This is an integrable spin chain with open boundary conditions, dual to non-local operators in $\mathcal{N} = 4$ SYM at *all loops* in a certain limit. This model represents a clear example of a non-perturbative duality between a spin chain and a subsector of $\mathcal{N} = 4$ SYM. We will see how the knowledge of this model lets us compute the scaling dimensions of operators in this subsector at any coupling. We finally briefly describe how the open Fishchain can provide a playground for the application of FSoV in the context of holographic CFTs.

Contents

This thesis is divided in two parts, each detailing one of the two topics we have described above:

- **Part 1** is dedicated to the study of integrable systems, and in particular integrable spin chains, from a mathematical point of view. The main goal of this part consists in showing how Functional Separation of Variables (FSoV) can be used to compute observables in these systems. In order to do so, we first need to introduce the key concepts on which FSoV is based. In particular, we will discuss the Yangian[3], a special symmetry group, and see how a certain class of integrable spin chains are naturally based on the representation theory of this object. Then, we will introduce the T-system and Q-system, which are the two fundamental functional descriptions of a spin chain, based on the Yangian. Finally, we will explore the Separation of Variables program, and introduce the author's original work [26] on FSoV. In particular, we show that Functional SoV can be used to obtain a complete set of observables for

[3] We will actually limit our discussion to a certain subset of Yangians, those related to the Lie algebra gl_N.

integrable spin chains built from the Yangian. We conclude this part by presenting the author's work in preparation [27] about the study of integrable spin chains with open boundary conditions.

- **Part 2** is dedicated to the development of connections between spin chains and High Energy Physics, and to the applications of the techniques discussed in the first part in this context. Basing on the author's original work [28], we will build the open Fishchain model, and rigorously prove its duality to non-local operators in $\mathcal{N} = 4$ SYM. We will implement a method based on the Quantum Spectral Curve to compute quantities in the open Fishchain that describe the scaling dimension of Wilson-Maldacena lines in $\mathcal{N} = 4$ SYM. We will then briefly talk about the possible application of FSoV to this model.

Part 1 - Integrability and Spin Chains

Chapter 1: Classical Integrability The defining feature of classical integrable systems is the existence of a large number of symmetries and conserved charges, which can be used to obtain exact solutions for their motion. This section is not intended to give a complete treatment of classical integrability. Rather, we will describe two special techniques, whose quantum version is at the core of our analysis of integrable spin chains. The first one is the *Lax representation*. This method, applicable only to some classes of classical integrable systems, lets us compute all their conserved charges. The second is the *Separation of Variables*, and can be used in any integrable system. It consists in the construction of a set of special coordinates, called separated, or action-angle, variables. In these coordinates, the equations of motion become exactly solvable.

Chapter 2: Quantum Integrability Quantum integrable systems are systems defined by the presence of factorised scattering. This means that any scattering process can be reduced to a series of $2 \to 2$ particle scatterings. Factorised scattering is naturally associated to symmetries known as quantum groups, which are a special class of non-commutative algebras. Quantum groups are defined in terms of a universal R-matrix satisfying the Yang-Baxter equation, a fundamental relation which imposes factorised scattering on the underlying integrable system. The quantum group we will analyse is the *Yangian*, which arises as the quantisation of a classical Lie algebra. We will specialise our treatment to the gl_N algebra, and introduce the Yangian $Y(gl_N)$ in terms of a set of generators satisfying the set of equations known as the RTT relations. Finally, we will briefly talk about the representation theory of $Y(gl_N)$.

Chapter 3: Integrable Spin Chains The quantum integrable spin chains we will analyse are periodic, one-dimensional, discrete models which can be defined as a representation of the Yangian $Y(gl_N)$. In this section, we will use the generators of $Y(gl_N)$ as the starting point to compute observables in these systems. First, we will introduce the Algebraic Bethe

Ansatz, a technique based on the *RTT* relations which lets us compute the eigenstates of integrable spin chains. We will then introduce more modern approaches for the study of spin chains, the *Q-system* and the *T-system*. These are based respectively on the Baxter Q-functions and the quantum Lax operators of the spin chain. The interplay between these two systems gives rise to the Baxter TQ equation, known for its use in the Quantum Spectral Curve formalism and which constitutes the key equation for FSoV.

Chapter 4: Sklyanin's Separation of Variables The Separation of Variables (SoV) program, initiated by Sklyanin [29], is the quantum version of the classical separated variables. We will show how we can use the T-system to build the so-called SoV basis for the Hilbert space of a certain class of integrable spin chains. The SoV basis is a special basis in which the spin chain states have a simple, separated form entirely given in terms of their Baxter Q-functions. We will then briefly mention how to compute some physical quantities, such as overlaps of states of the spin chain, using the SoV basis.

Chapter 5: Functional Separation of Variables The Functional Separation of Variables (FSoV) is a more direct and more powerful approach to Separation of Variables for spin chains. FSoV lets us compute observables in terms of Q-functions without having to know explicitly the SoV basis. This is a great advantage over Sklyanin's SoV, since the SoV basis cannot be easily built for a wide class of spin chains, including the ones appearing in gauge theories. We will prove that using FSoV we can compute a complete set of observables for integrable spin chains built from the Yangian $Y(gl_N)$. Furthermore, we will show that FSoV is equivalent to other SoV constructions, in the cases where the latter can be used, and can be used to reconstruct the SoV basis.

Chapter 6: Open Integrable Spin Chains Open integrable spin chains are a class of models obtained by adding special boundaries to the spin chains we have analysed so far. These boundaries do not fully break the Yangian symmetry - they preserve a subgroup known as Twisted Yangian. We will describe how integrability techniques developed for periodic spin chains can be adapted to this new setting, and in particular detail some initial progress in the FSoV program for open spin chains.

Part 2 - Spin Chains in Gauge Theories

Chapter 6: Conformal Field Theory and $\mathcal{N} = 4$ SYM In this section we briefly review some basics of Conformal Field Theories, explaining why a CFT is completely described by its spectrum and three-point structure constants. We will also review maximally supersymmetric Yang-Mills theory in 4 dimensions, providing its Lagrangian and describing how the spectrum of its $SU(2)$ subsector can be described by a spin chain.

Chapter 7: Cusps in $\mathcal{N} = 4$ SYM and the Open Fishchain In this section, we will prove that cusped Maldacena-Wilson lines in a double scaling limit of planar $\mathcal{N} = 4$ SYM are fully dual to an open spin chain, known as the open Fishchain. We will see how using the Baxter TQ equation for the open Fishchain we can fully compute the scaling dimension of a class of operators involving Maldacena-Wilson lines, at any coupling and with high numerical precision. Finally, we will mention how FSoV can be applied to this setting.

Conclusions

We will conclude by presenting other open questions and possible lines for future work.

Part I

Integrability and Spin chains

Chapter 1

Classical Integrability

Classical Integrability - also known as Liouville integrability - is a property of some classical systems that allows us to find exact solutions for their motion. In particular, their time evolution can be fully determined by solving a finite number of algebraic equations and by computing a finite number of integrals.

Liouville integrable systems form a large class of well-known solvable models. Some examples include the Kepler problem, the multi-dimensional harmonic oscillator, the Korteweg-de Vries equation, and so on. In this thesis, we will treat a classical integrable system in chapter 8, the classical open Fishchain.

Roughly speaking, a classical system with N degrees of freedom is *Liouville integrable* if it admits N independent conserved charges Q_i; one can then treat the N independent conservation laws $\dot{Q}_i = 0$ as the equations of motion to solve for the N degrees of freedom.

Liouville Integrability can be rigorously defined in the context of Hamiltonian systems, which are dynamical systems admitting a description via a Hamiltonian function on a phase space. After introducing the main definitions of Liouville integrability for Hamiltonian systems, we will describe the Separation of Variables and the Lax representation. These two techniques provide a direct way of solving some Liouville integrable systems. Furthermore, their quantum analog will be at the core of our study of integrable spin chains.

This chapter is mostly based on the book by Arutyunov [30], while the discussion on action-angle variables is taken from [31].

1.1 Hamiltonian systems

A Hamiltonian system is a dynamical system defined on a $2n$-dimensional phase space $\mathcal{H}$, equipped with:

- Canonical coordinates $(q_i, p_i),\ i = 1 \dots n$;
- A function $H : \mathcal{H} \to \mathbb{R}$ called Hamiltonian;

- A Poisson bracket $\{\,,\,\} : \mathcal{F}(\mathcal{H}) \times \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$, where $\mathcal{F}(\mathcal{H})$ is the space of smooth functions on the phase space.

The Poisson bracket has the structure of an (infinite dimensional) Lie algebra, and thus is determined on all the phase space $\mathcal{H}$ if we define its action on the canonical coordinates $(q_i, p_i),\ i = 1 \dots n$. This action is given by:

$$\{q_i, p_i\} = \delta_{ij}, \qquad \{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0\,. \tag{1.1.1}$$

It is easy to check that the *canonical* Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \tag{1.1.2}$$

satisfies all the properties required above.

The motion of a dynamical system is described by a trajectory on the phase space $(\vec{q}(t), \vec{p}(t))$; such trajectory can be determined (given initial conditions $(\vec{q}_0, \vec{p}_0)$) by solving the Hamilton's equations of motion:

$$\dot{q}_i = \{H, p_i\}, \qquad \dot{p}_i = \{H, q_i\}\,. \tag{1.1.3}$$

Therefore, the dynamics of a Hamiltonian systems are embedded in its Hamiltonian and its Poisson bracket. More generally, the time evolution of any function f on the phase space is determined by the Poisson bracket and the Hamiltonian:

$$\dot{f} = \{H, f\}\,. \tag{1.1.4}$$

The coordinates (q_i, p_i) are not the only ones that have Poisson brackets of form (1.1.1). In fact, one can apply a family of transformations on (q_i, p_i) such that the structure (1.1.1) is preserved. In particular, we define a *canonical transformation* as a change of variables on the phase space:

$$q_i \to q_i'(q_j, p_j), \qquad p_i \to p_i'(q_j, p_j)\,, \tag{1.1.5}$$

that does not change the form of the Poisson bracket (1.1.1).

1.2 Liouville integrability

If a Hamiltonian system with $\dim(\mathcal{H}) = 2n$ has n independent functions $f_i \in \mathcal{F}(\mathcal{H})$ (in the sense that the one-forms df_i, $i = 1 \dots n$ are linearly independent in each local coordinate patch of $\mathcal{H}$) such that:

$$\{f_i, f_j\} = 0,\ \forall i, j = 1 \dots n, \quad \text{and } \exists k \text{ such that } H = f_k\,, \tag{1.2.1}$$

then the Hamiltonian system is said to be *Liouville integrable*.

The functions f_i are said to be in involution with respect to the Poisson bracket $\{\,,\,\}$. Since they commute with the Hamiltonian, these functions are conserved in time, and we will refer to them as conserved charges or *Integrals of Motion* (IoMs).

The Arnold-Liouville theorem, whose formal statement and proof can be found in [30], ensures that any Liouville integrable system can be solved via quadratures, i.e. by solving algebraic equations and computing a finite number of one-variable integrals. In particular, it states that the motion of a Liouville integrable system can be described using the coordinates (f_i, ϕ_i), $i = 1 \dots n$, for which the equations of motion are linear in time. These coordinates however are in general not canonical.

In a Liouville integrable system it is always possible to build another set of canonical coordinates, called *action-angle variables*, in which Hamilton's equations of motion split into a set of $2n$ ordinary differential equations. This is why action-angles variables are also called *Separated Variables*.

1.2.1 Action-angle variables

One of the statements of the Arnold-Liouville theorem [30] is that we can foliate $\mathcal{H}$ via a set of surfaces, each isomorphic to an n-dimensional torus, where the conserved charges f_i take the constant values c_i.

To simplify our treatment, we will use the following shorthand notation: $\vec{a} \equiv (a_1 \dots a_n)$.

On each torus, we can invert the equations $f_i(\vec{p}, \vec{q}) = c_i$ to obtain $p_j = p_j(\vec{c}, \vec{q})$. The action variables are then defined by:

$$I_j(\vec{c}) = \frac{1}{2\pi} \oint_{\gamma_j} \sum_{i=1}^{n} p_i(\vec{q}, \vec{c}) dq_i \tag{1.2.2}$$

where γ_j is the j-th cycle of the n-torus. Given that I_j are functions only of the constants c_i, the action variables are clearly time-independent, hence their Hamilton's equation are:

$$\dot{I}_j = 0\,. \tag{1.2.3}$$

We may then define the angle variables θ_i by requiring that the transformation $(q_i, p_i) \to (\theta_i, I_i)$ is canonical, i.e. it preserves the canonical Poisson bracket (1.1.1).

Doing so results in the following definition of angle variables:

$$\theta_j = \frac{\partial S}{\partial I_j}, \quad \text{where } S(\vec{I}, \vec{q}) \equiv \int_{q_0}^{q} p_i(q', I) dq_i'\,. \tag{1.2.4}$$

An important feature of action-angle variables is that the Hamiltonian in these coordinates only depends on I_j variables. In fact, we have that:

$$\dot{I}_j = \{H, I_j\} = 0 \to \frac{\partial H}{\partial \theta_j} = 0 \tag{1.2.5}$$

Therefore, $H = H(\vec{I})$. The dynamics of the angle variables are therefore given by:

$$\dot{\theta}_j = \{H, \theta_j\} = \frac{\partial H}{\partial I_j} \equiv \omega_j(\vec{I}) \,, \tag{1.2.6}$$

where ω_j are constants since they only depend on the constants $\vec{I}$.

The equations of motion in action-angles variables are evidently separated and can be solved by direct integration, yielding:

$$I_j(t) = I_j^0 \,, \qquad \theta_j(t) = \omega_j(\vec{I}^0)t + \theta_j^0 \,, \tag{1.2.7}$$

where I_j^0 and θ_j^0 are integration constants which are determined from the initial conditions.

Clearly this procedure can be done in any Liouville integrable system, i.e. whenever we have as many integrals of motion c_i as half of the dimension of the phase space. Since the construction of $(\vec{I}, \vec{\theta})$ only involves algebraic operations and integrals, and solving the resulting equations of motion can be done by direct integrations in one variable, it follows that any Liouville integrable system can be solved via quadratures[1].

1.2.2 Lax representation

Due to the existence of the action-angle variables, classical integrable systems can always be solved via quadratures. However, in order to establish Liouville integrability and build action-angle variables, we need to know all the conserved charges of the integrable system. Their construction is in general a nontrivial problem.

In this section, we will introduce the so-called Lax representation for integrable systems. If a Hamiltonian system admits a Lax representation, it is a Liouville integrable system, and we can automatically build all its conserved charges.

A Hamiltonian system is said to possess a *Lax representation* if it is possible to recast its equations of motion as the Lax equation:

$$\dot{L} = [M, L] \,, \tag{1.2.8}$$

where L, M are two square matrices called respectively the Lax matrix and the auxiliary matrix, and $[\,.\,,\,.\,]$ is the usual commutator between matrices. (L, M) are known as a Lax pair for the Hamiltonian system.

The dimension of the matrices L, M is not fixed a priori, and in general the Lax Pair for an integrable system is not unique. Furthermore, even a given Lax pair possesses gauge freedom: the $d-$dimensional Lax pair (L, M) is equivalent to the Lax pair (L', M') where:

$$L' = fLf^{-1}, \qquad M' = fMf^{-1} + \dot{f}f^{-1}, \qquad f = \text{any } d \times d \text{ invertible matrix} \,. \tag{1.2.9}$$

[1]The construction of the action-angle variables is quite hard to do explicitly, especially for high dimensional systems. A few detailed examples of it can be found in [31].

It is always possible to build d independent conserved charges from the Lax matrix. This can be done by taking the trace of products of the Lax matrix:

$$I_k = \operatorname{tr} L^k, \ k = 1 \dots d. \tag{1.2.10}$$

I_k are (power sum) symmetric polynomials in the eigenvalues of L, and the number of independent symmetric polynomials is equal to the dimension of the Lax Matrix $\dim(L) = d$. Hence we can only build up to d independent conserved charges from a d-dimensional Lax matrix.

Proving that I_k are conserved is easy, in fact:

$$\dot{I}_k = k \operatorname{tr}(L^{k-1}\dot{L}) = k \operatorname{tr}(L^{k-1}[M, L]) = \operatorname{tr}\left[M, L^k\right] = 0 \tag{1.2.11}$$

where we have used (1.2.8) in the first passage.

Of course, the conserved charges I_k of an integrable model need also to be in Poisson involution, i.e. $\{I_i, I_j\} = 0$. Proving that this is the case for the charges (1.2.10) requires an ulterior condition on the Lax operators, given by the Babelon-Viallet theorem.

This theorem states that the eigenvalues of the Lax matrix are in involution if and only if there exist a tensor r_{12} over $\mathrm{Mat}_d\mathbb{C} \otimes \mathrm{Mat}_d\mathbb{C}$ with entries on $\mathcal{H}$ such that:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2] \tag{1.2.12}$$

where $L_1 \equiv L \otimes 1_d$ and $L_2 \equiv 1_d \otimes L$.

The tensor r_{ij} is known as the *classical R-matrix*. It also satisfies the Jacobi identity:

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{ cyclic perm.} = 0 \tag{1.2.13}$$

which becomes, in the case where the entries of r are constants in the phase space (i.e. they Poisson commute with the Lax matrices):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0 \tag{1.2.14}$$

which is also known as the *Classical Yang-Baxter Equation.*

As we have discussed, from the Lax representation one can only build a total of $d = \dim L$ independent integrals of motion. Thus, it seems that we might be forced to look for very big Lax matrices when dealing with high-dimensional integrable systems.

We can circumvent this limitation via the introduction of a *spectral parameter* $u \in \mathbb{C}$ in the Lax representation. A Hamiltonian system admits a Lax representation with a spectral parameter if its equations of motion are equivalent to:

$$\dot{L}(u) = [M(u), L(u)] \ , \tag{1.2.15}$$

where now both the Lax matrix and the auxiliary matrix are functions of the spectral parameter; in general, we may assume that they are polynomials[2] in u. The conserved quantities are given by:

$$I_k(u) = \mathrm{tr} L^k(u) \, . \tag{1.2.16}$$

[2]Depending on the notation, they can also be assumed to be rational functions of u.

These are also polynomials in u, and we can think of them as generating functions for the conserved charges of the model. This implies that the conserved charges are coefficients of powers of u in I_k[3]. Given that in principle we can set the degree in u of L to any number, we can generate as many integrals of motion as needed, independently from the dimension of the Lax Matrix.

We will use the Lax representation with spectral parameter in chapter 8 to build the classical open Fishchain, starting from its equations of motion. The quantum version of the Lax representation will play a fundamental role throughout the rest of this work, and will constitute one of the main tools that we use to study integrable spin chains.

[3]If we take I_k to be rational functions of u, the conserved charges will be defined as the coefficients of their Laurent expansion around $u = 0$.

Chapter 2

Quantum Integrability

In the previous chapter, we have defined the concept of classical integrability in terms of the existence of a certain number of independent conserved charges in an Hamiltonian system.

Quantum models can be constructed from classical Hamiltonian system via a procedure known as quantisation. A quantisation is a map between classical and quantum observables, defined in the following way:

- classical observables consists of the functions f on the phase space $\mathcal{H}$. These functions are commutative: $fg = gf,\ \forall g, f$;
- quantum observables are the set of Hermitian operators[1] $\mathbb{O}$ acting on the Hilbert space $\mathbb{H}$. These operators are not commutative: $AB - BA \neq 0$ for generic $A, B \in \mathbb{H}$.

A quantisation procedure consists in finding a one-to-one map $Q_\hbar$ from classical to quantum observables, depending on a parameter $\hbar$, with the requirements that $\hbar \to 0$ is a classical limit [30]:

$$\lim_{\hbar\to 0} \frac{1}{2} Q_\hbar^{-1}(Q_\hbar(f)Q_\hbar(g) + Q_\hbar(g)Q_\hbar(f)) = fg \tag{2.0.1}$$

$$\lim_{\hbar\to 0} Q_\hbar^{-1}\left(\frac{i}{\hbar}\left(Q_\hbar(f)Q_\hbar(g) - Q_\hbar(g)Q_\hbar(f)\right)\right) = \{f, g\} \tag{2.0.2}$$

This implies that as $\hbar \to 0$ the quantum system reduces to a classical Hamiltonian system, and in particular the quantum commutators $\frac{i}{\hbar}[A, B]$ reduce to classical Poisson brackets.

The most used quantisation map is the *canonical quantisation*, which maps canonical phase space variables (q_i, p_i) (having canonical Poisson brackets (1.1.1)) to Hilbert space operators $(\widehat{q}_i, \widehat{p}_i)$ satisfying the Heisenberg commutation relations:

$$[\widehat{q}_i, \widehat{q}_j] = 0, \quad [\widehat{p}_i, \widehat{p}_j] = 0, \quad [\widehat{p}_i, \widehat{q}_j] = -i\hbar\delta_{ij} \tag{2.0.3}$$

[1] In some cases, observables may be not Hermitian, see for example [32]. Another such example is the fishnet theory [24], where the dilatation operator has complex eigenvalues [33].

In addition to this quantisation map, one needs to choose a prescription for the ordering of non-commutative operators whenever we quantise products of classical observables.

Using a quantisation map, we can obtain quantum integrable models from Liouville integrable systems, provided that there are no anomalies. In particular, the conserved charges of the latter are functions on a phase space, and the quantisation map $Q_\hbar$ will give their quantum version as operators on a Hilbert space. We will see an example of this procedure with the Fishchain model in chapter 8.

However, the general definition of quantum integrability is non-trivial. We might in fact be tempted to define quantum integrability using the same requirements as the classical case, i.e. the presence of a sufficient number of mutually commuting conserved charges, which are a set of Hilbert space operators commuting with the quantum Hamiltonian.

The problem with this definition lies in the fact that any quantum system has an infinite number of independent conserved quantities, given by the Hermitian projectors on the eigenstates of the quantum Hamiltonian.

Formally, if $P_j e_i = \delta_{ij} e_j$, where e_j are eigenstates of H, i.e. $H e_j = h_j e_j$, then:

$$[P_j, H]\, e_i = (P_j h_i - H\delta_{ij}) e_i = 0,\ \forall i,j \quad \rightarrow \quad [P_j, H] = 0,\ \forall j \tag{2.0.4}$$

$$[P_i, P_j]\, e_k = (\delta_{ij}\delta_{jk} - \delta_{ji}\delta_{ik}) e_i = 0 \quad \rightarrow \quad [P_i, P_j] = 0,\ \forall i,j \tag{2.0.5}$$

Therefore any quantum system has many conserved charges in involution, and this phenomenon is not a sign of quantum integrability.

Quantum integrability is in fact characterised by a different feature: *factorised scattering*, the property that any multiparticle process in a infinite-volume quantum integrable theory can be always decomposed into a series of $2 \rightarrow 2$ scattering processes. This property is encoded in the so-called Yang-Baxter equation, which is in turn a manifestation of the symmetries encoded in a quantum group. In fact, the existence of at least two higher-spin conserved charges is needed for factorised scattering [34]. The quantum group of interest for the type of systems analysed in this thesis is known as Yangian.

In this chapter, we will define and study the main properties of quantum groups and Yangians. In particular, we will define them in terms of their generators t_{ij} and an R-matrix which imposes restrictions on them via the RTT relations.

The discussion of this chapter is based on [30, 35].

2.1 Hopf algebras and quantum groups

The algebras defined by the quantum operators on a Hilbert space are non-commutative[2]. Therefore, non-commutative algebras feature in any quantum theory.

Non-commutative algebras play an even bigger role in quantum integrable systems. In fact, quantum integrability can be traced to the presence of symmetries, which are deformations of Hopf algebras, a special type of non-commutative algebra.

[2] Although they are designed to reduce to a commutative algebra in the classical limit.

Hopf Algebras

A Hopf algebra is an associative algebra $\mathcal{A}$, possessing a unit $\mathbb{1}$ and a non-commutative product $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, with extra structures imposed on it. In particular, it possesses:

- a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \cdot \mathcal{A}$;
- a counit $\epsilon : \mathcal{A} \to \mathbb{C}$, which associates a complex number to each element of the algebra.
- an antipode $S : \mathcal{A} \to \mathcal{A}$, which is an algebra anti-homomorphism.

These operations must be compatible with each other, and this fact imposes a series of relations on them. They will not be used in this thesis, and the interested reader can find them in [30].

A well known example of a Hopf algebra is the Universal Enveloping Algebra (UEA) of a Lie algebra g, denoted as $U(g)$. A UEA is an associative algebra generated by the elements:

$$\{x_i\}_{i=1}^{dim(g)} \text{ subject to the relations } x_i x_j - x_j x_i = c_{ijk} x_k, \tag{2.1.1}$$

where c_{ijk} are the structure constants of the Lie algebra g, and x_i can be thought of as the generators of g. Thus, a UEA is constituted by the polynomials in the generators of g, modulo the commutator between its elements[3].

For a UAE, the Hopf algebra structure is the following: the product and the unit are the usual product and unit of the Lie algebra g, while the coproduct, the counit and the antipode are defined as:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad S(x) = -s, \qquad \epsilon(x) = 0. \tag{2.1.2}$$

Quantum groups

The special symmetries that define quantum integrable systems are deformations of Universal Enveloping Algebras by a parameter $\hbar$. Such symmetries reduce to the UAE in the classical limit $\hbar \to 0$, and are known as quantum groups.

The definition of quantum group we will use is based on the existence of an Universal R-matrix, $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, that satisfies the Yang-Baxter Equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,. \tag{2.1.3}$$

Here we have introduced a notation that we will use in the remainder of this thesis: $\mathcal{R}_{ij},\ i,j \in \{1,2,3\}$ is an operator defined on the triple tensor product $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, which

[3] In particular, the Poincare-Birkhoff-Witt theorem states that a basis of $U(g)$ is composed of all such polynomials.

acts trivially on the factor corresponding to the missing number in the set $\{1, 2, 3\}$. For example, we have that:

$$\mathcal{R}_{12} = \mathcal{R} \otimes 1, \qquad \mathcal{R}_{23} = 1 \otimes \mathcal{R}\,. \tag{2.1.4}$$

Using the R-matrix we have introduced, we can define a quantum group as an algebra over $\mathbb{C}$ generated by polynomials in the generators $t_{ij},\ i, j = 1 \dots N$ modulo the relations:

$$R_{12}T_1T_2 = T_2T_1R_{12}\,, \tag{2.1.5}$$

where T is a matrix whose entries are $(T)^i_j = t_{ij}$, and R is the R-matrix, an invertible complex matrix which is a realisation of the universal R-matrix defined in (2.1.3) acting on $\mathcal{A} = \mathbb{C}^n$. The equations (2.1.5) are known as the RTT relations.

The requirement that in the classical limit $\hbar \to 0$ the quantum group reduces to a Universal Enveloping Algebra can be imposed by assuming that $R = 1 + \hbar r + O(\hbar^2)$, where r is known as the classical r-matrix and satisfies the classical YBE (1.2.14).

Quantum affine groups By introducing a spectral parameter $u \in \mathbb{C}$ to a quantum group, we obtain a quantum affine group (QAG) [30]. In a QAG, the R-matrix will be a function of u and the generators T will be a formal Laurent series in u, $T(u) = \sum_{m\in\mathbb{Z}} T_m u^m$. The Yang-Baxter equation for a quantum affine group will depend on 3 complex parameters u, v, w and reads:

$$R_{12}(u, v)R_{13}(u, w)R_{23}(v, w) = R_{23}(v, w)R_{13}(u, w)R_{12}(u, v) \tag{2.1.6}$$

Furthermore, R will satisfy the unitarity condition $R_{12}(u, v)R_{21}(v, u) = \alpha(u)$, where $\alpha(u)$ is a scalar function of u.

To conclude, a quantum affine group is generated by the coefficients of $T(u)$ modulo the relations:

$$R_{12}(u, v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u, v)\,, \tag{2.1.7}$$

which are still known as the RTT relations.

2.2 Yangians

The Yangian is the quantum (affine) group that is the symmetry of the so-called rational integrable spin chain models, that are the main subject of this thesis. The Yangian $Y(g)$ is based on the Lie algebra g; in this work, we will only consider the case where $g = gl_N$ for simplicity.

There are three equivalent ways in which the Yangian can be defined, called Drinfeld's first, second and third realisation [36, 37].

The first realisation defines the Yangian in terms of commutation relations between the generators of g, $J_i,\ i = 1 \dots \dim(g)$ and some further generators $\hat{J}_i$. In particular one has:

$$[J_i, J_j] = c_{ijk}J_k\,, \tag{2.2.1}$$

$$\left[J_i, \hat{J}_j\right] = c_{ijk}\hat{J}_k\,, \tag{2.2.2}$$

where c_{ijk} are the structure constants of g and $\hat{J}_i = uJ_i + O(\hbar)$. This set of generators satisfies all the properties of a quantum group, as detailed in [35].

The second realisation is given in terms of Serre-Chevalley relations between a different set of generators of the Yangian, and we will not discuss it in this thesis.

Finally, the third realisation, also known as RTT formulation, is the most important for our scopes. It is very close in spirit to the definition of quantum groups we have introduced in the previous chapter, and we will discuss it in detail in the next section.

2.2.1 Drinfeld's third realisation of the Yangian

Drinfeld's third realisation of the Yangian is based on the definition of the rational R-matrix, a complex matrix which is one of the possible solutions of the Yang-Baxter equation (2.1.3).

Rational R-matrix The rational R-matrix $R \in Mat_{N^2}\mathbb{C}$, acting on the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$, is:

$$R(u,v) = (u-v)1 + \hbar P\,, \tag{2.2.3}$$

where 1 is the identity operator and P is the permutation operator on $\mathbb{C}^N \otimes \mathbb{C}^N$:

$$1(x \otimes y) = x \otimes y, \qquad P(x \otimes y) = y \otimes x, \quad \forall x, y \in \mathbb{C}^N\,. \tag{2.2.4}$$

The rational R-matrix is said to be of difference form, since it only depends on u and v via the combination $u - v$. It is also invariant under $GL(N)$ group transformations:

$$[R(u,v), G \otimes G] = 0, \quad \forall G \in GL(N). \tag{2.2.5}$$

Yangian $Y(gl_N)$ the Yangian $Y(gl_N)$ is the associative algebra over $\mathbb{C}$ with generators $T_{ij}^{(1)}, T_{ij}^{(2)} \ldots$, $i, j = 1 \ldots N$ satisfying the following defining relations:

$$\left[T_{ij}^{(r+1)}, T_{kl}^{(s)}\right] - \left[T_{ij}^{(r)}, T_{kl}^{(s+1)}\right] = \hbar(T_{kj}^{(r)} T_{il}^{(s)} - T_{kj}^{(s)} T_{il}^{(r)})\,, \tag{2.2.6}$$

valid for $r, s = 0, 1, 2 \ldots$ if we set $T_{ij}^{(0)} = \delta_{ij}$.

These defining relations can be written in a compact form using the rational R-matrix. By introducing the generating series:

$$T_{ij} = \sum_{n=0} u^n T_{ij}^{(n)}, \tag{2.2.7}$$

we may write (2.2.6) as:

$$(u-v)\,[T_{ij}(u), T_{kl}(v)] = \hbar(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)) \tag{2.2.8}$$

Equivalently, we can define the *monodromy matrix*:

$$T(u) \equiv \sum_{i,j=1}^{N} e_{ij} \otimes T_{ij}(u)\,, \tag{2.2.9}$$

where e_{ij} are the elements of the standard basis of $N \times N$ matrices; in this formula, and for the rest of this thesis, the first factor in the tensor product is called *auxiliary space*, and corresponds to the space of $N \times N$ complex matrices. The second factor is called *quantum or physical space*, and is the representation space of the Yangian generators T_{ij}.

Using the rational R-matrix, we may write (2.2.6) as an equation on the tensor product $Aux_1 \otimes Aux_2 \otimes Phys$:

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v) \tag{2.2.10}$$

where:

$$T_1(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes T_{ij}(u), \quad T_2(u) = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes T_{ij}(u). \tag{2.2.11}$$

Following our usual notation, the subscripts on T indicate that the monodromy matrix acts non-trivially only on the i-th copy of the auxiliary space.

Due to the form of the defining relations (2.2.10), Drinfeld's third realisation of the Yangian is also known as RTT realisation.

Since a Yangian is a quantum group, it is also a Hopf algebra. Therefore it possesses a coproduct, counit and antipode defined as:

$$\Delta(T_{ij}(u)) = \sum_{k=1}^{N} T_{ik}(u) \otimes T_{kj}(u), \qquad \epsilon(T(u)) = 1, \qquad S(T(u)) = T^{-1}(u)\,, \tag{2.2.12}$$

where $T^{-1}(u)$ is the matrix inverse of the monodromy matrix.

2.2.2 Representations of the Yangian

The integrable systems we will analyse in this thesis arise as representations of the Yangian $Y(gl_N)$. Hence, knowing the representation theory of the Yangian is crucial to study them. This is quite similar to the representation theory of the classical Lie algebra gl_N.

It is possible to classify all highest weight representations of $Y(gl_N)$ using the generators $T_{ij}(u)$. In particular, a irreducible representation of $Y(gl_N)$ on a vector space V is called *highest weight* if exist $|0\rangle \in V$ such that:

$$T_{ij}(u)|0\rangle = 0, \ \forall i > j\,, \tag{2.2.13}$$

$$T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle, \ \forall i = 1 \dots N\,. \tag{2.2.14}$$

Thus $T_{ij}(u)$, $i > j$ are the Yangian analog of raising operators, while $T_{ii}(u)$ are the analog of a Cartan subalgebra. The weights of the Yangian representation are given by $\lambda_i(u)$, which are polynomials in u.

Just like simple Lie algebras, all finite dimensional, irreducible representations of $Y(gl_N)$ are of highest weight type.

Finite-dimensional, highest weight representations of $Y(gl_N)$ can be classified uniquely by their Drinfeld polynomials $P_i(u)$ [38], which are monic polynomials in u satisfying:

$$\frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+\hbar)}{P_i(u)}, \quad \forall i = 1 \dots N\,. \tag{2.2.15}$$

Drinfeld polynomials play a similar role to the weights of the Lie algebra gl_N.

For the representations of the Yangian found in finite dimensional integrable spin chains, there is a simple isomorphism between Yangian weights and the weights of gl_N representations, as we will see in the next chapter.

Chapter 3

Integrable Spin Chains

Integrable spin chains are quantum models defined on a discrete one dimensional lattice. On each site of this lattice lives a Hilbert space. In this chapter and in the next few ones, we will assume that the lattice is *periodic*. The resulting spin chains are known as periodic or closed.

The first integrable spin chain model - known as the Heisenberg XXX spin chain - was proposed in the early 20th century, representing a one dimensional magnet. The Hilbert space at each site represents the spin of an electron in the magnet, and therefore is two-dimensional. This model was first solved by Bethe via his celebrated ansatz [39], who obtained its energy spectrum and the eigenstates of its Hamiltonian. We will see how to solve the same spectral problem in a more modern form, relying on the RTT relations (2.2.11), known as the Algebraic Bethe Ansatz (ABA) [40].

Integrable spin chains can be built from the Yangian via the so-called evaluation representation, which assigns a quantum Lax operator to each site of the spin chain. From the Lax operator, it is possible to immediately build the monodromy matrix (2.2.9). In particular, the Heisenberg XXX spin chain is built from the Yangian $Y(gl_2)$ in the fundamental representation. In general, we can define other integrable spin chains using the Yangian $Y(gl_N)$ in any representation. Via the ABA, one can compute the eigenstates (and even correlation functions) of the spin chain from the monodromy matrix. However, this technique becomes computationally heavy for higher ranks, and it is not applicable to spin chains in non-highest weight representations of the Yangian.

These facts led to the development of alternative techniques to compute observables in a spin chain, such as the Separation of Variables program, which will be introduced in the next chapters. In the rest of this chapter we will introduce the main tools for the modern description of integrable spin chains:

- The Q-system, composed of the Baxter Q-functions and the functional relations between them, encoded in the QQ-relations.
- The T-system, composed of the transfer matrices and the functional relations between them, encoded in the Hirota and Cherednik-Bazhanov-Reshetikhin (CBR) equations.

These two systems, together with the Baxter TQ equation, have been at the core of many advances in the study of integrable spin chains, and in particular form the basis of Separation of Variables.

This chapter is based on [40–43], with adaptations in terms of the notation used. For simplicity, we will consider as the reference model for this chapter spin chains in the fundamental representation of the $Y(gl_N)$ Yangian.

3.1 Evaluation representation of $Y(gl_N)$

The evaluation representation is a way to construct representations of $Y(gl_N)$ starting from representations of the gl_N Lie algebra, and naturally gives rise to integrable spin chains. Using this method, we can build spin chains using the well known representation theory of gl_N.

Concretely, the evaluation representation of $Y(gl_N)$ consists in picking up a representation π of gl_N, acting on the vector space V, and assigning to the Yangian generators $T_{ij}(u)$ (2.2.7) the values:

$$T_{ij}(u) \to (u-\theta)\delta_{ij}1 + \hbar\pi(E_{ji}) \tag{3.1.1}$$

where $\pi(E_{ji})$ are the generators of gl_N in a representation π, 1 is the identity operator on the representation space V, and θ are some complex-valued constants.

The Yangian weights $\lambda_i(u)$ (2.2.13) of the evaluation representation are related to the weights of the gl_N representation π. In fact, if π has highest weight vector $|0\rangle$ and weights λ_i, then $\pi(E_{ii})|0\rangle = \lambda_i|0\rangle$ and therefore:

$$T_{ii}(u)|0\rangle = (u-\theta+\hbar\lambda_i)|0\rangle \to \lambda_i(u) = u-\theta+\hbar\lambda_i \tag{3.1.2}$$

So $|0\rangle$ is also an highest weight for the evaluation representation of $Y(gl_N)$, and the Yangian weights depend linearly on the gl_N weights.

The evaluation representation of $Y(gl_N)$ is how we define rational integrable gl_N spin chains: for a spin chain of length L, we can use the coproduct of Drinfeld's third presentation (2.2.12) to tensor product L evaluation representations (3.1.1); each copy will represent a site of the spin chain.

We will now introduce some notation that will be used throughout the rest of this thesis. We define the *quantum Lax operator* at the site α as a matrix $L^{\alpha}_{\vec{\lambda}}(u)$ having components:

$$(L^{\alpha}_{\vec{\lambda}})_{ij}(u) = (u-\theta_\alpha)\delta_{ij}1 + \hbar\pi_{\vec{\lambda}}(E_{ji}),\ i,j = 1\dots N \tag{3.1.3}$$

where $\vec{\lambda}$ are the N weights of the gl_N representation π and θ_α are constants called inhomogeneities. The Lax operator is evidently a generator of the Yangian $Y(gl_N)$ built via the evaluation representation; therefore, it will satisfy the RTT relations (2.2.11).

The quantum Lax operator in the classical limit $\hbar \to 0$ becomes the classical Lax matrix, and the RTT relations become the Lax pair equations (1.2.12). Therefore, the quantum

Lax operator can be naturally viewed as the quantisation of a Lax matrix of a classical model.

As we have already mentioned, we can use the Yangian coproduct (2.2.12) to tensor copies of the Lax operator (which may have different inhomogeneities and/or be in different representations) to obtain the *quantum monodromy matrix* of the gl_N spin chain, whose elements are given by:

$$T_{ij}(u) = \sum_{k_1,k_2\ldots k_{L-1}=1}^{N} \left(L^1_{\vec{\lambda}_1}\right)_{ik_1}(u) \left(L^2_{\vec{\lambda}_2}\right)_{k_1k_2}(u) \ldots \left(L^L_{\vec{\lambda}_L}\right)_{k_{L-1}j}(u),\ i,j = 1\ldots N \quad (3.1.4)$$

By definition, the monodromy matrix is a $N \times N$ matrix, with entries being operator on the tensor product of L quantum spaces, each being the representation space of $\pi_{\vec{\lambda}_i},\ i = 1\ldots L$. For simplicity, we will assume that all sites of the spin chain have the same representation, and we will drop the labels $\vec{\lambda}$.

The elements of the quantum monodromy matrix are the Yangian generators (2.2.7) and therefore (3.1.4) satisfies the RTT relations:

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v) \quad (3.1.5)$$

where T_1 and T_2 were introduced under (2.2.11), and R is the rational R-matrix (2.2.3).

By construction, the monodromy matrix $T(u)$ is a monic polynomial in u of degree L. The action of its matrix elements on the highest weight state $|\Omega\rangle \equiv \bigotimes_{i=1}^{L} |0\rangle_{\vec{\lambda}_i}$ is given by:

$$T_{ij}(u)|\Omega\rangle = 0,\ \forall i > j, \qquad T_{ii}(u)|\Omega\rangle = \prod_{\alpha=1}^{L}(u - \theta_\alpha + \hbar\lambda_i^\alpha)|\Omega\rangle\,. \quad (3.1.6)$$

Finally, we notice that $T(u)$ is $GL(N)$ covariant, meaning that we have for any $G \in GL(N)$:

$$[G \otimes \Pi(G), T(u)] = 0 \quad (3.1.7)$$

where $\Pi(G)$ denotes the action of G on the physical Hilbert space.

3.2 Diagrammatic rules

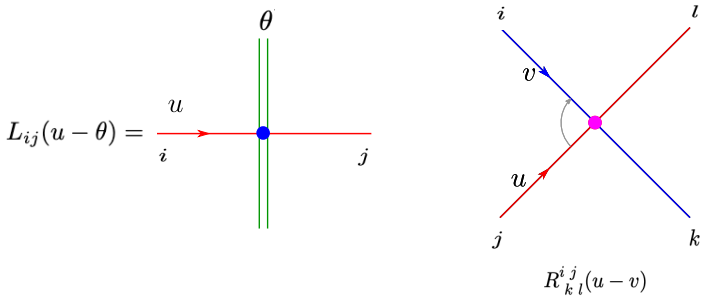
In this section, we introduce a set of diagrammatic rules that will help us in depicting some algebraic equations we will use.

For the periodic spin chains that we analyse in this section, they are quite simple:

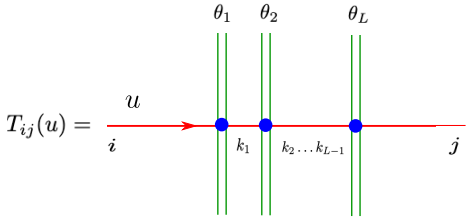
- We depict auxiliary spaces as horizontal, solid lines, equipped with a spectral parameter and an arrow. We follow the direction of the arrows to write equations.
- We depict physical spaces as vertical, double lines, equipped with inhomogeneities θ_α. We follow the double lines from the top and go down to write equations.

- R-matrices are pink dots at crossings of two auxiliary spaces. If A, B are the auxiliary spaces we get R_{AB}. The spectral parameter of R depends on the arrow directions on the auxiliary spaces.
- Lax operators $L_A(u-\theta_\alpha)$ appear at the crossing of an auxiliary space A and a physical space θ_α.

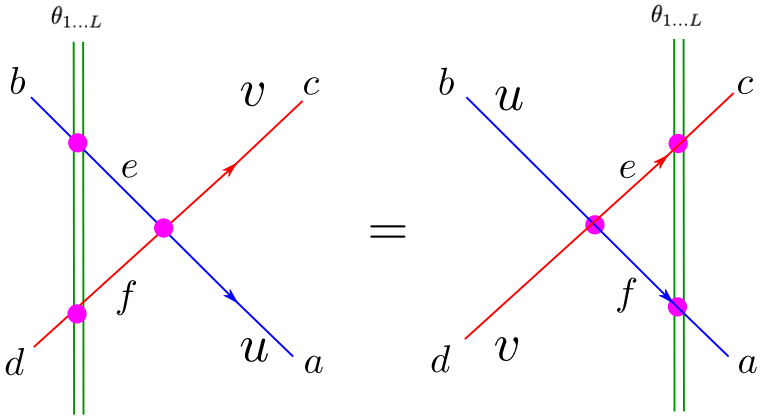
Using them, we draw the Lax operator and the R-matrix as:



The monodromy matrix (3.1.4) is drawn as:



As an example, we can easily draw the RTT relations (3.1.5) as follows:



where we assumed that the monodromy matrix is represented by one vertical line with multiple inhomogeneities.

3.3 Conserved charges from the monodromy matrix

Given an integrable spin chain with monodromy matrix (3.1.4), there is a natural procedure that allows to build a tower of quantum operators in involution with each other, which is very similar to the method we have described in section 1.2.2.

We start by defining the *transfer matrix* for the gl_N spin chain as:

$$t(u) = \mathrm{tr}(T(u)) \tag{3.3.1}$$

where T is the monodromy matrix, and tr is the trace on $N \times N$ matrices. Due to (3.1.7), it is immediate to see that the transfer matrix is $GL(N)$ invariant.

Since $T(u)$ is a monic polynomial in u, $t(u)$ is also a monic polynomial in u, whose coefficients are operators acting on the tensor product of L quantum spaces. We will now prove that these operators commute with each other; in fact, multiplying the RTT relation (3.1.5) by $R_{12}^{-1}(u,v)$ on the left and taking the trace on the auxiliary spaces 1 and 2, we get:

$$\begin{aligned}&\mathrm{tr}_{1,2}(T_1(u)T_2(v)) = \mathrm{tr}_{1,2}\left((R_{12}^{-1}(u,v)T_2(v)T_1(u)R_{12}(u,v)\right) = \mathrm{tr}_{1,2}\left(T_2(v)T_1(u)\right) \to \\ &\to t_1(u)t_2(v) - t_2(v)t_1(u) = 0\,,\end{aligned}$$

where in the first line we have used cyclicity of the trace and in the second line the fact that $\mathrm{tr}_{1,2}(A_1A_2) = \mathrm{tr}_1(A_1)tr_2(A_2)$. Therefore, the RTT relation for the Monodromy matrix implies that:

$$[t(u), t(v)] = 0,\ \forall u, v \in \mathbb{C}\,. \tag{3.3.2}$$

Due to the fact that $t(u)$ is a polynomial in u, this relation implies that the coefficients of u in $t(u)$ commute with each other. Explicitly, we define the $L-1$ integrals of motion (IoMs):

$$I_k \equiv \frac{1}{(k-1)!}\frac{d^{k-1}}{d^{k-1}u}t(u)|_{u=0}, \quad k = 1 \dots L. \tag{3.3.3}$$

These operators are in involution with each other:

$$[I_k, I_l] = 0,\ \forall k, l = 1 \dots L\,. \tag{3.3.4}$$

In this way, we can generate $L-1$ commuting independent operators, which will share the same basis of eigenstates. Since $t(u) = u^L + \sum_{k=1}^{L-1} u^{k-1} I_k$, the eigenstates of I_k are also eigenstates of the transfer matrix. Furthermore, if we interpret one of the I_k as the quantum Hamiltonian H of the spin chain, we can say that I_k are also conserved charges, since they commute with H[1].

[1] In general defining which integral of motion is the Hamiltonian is just a convention. In some cases, such as the Heisenberg XXX spin chain, there is a natural choice due to the fact that the quantum Hamiltonian is known from the corresponding physical model.

In general, the integrals of motion (3.3.3) do not represent all the conserved quantities in an integrable spin chain. As pointed out in the introduction to this chapter, this is not an issue since in a quantum model we can always build as many IoMs as needed. In any case, for a $Y(gl_N)$ spin chain there exists a procedure, called fusion, that allows to build more integrals of motion from the so-called N antisymmetric fused transfer matrices. We will see this in detail in section 3.7.1.

3.4 Algebraic Bethe Ansatz

The Algebraic Bethe Ansatz is a technique that lets us obtain eigenstates and eigenvalues of the Hamiltonian of an integrable spin chain from its transfer matrix (3.3.1). These eigenstates, which are also known as *Bethe states*, are obtained by diagonalisation of the transfer matrix, given that it commutes with the Hamiltonian and therefore they share a complete set of eigenstates.

For simplicity, we will examine in detail the rank $N = 2$ case where $Y(gl_N)$ is in fundamental representation. This model corresponds to the aforementioned Heisenberg XXX spin chain. We will then briefly discuss the generalisation to the ranks $N \geqslant 3$.

3.4.1 Heisenberg XXX spin chain

The Heisenberg spin chain is a model describing a 1-dimensional magnet. It is composed of L spin $\frac{1}{2}$-vectors placed on a periodic lattice, which represent the spins of the electrons in the magnet. Each spin vector $\vec{S}_\alpha$ interacts with only its nearest neighbors via the electromagnetic interaction $\vec{S}_\alpha \cdot \vec{S}_{\alpha+1}$, where $\vec{S} = \frac{\vec{\sigma}}{2}$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector containing the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.4.1}$$

The Hamiltonian governing this model is therefore given by:

$$H_{XXX} = \sum_{\alpha=1}^{L} H_{\alpha,\alpha+1} = \sum_{\alpha=1}^{L} \left(\vec{S}_\alpha \cdot \vec{S}_{\alpha+1} + \frac{1}{4} \right), \qquad \text{with } \vec{S}_{L+1} \equiv \vec{S}_1 \,. \tag{3.4.2}$$

The second term, corresponding to the vacuum energy level, has been chosen for simplicity: in fact, it is quite easy to see that

$$\left(\vec{S}_\alpha \cdot \vec{S}_{\alpha+1} + \frac{1}{4} \right) = P_{\alpha,\alpha+1} \,, \tag{3.4.3}$$

where P is the permutation operator between the sites α and $\alpha + 1$. Thus, an alternative form of (3.4.2) that will be very useful is:

$$H_{XXX} = \sum_{\alpha=1}^{L} P_{\alpha,\alpha+1} \,. \tag{3.4.4}$$

We would like to obtain the energies and the eigenstates associated with this Hamiltonian[2], who will organise themselves into irreducible representations of $gl(2)$ given that H commutes with the components of the total spin vector, $\mathbf{S}^i = \sum_{\alpha=1}^{L} S^i_\alpha$.

The Heisenberg XXX spin chain Hamiltonian was first diagonalised by Bethe [39], by virtue of a clever ansatz for the wavefunctions associated to the eigenstates of (3.4.2). This solution is known as the Coordinate Bethe Ansatz, and, given the form (3.4.4), consists in finding wavefunctions that are invariant under all L permutation operators.

In this section, we will present a different technique to diagonalise (3.4.2), that is intimately connected to the Yangian symmetry of the Heisenberg model, and heavily relies on the RTT relations. This technique was created by the Leningrad school in the '80s [40] and is known as the Algebraic Bethe Ansatz.

The Algebraic Bethe Ansatz for the Yangian $Y(gl_2)$ starts from the Monodromy Matrix (3.1.4), which we write in matrix form for convenience:

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & T_{22}(u) \end{pmatrix} . \tag{3.4.5}$$

The transfer matrix is then $t(u) = T_{11}(u) + T_{22}(u)$.

Hamiltonian from the transfer matrix

The transfer matrix contains integrals of motion of the model, and in particular the Hamiltonian of the Heisenberg spin chain. To retrieve the explicit form of the Hamiltonian (3.4.2), we need to tune the free parameters in the monodromy matrix (3.1.4), θ_α and $\hbar$. In particular, we need to set $\hbar = i$ and $\theta_\alpha = \frac{i}{2}$, $\forall \alpha = 1 \dots L$.

In terms of the Pauli matrices, the gl_2 algebra generators are given by:

$$E_{11} = \frac{1}{2} + \frac{\sigma_3}{2} \qquad E_{22} = \frac{1}{2} - \frac{\sigma_3}{2} \tag{3.4.6}$$

$$E_{12} = \frac{\sigma_1}{2} + \frac{i\sigma_2}{2} \qquad E_{21} = \frac{\sigma_1}{2} - \frac{i\sigma_2}{2} \tag{3.4.7}$$

Therefore, the Lax operators read:

$$L_\alpha(u) = \begin{pmatrix} u + iS^3_\alpha & i(S^1_\alpha - iS^2_\alpha) \\ i(S^1_\alpha + iS^2_\alpha) & u - iS^3_\alpha \end{pmatrix} . \tag{3.4.8}$$

It is immediate to see that:

$$L_\alpha(i/2) = P_{\alpha,a} \, , \tag{3.4.9}$$

where $P_{\alpha,a}$ is the permutation operator between vectors in the site α and vectors in the auxiliary space.

[2]Of course, for small L this can be done via direct diagonalisation of H_{XXX}. This becomes impractical very fast as L grows, even using a computer. This problem is exacerbated for higher rank models.

From this, we see that the transfer matrix evaluated at $u = \frac{i}{2}$ is given by:

$$t(i/2) = \mathrm{tr} \prod_{\alpha=1}^{L} P_{\alpha,a} = P_{L-1,L} \dots P_{23} P_{12} \,. \tag{3.4.10}$$

Taking the logarithmic derivative with respect to u we obtain:

$$\frac{d}{du} \log(t(u))|_{u=i/2} = \frac{t'(i/2)}{t(i/2)} = \frac{\sum_{i=1}^{L} P_{L_1,L} \dots \hat{P}_{i,i+1} \dots P_{12}}{P_{L-1,L} \dots P_{23} P_{12}} \tag{3.4.11}$$

where $\hat{P}_{ij}$ means that the permutation operator P_{ij} is missing. It is trivial to check that:

$$\frac{d}{du} \log(t(u))|_{u=i/2} = \sum_{\alpha=1}^{L} P_{\alpha,\alpha+1} \equiv H_{XXX} \,. \tag{3.4.12}$$

So, we have proven that we can indeed retrieve the Hamiltonian of the Heisenberg XXX spin chain from the monodromy matrix (3.4.5), albeit in a homogeneous limit where we set all parameters θ_α to be the same.

Diagonalization of transfer matrix

Since the Hamiltonian of the Heisenberg XXX spin chain is contained in the transfer matrix, the two quantities have a common basis of eigenvectors. The Algebraic Bethe Ansatz (ABA) [40, 44] allows to easily find eigenvectors of $t(u)$, mapping this problem to a set of algebraic equations known as Bethe equations. Knowing them, we can easily compute the eigenvalues of H: therefore, the ABA can be used to diagonalise the Hamiltonian of the Heisenberg XXX spin chain. In appendix (A.1), we give a complete derivation of this procedure. In this section, we will simply write the results.

Formally, the goal of the ABA is to solve the spectral problem:

$$t(u)|\Psi\rangle = \tau(u)|\Psi\rangle \,, \tag{3.4.13}$$

where τ are the eigenvalues of the transfer matrix (3.3.1) and $|\Psi\rangle$ are its eigenvectors, known as Bethe states. We write the monodromy matrix (3.4.5) as:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \,, \tag{3.4.14}$$

where A, B, C, D are operators on the quantum space of the spin chain. The transfer matrix in this notation is $t(u) = A(u) + D(u)$.

As detailed in appendix A.1, we can prove that the eigenstates of $t(u)$ take the form:

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_i)|\Omega\rangle \,, \tag{3.4.15}$$

where $|\Omega\rangle$ is the vacuum state for the spin chain, composed of spin ups in all the sites, and u_i are parameters known as the Bethe roots, solutions of the Bethe equations:

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,. \tag{3.4.16}$$

The eigenvalues of the transfer matrix on the Bethe states (3.4.15) are:

$$\tau(u) = \prod_{j=1}^{M} \frac{u - u_j - i}{u - u_j}(u + i/2)^L + (u - i/2)^L \prod_{j=1}^{M} \frac{u - u_j + i}{u - u_j}\,, \tag{3.4.17}$$

Now that we have obtained the spectrum of the transfer matrix, we may compute the energy eigenvalues of our Hamiltonian using (3.4.12). Indeed, since the second term in (3.4.17) drops out when evalutated at $u = i/2$, we obtain:

$$\left(\frac{d}{du}\log t(u)\right)_{u=i/2} = \sum_{j=1}^{M}\left(\frac{1}{-u_j - i/2} - \frac{1}{-u_j + i/2}\right)\,. \tag{3.4.18}$$

Hence:

$$E = \sum_{j=1}^{M} \frac{i}{u_j^2 + 1/4}\,. \tag{3.4.19}$$

To summarise, the Algebraic Bethe Ansatz lets us create eigenstates of the Hamiltonian $|\Psi\rangle$ via repeated applications to the ground state of a B operator, evaluated at the Bethe roots u_k. The energy levels of these states are then given by equation (3.4.19).

3.4.2 Rank $N = 3$ spin chains and Nested ABA

It is possible to extend the ABA to spin chains based on the Yangian $Y(gl_N)$, $N \geqslant 3$. In these cases, the method is known as Nested Algebraic Bethe Ansatz (NABA). The higher rank models do not have the same simple physical interpretation as the Heisenberg XXX spin chain, however they appear in the context of High Energy Physics and in particular they are at the core of integrability in AdS/CFT dualities. Studying them is therefore of fundamental importance.

For simplicity, in this section we will focus on the case where at each site of the chain we have the fundamental representation of the gl_3 algebra. We will also introduce the shorthand notation $f(\bar{u}) \equiv f(u_1 \dots u_L)$, where f is any function.

In the $N = 3$ case, the monodromy matrix is given by:

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}. \tag{3.4.20}$$

To simplify our treatment, we consider the case where the highest weight state is annihilated by $T_{23}(u)$; the final form of eigenvectors of $t(u)$ will not depend on this assumption, as only the explicit form of the Bethe equations will be modified. The general case is treated in [41]. We will also assume that the ground state $|\Omega\rangle$ is composed of spin up states in all the sites of the chain.

We now define a parametrisation of T in terms of A, B, C, D operators as:

$$T(u) = \begin{pmatrix} A(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \mathbb{D}(u) \end{pmatrix} = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) \end{pmatrix}. \tag{3.4.21}$$

Given that we have assumed $D_{12}(u)|\Omega\rangle = |\Omega\rangle$, the candidates for the role of creation operators for our states are $B_1(u)$ and $B_2(u)$.

Notice that now $\mathbb{B}, \mathbb{C}$ are vectors and $\mathbb{D}$ is a matrix; the gl_3 RTT relations imply that:

$$r_{12}(u-v)\mathbb{D}_1(u)\mathbb{D}_2(v) = \mathbb{D}_2(v)\mathbb{D}_1(u)r_{12}(u-v)\,, \tag{3.4.22}$$

where r is the gl_2 rational R-matrix: $\mathbb{D}$ is therefore the monodromy matrix of an auxiliary spin chain of symmetry $Y(gl_2)$, i.e. of an Heisenberg XXX chain.

The transfer matrix of the model is:

$$t(u) = \mathrm{tr}\, T(z) = A(z) + D_{11}(z) + D_{22}(z) = A(z) + \mathrm{tr}\mathbb{D}(z)\,. \tag{3.4.23}$$

In appendix A.2, we derive the eigenstates of (3.4.23) via the Nested Algebraic Bethe Ansatz. To summarise, they are built from the application of some operators to the vacuum state; these operators are built from the scalar product between two vectors:

$$|\Psi_M(\bar{u})\rangle = \prod_{i=1}^{M} \mathbb{B}_i(u_i)\mathbb{F}(\bar{u})|\Omega\rangle\,. \tag{3.4.24}$$

- The first vector is composed of $\mathbb{B}$ operators - the vectorial B operators of the gl_3 spin chain - evaluated at the Bethe roots $(u_1 \dots u_M)$ satisfying the Bethe equations (A.2.12).
- The second vector $\mathbb{F}$ is an eigenvector of an auxiliary gl_2 spin chain with transfer matrix (A.2.6). It is built via the B operators of the auxiliary spin chain, evaluated at the *auxiliary* Bethe roots $(v_1 \dots v_N)$, solutions of the auxiliary Bethe equations (A.2.8).

We have a set of $M + N$ equations for the $M + N$ total Bethe roots. The problem is that these equations are not separated - each of them depends on the full set of Bethe roots $(\bar{u}, \bar{v})$. This nesting of the Bethe roots is the origin of the name Nested Algebraic Bethe Ansatz.

Of course, given the states and the Bethe roots we may try to extract the Hamiltonian of the spin chain and compute its eigenvalues; this can be done but it is quite a lengthy calculation. We refer the interested reader to the review [41].

3.4.3 Higher rank models

In this section, we will briefly talk about how the NABA works for ranks $N \geqslant 4$. For a full treatment, we refer again to the review [41].

The procedure is iterative and quite similar to the rank $N = 3$ case: to build the eigenstates of the transfer matrix, we act with a linear combination of $N-1$ gl_N B-operators. Simlarly to the $N = 3$ case, the coefficients $\mathbb{F}$ need to be eigenstates of an auxiliary gl_{N-1} spin chain. To build the eigenstates of this gl_{N-1} spin chain, we need an auxiliary gl_{N-2} spin chain, and so on until we reach a gl_2 auxiliary spin chain, whose eigenstates we can build via the gl_2 Algebraic Bethe Ansatz.

Throughout this procedure, we will need to impose exactly $N-1$ sets of Bethe equations for the $N-1$ sets of Bethe roots $\bar{u}_1 \dots \bar{u}_{N-1}$. Each set of Bethe equations depends on all the Bethe roots. Solving this system is therefore extremely non-trivial even for very small spin chain lengths.

The transfer matrix eigenstates are built as combinations of the B-operators of all the nested spin chains $gl_2 \dots gl_N$, evaluated at the corresponding Bethe roots. From these states, it is possible to compute the spectrum of the Hamiltonian by extracting it from the transfer matrix.

3.4.4 Scalar products and form factors from the ABA

Having obtained the states of spin chains via the Algebraic Bethe Ansatz, we can use them to compute observables in the spin chain. These are either overlaps of these states, which we call *scalar products*, or matrix elements of operators acting on the spin chain Hilbert space, which we call *form factors*.

In order to compute these quantities, we need to build dual Bethe states $\langle\Psi|$; this can be done in a similar fashion to the Bethe states $|\Psi\rangle$, as detailed in [45].

Bethe states are orthogonal, hence overlaps between different states are trivial. Non-trivial observable that we can compute include the norms of Bethe states [46], and the overlaps of the so-called off-shell Bethe states, which are built via the same B operator as the Bethe states, but with B being evaluated at *generic* spectral parameter u:

$$|\Psi\rangle_{off-shell} = \prod_{i=1}^{M} B(u)|\Omega\rangle\,. \tag{3.4.25}$$

Bethe States (both on-shell and off-shell) can be also used to compute form factors of various operators in the spin chain [47]. Most literature is focused on *local* operators, i.e. operators who act non-trivially only on one site of the spin chain: for example, one might be interested in computing the form factor of the spin-flipping operator S_i^+ at the site i of an Heisenberg XXX spin chain.

Norms of Bethe states and form factors of local operators computed via the ABA have been shown to take the form of determinants [48–55]. These determinants expressions are valid both for overlaps of on-shell states and off-shell states, but are quite complicated.

An in-depth review can be found in [41]. In the next section, we will briefly discuss some limitations of this approach. In chapter 5, we will compute a similar, but wider class of observables using Funcional Separation of Variables. In particular, in section 5.7 we will make an explicit comparison between FSoV and the NABA.

3.4.5 Problems of the Bethe Ansatz techniques

From a computational point of view, the Bethe Ansatz is quite heavy - we need to solve many coupled algebraic equations to find all the Bethe roots. However, if we manage to do so, we have expressions for all the eigenstates and eigenvalues of the transfer matrix $t(u)$, starting from a vacuum eigenstate $|0\rangle$. Although we have seen the procedure for the defining representation of the Yangian $Y(gl_N)$, the NABA can be generalised to other finite-dimensional representations - only the explicit form of the Bethe equations will be modified.

There are still several difficulties that the Algebraic Bethe Ansatz can encounter. Although some can be solved (for example, via the introduction of *twist* in the spin chain, as we will see in section 3.5), others cannot. In particular:

- Not all the states that we obtain from the ABA are physical. In particular, some Bethe roots will give states that have non-polynomial in u eigenvalues $\tau(u)$[3]. There is no general method to distinguish a priori such solutions.
- There is no guarantee that the ABA gives us all physical states. This problem is known as the problem of *completeness*, and has only been solved via the ABA for a few finite-dimensional representations of the Yangian[4].
- Finally, the ABA is based on the existence of a vacuum state $|\Omega\rangle$. If the representation of $Y(gl_N)$ is highest-weight, we can use the highest weight vector as the vacuum state. However, for non-highest weight representations, such vacuum is not available. Therefore the ABA is not applicable in these cases, which are of crucial importance since they are extensively found in the study of integrable spin chains appearing in QFTs.

3.5 Twist

A twist in a $Y(gl_N)$ spin chain is equivalent to the application of two global $GL(N)$ transformations to the transfer matrix:

$$t(u) = \mathrm{tr}T(u) \to t'(u) = \mathrm{tr}\, Ht(u)G, \qquad \forall H, G \in GL(N)\,. \tag{3.5.1}$$

Adding a twist changes the physics of the model: in particular, it modifies the form of the Bethe equations, that will now include a twist-dependent term.

[3] Recall that in our construction $t(u)$ is a polynomial in u, so its eigenvalues should also be polynomial.
[4] This problem has been solved [56] via the QQ-relations of the Q-system, that we introduce in section 3.6.

Only one of the two matrices H, G in (3.5.1) is actually needed. In fact, since the transfer matrix is invariant under global $GL(N)$ similarity transformations, we can always set $H = 1$ while leaving G generic. In the rest of this work, we will always take H to be trivial.

Adding a twist G to the monodromy matrix $T(u)$ does not change the RTT relations: in particular, any $G \in GL(N)$ satisfies the RTT relations, and so we can view the twist matrix as a Lax operator in an extra site of the spin chain acting on a trivial physical space. In particular, we define the twisted monodromy matrix as $\tilde{T}(u) = T(u)G$.

Notice that, while not breaking the Yangian symmetry, a twist G with distinct eigenvalues λ_i, $i = 1 \dots N$ does break the global $GL(N)$ symmetry of the transfer matrix to the group $\bigotimes_{i=1}^{N} U(1)$. Therefore, it breaks the degeneracy of its states: this fact will be fundamental for the SoV construction.

For now, we will use a diagonal twist matrix:

$$G = \mathrm{diag}(\lambda_1 \dots \lambda_N), \qquad \lambda_i \neq \lambda_j, \ \forall i \neq j. \tag{3.5.2}$$

In section 4.3, we will introduce a different type of twist matrix that is fundamental for the Functional SoV construction.

3.6 Q-system

In this section, we will introduce one of the main concepts of this thesis - the Q-system. It is based on an alternative description of the states of an integrable spin chain, which are encoded in the *Baxter Q-functions*. These are polynomials in the spectral parameter u that can be defined in terms of the Bethe roots. The Baxter Q-functions are not independent, and the relations between them are encoded in a set of functional relations, the QQ-relations.

The full set of Baxter Q-functions and the QQ-relations form the Q-system of the integrable spin chain. While the rank of the symmetry of the spin chain is encoded in the form of the QQ-relations, all other details (such as the specific representation, inhomogeneities, twist etc.) only appear in the large u asymptotics of the Q-functions and in some simple extra factor in the QQ-relations. Therefore, the Q-system is quite general, and can be applied to any rational integrable spin chain.

Due to this universality, the Q-system has been used extensively to investigate various properties of spin chains. For example, it was used to solve the completeness problem for many spin chains [56]. Furthermore, it forms the backbone of the Quantum Spectral Curve [57], which is a Q-system for supersymmetric spin chains with non-trivial requirements on the analytic structure of its Q-functions.

Conventions From now on, we will assume that at each site of the spin chain there are generic inhomogeneities θ_i such that $\theta_i \neq \theta_j$, $\forall i \neq j$, and we will keep on using the convention $\hbar = i$.

3.6.1 Baxter Q-functions

In this section, we will define the Baxter Q-functions. The Q-functions describe a state of an integrable spin chain in terms of the Bethe roots u_j that are associated to it. In particular, for each solution of the Bethe equations, we can form a set of Q-functions describing a certain state by packaging the Bethe roots into monic polynomials in u.

3.6.2 Rank $N = 2$ Q-system

For the Heisenberg XXX spin chain that we have introduced in section 3.4.1, any state is described by the set of Bethe roots $u_k,\ j = 1 \dots M$, solution of the Bethe equations with inhomogeneities θ_i:

$$\prod_{i=1}^{L} \frac{u_k - \theta_i + i/2}{u_k - \theta_i - i/2} = \lambda_1^2 \prod_{j \neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,. \tag{3.6.1}$$

We may define the Q-function associated to an eigenstate of $t(u)$ characterised by the set of Bethe roots $\{u_k\}_{k=1}^{M}$ as the *twisted Baxter polynomial*:

$$q_1(u) = \lambda_1^{iu} \prod_{k=1}^{M} (u - u_k) \tag{3.6.2}$$

The same state can also be described by another Q-function in terms of a different set of Bethe roots $\tilde{u}_k$ as:

$$q_2(u) = \lambda_2^{iu} \prod_{k=1}^{\tilde{M}} (u - \tilde{u}_k) \tag{3.6.3}$$

There is a simple physical interpretation for this phenomenon: q_1 are associated to the Bethe roots needed to build states starting from the reference vacuum with all spin ups $|\Omega\rangle = \bigotimes_{i=1}^{L} |\uparrow\rangle_i$. q_2 are associated to the Bethe roots needed to build the *same* state starting from the other possible vacuum with all spin downs, $|\tilde{\Omega}\rangle = \bigotimes_{i=1}^{L} |\downarrow\rangle_i$.

While this simple interpretation only makes sense for the fundamental representation, we always have two possible sets of q functions q_1 and q_2 for any representation of $Y(gl_2)$. For a general non-compact representation, they will not be polynomials in u, nor will they be defined in terms of Bethe roots. However, for the type of representations that we will consider in this work, q_1 will always be a polynomial.

The two Q-functions introduced here are not fully independent; in fact, they must satisfy the following QQ-relation or Wronskian relation:

$$(\lambda_2^{-1} - \lambda_1^{-1}) \lambda_1^{iu} \lambda_2^{iu} Q_\theta^{[+2]}(u) = q_1(u) q_2^{[+2]}(u) - q_1^{[+2]}(u) q_2(u)\,, \tag{3.6.4}$$

where we have introduced the following notation:

$$Q_\theta(u) = \prod_{i=1}^{L} (u - \theta_i), \qquad f^{[+n]}(u) = f\left(u + \frac{i}{2} n\right)\,. \tag{3.6.5}$$

Assuming that q_a are twisted Baxter polynomials, the QQ relations (3.6.4) can be used to determine the Bethe roots without the need to resort to the Bethe ansatz. A simple example of how this work can be found in [14].

3.6.3 Rank $N \geqslant 3$ Q-system

As we have seen from the Nested Bethe Ansatz in section 3.4.3, for higher ranks the states of a spin chain in the defining representation of $Y(gl_N)$ are characterised by exactly $N-1$ sets of Bethe roots, that we will denote as $\{\{u_j^{(1)}\}_{j=1}^{M_1} \dots \{u_j^{(N-1)}\}_{j=1}^{M_{N-1}}\}$. We can therefore naturally define a set of $N-1$ Q-functions associated to a given state that will package these Bethe roots. Explicitly we define the Baxter polynomials:

$$q_{12\dots j}(u) = \prod_{i=1}^{M_j}(u - u_i^{(j)}),\ j = 1 \dots N-1 \tag{3.6.6}$$

Once again, since in the fundamental representation of gl_N there is not a unique vacuum, the Q-functions (3.6.6) are not the unique way of describing a state of the gl_N spin chain.

Recall that the Bethe roots come from building states using nested auxiliary spin chains $Y(gl_2) \subset Y(gl_3) \subset \dots Y(gl_{N-1}) \subset Y(gl_N)$. Intuitively, we may think that the Q-functions (3.6.6) correspond to a state built by choosing the gl_N vacuum with all spins in the 1 direction, the gl_{N-1} auxiliary vacuum with all spins in the 2 direction (i.e. the first of the remaining ones), and so on. We may however pick such vacua in any order we like - this will give us different Q-functions describing the same state, but built with different Bethe roots just like in the $Y(gl_2)$ case.

We can denote all the Q-functions associated to a given state in a $Y(gl_N)$ spin chain as:

$$q_A(u),\ A \subset \{1 \dots N\}, \qquad \text{where}\ \ q_0(u) = 1\ \ \text{and}\ \ q_{1\dots N} = 1\,. \tag{3.6.7}$$

These Q-functions are totally antisymmetric in their indices. By counting them and including the trivial Q-functions q_0 and $q_{1\dots N}$, the total number of Q-function for a gl_N state is given by $\sum_{i=0}^{N}\binom{n}{i} = 2^N$.

More generally, for spin chains in any highest weight representation of $Y(gl_N)$, it is possible to prove that there are always 2^N Q-functions. Their structure will have the following form:

$$Q_A(u) = \left(\prod_{a\in A}\lambda_a^{iu}\right) N_A q_A(u) \prod_{j=1}^{|A|} F_j(u)\,, \tag{3.6.8}$$

where λ_j are the twist matrix eigenvalues, N_A is a constant, $q_A(u)$ is a monic polynomial and $F_j(u)$ are some functions of the spectral parameter. We will refer to the quantities $q_A(u)$ as the *Baxter polynomials*, and to $\prod_{a\in A}\lambda_a^{iu} q_A$ as the *twisted Baxter polynomials*.

The Q-functions are not all independent - they will be subject to the QQ-relations:

$$Q_{Abc}Q_A^{[-2]} = Q_{Ab}Q_{Ac}^{[-2]} - Q_{Ac}Q_{Ab}^{[-2]} \qquad b, c = 1 \dots N-1^{5}\,. \tag{3.6.9}$$

[5] b and c can be contained in A, but in this case the QQ-relations are trivial since both sides are 0 due to the antisymmetricity of the Q-functions.

The set of 2^N Q-functions and the QQ-relations form the so-called gl_N Q-system.

3.6.4 Dual Q-functions and symmetries of the Q-system

Since Q_A is antisymmetric in its indices, we can define its Hodge dual, that we will denote Q^A, by:

$$Q^A(u) = \frac{\epsilon^{\bar{A}A} Q_{\bar{A}}(u)}{Q^{[-2]}_{1\dots N}(u)}\,, \tag{3.6.10}$$

where $\bar{A}$ is the complement of the set A in $\{1\dots N\}$. The dual Q-functions are the solution of the Dual Baxter TQ equation that we will see in section 3.8, and form the natural building blocks for the Functional Separation of Variables that we will see in chapter 5.

The Q-system is invariant under gauge transformations:

$$Q_A(u) \to f_{|A|}(u) Q_A(u)\,, \tag{3.6.11}$$

where f are generic analytic function of u that need to satisfy the set of finite difference equations $f_{|A|+1} f^{[-2]}_{|A|+2} = f_{|A|+2} f^{[-2]}_{|A|}$. Using this gauge freedom, the functions F_j in (3.6.8) can be fixed to different forms while mantaining the Q-system invariant. While it is possible to set such functions to be 1, it is not always the most convenient choice, as we will see later.

3.6.5 Quantum eigenvalues and transfer matrix from the Q-functions

From the knowledge of the Q-functions for a given state, it is possible to derive the eigenvalue of the transfer matrix $t(u)$ on that same state. As a byproduct of this process, we will compute the so-called *quantum eigenvalues*, $\Lambda_j(u)$. These are defined in terms of the Q-functions (3.6.8) as:

$$\Lambda_j(u) = \frac{Q^{[2]}_{1\dots j-1}(u)}{Q_{1\dots j-1}(u)} \frac{Q^{[-2]}_{1\dots j}(u)}{Q_{1\dots j}(u)}, \quad j = 1\dots N\,. \tag{3.6.12}$$

Notice that this definition only involves a subset of the Q-functions, those where A is an ascending-ordered subset of $\{1\dots N\}$.

The eigenvalue $\tau(u)$ of the transfer matrix $t(u)$ corresponding to the state described by the given set of Q-functions can be expressed in terms of the quantum eigenvalues as:

$$\tau(u) = \sum_{i=1}^{N} \Lambda_i(u)\,. \tag{3.6.13}$$

3.7 T-system

We have seen that the Q-system contains information about the transfer matrix. The opposite is also true: we can obtain the Q-functions from the knowledge of the transfer

matrix, or more specifically the T-system, without needing to solve the Bethe equations. This is done via the so-called Baxter TQ equation, a central concept for this thesis which we define in section 3.8.

In this section we describe the T-system, composed of an extended set of transfer matrices, which includes the fundamental transfer matrix (3.3.1), and two functional relations between them, the Hirota [58] and CBR equations [59–61]. These functional relations are not fundamental for the scopes of this thesis, and we leave their discussion to the appendix A.3.1.

The transfer matrices in the T-system can all be built starting from the monodromy matrix (3.1.4), using a procedure known as *fusion* [62]. Fusion is the Yangian equivalent of the procedure that lets us build irreducible representations of gl_N from the tensor product of multiple copies of its fundamental representation. In particular, fusion can be described in terms of the Young tableaux appearing in the representation theory of gl_N. For the scopes of this thesis, we only need a subset of the fused transfer matrices, corresponding to the Young tableaux with a single column and known as fused antisymmetric transfer matrices. We describe how to build them from the so-called quantum minors in the next section, while leaving the general description of the fusion procedure to the appendix A.3.

3.7.1 Quantum minors and quantum determinant

In this section, we will introduce the quantum minors and quantum determinant of a spin chain. These are a special case of the fusion procedure, but are fundamental for the scopes of this thesis so we will describe them separately from the general case, which can be found in appendix A.3. We will not give formal proofs for most of the statements of this section; the interested reader can find them in [43].

We will use the convention that objects with low indices $A_{i,j\dots}$ are tensors acting non-trivially on the $i, j\dots$-th copies of the auxiliary space, while objects with the same number of up and down indices $T^{ij\dots}_{kl\dots}$ will always be intended as matrix elements of the corresponding tensor.

To define the quantum minors, we first need to introduce the generalised R-matrix, acting on m copies of the auxiliary space $\mathbb{C}^N$:

$$R(u_1\dots u_m) = (R_{12}\dots R_{1m})\dots(R_{m-2,m-1}R_{m-2,m})(R_{m-1,m})\,, \tag{3.7.1}$$

where $u_1\dots u_m$ are generic complex parameters, $R_{i,j} \equiv R_{ij}(u_i - u_j)$ is the usual rational R-matrix (2.2.3) acting non-trivially on the auxiliary spaces i, j. We can easily represent the matrix elements of the generalised R-matrix using the diagrammatic rules. As a simple example, in fig. 3.1, we depict the matrix elements of $R(u_1, u_2, u_3)$, which are given by:

$$R^{i_1,i_2,i_3}_{j_1,j_2,j_3}(u_1,u_2,u_3) = \sum_{k_1,k_2,k_3} R^{i_1,i_2}_{k_1,k_2}(u_1-u_2)R^{k_1,i_3}_{j_1,k_3}(u_1-u_3)R^{k_2,k_3}_{j_2,j_3}(u_2-u_3)\,. \tag{3.7.2}$$

The generalised R-matrix (3.7.1) satisfies the following relation:

$$R(u_1\dots u_m)T_1(u_1)\dots T_m(u_m) = T_m(u_m)\dots T_1(u_1)R(u_1\dots u_m) \tag{3.7.3}$$

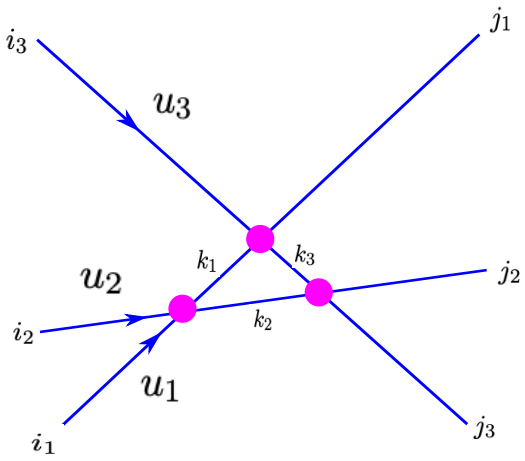


Figure 3.1: Matrix elements of the generalised R-matrix $R(u_1, u_2, u_3)$, acting on $\otimes_{i=1}^{3}\mathbb{C}^N$.

where T_i is the monodromy matrix for $Y(gl_N)$ acting on m copies of the auxiliary space $\bigotimes_{i=1}^{m}\mathbb{C}^N$ as:

$$T_i(u) = 1 \otimes \ldots 1 \otimes \underbrace{T(u)}_{\text{i-th space}} \otimes 1 \cdots \otimes 1\,. \tag{3.7.4}$$

If we take the parameters u_j in (3.7.1) to be $u_j = u + ji - i$, then $u_j - u_{j+1} = -i$ and:

$$R(u_1 \ldots u_m) \sim A_m\,, \tag{3.7.5}$$

where A_m is the totally antisymmetric projector (or antisymmetriser) over $\bigotimes_{i=1}^{m}\mathbb{C}^N$; the proportionality constant can be fixed by rescaling $R(u_1 \ldots u_m)$ and imposing that the resulting projector is idempotent, i.e. $A_m^2 = A_m$. As an example, we use our diagrammatic rules to depict the antisymmetriser A_3 in figure 3.2. Its matrix elements can be obtained by setting $u_j = u + ji - i$ in (3.7.2).

Note that if we choose $u_j - u_{j+1} = i$, then $R(u_1 \ldots u_m) = Sym_m$, the totally symmetric projector over $\bigotimes_{i=1}^{m}\mathbb{C}^N$.

Setting $u_j = u + ji - i$ in the generalised RTT relation (3.7.3), the R-matrices become antisymmetrisers over m copies of the auxiliary space of the monodromy matrix, and we get the following relation:

$$T^m(u) \equiv A_m T_1(u) \ldots T_m(u + im - i) = T_m(u + im - i) \ldots T_1(u) A_m \tag{3.7.6}$$

The matrix elements of this equation are known as the *quantum minors*, and are explicitly given by:

$$\begin{aligned} T^{i_1 \ldots i_m}_{j_1 \ldots j_m}(u) &= \sum_{p \in \mathfrak{S}_m} \operatorname{sgn} p \cdot T^{i_{p(1)}}_{j_1}(u) \ldots T^{i_{p(m)}}_{j_m}(u + im - i) \\ &= \sum_{p \in \mathfrak{S}_m} \operatorname{sgn} p \cdot T^{i_1}_{j_{p(1)}}(u + im - i) \ldots T^{i_m}_{j_{p(m)}}(u). \end{aligned} \tag{3.7.7}$$

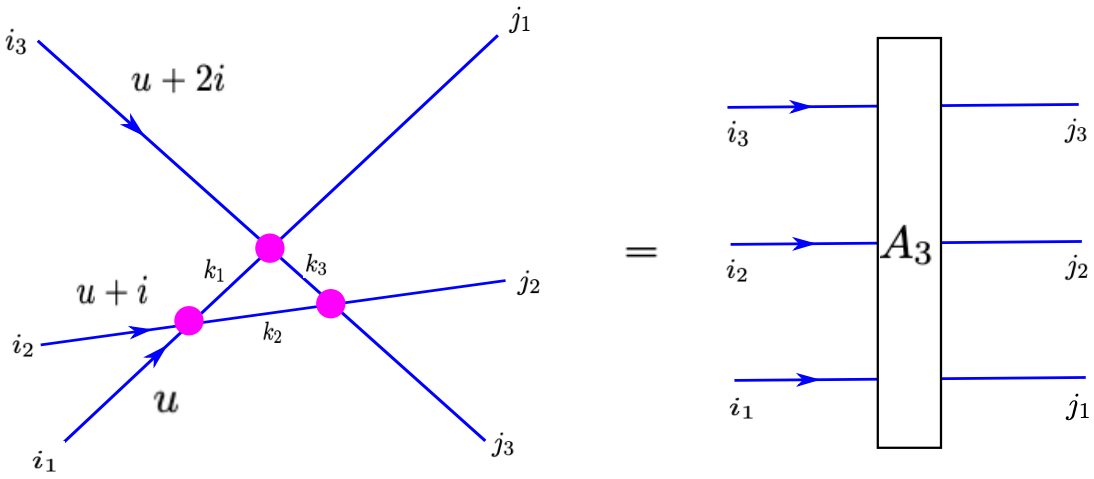


Figure 3.2: Antisymmetriser on $\bigotimes_{i=1}^{3}\mathbb{C}^N$ built via the R-matrices.

where p is a permutations of m elements and T^i_j are the monodromy matrix elements (3.1.4).

The quantum minors are still a representation of the Yangian $Y(gl_N)$. In particular, they satisfy the RTT relations (with the R-matrix being given by an antisymmetrised tensor product of the usual rational R-matrix), and they have coproduct given by:

$$\Delta\left(T^{i_1\dots i_m}_{j_1\dots j_m}(u)\right) = \sum_{k_1\cdots k_m=1}^{N} T^{i_1\dots i_m}_{k_1\dots k_m}(u)\otimes T^{k_1\dots k_m}_{j_1\dots j_m}(u)\,. \tag{3.7.8}$$

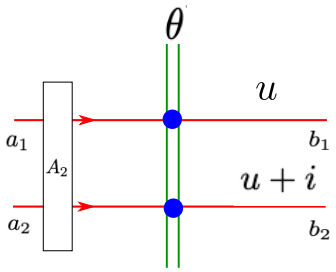
This means that we can build the m-th quantum minors of a spin chain of length L by taking the coproduct of L m-th fused Lax operators, whose matrix elements are defined in terms of the matrix elements of the Lax operator (3.1.3) by:

$$L^{i_1\dots i_m}_{j_1\dots j_m}(u) = \sum_{p\in\mathfrak{S}_m} \operatorname{sgn} p\cdot L^{i_{p(1)}}_{j_1}(u)\dots L^{i_{p(m)}}_{j_m}(u+im-i)\,. \tag{3.7.9}$$

This construction is convenient because we can easily depict the fused Lax operators using the diagrammatic rules introduced in section 3.2, where the antisymmetriser A_m built from the R-matrices will be represented as a rectangle passing through all the copies of the auxiliary space.

As an example, we depict the fused Lax operator that can be used to define the quantum minor $T^2(u)$:

$$L^{a1,a2}_{b1,b2}(u-\theta) =$$



Its matrix elements are given by:

$$L^{a_1,a_2}_{b_1,b_2}(u) = L^{a_1}_{\ b_1}(u)L^{a_2}_{\ b_2}(u+i) - L^{a_2}_{\ b_1}(u)L^{a_1}_{\ b_2}(u+i)\,. \tag{3.7.10}$$

Using the fused Lax operators, the m-th quantum minor of a spin chain of length L is then:

$$T^{i_1\dots i_m}_{j_1\dots j_m}(u) = \sum_{k_i}(L^1)^{i_1\dots i_m}_{k_1\dots k_m}(u-\theta_1)\dots(L^L)^{k_1\dots k_m}_{j_1\dots j_m}(u-\theta_L)\,, \tag{3.7.11}$$

where the fused Lax operator $(L^k)^{i_1\dots i_m}_{j_1\dots j_m}$ acts on the tensor product of L physical spaces as $1\otimes\dots 1\otimes \underbrace{L^{i_1\dots i_m}_{j_1\dots j_m}}_{k\text{-th copy}}\otimes 1\cdots\otimes 1$.

If $I, J \subset \{1\dots N\}$ are m dimensional and $i\in I, j\in J$, then the following commutation relation for the quantum minors is satisfied:

$$\left[T^i_{\ j}(u), T^I_J(v)\right] = 0 \tag{3.7.12}$$

This implies that the N-th quantum minor $T^{1\dots N}_{1\dots N}(u)$ commutes with all elements of the monodromy matrix T^i_j:

$$\left[T^i_{\ j}(u), T^{1\dots N}_{1\dots N}(v)\right] = 0,\ \ \forall i,j\in\{1\dots N\}\,. \tag{3.7.13}$$

Therefore, $T^{1\dots N}_{1\dots N}(u)$, also known as the *quantum determinant*, is a central element of $Y(gl_N)$. This name comes from the fact that, in the classical limit $i\to 0$, the quantum determinant becomes the determinant of the monodromy matrix (3.1.4).

From the quantum minors, we can define the N *fused antisymmetric transfer matrices*:

$$t_a(u) = \mathrm{tr}_a(T^a(u)),\ \ a = 1\dots N \tag{3.7.14}$$

These quantities commute with each other, as we prove in appendix A.3, and provide the integrals of motion for the spin chain that we mentioned in section 3.3. In particular, we have the expansion:

$$t_a(u) = u^L + \sum_{\alpha=1}^{L-1} u^{\alpha-1} I_{a,\alpha}\,, \tag{3.7.15}$$

where $I_{a,\alpha},\ a=1\dots N-1, \alpha = 1\dots L$ are the integrals of motion[6].

[6]The quantum determinant does not contain IoMs since it is a central element of $Y(gl_N)$ and therefore proportional to the identity operator.

3.8 Baxter TQ equation

The Q-system and the T-system are two equivalent ways to describe a rational integrable spin chain. These two systems are not completely independent; they are connected by the Baxter TQ equation [63–65]. This equation, and its dual version, form the foundation of the Functional Separation of Variables method described in section 5. For now, we will introduce them, and see how we can exploit them to obtain all the Q-functions from the T-system.

We define the Baxter TQ equation (or Baxter equation) as the eigenvalue equation for the Baxter operator $\mathcal{O}$, which is constituted by fused antisymmetric transfer matrices (3.7.14) and finite shift operators $\mathcal{D}$, and whose eigenfunctions are the Q-functions.

Similarly, the Dual Baxter equation is the eigenvalue equation for the dual Baxter operator $\mathcal{O}^\dagger$, whose eigenfunctions are the dual Q-functions (3.6.10).

We define the *Baxter Operator* as:

$$\mathcal{O} = \sum_{a=0}^{n} (-1)^a \tau_a(u) \mathcal{D}^{2a}\,, \tag{3.8.1}$$

where $\tau_{a,1}$ are the eigenvalues of the antisymmetric transfer matrices t_a, and $\mathcal{D}f(u) = f(u + i/2)$. We define the *Dual Baxter operator* as:

$$\mathcal{O}^\dagger = \sum_{a=0}^{n} (-1)^a \tau_a(u - ia) \mathcal{D}^{-2a}\,. \tag{3.8.2}$$

The Baxter and Dual Baxter equations are given by:

$$\mathcal{O} Q_i^{[-2]}(u) = 0 \tag{3.8.3}$$

$$\mathcal{O}^\dagger Q^i(u) = 0\,, \tag{3.8.4}$$

where Q_i are the Q-functions (3.6.8) with a single index, and Q^i are the dual Q-functions (3.6.10) with a single index. Explicitly:

$$Q_i(u) = N_i\, F_1(u)\, \lambda_i^{iu}\, q_i(u), \qquad Q^i(u) = \frac{\epsilon^{1\dots\hat{i}\dots N} Q_{1\dots\hat{i}\dots N}(u)}{Q^{[-2]}_{1\dots N}(u)}\,. \tag{3.8.5}$$

For these equations to hold, we need to choose the factors $F_j(u)$ in the Q-functions appropriately - in particular, the Q-functions have to be compatible with the T-system in the sense we explain now.

The Baxter operator can be expressed in terms of quantum eigenvalues (3.6.12):

$$\mathcal{O} = \left(1 - \Lambda_n(u)\mathcal{D}^{-2}\right) \dots \left(1 - \Lambda_1(u)\mathcal{D}^{-2}\right)\,. \tag{3.8.6}$$

By comparing this expression with (3.8.1), one can immediately read off the expression of $\tau_a(u)$ in terms of the quantum eigenvalues and therefore in terms of the Q-functions. For

example for a gl_3 spin chain we have:

$$\tau_1(u) = \Lambda_1(u) + \Lambda_2(u) + \Lambda_3(u)\,, \tag{3.8.7}$$

$$\tau_2(u) = \Lambda_1(u+i)\Lambda_2(u) + \Lambda_1(u+i)\Lambda_3(u) + \Lambda_2(u+i)\Lambda_3(u)\,, \tag{3.8.8}$$

$$\tau_3(u) = \Lambda_1(u+2i)\Lambda_2(u+i)\Lambda_3(u)\,. \tag{3.8.9}$$

We say that the Q-functions are compatible with the T-system if these τ_a are the eigenvalues of the antisymmetric transfer matrices t_a built via fusion in (3.7.14). This fixes completely the functions $F_j(u)$ in the Q-functions (3.6.8).

Since $F_j(u)$ do not depend on the state of the spin chain, we can fix them by choosing the simplest possible state. For compact representations, this will be the ground state, where the Q-functions in (3.6.12) have Baxter polynomials $q_A(u)$ of degree 0.

Once the functions F_j are fixed using the ground state, we can use the other eigenvalues of the T-system as a set of $N-1$ equations for each excited state, where the unknowns are the Bethe roots of the corresponding Q-functions. Even though such equations will involve polynomials in u of degree proportional to L, at least for low lengths they can be solved explicitly to obtain all the Q-functions of the spin chain, establishing the full equivalence between T and Q systems.

Chapter 4

Separation of Variables for Integrable Spin Chains

In this chapter, we will discuss Separation of Variables (SoV) for integrable spin chains. This technique can be used to compute a variety of observables in the spin chain, and constitutes a modern alternative to the Bethe Ansatz described in section 3.4.

The approach to SoV we will describe in this section was first proposed by Sklyanin [29, 66–68], and is based on the introduction of a special basis for the Hilbert space of the spin chain, known as SoV basis. The SoV basis is an analog of the action-angle variables of chapter 1: in it, the wavefunctions corresponding to the Bethe states are separated, i.e. they become a product of functions of one variable. These building blocks turn out to be the Baxter Q-functions, evaluated at a special set of points that we call the separated variables x.

To build the SoV basis, we follow Sklyanin's recipe. It is based on a quantisation of the separated variables for classical spin chains, which we define as the limit $\hbar \to 0$ of the quantum integrable spin chains treated so far. The quantum separated variables x are defined as the zeros of a special operator in the spin chain, the B operator. The SoV basis is defined as the set of eigenstates of the B operator.

This construction was first worked out for rank $N = 2$ spin chain [29], and was then generalised to any rank [69], and we will review it in this chapter. We will describe the correspondance between the separated variables for compact spin chains and the so-called Gelfand-Tsetlin (GT) patterns, introduced in Appendix B.1.

We will finally discuss how the Q-functions appear as wavefunctions in the SoV basis, and briefly describe how to compute observables using them.

It is worth mentioning that Sklyanin's procedure, based on the SoV operator, is not the unique way to build the SoV basis. Another method was proposed in [70], and does not use a B operator. While these two techniques have slightly different ranges of application, they have been proved to be equivalent in the cases where they are both applicable [42], including the spin chains in compact representations that we describe in this chapter.

For the classical SoV, the main reference used in this chapter is [71]. The quantum

SoV discussion is a simplified version of the one found in [42], with a small inclusion of the author's publication [26] that extends it to the non-compact spin-s representations that we will use in the next chapter.

4.1 Separation of variables for classical spin chains

Sklyanin's quantum SoV is based on a quantisation of the separated variables for a classical spin chain. Therefore, in this section we will briefly review how to build these classical separated variables.

To obtain classical separated variables, we need to find $2n$ canonical coordinates x_i, p_j who satisfy the canonical commutation relations:

$$\{x_i, x_j\} = 0 \qquad \{p_i, x_j\} = \delta_{ij} \qquad \{p_i, p_j\} = 0 \qquad i, j = 1 \dots n \tag{4.1.1}$$

and n functions ϕ_i, that will play the role of the separated equations of motion, such that:

$$\phi_j\left(x_j, p_j, f_1, \dots\dots, f_n\right) = 0, \quad (j = 1, \dots, n) \tag{4.1.2}$$

where f_i are the n integrals of motion.

To do so, we use the algebraic spectral curve. If T is the classical monodromy matrix built from the classical limit $\hbar \to 0$ of the Lax operators (3.1.3), then the algebraic spectral curve is the solutions to the eigenvalue equation for T:

$$det(l_i(u) - T(u)) = 0\,, \tag{4.1.3}$$

where l_i are the eigenvalues of $T(u)$.

For simplicity, we focus on the rank 2 case in the fundamental representation, for which T is a 2×2 matrix. Expanding equation (4.1.3), we obtain:

$$l(u)^2 - t(u)l(u) - \det T(u) = 0, \quad t(u) \equiv \mathrm{tr}T(u)\,. \tag{4.1.4}$$

The two solutions to this quadratic equation are $l_\pm(u) = \frac{1}{2}\left(t(u) \pm \sqrt{t(u)^2 - 4\det T(u)}\right)$, and define the algebraic spectral curve.

Now we build separated variables from the spectral curve. This can be done via the eigenvectors of T, which we define as the two-component vector $\Omega^\pm$ such that:

$$T(u)\Omega^\pm = l_\pm(u)\Omega^\pm\,. \tag{4.1.5}$$

Choosing the normalisation such that $\Omega_1^\pm = 1$, the eigenvalue equation (4.1.5) is solved by:

$$\Omega_2^\pm = \frac{-T_{11}(u) + l_\pm(u)}{T_{12}(u)} = -\frac{T_{21}(u)}{T_{22}(u) - l_\pm(u)} \tag{4.1.6}$$

So the eigenvectors $\Omega_\pm$ become singular at the points x_α where $T_{12}(x_\alpha) = 0$, or z_α where $l_\pm(z_\alpha) - T_{22}(z_\alpha) = 0$.

At the points x_α, $T_{12} = 0$ and $T(x_\alpha)$ is triangular. Thus the spectral curve equation becomes:

$$(l(x_\alpha) - T_{11}(x_\alpha))(l(x_\alpha) - T_{22}(x_\alpha)) = 0 \to l_1 = T_{11} \text{ and } l_2 = T_{22}. \tag{4.1.7}$$

In the equations (4.1.6) evaluated at x_α, the one on the left has no poles and the only remaining ones are those for which $l_\pm(x_\alpha) = T_{22}(x_\alpha)$.

We will now define the A and B operators as $A(u) = T_{11}(u)$ and $B(u) = T_{12}(u)$; in this notation, x_α are the zeros of the B operator.

Using the classical RTT relation, it is possible to prove that the coordinates defined by x_α and $\log p_\alpha$, where $p_\alpha \equiv A(x_\alpha)$, are canonically conjugated. Furthermore, these coordinates can be used to separate the equations of motion of the classical spin chain as in equation (4.1.2). For this reason, x_α and $\log p_\alpha$ form the separated variables for a classical spin chain. We will not be describing in detail how this procedure works: we invite the interested reader to check [71].

Finding the separated variables for higher rank cases can be done in a similar fashion. One needs to find a similarity transformation that makes the monodromy matrix triangular, and look for zeroes of the non-diagonal entries, which will define the A and B operators. x will be the zeros of the B operator, while p will be given by $A(x)$.

As an example, in the $N = 3$ case the A and B operators are given by:

$$A(u) = \frac{T_{11}(u)T_{32}(u) - T_{12}(u)T_{31}(u)}{T_{23}(u)}, \tag{4.1.8}$$

$$B(u) = T_{23}(u)\left(T_{21}(u)T_{32}(u) - T_{22}(u)T_{31}(u)\right) - \tag{4.1.9}$$

$$-T_{13}(u)\left(T_{12}(u)T_{31}(u) - T_{11}(u)T_{32}(u)\right). \tag{4.1.10}$$

4.2 Sklyanin's quantum separation of variables

Sklyanin's quantum SoV is based on a direct quantisation of the procedure we have just presented. The central tool to this construction is the quantum B operator: its zeros will form the separated variables, while its eigenstates are the so-called SoV basis.

The construction is slightly different for $Y(gl_2)$ and higher ranks, so we will review them separately.

4.2.1 Rank $N = 2$

For the rank 2 case, the quantisation of the classical separated variables is straightforward. We define the A and B operators as:

$$A(u) = T_{11}(u) \qquad B(u) = T_{12}(u) \tag{4.2.1}$$

where T is the quantum monodromy matrix (3.1.4).

We need two properties from the B operator [69]. The first is that it needs to commute with itself:

$$[B(u), B(v)] = 0\,, \tag{4.2.2}$$

which is ensured from the RTT relations. The second is that B can be used to construct the eigenvectors of the quantum transfer matrix $t(u)$; as we have seen in section 3.4, this can be done via the Algebraic Bethe Ansatz, obtaining the Bethe states:

$$|\Psi\rangle = \prod_{i=1}^{M} B(u_i)|\Omega\rangle, \quad u_i \text{ are the Bethe roots}\,. \tag{4.2.3}$$

Just like the classical case, the zeroes of B are the separated variables. In particular, if we could write B as a polynomial in u with zeros at x_α:

$$B(u) = B_0 \prod_{\alpha=1}^{n} (u - x_\alpha)\,, \tag{4.2.4}$$

then the left eigenvectors of B, labelled by the set $\{x_\alpha\}_{\alpha=1}^n$ and denoted as $\langle x_1 \dots x_n|$, would form the *SoV basis*.

The SoV basis forms a basis of the Hilbert space in which the wavefunctions of the Bethe states separate - i.e. they become products of functions of a single separated variable x_α. In fact, we have that:

$$\langle x_1 \dots x_n|\Psi\rangle = \langle x_1 \dots x_n| \prod_{i=1}^{M} B(u_i)|\Omega\rangle = B_0 \prod_{\alpha=1}^{n} \prod_{i=1}^{M} (u_i - x_\alpha)\langle x_1 \dots x_n|\Omega\rangle\,. \tag{4.2.5}$$

Choosing a normalisation of the SoV basis such that $B_0\langle x_1 \dots x_n|\Omega\rangle = 1$, we see that:

$$\langle x_1 \dots x_n|\Psi\rangle = (-1)^n \prod_{\alpha=1}^{n} Q(x_\alpha) \tag{4.2.6}$$

where we have used the definition of Baxter Q-functions associated to the state $|\Psi\rangle$ (3.6.2):

$$Q(u) = \prod_{\alpha=1}^{M} (u - u_\alpha)\,. \tag{4.2.7}$$

To compute observables, such as the form factors of operators, using SoV, we also need a right SoV basis. For rank $N = 2$, it can be simply defined as the *right* eigenvectors of the B operator $|x_1 \dots x_m\rangle$ [42].

Since the SoV bases are complete bases, we have the completeness relation $1 = \sum_{x_\alpha} |x\rangle\langle x|\mu(x)$, where μ is the *SoV measure* defined as $\mu(x) = (\langle x|x\rangle)^{-1}$. As an example, we show how to compute the scalar product of two Bethe states using the SoV basis:

$$\langle \Phi|\Psi\rangle = \langle 0|B(u_1) \dots B(u_M)|0\rangle \sum_{x_\alpha} |x\rangle\langle x|\mu(x) = \sum_{x_\alpha} \prod_{m=1}^{M} \prod_{\alpha=1}^{n} (u_m - x_\alpha)\,. \tag{4.2.8}$$

However, there is a problem in this procedure: the B operator is nilpotent [69], and cannot be diagonalised to build the SoV basis. This is expected for any compact representation of $Y(gl_2)$: B creates states from the highest weight vector, so by applying it repeatedly we must end at some point on the lowest weight vector, which is annihilated by the action of B (recall that $B = T_{12}$ is the Yangian raising operator).

Not everything is lost: the B operator can be made diagonalisable by introducing a special *twist* in the spin chain [69].

4.3 Companion twist frame

We have seen in section 3.5 that the addition of a twist to the spin chain breaks the global symmetry of the transfer matrix, and thus makes the Bethe states non-degenerate. However, even for a gl_2 spin chain, a diagonal twist is not sufficient to make the B operator diagonalisable. It is possible to act with further global transformations on (3.5.2) to solve this issues, as was argued in [69]. In this thesis, we will instead introduce a non-diagonal twist matrix, the *companion twist matrix*, which by itself makes B diagonalisable, and has other useful features.

The companion twist matrix, introduced in [72], has the following form in the $Y(gl_2)$ case:

$$G_{\mathrm{C}} = \begin{pmatrix} \chi_1 & -\chi_2 \\ 1 & 0 \end{pmatrix} \tag{4.3.1}$$

where χ_i are the characters associated to the diagonal twist (3.5.2):

$$\chi_1 = \mathrm{tr}(G) = \lambda_1 + \lambda_2 \qquad \chi_2 = \det(G) = \lambda_1\lambda_2 \tag{4.3.2}$$

It is possible to check that G_C is related to (3.5.2) by a similarity transformation: this implies that transfer matrices built with G and G_C are physically equivalent and have the same eigenvalues λ_1, λ_2. Furthermore, the Q-functions are the same for the two twist matrices.

The usefulness of this choice of twist in the SoV framework has now been extensively demonstrated [72–74]:

- The transfer matrix $t(u) = \mathrm{tr}(T(u)G_C)$ is linear in the characters χ_i, and in particular $t(u) = T_{12}(u) + \chi_1 T_{11}(u) - \chi_2 T_{21}(u)$;
- the B operator $B(u)$ is explicitly diagonalisable and with non-degenerate spectrum;
- The SoV basis is independent of the twist eigenvalues.

With a diagonalisable B operator, we can obtain the SoV basis for any highest-weight representation as seen in the previous section. The SoV basis will depend on the length and the specific representation of $Y(gl_2)$ we are using. For example, for a representation of weights $(\mathbf{s}, 0)$, it will have the form:

$$\langle \mathsf{x}|\mathbf{b}(u) = \langle \mathsf{x}| \prod_{\alpha=1}^{L} (u - \mathsf{x}_\alpha)\,, \tag{4.3.3}$$

where $\mathbf{b}(u) \equiv -\dfrac{B(u)}{Q_\theta^{[2\mathbf{s}-2]}}$ and:

$$\mathsf{x}_\alpha = \theta_\alpha + i(\mathbf{s} + n_\alpha), \quad \alpha = 1 \dots L\,. \tag{4.3.4}$$

Here θ_α are the inhomogeneities of the spin chain, while n_α are non-negative integers which are in one-to-one correspondence with the SoV basis states. A systematic description of how to obtain these numbers involves the Gelfand-Tsetlin patterns, defined in the Appendix B.1.

Notations Due to the presence of the non-diagonal companion twist matrix, we need to introduce some new notation. The monodromy matrix elements T_{ij} will always refer to the *untwisted* monodromy matrix, while the B operator is defined in terms of the *twisted* monodromy matrix $\tilde{T}(u) = T(u).G_C$, i.e. $B(u) \equiv \tilde{T}_{12}(u) = T_{11}(u)$.

Thus, the name 'B operator' will be reserved for the twisted SoV B operator, which is different from the ABA B operator introduced in section 3.4!

4.4 Rank $N \geqslant 3$ SoV

4.4.1 The B and C operators

In the case $N = 3$, Sklyanin managed to obtain the quantum B operator by direct quantisation of the classical case. Sklyanin's expression is given by:

$$B(u) = T_{23}(u)T_{23}^{12\,[2]}(u) + T_{13}(u)T_{13}^{12\,[2]}(u) \tag{4.4.1}$$

where T_{kl}^{ij} are the quantum minors matrix elements defined in (3.7.7). It was shown analytically [75] that for the fundamental representation, Sklyanin's B operator, evaluated at the Bethe roots of Q_1, creates Bethe states i.e. eigenstates of the transfer matrix:

$$|\Psi\rangle = \prod_j B(u_j)|0\rangle\,. \tag{4.4.2}$$

This fact can be generalised to any rank [69], by defining the B operator as:

$$B(u) = \sum_{J_1 \dots J_{n-1}} T_n^{J_1}(u)T_{J_1 n}^{J_2\,[2]}(u)T_{J_2 n}^{J_3\,[4]}(u)\dots T_{J_{n-2},n}^{J_{n-1}\,[2n-4]}(u)\,, \tag{4.4.3}$$

where $J_k = (j_k^1 \dots j_k^k)$ and we sum over configurations such that $1 \leqslant j_k^1 \leqslant \dots \leqslant j_k^k \leqslant n$.

Just like the $N = 2$ case, Sklyanin's B operator is nilpotent, and therefore not diagonalisable.

This problem can again be solved by twisting. We can introduce a diagonal twist $G = diag(\lambda_1, \lambda_2, \lambda_3)$ and act with a similarity transformation on the twisted spin chain to get a 'good' B operator, as argued in [69].

We will instead use again the companion twist matrix. Note that both procedures are generalisable to any rank. We start by defininining the $GL(N)$ companion twist matrix:

$$G_{ij} = (-1)^{j-1}\chi_j\delta_{i1} + \delta_{i,j+1}\,, \tag{4.4.4}$$

where χ_j are the elementary symmetric polynomials in the twist eigenvalues λ_i:

$$\prod_{j=1}^{N}(t+\lambda_j) = \sum_{r=0}^{N} t^{N-r}\chi_r\,. \tag{4.4.5}$$

χ_r are also the characters of the totally anti-symmetric representations of $GL(N)$. We remark that the companion twist matrix has the same eigenvalues $(\lambda_1 \dots \lambda_N)$ as the usual diagonal twist matrix.

For example, in the $N = 3$ case the Companion Twist Matrix is:

$$G = \begin{pmatrix} \chi_1 & -\chi_2 & \chi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.4.6}$$

The twisted transfer matrix $t(u) = \mathrm{tr}(T(u)G)$ is linear in the characters. In particular, we can see that:

$$t(u) = \mathrm{tr}(T(u)G) = \sum_{j=1}^{N-1}\chi_0 T_{j,j+1}(u) + \sum_{r=1}^{N}\chi_r(-1)^{r-1}T_{r1}(u)\,, \tag{4.4.7}$$

where we have defined $\chi_0 = 1$.

In the companion twist frame, the B operator becomes:

$$B(u) = \sum_{J_1\dots J_{n-1}} T_1^{J_1}(u)T_{1,J_1+1}^{J_2}{}^{[2]}(u)\dots T_{1,J_{n-2}+1}^{J_{n-1}}{}^{[2n-4]}(u)\,. \tag{4.4.8}$$

This B operator is diagonalisable, and its eigenvectors constitute the *left* SoV basis $\langle x|$.

This SoV basis factorises right Bethe states $|\Psi\rangle$ in terms of products of the corresponding Q-functions; however, if we were to use it to try to factorise left Bethe states $\langle\Psi|$, we would not get a simple expression. It is possible to factorise $\langle\Psi|$ nicely by introducing a *right* SoV basis $|y\rangle$, who is defined as the eigenvectors of the SoV C operator [73], which for $Y(gl_N)$ reads:

$$C(u) = \sum_{J_1\dots J_{n-1}} T_{1,J_{n-2}+1}^{J_{n-1}}(u)\dots T_{1,J_1+1}^{J_2}(u)T_1^{J_1}(u)\,. \tag{4.4.9}$$

The C operator can be used to generate Bethe states in the antifundamental representation of gl_N [73]:

$$\langle\Psi| = \langle 0|\prod_{i=1}^{M} C(u_j^*)\,, \tag{4.4.10}$$

where u_j^* is a certain set of Bethe roots for this representation.

The B and C operators are related by the so-called $*$ anti-automorphism of the Yangian:

$$T_{ij}(u) \quad \stackrel{*}{\to} \quad T_{ij}(-u) \tag{4.4.11}$$

$$T_{ij}(u)T_{kl}(v) \quad \stackrel{*}{\to} \quad T_{kl}(-v)T_{ij}(-u) \tag{4.4.12}$$

In particular, $B(u) \stackrel{*}{\to} C(u)$, and thus we can use most of the technology developed for the B operator to treat C as well.

4.4.2 Building the SoV basis

In this section, we will review how to obtain the explicit expression for the separated variables $x_{\alpha,a}$ and $y_{\alpha,a}$ labelling the left and right SoV bases.

The starting point is a slightly different construction for the SoV bases, due to Maillet and Niccoli [70, 76–80]. The SoV basis can be obtained via the action on some reference vector $\langle S|$ of the transfer matrix $t(u)$ evaluated at the inhomogeneities of the spin chain θ_α. For example, for a $Y(gl_2)$ chain in the fundamental representation, we have:

$$\langle n_1 \dots n_L| = \langle S| \prod_{\alpha=1}^{L} t(\theta_\alpha)^{n_\alpha}, \;\; n_\alpha = 0, 1\,. \tag{4.4.13}$$

For higher ranks, to build the SoV basis we also need to apply fused transfer matrices to $\langle S|$, and we also need to shift the inhomogeneities by multiples of i.

It is evident that this SoV basis automatically factorises Bethe states, since the latter are defined as eigenvectors of $t(u)$. The Maillet-Niccoli SoV basis is supplemented with some closure relations. These ensure that we can obtain SoV basis vectors with lower n_α from those with higher n_α by application of transfer matrices evaluated at some special points. Such closure relations are based on the Hirota and the CBR equations of the T-system.

Although the Maillet-Niccoli method requires the arbitrary choice of a vector $\langle S|$, which is not natural in non-compact cases, it can give a good insight on the spectrum of the separated variables for compact spin chains. In particular, it can be proven [73] that the eigenvectors of the B operators can be built by successive action of fused transfer matrices evaluated at some special point, who constitute the values of the separated variables $x_{\alpha,a}$. These special points can be determined by the so-called Gelfand-Tsetlin patterns for the Yangian $Y(gl_N)$, which we introduce in Appendix B.1. We also point out that the Gelfand-Tsetlin basis and patterns are fundamental to prove the link between the SoV basis construction via the B-operator of [69] and the Maillet-Niccoli one [70]. We refer the interested reader to [42] for a detailed explanation.

4.4.3 Separated variables and factorised wavefunctions

We now come to the main point of this section - given a compact $Y(gl_N)$ spin chain in a representation with weights $\vec{\nu}_\alpha$, $\alpha = 1 \dots L$, the Gelfand-Tsetlin patterns label the SoV

bases $|x\rangle$ and $\langle y|$ as follows [73]:

$$|x\rangle \quad \text{is labelled by} \quad x^\alpha_{k,j} = \theta_\alpha + i(\mu^\alpha_{k,j} - j + 1) \tag{4.4.14}$$

$$\langle y| \quad \text{is labelled by} \quad y^\alpha_{k,j} = \theta_\alpha + i(\nu_{N-k} - \bar{\mu}^\alpha_{k,j} + j - 1) \tag{4.4.15}$$

where $\mu^\alpha_{k,j}$ are the (finite) possible values in the dual diagonals of the GT patterns and $\bar{\mu}^\alpha_{k,j}$ measures how much the element of the dual diagonal has been excited above its minimum value allowed by the branching rules (B.1.6), and is defined by $\bar{\mu}^\alpha_{k,j} = \mu^\alpha_{k,j} - \nu^\alpha_{k+1}$. Note that the number of these states is finite and corresponds to the dimension of the Hilbert space of the spin chain.

Furthermore, the same $x^\alpha_{k,j}$ and $y^\alpha_{k,j}$ appear in the factorised wavefunctions. We define as *factorised wavefunctions* the overlaps between the SoV bases and the Bethe states.

Factorised wavefunctions are built as follows: first, we build eigenvectors of B (C) by applying transfer matrices in the T-system evaluated at $x^\alpha_{k,j}$ ($y^\alpha_{k,j}$) [73], in a similar way as (4.4.13). Then, we take the overlap between these eigenvectors and Bethe states. But Bethe states by definition diagonalise the transfer matrices we use to build the SoV bases, so we end up with transfer matrices eigenvalues τ evaluated at $x^\alpha_{k,j}$ ($y^\alpha_{k,j}$). Finally, we write the transfer matrix eigenvalues in terms of Q-functions, as delined in section 3.8.

The final result is given in terms of products of Q-functions evaluated at the separated variables $x^\alpha_{k,j}$ ($y^\alpha_{k,j}$). In particular, after an appropriate normalisation one obtains the following expressions for left and right wavefunctions:

$$\Psi(x) = \langle x|\Psi\rangle = \prod_{\alpha=1}^{L}\prod_{k=1}^{N-1} \det_{i,j\leqslant k} q_i(x^\alpha_{kj})\,, \tag{4.4.16}$$

$$\Psi(y) = \langle \Psi|y\rangle = \prod_{\alpha=1}^{L}\prod_{k=1}^{N-1} \det_{i,j\leqslant k} q^i(y^\alpha_{kj})\,, \tag{4.4.17}$$

where q_i are twisted Baxter polynomials with a single index introduced in (3.6.8), and q^i are the twisted dual Baxter polynomials.

We also mention that in the case of gl_N representations of weights $(s, 0, \ldots, 0)$, it is always possible to normalise the SoV basis so that the right wavefunction $\Psi(x)$ becomes a product of Q-functions [42]. In particular we get the following simple expression:

$$\Psi(x) = \prod_{\alpha=1}^{L}\prod_{j=1}^{N} q_1\left(x^\alpha_{j1}\right)\,. \tag{4.4.18}$$

Non-compact spin chains

The SoV basis can also be introduced for non-compact spin chains. In the next chapter, we will consider a spin-**s** highest-weight representation of $Y(gl_N)$, with highest-weight state $|0\rangle$ satisfying

$$\begin{aligned} &\mathbb{E}^\alpha_{ij}|0\rangle = 0, \quad i < j \\ &\mathbb{E}^\alpha_{ii}|0\rangle = \omega_i|0\rangle\,, \end{aligned} \tag{4.4.19}$$

where $\mathbb{E}$ are generators of gl_N and $\omega_1 = -\mathbf{s}$ and $\omega_i = +\mathbf{s}$ for $i \geqslant 2$. This is the simplest non-compact representation which can be considered and we have chosen it for simplicity to illustrate our main results, but we believe all the main statements can be easily extended to more general representations. In this representation, the wavefunctions are given by:

$$\Psi(x) := \langle x|\Psi\rangle = \prod_{\alpha=1}^{L}\prod_{a=1}^{N-1} Q_1(x_{\alpha,a}) \tag{4.4.20}$$

$$\Psi(y) := \langle \Psi|y\rangle = \prod_{\alpha=1}^{L}\prod_{a=1}^{N-1} \det_{1\leqslant a,b\leqslant N-1} Q^{a+1}\left(y_{\alpha,b} + \frac{i}{2}(N-2)\right) . \tag{4.4.21}$$

The separated variables are labelled by:

$$x_{\alpha,a} = \theta_\alpha + i\,(\mathbf{s} + n_{\alpha,a})\,, \quad y_{\alpha,a} = \theta_\alpha + i\,(\mathbf{s} + m_{\alpha,a} + 1 - a) \tag{4.4.22}$$

where $n_{\alpha,a}$ and $m_{\alpha,a}$ are non-negative integers subject to the constraints $n_{\alpha,1} \geqslant \cdots \geqslant n_{\alpha,N-1} \geqslant 0$ and $m_{\alpha,1} \geqslant \cdots \geqslant m_{\alpha,N-1} \geqslant 0$, with each possible configuration corresponding to a basis state. Note that these restrictions give infinite dimensional SoV bases, in agreement with the fact that the Hilbert space of this non-compact spin chain is infinite dimensional.

4.4.4 SoV charge operator

From the knowledge of the SoV bases it is possible to compute the so-called SoV measure:

$$\mathcal{M}_{\mathsf{y},\mathsf{x}} = \langle y|x\rangle\,. \tag{4.4.23}$$

For $Y(gl_2)$, where the left and right SoV bases are the same, this measure is diagonal. For higher ranks it is non-diagonal and highly nontrivial. An example of it for a $Y(gl_3)$ spin chains of short length in the fundamental representation can be found in [69].

A useful object proposed in [81] is the so-called SoV charge operator $\mathbf{N}$. It commutes with the B and C operators and is diagonalised in both SoV bases $|\mathsf{y}\rangle$ and $\langle\mathsf{x}|$ and counts the number of "excitations" above the SoV ground state. More precisely:

$$\mathbf{N}|\mathsf{y}\rangle = \left(\sum_{\alpha,a} m_{\alpha,a}\right)|\mathsf{y}\rangle, \quad \langle\mathsf{x}|\mathbf{N} = \langle\mathsf{x}|\left(\sum_{\alpha,a} n_{\alpha,a}\right) . \tag{4.4.24}$$

The SoV charge operator can be obtained as the first non-trivial coefficient in the large u expansion of B or C, and imposes useful selection rules on the SoV measure, as we will see in the next section.

4.4.5 SoV measure

The explicit form of the measure for $Y(gl_N)$ spin chains in the spin-**s** representations defined above, worked out in [74], is given by[1]

$$\mathcal{M}_{\mathsf{y},\mathsf{x}} = s_{\mathbf{L}} \sum_{k} \mathrm{sign}(\sigma) \left(\prod_{a=1}^{N-1} \frac{\Delta_a}{\Delta_\theta} \right) \prod_{\alpha=1}^{L} \prod_{a=1}^{N-1} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \Bigg|_{\sigma_{\alpha,a}=k_{\alpha,a}-m_{\alpha,a}+a} . \tag{4.4.25}$$

In Appendix C.1, we derive a more general formula that also gives this expression. We will now summarise the notations we use, following [74]. $s_{\mathbf{L}}$ is a simple sign factor

$$s_{\mathbf{L}} = (-1)^{\frac{L}{4}(L-1)(N^2+N-2)} . \tag{4.4.26}$$

σ denotes a permutation of L copies of the numbers $\{1, 2, \ldots, N-1\}$

$$\{\underbrace{1, \ldots, 1}_{L}, \ldots, \underbrace{N-1, \ldots, N-1}_{L}\} \tag{4.4.27}$$

with $\sigma_{\alpha,a}$ denoting the number at position $a + (\alpha - 1)(N-1)$. σ^0 denotes the identity permutation on this set and so $\sigma^0_{\alpha,a} = a$. The signature of the permutation $\mathrm{sign}(\sigma)$ is ± 1 depending on the number of elementary permutations needed to bring the ordered set $u_{\sigma^{-1}(1)} \cup u_{\sigma^{-1}(2)} \cdots \cup u_{\sigma^{-1}(N-1)}$ to the canonical order $u_{1,1}, u_{1,2}, \ldots, u_{L,N-1}$ where $u_{\sigma^{-1}(a)} = \{u_{\alpha,b} : \sigma_{\alpha,b} = a\}$. Whereas $\mathrm{sign}(\sigma)$ could be ambiguous due to different possible orderings inside $\sigma^{-1}(a)$, the combination with the Vandermonde determinants Δ_b is well defined. There are $\frac{(N-1)L!}{L!^{N-1}}$ possible permutations σ, and if σ is not such a permutation we define $\mathrm{sign}(\sigma) = 0$.

Since the SoV charge operator (4.4.24) commutes with both B and C, $\mathcal{M}_{\mathsf{y},\mathsf{x}}$ is only non-zero if the states $\langle \mathsf{x}|$ and $|\mathsf{y}\rangle$ have the same SoV charge eigenvalue. Furthermore, $\mathcal{M}_{\mathsf{y},\mathsf{x}}$ is only non-zero if there exists a permutation σ of the number (4.4.27) such that

$$m_{\alpha,a} = n_{\alpha,a} - \sigma_{\alpha,a} + a \tag{4.4.28}$$

for each α, a. There are distinct dual basis states $|\mathsf{x}\rangle$ with the same value of $n_{\alpha,a}$ and hence there are multiple permutations satisfying (4.4.28). We denote such inequivalent permutations (within each α) by k which we then sum over. The sum over k is needed only in a limited number of cases, for example in the gl_3 case only $k = n$ is possible.

In (4.4.25), Δ_b, which depends on σ, denotes the Vandermonde determinant constructed from all $\mathsf{x}_{\alpha,a}$ for which $\sigma_{\alpha,a} = b$ and Δ_θ denotes the Vandermonde determinant built from θ's

$$\Delta_\theta = \prod_{\alpha<\beta} (\theta_\alpha - \theta_\beta) . \tag{4.4.29}$$

[1]There is a typo in [74] where the sign factor $s_{\mathbf{L}}$ does not appear. However, it is correctly included in the Mathematica code contained in that paper.

Finally, the function $r_{\alpha,n}$ is defined as

$$r_{\alpha,n} = -\frac{1}{2\pi}\prod_{\beta=1}^{L}(n+1-i\theta_\alpha+i\theta_\beta)_{\mathbf{2s}-1}\,, \tag{4.4.30}$$

where $(z)_s = \frac{\Gamma(s+z)}{\Gamma(z)}$ is the Pochhammer symbol.

4.4.6 Computing observables using the SoV basis

We now have all the tools to compute observables using Separation of Variables. Suppose, for example, we want to compute the overlap of two Bethe states. Since Bethe states are orthogonal, this should be a trivial calculation. However, obtaining the explicit result via SoV is not simple, especially for high rank spin chain. In fact, we have to compute the following expression:

$$\langle \Psi_A|\Psi_B\rangle = \sum_{\mathsf{x},\mathsf{y}} \Psi_A(\mathsf{y})\mathcal{M}_{\mathsf{y},\mathsf{x}}\Psi_B(\mathsf{x})\,. \tag{4.4.31}$$

while we do in principle know all the elements on the RHS, this is indeed a hard calculation to perform in most cases. This is especially true for spin chains in non-compact, non-highest weight representations, such as the ones found in $\mathcal{N}=4$ SYM.

In the next section, we will introduce an alternative SoV construction, that allows to bypass these problems and is particulary adapt for applications to integrable CFTs, the Functional SoV.

Chapter 5

Functional Separation of Variables

The operator-based SoV (OSoV) construction (i.e. based on the B and C operators) that we analysed in the previous chapter has recently been supplemented with a *functional* SoV (FSoV) construction [82] allowing us to compute highly non-trivial quantities such as scalar products and form factors directly in separated variables, bypassing the explicit operator-based construction of the SoV bases. While being completely equivalent for spin chains in compact representations, the functional approach is particularly attractive in settings where an explicit construction of the SoV bases is complicated. For example, this is the case for non-compact spin chains without a highest-weight state, that are commonly found in High Energy Physics. Functional SoV has been already used to compute non-trivial observables in these systems despite its recent concoction [21, 25].

The functional SoV approach allows one to naturally compute the overlaps of Bethe states in terms of a *determinant* of the corresponding Q-functions. These observables can be immediately enhanced to a family of diagonal form factors $\langle\Psi|\partial_p\hat{I}|\Psi\rangle$, where p is some parameter of the model and $\hat{I}$ is an integral of motion, via the simple use of quantum mechanical perturbation theory [74, 82]. From this, we can extract the form-factors of some *local* operators.

The study of correlators via the FSoV approach has been advanced in [26] by the *character projection* technique and by identifying a set of $(N-1)\times(N+1)$ distinguished operators $\mathrm{P}_{a,r}(u)$ acting on the spin chain Hilbert space, which we call *principal*.

The main feature of the principal operators is that their off-diagonal matrix elements can be computed in a simple determinant form in terms of the Q-functions, similarly to overlaps of Bethe states. Even more generally, we show that the same determinant form holds true for the form factor $\langle\Psi_A|\mathrm{P}_{a,r}(u)|\Psi_B\rangle$, where $|\Psi_B\rangle$ and $\langle\Psi_A|$ are two general *factorisable states*, a class that includes both on-shell and off-shell Bethe states.

Furthermore, the form-factors of certain anti-symmetric combinations of the principal operators also take a determinant form. A particular case of such combinations is the SoV B and C operators that are used to build the SoV bases. Thus the FSoV construction allows to derive from first principles the form of the B and C operators.

Finally, we also compute the SoV basis representation of all the principal operators, which allows one to construct arbitrary combinations of these operators (not only antisymmetric). In particular we show that those operators generate the complete set of the spin chain Monodromy matrix elements $T_{ij}(u)$. Note that at least in the finite dimensional case, this implies, via the "quantum inverse transform" [83] that we have access to all local symmetry generators $\mathbb{E}_{ij}^{(\alpha)}$ from which one can in turn build any physical observables in this system. We also believe this to be the case in general but we do not have a simple proof of this.

In this chapter, we will assume that the spin chain is of length L and is in the spin-$\mathbf{s}$ representation defined in (4.4.19). However, the FSoV formalism can also be easily adapted to study spin chains in any highest-weight representations, although some expressions would become more complicated. Therefore, we choose the spin-$\mathbf{s}$ representation for simplicity's sake.

This chapter is based on the author's work [26].

5.1 Principal operators

A major goal in this chapter will be to compute the matrix elements of (sums of) certain monodromy matrix entries between two transfer matrix eigenstates and their generalisation to arbitrary factorisable states. We will refer to these particular monodromy matrix entries as *principal operators*.

The principal operators are defined as follows. It is easy to check that the fused companion twist matrices $G^{(a)}$, obtained by doing fusion on the companion twist matrix introduced in section 3.5, are linear in the characters χ_r. As such, each of the totally antisymmetric transfer matrices $t_a(u)$ admits the expansion:

$$t_a(u) \equiv \sum_{r=0}^{N} \chi_r \, \mathbb{P}_{a,r}(u) \,. \tag{5.1.1}$$

We call the operators $\mathbb{P}_{a,r}(u)$ principal and the reason for their importance will become clear in section 5.3. Note that they are independent of the twist eigenvalues λ_j as all twist dependence of the transfer matrices is contained in the characters χ_r.

For example, the transfer matrix $t_1(u)$ can be expanded as

$$t_1(u) = \sum_{j=1}^{N-1} \chi_0 T_{j,j+1}(u) + \sum_{r=1}^{N} \chi_r (-1)^{r-1} T_{r1}(u) \,, \tag{5.1.2}$$

where $\chi_0 = 1$.

Similar expansions can be performed for the totally antisymmetric transfer matrices $t_a(u)$, built from the quantum minors (3.7.7). These are given by:

$$t_a(u) = \sum_{1 \leqslant i_1 < \cdots < i_a \leqslant N} T^{i_1 \dots i_a}_{j_1 \dots j_a}(u) G_{j_1 i_1} \dots G_{j_a i_a} \,, \tag{5.1.3}$$

where G_{ij} are matrix elements of the companion twist matrix. As a result of the summation condition $1 \leqslant i_1 < \cdots < i_a \leqslant N$ the coefficient of each χ_r is a sum of quantum minors with distinct upper indices which cannot cancel each other and as a result the coefficient of each χ_r is non-zero as long as $1 \leqslant a \leqslant N-1$.

While most principal operators are given by large sums over quantum minors things simplify for $a = N-1$ as the $N-1$-th anti-symmetric representation monodromy matrix is simply equal to the quantum-inverse matrix of $T(u)$ divided by a trivial factor [43]. We introduce the notation T^{ij} for these operators, defined by

$$T^{ij}(u) \prod_{k=1}^{N-1} Q_\theta^{[2(\mathbf{s}-k)]}(u) = T^{1\,\dots\,\hat{j}\,\dots\,N}_{1\,\dots\,\hat{i}\,\dots\,N}\left(u - \frac{i}{2}(N-2)\right), \tag{5.1.4}$$

where the notation $\hat{i}$, $\hat{j}$ means that the corresponding index is missing, and we multiply the LHS by Q_θ (defined in (3.6.5)) to remove the non-dynamical factors. It is then easy to derive that:

$$t_{N-1}(u) = \sum_{r=0}^{N-1} \chi_r\, T^{r+1,N}(u) - \chi_N \sum_{j=1}^{N-1} T^{j+1,j}(u)\,. \tag{5.1.5}$$

We will write out explicitly the principal operators in terms of monodromy matrix elements T_{ij} for the special cases of gl_2 and gl_3.

gl_2 case. In this case we have already seen in section 3.5 that:

$$t_1(u) = T_{12}(u) + \chi_1 T_{11}(u) - \chi_2 T_{21}(u) \tag{5.1.6}$$

and hence

$$\mathrm{P}_{1,0}(u) = T_{12}(u), \quad \mathrm{P}_{1,1}(u) = T_{11}(u), \quad \mathrm{P}_{1,2}(u) = -T_{21}(u)\,. \tag{5.1.7}$$

gl_3 case. For the special case of gl_3 there are only two antisymmetric transfer matrices $t_1(u)$ and $t_2(u)$ which in the notations described above admit the expansions of Table 5.1, where t_2 is written both in terms of the original monodromy elements T_{ij} and the elements T^{ij} defined in (5.1.4).

Since the transfer matrices $t_a(u)$ admits the expansion (3.7.15) into integrals of motion $I_{a,\alpha}$ it clearly follows that each $I_{a,\alpha}$ also admits a linear expansion into characters χ_r. We will denote the coefficients of the characters in this expansion $I^{(r)}_{a,\alpha}$ and so

$$\hat{I}_{a,\alpha} = \sum_{r=0}^{N} \chi_r\, \hat{I}^{(r)}_{a,\alpha}\,. \tag{5.1.8}$$

Finally, since the transfer matrices commute for different values of the spectral parameters $[t_a(u), t_b(v)] = 0$ we see that by expanding into principal operators we obtain the relation

$$\sum_{r,s} \chi_r \chi_s [\mathrm{P}_{a,r}(u), \mathrm{P}_{b,s}(v)] = 0\,. \tag{5.1.9}$$

$\mathrm{P}_{1,0}(u) =$	$+T_{12} + T_{23}$	
$\mathrm{P}_{1,1}(u) =$	$+T_{11}$	
$\mathrm{P}_{1,2}(u) =$	$-T_{21}$	
$\mathrm{P}_{1,3}(u) =$	$+T_{31}$	
$\mathrm{P}_{2,0}(u) =$	$\left(T_{12}T_{23}^{--} - T_{13}T_{22}^{--}\right)/Q_\theta^{[2\mathbf{s}-2]}$	$+T^{13}/Q_\theta^{[2\mathbf{s}-2]}$
$\mathrm{P}_{2,1}(u) =$	$\left(T_{11}T_{23}^{--} - T_{13}T_{21}^{--}\right)/Q_\theta^{[2\mathbf{s}-2]}$	$+T^{23}/Q_\theta^{[2\mathbf{s}-2]}$
$\mathrm{P}_{2,2}(u) =$	$\left(T_{11}T_{22}^{--} - T_{12}T_{21}^{--}\right)/Q_\theta^{[2\mathbf{s}-2]}$	$+T^{33}/Q_\theta^{[2\mathbf{s}-2]}$
$\mathrm{P}_{2,3}(u) =$	$\left(-T_{11}T_{32}^{--} + T_{12}T_{31}^{--} - T_{21}T_{33}^{--} + T_{23}T_{31}^{--}\right)/Q_\theta^{[2\mathbf{s}-2]}$	$-(T^{21} + T^{32})/Q_\theta^{[2\mathbf{s}-2]}$

Table 5.1: gl_3 Principal Operators in terms of Monodromy Matrix elements.

As this should hold for arbitrary twist eigenvalues λ it is easy to see[1] that the above expression implies $[\mathrm{P}_{a,r}(u), \mathrm{P}_{b,r}(v)] = 0$, that is principal operators corresponding to the same character index r form a commutative family.

As a final note, we show that the B and C operators can be written in terms of Principal operators. For the $Y(gl_3)$ case, using the RTT relations it is possible to rewrite the expressions for B (4.4.1) and C (4.4.9) in a slightly different form:

$$B(u) = -T_{11}(T_{11}T_{22}^{--} - T_{22}T_{21}^{--}) - (T_{11}T_{23}^{--} - T_{13}T_{21}^{--})T_{21} \ , \tag{5.1.10}$$

$$C(u) = -T_{11}(T_{11}^{++}T_{22} - T_{22}^{++}T_{21}) - (T_{11}^{++}T_{23} - T_{13}^{++}T_{21})T_{21} \ . \tag{5.1.11}$$

This simple rewriting allows us to express the B and C operators in terms of the principal operators (after removing the trivial non-dynamical factor) in an ordering which will be convenient later

$$-\frac{B(u)}{Q_\theta^{[2\mathbf{s}-2]}} \equiv \mathbf{b}(u) = \mathrm{P}_{1,1}\mathrm{P}_{2,2} - \mathrm{P}_{2,1}\mathrm{P}_{1,2} \ , \quad -\frac{C(u)}{Q_\theta^{[2\mathbf{s}]}} \equiv \mathbf{c}(u) = \mathrm{P}_{1,1}\mathrm{P}_{2,2}^{++} - \mathrm{P}_{2,1}^{++}\mathrm{P}_{1,2} \ . \tag{5.1.12}$$

5.2 Functional Separation of Variables method

In this section we review the key idea of the functional separation of variables method of [82]. We will then extend this method in section 5.3 by introducing the character projection tool.

[1] For example one can change variables from λ_i, $i = 1, \ldots, N$ to χ_i, $i = 1, \ldots, N$. The Jacobian of such transformation is simply a Vandermonde determinant of λ's so this is always possible for generic λ's. After that (5.1.9) becomes a quadratic polynomial in N independent variable χ_i, $i = 1, \ldots, N$ which is identically zero, which is only possible if all coefficients vanish.

We will use slightly different conventions compared to chapter 3. First, we will use the Q-functions:

$$Q_A(u) = \prod_{j=1}^{|A|} \lambda_j^{iu} q_A(u), \qquad q_A \sim u^{M_i},\ M_i \in \mathbb{N}\,. \tag{5.2.1}$$

These Q-functions are analytic in all the complex plane and will render the calculations in this chapter much simpler.

With this choice for the Q-functions, the Baxter and Dual Baxter operators are:

$$\mathcal{O}^\dagger = \sum_{a=0}^{N} (-1)^a \tau_a(u) \mathcal{D}^{N-2a} \ ,\ \ \mathcal{O} = \sum_{a=0}^{N} (-1)^a \mathcal{D}^{2a-N} \tau_a(u) \varepsilon(u) \tag{5.2.2}$$

where $\mathcal{D}$ is the shift operator satisfying $\mathcal{D}\, f(u) = f(u + \frac{i}{2})$, $\tau_a,\ a = 1, \dots, N-1$ are the eigenvalues of the totally antisymmetric transfer matrices t_a and we have denoted:

$$\tau_0(u) = Q_\theta^{[2\mathbf{s}]}, \quad \tau_N(u) = \chi_N Q_\theta^{[-2\mathbf{s}]}, \quad Q_\theta(u) = \prod_{\alpha=1}^{L} (u - \theta_\alpha)\,. \tag{5.2.3}$$

Here, $\chi_N = \det(G)$, where G is the twist introduced in section 3.5. Finally $\varepsilon(u)$ is the function

$$\varepsilon(u) = \prod_{\beta=1}^{L} \frac{\Gamma(\mathbf{s} - i(u - \theta_\beta))}{\Gamma(1 - \mathbf{s} - i(u - \theta_\beta))}\,. \tag{5.2.4}$$

5.2.1 Functional orthogonality and scalar product

The key relation in the FSoV approach is the adjointness condition [74, 81, 82]

$$\Big(\!\Big(f \mathcal{O}^\dagger g \Big)\!\Big)_\alpha = \Big(\!\Big(g M_\alpha \mathcal{O} f \Big)\!\Big)_\alpha\,, \tag{5.2.5}$$

where $\mathcal{O}$ and $\mathcal{O}^\dagger$ are the Baxter and the dual Baxter operators defined in (5.2.2), the bracket $\Big(\!\Big(f(w) \Big)\!\Big)_\alpha$ is defined by

$$\Big(\!\Big(f(w) \Big)\!\Big)_\alpha = \int_{-\infty}^{\infty} \mathrm{d}w\, \mu_\alpha(w) f(w)\,, \tag{5.2.6}$$

the measure factor μ_α is given by [74]

$$\mu_\alpha(w) = \frac{1}{1 - e^{2\pi(w - \theta_\alpha - i\mathbf{s})}} \prod_{\beta=1}^{L} \frac{\Gamma(\mathbf{s} - i(w - \theta_\beta))}{\Gamma(1 - \mathbf{s} - i(w - \theta_\beta))}\,, \tag{5.2.7}$$

and M_α is some unimportant factor which does not depend on the functions f and g that can be set to 1 by changing our conventions.

We will be interested in particular in the case where the functions f and g are the twisted Q-functions Q_1 (5.2.1) and the dual Q-functions $Q^2, \dots, Q^N$ (defined in (3.6.10)),

or functions with similar analytic properties. The way to compute these integrals is to close the contour in the upper half plane and write them as a sum of residues. However, we need to ensure that the integrals actually converge and that the contour can be closed in this way without changing the result. In order to do so, it is sufficient to impose constraints on the twist eigenvalues that we find inside the Q-functions, as in [74], which read

$$0 < \arg\lambda_a - \arg\lambda_1 < 2\pi, \quad a = 2, \dots, N\,. \tag{5.2.8}$$

Once we do this, we can replace the integral by the sum of the residues in the upper half-plane. Since the Q-functions (5.2.1) are analytic everywhere, the only contribution comes from the simple poles of the measure factor (5.2.7). These poles are situated at $w = \theta_\alpha + i\mathbf{s} + in$, $n \in \mathbb{Z}_{\geqslant 0}$. As such we can write the bracket as an infinite sum of the residues at the poles of the measure:

$$\Big\langle\!\Big\langle\, f(w) \,\Big\rangle\!\Big\rangle_\alpha = \sum_{n=0}^{\infty} \frac{r_{\alpha,n}}{r_{\alpha,0}} f(\theta_\alpha + i\mathbf{s} + in)\,, \tag{5.2.9}$$

with $r_{\alpha,n}$ being the residue of μ_α at the pole $\theta_\alpha + i\mathbf{s} + in$:

$$r_{\alpha,n} = -\frac{1}{2\pi} \prod_{\beta=1}^{L} (n + 1 - i\theta_\alpha + i\theta_\beta)_{2\mathbf{s}-1}\,, \tag{5.2.10}$$

where $(z)_s = \frac{\Gamma(s+z)}{\Gamma(z)}$ denotes the Pochhammer symbol and we have included the overall normalisation $r_{\alpha,0}$ for convenience.

5.2.2 Basic idea of Functional SoV

To demonstrate the basic idea of the FSoV notice that the adjointness condition (5.2.5) implies in particular

$$\Big\langle\!\Big\langle\, f\mathcal{O}^\dagger Q^{1+a} \,\Big\rangle\!\Big\rangle_\alpha = 0 = \Big\langle\!\Big\langle\, Q_1 \mathcal{O}^\dagger g \,\Big\rangle\!\Big\rangle_\alpha = 0, \quad \alpha = 1, \dots, L, \quad a = 1, \dots, N-1 \tag{5.2.11}$$

and so if we pick Q_A^{1+a} and Q_1^B to be the Q-functions associated to two transfer matrix eigenstates $|\Psi_A\rangle$ and $|\Psi_B\rangle$ we have:

$$\Big\langle\!\Big\langle\, Q_1^B (\mathcal{O}_A^\dagger - \mathcal{O}_B^\dagger) Q_A^{1+a} \,\Big\rangle\!\Big\rangle_\alpha = 0,\ \alpha = 1, \dots, L,\ a = 1, \dots, N-1\,. \tag{5.2.12}$$

Now if we insert the explicit form of $\mathcal{O}^\dagger$ (5.2.2) for the two states A and B we obtain the following system of equations:

$$\sum_{\beta=1}^{L} \sum_{b=1}^{N-1} \Big\langle\!\Big\langle\, Q_1^B u^{\beta-1} Q_A^{1+a\,[N-2b]} \,\Big\rangle\!\Big\rangle_\alpha I_{b,\beta}^{AB} = 0,\ \alpha = 1, \dots, L,\ a = 1, \dots, N-1 \tag{5.2.13}$$

where we have defined $I^{AB}_{b,\beta} = (-1)^b(I^A_{b,\beta} - I^B_{b,\beta})$. Here $I^A_{b,\beta}$ ($I^B_{b,\beta}$) are the eigenvalues of the integrals of motion $\hat{I}_{b,\beta}$ (defined in (3.7.15)) evaluated on the state $|\Psi_A\rangle$ ($|\Psi_B\rangle$). All other terms of the dual Baxter operator cancel out since they do not depend on the state. Since the collection of integrals of motion $I_{b,\beta}$ has non-degenerate spectrum at least one of the differences $I^{AB}_{b,\beta}$ must be non-zero for the two distinct states and so in order for the linear system (5.2.13) to have a non-trivial solution we must have[2]

$$\det_{(a,\alpha),(b,\beta)} \left(\!\!\left(Q^B_1 u^{\beta-1} Q^{1+a\,[N-2b]}_A \right)\!\!\right)_\alpha \propto \delta_{AB}\,. \tag{5.2.14}$$

This is the *functional orthogonality relation*, and is the cornerstone of the Functional SoV program. It reproduces a crucial feature of the scalar product between two Bethe states, namely that it vanishes for two distinct states. In fact, it can be shown [74] to be exactly *identical* to the scalar product (4.4.31) by including a state-independent normalisation $\mathcal{N}$ which should be chosen to ensure that $\mathcal{M}_{0,0} = 1$ and so we have

$$\langle \Psi_A | \Psi_B \rangle = \frac{1}{\mathcal{N}} \det_{(a,\alpha),(b,\beta)} \left(\!\!\left(Q^B_1 u^{\beta-1} Q^{1+a\,[N-2b]}_A \right)\!\!\right)_\alpha , \tag{5.2.15}$$

where the normalisation factor $\mathcal{N}$ is given by

$$\mathcal{N} = \prod_{\alpha>\beta} (\theta_\alpha - \theta_\beta)^{N-1} = (-1)^{\frac{L}{2}(L-1)(N-1)} \Delta_\theta^{N-1} \tag{5.2.16}$$

where Δ_θ is the Vandermonde determinant in the spin chain inhomogeneities:

$$\Delta_\theta := \prod_{\alpha<\beta} (\theta_\alpha - \theta_\beta)\,. \tag{5.2.17}$$

5.2.3 Scalar product between arbitrary factorisable states

The functional orthogonality relation (5.2.14), together with the orthogonality conditions for the SoV vacuum states $\mathcal{M}_{0,\mathsf{x}} = \delta_{0,\mathsf{x}}$ and $\mathcal{M}_{\mathsf{y},0} = \delta_{\mathsf{y},0}$, allows one to completely determine all matrix elements $\mathcal{M}_{\mathsf{y},\mathsf{x}}$ of the SoV measure (4.4.25) from the knowledge of the determinant form of the scalar product (5.2.15). In fact, by considering all possible pairs of different Bethe states A and B, we obtain a system of linear equations for every matrix element. A rigorous counting can even be carried out in the infinite-dimensional case, as detailed in [74].

As was noticed in [74] the fact that the determinant (5.2.15) reproduces the sum (4.4.31) is independent of whether or not the functions Q_1 and Q^{1+a} actually solve the Baxter equation. As a result, we can consider any so-called *factorisable* states $|\Phi\rangle$ and $\langle\Theta|$ with wave functions

$$\Phi(\mathsf{x}) = \prod_{\alpha=1}^{L} \prod_{a=1}^{N-1} F_\alpha(\mathsf{x}_{\alpha,a}), \quad \Theta(\mathsf{y}) = \prod_{\alpha=1}^{L} \det_{1\leqslant a,b\leqslant N-1} G^{1+a}_\alpha \left(\mathsf{y}_{\alpha,b} + \frac{i}{2}(N-2) \right), \tag{5.2.18}$$

[2] A row in this matrix is labelled by the pair (a,α) and a column is labelled by the pair (b,β). The pairs of indices (a,α) and (b,β) are ordered lexicographically.

where F_α and G_α^{1+a} can be any functions (chosen such that the infinite sum over SoV states converges) and their scalar product will still be given by the determinant (5.2.15), where the bracket is understood as the sum over residues (5.2.9).

A useful and non-trivial example to consider is the case of the scalar product between eigenstates of two transfer matrices built with different twists. Concretely, we consider a family of transfer matrices t_a of twist G and another family of transfer matrices $\tilde{t}_a$ with G replaced by $\tilde{G}$, obtained by replacing the twist eigenvalues λ_i of G with a new set $\tilde{\lambda}_i$. In fact as we mentioned in section 3.5, the SoV bases are independent of the twist parameters λ_j. As a result, the same SoV bases serve to factorise the wave functions of transfer matrices built with *any* twist matrix of the form (4.4.4) such as $\tilde{G}$ and so we have

$$\langle \Psi_A | \tilde{\Psi}_B \rangle = \sum_{\mathsf{x},\mathsf{y}} \Psi_A(\mathsf{y}) \mathcal{M}_{\mathsf{y},\mathsf{x}} \tilde{\Psi}_B(\mathsf{x}) \,, \tag{5.2.19}$$

where we have denoted a right eigenstate of the transfer matrices $\tilde{t}_a$ by $|\tilde{\Psi}_B\rangle$. This means that we can easily compute scalar products between eigenstates of transfer matrices with different twists via determinants of Q-functions. In particular we get:

$$\langle \Psi_A | \tilde{\Psi}_B \rangle = \frac{1}{\mathcal{N}} \det_{(a,\alpha),(b,\beta)} \left(\!\!\left(\tilde{Q}_1^B u^{\beta-1} Q_A^{1+a\,[N-2b]} \right)\!\!\right)_\alpha \tag{5.2.20}$$

where $\tilde{Q}_1^B$ are the Q-functions associated to the state $|\tilde{\Psi}_B\rangle$ and the transfer matrices $\tilde{t}_a(u)$.

5.2.4 Correlators from variation of spin chain parameters

The functional SoV approach allows one to extract a host of diagonal form-factors by varying the integrals of motion with respect to some parameter p of the spin chain, such as twists λ_j or inhomogenities θ_α or even the local representation weights. The construction is based on standard quantum mechanical perturbation theory and we review it here.

The starting point is the trivial relation $\left(\!\!\left(Q_1 \mathcal{O}^\dagger Q^{1+a} \right)\!\!\right) = 0$ with Q^{1+a} being on-shell Q-function, i.e. satisfying the dual Baxter equation $\mathcal{O}^\dagger Q^{1+a} = 0$. This obviously remains true if we consider a variation $p \to p + \delta p$ of the parameter p in Q^{1+a} and $\mathcal{O}$ resulting in:

$$\left(\!\!\left(Q_1 (\mathcal{O}^\dagger + \delta_p \mathcal{O}^\dagger)(Q^{1+a} + \delta_p Q^{1+a}) \right)\!\!\right) = 0. \tag{5.2.21}$$

Expanding to first order in δ, using the adjointness property of $\mathcal{O}^\dagger$ and also assuming that $\mathcal{O} Q_1 = 0$ we obtain at the leading order in perturbation theory:

$$\left(\!\!\left(Q_1 \partial_p \mathcal{O}^\dagger Q^{1+a} \right)\!\!\right)_\alpha = 0 \,. \tag{5.2.22}$$

By expanding out $\partial_p \mathcal{O}^\dagger$, this relation allows one to obtain an inhomogeneous linear system for the derivatives $\partial_p I_{b,\beta}$ of the integral of motion eigenvalues $I_{b,\beta}$. As a result we have the relation, following from Cramer's rule,

$$\frac{\langle \Psi | \partial_p \hat{I}_{b',\beta'} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \partial_p I_{b',\beta'} = \frac{\det\limits_{(a,\alpha),(\beta,b)} m'_{(a,\alpha),(b,\beta)}}{\det\limits_{(a,\alpha),(\beta,b)} m_{(a,\alpha),(b,\beta)}} \,, \tag{5.2.23}$$

where $m_{(a,\alpha),(b,\beta)} = \Big(\!\Big(Q_1 u^{\beta-1}\mathcal{D}^{N-2b}Q^{1+a} \Big)\!\Big)_\alpha$ and m' is obtained from m by replacing the column (b',β') with $y_{(a,\alpha)} \equiv \Big(\!\Big(Q_1 \hat{Y}_p \circ Q^{1+a} \Big)\!\Big)_\alpha$, where $\hat{Y}_p$ is the part of $\partial_p \mathcal{O}^\dagger$ which does not depend on the integrals of motion, given by:

$$\hat{Y}_p = -\left(\partial_p Q_\theta^{[-2\mathbf{s}]}\mathcal{D}^{-N} + (-1)^N \partial_p Q_\theta^{[+2\mathbf{s}]}\mathcal{D}^{+N}\right) - \sum_{b=1}^{N-1}(-1)^{b+1}\partial_p \chi_b u^L \mathcal{D}^{-2b+N}\,. \tag{5.2.24}$$

We introduce the short-hand notation for the determinants as follows

$$[o_{b,\beta}] \equiv \det_{(a,\alpha),(b,\beta)} \Big(\!\Big(\tilde{Q}_1^B o_{b,\beta} Q^A_{1,1+a} \Big)\!\Big)\,, \tag{5.2.25}$$

where $o_{b,\beta}$ is some finite difference operator. Since the LHS makes no reference to the twists or indices A and B used on the Q-functions these should be inferred from context. As such the scalar product in this notation is given by

$$\langle \Psi_A | \tilde{\Psi}_B \rangle = \frac{1}{\mathcal{N}}[w^{\beta-1}\mathcal{D}^{3-2b}]\,. \tag{5.2.26}$$

We will also use the replacement notation

$$[(b',\beta') \to o]\,, \tag{5.2.27}$$

which corresponds to replacing $w^{\beta'-1}\mathcal{D}^{3-2b'}$ in the determinant $[w^{\beta-1}\mathcal{D}^{3-2b}]$ with the finite difference operator o. For instance the numerator of (5.2.23) becomes

$$\det_{(a,\alpha),(b,\beta)} m'_{(a,\alpha),(b,\beta)} \equiv \big[(b',\beta') \to \hat{Y}\big]\,. \tag{5.2.28}$$

Since the scalar product $\langle\Psi|\Psi\rangle$ in our normalisation is proportional to the denominator of the right hand side (see (5.2.19)) we have

$$\langle \Psi | \partial_p \hat{I}_{b',\beta'} | \Psi \rangle = \frac{1}{\mathcal{N}}[(b',\beta') \to \hat{Y}]\,. \tag{5.2.29}$$

It is appealing to assume that the operator $\partial_p \hat{I}_{b',\beta'}$ can be characterised by this particular modification of the structure of the determinant as compared to the identity operator given by (5.2.19). One can also notice that for the identity operator in (5.2.19) we managed to obtain a more general relation with the left and right states corresponding to two different eigenvalues of the transfer matrix or, even more generally, to the transfer matrices with different twists. It is thus very tempting to upgrade the relation (5.2.29) by replacing $\langle\Psi|$ and Q^{1+a} accordingly by those corresponding to a different state. Whereas this does give the right result in some cases, as was noticed in [25], in general this strategy, unfortunately, fails as we verified explicitly for some small length cases. However, for the case when the parameters p are the twist angles this naive approach gives the correct result as we prove in the next section where we also provide generalisations of this result.

However, we noticed that for the case of $p = \lambda_a$ the equation (5.2.29) survives a series of upgrades. Firstly, it works for two arbitrary left and right factorisable states. Secondly, and probably the most surprising, it still works for multiple derivatives in the twist parameters:

$$\langle \Psi^A | \partial_{\lambda_{a_1}} \dots \partial_{\lambda_{a_k}} \hat{I}_{b',\beta'} | \Psi^B \rangle = \frac{1}{\mathcal{N}} \left[(b',\beta') \to - \sum_{b=1}^{N-1} (-1)^{b+1} \partial_{\lambda_{a_1}} \dots \partial_{\lambda_{a_k}} \chi_b u^L \mathcal{D}^{-2b+N} \right] . \tag{5.2.30}$$

In the next section we will derive this identity using character projection. We will also see more explicitly that the operators of type (5.2.30) are closely related with the principal operators introduced earlier.

5.3 Character projection

In this section we extend the FSoV method, introduced in the previous section, in order to obtain form-factors of non-trivial operators between two arbitrary factorisable states. We will use these results in the next section to extract the matrix elements of a set of observables in the SoV bases in a similar way to the measure, which then allows us to efficiently compute the expectation values of a complete set of physical observables. For simplicity in this section we only analyse the gl_3 case.

5.3.1 Derivation

We start from the conjugate Baxter operator $\mathcal{O}^\dagger$. We define the gl_3 dual Q-functions as $Q^{1+a} := Q_{1,1+a}$. $\mathcal{O}^\dagger$ gives 0 when applied to the $Q_{1,1+a}$ functions as they satisfy the dual Baxter equation (5.2.2), which in the gl_3 case becomes[3]:

$$\mathcal{O}^\dagger = Q_\theta^{[2\mathbf{s}]} D^3 - \tau_1 D^1 + \tau_2 D^{-1} - \chi_3 Q_\theta^{[-2\mathbf{s}]} D^{-3} \ , \ \mathcal{O}^\dagger Q_{1,1+a} = 0 \, . \tag{5.3.1}$$

This implies that for any g, chosen such that the integral in the scalar product is convergent, we have:

$$\Big(g \mathcal{O}_A^\dagger Q_{1,a+1}^A \Big)_\alpha = 0 \ , \ \alpha = 1, \dots, L \ , \ a = 1, 2 \, . \tag{5.3.2}$$

For definiteness we take $g = \tilde{Q}_1^B$, which is a Q-function corresponding to a state of a transfer matrix with generic twist eigenvalue $\tilde{\lambda}_a$, different from that of the state A, which we denote as λ_a. The corresponding characters are denoted as $\tilde{\chi}_r$ and χ_r. We consider the set of $2L$ equations in (5.3.2) as equations on the $2L$ integrals of motion $I_{b,\beta}^A$, $b = 1, 2$, $\beta = 1, \dots, L$, which are the non-trivial coefficients in $\tau_2(u)$ and $\tau_1(u)$. More explicitly we have

$$\sum_{\beta,b} (-1)^b \Big(\tilde{Q}_1^B u^{\beta-1} \mathcal{D}^{3-2b} Q_{1,a+1}^A \Big)_\alpha I_{b,\beta}^A = - \sum_{r=0}^{3} \chi_r \Big(\tilde{Q}_1^B \mathcal{O}_{(r)}^\dagger Q_{1,a+1}^A \Big)_\alpha \, , \tag{5.3.3}$$

[3] In the gl_3 case, the dual Q-functions Q^{a+1} are by definition equivalent, up to a sign, to the Q-functions with two indices $Q_{1,a+1}$ for $a = 1, 2$.

where we introduced the following notations for the non-dynamical terms in the dual Baxter equation (5.2.2):

$$\mathcal{O}^{\dagger}_{(0)} = Q_{\theta}^{[2\mathbf{s}]}\mathcal{D}^{3} \ , \ \mathcal{O}^{\dagger}_{(1)} = -u^{L}\mathcal{D} \ , \ \mathcal{O}^{\dagger}_{(2)} = u^{L}\mathcal{D}^{-1} \ , \ \mathcal{O}^{\dagger}_{(3)} = -Q_{\theta}^{[-2\mathbf{s}]}\mathcal{D}^{-3} \ . \tag{5.3.4}$$

The solution to (5.3.3) can be written as a ratio of determinants. In the notations of section 5.2.4 we have

$$I^{A}_{b',\beta'} = (-1)^{b'+1}\sum_{r=0}^{3}\chi_{r}\frac{[(b',\beta')\to\mathcal{O}^{\dagger}_{(r)}]}{[w^{\beta-1}\mathcal{D}^{3-2b}]} \ . \tag{5.3.5}$$

At the same time, since $I^{A}_{b',\beta'}$ is the eigenvalue of the operator $\hat{I}_{b',\beta'}$ on the left eigenstate $\langle\Psi^{A}|$ we have

$$I^{A}_{b',\beta'} = \frac{\langle\Psi^{A}|\hat{I}_{b',\beta'}|\tilde{\Psi}^{B}\rangle}{\langle\Psi^{A}|\tilde{\Psi}^{B}\rangle} = \mathcal{N}\frac{\langle\Psi^{A}|\hat{I}_{b',\beta'}|\tilde{\Psi}^{B}\rangle}{[w^{\beta-1}\mathcal{D}^{3-2b}]} \tag{5.3.6}$$

where in the last identity we used the expression for the scalar product of two factorisable states (5.2.20). Comparing (5.3.5) and (5.3.6) we get

$$\langle\Psi^{A}|\hat{I}_{b',\beta'}|\tilde{\Psi}^{B}\rangle = \frac{(-1)^{b'+1}}{\mathcal{N}}\sum_{r=0}^{3}\chi_{r}\ [(b',\beta')\to\mathcal{O}^{\dagger}_{(r)}] \ . \tag{5.3.7}$$

The next step, which we call *character projection*, is quite crucial. As we discussed in Section 5.1 the IoMs, as operators, depend non-trivially on the twist of the spin chain λ_a, but when expressed in terms of the characters this dependence is linear in χ_r, see (5.1.8). We also notice that the RHS of (5.3.7) has explicit linear dependence on χ_r. However, notice that both sides of (5.3.7) have an additional implicit dependence on the twists through the eigenstate $\langle\Psi^{A}|$ and the corresponding Q-function $Q^{A}_{1,1+a}$. In order to remove this dependence we use the result of section 5.2.3, which states that the determinants in the RHS of (5.3.7) can be written in the form

$$\frac{(-1)^{b'+1}}{\mathcal{N}}[(b',\beta')\to\mathcal{O}^{\dagger}_{(r)}] = \sum_{\mathsf{x},\mathsf{y}}\Psi^{A}(\mathsf{y})M^{(r);b',\beta'}_{\mathsf{y},\mathsf{x}}\tilde{\Psi}^{B}(\mathsf{x}) \tag{5.3.8}$$

which is analogous to (5.2.19), with $M^{(r);\beta',a'}_{\mathsf{y},\mathsf{x}}$ being independent of the states A and B. In section 5.4.2 we compute the coefficients $M^{(r);\beta',a'}_{\mathsf{x},\mathsf{y}}$ explicitly. The expression (5.3.8) is obtained by expanding the determinant and comparing the combinations of the Q-functions with those appearing in $\tilde{\Psi}^{B}(\mathsf{x})$ and $\Psi^{A}(\mathsf{y})$ as shown in (4.4.20).

At the same time for the LHS of (5.3.7) we have

$$\langle\Psi^{A}|\hat{I}_{b',\beta'}|\tilde{\Psi}^{B}\rangle = \sum_{\mathsf{x},\mathsf{y}}\langle\Psi^{A}|\mathsf{y}\rangle\langle\mathsf{y}|\hat{I}_{b',\beta'}|\mathsf{x}\rangle\langle\mathsf{x}|\tilde{\Psi}^{B}\rangle \tag{5.3.9}$$

by using completeness of SoV bases. The operator $\hat{I}_{b',\beta'}$ can be decomposed into terms corresponding to different characters χ_r as $\hat{I}_{b',\beta'} = \sum_{r=0}^{3} \chi_r \hat{I}^{(r)}_{b',\beta'}$, see (5.1.8). By comparing (5.3.8) and (5.3.9) we get

$$\sum_{\mathsf{x},\mathsf{y}} \langle \Psi^A | \mathsf{y} \rangle \langle \mathsf{x} | \tilde{\Psi}^B \rangle \left[\sum_{r=0}^{3} \chi_r \left(\langle \mathsf{x} | \hat{I}^{(r)}_{b',\beta'} | \mathsf{y} \rangle - M^{(r);b',\beta'}_{\mathsf{x},\mathsf{y}} \right) \right] = 0 \,. \tag{5.3.10}$$

Note that the expression in the square brackets does not depend on the state A and only carries the information on the twist of this state in the characters χ_r. For simplicity, consider an arbitrary finite dimensional case with representation of dimension D per site. Considering the expression in the square bracket as a collection of $D^L \times D^L$ numbers computed for different x and y we get a system of linear equations on those coefficients. There are D^L states $\langle \Psi^A |$ and D^L states $|\tilde{\Psi}^B\rangle$ so we have as many equations as unknowns and furthermore the matrix $\langle \Psi^A | \mathsf{y} \rangle \langle \mathsf{x} | \tilde{\Psi}^B \rangle$ can be considered as an overlap matrix between two complete bases $\langle \Psi^A | \otimes | \tilde{\Psi}^B \rangle$ to $\langle \mathsf{x} | \otimes | \mathsf{y} \rangle$ in the double copy of the initial Hilbert space $H \otimes H^\dagger$, and thus is not degenerate. In fact we have many more of the equations as $|\tilde{\Psi}^B\rangle$ contains its own set of independent continuous twist parameters. We see that as a consequence of the consistency of the linear system it should have a trivial solution and thus we should have that the square bracket is identically zero

$$\sum_{r=0}^{3} \chi_r \left(\langle \mathsf{x} | \hat{I}^{(r)}_{b',\beta'} | \mathsf{y} \rangle - M^{(r);b',\beta'}_{\mathsf{x},\mathsf{y}} \right) = 0 \,. \tag{5.3.11}$$

The above equation also stays true for the infinite dimensional case and this will be argued in Appendix C.1 where the coefficients $M^{(r);b',\beta'}_{\mathsf{x},\mathsf{y}}$ are explicitly computed.

Another way to arrive to (5.3.11) from (5.3.10) is by multiplying the LHS by $\langle \mathsf{y}' | \Psi^A \rangle \langle \tilde{\Psi}^B | \mathsf{x}' \rangle$ and summing over complete basis of eigenstates Ψ^A and Ψ^B with the completeness relation[4]

$$1 = \sum_A |\Psi_A\rangle\langle\Psi_A| \tag{5.3.12}$$

As a result we have $\sum_A \langle \mathsf{y}' | \Psi^A \rangle \langle \Psi^A | \mathsf{y} \rangle = \delta_{y y'}$ which removes the dependence on the wave functions and leads to (5.3.11).

Next, the round bracket in (5.3.11) does not depend on the twists, and the only way the above identity stays true for arbitrary values of twists is if

$$\langle \mathsf{x} | I^{(r)}_{b',\beta'} | \mathsf{y} \rangle = M^{(r);b',\beta'}_{\mathsf{x},\mathsf{y}} \,. \tag{5.3.13}$$

Thus we get a set of $4 \times 2 \times L$ observables $\hat{I}^{(r)}_{a,\alpha}$ explicitly in the SoV basis, which are precisely the coefficients of the principal operators $\mathrm{P}_{a,r}(u)$

$$\mathrm{P}_{a,r}(u) = \sum_{\beta=1}^{L} I^{(r)}_{a,\beta} u^{\beta-1} + u^L \delta_{a,r} \,. \tag{5.3.14}$$

[4]See Appendix D.3 for a proof of the existence of this relation for our family of infinite-dimensional representations.

In section 5.6.1 we prove that this set of non-local observables is complete and we will explicitly compute the SoV matrix elements for $\mathrm{P}_{a,r}(u)$ in section 5.3.4.

Finally, after obtaining the relations (5.3.13) for the individual operators in the SoV basis we can revert the logic and multiply (5.3.13) by $\sum_{\mathsf{x},\mathsf{y}}\langle\Psi^A|\mathsf{y}\rangle\langle\mathsf{x}|\tilde{\Psi}^B\rangle$ to obtain the *character projected* version of the equation (5.3.7)

$$\boxed{\langle\Psi^A|\hat{I}^{(r)}_{b',\beta'}|\tilde{\Psi}^B\rangle = \frac{(-1)^{b'+1}}{\mathcal{N}}[(b',\beta')\to\mathcal{O}^{\dagger}_{(r)}]}\,, \tag{5.3.15}$$

which constitutes the main result of this section. To summarise, we obtained a determinant form of form-factors of all operators $\hat{I}^{(r)}_{b,\beta}$ between two arbitrary factorisable states. It is easy to see that (5.3.15) is equivalent to (5.2.30).

Before closing this subsection a comment is in order. A key step in our derivation relied on the denominator in (5.3.6) being non-zero. This is indeed non-zero as long as $|\tilde{\Psi}_B\rangle$ is not orthogonal to $\langle\Psi_A|$ which is true as long as $|\tilde{\Psi}_B\rangle$ is a generic factorisable state or as long as the twists in $|\tilde{\Psi}_B\rangle$ are independent from those in $\langle\Psi_A|$. The expressions (5.3.15) for the form-factors are then valid for any choice of twists or indeed any factorisable states. However, it is possible to recast the derivation in an alternate way which avoids this step completely and we present it in Appendix D.1: the above derivation, which may be singular in certain degenerate cases, is presented to highlight the determinant origin of our result as a consequence of Cramer's rule. Finally, the counting argument presented above relied on the representation being finite dimensional. The results remain true even when extended to the infinite-dimensional case as is discussed in Appendix C.1.

5.3.2 Form-factors for gl_3 principal operators

In the previous section we found the form-factors of the coefficients $\hat{I}^{(r)}_{a,\alpha}$ of the u-expansion of the principal operators $\mathrm{P}_{a,r}(u)$. In this section we derive compact determinant expressions for the form-factors of $\mathrm{P}_{a,r}(u)$ themselves as functions of the spectral parameter u. We will use w for the dummy spectral parameter appearing inside the determinants to avoid confusion with u – the argument of $\mathrm{P}_{a,r}(u)$.

Let us start from $\mathrm{P}_{1,1}(u) = T_{11}(u)$. From (5.3.14) we see this principal operator is a generating function for the set of operators $\hat{I}^{(1)}_{1,\alpha}$ with $\alpha = 1,\dots,L$. From (5.3.15) we thus have

$$\langle\Psi^A|T_{11}(u)|\tilde{\Psi}^B\rangle \;=\; u^L\langle\Psi^A|\tilde{\Psi}^B\rangle - \frac{1}{\mathcal{N}}\sum_{\beta'=1}^{L} u^{\beta'-1}[(1,\beta')\to w^L\mathcal{D}]\,. \tag{5.3.16}$$

This expression appears to be a sum over determinants. Let us show that it can be compressed into a single determinant. Let us write the determinants in the sum (5.3.16) more explicitly by introducing the notation

$$[o_{b,\beta}] = [o_{1,1},\dots,o_{1,L},o_{2,1},\dots,o_{2,L}]\,, \tag{5.3.17}$$

obtaining

$$\begin{aligned}\mathcal{N}\langle\Psi^A|\mathbb{P}_{1,1}(u)|\tilde{\Psi}^B\rangle &= \\ &- u^0[w^L\mathcal{D}, w\mathcal{D}, w^2\mathcal{D}, \dots, w^{L-1}\mathcal{D}, \mathcal{D}^{-1}, w\mathcal{D}^{-1}, \dots, w^{L-1}\mathcal{D}^{-1}] \\ &- u^1[\mathcal{D}, w^L\mathcal{D}, w^2\mathcal{D}, \dots, w^{L-1}\mathcal{D}, \mathcal{D}^{-1}, w\mathcal{D}^{-1}, \dots, w^{L-1}\mathcal{D}^{-1}] \\ &- u^2[\mathcal{D}, w\mathcal{D}, w^L\mathcal{D}, \dots, w^{L-1}\mathcal{D}, \mathcal{D}^{-1}, w\mathcal{D}^{-1}, \dots, w^{L-1}\mathcal{D}^{-1}] \\ &\dots \\ &+ u^L[\mathcal{D}, w\mathcal{D}, w^2\mathcal{D}, \dots, w^{L-1}\mathcal{D}, \mathcal{D}^{-1}, w\mathcal{D}^{-1}, \dots, w^{L-1}\mathcal{D}^{-1}]\,,\end{aligned} \tag{5.3.18}$$

where in the last term we also wrote the overlap of the states in the determinant form (5.2.20). By a simple rearrangement of the columns we get:

$$(-1)^L[\{(w^j - u^j)\mathcal{D}\}_{j=1}^L, \{w^{j-1}\mathcal{D}^{-1}\}_{j=1}^L] \tag{5.3.19}$$

or equivalently:

$$(-1)^L[\{(w-u)w^{j-1}\mathcal{D}\}_{j=1}^L, \{w^{j-1}\mathcal{D}^{-1}\}_{j=1}^L]. \tag{5.3.20}$$

Hence we arrive to the following expression as a single determinant

$$\boxed{\langle\Psi^A|\mathbb{P}_{1,1}(u)|\tilde{\Psi}^B\rangle = \frac{(-1)^L}{\mathcal{N}}[\{(w-u)w^{j-1}\mathcal{D}\}_{j=1}^L, \{w^{j-1}\mathcal{D}^{-1}\}_{j=1}^L]\,.} \tag{5.3.21}$$

We will now introduce a very convenient shorthand notation. For ordered sets $\mathbf{u}_a$ and 4 integers L_a, $a = 0, 1, 2, 3$ we define the following object

$$\begin{aligned}&\left[L_0; \mathbf{u}_0\middle|L_1; \mathbf{u}_1\middle|L_2; \mathbf{u}_2\middle|L_3; \mathbf{u}_3\right]_\Psi = \frac{1}{\mathcal{N}} \times \\ &\left[\left\{\frac{\Delta_{\mathbf{u}_0\cup w}}{\Delta_{\mathbf{u}_0}}w^j\mathcal{D}^3\right\}_{j=0}^{L_0-1}, \left\{\frac{\Delta_{\mathbf{u}_1\cup w}}{\Delta_{\mathbf{u}_1}}w^j\mathcal{D}^1\right\}_{j=0}^{L_1-1}, \left\{\frac{\Delta_{\mathbf{u}_2\cup w}}{\Delta_{\mathbf{u}_2}}w^j\mathcal{D}^{-1}\right\}_{j=0}^{L_2-1}, \left\{\frac{\Delta_{\mathbf{u}_3\cup w}}{\Delta_{\mathbf{u}_3}}w^j\mathcal{D}^{-3}\right\}_{j=0}^{L_3-1}\right],\end{aligned} \tag{5.3.22}$$

where $\Delta_{\mathbf{v}}$ for some ordered set $\mathbf{v}$ is a Vandermonde determinant

$$\Delta_{\mathbf{v}} = \prod_{i<j}(v_i - v_j) \tag{5.3.23}$$

and $\mathbf{v}\cup w$ means that we add one element w to the ordered set $\mathbf{v}$ at the end. For example equation (5.3.21) can be written as

$$\langle\mathbb{P}_{1,1}(u)\rangle = \left[0; \middle|L; u\middle|L; \middle|0; \right]_\Psi\,. \tag{5.3.24}$$

Here and below we will systematically omit Ψ^A and $\tilde{\Psi}^B$. Note that the determinant in the RHS of (5.3.24) implicitly contains the Q-functions of the corresponding states.

Using a similar strategy as above we derived the following single determinant expressions for the principal operators between two arbitrary factorisable states

$$
\begin{array}{lccccccc}
\langle \hat{1} \rangle = & \Big[& 0; & \Big| & L; & \Big| & L; & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{1,0}(u) \rangle = - & \Big[& 1; \theta - i\mathbf{s} & \Big| & L-1; u & \Big| & L; & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{1,1}(u) \rangle = & \Big[& 0; & \Big| & L; u & \Big| & L; & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{1,2}(u) \rangle = (-1)^L & \Big[& 0; & \Big| & L-1; u & \Big| & L+1; & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{1,3}(u) \rangle = - & \Big[& 0; & \Big| & L-1; u & \Big| & L; & \Big| & 1; \theta + i\mathbf{s} & \Big]_\Psi \\
\langle \mathrm{P}_{2,0}(u) \rangle = (-1)^L & \Big[& 1; \theta - i\mathbf{s} & \Big| & L; & \Big| & L-1; u & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{2,1}(u) \rangle = (-1)^{L-1} & \Big[& 0; & \Big| & L+1; & \Big| & L-1; u & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{2,2}(u) \rangle = & \Big[& 0; & \Big| & L; & \Big| & L; u & \Big| & 0; & \Big]_\Psi \\
\langle \mathrm{P}_{2,3}(u) \rangle = (-1)^L & \Big[& 0; & \Big| & L; & \Big| & L-1; u & \Big| & 1; \theta + i\mathbf{s} & \Big]_\Psi
\end{array}
\tag{5.3.25}
$$

Here we have defined $\theta \pm i\mathbf{s} := \{\theta_1 \pm i\mathbf{s}, \dots, \theta_L \pm i\mathbf{s}\}$. In the next section we will use these expressions to obtain the matrix elements in the SoV basis of the principal operators.

5.3.3 Form-factors for gl_2 principal operators

In order to compare with previous results in the literature we also write form-factors for the principal operators in the case of the gl_2 spin chain in a form similar to those of the previous section.

We start from the gl_2 dual Baxter operator $\mathcal{O}^\dagger = Q_\theta^{[2s]}\mathcal{D}^2 - \tau_1 + \chi_2 Q_\theta^{[-2s]}\mathcal{D}^{-2}$. For the gl_2 spin chain, we only have the fundamental transfer matrix $t_1(u)$, so we only have the principal operators $\mathrm{P}_{1,r}(u)$, $r = 0, 1, 2$. The notation (5.3.22) in the gl_2 case becomes

$$
\begin{aligned}
&\Big[L_0; \mathbf{u}_0 \Big| L_1; \mathbf{u}_1 \Big| L_2; \mathbf{u}_2 \Big]_\Psi = \frac{1}{\mathcal{N}} \times \\
&\Big[\Big\{ \frac{\Delta_{\mathbf{u}_0 \cup w}}{\Delta_{\mathbf{u}_0}} w^j \mathcal{D}^2 \Big\}_{j=0}^{L_0-1}, \Big\{ \frac{\Delta_{\mathbf{u}_1 \cup w}}{\Delta_{\mathbf{u}_1}} w^j \Big\}_{j=0}^{L_1-1}, \Big\{ \frac{\Delta_{\mathbf{u}_2 \cup w}}{\Delta_{\mathbf{u}_2}} w^j \mathcal{D}^{-2} \Big\}_{j=0}^{L_2-1}, \Big] .
\end{aligned}
\tag{5.3.26}
$$

Following exactly the same steps as for gl_3 we find that the matrix elements for the principal

operators and the identity operator are given by

$$\begin{aligned}
\langle \hat{1} \rangle &= && \Big[0; && \Big| && L; && \Big| && 0; && \Big]_\Psi \\
\langle \mathbb{P}_{1,0}(u) \rangle &= +\langle T_{12}(u) \rangle = - && \Big[1; \theta - i\mathbf{s} && \Big| && L-1; u && \Big| && 0; && \Big]_\Psi \\
\langle \mathbb{P}_{1,1}(u) \rangle &= +\langle T_{11}(u) \rangle = && \Big[0; && \Big| && L; u && \Big| && 0; && \Big]_\Psi \\
\langle \mathbb{P}_{1,2}(u) \rangle &= -\langle T_{21}(u) \rangle = (-1)^L && \Big[0; && \Big| && L-1; u && \Big| && 1; \theta + i\mathbf{s} && \Big]_\Psi .
\end{aligned} \tag{5.3.27}$$

Here we used (5.1.7) to relate principal operators with the elements of the monodromy matrix. From these equations it is already easy to see that $T_{11}(u) = B(u)$ is the SoV B-operator, which acting on the factorised wave function, replaces $Q(w) \to (u-w)Q(w)$. We will analyse the action of the remaining operators on the SoV basis in the next section.

5.3.4 Principal operators in SoV basis

The goal of this section is to convert the form factors we have derived in section 5.3.2 to the SoV basis. The general strategy is simple: starting from a form factor $\langle \Psi^A | \hat{O} | \tilde{\Psi}^B \rangle$, for some operator $\hat{O}$, which we assume can be expressed as

$$\langle \Psi^A | \hat{O} | \tilde{\Psi}^B \rangle = \Big[L_0; \mathbf{u}_0 \Big| L_1; \mathbf{u}_1 \Big| L_2; \mathbf{u}_2 \Big| L_3; \mathbf{u}_3 \Big]_\Psi \tag{5.3.28}$$

we insert two resolutions of the identity $\sum_{\mathsf{x}} |\mathsf{x}\rangle\langle\mathsf{x}| = \sum_{\mathsf{y}} |\mathsf{y}\rangle\langle\mathsf{y}| = 1$:

$$\langle \Psi^A | \hat{O} | \tilde{\Psi}^B \rangle = \sum_{\mathsf{x},\mathsf{y}} \langle \Psi^A | \mathsf{y} \rangle \langle \mathsf{x} | \tilde{\Psi}^B \rangle \langle \mathsf{y} | \hat{O} | \mathsf{x} \rangle . \tag{5.3.29}$$

We then use (4.4.20) to write the RHS in terms of Q-functions. Since the LHS can be written in terms of determinants of Q-functions as proven in section 5.3.2, we can treat (5.3.29) as a linear system, where the unknowns are precisely the form factors in the SoV basis. It is then immediate to read off the matrix elements $\langle \mathsf{y} | \hat{O} | \mathsf{x} \rangle$.

It is straightforward to deduce a general formula, which we derive in Appendix C.1, which reads

$$\Big[L_0; \mathbf{u}_0 \Big| L_1; \mathbf{u}_1 \Big| L_2; \mathbf{u}_2 \Big| L_3; \mathbf{u}_3 \Big]_\Psi = \sum_{\mathsf{x}\mathsf{y}} \tilde{\Psi}_B(\mathsf{x}) \Psi_A(\mathsf{y}) \Big[L_0; \mathbf{u}_0 \Big| L_1; \mathbf{u}_1 \Big| L_2; \mathbf{u}_2 \Big| L_3; \mathbf{u}_3 \Big]_{\mathsf{x}\mathsf{y}} \tag{5.3.30}$$

where we have introduced the notation

$$\Big[L_0; \mathbf{u}_0 \Big| L_1; \mathbf{u}_1 \Big| L_2; \mathbf{u}_2 \Big| L_3; \mathbf{u}_3 \Big]_{\mathsf{x}\mathsf{y}} := \frac{s_{\mathbf{L}}}{\Delta_\theta^2} \sum_k \mathrm{sign}(\sigma) \prod_{\alpha,a} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \frac{\Delta_{\mathbf{u}_b \cup \mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \Bigg|_{\sigma_{a,\alpha} = k_{a,\alpha} - m_{\alpha,a} + a} . \tag{5.3.31}$$

The notation used here is identical to that used for the measure (6.7.15), with the only difference now being the sign factor $s_{\mathbf{L}}$ is defined as, for gl_N,

$$s_{\mathbf{L}} := (-1)^{\frac{LN}{4}(L-1)(N-1) + \sum_{n=0}^{N} \frac{L_n}{2}(L_n - 1)} \tag{5.3.32}$$

and now σ in (5.3.31) is a permutation of the set

$$\{\underbrace{0,\dots,0}_{L_0},\underbrace{1,\dots,1}_{L_1},\underbrace{2,\dots,2}_{L_2},\underbrace{3,\dots,3}_{L_3}\} \tag{5.3.33}$$

and as before $\sigma_{\alpha,a}$ denotes the number in position $a+2(\alpha-1)$.

Selection rules One can show that the SoV charge operator (4.4.24) imposes selection rules on the states $\langle \mathsf{y}|$ and $|\mathsf{x}\rangle$ for which the matrix elements $\langle \mathsf{y}|\hat{O}|\mathsf{x}\rangle$ can be non-zero. As we explain in Appendix C.1, the overlap can only be non-zero if there exists some permutation ρ^α of $\{1,2\}$ such that

$$m_{\alpha,a} = n_{\alpha,\rho^\alpha_a} - \sigma_{\alpha,\rho^\alpha_a} - a \tag{5.3.34}$$

for some fixed σ. We now sum over all values of (α,a) and denote the SoV charge of the state $\langle \mathsf{y}|$ ($|\mathsf{x}\rangle$) by $\mathbf{N}_\mathsf{y}$ ($\mathbf{N}_\mathsf{x}$). We obtain

$$\mathbf{N}_\mathsf{y} - \mathbf{N}_\mathsf{x} = 3L - \sum_{\alpha,a}\sigma_{\alpha,\rho^\alpha_a}\,. \tag{5.3.35}$$

Since σ is a permutation of (5.3.33) the sum $\sum_{\alpha,a}\sigma_{\rho^\alpha_a,\alpha}$ simply equates to $L_1+2L_2+3L_3$ and hence we see that $\langle \mathsf{y}|\hat{O}|\mathsf{x}\rangle$ is only non-zero if

$$\mathbf{N}_\mathsf{y} - \mathbf{N}_\mathsf{x} = 3L - \sum_{n=0}^{3} n\,L_n\,. \tag{5.3.36}$$

Notice that this reproduces the observation of [74] that the measure $\mathcal{M}_{\mathsf{yx}} = \langle \mathsf{y}|\mathsf{x}\rangle$ is only non-zero if $\mathbf{N}_\mathsf{x} = \mathbf{N}_\mathsf{y}$. Indeed, for the measure we have $L_0 = L_3 = 0$ and $L_1 = L_2 = L$. Plugging into (5.3.36) we immediately find $\mathbf{N}_\mathsf{x} = \mathbf{N}_\mathsf{y}$.

gl_2 matrix elements

Using the general formula (5.3.30) we will compute the SoV matrix elements of the gl_2 principal operators in order to make contact with existing results in literature.

Modifying the notation (5.3.31) to the case of gl_2 we define

$$\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big]_\Psi = \sum_{\mathsf{xy}}\tilde{\Psi}_B(\mathsf{x})\Psi_A(\mathsf{y})\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big]_{\mathsf{xy}} \tag{5.3.37}$$

with

$$\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big]_{\mathsf{xy}} = \frac{s_\mathbf{L}}{\Delta_\theta}\mathrm{sign}(\sigma)\prod_{\alpha,a}\frac{r_{\alpha,n_\alpha}}{r_{\alpha,0}}\prod_b \frac{\Delta_{\mathbf{u}_b\cup\mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}}\Bigg|_{\sigma_\alpha=n_\alpha-m_\alpha+1} \tag{5.3.38}$$

σ is a permutation of the set

$$\{\underbrace{0,\ldots,0}_{L_0},\underbrace{1,\ldots,1}_{L_1},\underbrace{2,\ldots,2}_{L_2}\} \tag{5.3.39}$$

with σ_α denoting the number at position α. Notice that unlike in the higher rank case there is no sum over k as only $k_\alpha = n_\alpha$ is possible.

We will now use this general formula to derive the SoV matrix elements of the gl_2 principal operators. We will begin with the operator $\mathrm{P}_{1,1}(u) = T_{11}(u)$ for which we have

$$\langle \mathrm{P}_{1,1}(u)\rangle = \Big[0;\Big|L;u\Big|0;\Big]_\Psi\,. \tag{5.3.40}$$

In this case σ is simply a permutation of $\{1,\ldots,1\}$ and the only possibility is that it is the identity permutation with $\sigma_\alpha = 1$. As a result we find that the non-zero matrix elements $\langle \mathsf{y}|\mathrm{P}_{1,1}(u)|\mathsf{x}\rangle$ are given by

$$\langle \mathsf{y}|\mathrm{P}_{1,1}(u)|\mathsf{x}\rangle = \frac{1}{\Delta_\theta}\prod_{\alpha=1}^{L}(u-\mathsf{x}_\alpha)\prod_{\alpha>\beta}(\mathsf{x}_\alpha-\mathsf{x}_\beta)\prod_{\alpha=1}^{L}\frac{r_{\alpha,n_\alpha}}{r_{\alpha,0}}\Bigg|_{m_\alpha=n_\alpha}\,. \tag{5.3.41}$$

We then read off that[5]

$$\langle \mathsf{y}|\mathrm{P}_{1,1}(u)|\mathsf{x}\rangle = \prod_{\alpha=1}^{L}(u-\mathsf{x}_\alpha)\langle \mathsf{y}|\mathsf{x}\rangle \tag{5.3.42}$$

and hence the operator $\mathrm{P}_{1,1}(u) = T_{11}(u)$ is diagonalised in the basis $|\mathsf{x}\rangle$. This is not surprising as $T_{11}(u)$ coincides with the Sklyanin's B operator when the twist is taken to be of the form (4.4.4). What is remarkable is that we *derived* that this operator acts diagonally on the SoV basis directly from the FSoV construction. We will later see that this persists at higher rank.

Next we examine $\mathrm{P}_{1,0}(u) = T_{12}(u)$ and have

$$\langle \mathrm{P}_{1,0}(u)\rangle = -\Big[1;\theta-i\mathbf{s}\Big|L-1;u\Big|0;\Big]_\Psi. \tag{5.3.43}$$

Using the relation (5.3.37) we obtain

$$\Big[1;\theta-i\mathbf{s}\Big|L-1;u\Big|0;\Big]_{\mathsf{xy}} = \frac{s_{\mathbf{L}}}{\Delta_\theta}\mathrm{sign}(\sigma)\frac{\Delta_{\theta-i\mathbf{s}\cup\mathsf{x}_{\sigma^{-1}(0)}}}{\Delta_{\theta-i\mathbf{s}}}\Delta_{u\cup\mathsf{x}_{\sigma^{-1}(1)}}\prod_\alpha\frac{r_{\alpha,n_\alpha}}{r_{\alpha,0}}\Bigg|_{\sigma_\alpha=n_\alpha-m_\alpha+1} \tag{5.3.44}$$

where now σ is a permutation of the set

$$\{0,1,\ldots,1\}\,. \tag{5.3.45}$$

[5]For gl_2 we see that the measure is diagonal and so $\langle \mathsf{y}| \propto \langle \mathsf{x}|$. We keep the notation $\langle \mathsf{y}|$ in order to be consistent with higher rank.

We can characterise each σ by the property $\sigma_\gamma = 0$ for some $\gamma = 1, \dots, L$ and there are L such permutations. Hence, we obtain

$$\langle \mathsf{y}|\mathbb{P}_{1,0}(u)|\mathsf{x}\rangle = \frac{Q_\theta^{[2\mathbf{s}]}(\mathsf{x}_\gamma)}{\Delta_\theta} \prod_{\alpha\neq\gamma} \frac{u-\mathsf{x}_\alpha}{\mathsf{x}_\gamma-\mathsf{x}_\alpha} \prod_{\alpha>\beta} (\mathsf{x}_\alpha - \mathsf{x}_\beta) \prod_\alpha \frac{r_{\alpha,n_\alpha}}{r_{\alpha,0}} \Bigg|_{m_\gamma = n_\gamma - 1, m_\alpha = n_\alpha} \tag{5.3.46}$$

where we have used that $|\sigma| = \gamma - 1$. The situation with $\mathbb{P}_{1,2}(u) = -T_{21}(u)$ is identical. We have

$$\langle \mathsf{y}|\mathbb{P}_{1,2}(u)|\mathsf{x}\rangle = (-1)^L \Big[0; \big|L-1; u\big|1; \theta + i\mathbf{s}\Big]_{\mathsf{xy}}. \tag{5.3.47}$$

Now, σ_γ is a permutation of

$$\{1, \dots, 1, 2\} \tag{5.3.48}$$

Up to the fact that now $|\sigma_\gamma| = L - \gamma$ the situation is identical to the previous case and we find

$$\langle \mathsf{y}|\mathbb{P}_{1,2}(u)|\mathsf{x}\rangle = -\frac{Q_\theta^{[-2\mathbf{s}]}(\mathsf{x}_\gamma)}{\Delta_\theta} \prod_{\alpha\neq\gamma} \frac{u-\mathsf{x}_\alpha}{\mathsf{x}_\gamma-\mathsf{x}_\alpha} \prod_{\alpha>\beta} (\mathsf{x}_\alpha - \mathsf{x}_\beta) \prod_\alpha \frac{r_{\alpha,n_\alpha}}{r_{\alpha,0}} \Bigg|_{m_\gamma = n_\gamma + 1, m_\alpha = n_\alpha} \tag{5.3.49}$$

which perfectly reproduces the well-known gl_2 results [84].

gl_3 matrix elements - explicit example

We now turn our attention to the matrix elements of the gl_3 principal operators. Since we have access to the general formula (5.3.30) we will not present the matrix elements $\langle \mathsf{y}|\mathbb{P}_{a,r}(u)|\mathsf{x}\rangle$ for each principal operator explicitly. Instead we will demonstrate an explicit computation showing the formula (5.3.31) being used in practice.

We consider an gl_3 spin chain of length $L = 2$. The bases $\langle \mathsf{y}|$ and $|\mathsf{x}\rangle$ are labelled by non-negative integers $m_{\alpha,a}$ and $n_{\alpha,a}$ respectively, with $a, \alpha \in \{1, 2\}$. Hence, we will use the notation

$$\langle \mathsf{y}| := \langle m_{1,1}, m_{1,2}; m_{2,1}, m_{2,2}|, \quad |\mathsf{x}\rangle = |n_{1,1}, n_{1,2}; n_{2,1}, n_{2,2}\rangle. \tag{5.3.50}$$

We will compute the following matrix element

$$\langle 3, 2; 0, 0|\mathbb{P}_{1,0}(u)|2, 1; 1, 0\rangle. \tag{5.3.51}$$

The starting point is the expression

$$\langle \Psi_A|\mathbb{P}_{1,0}(u)|\tilde{\Psi}_B\rangle = -\Big[1; \theta - i\mathbf{s}\Big|L-1; u\Big|L; \Big|0; \Big]_\Psi. \tag{5.3.52}$$

As a result of (5.3.30) we see that the SoV matrix elements are given by

$$\langle \mathsf{y}|\mathbb{P}_{1,0}(u)|\mathsf{x}\rangle = -\Big[1; \theta - i\mathbf{s}\Big|L-1; u\Big|L; \Big|0; \Big]_{\mathsf{y},\mathsf{x}}. \tag{5.3.53}$$

We will use the expression obtained in (5.3.31) to explicitly compute (5.3.51). Repeating it here for convenience, (5.3.31) reads

$$\langle \mathsf{y}|\hat{O}|\mathsf{x}\rangle = s_{\mathbf{L}} \sum_{k} \frac{(-1)^{|\sigma|}}{\Delta_\theta^2} \prod_{\alpha,a} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \prod_{b} \frac{\Delta_{\mathbf{u}_b \cup \mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \Bigg|_{\sigma_{\alpha,a}=k_{\alpha,a}-m_{\alpha,a}+a} . \tag{5.3.54}$$

For the case at hand, we have $L = 2$ and $L_0 = L_1 = 1$, $L_2 = 2$ and $L_3 = 0$. Furthermore,

$$\mathbf{u}_0 = \theta - i\mathbf{s} := \{\theta_1 - i\mathbf{s}, \theta_2 - i\mathbf{s}\}, \quad \mathbf{u}_1 = \{u\} \tag{5.3.55}$$

with both $\mathbf{u}_2$ and $\mathbf{u}_3$ empty.

First, in order to obtain a non-zero matrix element we need to check that the SoV charges of $\langle \mathsf{y}|$ and $|\mathsf{x}\rangle$ satisfy the SoV charge selection rule (5.3.36) which reads

$$\mathbf{N}_{\mathsf{y}} - \mathbf{N}_{\mathsf{x}} = 3L - \sum_{n=0}^{3} n\, L_n \tag{5.3.56}$$

with $\mathbf{N}_{\mathsf{x}} = \sum_{\alpha,a} n_{\alpha,a}$ and $\mathbf{N}_{\mathsf{y}} = \sum_{\alpha,a} m_{\alpha,a}$ and $L = 2$. We have

$$\mathbf{N}_{\mathsf{x}} = 2 + 1 + 1 = 4, \quad \mathbf{N}_{\mathsf{y}} = 3 + 2 = 5\,. \tag{5.3.57}$$

For the operator $\text{P}_{1,0}(u)$ we have $L_0 = 1$, $L_1 = 1$ $L_2 = 2$ and $L_3 = 3$ and hence (5.3.56) is satisfied. As such, σ in (5.3.31) corresponds to a permutation of

$$\{0, 1, 2, 2\}\,. \tag{5.3.58}$$

We now need to construct permutations of the set $\{n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}\}$ for fixed α. In general there are 4 possible permutations which read

$$\begin{aligned} &\{n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}\}, \quad \{n_{1,2}, n_{1,1}, n_{2,1}, n_{2,2}\}, \\ &\{n_{1,1}, n_{1,2}, n_{2,2}, n_{2,1}\}, \quad \{n_{1,2}, n_{1,1}, n_{2,2}, n_{2,1}\} \end{aligned} \tag{5.3.59}$$

but if there are degeneracies in $n_{\alpha,a}$ for fixed α there can be fewer permutations. In our case there are no degeneracies and we have the following permutations

$$\{2, 1, 1, 0\}, \quad \{1, 2, 1, 0\}, \quad \{2, 1, 0, 1\}, \quad \{1, 2, 0, 1\}\,. \tag{5.3.60}$$

The formula (5.3.31) requires summing over all permutations in (5.3.60) for which $\sigma_{\alpha,a} = k_{\alpha,a} - m_{\alpha,a} + a$ produces a valid permutation of (5.3.58). For each of the permutations in (5.3.60) the corresponding $\sigma_{\alpha,a}$ are given by

$$\{0, 1, 2, 2\}, \quad \{0, 1, 3, 3\}, \quad \{-1, 2, 2, 2\}, \quad \{-1, 2, 1, 3\}\,. \tag{5.3.61}$$

Only the first set corresponds to a permutation of $\{0, 1, 2, 2\}$, which has $|\sigma| = 1$, and hence the only term in the sum over permutations of $n_{\alpha,a}$ for fixed α comes from $\{2, 1, 1, 0\}$. Of

course, in general there can be multiple such permutations which need to be taken into account.

From here, for this single σ, we can read off

$$\mathsf{x}_{\sigma^{-1}(0)} = \mathsf{x}_{1,1}, \quad \mathsf{x}_{\sigma^{-1}(1)} = \mathsf{x}_{1,2}, \quad \mathsf{x}_{\sigma^{-1}(2)} = \{\mathsf{x}_{2,1}, \mathsf{x}_{2,2}\} \tag{5.3.62}$$

which results in

$$\prod_b \frac{\Delta_{\mathbf{u}_b \cup \mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} = Q_\theta^{[2\mathbf{s}]}(\mathsf{x}_{1,1})(u - \mathsf{x}_{1,2})(\mathsf{x}_{2,1} - \mathsf{x}_{2,2})\,. \tag{5.3.63}$$

Finally we plug everything in, obtaining

$$\langle 3,2;0,0|\mathbb{P}_{1,0}(u)|2,1;1,0\rangle = -i(u-\theta_1-i(\mathbf{s}+1))\frac{Q_\theta^{[2\mathbf{s}]}(\theta_1+i(\mathbf{s}+2))}{(\theta_1-\theta_2)^2}\frac{r_{1,2}}{r_{1,0}}\frac{r_{1,1}}{r_{1,0}}\frac{r_{2,1}}{r_{2,0}}\,. \tag{5.3.64}$$

or more explicitly

$$-\frac{8\mathbf{s}^3(\mathbf{s}+1)(2\mathbf{s}+1)\,(2\mathbf{s}-i\theta_{12})^{\,2}\,(1-i\theta_{12}+2\mathbf{s})\,(2-i\theta_{12}+2\mathbf{s})\,(2\mathbf{s}+i\theta_{12})\,(1-i\theta_1+\mathbf{s}+iu)}{\theta_{12}^2\,(i-\theta_{12})\,(i+\theta_{12})^{\,2}\,(\theta_{12}+2i)} \tag{5.3.65}$$

where we have defined $\theta_{12} = \theta_1 - \theta_2$.

5.4 Form-factors of Multiple Insertions

In the previous sections we derived various matrix elements of the principal operators. In this section we will extend this consideration to multiple insertions of the principal operators.

The most general case can be obtained by using the matrix elements in the SoV basis, however, this does not guarantee that the form-factor will have a simple determinant form. We consider this general case in section 5.4.2. At the same time, for a large number of combinations of the principal operators we still managed to obtain determinant representations as we explain now.

5.4.1 Antisymmetric combinations of principal operators

The set-up in this section is similar to that of section 5.3.1. We consider the gl_3 case with two factorisable states $\langle\Psi^A|$ and $|\tilde{\Psi}^B\rangle$. In addition we assume that the state $\langle\Psi^A|$ is on-shell meaning that it is an actual wave function of a spin chain and that it diagonalises the transfer matrix with twists λ_a.

Let us try to extend the previous method to general multiple insertions. The starting point is again from (5.3.3), which we write below for convenience

$$\sum_{b,\beta}(-1)^b\Big(\!\!\Big(\,\tilde{Q}_1^B u^{\beta-1}\mathcal{D}^{3-2b}Q_{1,a+1}^A\,\Big)\!\!\Big)_\alpha I_{b,\beta}^A = -\sum_{r=0}^{3}\chi_r\Big(\!\!\Big(\,\tilde{Q}_1^B\mathcal{O}_{(r)}^\dagger Q_{1,a+1}^A\,\Big)\!\!\Big)_\alpha\,. \tag{5.4.1}$$

We rewrite the above equation by modifying one term in the sum in the LHS at $b, \beta = b'', \beta'''$. Namely, we replace $\Big(\!\Big(\tilde{Q}_1^B (w^{\beta''-1}\mathcal{D}^{3-2b''}) Q^A_{1,a+1} \Big)\!\Big)_\alpha$ by $\Big(\!\Big(\tilde{Q}_1^B \mathcal{O}^\dagger_{(s)} Q^A_{1,a+1} \Big)\!\Big)_\alpha$. In order for the equality to hold we also have to change the RHS accordingly

$$\sum_{\beta,b}(-1)^b \Big(\!\Big(\tilde{Q}_1^B \ (w^{\beta-1}\mathcal{D}^{3-2b})\Big|_{w^{\beta''-1}\mathcal{D}^{3-2b''}\to\mathcal{O}^\dagger_{(s)}} Q^A_{1,a+1} \Big)\!\Big)_\alpha I^A_{b,\beta} \tag{5.4.2}$$

$$= -\sum_{r=0}^{3}\chi_r \Big(\!\Big(\tilde{Q}_1^B \mathcal{O}^\dagger_{(r)} Q^A_{1,a+1} \Big)\!\Big)_\alpha + (-1)^{b''}\Big(\!\Big(\tilde{Q}_1^B \Big[\mathcal{O}^\dagger_{(s)} - (w^{\beta''-1}\mathcal{D}^{3-2b''})\Big] Q^A_{1,a+1} \Big)\!\Big)_\alpha I^A_{b'',\beta''} \ .$$

So far this is just an innocent rewriting. Next we treat the RHS as an inhomogeneous part of the linear system on $I^A_{b,\beta}$ and apply Cramer's rule. As we have two terms in the RHS of (5.4.2) we obtain a sum of two ratios of determinants. As a result, for $b', \beta' \neq b'', \beta''$ we have

$$\begin{aligned} I^A_{b',\beta'} &= (-1)^{b'+1}\frac{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}, (b',\beta')\to\sum_r\chi_r\mathcal{O}^\dagger_{(r)}]}{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}]} \\ &- (-1)^{b'+b''}I^A_{b'',\beta''}\frac{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}, (b',\beta')\to w^{\beta''-1}\mathcal{D}^{3-2b''}]}{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}]} \ . \end{aligned} \tag{5.4.3}$$

Notice that the term with $\mathcal{O}^\dagger_{(s)}$ in the RHS of (5.4.2) disappears as it produces a zero determinant in the numerator. The last term in (5.4.3) can be simplified a bit as we first replace the (b'',β'') column with $\mathcal{O}^\dagger_{(s)}$ and then insert into the column (b',β') the exact expression which was previously at the column (b'',β'')

$$\begin{aligned} I^A_{b',\beta'} &= (-1)^{b'+1}\frac{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}, (b',\beta')\to\sum_r\chi_r\mathcal{O}^\dagger_{(r)}]}{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}]} \\ &+ (-1)^{b'+b''}I^A_{b'',\beta''}\frac{[(b',\beta')\to\mathcal{O}^\dagger_{(s)}]}{[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}]} \ . \end{aligned} \tag{5.4.4}$$

Next we use the previously derived (5.3.15), which in the new notations becomes $\Big[(b',\beta')\to\mathcal{O}^\dagger_{(r)}\Big] = (-1)^{b'+1}\mathcal{N}\langle\Psi^A|\hat{I}^{(r)}_{b',\beta'}|\tilde{\Psi}^B\rangle$. We get

$$\begin{aligned} &I^A_{b',\beta'}\langle\Psi^A|\hat{I}^{(s)}_{b'',\beta''}|\tilde{\Psi}^B\rangle - I^A_{b'',\beta''}\langle\Psi^A|\hat{I}^{(s)}_{b',\beta'}|\tilde{\Psi}^B\rangle \\ &= \sum_r\chi_r\frac{(-1)^{b'+b''}}{\mathcal{N}}[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}, (b',\beta')\to\mathcal{O}^\dagger_{(r)}] \ . \end{aligned} \tag{5.4.5}$$

Then we use that $I^A_{b',\beta'}\langle\Psi^A| = \langle\Psi^A|\hat{I}_{b',\beta'}$ to plug the LHS under one expectation value

$$\langle\Psi^A|\hat{I}_{b',\beta'}\hat{I}^{(s)}_{b'',\beta''} - \hat{I}_{b'',\beta''}\hat{I}^{(s)}_{b',\beta'}|\tilde{\Psi}^B\rangle = \sum_r\chi_r\frac{(-1)^{b'+b''}}{\mathcal{N}}[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)}, (b',\beta')\to\mathcal{O}^\dagger_{(r)}] \ .$$

Finally, we apply the character projection trick to obtain

$$\boxed{\langle\Psi^A|\hat{I}^{(r)}_{b',\beta'}\hat{I}^{(s)}_{b'',\beta''}-\hat{I}^{(r)}_{b'',\beta''}\hat{I}^{(s)}_{b',\beta'}|\tilde{\Psi}^B\rangle=\frac{(-1)^{b'+b''}}{\mathcal{N}}[(b'',\beta'')\to\mathcal{O}^\dagger_{(s)},(b',\beta')\to\mathcal{O}^\dagger_{(r)}]}\,.$$

As before, once we have this expression we can remove the assumption that Ψ^A is an on-shell and replace it by a generic factorisable state following the same argument as in section 5.2.

Finally, the derivation we outlined above can be iterated to get the following general expression for the multiple insertions of the principal operators antisymmetrised w.r.t. the multi-indices (b,β)

$$\boxed{\langle\Psi^A|\hat{I}^{(s_1)}_{[b_1,\beta_1}\dots\hat{I}^{(s_k)}_{b_k,\beta_k]}|\tilde{\Psi}^B\rangle=\frac{(-1)^{b_1+\dots+b_k+k}}{k!\,\mathcal{N}}[(b_1,\beta_1)\to\mathcal{O}^\dagger_{(s_1)},\dots,(b_k,\beta_k)\to\mathcal{O}^\dagger_{(s_k)}]}\quad(5.4.6)$$

Note that the RHS vanishes if any of the character indices (s_i) coincide. Thus in order to get a nontrivial RHS we can have at most 4 antisymmetrised principal operators for the gl_3 case and $N+1$ for general gl_N. The fact that the RHS is antisymmetric in the character indices is also reflected on the LHS, where this is a consequence of the commutativity of transfer matrices. In fact, expanding the relation (5.1.9) in u and v we immediately get that $\hat{I}^{(r)}_{[b',\beta'}\hat{I}^{(s)}_{b'',\beta'']}=-\hat{I}^{(s)}_{[b',\beta'}\hat{I}^{(r)}_{b'',\beta'']}$. Since this can be done for any consecutive pair of character indices in the LHS of (5.4.6), it follows that this quantity is completely antisymmetric in the character indices as a consequence of the RTT relations.

Finally, like in section 5.3.2 we can convert the expression for the form-factor of the coefficients of the principal operators into the form-factor of the principal operators themselves. For example, we have:

$$\begin{aligned}
(-1)^L\frac{\langle\mathbb{P}_{1,1}(u)\mathbb{P}_{1,2}(v)-\mathbb{P}_{1,1}(v)\mathbb{P}_{1,2}(u)\rangle}{u-v}&=\left[\;0;\;\middle|\;L-1;u,v\;\middle|\;L+1;\;\middle|\;0;\;\right]_\Psi\\
\langle\mathbb{P}_{1,1}(u)\mathbb{P}_{2,2}(v)-\mathbb{P}_{2,1}(v)\mathbb{P}_{1,2}(u)\rangle&=\left[\;0;\;\middle|\;L;u\;\middle|\;L;v\;\middle|\;0;\;\right]_\Psi\\
-\langle\mathbb{P}_{1,0}(u)\mathbb{P}_{2,2}(v)-\mathbb{P}_{2,0}(v)\mathbb{P}_{1,2}(u)\rangle&=\left[\;1;\theta-i\mathbf{s}\;\middle|\;L-1;u\;\middle|\;L;v\;\middle|\;0;\;\right]_\Psi\\
(-1)^{L-1}\langle\mathbb{P}_{1,0}(u)\mathbb{P}_{2,3}(v)-\mathbb{P}_{2,0}(v)\mathbb{P}_{1,3}(u)\rangle&=\left[\;1;\theta-i\mathbf{s}\;\middle|\;L-1;u\;\middle|\;L-1;v\;\middle|\;1;\theta+i\mathbf{s}\;\right]_\Psi.
\end{aligned}\quad(5.4.7)$$

For a more complicated but nice looking example of a triple insertion we get:

$$\frac{\epsilon^{ijk}\langle\mathbb{P}_{1,1}(u_i)\mathbb{P}_{1,2}(u_j)\mathbb{P}_{1,3}(u_k)\rangle}{(u_1-u_2)(u_1-u_3)(u_2-u_3)}=(-1)^L\left[\;0;\;\middle|\;L-2;u_1,u_2,u_3\;\middle|\;L+1;\;\middle|\;1;\theta+i\mathbf{s}\;\right]_\Psi\quad(5.4.8)$$

Notice that the second form-factor in (5.4.7) contains exactly the same combination that we found for the expressions for B and C operators in (5.1.12)! We will discuss the implications of this observation in section 5.4.3.

5.4.2 Via Matrix elements in SoV basis

In the above subsection we demonstrated how it is possible to write a large family of correlation functions with anti-symmetrised insertions of principal operators. However, this

does not exhaust all possible correlators. On the other hand, we can in principal reduce the computation of correlators with any number of insertions to sums over products of form-factors with a single insertion by inserting a resolution of the identity over transfer matrix eigenstates. In practice this is not very useful as one would need to know the Q-functions for every state and not just those appearing in the wave functions.

This issue can be resolved by using the matrix elements of the principal operators in the SoV bases instead. Consider the double insertion

$$\langle \Psi_A | \mathbb{P}_{a,r}(u) \mathbb{P}_{b,s}(v) | \tilde{\Psi}_B \rangle \,. \tag{5.4.9}$$

We now consider three resolutions of the identity

$$1 = \sum_{\mathsf{x}} |\mathsf{x}\rangle\langle\mathsf{x}| = \sum_{\mathsf{y}} |\mathsf{y}\rangle\langle\mathsf{y}| = \sum_{\mathsf{x},\mathsf{y}} |\mathsf{x}\rangle\langle\mathsf{y}| (\mathcal{M}^{-1})_{\mathsf{y},\mathsf{x}} \tag{5.4.10}$$

where $(\mathcal{M}^{-1})_{\mathsf{y},\mathsf{x}}$ denotes the components of the inverse SoV measure $\mathcal{M}$ (6.7.15) which appears in the resolution of the identity

$$1 = \sum_{\mathsf{x},\mathsf{y}} |\mathsf{y}\rangle\langle\mathsf{x}| \mathcal{M}_{\mathsf{y},\mathsf{x}} \,. \tag{5.4.11}$$

We insert the three resolutions into the above correlator, obtaining

$$\langle \Psi_A | \mathbb{P}_{a,r}(u) \mathbb{P}_{b,s}(v) | \Psi_B \rangle = \sum_{\mathsf{x},\mathsf{x}',\mathsf{y},\mathsf{y}'} \Psi_A(\mathsf{y}) \, \langle \mathsf{y} | \mathbb{P}_{a,r}(u) | \mathsf{x}' \rangle \, \langle \mathsf{y}' | \mathbb{P}_{b,s}(v) | \mathsf{x} \rangle \, (\mathcal{M}^{-1})_{\mathsf{y}',\mathsf{x}'} \Psi_B(\mathsf{x}) \,. \tag{5.4.12}$$

At this point we see that the computation of multi-insertions becomes quite complicated. Indeed, for the rank 1 gl_2 case the measure $\mathcal{M}_{\mathsf{y},\mathsf{x}}$ is diagonal and so the computation of the inverse measure $(\mathcal{M}^{-1})_{\mathsf{y}',\mathsf{x}'}$ is trivial. For higher rank the measure is no longer diagonal and $(\mathcal{M}^{-1})_{\mathsf{y}',\mathsf{x}'}$ needs to be computed. Nevertheless, it can be computed since $\mathcal{M}_{\mathsf{y},\mathsf{x}}$ is explicitly known (6.7.15) and furthermore $\mathcal{M}_{\mathsf{y},\mathsf{x}}$, in an appropriate order of x and y, is an upper-triangular block diagonal matrix where each block is finite-dimensional even in the case of non-compact gl_N [74].

5.4.3 SoV B and C operators

In this section we will demonstrate that our results allow one to derive that the SoV B and C operators (5.1.10) are diagonalised in the SoV bases $|\mathsf{x}\rangle$ and $\langle\mathsf{y}|$ respectively. Structurally, the **B** and **C** operators are very similar. We recall the expressions (5.1.12) which read

$$\begin{aligned} B(u) &= \mathbb{P}_{1,1}(u)\mathbb{P}_{2,2}(u) - \mathbb{P}_{2,1}(u)\mathbb{P}_{1,2}(u) \\ C(u) &= \mathbb{P}_{1,1}(u)\mathbb{P}_{2,2}(u+i) - \mathbb{P}_{2,1}(u+i)\mathbb{P}_{1,2}(u) \,. \end{aligned} \tag{5.4.13}$$

Both of these expressions are special cases of the general double insertion $\mathbb{P}_{1,1}(u)\mathbb{P}_{2,2}(v) - \mathbb{P}_{2,1}(v)\mathbb{P}_{1,2}(u)$ appearing in (5.4.7). We will denote this operator as $B(u,v)$, that is

$$B(u,v) = \mathbb{P}_{1,1}(u)\mathbb{P}_{2,2}(v) - \mathbb{P}_{2,1}(v)\mathbb{P}_{1,2}(u) \,. \tag{5.4.14}$$

By using the relation (5.3.30) we can convert its matrix elements in the Ψ basis in (5.4.7) to matrix elements in the x, y basis. The result simply reads

$$\langle \mathsf{y}|B(u,v)|\mathsf{x}\rangle = \frac{s_{\mathbf{L}}}{\Delta_\theta^2}\sum_k \mathrm{sign}(\sigma)\prod_{\alpha,a}\frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}}\Delta_{u\cup \mathsf{x}_{\sigma^{-1}(1)}}\Delta_{v\cup \mathsf{x}_{\sigma^{-1}(2)}}\Big|_{\sigma_{a,\alpha}=k_{a,\alpha}-m_{\alpha,a}+a} \tag{5.4.15}$$

where σ is a permutation of

$$\{\underbrace{1,\dots,1}_{L},\underbrace{2,\dots,2}_{L}\}\,. \tag{5.4.16}$$

We now examine the special cases $v = u$ and $v = u+i$, relevant for B and C respectively.

B operator The crucial point is that in (5.4.15) we have that $\Delta_{u\cup \mathsf{x}_{\sigma^{-1}(1)}}\Delta_{v\cup \mathsf{x}_{\sigma^{-1}(2)}} = (u-\mathsf{x}_{\sigma^{-1}(1)})(v-\mathsf{x}_{\sigma^{-1}(2)})\Delta_1\Delta_2$ and hence, we see that, for $v = u$, we have

$$(u-\mathsf{x}_{\sigma^{-1}(1)})(u-\mathsf{x}_{\sigma^{-1}(2)}) = \prod_{\alpha,a}(u-\mathsf{x}_{\alpha,a}) \tag{5.4.17}$$

which is independent of σ. Hence, this factor can be pulled outside the sum over permutations and we obtain

$$\begin{aligned}\langle \mathsf{y}|B(u,u)|\mathsf{x}\rangle &= \prod_{\alpha,a}(u-\mathsf{x}_{\alpha,a})\ \frac{s_{\mathbf{L}}}{\Delta_\theta^2}\sum_k \mathrm{sign}(\sigma)\prod_{\alpha,a}\frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}}\Delta_1\Delta_2\Big|_{\sigma_{a,\alpha}=k_{a,\alpha}-m_{\alpha,a}+a}\\ &= \prod_{\alpha,a}(u-\mathsf{x}_{\alpha,a})\langle \mathsf{y}|\mathsf{x}\rangle\,.\end{aligned} \tag{5.4.18}$$

Hence the operator $B(u) := B(u,u)$ acts diagonally on $|\mathsf{x}\rangle$ with eigenvalue $\prod_{\alpha,a}(u-\mathsf{x}_{\alpha,a})$. This coincides precisely with the spectrum of Sklyanin's $B(u)$ operator [74].

C operator We will now show that C is diagonalised in the $|\mathsf{y}\rangle$ basis in the same manner as we did for B. We start again from the expression:

$$\langle \mathsf{y}|B(u,u+i)|\mathsf{x}\rangle = \frac{s_{\mathbf{L}}}{\Delta_\theta^2}\sum_k \mathrm{sign}(\sigma)\prod_{\alpha,a}\frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}}\Delta_{u\cup \mathsf{x}_{\sigma^{-1}(1)}}\Delta_{u+i\cup \mathsf{x}_{\sigma^{-1}(2)}}\Big|_{\sigma_{a,\alpha}=k_{a,\alpha}-m_{\alpha,a}+a}\,. \tag{5.4.19}$$

We will now show that $\Delta_{u\cup \mathsf{x}_{\sigma^{-1}(1)}}\Delta_{u+i\cup \mathsf{x}_{\sigma^{-1}(2)}} = \prod_{\alpha,a}(u-\mathsf{y}_{\alpha,a})\Delta_1\Delta_2$. We have

$$\Delta_{u\cup \mathsf{x}_{\sigma^{-1}(1)}}\Delta_{u+i\cup \mathsf{x}_{\sigma^{-1}(2)}} = (u-\mathsf{x}_{\sigma^{-1}(1)})(u+i-\mathsf{x}_{\sigma^{-1}(2)})\Delta_1\Delta_2 \tag{5.4.20}$$

We now examine the factor $(u-\mathsf{x}_{\sigma^{-1}(1)})(u+i-\mathsf{x}_{\sigma^{-1}(2)})$ which can be rewritten as

$$\prod_{\alpha,a:\sigma_{a,\alpha}=1}(u-\mathsf{x}_{\alpha,a})\prod_{\alpha,a:\sigma_{a,\alpha}=2}(u+i-\mathsf{x}_{\alpha,a})\,. \tag{5.4.21}$$

Next, we use that $\mathsf{x}_{\alpha,a} = \theta_\alpha + i(\mathbf{s} + n_{\alpha,a})$ and $\mathsf{y}_{\alpha,a} = \theta_\alpha + i(\mathbf{s} + m_{\alpha,a} - a)$ with $n_{\alpha,a} = m_{\alpha,a} - \sigma_{a,\alpha} + a$ to obtain

$$\prod_{\alpha,a:\sigma_{a,\alpha}=1}(u-\mathsf{x}_{\alpha,a})\prod_{\alpha,a:\sigma_{a,\alpha}=2}(u+i-\mathsf{x}_{\alpha,a}) = \prod_{\alpha,a}(u-\theta_\alpha - i\mathbf{s} - m_{\alpha,a} + 1 - a)\,. \tag{5.4.22}$$

The final expression coincides with $\prod_{\alpha,a}(u-\mathsf{y}_{\alpha,a})$ which is independent of σ. Hence we obtain

$$\langle \mathsf{y}|C(u)|\mathsf{x}\rangle := \langle \mathsf{y}|B(u,u+i)|\mathsf{x}\rangle = \prod_{\alpha,a}(u-\mathsf{y}_{\alpha,a})\langle \mathsf{y}|\mathsf{x}\rangle\,, \tag{5.4.23}$$

meaning that the operator $C(u)$ acts diagonally on the $\langle \mathsf{y}|$ basis with eigenvalue $\prod_{\alpha,a}(u-\mathsf{y}_{\alpha,a})$.

5.5 Extension to gl_N spin chains

In this section we will extend our results from the previous sections to the gl_N case. The construction is a simple generalisation of the results in the previous sections, where we focused mainly on gl_2 and gl_3 cases. We will briefly go through the main steps of the derivations.

5.5.1 Determinant representation of form-factors

We start again from the dual Baxter operator

$$\mathcal{O}_A^\dagger = \sum_{a=0}^{N}(-1)^a \tau_a^A(u)\mathcal{D}^{N-2a}, \quad \mathcal{O}_A^\dagger Q_A^{1+a} = 0\,. \tag{5.5.1}$$

Now we consider the usual trivial identity, where $\mathcal{O}_A^\dagger$ is applied to Q_A^{1+a}:

$$\Big(\!\!\Big(Q_1^B \mathcal{O}_A^\dagger Q_A^{1+a}\Big)\!\!\Big)_\alpha = 0\ , \quad a=1,\dots,N-1,\ \ \alpha=1,\dots,L \tag{5.5.2}$$

Now we first expand the Baxter operator and the eigenvalues of the transfer matrices τ_a^A in the spectral parameter u, obtaining

$$\sum_{b,\beta}(-1)^b \Big(\!\!\Big(Q_1^B u^{\beta-1}\mathcal{D}^{N-2b}Q_A^{1+a}\Big)\!\!\Big)_\alpha I^A_{b,\beta} = -\sum_{r=0}^{N}\chi_r^A \Big(\!\!\Big(Q_1^B \mathcal{O}^\dagger_{(r)} Q_A^{1+a}\Big)\!\!\Big)_\alpha\,, \tag{5.5.3}$$

where we have defined

$$\mathcal{O}^\dagger_{(0)} = Q_\theta^{[2\mathbf{s}]}\mathcal{D}^N\ , \ \ \mathcal{O}^\dagger_{(r)} = (-1)^r u^L \mathcal{D}^{N-2r},\ r=1,\dots,N-1\ , \ \ \mathcal{O}^\dagger_{(N)} = (-1)^N Q_\theta^{[-2\mathbf{s}]}\mathcal{D}^{-N}\,. \tag{5.5.4}$$

Using Cramers' rule, we can compute the matrix elements of the integrals of motion exactly as in the gl_3 case leading to

$$I_{b',\beta'} = (-1)^{b'+1}\frac{[(b',\beta') \to \sum_{r=0}^{N}\chi_r\,\mathcal{O}^{\dagger}_{(r)}]}{[w^{\beta-1}\mathcal{D}^{N-2b}]}\,. \tag{5.5.5}$$

Since $\langle\Psi_A|$ is an eigenvector of $\hat{I}_{b,\beta}$ with eigenvalue $I_{b,\beta}$ we can rewrite the above as

$$\langle\Psi_A|\hat{I}_{b',\beta'}|\tilde{\Psi}_B\rangle = \frac{(-1)^{b'+1}}{\mathcal{N}}\frac{[(b',\beta') \to \sum_{r=0}^{N}\chi_r\,\mathcal{O}^{\dagger}_{(r)}]}{[w^{\beta-1}\mathcal{D}^{N-2b}]}\,. \tag{5.5.6}$$

The principal operator coefficients $\hat{I}^{(r)}_{b,\beta}$ are then introduced via the expansion into characters of the integrals of motion $\hat{I}_{b,\beta}$

$$\hat{I}_{b',\beta'} = \sum_{r=0}^{N}\chi_r\hat{I}^{(r)}_{b',\beta'}\,. \tag{5.5.7}$$

Performing character projection we then obtain the form-factors

$$\langle\Psi_A|\hat{I}^{(r)}_{b',\beta'}|\tilde{\Psi}_B\rangle = \frac{(-1)^{b'+1}}{\mathcal{N}}[(b',\beta') \to \mathcal{O}^{\dagger}_{(r)}]\,. \tag{5.5.8}$$

We see that this relation is identical to that of the gl_3 case (5.3.15).

In the same way as in gl_3 we can assemble the operators $\hat{I}^{(r)}_{b,\beta}$ into the generating functions $\mathrm{P}_{b,r}(u)$ – the principal operators. The form-factor of the generating function $\mathrm{P}_{b',r}(u)$ defined by (5.1.1) is then given by

$$\langle\Psi_A|\mathrm{P}_{b',r}(u)|\tilde{\Psi}_B\rangle = \delta_{b'r}u^L[w^{\beta-1}\mathcal{D}^{N-2b}] + \sum_{\beta'=1}^{L}(-1)^{b'+1}u^{\beta'-1}[(b',\beta') \to \mathcal{O}^{\dagger}_{(r)}]\,. \tag{5.5.9}$$

This result can be easily recast in determinant form using the same arguments as the gl_3 case. We introduce the notation:

$$\begin{aligned}&\Big[L_0;\mathbf{u}_0\Big|\dots\Big|L_N;\mathbf{u}_N\Big]_{\Psi} = \frac{1}{\mathcal{N}}\times\\ &\Big[\Big\{\frac{\Delta_{\mathbf{u}_0\cup w}}{\Delta_{\mathbf{u}_0}}w^j\mathcal{D}^N\Big\}_{j=0}^{L_0-1},\dots,\Big\{\frac{\Delta_{\mathbf{u}_N\cup w}}{\Delta_{\mathbf{u}_N}}w^j\mathcal{D}^{-N}\Big\}_{j=0}^{L_N-1}\Big]\,.\end{aligned} \tag{5.5.10}$$

We will write explicit expression for the form factors of type $\langle\mathrm{P}_{b',r}(u)\rangle$. We have that:

$r=b'$		$\Big[0;$	$\dots$	$(L)^r;u$	$\dots$	$\Big\vert 0;$	$\Big]_{\Psi}$
$r=0$	$(-1)^{b'L+b'+L}$	$\Big[1;\theta-i\mathbf{s}$	$\dots$	$(L-1)^{b'};u$	$\dots$	$\Big\vert 0;$	$\Big]_{\Psi}$
$r=N$	$(-1)^{b'+L(N-b')+N+1}$	$\Big[0;$	$\dots$	$(L-1)^{b'};u$	$\dots$	$\Big\vert 1;\theta+i\mathbf{s};$	$\Big]_{\Psi}$

(5.5.11)

$r > b'$	$(-1)^{b'+r+1+L(r-b')}$	$\Big[0;$	$\dots$	$(L-1)^{b'};u$	$\dots$	$(L+1)^{r};$	$\dots$	$0;\Big]_\Psi$
$r < b'$	$(-1)^{b'+r+L(b'-r)}$	$\Big[0;$	$\dots$	$(L+1)^{r};$	$\dots$	$(L-1)^{b'};u$	$\dots$	$0;\Big]_\Psi$

(5.5.12)

Multiple insertions The expression (5.4.6) for multiple insertions generalise without modification from the gl_3 case and we have

$$\langle\Psi^A|\hat{I}^{(s_1)}_{[b_1,\beta_1}\dots\hat{I}^{(s_k)}_{b_k,\beta_k]}|\tilde{\Psi}^B\rangle=\frac{(-1)^{b_1+\dots+b_k+k}}{k!\,\mathcal{N}}[(\beta_1,b_1)\to\mathcal{O}^\dagger_{(s_1)},\dots,(\beta_k,b_k)\to\mathcal{O}^\dagger_{(s_k)}] \tag{5.5.13}$$

As mentioned in the gl_3, the LHS is anti-symmetric in character indices and so in order to get a non-zero correlator we require that $k\leqslant N+1$.

5.5.2 Matrix elements in SoV bases

We can repeat the arguments from the gl_3 section to compute all form-factors of the form $\langle\mathsf{y}|\mathbb{P}_{a,r}(u)|\mathsf{x}\rangle$. We introduce the notation

$$\Big[L_0;\mathbf{u}_0\Big|\dots\Big|L_N;\mathbf{u}_N\Big]_{\mathsf{y},\mathsf{x}} \tag{5.5.14}$$

defined by the property

$$\Big[L_0;\mathbf{u}_0\Big|\dots\Big|L_N;\mathbf{u}_N\Big]_{\Psi}=\sum_{\mathsf{x},\mathsf{y}}\Psi_A(\mathsf{y})\Big[L_0;\mathbf{u}_0\Big|\dots\Big|L_N;\mathbf{u}_N\Big]_{\mathsf{y},\mathsf{x}}\tilde{\Psi}_B(\mathsf{x}) \tag{5.5.15}$$

where we remind the reader that the SoV wave functions are given by

$$\Psi_A(\mathsf{y})=\prod_{\alpha=1}^{L}\det_{1\leqslant a,a'\leqslant N-1}Q_A^{1+a}(\mathsf{y}_{\alpha,a'}+\tfrac{i}{2}),\quad \tilde{\Psi}_B(\mathsf{x})=\prod_{\alpha=1}^{L}\prod_{a=1}^{N-1}\tilde{Q}_1^B(\mathsf{x}_{\alpha,a})\,. \tag{5.5.16}$$

The explicit expression for (5.5.15) is worked out to be

$$\frac{s_{\mathbf{L}}}{\Delta_\theta^{N-1}}\sum_k(-1)^{|\sigma|}\prod_{\alpha,a}\frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}}\prod_b\frac{\Delta_{\mathbf{u}_b\cup\mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}}\Bigg|_{\sigma_{a,\alpha}=k_{a,\alpha}-m_{\alpha,a}+a}. \tag{5.5.17}$$

The index b takes values in the set $\{0,1,\dots,N\}$, $a\in\{1,\dots,N-1\}$ and $\alpha\in\{1,\dots,L\}$ and the summation is over all permutations k of the set $\{n_{\alpha,a}\}$ for fixed α for which σ defined by $\sigma_{\alpha,a}=k_{\alpha,a}-m_{\alpha,a}+a$ defines a permutation of the set

$$\{\underbrace{0,\dots,0}_{L_0},\dots,\underbrace{N,\dots,N}_{L_N}\}\,. \tag{5.5.18}$$

The matrix element (5.5.17) is only non-zero if the SoV charges $\mathbf{N}_\mathsf{x}$ and $\mathbf{N}_\mathsf{y}$ satisfy the relation

$$\mathbf{N}_\mathsf{y}-\mathbf{N}_\mathsf{x}=\frac{N}{2}(N-1)L-\sum_{n=0}^{N}n\,L_n\,. \tag{5.5.19}$$

The details of the derivation are exactly the same as in the gl_3 case described in Appendix C.1.

B and C operators. Having access to the complete set of SoV matrix elements it is now easy to determine which operators correspond to the SoV B and C operators. Following the derivation in the gl_3 case it is trivial to work out that $B(u)$ corresponds to the operator with

$$\mathbf{u}_0 = \mathbf{u}_N = \{\}, \quad \mathbf{u}_r = \{u\}, \quad r = 1, \dots, N-1 \tag{5.5.20}$$

whereas $C(u)$ corresponds to the operator with

$$\mathbf{u}_0 = \mathbf{u}_N = \{\}, \quad \mathbf{u}_r = \{u + i(r-1)\}, \quad r = 1, \dots, N-1\,. \tag{5.5.21}$$

Indeed, by examining the matrix element (5.5.17) as in the gl_3 case we immediately read off that the operator defined by (5.5.20) ((5.5.21)) acts diagonally on $|\mathsf{x}\rangle$ ($\langle\mathsf{y}|$) with eigenvalue given by $\prod_{\alpha,a}(u-\mathsf{x}_{\alpha,a})$ ($\prod_{\alpha,a}(u-\mathsf{y}_{\alpha,a})$) and hence coincides with $B(u)$ ($C(u)$) respectively due to the non-degeneracy of these operators' spectra. It is possible to work out what these operators correspond to in terms of principal operators $\mathrm{P}_{a,r}(u)$. They are given by

$$\begin{aligned} B(u) &= (N-1)!\varepsilon^{a_1\dots a_{N-1}}\mathrm{P}_{a_1,1}(u)\dots\mathrm{P}_{a_{N-1},N-1}(u) \\ C(u) &= (N-1)!\varepsilon^{a_1\dots a_{N-1}}\mathrm{P}_{a_1,1}(u)\dots\mathrm{P}_{a_{N-1},N-1}(u+i(N-2))\,. \end{aligned} \tag{5.5.22}$$

The fact that these operators coincide with the B and C operators of [74] is not manifest – application of the RTT relations is required as was already demonstrated in the gl_3 case. Nevertheless, the fact that their spectra and eigenstates coincide guarantees that they are equal.

5.6 Properties of principal operators

The main goal of this section is to demonstrate the completeness of the set of the principal operators. We show that any element of the Yangian can be obtained as a combination of the principal operators, which in at least finite dimensional cases guarantees that all physical observables can be obtained in this way. In the last section we also give explicit expressions for the principal operators in the diagonal frame – i.e. in the case when the twist matrix becomes diagonal.

5.6.1 Completeness

In this section we will demonstrate a crucial property of the operator basis, namely that knowledge of the matrix elements of each of our principal operators is equivalent to the knowledge of the matrix elements of every operator $T_{ij}(u)$ in the Yangian algebra. More precisely we will show that any monodromy matrix element $T_{ij}(u)$ can be constructed as a polynomial of degree at most $N+1$ in principal operators.

Knowing all $T_{ij}(u)$ is essentially equivalent to the full algebra of observables. For example, in the finite dimensional case i.e. when $\mathbf{s} = -n/2$, $n \in \mathbb{Z}_+$ one can use the "inverse scattering transform" [85] to construct local symmetry generators acting on a single site of the chain in terms of $T_{ij}(u)$. The precise notion of completeness could be ambiguous – in

order to be precise in this chapter when referring to completeness of the system of principal operators we understand that any element of the Yangian can be generated in finitely many steps (independently of the length of the chain). Note that while we are not aware of any simple way to extract local operators in the infinite-dimensional case in terms of $T_{ij}(u)$ we would like to stress that these operators still contain all information about the system. For example, consider the spin-**s** representation used in this section and consider some local operator $\mathbb{E}^{(\alpha)}$. The key point is the existence of the SoV basis $\langle \mathsf{x}|$ which is constructed by action of polynomials in $T_{ij}(u)$ on the SoV ground state $\langle 0|$ [74]. Hence, the action of $\mathbb{E}^{(\alpha)}$ on the SoV basis can be re-expressed as a sum over (finitely many[6]) SoV basis states $\langle \mathsf{x}'|$ and hence the matrix elements of $\mathbb{E}^{\alpha}$ are completely fixed by the SoV matrix elements of the monodromy matrix $T_{ij}(u)$.

We now show that the principal operators generate the full Yangian. Our starting point is the large u expansion of the operators $T_{ij}(u)$

$$T_{ij}(u) = u^L \delta_{ij} + u^{L-1} \left(i\mathcal{E}_{ji} - \delta_{ij}\Theta \right) + \mathcal{O}\left(u^{L-2}\right), \quad \Theta := \sum_{\alpha=1}^{L} \theta_\alpha \,. \tag{5.6.1}$$

Note that the indices on $\mathcal{E}$ are swapped compared to those on T. The operators $\mathcal{E}_{ij}$ are generators of the global $\mathfrak{gl}(N)$ algebra

$$\mathcal{E}_{ij} = \sum_{\alpha=1}^{L} \mathbb{E}_{ij}^{(\alpha)} \tag{5.6.2}$$

and satisfy the $\mathfrak{gl}(N)$ commutation relations

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{jk}\mathcal{E}_{il} - \delta_{li}\mathcal{E}_{kj} \,. \tag{5.6.3}$$

We will now prove the following property: that any $T_{ij}(v)$ can be expressed as a commutator of a global $\mathfrak{gl}(N)$ generator and a principal operator $T_{k1}(v)$. The key point are the RTT relations expanded at large u which read

$$[\mathcal{E}_{ji}, T_{kl}(v)] = T_{kj}(v)\delta_{il} - T_{il}(v)\delta_{kj} \,. \tag{5.6.4}$$

From here it is clear that we can write any operator $T_{ij}(v)$ as

$$T_{ij}(v) = T_{11}(v)\delta_{ij} + [\mathcal{E}_{j1}, T_{i1}(v)] = \mathrm{P}_{1,1}(v)\delta_{ij} + (-1)^{i-1}[\mathcal{E}_{j1}, \mathrm{P}_{1,i}(v)] \tag{5.6.5}$$

where the RHS only contains principal operators and global $\mathfrak{gl}(N)$ generators.

The family of principal operators includes the following global Lie algebra generators: $\mathcal{E}_{1j}$ and $\mathcal{E}^- = \sum_{j=1}^{n-1} \mathcal{E}_{j+1,j}$. These appear in the asymptotics of the generating functions

$$\mathrm{P}_{1,0}(u) = iu^{L-1}\mathcal{E}^- + \mathcal{O}(u^{L-2}), \quad (-1)^j \mathrm{P}_{1,j}(u) = u^L \delta_{j1} + u^{L-1}\left(i\mathcal{E}_{1j} - \delta_{ij}\Theta\right) + \mathcal{O}\left(u^{L-2}\right) . \tag{5.6.6}$$

[6]There are only finitely many states of a given SoV charge, and each local Lie algebra generator raises or lowers the SoV charge by some finite amount.

Hence, if we can prove that these operators can be used to generate the set of $\mathcal{E}_{j1}$ then it follows from (5.6.5) that knowing the matrix elements of all principal operators implies knowledge of the matrix elements of all $T_{ij}(u)$. From the commutation relations (5.6.3) it is easy to see that

$$\mathcal{E}_{j+1,1} = [\mathcal{E}^-, \mathcal{E}_{j1}] \,. \tag{5.6.7}$$

Thus, we have

$$\mathcal{E}_{j1} = \underbrace{[\mathcal{E}^-, [\mathcal{E}^-, [\ldots, [\mathcal{E}^-, \mathcal{E}_{11}]]}_{j-1}] \,, \tag{5.6.8}$$

where the RHS contains only principal operators. After that from (5.6.5) we get all operators $T_{ij}(u)$ generated, which completes the proof.

Let us remark that despite the abundance of literature on SoV in gl_2 spin chains the relation (5.6.5) does not seem to have been exploited. Indeed, the standard approach is to obtain the matrix elements of the one non-principal operator $T_{22}(u)$ in terms of the principal operators via the quantum determinant relation

$$\mathrm{qdet} T(u) = T_{11}^- T_{22}^+ - T_{21}^- T_{12}^+ \tag{5.6.9}$$

together with the known eigenvalue of the quantum determinant and the fact that $T_{11}(u)$ is invertible, see for example [86]. This produces a rather complicated expression for $T_{22}(u)$. On the other hand, using the relation (5.6.5) we see that $T_{22}(u)$ can be written in terms of principal operators simply as

$$T_{22}(u) = \mathbb{P}_{1,1}(u) - [\mathcal{E}_{21}, \mathbb{P}_{2,1}(u)] \,. \tag{5.6.10}$$

5.6.2 Principal operators in the diagonal frame

In the main part of this chapter, we used the frame with the twist matrix G being of the special form (4.4.4). Whereas for SoV approach this choice is extremely beneficial, as the SoV basis does not depend on the twist eigenvalues λ_a, it is not the most commonly used in the literature. A more standard choice is the diagonal twist $G = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. In this section we give an explicit way to relate those two conventions. As we will see the basic consequence of changing the frame is that the explicit expressions for the principal operators $\mathrm{P}_{r,s}$ in terms of the monodromy matrix elements T_{ij} will slightly change in the frame where the twist matrix is diagonal.

In the companion twist frame the transfer matrix $t(u)$ is given by $t(u) = \mathrm{tr}\,(T(u)G)$ where G is the companion twist matrix (4.4.4). We want to perform a similarity transformation $\Pi(S)$ on the Hilbert space of the spin chain where S is some $GL(N)$ group element and $\Pi(S)$ denotes its representative on the spin chain so that the transfer matrix transforms as $\mathrm{tr}\,(T(u)G) \to \mathrm{tr}\,(T(u)g)$ where g is the diagonal twist matrix with the same eigenvalues as G. As was established in [74] a possible choice for S is given by the Vandermonde matrix

$$(S^{-1})_{ij} = \lambda_j^{N-i} \,. \tag{5.6.11}$$

Under this transformation the monodromy matrix elements $T_{ij}(u)$ transform as

$$T_{ij}(u) \to \Pi^{-1} T_{ij}(u) \Pi = (S^{-1} T(u) S)_{ij} \ , \ \ \Pi \equiv \Pi(S) \tag{5.6.12}$$

with similar expressions holding for anti-symmetric monodromy matrices.

To summarise we have the wave-functions in the diagonal frame related to the wave-functions in the companion frame by

$$|\Psi^{\rm diag}\rangle = \Pi^{-1} |\Psi\rangle \ , \ \ \langle \Psi^{\rm diag}| = \langle \Psi | \Pi \, . \tag{5.6.13}$$

and they diagonalise the transfer matrices $t^{\rm diag}(u)$ and $t(u)$ correspondingly related as

$$t^{\rm diag}(u) = \Pi^{-1} t_1(u) \Pi = {\rm tr}(S^{-1} T(u) S G) = {\rm tr}(T(u) g) \, . \tag{5.6.14}$$

Similarly we define $\mathbb{P}^{\rm diag}_{a,r} = \Pi^{-1} \mathbb{P}_{a,r} \Pi$ so that

$$\langle \Psi^{\rm diag}_A | \mathbb{P}^{\rm diag}_{a,r} | \Psi^{\rm diag}_B \rangle = \langle \Psi_A | \mathbb{P}_{a,r} | \Psi_B \rangle = \text{determinant} \, . \tag{5.6.15}$$

Note that the above expression only holds for the states with the same twist unlike the expressions in the companion twist frame which hold for any twist on either state.

In general the expressions for the principal operators in the diagonal frame in terms of T_{ij} are quite bulky, but straightforward to work out from (5.6.12). For example for gl_3 we have $\mathbb{P}^{\rm diag}_{1,1} = (S^{-1} T S)_{1,1}$

$$\begin{aligned} \mathbb{P}^{\rm diag}_{1,1} \ &= \ \frac{\lambda_1^2 T_{11}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{\lambda_1^2 T_{12}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} + \frac{\lambda_1^2 T_{13}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \\ &+ \ \frac{\lambda_2^2 T_{21}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{\lambda_2^2 T_{22}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} + \frac{\lambda_2^2 T_{23}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \\ &+ \ \frac{\lambda_3^2 T_{31}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{\lambda_3^2 T_{32}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} + \frac{\lambda_3^2 T_{33}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \, . \end{aligned} \tag{5.6.16}$$

Note that whereas in the companion twist frame the principal operators by definition where independent of the twist eigenvalues, in the diagonal frame they explicitly depend on λ_i. In order to get nice looking expressions is it better to introduce the notation $T^{\rm good}(u) = S^{-1} T(u) S G = S^{-1} T(u) g S$ going back to [69]. It obeys $t^{\rm diag} = {\rm tr}(T^{\rm good})$ and is related in a simple way to the principal operators in the diagonal frame (4.4.4) so that

$$\begin{aligned} \mathbb{P}^{\rm diag}_{1,i}(u) &= (-1)^{N-i} \frac{T^{\rm good}_{i,N}(u)}{\chi_N}, \quad i = 1, 2, \ldots, N \, , \\ \mathbb{P}^{\rm diag}_{1,0}(u) &= \sum_{i=1}^{N-1} \left(T^{\rm good}_{i,i} - (-1)^{N-i} \frac{\chi_i}{\chi_N} T^{\rm good}_{i,N} \right) \, . \end{aligned} \tag{5.6.17}$$

One can check that $\sum_{r=0}^{N} \mathbb{P}^{\rm diag}_{1,r}(u) \chi_r = {\rm tr}(T^{\rm good})$. In particular, from (5.6.17) the above we see that the form-factor of any $T^{\rm good}_{i,N}$ in the diagonal frame is a determinant. For the particular case of gl_2 these operators generalise the well-known operators $T^{\rm good}_{11}$ and $T^{\rm good}_{22}$ which act as conjugate momenta of the separated variables encoded in $T^{\rm good}_{12}$, see [87].

5.7 Comparison between FSoV and NABA

In this chapter we used the functional separation of variables (FSoV) technique in combination with the novel character projection (CP) method to compute all matrix elements of the set of principal operators which in particular includes some individual monodromy matrix elements T_{ij} and their combinations in a concise determinant form. We also showed that they generate a complete basis of observables of the spin chain and contain the SoV B operator as a particular case. Thus we gained access to the matrix elements of a set of operators which generates a complete set of observables in high-rank integrable gl_N spin chains.

Determinant representations for form factors of some T_{ij} have appeared in the literature before for the gl_3 case in the Nested Bethe Ansatz approach [52, 88, 89]. However, in addition to giving an alternative form for those objects, the results presented in this chapter have a number of advantages and conceptual differences:

- Firstly, the form factors are expressed directly in terms of Baxter Q-functions instead of Bethe roots. From a direct calculational perspective Q-functions offer a significant advantage [90].

- Secondly, the FSoV approach does not require the existence of a highest-weight state. As such our approach is applicable to models which do not have the highest-weight state, for example the Fishchain described in this thesis.

- Thirdly, as demonstrated, our approach is valid for any rank gl_N with general formulas being almost equally simple to write down as for gl_3. The NABA becomes much harder for higher ranks, due to the presence of extra levels of nesting.

- Fourthly, our formulas are applicable to the set-up where the transfer matrix eigenstates are constructed with two distinct twists, which have attracted attention recently [91], or in fact any two arbitrary off-shell states (to which we refer to as factorisable). In addition to being a new result, this is a very important technical advantage for example in non-highest-weight models where the scalar product between states built with the same twist is divergent [92] and so deforming one set of twists serves as a natural regulator[7].

- Finally, using our approach we were able to compute the matrix elements of the principal operators in the SoV bases meaning one can compute the matrix elements of any number of insertions. Currently and to the best of our knowledge, this has been done via the NABA only for a single insertion of the monodromy matrix elements T_{ij} and for ranks $N \leqslant 3$.

[7] See [93] for the explicit realisation of the twist in Fishnet CFT.

Chapter 6

Open Integrable Spin Chains

So far, this thesis has been focused on periodic integrable spin chains, which arise as representations of the Yangian $Y(gl_N)$. The periodicity condition however neglects any possible boundary effect, which arise in applications to condensed matter physics. Even twisting the spin chain does not help in modelling them.

Therefore, it is natural to try and add boundaries to a spin chain and make it non-periodic. We would however like to preserve the underlying integrability of the model. This fact greatly reduces the freedom in the choice of the possible boundaries. In fact, the way to add boundaries to a periodic spin chain is by 'opening' it and putting two matrices representing the boundaries at its two new end points. Using generic matrices as boundaries breaks the integrability of the spin chain, i.e. it breaks the Yangian symmetry.

In [94], Sklyanin found a set of algebraic equations that a boundary matrix should satisfy to preserve integrability; these conditions can be packaged in a matrix equation that involves the rational R-matrix (2.2.3) and is known as the Boundary Yang-Baxter Equation (BYBE). The solutions of the BYBE are matrices that we denote as K. Using as the boundary matrices of a periodic spin chain two solutions to the BYBE, we obtain an integrable spin chain with open boundary conditions, known as *open integrable spin chain*.

From the Yangian symmetry point of view, the addition of integrable boundary matrices breaks $Y(gl_N)$ to a subgroup, the *Twisted Yangian*, that we will denote as $Y_C(gl_N)$. Here, C denotes the class of boundary matrices allowed.

Two boundary matrices K and K' that are related by a similarity transformation will give rise to the same Twisted Yangian, and they are in the same class C. All the (complex-valued) solutions of the BYBE will give rise to a certain Twisted Yangian $Y_C(gl_N)$. Recently there has been great progress in classifying twisted Yangians [95–97], and in particular the general form of the boundary matrices for many different classes of twisted Yangians C is known. Furthermore, just like Yangians arise from a quantisation of classical Lie algebras, it has been understood that the Twisted Yangians arise as quantisations of the symmetric pairs given by Cartan's classification of symmetric spaces (see [98], Chapter X).

In this chapter, we will treat twisted Yangians as arising from a periodic spin chain and boundary matrices satisfying the BYBE. We will focus on a single class of boundary

matrices, which are associated to the Twisted Yangian known as $Y^+(2N)$ in the literature. This twisted Yangian is the basic symmetry of the open Fishchain model described in the next section.

Many techniques that we described in chapter 3 for the Yangian carry over to the twisted Yangian, and we will describe them here. In particular, after introducing the basic notation, we will describe the open T-system, the open Q-system and the open Baxter TQ relation. The spin chain we will use as reference model in this chapter is an open spin chain in a highest-weight representation of weights $(-2s, -2s, 0, 0)$, $s \in -\frac{\mathbb{N}}{2}$ of $Y^+(2N)$, as this is a simple model that arises naturally as a strong coupling limit of the open Fishchain.

To conclude this section, we will start the Separation of Variables program for $Y^+(2N)$. Some results for SoV have been obtained in [99] for the twisted Yangian based on the so-called Reflection Equation [100]. However, a big limitation of this approach is that the SoV B operator is not known for any open spin chain, and many other gaps need to be filled. We will present results that adapt the Functional approach of chapter 5 to $Y^+(2N)$. While some technical difficulties still need to be solved, this is a promising road that will lead to a much better understanding of open integrable spin chains: as we have seen in chapter 5, even the trivial Functional scalar product provides access to non-trivial form factors.

This chapter is based on the author's work in preparation [27].

6.1 Open spin chains and twisted Yangians

The symmetry that defines rational spin chains with open boundary conditions is known as the twisted Yangian [101]. The twisted Yangian $Y_C(gl_N)$ is a subalgebra of the Yangian $Y(gl_N)$, and is characterised by the boundary matrices K. While the Yangian $Y(gl_N)$ arises as the quantisation of the classical gl_N algebra, the twisted Yangian $Y_C(gl_N)$ arises as the quantisation of the classical symmetric pair (gl_N, g^ρ), which we define in the next paragraph.

Symmetric pairs A *symmetric pair* is a pair composed of a (semi)simple Lie algebra g and an involutive automorphism ρ acting on it, such that $\rho^2(X) = X$, $\forall X \in g$. The subalgebra g^ρ corresponds to the fixed points of the reflection ρ, i.e. $g^\rho = \{Z \,|\, \rho(Z) = Z, Z \in g\}$. Usually, g^ρ can be identified with another semisimple Lie algebra contained in g.

Being involutive, ρ can only have eigenvalues ± 1, therefore its eigenvectors which are not in g^ρ are those with eigenvalue -1. Symmetric pairs were classified by Cartan, and a table can be found in [98].

Twisted Yangians Twisted Yangians arise as the quantisation of classical symmetric pairs. To the best of our knowledge, not all symmetric pairs have been used yet to define a twisted Yangian. The ones that have been built so far correspond to symmetric pairs involving the Lie algebras in the A, B, C, D series [95].

In this thesis, we will be dealing with the family of Twisted Yangians known as $Y^+(2N)$. These are the quantisation of the classical symmetric pair (gl_{2N}, so_{2N}), originally introduced in [101].

For the family $Y^+(2N)$, the automorphism ρ is given explicitly by $\rho(X) = X^t$, where ^{t} is the generalised transposition:

$$X^t = VX^TV^{-1}, \quad V = \text{antidiag}(1, \dots, 1)\,. \tag{6.1.1}$$

This automorphism actually leaves invariant two subalgebras of gl_{2N}, so_{2N} and sp_N. To select so_{2N} at the level of the twisted Yangian, we will need to impose further restrictions on the boundary K matrices characterising $Y^+(2N)$.

6.1.1 Boundary Yang-Baxter equation and open transfer matrix

The twisted Yangian can be defined via the introduction of boundary matrices K in a periodic spin chain. We define the boundary matrices K_- as a solution of the Boundary Yang-Baxter equation (BYBE):

$$R_{12}(u-v)K_1(u)R^t_{12}(-u-v)K_2(v) = K_2(v)R^t_{12}(-u-v)K_1(u)R_{12}(u-v) \tag{6.1.2}$$

where we are using the rational R-matrix (2.2.3) and denote by t the generalised transposition (6.1.1) applied to the auxiliary space 1.

The BYBE and the usual RTT relations (2.2.11) are the fundamental algebraic relations defining integrable open spin chains.

We now describe how to build open spin chains from solutions of the BYBE (6.1.2) and a periodic rational spin chain. Given the monodromy matrix (3.1.4) associated to a $Y(gl_N)$ periodic spin chain in some representation, we define:

$$U_-(u) = T(u)K_-(u)T^t(-u)\,. \tag{6.1.3}$$

This is the analog of the monodromy matrix $T(u)$ for open spin chains. Therefore, we will call $U_-(u)$ the *boundary monodromy matrix*.

The boundary monodromy matrix (6.1.3) satisfies the BYBE:

$$R_{12}(u-v)U^-_1(u)R^t_{12}(-u-v)U^-_2(v) = U^-_2(v)R^t_{12}(-u-v)U^-_1(u)R_{12}(u-v)\,. \tag{6.1.4}$$

This is due to the coproduct property of the Twisted Yangian:

$$\Delta(K_{ij}(u)) = \sum_{k,l=1}^{N} T_{ik}(u) \otimes K_{kl}(u) \otimes (T^t)_{lj}(-u)\,, \tag{6.1.5}$$

where $T_{ij} \in Y(gl_N)$ are monodromy matrix elements of a periodic spin chain. This coproduct does not satisfy all the axioms of the coproduct of a quantum group seen in chapter 2:

this implies that the twisted Yangian is not a quantum group itself. Nevertheless, many techniques developed for the Yangian continue to work when adapted to the twisted Yangian.

The generating function of the conserved charges for an open spin chain is called *open transfer matrix*. It is built by multiplying the boundary monodromy matrix (6.1.3) by another boundary matrix K_+, which satisfies the following BYBE:

$$R_{12}(-u+v)K_1^+(-u)R_{12}^t(u+v)K_2^+(-v) = K_2^+(-v)R_{12}^t(u+v)K_1^+(-u)R_{12}(-u+v)\,. \quad (6.1.6)$$

Then, we define the open transfer matrix by:

$$\tilde{\mathbb{T}}(u) = \mathrm{tr}(K_+(u)U_-(u))\,. \quad (6.1.7)$$

If K_+ and K_- are polynomials in u, the transfer matrix will also be a polynomial in u. The open transfer matrix (6.1.7) possesses a global symmetry G^ρ, which is the group associated to g^ρ, the conserved subalgebra of the underlying symmetric pair.

The transfer matrix (6.1.7) forms a commuting family of operators $\left[\tilde{\mathbb{T}}(u), \tilde{\mathbb{T}}(v)\right] = 0$. This can be proven explicitly using the BYBE equations as in [94]. In section 6.4, we give instead a diagrammatic proof. The coefficient of the powers of u in $\tilde{\mathbb{T}}(u)$ are a subset of the conserved charges of the open spin chain.

To select a specific twisted Yangian $Y_C(gl_N)$, we need to impose further conditions on the boundary matrices. In particular, to get $Y^+(2N)$, the quantisation of the symmetric pair (gl_{2N}, so_{2N}), we need to impose the following identities on the K matrices [102]:

$$K_-(u) = K_-^t(-u) \quad (6.1.8)$$
$$K_+(u) = -K_+^t(-u)\,. \quad (6.1.9)$$

It can be checked [97] that the BYBE (6.1.2) and (6.1.6) with either of the conditions (6.1.8) do not have u-dependent solutions whose entries are complex numbers. In this chapter, we will take as particular solutions the following boundary matrices:

$$K^+(u) = \mathrm{diag}(\zeta_1, \ldots, \zeta_N, -\zeta_N \cdots - \zeta_1)\,, \quad (6.1.10)$$
$$K^-(u) = I_{2N}\,, \quad (6.1.11)$$

where ζ_i are complex parameters that play the role of the twist in the open spin chain, and I_{2N} is the identity matrix in $2N$ dimensions. In the next section, we will prove that this choice of the boundaries can always be taken for $Y^+(2N)$.

6.1.2 Properties of the open transfer matrix

$SO(2N)$ symmetry

In this chapter, we will see why the properties (6.1.8) let us obtain a non-degenerate transfer matrix. In particular, we will see how the form of the boundary matrices (6.1.10) breaks the global $SO(2N)$ symmetry of the open transfer matrix.

Suppose that $\tilde{t}(u) = \operatorname{tr} T(u)T^t(-u)$. This transfer matrix has a manifest global $SO(2N)$ symmetry. This is due to the fact that $T(u)$ is $GL(2N)$ covariant, meaning we have for any $G \in GL(2N)$:

$$[G \otimes \Pi(G), T(u)] = 0 \tag{6.1.12}$$

where $\Pi(G)$ denotes the action of G on the physical Hilbert space. As a result of this, we have

$$\Pi(G)^{-1}T(u)\Pi(G) = GT(u)G^{-1}\,. \tag{6.1.13}$$

Hence,

$$\begin{aligned}\Pi(G)^{-1}\tilde{t}(u)\Pi(G) &= \operatorname{tr}(\Pi(G)^{-1}T(u)T^t(-u)\Pi(G)) \\ &= \operatorname{tr}(\Pi(G)^{-1}T(u)\Pi(G)(\Pi(G)^{-1}T(-u)\Pi(G))^t)\end{aligned} \tag{6.1.14}$$

where we used that the trace and transpose only act on the auxiliary space. Now we use the $GL(2N)$ covariance to get

$$\operatorname{tr}(GT(u)G^{-1}(GT(-u)G^{-1})^t) = \operatorname{tr}(G^tGT(u)(GG^t)^{-1}T^t(-u))\,. \tag{6.1.15}$$

Clearly, for $G \in SO(2N)$, we have that $G^tG = 1$. Hence the transfer matrix $\tilde{t}(u)$ is $SO(2N)$ invariant.

This symmetry means that the eigenspaces of the transfer matrix will organise themselves into multiplets of $SO(2N)$ and hence we will have a degenerate spectrum. Just like the periodic case, in order to lift these degeneracies we can add a twist by considering $\mathbb{T}(u) = KT(u)T^t(-u)$ where $K^t = -K$, while leaving the other boundary trivial. This is exactly what we imposed in (6.1.8) for the boundary matrices K_- and K_+ (notice that $I_{2N}^t = I_{2N}$).

Finally, note that any matrix K with $K^t = -K$ with distinct non-zero eigenvalues can be diagonalised by an orthogonal matrix P, i.e. with $P^tP = 1$. By applying this transformation inside the transfer matrix, we can transform any K such that $K^t = -K$ to the diagonal matrix K_+ in (6.1.10).

Parity of the transfer matrix

The transfer matrix (6.1.7) is a polynomial in u of degree $2L$. Since the integrals of motion are the coefficients of its powers of u, one could think that $\tilde{\mathbb{T}}(u)$ has $2L$ IoMs - twice the number of a periodic spin chain. However, this is not the case: the transfer matrix of an open spin chain of length L contains L integrals of motion.

This is due to the fact that the open transfer matrix has simple parity properties in the spectral parameter, and can be made manifestly even in u by slightly modifying the conventions we have used so far. In our new conventions, we shift the spectral parameter in the Lax operators (3.1.3):

$$L_{ij}^{\alpha}(u) = (u - \theta_\alpha + is)\delta_{ij} + i\pi(E_{ji})\,. \tag{6.1.16}$$

After building the transfer matrix (6.1.7), we normalise it by a function of u:

$$\mathbb{T}(u) = \frac{\tilde{\mathbb{T}}(u)}{u + i/2}\,. \tag{6.1.17}$$

As we prove in Appendix E.1 doing so ensures that:

$$\mathbb{T}(u) = \mathbb{T}(-u)\,, \tag{6.1.18}$$

meaning that the transfer matrix of an open spin chain is even in the spectral parameter.

6.2 Open Q-system

The spectrum and eigenstates of the open transfer matrix can be reconstructed via the Algebraic Bethe Ansatz [103], in a very similar way to the periodic spin chain described in detail in section 3.4. In particular, states can be built by applications on a vacuum vector of some B operators, evaluated at the $2N-1$ sets of Bethe roots $\{u_k^i\}_{i=1}^{2N-1}$, which are the solutions to the nested open Bethe equations.

We will not present the detailed calculations here. The key point is that we can package the $2N-1$ sets of Bethe roots into $2N-1$ sets of Q-functions, who form a *open Q-system* analogous to the periodic one described in section 3.6.

Let us define $\zeta_{N+a} \equiv -\zeta_{N-a},\ \forall a > N$. We define the open Q-functions as:

$$Q_A(u) = N_A \left(\prod_{a \in A} \zeta_a^{iu} \right) F_{|A|}(u) q_A(u)\,, \tag{6.2.1}$$

where $F_{|A|}$ are some functions that can be changed using the gauge freedom of the open Q-system, and q_A are the Baxter polynomials, monic polynomials in u containing the Bethe roots. The structure of the open Q-functions is very similar to the periodic Q-functions (3.6.8); in particular, the boundary parameters ζ_i play the same role as the twist eigenvalues of the periodic case.

There are 2^{2N} Q-functions in total, and the QQ-relations still hold [104]:

$$Q_{Aab}(u) Q_A^{[-2]}(u) = Q_{Aa}(u) Q_{Ab}^{[-2]}(u) - Q_{Ab}(u) Q_{Aa}^{[-2]}(u)\,. \tag{6.2.2}$$

As argued in [102], the main feature of the Q-functions for open spin chains are the following conditions on the Baxter polynomials $q_A(u)$:

$$q_{1\ldots l}(u) = q_{1\ldots 2N-l}(-u), \qquad l \leqslant N \tag{6.2.3}$$

This holds for other choices of the indices as well; in particular, if $l = N$, this relation implies that the *middle node* Baxter polynomial $q_{1\ldots N}$ is even in u[1].

[1] In the paper [102], the authors claim that all the open Q-functions should be even. This is not true, as can be verified from explicit calculations even in the simplest examples: only the middle node Baxter polynomials are even in u.

6.3 Open T-system

We now construct the family of mutually commuting, totally antisymmetric transfer matrices $\mathbb{T}_i(u)$, $i = 1 \dots 2N$ for the twisted Yangian, which will form the *open T-system*. Although conceptually similar to the periodic case, fusion for the open spin chain is more complicated, since the underlying algebra is the BYBE (6.1.2) rather than the simpler RTT relations.

In order to remove some overall factors that would otherwise appear throughout this section, we introduce the normalised rational R-matrix via a rescaling of (2.2.3):

$$S(u) = \frac{R(u)}{u+i} = \frac{1}{u+i}(u1 + iP)\,. \tag{6.3.1}$$

This rescaling leaves invariant the RTT relations and is an isomorphism of the Yangian $Y(gl_N)$.

In the periodic case, we obtained the totally antisymmetric transfer matrices via the quantum minors (3.7.7). We will use a similar approach here: we define the Sklyanin minors [43] as the matrix elements of the following equation:

$$U^m(u) \equiv A_m U_1 S^t_{12} U_2 \dots U_{m-1} \left(\prod_{k=1}^{m-1} S^t_{km} \right) U_m\,. \tag{6.3.2}$$

In this equation, we have used the antisymmetriser A_m, the normalised R^t-matrix $S^t(u) = \frac{R^t(u)}{u+i}$ and introduced the following shorthand notation:

$$U_j \equiv U_j^- \left(u - i\frac{m+1}{2} + ij \right), \tag{6.3.3}$$

$$S^t_{kl} \equiv S^t_{kl}(-2u + i(m+1-l-k))\,. \tag{6.3.4}$$

Explicitly the Sklyanin minors are given by:

$$U^{a_1 \dots a_m}_{b_1 \dots b_m}(u) = \sum_{p \in \mathfrak{S}_m} \operatorname{sgn} p \cdot (U^m(u))^{a_{p(1)} \dots a_{p(m)}}_{b_1 \dots b_m}\,. \tag{6.3.5}$$

The Sklyanin minors satisfy a generalised BYBE, where the R-matrix is substituted with the fused R-matrix in a similar way as (A.3.4) [43].

In contrast with the quantum minors, the Sklyanin minors are not enough to define the open antisymmetric transfer matrices. In fact, we also need the fused K^+ matrix as the matrix elements of the following equation:

$$K^m_+(u) \equiv A_m K^+_1 \tilde{S}^t_{12} K^+_2 \dots K^+_{m-1} \left(\prod_{k=1}^{m-1} \tilde{S}^t_{km} \right) K^+_m\,, \tag{6.3.6}$$

where $\tilde{S}^t_{kl} \equiv S^t_{kl}(2u + i(m+1-l-k))$ and $K^+_j \equiv K_j(u + i\frac{m+1}{2} - ij)$. Explicitly we define:

$$K^{a_1 \dots a_m}_{b_1 \dots b_m}(u) = \sum_{p \in \mathfrak{S}_m} \operatorname{sgn} p \cdot (K^m(u))^{a_{p(1)} \dots a_{p(m)}}_{b_1 \dots b_m}\,. \tag{6.3.7}$$

Notice that we can also define the boundary matrix $K_-^m(u)$ by substituting U with K_- in (6.3.5). This will be used in chapter 8.

Using both the Sklyanin minors (6.3.5) and (6.3.7), we can define the m-th fused antisymmetric transfer matrix as:

$$\tilde{\mathbb{T}}_m(u) = \mathrm{tr}_m(K^m(u)U^m(u)) = K^{a_1\dots a_m}_{b_1\dots b_m}(u)U^{b_1\dots b_m}_{a_1\dots a_m}(u), \qquad \forall m \leqslant 2N\,, \tag{6.3.8}$$

where the trace is taken on m antisymmetrised copies of the auxiliary space. Just like the periodic case, $\tilde{\mathbb{T}}_{2N}$ is the center of the twisted Yangian Y^+_{2N} and is known as the *Sklyanin determinant*.

It is possible to make the fused antisymmetric transfer matrices even in u by a simple rescaling. In particular, we have that:

$$\mathbb{T}_m(u) = \frac{\tilde{\mathbb{T}}_m(u)}{\prod_{k=1}^m \left(u - i\frac{m}{2} + ik\right)}, \qquad \mathbb{T}_m(u) = \mathbb{T}_m(-u)\ \ \forall m \leqslant 2N\,. \tag{6.3.9}$$

The fused antisymmetric transfer matrices form a commuting family of quantum operators:

$$[\mathbb{T}_i(u), \mathbb{T}_j(v)] = 0, \qquad \forall i,j \leqslant 2N\,. \tag{6.3.10}$$

Just like in the periodic case, it is possible to introduce further fused transfer matrices using Young diagrams; we will not do so explicitly, as we will not be using them in this work. The open T-system is formed by these fused transfer matrices, plus the open Hirota and CBR equations, which have a similar form to the periodic ones.

6.4 Diagrammatic rules for open spin chains

As in section 3.2, we can implement diagrammatic rules to express many algebraic equations for open spin chains. These will essentially be the same as the one presented in section 3.2 with a few new ingredients:

- The dashed auxiliary horizontal lines, which is obtained after an horizontal line is reflected on the boundary. It is associated with a negative spectral parameter. A physical space vertical line crossing a dashed horizontal line gives the Lax operator L^t.
- The transposed R-matrix R^t, which is obtained when a solid and a dashed horizontal line cross.
- The boundary matrices K_+ and K_-, which are put on a black vertical line representing the "boundary" physical space (which is trivial in this section).

We draw the diagrammatic rules in figure 6.1.

As an example of their use, we draw the BYBE equations (6.1.2) and (6.1.6) in figures 6.2 and 6.3 respectively.

Furthermore, we use them to draw the proof of the statement that the open transfer matrices commute with each other, i.e. $[\mathbb{T}(u), \mathbb{T}(v)]$, in figure 6.4.

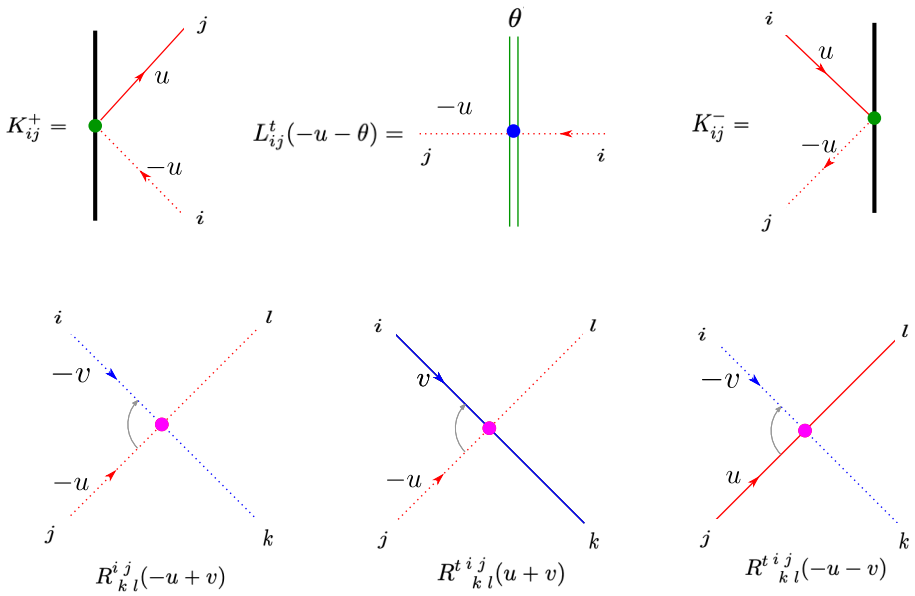


Figure 6.1: Diagrammatic rules

6.5 Transfer matrix eigenvalues from Q-functions

Based on the parity properties of the transfer matrix $\mathbb{T}(u)$ a proposal for its eigenvalues in terms of a handful of Bethe roots was proposed in [102]. We recast this here in terms of Q-functions in the following way:

$$\mathbb{T}_1(u) = \sum_{k=1}^{N} \Lambda_k(u) \tag{6.5.1}$$

where we have introduced the quantum eigenvalues $\Lambda_k(u)$ defined by

$$\Lambda_k(u) = \frac{Q^{[2]}_{1\ldots k-1}}{Q_{1\ldots k-1}} \frac{Q^{[-2]}_{1\ldots k}}{Q_{1\ldots k}} \,. \tag{6.5.2}$$

By exploiting the analytic structure of the Q-functions (6.2.1) the quantum eigenvalues can be explicitly written as

$$\Lambda_k(u) = \frac{F_{k-1}(u+i)}{F_{k-1}(u)} \frac{F_k(u-i)}{F_k(u)} \frac{\tilde{q}_{1\ldots k-1}(u+i)}{\tilde{q}_{1\ldots k-1}(u)} \frac{\tilde{q}_{1\ldots k}(u-i)}{\tilde{q}_{1\ldots k}(u)} \,, \tag{6.5.3}$$

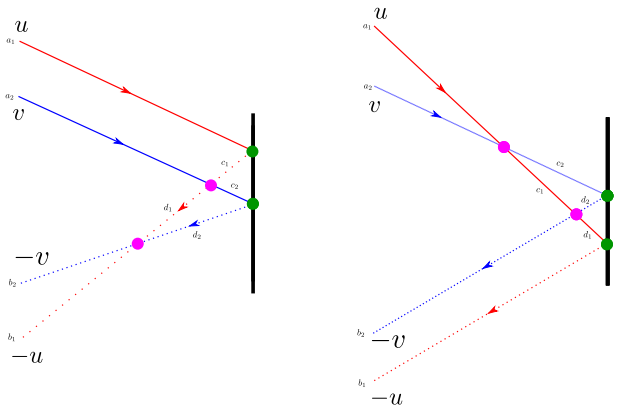


Figure 6.2: The BYBE (6.1.2)

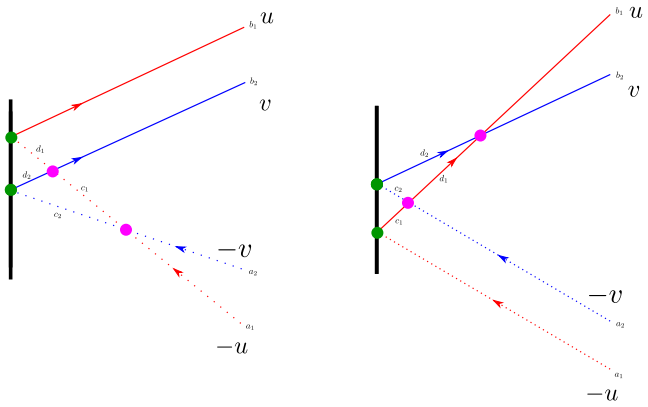


Figure 6.3: The BYBE (6.1.6)

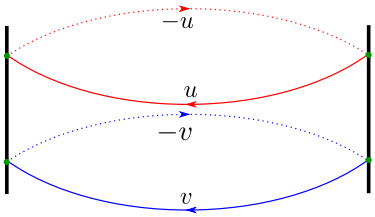
where $\tilde{q}_A$ indicates the twisted Baxter polynomial $\prod_{a\in A}\zeta_a^{iu}q_A$. From this equation it is possible to fix uniquely the functions F_k by confronting (6.5.1) with the eigenvalues of $\mathbb{T}(u)$ obtained from the boundary monodromy matrix.

Just like the periodic case, we can use the quantum eigenvalues to build the eigenvalues of all antisymmetric transfer matrices. Since we know the latter from the fusion procedure, we can treat the quantum eigenvalues as unknown in a linear system. Solving it lets us reconstruct the Q-functions, i.e. the Bethe roots, without the need of the Bethe equations.

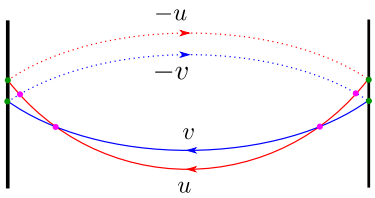
Note that the asymptotics of the antisymmetric open transfer matrices are directly related to the elementary symmetric polynomials in the K_+ boundary matrix eigenvalues.

6.6 Open Baxter TQ equation

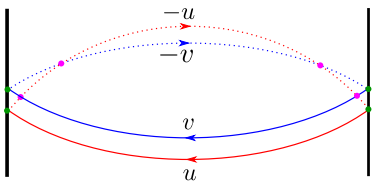
The key equations for the rest of this work describing open spin chains are the Baxter and the Dual Baxter equation.



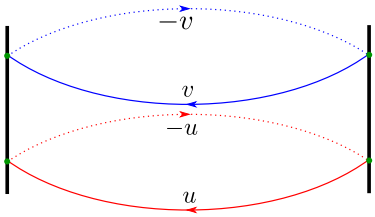
(a) Step 1 — We start off with $\mathbb{T}(u)\mathbb{T}(v)$, which acts as an operator on the quantum space.



(b) Step 2 — By introducing the identity as a product of S-matrices from the unitarity condition $S(u)S(-u) = 1$, we can pass the auxiliary space line of $\mathbb{T}(u)$ through $\mathbb{T}(v)$.



(c) Step 3 — Now we apply the boundary Yang-Baxter equations (6.1.2) and (6.1.6).



(d) Step 4 — Finally we resolve the identity again using unitarity, to obtain $\mathbb{T}(v)\mathbb{T}(u)$ hence proving that indeed the transfer matrices commute with each other for arbitrary values of the spectral parameter.

Figure 6.4: Diagrammatic proof of $[\mathbb{T}(u), \mathbb{T}(v)] = 0$

In the conventions of this chapter, we define the open Baxter and open dual Baxter operators as:

$$\mathcal{O} = \sum_{a=0}^{2N} (-1)^a \, \mathbb{T}_a^{[-2N+a+1]} \mathcal{D}^{-2N+2a}(u)\,, \tag{6.6.1}$$

$$\mathcal{O}^\dagger = \sum_{a=0}^{2N} (-1)^a \mathcal{D}^{2N-2a} \mathbb{T}_a^{[a-1]}(u)\,. \tag{6.6.2}$$

Then the Baxter and Dual Baxter equation are respectively:

$$\mathcal{O} Q_i(u) = 0\,, \tag{6.6.3}$$

$$\mathcal{O}^\dagger Q^i(u) = 0\,. \tag{6.6.4}$$

The Baxter and Dual Baxter equation described here can be simplified - the antisymmetric transfer matrices of an open spin chain contain non-dynamical prefactors that can be

stripped out, and furthermore we can reduce the Q-functions to be twisted Baxter polynomials. We will see an example of this in the rank 4 case in the next section, and also in chapter 8.

6.7 Functional Separation of Variables for the twisted Yangian

In this section, we will work for simplicity on the rank 4 case $Y^+(4)$. The motivations for choosing this as our example are two:

- The rank 2 case $Y^+(2)$ is isomorphic to other types of twisted Yangian [97], such as the Reflection Algebra [100]. Thus the rank 4 case is the simplest case which shows the unique features of the family $Y^+(2N)$;
- $Y^+(4)$ is the symmetry of the open Fishchain model described in chapter 8.

We will write down the Baxter and Dual Baxter equation for twisted Baxter polynomials, and use them to obtain the functional SoV measure by imposing functional orthogonality. Finally, we will work out the functional scalar product between Bethe states.

First, we need to rescale the antisymmetric transfer matrices to remove some overall non-dynamical factors. In particular, we define:

$$\tau_1(u) = \mathbb{T}_1(u)\,, \tag{6.7.1}$$
$$\tau_2(u) = \mathbb{T}_2(u)(u+i/2)(u-i/2)\,, \tag{6.7.2}$$
$$\tau_3(u) = \mathbb{T}_3(u)\frac{(u+i)(u-i)}{Q_\theta^{[2s-2]}(u)Q_\theta^{[-2s+2]}(u)}\,, \tag{6.7.3}$$

where $Q_\theta(u) \equiv \prod_{\alpha=1}^{L}(u-\theta_\alpha)(-u-\theta_\alpha)$.

By stripping away the normalisation N_A and F_k from the Q-functions (6.2.1), we get the following form for the Baxter operator:

$$\mathcal{O}_4 = \mathcal{O}_- + \tau_2^- + \mathcal{O}_+\,, \tag{6.7.4}$$

where

$$\mathcal{O}_- = Q_\theta^{[-2(s+1)]}Q_\theta^{[-2s]}\mathcal{D}^{-2} - (u-i)Q_\theta^{[-2s]}\tau_1^{[-2]}\mathcal{D}^{-1} \tag{6.7.5}$$
$$\mathcal{O}_+ = -uQ_\theta^{[2(s-1)]}\tau_3\mathcal{D} + (\zeta_1\zeta_2)^2Q_\theta^{[2(s-1)]}Q_\theta^{[2s]}\mathcal{D}^2 \tag{6.7.6}$$

The dual Baxter operator is:

$$\mathcal{O}^\dagger{}_4 = \mathcal{O}_+^\dagger + \tau_2^+ + \mathcal{O}_-^\dagger \tag{6.7.7}$$

where:

$$\mathcal{O}_+^\dagger = Q_\theta^{[2s]}Q_\theta^{[2(s+1)]}\mathcal{D}^2 - (u+i)Q_\theta^{[2s]}\tau_1^{[2]}\mathcal{D} \tag{6.7.8}$$
$$\mathcal{O}_-^\dagger = -uQ_\theta^{[-2(s-1)]}\tau_3\mathcal{D}^{-1} + (\zeta_1\zeta_2)^2Q_\theta^{[-2(s-1)]}Q_\theta^{[-2s]}\mathcal{D}^{-2} \tag{6.7.9}$$

The Baxter and Dual Baxter equations are given by:

$$\begin{aligned} \mathcal{O}_4 \tilde{q}_a(u) &= 0 && (6.7.10)\\ \mathcal{O}_4^\dagger \tilde{q}^a(u) &= 0 && (6.7.11)\\ & && (6.7.12) \end{aligned}$$

where $\tilde{q}_a = \zeta_a^{iu} q_a(u)$ are the twisted Q-functions and $\tilde{q}^a$ are their duals, defined exactly as in the periodic case (3.6.10).

6.7.1 Functional measure and scalar product

Functional Separation of Variables works for the twisted Yangian in the same way as for the periodic case, which we analysed in chapter 5.

The first step consists in defining the functional bracket:

$$\Big(\!\Big(f, g \Big)\!\Big)_\alpha = \int_C du\, \mu_\alpha(u) f(u) g(u) \tag{6.7.13}$$

where μ_α is the functional measure, and C is a contour that contains all the poles of μ_α and of the dual Baxter equation, if the latter has any. In our conventions where the Q-functions are Baxter polynomials, the poles only come from the measure.

We can bootstrap the family of measures μ_α, $\alpha = 1 \dots L$ by imposing the functional orthogonality relation:

$$\Big(\!\Big(f \mathcal{O}_4^\dagger g \Big)\!\Big)_\alpha = \Big(\!\Big(g \mathcal{O}_4 f \Big)\!\Big)_\alpha \tag{6.7.14}$$

By using our gauge choices, we get that:

$$\mu_\alpha(u) = \prod_{\beta=1}^{L} \frac{\Gamma(u-\theta_\beta+i(s-1/2))}{\Gamma(u-\theta_\beta-i(s-3/2))} \frac{\Gamma(u+\theta_\beta+i(s-1/2))}{\Gamma(u+\theta_\beta-i(s-3/2))} \rho_\alpha(u), \quad \alpha = 1 \dots L\,, \tag{6.7.15}$$

where $\rho_\alpha(u)$ are any i-periodic functions. Notice that μ_α have infinite poles if the weights of the representation are non-integer, i.e. for non-compact open spin chains. For the simple representation $s = -1/2$, we get:

$$\mu_\alpha(u) = \frac{1}{Q_\theta^{[2]}(u) Q_\theta(u) Q_\theta^{[-2]}(u)} \rho_\alpha(u)\,, \tag{6.7.16}$$

where ρ_α is a rational function. If $\rho_\alpha = 1$, the measure has poles at:

$$\{-\theta_\beta - i, -\theta_\beta + i, -\theta_\beta, \theta_\beta, \theta_\beta - i, \theta_\beta + i\}, \quad \beta = 1 \dots L\,. \tag{6.7.17}$$

By choosing the functions ρ_α appropriately, it is possible to cancel out the poles at $-\theta_\beta + ni$, while leaving the functional scalar product of next section unchanged. The functional bracket can be computed by summing over the residues at these poles, provided the asymptotics of the twisted Baxter polynomials are chosen appropriately.

6.7.2 Scalar products from functional orthogonality

Having defined the functional bracket (6.7.13), we can use it to obtain an expression for the scalar product of states of the open spin chain. The procedure is quite similar to the one derived for periodic spin chains.

The starting point is the system of trivial equalities:

$$\left\langle\!\!\left\langle \tilde{q}_j^B(u) \left(\mathcal{O}_{4,A}^\dagger - \mathcal{O}_{4,B}^\dagger\right) \tilde{q}_A^i(u) \right\rangle\!\!\right\rangle_\alpha = 0\,, \tag{6.7.18}$$

where the subscript A (B) in the Dual Baxter operator imply that it is applied to the state labelled by A (B).

Notice that these equalities are all trivial, since $\mathcal{O}_4^\dagger \tilde{q}^i = 0$ and the functional orthogonality condition implies that $\left\langle\!\!\left\langle f\mathcal{O}_4^\dagger \tilde{q}_j \right\rangle\!\!\right\rangle = 0 \ \forall j$, for any function f that does not spoil the convergence of the functional bracket.

The next step is to use the explicit form of the dual Baxter operator (6.7.7), and expand the transfer matrices in integrals of motion. The integrals of motion will be the variables for which we solve the linear system composed by the equations (6.7.18).

For the class of representations we consider, of weights $(-2s, -2s, 0, 0)$, $s \in -\frac{\mathbb{N}}{2}$, $\mathbb{T}_1$ and $\mathbb{T}_3$ contain L Integrals of motion each, while $\mathbb{T}_2$ contains $2L$ of them. Therefore, we will need $4L$ equations of type (6.7.18).

A natural way to pick this number of equations is by taking $\alpha = 1 \dots L$, $j = 1, 2$ and $i = 3, 4$ in (6.7.18). This is done in analogy to the periodic case, where such choice would reproduce the wavefunctions $\Psi(x) = \langle x|\Psi\rangle$ present in the scalar product computed via the B operator. We hope that this is the case here as well, although we do not possess an SoV B operator for this open spin chain.

We will now compute the scalar product explicitly. For simplicity, we set $L = 1$ and $s = -1/2$. To do so, we need to expand the functional orthogonality relation (6.7.18), and we get:

$$\begin{aligned}
\Big\langle\!\!\Big\langle \Big(& \tilde{q}_A^i(u)^{[-1]}(u-\theta_1-i)(-u-\theta_1-i) I_{2,1}^{AB}(u), \\
& - \tilde{q}_A^i(u)^{[-1]}(u-\theta_1+i)(-u-\theta_1+i) I_{2,2}^{AB}(u), \\
& - \tilde{q}^i(u)^{[+1]}(u+\frac{i}{2}) Q_\theta^{[-2]} I_1^{AB}(u)^{[+1]}, \\
& - \tilde{q}^i(u)^{[-3]}(u-\frac{i}{2})(u-\theta_1+i)(-u-\theta_1-h) I_3^{AB}(u)^{[-1]} \Big) \tilde{q}_j^B(u) \Big\rangle\!\!\Big\rangle_\alpha = 0\,.
\end{aligned}$$

Here, we have introduced the IoMs contained in the antisymmetric transfer matrices eigenvalues τ_i as I_i. We splitted the ones in τ_2 into two factors, $I_{2,1}$ and $I_{2,2}$, each containing $L = 1$ IoMs. Furthermore, I_i^{AB} indicates that we are subtracting the IoMs evaluated on the state B from those evaluated on the state B. Some of these quantities must be non-zero if $A \neq B$.

As in the periodic case, in order to have a non-trivial solution of this linear system, we must have that:

$$\det_{j=1,2;i=3,4} \left(\!\!\left(\tilde{q}_j^A L \tilde{q}_B^i \right)\!\!\right) \propto \delta_{AB} \,, \tag{6.7.19}$$

where L is the following vector applied to the right Q-function:

$$\begin{aligned} L = \Big\{ &\left(u + \frac{i}{2}\right) Q_\theta^{[-2]} \mathcal{D}, (u - \theta_1 + i)(-u - \theta_1 + i)\mathcal{D}^{-1}, \\ &(u - \theta_1 - i)(-u - \theta_1 - i)\mathcal{D}^{-1}, \left(u - \frac{i}{2}\right) Q_\theta^{[+2]} \mathcal{D}^{-3} \Big\} \,. \end{aligned} \tag{6.7.20}$$

This formula can be easily generalised to any L and $s \in -\frac{\mathbb{N}}{2}$, and forms the first step for the application of FSoV to open spin chains. In particular, we hope to be able to apply this construction to Wilson loops in $\mathcal{N} = 4$ SYM, via the open Fishchain that we will introduce in Chapter 7.

Part II

Spin Chains in Gauge Theories

Chapter 7

Conformal Field Theory and $\mathcal{N} = 4$ SYM

In Part 1 of this thesis, we have introduced the tools to analyse integrable systems, with a particular focus on integrable spin chains. In this Part, we will see how these systems arise in High Energy Physics, and how they can be used to compute physical observables in Quantum Field Theories. In particular, we will focus on the context of the AdS_5/CFT_4 correspondence, a gauge/gravity duality formulated between type II string theory on a $AdS_5 \times S^5$ background and 4-dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory with $SU(N)$ gauge symmetry [4]. This correspondence gave a huge impulse to the study of integrability in the context of high energy physics, due to the fact that the 2-dimensional sigma model describing strings moving in AdS_5 spacetime was proven to be classically integrable [105].

Supported by this fact, the search of integrable structures in the CFT side of the duality, i.e. $\mathcal{N} = 4$ SYM, turned out to be extremely fruitful. The first breakthrough happened when it was noticed that [3] in the large N (planar) limit the one-loop mixing matrix of operators composed by a single trace of scalar fields can be identified with the Hamiltonian of a periodic integrable $SO(6)$ spin chain. This Hamiltonian can then be diagonalized using the tools of integrability seen in Part 1 of this thesis. Its eigenvalues correspond to the one-loop anomalous dimensions of the single-trace operators of $\mathcal{N} = 4$ SYM theory.

The work by Minahan and Zarembo opened up a line of research that culminated in a method that lets us obtain the full, non-perturbative spectrum of planar $\mathcal{N} = 4$ SYM, known as the Quantum Spectral Curve (QSC) [57]. The QSC computes the Baxter Q-functions of the integrable spin chain dual to $\mathcal{N} = 4$ SYM, and extracts the spectrum of single-trace operators from them. Hopefully, this line of research will lead us to the calculation of the three-point structure constants too, in light of the promising results from recent works [15, 16, 18, 21, 22, 106–109]. If this task is completed, integrability will have played the main role in the solution of an interacting gauge theory, a feat that looked impossible just twenty years ago.

In this chapter, we will review the initial part of this story, introducing the theory of

$\mathcal{N} = 4$ SYM and describing how Minahan and Zarembo found a spin chain in it. While their construction is perturbative, it inspired non-perturbative dualities as well, such as the one that we will see in the next chapter between cusped Wilson lines and the open Fishchain.

7.1 Brief overview of $\mathcal{N} = 4$ SYM theory

In this section, we will give a brief overview of $\mathcal{N} = 4$ SYM. We will first introduce Conformal Field Theories and define fundamental concepts such as the scaling dimensions, three-point correlation functions. We will end this section with the Lagrangian for $\mathcal{N} = 4$ SYM, which will be useful in the next chapter.

7.1.1 CFT basics

A conformal field theory (CFT) in D dimensions is a quantum field theory enjoying conformal symmetry, which in Euclidean space is represented by the global symmetry group $SO(1, D + 1)$. Being a QFT, a CFT is endowed with an Hilbert space (one for each time slice[1]), and operators acting on it. The operators acting at a single point in space, or *local operators*, of a CFT are its quantum fields.

The observables in a CFT are the correlation functions of its operators, i.e. their expectation values on the vacuum state, where each operator is evaluated at a different spatial point. Conformal symmetry makes correlation functions extremely constrained: in a CFT, *any* correlation function of local operators can be fully determined by two sets of numbers, the conformal (or scaling) dimensions and the structure constants of its primary operators. Here, primary operators are the operators that act as "highest weight states" for the conformal group, i.e. they are annihilated by the action of a subset of its generators, the special conformal transformations. Primary operators are also eigenstates of the dilatation operator D of the conformal group, with eigenvalues being their scaling dimension Δ.

Scaling dimension Given two scalar primary operators O_A, their two point correlation function can be constrained by the Ward identities associated to conformal symmetry. This implies that, if O_A are both scalar primary operators, their two point correlation function is:

$$\langle O_A(x) O_B(y) \rangle = \frac{\delta_{AB}}{|x - y|^{\Delta_A + \Delta_B}} , \tag{7.1.1}$$

where Δ_A is the scaling dimension of the operator O_A, Δ_B is the scaling dimension of O_B, x and y are D-dimensional Euclidean vectors and $|\quad|$ is the Euclidean norm on $\mathbb{R}^D$. If O are not scalar, their two point function will depend on Δ, plus an extra, non-dynamical factor that depends on the spin of the operators.

[1] In Euclidean space, we can just select one dimension and treat it as "time".

Structure constants The Ward identities constrain the three point correlation function of 3 scalar primary operators to be:

$$\langle O_A(z_1)O_B(z_2)O_C(z_3)\rangle = \frac{\lambda_{ABC}}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}}\,, \tag{7.1.2}$$

where $z_{ij} \equiv z_i - z_j$ and λ_{ABC} are $\mathbb{C}$-numbers called structure constants. If O are not scalars, their three point function will have extra non-dynamical factors that depend on the spin of the operators.

The form of the 2 and 3 point functions are completely fixed by conformal invariance, and the only dynamical dependence in them are the CFT data, the set of all possible Δ and λ in a given theory.

OPE A CFT has a convergent Operator Product Expansion (OPE): given two operators O_A and O_B at two different points, we can expand their product in a convergent series around one of these two points:

$$O_A(x)O_B(x') = \sum_C \lambda_{ABC}(x-x')^{\Delta_C-\Delta_A-\Delta_B}O_C(x)\,. \tag{7.1.3}$$

Using this property, we can always express an N-point correlation function in terms of a $(N-1)$-point one plus the CFT data. This lets us always reduce the calculation of any correlation function to the knowledge of the CFT data. Therefore from the knowledge of CFT data we can determine any observable in a CFT, i.e. solve it.

7.1.2 $\mathcal{N} = 4$ SYM theory

Maximally supersymmetric Yang-Mills theory in $D = 4$, or $\mathcal{N} = 4$ SYM, is a CFT in 4-dimensional Euclidean space, possessing the maximal amount of supercharges allowed for this spacetime. It is an interacting gauge theory, with gauge symmetry $SU(N)$. Crucially, its conformal symmetry is unbroken at any order in perturbation theory [110]. Its Lagrangian is given by:

$$\mathcal{L} = \frac{1}{g^2_{\rm YM}}\,\mathrm{Tr}\left[\frac{1}{2}\left[D_\mu, D_v\right]^2 + \left(D_\mu\Phi_i\right)^2 - \frac{1}{2}\left[\Phi_i,\Phi_j\right]^2 + i\bar{\Psi}\left(\Gamma^\mu D_\mu\Psi + \Gamma^i\left[\Phi_i,\Psi\right]\right) + \partial_\mu\bar{c}D_\mu c + \zeta\left(\partial_\mu A_\mu\right)^2\right] \tag{7.1.4}$$

Here, g_{YM} is the Yang-Mills coupling, and all the fields are in the adjoint representation of $SU(N)$. These fields are:

- Φ_i, $i = 1\ldots 6$ are scalar fields;
- Ψ_a, $a = 1\ldots 4$ are Dirac spinor fields;
- A_μ is the non-abelian gauge field;
- $c, \bar{c}$ are the ghost fields.

We will be mostly interested in the planar limit of $\mathcal{N} = 4$ SYM, which is obtained by sending the rank N of the gauge group $SU(N)$ to infinity while the 't Hooft coupling $g = g_{YM}^2 N$ stays fixed. In this limit, the Feynman diagrams that cannot be drawn on a spherical surface become subleading, as they will be multiplied by overall negative powers of N. Hence, only the so-called *planar diagrams*, that can be drawn on spherical surfaces, will contribute to the correlation functions in the planar limit.

7.1.3 The $SU(2)$ subsector of $\mathcal{N} = 4$ SYM and spin chains

In this section, we will review the seminal work by Minahan and Zarembo [3].

We start by noticing that the 2-point correlation function (7.1.1) is not as simple as it looks. In fact, the operators O_A, eigenstates of D, are actually *renormalized* operators, which are related to a naive set of bare operators (i.e. the quantum fields in the Lagrangian (7.1.4)) via the so-called mixing matrix $\hat{H}$ as:

$$O_A(x) = \left(e^{\hat{H}\log\Lambda}\right)_{AB} O_0^B(x)\,, \tag{7.1.5}$$

where Λ is a momentum cutoff. In fact, under a RG flow the bare operators mix between each other, while the renormalized ones have simple scaling:

$$O_A(x) \to \lambda^{\Delta} O_A(\lambda^{-1}x)\,. \tag{7.1.6}$$

This means that the latter must be eigenvectors of the mixing matrix $\hat{H}$, with eigenvalue Δ: therefore, by diagonalizing $\hat{H}$ we can find the scaling dimensions of our theory. Of course, this is easier said than done: since $\mathcal{N} = 4$ SYM is a very complicated interacting theory, even taking simple operators and computing $\hat{H}$ at 1-loop level is a very hard task. Then we would still need to diagonalize it and find its eigenstates!

Minahan and Zarembo in [3] noticed that the 1-loop expansion of the mixing matrix for operators which are made by a single trace of scalar fields of $\mathcal{N} = 4$ SYM is the Hamiltonian of a spin chain, for which the Algebraic Bethe Ansatz (3.4) can be used to find eigenstates and eigenvalues.

We will now review in detail how this is done for the so-called $SU(2)$ sector of $\mathcal{N} = 4$ SYM.

We start by considering single-trace operators of the 6 scalars Φ_i of $\mathcal{N} = 4$ SYM:

$$O_A^0(x) = \mathrm{tr}\,(\Phi_{i_1} \dots \Phi_{i_L}) \tag{7.1.7}$$

To compute $\hat{H}$ at 1-loop, we need to compute a huge number of planar diagrams, which will be of self energy type and of "interaction" type between two scalars, as in figure 7.1. Since we are interested only in the UV divergent part, we only need to evaluate them as the loop momentum $k \to \infty$. In this limit, we only obtain two types of integral, which give rise to a logarithmic and a quadratic divergence, given by:

$$I_1 \equiv g^2 \log\Lambda = \frac{\lambda}{2}\int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}, \qquad I_2 \equiv \frac{g^2}{2}\frac{\Lambda^2}{p^2} = \frac{\lambda}{2}\int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 p^2}\,, \tag{7.1.8}$$

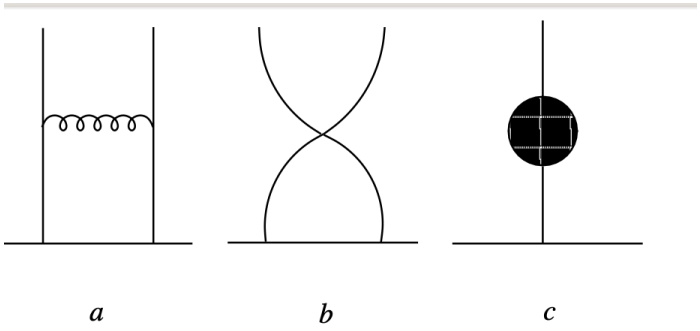


Figure 7.1: Planar diagrams for the 1-loop mixing matrix of single trace scalar operators, adapted from [3]

.

where $g^2 \equiv \dfrac{\lambda}{16\pi^2}$. By summing over all diagrams, the second contribution drops out and we obtain a logarithmic divergence, as we would expect from the loop expansion of the mixing operator in (7.1.5). Explicitly we get:

$$\left(e^{\hat{H}\log\Lambda}\right)^{\dots,i_n,i_{n+1},\dots}_{\dots,j_n,j_{n+1},\dots} \simeq 1 - \frac{g^2 N}{16\pi^2}\log(\Lambda)\left(2\delta^{j_n}_{i_n}\delta^{j_{n+1}}_{i_{n+1}} + \delta_{i_n i_{n+1}}\delta^{j_n j_{n+1}} - 2\delta^{j_{n+1}}_{i_n}\delta^{j_n}_{i_{n+1}}\right) , \quad (7.1.9)$$

where we only have non-zero elements for pairs of consecutive indices. This operator can be written in a form which resembles a spin chain Hamiltonian acting on the following states:

$$|s_{i_1}\dots s_{i_L}\rangle \quad (7.1.10)$$

where s_{i_n} are 6-dimensional spins. These correspond to the operators (7.1.7), with the obvious identification of $s_i \leftrightarrow \Phi_i$, $i = 1\dots L$. It is easy to then rewrite (7.1.9) acting on these state by introducing the identity, the permutation and the trace operators:

$$\begin{aligned} I\,|\dots s_i s_j \dots\rangle &= |\dots s_i s_j \dots\rangle \\ P\,|\dots s_i s_j \dots\rangle &= |\dots s_j s_i \dots\rangle \\ K\,|\dots s_i s_j \dots\rangle &= \delta_{ij}\sum_{k=1}^{6}|\dots s_k s_k \dots\rangle\ . \end{aligned} \quad (7.1.11)$$

We obtain that the spin chain Hamiltonian corresponding to (7.1.9) is:

$$\hat{H} = g^2\sum_{n=1}^{L}\left(2I_{n,n+1} + K_{n,n+1} - 2P_{n,n+1}\right)\ . \quad (7.1.12)$$

This is actually the Hamiltonian of a rational integrable spin chain in the fundamental representation of $Y(so_6)$. It is possible to apply a version of the Nested Bethe Ansatz to find

eigenstates and eigenvalues of this Hamiltonian, which will correspond to the eigenstates of the mixing matrix and their 1-loop anomalous dimensions. This would be computationally heavy, and we prefer to focus on the $SU(2)$ subsector, which is much simpler but can still illustrate the power of integrability. In this subsector, we will take only the complex combinations of 4 scalar fields, say $Z = \Phi_1 + i\Phi_2$ and $X = \Phi_3 + i\Phi_4$. Then the states we need to consider will have the form $|ZZXXZXZ\dots\rangle$. The main simplification for this model is that the trace operator K vanishes identically when applied to any such state. In fact, we have that applying K to any state with two equal consecutives spins, say ZZ, gives:

$$\begin{aligned} K|\dots ZZ\dots\rangle = & \\ &= K\,\mathrm{tr}(\dots(\Phi_1+i\Phi_2)(\Phi_1+i\Phi_2)\dots) \\ &= K\,\mathrm{tr}(\dots(\Phi_1\Phi_1+i\Phi_1\Phi_2+i\Phi_2\Phi_1-\Phi_2\Phi_2)\dots) \\ &= \mathrm{tr}(\dots(2\Phi_1\Phi_1+2\Phi_2\Phi_2-2\Phi_1\Phi_1-2\Phi_2\Phi_2)\dots) = 0\,. \end{aligned} \qquad . \tag{7.1.13}$$

Acting on consecutives different spins with K gives 0, so we conclude that $K|\dots\rangle = 0$ for any state. Therefore our spin chain Hamiltonian for the $SU(2)$ subsector is simply:

$$\hat{H} = 2g^2 \sum_{n=1}^{L} (I_{n,n+1} - P_{n,n+1})\ . \tag{7.1.14}$$

This is just the XXX Heisenberg spin chain Hamiltonian described extensively in chapter 2. We can easily diagonalise it using the ABA of section 3.4.

For higher loops, one could perform similar calculations and obtain that $\hat{H}$ is still dual to an integrable spin chain. These spin chains have long-range interactions, and most of the techniques studied in this thesis cannot be easily applied yet to them. The modern method to study the non-perturbative conformal dimension is the Quantum Spectral Curve, which does not need to deal directly with these long-range spin chains.

The fact that Feynman diagrams contributing to observables of $\mathcal{N} = 4$ SYM can be described in terms of an integrable spin chain, or that they exhibit Yangian symmetry, is a concept that has been used extensively in literature, despite its quite recent conception: a few non-exhaustive examples can be found in [33, 111–120]. In the next chapter, our analysis will use it to derive an integrable spin chain that is dual to *all loops* Feynman diagrams describing cusped Maldacena-Wilson loops, in a special limit of planar $\mathcal{N} = 4$ SYM.

Chapter 8

Cusps in $\mathcal{N} = 4$ SYM and Open Fishchain

In this chapter, we consider a cusped supersymmetric Wilson loop (also known as Maldacena-Wilson loop), with insertions of local operator at the cusp, in planar $\mathcal{N} = 4$ SYM. We will adapt methods developed for the Fishnet theories [33, 93, 121], a simple, non-unitary CFT in $D = 4$, to develop an integrability based description, in the form of an open spin chain, of these observables in the so-called ladders limit of $\mathcal{N} = 4$ SYM. This spin chain, called the open Fishchain, is built upon the twisted Yangian $Y^+(4)$ in a very non-trivial, non-compact representation. We use this spin chain description to obtain a solution for the spectrum of cusped supersymmetric Wilson loop, using the Baxter TQ equation supplemented with a particular quantisation condition.

The setup that we will consider in this chapter is the following: we have a cusped Maldacena-Wilson line [23, 122] with internal cusp angle φ, as in figure 8.1. This Maldacena-Wilson lines has two rays, one to the left and one to the right of the cusp, to which there are coupled the gauge field and a scalar Φ_i of $\mathcal{N} = 4$ SYM.

We pick the scalar Φ_1 to couple to the left ray and $\Phi_1 \cos\theta + \Phi_2 \sin\theta$ to couple to the right ray, thus θ is the angle between the two scalar fields. J[1] $\mathcal{N} = 4$ SYM complex scalars $Z = \frac{1}{\sqrt{2}}(\Phi_5 + i\,\Phi_6)$, orthogonal to the ones that couple to the rays, are inserted at the cusp. In addition, we can include in our description the excited states, in analogy with [25, 123], which corresponds to insertions on the cusp of linear combinations of the scalars coupled to the lines. This observable has a well defined anomalous dimension, which was studied in [124–126] by means of Thermodynamic Bethe Ansatz and then QSC methods. We will instead carry out a first principles derivation in the so-called *ladders limit*, which we describe below. The only insight we borrow from the QSC approach is a simple quantisation condition, that can be naturally imposed using FSoV arguments as seen in [92].

The *ladders limit* which we study in this chapter was first introduced for the case $J = 0$ in [127] and then used in [128]. This is obtained by taking the 't Hooft coupling $g \to 0$

[1]The name of this parameter comes from the Wilson loops literature. In our construction, it will correspond to the length of the spin chain, thus for this chapter we will use the letter J instead of L.

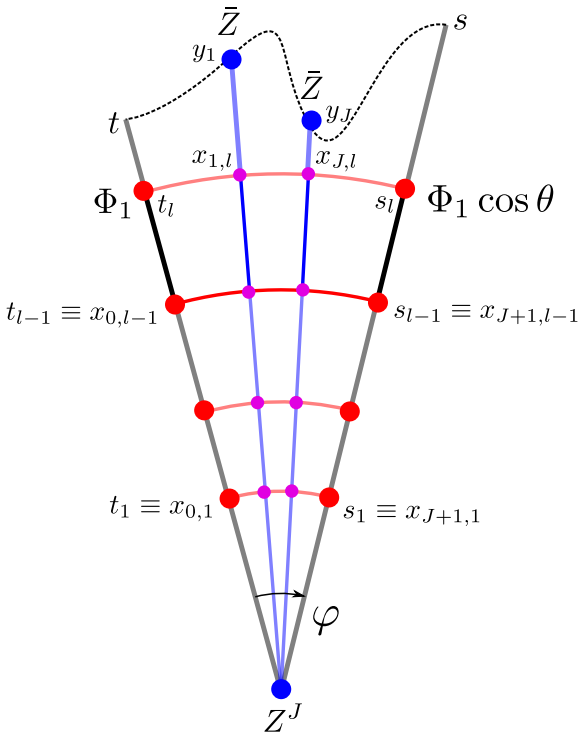


Figure 8.1: The CFT wavefunction for $J = 2$ is a sum of the fishnet diagrams with any number of bridges. This figure shows one such diagram with $l = 4$ bridges. The graph building operator is highlighted.

and $\theta \to i\,\infty$, in such a way that $\hat{g} \equiv g\left(\frac{\exp(-i\theta/2)}{2}\right)^{\frac{1}{J+1}}$ is kept constant. For the case $J = 0$ it was noticed in [127, 128] that only the ladder graphs contribute to the anomalous dimensions and the correlation functions. In this chapter we show that for the general $J > 0$ case the diagrams which survive are those of the fishnet type with a boundary corresponding to the two rays of the Wilson line (see figure 8.1). This drastic simplification in Feynman graphs allows us to construct a resummation procedure for them involving a *graph-building* operator. Such an operator was first constructed in the case of a Wilson-Maldacena loop with no scalar insertions in [25] and for the Fishnet theory in [33]. A new ingredient in our construction is the boundary of the Fishnet, which itself carries a 1D dynamics. This corresponds to the need to build an open spin chain, with two boundary matrices that are not constant matrices but nontrivial operators. This novel feature has not been yet fully investigated in the literature of open spin chains.

In this chapter we first explore the integrability in the classical (strong coupling) limit $\hat{g} \to \infty$ and then quantise this system and develop the full quantum integrability. The integrability description comes from a chain of particles living on AdS_5 (with radius going

to zero at strong coupling) also known as "Fishchain" [121, 129, 130]. This time, however, we have two "particles" with zero conformal weight at the ends of the chain whose motion is additionally restricted to the Wilson lines. In the quantum construction we identify explicitly the conserved charges of the system in the commuting family of operators, and prove that the graph-building operator of the Feynman graph in the perturbation theory is one of them. In this way we obtain a full quantum non-perturbative description for the spectrum.

We also briefly discuss an interesting limit when the cusp becomes a straight line. In this limit the insertion becomes an operator in 1D defect CFT, which has been intensively studied in recent years [123, 131–138].

8.1 Ladders limit and graph building operator

In this section we will describe the Feynman diagrams contributing to the expectation value of the cusped Wilson line. We show that in the ladders limit it gets an iterative Dyson-Schwinger structure, governed by a graph building operator. The graph building operator is a hybrid between that obtained for $J = 0$ in [127, 128] for the cusp without insertion and the one for the fishnet theory [24, 111]. In the rest of the chapter we develop the integrability based method to diagonalise this operator.

8.1.1 Graph building operator

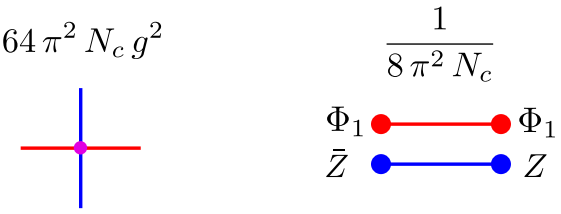


Figure 8.2: We only need a subset of all Feynman diagrams. Above are the conventions for the scalar propagators and the interaction vertex between Φ_1 and $Z = \frac{\Phi_5+i\Phi_6}{\sqrt{2}}$. We use the standard definition $g = \frac{\sqrt{\lambda}}{4\pi}$ with the 't Hooft coupling $\lambda = g_{YM}^2 N$.

The Maldacena-Wilson Loop with J insertions of scalar fields and cusp angle φ is given by:

$$W = \frac{1}{N} tr \mathrm{P} \exp \int_0^\infty dt\, 4\,\pi g \left(i\, A \cdot x'(t) + \Phi_1 |x'(t)| \right) \\ \times Z(0)^J \times \mathrm{P} \exp \int_0^\infty ds\, 4\,\pi g \left[-i\, A \cdot x'(s) + (\Phi_1 \cos\theta + \Phi_2 \sin\theta) |x'(s)| \right] , \quad (8.1.1)$$

where $x'(t) \equiv \frac{\partial x(t)}{\partial t}$ and $x'(s) \equiv \frac{\partial x(s)}{\partial s}$, g is the 't Hooft coupling of planar $\mathcal{N}=4$ SYM, P is the path-ordering operator and the trace is taken over the full expression. The two scalars that couple to the individual Wilson rays form an angle θ between each other. The expectation value of this quantity is divergent in both the IR and the UV, with the divergence controlled by the dimension Δ[2]:

$$\langle W \rangle \sim \left(\frac{R^{IR}}{\epsilon^{UV}} \right)^{-\Delta} , \tag{8.1.2}$$

where Δ corresponds to the overall scaling dimension of W and is also known as the cusp anomalous dimension in the $J=0$ case. The cusp anomalous dimension was studied intensively in perturbation theory and integrability [122, 124, 125, 127, 128, 139–155]. In this chapter we will study a more general observable which is the expectation value of W with J additional insertions (under the trace) of complex scalar fields $\bar{Z} = \frac{1}{\sqrt{2}}(\Phi_5 - i\Phi_6)$ at points y_i which lie outside of the contour, and also truncate the upper limit in Wilson lines at some finite t and s. In analogy with [121] we call this object the CFT wavefunction $\psi(t, s, y_i)$. At first sight this object is not gauge invariant, however in the *ladders limit* it is well defined. In fact, it can be made gauge invariant by closing the Wilson loop by introducing additional segments of non-supersymmetric Wilson lines running through the $\bar{Z}$ insertions, which will decouple in the ladders limit, as in fig. 8.1. As we will see, the role of the effective coupling in the ladders limit is played by

$$\hat{g} \equiv g \left(\frac{\exp\left(-i\theta/2\right)}{2} \right)^{\frac{1}{J+1}} , \tag{8.1.3}$$

which we will assume finite while $g \to 0$ and $\theta \to i\,\infty$. In this limit we will get the following simplifications:

- First of all, since we are taking the 't Hooft coupling to zero, the gluons and fermions decouple, and we are left with a theory of interacting scalar fields. Hence, we can drop out the gauge field A from the definition (8.1.1).

- In a Feynman diagram expansion, only the contributions with the highest power of $\cos\theta$ will survive. For $J=0$, the only diagrams at l-loop order correspond to ladder diagrams, that is, diagrams that contain l scalar propagators beginning on one of the Wilson lines and ending on the other [128].

- For $J>0$, the scalars at the cusp Z can only contract with the external insertions of $\bar{Z}$. This means that only one type of vertex is allowed, i.e. the one in figure 8.2. This is analogous to what one finds in the simplest fishnet CFT. Consequently, only "fishnet" diagrams contribute.

[2]Strictly speaking for $J>0$ it is only divergent for large enough coupling as at tree level we have $\Delta = J$. For $J=0$ it is divergent for any $g>0$.

Using these simplifications, we can define the CFT wavefunction in the ladders limit as:

$$\psi(t,y_1,\ldots,y_J,s) \equiv \frac{1}{N}\Big\langle \operatorname{tr}\prod_{j=1}^{J}\bar{Z}(y_j)\times \mathrm{P}\exp\int_0^t dt'\,(4\pi g)\,|x'(t')|\Phi_1 \\ \times Z(0)^J \times \mathrm{P}\exp\int_0^s ds'\,(4\pi g)\,|x'(s')|\Phi_1\cos\theta\Big\rangle\,. \quad (8.1.4)$$

The CFT wavefunction is obtained expanding the path-ordered exponentials

$$\begin{aligned}&\psi(t,y_1,\ldots,y_J,s) = \\ &\sum_{l=0}^{\infty}\psi_l(t,y_1,\ldots,y_J,s) = \sum_{l=0}^{\infty}\operatorname{tr}\int_0^t dt_l|x'(t_l)|\int_0^{t_l} dt_{l-1}|x'(t_{l-1})|\,\ldots\int_0^{t_2} dt_1|x'(t_1)| \\ &\int_0^s ds_l|x'(s_l)|\int_0^{s_l} ds_{l-1}|x'(s_{l-1})|\,\ldots\int_0^{s_2} ds_1|x'(s_1)|F_l(y_j,t_i,s_i)\,.\end{aligned} \quad (8.1.5)$$

Here, $\psi_l(y_j,t_i,s_i)$ represents the contribution of the l-bridge fishnet Feynman graph, where a bridge is defined as a series of $J+1$ propagators connecting the left and right Wilson rays, as can be seen in figure 8.1. Note that the sum is over the number of "bridges" l. The integrand in $s_i,\ t_i$ is given by:

$$F_l(y_j,t_i,s_i) = \frac{1}{N}\left(\frac{1}{8\,\pi^2\,N}\right)^{l(J+1)+J(l+1)}(64\,\pi^2\,N\,g^2)^{l\,J}(16\,\pi^2\,g^2\cos\theta)^l N^{(l+1)(J+1)+1} \\ \int\left(\prod_{i=1}^{J}\prod_{j=1}^{l}d^4x_{i,j}\right)\left(\prod_{r=0}^{J}\prod_{k=1}^{l}\frac{1}{(x_{r+1,k}-x_{r,k})^2}\right)\left(\prod_{m=1}^{J}\prod_{n=0}^{l}\frac{1}{(x_{m,n+1}-x_{m,n})^2}\right)\,. \quad (8.1.6)$$

Here we have defined $x_{k,0}\equiv y_0\equiv\left(e^{t_k},0,0,0\right)$, $x_{k,J+1}\equiv y_{J+1}\equiv(e^{s_k}\cos\varphi,e^{s_k}\sin\varphi,0,0)$ $\forall k=1\ldots l$, and $x_{l+1,j}\equiv y_j$, $x_{0,j}\equiv 0\ \forall j=1\ldots J$. In the formula (8.1.6), the second factor in the first line of the r.h.s contains the contribution from the propagators, the third the one from the vertices, the fourth comes from the expansion of the path-ordered exponentials, while the fifth represents the contribution from the closed index loops of the planar diagram. In the second line we first integrate over all positions of the vertices, the second term is a collection of all vertical propagators, while the third contains that of the horizontal ones (see figure 8.1 for the case of $J=2$ and $l=4$). Notice that at any loop order these graphs have the same order in N, consistent with the fact that we are using a planar diagram expansion. Instead of computing this integral we notice that we can define it recursively in terms of the inverse of a graph building operator as we illustrate below. First, notice that $\Box_{y_i}$ acts on scalar propagators as:

$$\Box_{y_j}\frac{1}{(y_j-x_{j,l})^2} = -4\pi^2\,\delta(y_j-x_{j,l})\,. \quad (8.1.7)$$

Moreover, acting with $\partial_t\partial_s$ on the contour of a Wilson line brings down the expansion of the path ordered exponential by one step, at the cost of a factor $|y_0'(t)||y_{J+1}'(s)|$. Therefore acting on ψ with a string of $\square_{y_j}$, followed by $\partial_t\partial_s$, we get back ψ expanded to one less bridge, up to a multiplicative factor:

$$\partial_t\partial_s \prod_{j=1}^{J} \square_{y_j}\psi_l = (-1)^J(4\hat{g}^2)^{J+1}\frac{|y_0'||y_{J+1}'|}{\prod_{i=0}^{J}(y_i-y_{i+1})^2}\psi_{l-1} \ , \tag{8.1.8}$$

where we use the definition of $\hat{g}$ from (8.1.3). From this we can extract an operator annihilating the CFT wavefunction:

$$(\hat{B}^{-1}-1)\psi = 0 \ \ , \ \ \hat{B}^{-1} \equiv \frac{(-1)^J}{(4\hat{g}^2)^{J+1}}\frac{\prod_{i=0}^{J}(y_i-y_{i+1})^2}{|y_0'||y_{J+1}'|}\,\partial_t\partial_s \prod_{j=1}^{J}\square_{y_j} \ . \tag{8.1.9}$$

We refer to $\hat{B}^{-1}$ as an inverted graph-building operator. The role of $\hat{B}^{-1}-1$ was realised in [129] to be the analogue of the world-sheet Hamiltonian of a string theory [3]. We will explore this further in the next section.

The Wilson loop with insertion W is invariant under dilatations, which stretches the space around the origin (which we take to be the position of the cusp). Thus we can use the following dilatation operator, acting on the CFT wavefunction

$$\hat{D} = -i\left(\partial_t+\partial_s+\sum_{i=1}^{J}(y_i\cdot\partial_{y_i}+1)\right) \ , \tag{8.1.10}$$

to measure the dimension Δ of the initial cusped Wilson line. More precisely, the eigenvalue of $\hat{D}$ is $i\,\Delta$. This operator commutes with $\hat{B}$ as it is easy to see. Another operator which commutes with $\hat{B}$ is the generator of rotations in the orthogonal plane to the Wilson line:

$$\hat{S} = i\sum_{i=1}^{J}\left(y_i^3\,\partial_{y_i^4}-y_i^4\,\partial_{y_i^3}\right) \ . \tag{8.1.11}$$

This operator measures the spin of W. For Z^J scalar insertions $S=0$, but one can also study more general insertions with derivatives in the orthogonal plane, corresponding to $S\neq 0$, which are also described by our construction.

In analogy with the fishnet [130] one should diagonalise both $\hat{S}$ and $\hat{D}$. After doing so, the equation $(\hat{B}^{-1}-1)\,\psi=0$ should restrict us to the discrete spectrum of eigenvalues of the dilatation operator, which would give us all the anomalous dimensions of the operators with given quantum numbers. Indeed we will find that there are infinitely many (but a discrete set) of such ψ's diagonalising all the 3 operators. In analogy with [25, 123] we expect each

[3] It's important to notice that the graph-building operator is diagonalisable and its spectrum contains only operators with non-trivial anomalous dimension. All operators which form Jordan blocks, like those studied in [156], seem to be absent from its spectrum.

of them to correspond to a particular insertion of operators, which could include derivatives and extra Φ_1, Φ_2 fields in addition to Z^J, whose number is fixed by the R-charge. These type of insertions at the cusp will not modify the iterative structure of the diagrams, instead just adding a finite number of propagators close to the cusp (cf. figure 8.3). Therefore, all these states should be governed by the same equation (8.1.8).

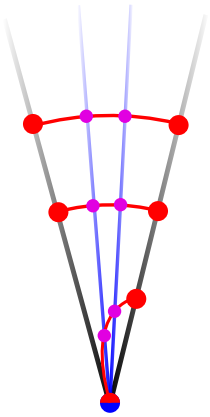


Figure 8.3: An example of an "excited state" for $J = 2$. Here, propagators from the extra insertion of Φ_1 at the cusp contract with the Wilson line without crossing any other propagators of Φ_1 (shown in red), as such diagrams would be subleading in the ladders limit.

In the next sections we will explore how the integrability arises explicitly in this system. In particular, we will show first in the classical (strong coupling) limit and then in general that the operators $\hat{B}, \hat{D}$ and $\hat{S}$ are part of a bigger commuting family of operators.

8.2 Classical open fishchain

In this section, following [129], we interpret the inverse of the graph-building operator as a Hamiltonian of a quantum system of particles. Then we take the quasi-classical strong coupling limit of the system, deriving the classical fishchain with specific open boundary conditions. We analyse in detail the classical system and find some of the solutions of the equations of motion.

8.2.1 Strong coupling limit

The starting point for the strong coupling $\hat{g} \to \infty$ analysis is equation (8.1.8). By re-writing (8.1.9) in terms of the conjugate momenta:

$$p_i = -i\partial_{y_i}, \qquad \pi_t = -i\partial_t, \qquad \pi_s = -i\partial_s, \tag{8.2.1}$$

we obtain the Hamiltonian $\hat{H}$, governing a system with $4J+2$ degrees of freedom, given by:

$$\hat{H} = \pi_t\, \pi_s \prod_{i=1}^{J} p_i^2 + (4\hat{g}^2)^{J+1} \frac{|y_0'(t)||y_{J+1}'(s)|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}\,. \tag{8.2.2}$$

In this section we will be treating this Hamiltonian as the one of a classical system. In analogy with [129] we will see that the classical limit corresponds to the strong coupling $\hat{g} \to \infty$ limit of the original quantum system (8.1.9). We will now demonstrate the classical integrability of this system and then describe its quantisation in section 8.4.

We remark that y_i, $i = 1 \dots J$ are 4D vectors with four bulk degrees of freedom for each one, while y_0 and y_{J+1} are 4D vectors having one boundary degree of freedom each. Therefore, without loss of generality, we parametrise the latter as:

$$y_0(t) = \left(e^t,\, 0,\, 0,\, 0\right)\,, \qquad y_{J+1}(s) = \left(e^s \cos\varphi,\, e^s \sin\varphi,\, 0,\, 0\right)\,. \tag{8.2.3}$$

We will find it beneficial to embed the system in $6D$ space, which will allow to make the conformal symmetry of the system manifest, but will also result in a local action with nearest neighbour interaction.

In the rest of this section we will deduce the classical equations of motion of this system, using the Lagrangian formalism. First, by performing a Legendre transformation on (8.2.2), we find the Lagrangian to be:

$$L = 2^{\frac{-2J}{2J+1}}(2J+1)\left(\dot{t}\,\dot{s}\prod_{i=1}^{J}\dot{y}_i^2\right)^{\frac{1}{2J+1}} - (4\hat{g}^2)^{J+1}\,\frac{|y_0'(t)||y_{J+1}'(s)|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}\,, \tag{8.2.4}$$

where $\dot{f} \equiv \frac{d}{d\tau} f$, with τ being a "world-sheet" time variable (conjugate to the Hamiltonian (8.2.2)). The action $S = \int L\, d\tau$ is not invariant under time reparametrisation symmetry $\tau \to f(\tau)$, which is needed to ensure $\hat{H}\psi = 0$. In order to enforce this symmetry we introduce an auxiliary field γ transforming as $\gamma \to \dfrac{\gamma}{\dot{f}}$ when $\tau \to f(\tau)$ which gives

$$L = 2^{\frac{-2J}{2J+1}}(2J+1)\left(\frac{1}{\gamma}\dot{t}\,\dot{s}\prod_{i=1}^{J}\dot{y}_i^2\right)^{\frac{1}{2J+1}} - \gamma\,(4\hat{g}^2)^{J+1}\,\frac{|y_0'(t)||y_{J+1}'(s)|}{\prod_{i=0}^{J}(y_i - y_{i+1})^2}\,. \tag{8.2.5}$$

This is now time-reparametrisation invariant. We then eliminate the auxiliary field setting it to its extremum (by a suitable time reparametrisation). We have to remember that y_0 (and y_{J+1}) is not itself a canonical coordinate, but depends on the world-sheet time through $t(\tau)$ (and $s(\tau)$ respectively). Thus we can use $\dot{y}_0 = y_0'\dot{t}$ and similarly $\dot{y}_{J+1} = y_{J+1}'\dot{s}$. After that we get:

$$L = (2J+2)(2i)^{\frac{1}{J+1}}\,\hat{g}\left[\frac{|\dot{y}_0||\dot{y}_{J+1}|\prod_{i=1}^{J}\dot{x}_i^2}{\prod_{i=0}^{J}|y_i - y_{i+1}|^2}\right]^{\frac{1}{2(J+1)}}\,. \tag{8.2.6}$$

We now embed the system in $6D$ Minkowksi spacetime, using lightcone coordinates in the Poincare' slice:

$$y_i^\mu = \frac{X_i^\mu}{X_i^+}, \qquad X_i^2 = 0, \qquad X_i^+ = X_i^0 + X_i^{-1}\,. \tag{8.2.7}$$

Hence we get:

$$L = (2J+2)(2i)^{\frac{1}{J+1}}\,\hat{g}\left[\frac{|\dot{X}_0||\dot{X}_{J+1}|\prod_{i=1}^J \dot{X}_i^2}{\prod_{i=0}^J(-2X_i.X_{i+1})}\right]^{\frac{1}{2(J+1)}}\,. \tag{8.2.8}$$

Furthermore, we can disentangle this action to bring it to a Polyakov-like form, by introducing auxiliary fields α_i, getting:

$$L = \xi\left(\alpha_0\frac{|\dot{X}_0||\dot{X}_{J+1}|}{2} + \sum_{i=1}^J\left(\alpha_i\frac{\dot{X}_i^2}{2} + \eta_i X_i^2\right) + (J+1)\prod_{k=0}^J(-\alpha_k X_k.X_{k+1})^{-\frac{1}{J+1}}\right)\,, \tag{8.2.9}$$

where

$$\boxed{\xi \equiv (2i)^{\frac{1}{J+1}}\hat{g}}\,. \tag{8.2.10}$$

In (8.2.9) we also introduced the light-cone constraint $X_i^2 = 0$ via the Lagrange multiplier η_i. In order to get back the Nambu-Goto-like form (8.2.8), we have to extremise the fields α_i and plug these values back into (8.2.9). It is possible to do this due to the new re-scaling symmetry of X_i. More precisely, the Lagrangian (8.2.9) has $J+3$ gauge symmetries: time-dependent rescaling $X_i \to g_i(\tau)X_i$, $\alpha_i \to \alpha_i g_i^{-1/2}(\tau)$, $\eta_i \to \eta_i g_i^{-1/2}(\tau)$, $i = 0\dots J+1$ and time reparametrisation $\tau \to f(\tau)$, under which fields transform as $X_i \to \frac{X_i}{f}$, $\alpha_i \to \dot{f}\alpha_i$, $\eta_i \to \frac{\eta_i}{\dot{f}}$. Instead of setting α_i's to their extreme values we can use the symmetries to impose $\alpha_i = 1$, $\forall i = 0,\dots,J$. This would lead to the following constraints (the same way as one gets Virasoro constraints):

$$\dot{X}_k^2 = \mathcal{L}\,, \tag{8.2.11}$$

where

$$\mathcal{L} \equiv 2\prod_{i=0}^J(-X_i\cdot X_{i+1})^{-\frac{1}{J+1}}\,, \tag{8.2.12}$$

with $k = 1,\dots,J$ in the first equation. Furthermore, from the equation of motion for α_0 we get $|\dot{X}_0||\dot{X}_{J+1}| = \mathcal{L}$: this still leaves us with the freedom to rescale $X_0 \to h(\tau)X_0$ and simultaneously $X_{J+1} \to \frac{1}{h(\tau)}X_{J+1}$, which we can fix by imposing additionally $|\dot{X}_0| = |\dot{X}_{J+1}|$. Hence, we can just extend the range of k in (8.2.11) to $k = 0,\dots,J+1$. Finally, to fix the remaining time-reparametrisation gauge freedom we can set:

$$\mathcal{L} = 1\,, \tag{8.2.13}$$

which is a convenient gauge to work with. We have imposed $J+3$ conditions, so all gauge degrees of freedom are fixed. The gauge fixed Lagrangian is then:

$$L = \xi\left(\frac{|\dot{X}_0||\dot{X}_{J+1}|}{2} + \sum_{i=1}^J\frac{\dot{X}_i^2}{2} + (J+1)\prod_{k=0}^J(-X_k.X_{k+1})^{-\frac{1}{J+1}}\right)\,, \tag{8.2.14}$$

Finally, by noticing that $2|\dot{X}_0||\dot{X}_{J+1}| = \dot{X}_0^2 + \dot{X}_{J+1}^2 - (|\dot{X}_0| - |\dot{X}_{J+1}|)^2$ we can replace $|\dot{X}_0||\dot{X}_{J+1}| \to \frac{\dot{X}_0^2}{2} + \frac{\dot{X}_{J+1}^2}{2}$ in (8.2.14), modulo terms quadratic in constraints. Similarly, defining $y = 2\prod_{i=0}^{J}(-X_i \cdot X_{i+1})^{-\frac{1}{J+1}} \simeq 1$ on constraints, we have $y = e^{\log y} = 1 + \log y + \mathcal{O}(\log^2 y)$, which allows us to replace the potential term by $\sum_{k=0}^{J} \frac{1}{2} \log \frac{-X_k.X_{k+1}}{2e}$. Therefore we get the gauge fixed Lagrangian:

$$L = \xi \left(\frac{\dot{X}_0^2}{4} + \sum_{i=1}^{J} \frac{\dot{X}_i^2}{2} + \frac{\dot{X}_{J+1}^2}{4} - \sum_{i=0}^{J} \frac{1}{2} \log \frac{-X_i.X_{i+1}}{2e} \right) , \tag{8.2.15}$$

with constraints given by:

$$\prod_{i=0}^{J} \frac{-X_i.X_{i+1}}{2} = 1 , \tag{8.2.16}$$

$$X_i^2 = 0 \ , \ \dot{X}_i^2 = 1 \ , \ i = 0, \ldots, J+1 . \tag{8.2.17}$$

Note that on the constraints we also have $L \simeq \xi(J+1)$. In this form the Lagrangian is explicitly local and the interaction is only between the nearest neighbours. It may appear a bit strange that the boundary particles have mass 1/2 w.r.t. to the particles in the bulk, however we will see in the next section that in this way the equations of motion are more uniform. The reason is that the bulk particles have to be split in two and reflected, unlike those at the boundaries. In (8.2.14) the $6D$ variables X_i, $i = 1, \ldots, J$ are independent canonical coordinates, constrained by (8.2.11) and (8.2.13). At the same time the boundary particles X_0 and X_{J+1} are encoded in terms of one variable each $t(\tau)$ and $s(\tau)$, due to (8.2.3). Explicitly:

$$X_i(\tau) = r_i(\tau)\,(\cosh w_i(\tau), -\sinh w_i(\tau), \cos\phi_i, \sin\phi_i, 0, 0) \quad i = 0, J+1, \tag{8.2.18}$$

where $\phi_0 = 0$, $\phi_{J+1} = \varphi$, $w_0(\tau) = t(\tau)$ and $w_{J+1}(\tau) = s(\tau)$. On the constraint $\dot{X}_i^2 = 1$ we also have $r_i(t) = \frac{1}{\dot{w}_i(\tau)}$. Apart from this, the Lagrangian (8.2.14) is very similar to the one found in the classical limit of the fishnet graphs in [129]. It can be interpreted as the one of a discretised string with string-bits having a nearest neighbour interaction. However, due to the difference in the boundary DOFs it still remains to be seen whether the system is classically integrable, as it was in the original case [129].

8.2.2 Equations of motion

We now compute the Euler-Lagrange equations starting from (8.2.15). The J equations for bulk variables are interpreted as equations of motion for J bulk particles, while the equations for X_0 and X_{J+1} are interpreted as equations of motion for two particles constrained on the two Wilson lines. Since the Lagrangian (8.2.15) has nearest neighbour interactions, only the first and last bulk particles in the spin chain will feel the presence of the particles on the Wilson lines. For example, for the bulk particle j we have [129]:

$$\ddot{X}_j = 2\eta_j X_j - \frac{1}{2} \left(\frac{X_{j+1}}{X_{j+1}.X_j} + \frac{X_{j-1}}{X_j.X_{j-1}} \right) , \quad j = 1, \ldots, J \tag{8.2.19}$$

For the particles on the Wilson lines, we only have one physical degree of freedom for each, $t(\tau)$ and $s(\tau)$. The relative equations of motion are given by:

$$\frac{\ddot{t}}{\dot{t}^2} = \frac{X_1.\partial_{t(\tau)}X_0}{X_1.X_0} \ , \ \frac{\ddot{s}}{\dot{s}^2} = \frac{X_J.\partial_{s(\tau)}X_{J+1}}{X_J.X_{J+1}} \ . \tag{8.2.20}$$

These two equations can be written in the form (8.2.19) by introducing the reflected particles X_{-1} and X_{J+2} as the reflection of the particles X_1 and X_J w.r.t. the ray parametrised by t and s respectively. More precisely we introduce the reflection matrix and rotation matrices:

$$C^M{}_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{MN} \ , \ G^M{}_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi & 0 & 0 \\ 0 & 0 & \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{MN} \ . \tag{8.2.21}$$

Then we define the images of the particles 1 and J by the reflection about the ray parametrised by t and s respectively as $X_{-1} = C.X_1$ and $X_{J+2} = G.C.G^{-1}.X_J = G^2.C.X_J$. With these definitions the equations (8.2.20) coincide with (8.2.19) for $j = 0$ and $j = J+1$ correspondingly.

Thus, we conclude that at the level of the classical equations of motion the open version of the fishchain we consider here is identical to the double-size closed fishchain of [129], with length $2J + 2$ and quasi-periodic boundary condition twisted by a 2φ rotation (see figure 8.4). However, there are some important differences in the Poisson structure and consequently the quantisation is different.

8.2.3 Conserved charges

The presence of boundaries in the open fishchain breaks the $SO(1,5)$ symmetry that its closed counterpart enjoyed to the subgroup $SO(2) \times SO(1,1)$. Nevertheless it is useful to define

$$q_j^{MN} \equiv \dot{X}_j^M X_j^N - \dot{X}_j^N X_j^N = 2\dot{X}_j^{[M} X_j^{N]} \ , \tag{8.2.22}$$

for $j = 0, \dots J+1$, which are the local $SO(1,5)$ generators for $j = 1, \dots, J$. We also define the total charge

$$Q^{MN} = \xi \left(\frac{q_0^{MN}}{2} + \sum_{i=0}^{J} q_j^{MN} + \frac{q_{J+1}^{MN}}{2} \right) \ . \tag{8.2.23}$$

As the $SO(1,5)$ symmetry is broken, only the components of Q^{MN} corresponding to the unbroken symmetry subgroup will remain conserved in time. Thus we only have two Noether charges, corresponding to the $SO(2)$ angular momentum and to the scaling dimension:

$$S = Q_{3,4} \ , \quad D = Q_{-1,0} = i\,\Delta \ . \tag{8.2.24}$$

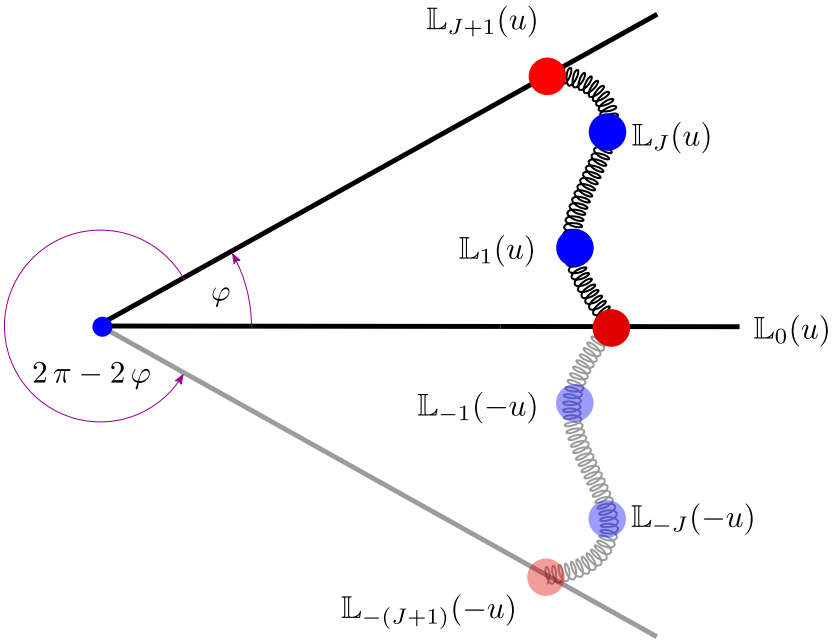


Figure 8.4: Schematic representation of the open Fishchain

8.2.4 Example of solution of the classical equations of motion

Now we proceed to the numerical solution of the system (8.2.19). To do so, we introduce the following parametrisation for the bulk particles, which is similar to the one used for the boundary particles (8.2.18):

$$X_a(\tau) = \frac{1}{\sqrt{\dot{w}_a^2(\tau) + \dot{\phi}_a^2(\tau)}} \Big(\cosh w_a(\tau), \quad -\sinh w_a(\tau), \quad \cos\phi_a(\tau), \quad \sin\phi_a(\tau), \quad 0, \quad 0 \Big) \ . \tag{8.2.25}$$

where $a = 1, \ldots, J$. This resolves the $X^2 = 0$ and $\dot{X}^2 = 1$ constraints. For the ansatz (8.2.25) the particles are all in the same plane. The boundary particles are constrained to

move on the Wilson rays, so their angular position is fixed

$$\phi_0(\tau) = 0 \,, \tag{8.2.26}$$

$$\phi_{J+1}(\tau) = \varphi \,. \tag{8.2.27}$$

We can imagine the simplest solution would be when these particles move along straight lines. For that we need to compensate the interaction with neighbours, which could otherwise bend the trajectory, so we require

$$\phi_k(\tau) = \frac{k}{J+1}\varphi \,. \tag{8.2.28}$$

To simplify our ansatz further we can assume that $w_k(\tau) = W(\tau)$. Plugging this ansatz into the EOMs (8.2.19) we obtain

$$w_k(\tau) = \beta\,\tau \,. \tag{8.2.29}$$

Finally, constraint (8.2.16) gives

$$\left(\frac{\sin\left(\frac{\varphi}{2J+2}\right)}{\beta}\right)^{2J+2} = 1 \,, \tag{8.2.30}$$

which has $2J+2$ different solutions

$$\beta = e^{2\pi i \frac{n}{2J+2}} \sin\left(\frac{\varphi}{2\,J+2}\right) \quad , \quad n = 1 \ldots 2J+2 \,. \tag{8.2.31}$$

To get an interpretation of this, we also compute the anomalous dimension, using (8.2.24)

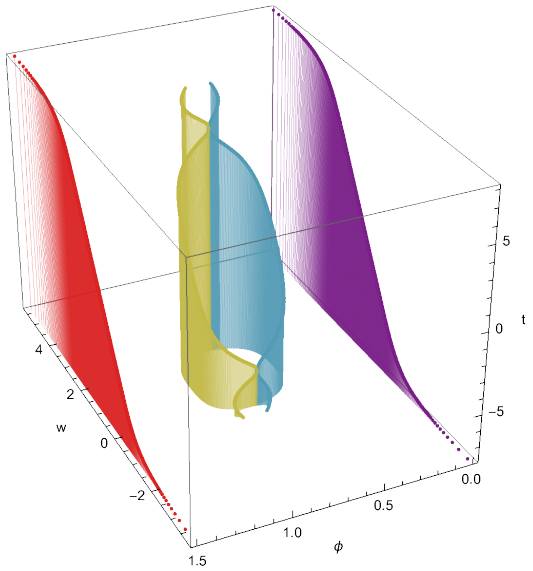
$$\Delta = -\frac{(J+1)\,i}{\beta}\xi \,. \tag{8.2.32}$$

We see that the ambiguous factor can be absorbed into ξ. In fact, the initial graph building operator only depends on ξ^{2J+2}, thus this type of ambiguity is expected. In fact this is the same as in the case of the closed fishchain [129], where the solutions were found to multiply in a similar way and were responsible for the different asymptotics of a 4 point correlator. We can check our classical solution by comparing with some known results for $J = 0$ case. From (8.2.32) for $J = 0$ we obtain:

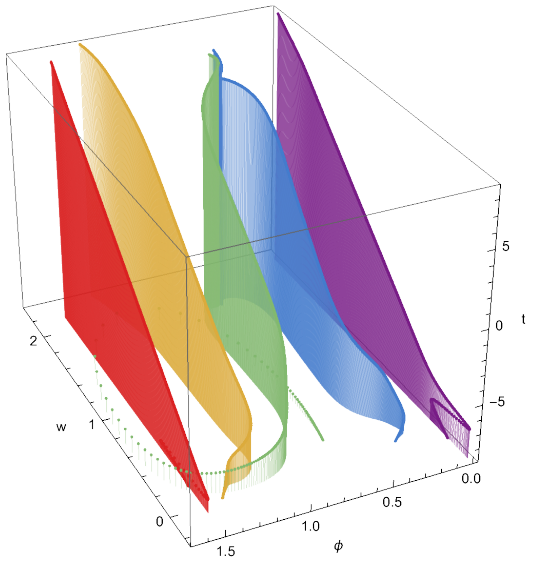
$$\Delta = \pm\frac{2\hat{g}}{\sin\frac{\varphi}{2}} \tag{8.2.33}$$

which agrees perfectly with the equation (E.6) in [25]. We note that for $\hat{g} > 0$ only the minus sign solution appears in the spectrum whereas the plus sign solution corresponds to large and negative $\hat{g}$.

More general solutions can be obtained numerically. We generated a couple of solutions, obtained by perturbing the analytic solution we just presented. These can be found in figure 8.5a and figure 8.5b.



(a) $J = 2$ — Notice that the boundary particles run away to infinity in a finite amount of time, while the particles in the bulk have only moved a finite amount.



(b) $J = 3$ — Notice that one of the bulk particles proceeds to make a complete loop.

Figure 8.5: Plot of the motion of particles obtained by a numerical solution of the classical equations of motion. In these solutions, motion is restricted to the plane of the Wilson loop. As expected, the boundary particles are confined to fixed rays whereas the bulk particles are free to move anywhere in the plane.

8.3 Classical integrability

In this section we prove that the dual model is integrable at the classical level by studying its Poisson structure. We will construct the Lax matrices, corresponding to the particles in the bulk, and the dynamical reflection matrix will represent the boundary particles. Using these building blocks we will construct a family of mutually Poisson-commuting objects.

The main purpose of this section is to establish the grounds for quantisation. For this reason we will only build here a subset of all commuting integrals of motion, as they will anyway appear in the quantum case in full generality.

8.3.1 Poisson brackets

In this section we discuss the Poisson structure following from the Lagrangian (8.2.15). For the bulk DOFs the Poisson structure is identical to the closed fishchain case already studied in [130]. One can find the conjugate momenta and define the Poisson bracket in the standard way. In particular, for the bulk particles the momentum conjugate to $X_{i,M}$ is

$$P_i^M = \xi \dot{X}_i^M \,, \tag{8.3.1}$$

and the Poisson bracket is defined as $\{X_{i,M}, P_j^N\} = \delta_{ij}\delta_M^N$. Due to the constraints the Poisson brackets is ambiguous, and we could define a Dirac bracket. Alternatively, one can work with gauge invariant quantities. The gauge invariant combination of phase space coordinates for the bulk particles are the local symmetry generators

$$q_i^{MN} = \frac{1}{\xi}\left(X_i^N P_i^M - X_i^M P_i^N\right) = X_i^N \dot{X}_i^M - X_i^M \dot{X}_i^N \ , \tag{8.3.2}$$

which form the $SO(1,5)$ algebra under the Poisson bracket:

$$\{q_k^{MN}, q_k^{KL}\} = \frac{1}{\xi}\left(-\eta^{MK} q_k^{NL} + \eta^{NK} q_k^{ML} + \eta^{ML} q_k^{NK} - \eta^{NL} q_k^{MK}\right) \ , \quad k = 1, \ldots, J \, . \tag{8.3.3}$$

Similarly, one can proceed with the boundary degrees of freedom t and s. The canonically conjugate momenta to $t(\tau)$ and $s(\tau)$ are

$$\Pi_t = \frac{\xi}{2t'(\tau)} \ , \ \Pi_s = \frac{\xi}{2s'(\tau)} \ . \tag{8.3.4}$$

Even though the boundaries explicitly break down the $SO(1,5)$ symmetry, it is still useful to define q_0^{MN} and q_{J+1}^{MN} in a similar to (8.3.2) way

$$q_0^{NM} = \frac{2}{\xi}\left(Y_0^M Y_0'^N - Y_0^N Y_0'^M\right)\Pi_t \, , \ \ q_{J+1}^{NM} = \frac{2}{\xi}\left(Y_{J+1}^M Y_{J+1}'^N - Y_{J+1}^N Y_{J+1}'^M\right)\Pi_s \, , \tag{8.3.5}$$

where

$$Y_0 = \{\cosh t, -\sinh t, 1, 0, 0, 0\} \ , \ \ Y_{J+1} = \{\cosh s, -\sinh s, \cos\varphi, \sin\varphi, 0, 0\} \, . \tag{8.3.6}$$

Since the Wilson lines explicitly break conformal symmetry, the Poisson bracket of q_0 is modified to

$$\{q_0^{MN}, q_0^{KL}\} = \frac{1}{\xi}\left(-\tilde{\eta}^{MK} q_0^{NL} + \tilde{\eta}^{NK} q_0^{ML} + \tilde{\eta}^{ML} q_0^{NK} - \tilde{\eta}^{NL} q_0^{MK}\right) \ , \ \ \tilde{\eta} = \eta\,(\mathbb{1} + C) \ , \tag{8.3.7}$$

where C is the reflection matrix defined in (8.2.21). Similarly for the right boundary we get

$$\{q_{J+1}^{MN}, q_{J+1}^{KL}\} = \frac{1}{\xi}\left(-\tilde{\eta}_\phi^{MK} q_{J+1}^{NL} + \tilde{\eta}_\phi^{NK} q_{J+1}^{ML} + \tilde{\eta}_\phi^{ML} q_{J+1}^{NK} - \tilde{\eta}_\phi^{NL} q_{J+1}^{MK}\right) \ , \ \ \tilde{\eta}_\phi = \eta\,(\mathbb{1} + G.G.C) \ , \tag{8.3.8}$$

where G is the rotation matrix defined in (8.2.21).

Finally, let us write the Hamiltonian H, corresponding to the lagrangian (8.2.15) in terms of the local symmetry generators q_i. For this we introduce

$$H_q \equiv \frac{1}{2^{2J+2}}\mathrm{tr}\left(q_0^2.q_1^2.\ldots.q_J^2.q_{J+1}^2.G.G.C.q_J^2.\ldots.q_1^2.C\right) - 1 \, . \tag{8.3.9}$$

Then we find that H_q is proportional to the Hamiltonian H up to a constant multiplier and up to second order in constraints

$$H_q = \exp\left(\tfrac{4}{\xi}H\right) - 1 \simeq \tfrac{4}{\xi}H + \mathcal{O}(H^2) \, . \tag{8.3.10}$$

As our constraint implies $H=0$ we can equivalently use $\frac{4}{\xi}H_q$ instead. The advantage of H_q over H is that it is written explicitly in terms of q_i's. At explained in [130] in the case of q_i's there is no difference between the Poisson and Dirac brackets and so they are more convenient for the quantization.

Next, we will build the Lax representation based on the Poisson structure explained here and develop the integrability construction.

8.3.2 Lax representation

In this section we will build the the Lax representation for the equations of motion of the open Fishchain. It is useful to introduce the local current $j_i^{MN}=-2\dfrac{X_{i-1}^{[M}X_i^{N]}}{X_{i-1}.X_i}$, satisfying

$$\dot{q}_i^{MN}=\{q_i^{MN},H\}=-\frac{1}{2}\left(j_{i+1}^{MN}-j_i^{MN}\right)\ \ ,\ \ i=0,\dots,J+1\ . \tag{8.3.11}$$

This allows us to define the Lax pair of matrices $\mathbb{L}_i$ and $\mathbb{V}_i$:

$$\mathbb{L}_i=u\,1_{4x4}+\frac{i}{2}q_i^{MN}\Sigma_{MN}\ \ ,\ \ \mathbb{V}_i=-\frac{i}{4u}\,j_i^{MN}\Sigma_{MN}\,, \tag{8.3.12}$$

where Σ_{MN} are the $6D$ σ matrices, giving a $4D$ representation of $SO(1,5)$. The explicit form we are using can be found in [130]. Both $\mathbb{L}_i$ and $\mathbb{V}_i$ are complex 4×4 matrices with entries that are functions on the phase space of the i-th site of the spin chain. One can show [129] from (8.2.19) that $\mathbb{L}_i$ and $\mathbb{V}_i$ satisfy the flatness condition

$$\dot{\mathbb{L}}_i=\mathbb{L}_i.\mathbb{V}_{i+1}-\mathbb{V}_i.\mathbb{L}_i=\mathbb{V}_{i+1}.\mathbb{L}_i-\mathbb{L}_i.\mathbb{V}_i\,. \tag{8.3.13}$$

From this we have that the open transfer matrix:

$$\mathbb{T}(u)=\mathrm{tr}\mathbb{L}_J^t(-u).\mathbb{L}_1^t(-u).\mathbb{L}_0(u).\mathbb{L}_1(u)\cdots\mathbb{L}_J(u).\mathbb{L}_{J+1}(u).G^{\mathbf{4}}.G^{\mathbf{4}}\ , \tag{8.3.14}$$

is conserved in time for any value of u, i.e. $\{\mathbb{T}(u),H\}=0$. In the above expression we have defined the twist matrix:

$$G^{\mathbf{4}\,a}{}_f=\begin{pmatrix}e^{i\frac{\varphi}{2}} & 0 & 0 & 0\\ 0 & e^{-i\frac{\varphi}{2}} & 0 & 0\\ 0 & 0 & e^{-i\frac{\varphi}{2}} & 0\\ 0 & 0 & 0 & e^{i\frac{\varphi}{2}}\end{pmatrix}^a_f\ . \tag{8.3.15}$$

To prove that the model is classically integrable, we also need to show that the integrals of motion are in convolution, i.e. that $\{\mathbb{T}(u),\mathbb{T}(v)\}=0$. This is not automatic, since the Lax pair construction we have reviewed in section 1 does not apply to (classical) open spin chain.

In order to prove the convolution property of integrals of motion one can use (8.3.3) and (8.3.12) to show that:

$$\xi \{\mathbb{L}_n(u), \mathbb{L}_m(v)\} = [r(u,v), \mathbb{L}_n(u) \otimes \mathbb{L}_m(v)]\delta_{nm} \,, \tag{8.3.16}$$

where r is the classical r-matrix.

For the boundary matrices we have a different relation due to the modifications in the Poisson brackets (8.3.7) and (8.3.8). Denoting

$$\mathbb{K}(u) \equiv C.\mathbb{L}_0(u) \;\;,\;\; \bar{\mathbb{K}}(u) \equiv G^{-1}.\mathbb{L}_{J+1}(u).G.C \,, \tag{8.3.17}$$

we have:

$$\xi \{\mathbb{K}_{ab}(u), \mathbb{K}_{cd}(v)\} = \frac{\mathbb{K}_{ad}(u)\mathbb{K}_{cb}(v) - \mathbb{K}_{ad}(v)\mathbb{K}_{cb}(u)}{u-v} - \frac{\mathbb{K}_{db}(u)\mathbb{K}_{ca}(v) - \mathbb{K}_{bd}(v)\mathbb{K}_{ac}(u)}{u+v} \,. \tag{8.3.18}$$

and the same for $\bar{\mathbb{K}}$. In Appendix F.1 we use these identities to show that indeed

$$\{\mathbb{T}(u), \mathbb{T}(v)\} = 0 \,. \tag{8.3.19}$$

Therefore, the open Fishchain is classically integrable - its equation of motion are equivalent to a Lax pair equation, and from it we can construct mutually commuting transfer matrices.

8.4 Quantum integrability

In order to demonstrate the integrability at the quantum level we will have to embed the graph building operator into a family of commuting operators. To first approximation, one can replace the local $SO(1,5)$ generators q_i by the operators $\hat{q}_i$. However, there are some quantum corrections to work out due to non-commutativity of various components of $\hat{q}_i^{MN}$, and this is what we will do in this section.

We will define the Lax operator $\hat{\mathbb{L}}$ and the boundary matrices $\hat{\mathbb{K}}$ as quantum versions of the classical ones. They will continue to be 4×4 matrices, but now each component will become a differential operator. Thus we will treat them as tensors acting on a tensor product of a finite, 4-dimensional vector space and an infinite dimensional functional space. We will refer to these spaces as auxiliary and physical spaces as usual. In particular, the physical space for the bulk $\mathbb{L}_i$ matrices will be made of functions of 6D projective space variables X_i^μ, while the physical space for the boundary $\mathbb{K}$ and $\bar{\mathbb{K}}$ matrices will be made of functions of the boundary degrees of freedom t and s.

8.4.1 Quantisation of the integrability relations

We need to build the quantum analogue of (8.3.16), which is the Yang-Baxter equation, and of (8.3.18), which is given by the boundary Yang-Baxter equation.

Quantum Lax matrix The quantum version of the Lax matrix is[4]

$$\hat{\mathbb{L}}_i{}^a{}_b(u) = u\,\delta^a_b + \frac{i}{2}\hat{q}_i^{MN}\Sigma_{MN}{}^a{}_b\,, \tag{8.4.1}$$

where $\hat{q}_i^{MN}$ is the local generator of $SO(1,5)$, obtained as a quantisation of (8.3.2), i.e. by replacing $P_j^K \to \hat{P}_j^K = -i\partial_{X_{j,K}}$:

$$\hat{q}_j^{MN} = -\frac{i}{\xi}\left(X_j^N\frac{\partial}{\partial X_{j,M}} - X_j^M\frac{\partial}{\partial X_{j,N}}\right)\,. \tag{8.4.2}$$

It satisfies the $SO(1,5)$ commutation relation:

$$\left[\hat{q}_k^{MN},\hat{q}_k^{KL}\right] = \frac{i}{\xi}\left(-\eta^{MK}\hat{q}_k^{NL} + \eta^{NK}\hat{q}_k^{ML} + \eta^{ML}\hat{q}_k^{NK} - \eta^{NL}\hat{q}_k^{MK}\right)\quad,\quad k=1,\dots,J\,. \tag{8.4.3}$$

As explained in [130] $\hat{q}_i$ can be understood as acting on the functions of 4-dimensional variables y_i (e.g. CFT wave function) as if it was the corresponding conformal generator in $4D$. In other words one can use the following map between the functions of $4D$ coordinates y_m and functions of $6D$ coordinates X^M

$$f(y_1,\dots,y_m) \to \frac{1}{X^{-1}+X^0}f\left(\frac{X^1}{X^{-1}+X^0},\dots,\frac{X^4}{X^{-1}+X^0}\right) \tag{8.4.4}$$

as q_i preserves the 6 interval $X^M X_M$ we can set it to zero consistently. Note the action on the $4D$ is only well defined for observables build out of q_i's. In particular $\hat{P}_j$ and $\hat{X}_j$ themselves are operators living in AdS_5 [130].

The Lax operator (8.4.1) satisfies the usual RTT relations with the rational R-matrix if we set the quantum parameter $\hbar$ to $\frac{i}{\xi}$. This Lax operator is built on a representation of gl_4 that is non-compact and non-highest weight, as evident from the fact that the generators q_i^{MN} are differential operators acting on an infinite-dimensional space. This implies that the Bethe Ansatz approaches are not available for this spin chain, and the Q-functions cannot be built in terms of Bethe roots and Baxter polynomials. However, the FSoV techniques discussed in chapter 6 would still be available.

8.4.2 Boundary reflection operator

In the classical case at the boundary we found that q_0 and q_{J+1} satisfied the modified Poisson brackets (8.3.7). This in turn results in $\mathbb{K}$ satisfying the boundary Yang-Baxter equation.

The quantum version of q_0 and q_{J+1} are again obtained by replacing $\Pi_t \to -i\partial_t$ and $\Pi_s \to -i\partial_s$, and read:

$$\hat{q}_0^{NM} \equiv -i\frac{2}{\xi}(Y_0^M\dot{Y}_0^N - Y_0^N\dot{Y}_0^M)\partial_t\quad,\quad \hat{q}_{J+1}^{NM} \equiv -i\frac{2}{\xi}(Y_{J+1}^M\dot{Y}_{J+1}^N - Y_{J+1}^N\dot{Y}_{J+1}^M)\partial_s\,, \tag{8.4.5}$$

[4]Note that our conventions differ by sign in comparison with [130].

where $Y's$ are explicit functions of s and t, parameterising the Wilson rays defined in (8.3.6). Following the classical case, we also introduce:

$$\hat{\mathbb{L}}_0{}^a{}_b(u) = u\,\delta^a_b + \frac{i}{2}\hat{q}_0^{MN}\Sigma_{MN}{}^a{}_b \ , \ \hat{\mathbb{L}}_{J+1}{}^a{}_b(u) = u\,\delta^a_b + \frac{i}{2}\hat{q}_{J+1}^{MN}\Sigma_{MN}{}^a{}_b \ . \tag{8.4.6}$$

These will form the operatorial part of the boundary matrices.

Next we need to identify the quantisation of $\mathbb{K}$ (8.3.17), such that (8.3.18) becomes the BYBE (6.1.2). We find that at the quantum level there is a quantum correction to the spectral parameter, invisible in the classical $\xi \to \infty$ limit. Namely the BYBE is solved by

$$\hat{\mathbb{K}}(u) = C.\hat{\mathbb{L}}_0(u - \tfrac{i}{2\xi}) \ , \tag{8.4.7}$$

where C is the same reflection matrix as the classical case (8.3.15). For the other boundary matrix, we need to find a solution of (6.1.6). One can easily verify that the solution to this equation has the following form:

$$\hat{\bar{\mathbb{K}}}(u) = G^{-1}.\mathbb{L}_{J+1}(u + \tfrac{i}{2\xi}).G.C \ , \tag{8.4.8}$$

where G is the twist matrix defined in (8.3.15)[5]. This expression is again identical to the classical expression up to a quantum correction in the spectral parameter.

The R-matrix itself is defined up to an arbitrary scalar factor, which does not affect any of the previous relations. However, in the next sections we will be using the fusion procedure for the boundary reflection matrix which is sensitive to the normalisation.

In the next sections we will use the normalised R-matrix (6.3.1) S, and denote for future convenience:

$$A(u) \equiv \frac{u^2\xi^2}{1-u^2\xi^2} \ . \tag{8.4.9}$$

8.4.3 Transfer matrix

In this section we will build the family of antisymmetric transfer matrices for the open Fishchain out of the building blocks discussed above and the fusion procedure.

We will use the index $\mathbf{4}$ to indicate the fundamental open transfer matrix of the open Fishchain, which is a representation of the twisted Yangian $Y^+(4)$, and define it as:

$$\hat{\mathbb{T}}^{\mathbf{4}}(u) = \mathrm{tr}\left[\mathbb{L}^t_J(-u)\cdots\mathbb{L}^t_2(-u)\mathbb{L}^t_1(-u)\mathbb{K}(u)\mathbb{L}_1(u)\ldots\mathbb{L}_{J-1}(u)\mathbb{L}_J(u)G\bar{\mathbb{K}}(u)G^t\right] \ . \tag{8.4.10}$$

The boundary monodromy matrix of the model can be defined as:

$$\mathbb{U}(u) = \mathbb{L}_1(u)\ldots\mathbb{L}_{J-1}(u)\mathbb{L}_J(u)G\bar{\mathbb{K}}(u)G^t\mathbb{L}^t_J(-u)\ldots\mathbb{L}^t_2(-u)\mathbb{L}^t_1(-u) \, , \tag{8.4.11}$$

but we will not use it in this chapter.

As showed in chapter 6, open transfer matrices form a family of mutually commuting operators:

$$[\hat{\mathbb{T}}^{\mathbf{4}}(u), \hat{\mathbb{T}}^{\mathbf{4}}(v)] = 0 \ . \tag{8.4.12}$$

[5]Notice that the twist is included for convenience in this solution via the twisted Yangian coproduct (6.1.5).

First fused transfer matrix

In order to build $\mathbb{T}^{\mathbf{6}}$ - the transfer matrix acting on the auxiliary space $\mathbb{C}^4 \wedge \mathbb{C}^4 \sim \mathbb{C}^6$, or $\mathbb{T}_2$ in the notations of chapter 6 - we need the fuse the corresponding building blocks, $\mathbb{K}$ and $\mathbb{L}^6$.

The $\mathbb{L}^{\mathbf{6}}$ needed in this section has auxiliary space being the 6-dimensional Minkowski space with metric η_{MN}. The fused Lax operator is a quadratic polynomial in u with coefficients built out of local charge operators $\hat{q}_i$ as follows:

$$\hat{\mathbb{L}}_i^{\mathbf{6}\,MN}(u) = \left(u^2 - \frac{1}{8}\mathrm{tr}\hat{q}_i^2\right)\eta^{MN} - u\hat{q}_i^{MN} + \left(\frac{1}{2}\hat{q}_i^{2\,MN} - \frac{i}{\xi}\hat{q}_i^{MN} + \frac{1}{4\xi^2}\eta^{MN}\right) . \quad (8.4.13)$$

Note that the fused Lax operator $\hat{\mathbb{L}}_i^{\mathbf{6}}(u)$ is invariant under generalised transposition t plus flipping of the spectral parameter $u \to -u$.

Boundary reflection operator What remains to be done is fusing the reflection operators. This can be done using the procedure described in chapter 6.

In terms of $Y = Y_0$, defined in (8.3.6), we get:

$$\begin{aligned}\hat{\mathbb{K}}^{\mathbf{6}}_{MN}(u) &= C^{\mathbf{6}}_{MN}u\left(u - \frac{i}{\xi}\right) + u\frac{2i}{\xi}(Y_N\dot{Y}_M - Y_M\dot{Y}_N)\partial_t \\ &+ \frac{2}{\xi^2}Y_N\hat{\partial}_t Y_M\hat{\partial}_t - \frac{2i}{\xi^3 u}Y_M\hat{\partial}_t Y_N\hat{\partial}_t\ .\end{aligned} \quad (8.4.14)$$

As we can see, it is a second order differential operator in t and a second order polynomial in the spectral parameter u. Similarly for the right boundary we get (replacing t by s):

$$\begin{aligned}\hat{\bar{\mathbb{K}}}^{\mathbf{6}}_{MN}(u) &= C^{\mathbf{6}}_{MN}u\left(u + \frac{i}{\xi}\right) + u\frac{2i}{\xi}(Y_N\dot{Y}_M - Y_M\dot{Y}_N)\partial_s \\ &+ \frac{2}{\xi^2}Y_M\hat{\partial}_s Y_N\hat{\partial}_s + \frac{2i}{\xi^3 u}Y_N\hat{\partial}_s Y_M\hat{\partial}_s\ .\end{aligned} \quad (8.4.15)$$

In the equations above we are using the reflection matrix $C^{\mathbf{6}}$ in vector representation is the matrix C we introduced in (8.2.21). An important property, which follows directly from the definitions (8.4.14) and (8.4.15), is that $\mathbb{K}^{\mathbf{6}}(+i/\xi) = 0$ and $\bar{\mathbb{K}}^{\mathbf{6}}(-i/\xi) = 0$.

Using $\hat{q}_0$ (8.4.5) we can write

$$\hat{\mathbb{K}}^{\mathbf{6}}(u) = C^{\mathbf{6}}u\left(u - \frac{i}{\xi}\right) - u\hat{q}_0 + \frac{1}{2}\hat{q}_0^2 - \frac{i}{2\xi u}(\hat{q}_0^2)^T\ . \quad (8.4.16)$$

For the right boundary we have very similar expression

$$\hat{\bar{\mathbb{K}}}^{\mathbf{6}}(u) = (G^{\mathbf{6}})^{-1}.\left(u\left(u + \frac{i}{\xi}\right) - u\hat{q}_{J+1} + \frac{1}{2}(\hat{q}_{J+1}^2)^T + \frac{i}{2\xi u}\hat{q}_{J+1}^2\right).G^{\mathbf{6}}.C^{\mathbf{6}}\ . \quad (8.4.17)$$

[6] We could also fuse the boundary monodromy matrix $\mathbb{U}$ directly, but this seems to be computationally heavier to do.

where $G^{\mathbf{6}}$ in vector representation is the twist matrix G from (8.2.21). The twist matrix $G^{\mathbf{6}}$ appears in the expression (8.4.17) for the right boundary reflection operator, as it is defined in a way that does not depend on φ.

Having all the needed ingredients we can compute $\mathbb{T}^{\mathbf{6}}$ by replacing in the RHS of (8.4.10) all the operators and matrices by their fused counterpart.

Hamiltonian from the transfer matrices

In this section we will show that the Hamiltonian of the system is a part of the commuting family of operators. For that consider:

$$\mathbb{T}^{\mathbf{6}}(0) = 4 \lim_{u\to 0} u^2\xi^2 \mathrm{tr}\left[\mathbb{L}_J(0)\cdots\mathbb{L}_2(0)\mathbb{L}_1(0)\mathbb{K}(u)\mathbb{L}_1(u)\ldots\mathbb{L}_{J-1}(0)\mathbb{L}_J(0)G\bar{\mathbb{K}}(u)G^t\right] . \tag{8.4.18}$$

First we can use that:

$$\hat{\mathbb{L}}_i^{\mathbf{6}\,MN}(0) = \frac{\hat{q}_i^{2\,MN}}{2} - \frac{i}{\xi}\hat{q}_i^{MN} - \frac{\eta^{MN}}{8}\mathrm{tr}\hat{q}_i^2 + \frac{\eta^{MN}}{4\xi^2} = \frac{:\hat{q}_i^2:^{MN}}{2} = \frac{1}{2\xi^2}X_i^M X_i^N \partial^2_{X_i^K} . \tag{8.4.19}$$

where in the last equality we used the identity from [130]. Also from (8.4.14) and (8.4.15) we have

$$u\,\xi\,\hat{\mathbb{K}}^{\mathbf{6}\,MN}(u)\Big|_{u=0} = -\frac{i}{2}(\hat{q}_0^2\quad)^{NM} = -\frac{2i}{\xi^2}Y_0^M\hat{\partial}_t Y_0^N\hat{\partial}_t \ , \tag{8.4.20}$$

$$u\xi\left(G^{\mathbf{6}}\hat{\bar{\mathbb{K}}}^{\mathbf{6}}(u)(G^{\mathbf{6}})^{-1}\right)^{MN}\Big|_{u=0} = +\frac{i}{2}(\hat{q}_{J+1}^2)^{MN} = +\frac{2i}{\xi^2}Y_{J+1}^N\hat{\partial}_s Y_{J+1}^M\hat{\partial}_s . \tag{8.4.21}$$

Combining all parts together, up to sub-leading terms in $1/\xi$ we get the quantum version of $H_q + 1$, where H_q is defined in (8.3.9). In order to check that this produces the correct quantisation of H_q, i.e. the one related to the graph building operator, we have to analyse the expression (8.4.19) more carefully. Paying attention to the order of the operators we get

$$\begin{aligned}\mathbb{T}^{\mathbf{6}}(0) &= 4\frac{4}{2^{2J}\xi^{4J+4}}\eta_{NM}X_J^M X_J.X_{J-1}\ldots X_1.Y_0\partial_t\prod_{i=1}^{J}\Box_i^{(6)} \\ &\times\ Y_0.X_1X_1.X_2\ldots X_J.Y_{J+1}\partial_s Y_{J+1}^N\partial_s\partial_t\prod_{i=1}^{J}\Box_i^{(6)} ,\end{aligned} \tag{8.4.22}$$

where all derivatives are understood as operators acting on the CFT wavefunction embedded in the lightcone of $6D$ Minkowski spacetime. In order to relate the above expression with the graph building operator (8.1.9), which is expressed in terms of derivatives acting on functions in $4D$ Euclidean spacetime, we recall that $\mathbb{T}^{\mathbf{6}}$ is built out of q_i's and as such we can act with it, in a consistent way, on functions of $4D$ coordinates, following the prescription (8.4.4). Furthermore, one can just replace the $6D$ d"Alembertian operator in $4D$ d"Alembertian due to the identity

$$\Box^{(6)} = \Box^{(4)} + \partial_{X_+}\partial_{X_-} , \tag{8.4.23}$$

and the fact that there is no dependence on X^- in the $4D$ functions, by construction (8.4.4). Therefore,

$$Y_0.X_1X_1.X_2\dots X_J.Y_{J+1}\partial_s\partial_t\prod_{i=1}^{J}\Box_i^{(6)} \quad = \tag{8.4.24}$$

$$\left(-\tfrac{1}{2}\right)^{J+1}\frac{\partial_s\partial_t}{|y_0'||y_{J+1}'|}\prod_{i=0}^{J}(y_i-y_{i+1})^2\prod_{i=1}^{J}\Box_i^{(4)} \quad = \quad \left(\tfrac{1}{2}\right)^{J+1}(4\hat{g}^2)^{J+1}\hat{B}^{-1}\,, \tag{8.4.25}$$

where we used that $X_1.X_2 = -\frac{1}{2}(x_1-x_2)^2$ and $Y_0.X_1 = -\frac{e^{-t}}{2}(x_0-x_1)^2$. We use the expression for the inverse of the graph-building operator $\hat{B}^{-1}$ from (8.1.9). Then for $\mathbb{T}^{\mathbf{6}}(0)$ one gets precisely

$$\mathbb{T}^{\mathbf{6}}(0) = 4\hat{B}^{-2}\,. \tag{8.4.26}$$

Where we used (8.2.10) to relate ξ and $\hat{g}$. We see that all factors cancel exactly, implying that at the quantum level we also have $\mathbb{T}^{\mathbf{6}}(0)\psi = 4\psi$ as it follows from (8.1.9). At the same time we see that the quantum graph building operator $\hat{B}$ is indeed a part of the commuting family of operators, which demonstrates the integrability of the initial system of Feynman graphs.

Second fused transfer matrix

Here we compute the $\bar{\mathbf{4}}$ transfer matrix (or $\mathbb{T}_3$ in the notation of chapter 6), corresponding to the antisymmetrisation of the antisymmetric tensor product of three copies of $\mathbf{4}$ irrep. ingredients with the corresponding shifts in the spectral parameters, dictated by the fusion procedure. The calculation for $\mathbb{L}^{\mathbf{4}}$ was done in [130]. The result for $\mathbb{L}^{\bar{\mathbf{4}}}$ can be re-expressed in terms of one $\mathbb{L}^{\mathbf{4}}$ times a scalar polynomial factor

$$\hat{\mathbb{L}}_k^{\bar{\mathbf{4}}}(u) = \left(u^2 - \frac{\mathrm{tr}\hat{q}_k^2}{8} + \frac{1}{\xi^2}\right)(\hat{\mathbb{L}}_k^{\mathbf{4}})^t(-u)\,, \tag{8.4.27}$$

Doing the fusion of the boundary reflection operator and projecting onto the $\mathbf{4}$ auxiliary space, we get:

$$\hat{\mathbb{K}}^{\bar{\mathbf{4}}\,ab}(u) = -\left(u^2 - \frac{iu}{\xi} + \frac{3}{4\xi^2}\right)\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,ba}(-u)\ ,\ \hat{\bar{\mathbb{K}}}^{\bar{\mathbf{4}}}_{ab}(u) = -\left(u^2 + \frac{iu}{\xi} + \frac{3}{4\xi^2}\right)\bar{\mathbb{K}}^{\mathbf{4}}_{ba}(-u)\,. \tag{8.4.28}$$

Finally, the twist matrix is the inverse of the one for the $\mathbf{4}$ irrep. (8.3.15). The polynomial factors in (8.4.27) and (8.4.28) will play an important role in the Baxter TQ equation.

Sklyanin determinant

Here we compute the ingredients of the transfer matrix in the representation $\bar{\mathbf{1}}$, also known as the Sklyanin determinant. Like in the previous sections, this can be computed as an antisymmetrisation of the tensor product of four copies of $\mathbb{L}$ and $\mathbb{K}$ in the $\mathbf{4}$ irrep. For both

$\mathbb{L}^{\bar{1}}$ and $\mathbb{K}^{\bar{1}}$ we find that they are just fourth order polynomials in u acting trivially on the physical space. Again the calculation of $\mathbb{L}^{\bar{1}}$ was already performed in [130] and the result reads:

$$\hat{\mathbb{L}}_i^{\bar{1}}(u) = \left(u^2 - \frac{\operatorname{tr} \hat{q}_i^2}{8} + \frac{5}{4\xi^2}\right)^2 + \frac{\operatorname{tr} \hat{q}_i^2}{8\xi^2} - \frac{1}{\xi^4} \,. \tag{8.4.29}$$

For the boundary reflection operator, we obtain:

$$\mathbb{K}^{\bar{\mathbf{1}}}(u) = \left(u - \frac{2i}{\xi}\right)\left(u - \frac{i}{\xi}\right) u \left(u + \frac{i}{\xi}\right) \quad , \quad \bar{\mathbb{K}}^{\bar{\mathbf{1}}}(u) = \left(u + \frac{2i}{\xi}\right)\left(u + \frac{i}{\xi}\right) u \left(u - \frac{i}{\xi}\right) \,. \tag{8.4.30}$$

$J = 0$ example

Before discussing the general case we first give the explicit result for the simplest case of a chain of zero length. This means that we are only left with the boundary reflection operators. Furthermore, the graph-building operator is a second order differential operator in s and t, as it should commute with the dilatation operator only one variable remains. If we further impose $\mathbb{T}^{\mathbf{6}}(0) = 4$, we will automatically diagonalise all transfer matrices obtaining the following results for their eigenvalues:

$$\begin{aligned}
\mathbb{T}^{\mathbf{4}}(v) &= \frac{1}{\xi^2}\left(4v^2\cos\varphi + \cos\varphi + 8\xi^2\right) \,, \\
\mathbb{T}^{\mathbf{6}}(v) &= A(2v)\frac{v^2+1}{\xi^4 v^2}\left(v^4(2\cos(2\varphi)+4) + v^2\left(16\xi^2\cos\varphi - 4\Delta^2\sin^2\varphi\right) + 16\xi^4\right) \,, \\
\mathbb{T}^{\bar{\mathbf{4}}}(v) &= A(2v)A(2v+i)A(2v-i)\frac{\left(4v^2+1\right)\left(4v^2+9\right)}{16\xi^6}\left(4v^2\cos\varphi + \cos\varphi + 8\xi^2\right) \,, \\
\mathbb{T}^{\bar{\mathbf{1}}}(v) &= A^2(2v)A(2v+i)A(2v-i)A(2v+2i)A(2v-2i)\frac{v^2\left(v^2+1\right)^2\left(v^2+4\right)}{\xi^8} \,,
\end{aligned} \tag{8.4.31}$$

where we introduced the rescaled spectral parameter $v = u\,\xi$. The factors A, where $A(v) = \frac{v^2}{1+v^2}$, are due to the R-matrix normalisation as discussed in section 8.4.2. We also worked out the form of the transfer-matrix eigenvalues for $J = 1$ case in Appendix F.3 in terms of a few unknown constants. We have explicitly verified that all the $\mathbb{T}$-operators for $J = 0$ and $J = 1$ commute between themselves and with the charges Δ, S, H as expected.

In the next section we will extend these results to the general J case.

Eigenvalues of the transfer matrices

Here we deduce the general form of the eigenvalues of the transfer matrices. First, one can notice explicitly that for $J = 0$ and $J = 1$ case they are even functions of the spectral parameter. In Appendix F.2 we prove that this is true for any J. Some other properties of the transfer matrices are:

- $\mathbb{T}^{\mathbf{4}}$ is a polynomial of degree $2J + 2$ in v, as it follows from its definition (8.4.10).

- $\mathbb{T}^{\mathbf{6}}$ is a rational function with two poles at $v=\pm i/2$, coming from the normalisation factor $A(2v)$. Another potential pole at $v=0$, coming from the boundary reflection operators, is cancelled by the same $A(2v)$. At large v it behaves as $\sim v^{4J+4}$.
- Previously we noticed that $\mathbb{K}^{\mathbf{6}}(u=+i/\xi)=0$ and $\bar{\mathbb{K}}^{\mathbf{6}}(u=-i/\xi)=0$, therefore we can see that $\mathbb{T}^6$ should have a prefactor of $v^2+1=\xi^2u^2+1$.
- Finally, in section 8.4.3 we have showed that $\mathbb{T}^{\mathbf{6}}(0)=4$ due to (8.4.26).
- The properties of $\mathbb{T}^{\bar{\mathbf{4}}}$ are very similar to those of $\mathbb{T}^{\mathbf{4}}$, apart from the trivial factors of A's and additional trivial factors coming from $\mathbb{L}^{\bar{\mathbf{4}}}$ and $\mathbb{K}^{\bar{\mathbf{4}}}$.
- Finally, $\mathbb{T}^{\bar{\mathbf{1}}}$ (the Sklyanin determinant) contains only trivial factors and can be computed explicitly for any J.

Based on these observations we can write the transfer matrices in terms of the polynomials $P_k^\lambda(v^2)$ as:

$$\begin{aligned}
\mathbb{T}^{\mathbf{1}}(v) &= 1\ ,\\
\mathbb{T}^{\mathbf{4}}(v) &\equiv \frac{P^{\mathbf{4}}_{J+1}(v^2)}{\xi^{2J+2}}\ ,\\
\mathbb{T}^{\mathbf{6}}(v) &\equiv A(2v)\frac{v^2+1}{v^2}\frac{P^{\mathbf{6}}_{2J+2}(v^2)}{\xi^{4J+4}}\ ,\\
\mathbb{T}^{\bar{\mathbf{4}}}(v) &= A(2v)A(2v+i)A(2v-i)\frac{(v^2+\frac{9}{4})(v^2+\frac{1}{4})^{2J+1}P^{\bar{\mathbf{4}}}_{J+1}(v^2)}{\xi^{6J+6}}\ ,\\
\mathbb{T}^{\bar{\mathbf{1}}}(v) &= A^2(2v)A(2v+i)A(2v-i)A(2v+2i)A(2v-2i)\frac{(v^2+4)(v^2+1)^{2J+2}v^{4J+2}}{\xi^{8J+8}}\ .
\end{aligned} \tag{8.4.32}$$

Here, P_k^λ is a polynomial of degree k, labelled by the representation λ in the auxiliary space. The eigenvalues of the conserved charges of the system are the coefficients of the powers of v^2 in these polynomials. We will denote them as (defining $w\equiv v^2$):

$$\begin{aligned}
P^{\mathbf{4}}_{J+1}(w) &= \sum_{i=0}^{J+1} a_i w^{-i+J+1}\ ,\\
P^{\mathbf{6}}_{2J+2}(w) &= \sum_{i=0}^{2J+2} b_i w^{-i+2J+2}\ ,\\
P^{\bar{\mathbf{4}}}_{J+1}(w) &= \sum_{i=0}^{J+1} c_i w^{-i+J+1}\ .
\end{aligned} \tag{8.4.33}$$

In our definition, a_0 will represent the coefficient of the highest power in v^2 in $P^{\mathbf{4}}_{J+1}$, a_1 the second highest etc. The leading coefficients are easy to compute explicitly directly from the

definition

$$a_0 = c_0 = 4\cos\varphi \ , \ \ b_0 = 2\cos 2\varphi + 4 \ . \tag{8.4.34}$$

They just give the twisted (or q-)dimension of the corresponding representations. Since the leading coefficients are trivial, in total we get $4J + 4$ non-trivial coefficients in the polynomials P. As our system has $4J + 2$ degrees of freedom it may suggest that there are 2 more relations between the coefficients of the polynomials P. Indeed, in the $J = 0$ and $J = 1$ cases we found them by computing the differential operators explicitly, but it is rather hard to deduce the general relations. In the $J = 1$ case we found exactly 6 independent operators guarantees integrability of the system.

We found that the global charges Δ and S are encoded into the sub-leading coefficients in the following way:

$$\begin{aligned} c_1 - a_1 &= 8\, i\, S\, \Delta\, \sin\varphi \ , \\ (a_1 + c_1)\cos\varphi - b_1 &= 2\, \left(2\, S^2 + 2\, \Delta^2 + J\right)\, \sin^2\varphi + 2\, \cos^2\varphi \ . \end{aligned} \tag{8.4.35}$$

These relations are also quite hard to derive in general, but we explicitly verified the first relation up to $J = 3$ and the second up to $J = 2$.

Finally, the condition $\mathbb{T}^{\mathbf{6}}(0) = 4$ implies:

$$b_{2J+2} = \xi^{4J+4} = 16\hat{g}^{4J+4} \ . \tag{8.4.36}$$

In order to find the eigenvalues of all coefficients of the transfer matrices we will have to develop a numerical procedure. For that we will first build the TQ-relations in the next section.

8.5 Baxter TQ equation

In this section we follow the derivation of [121] to deduce the general simplified form of the TQ-relations and deduce asymptotic of the Q-functions. The starting point is the TQ-relation, valid in the gauge used in this chapter[7]:

$$Q(v+2i) + \mathbb{T}^{\mathbf{4}}(v+i/2)Q(v+i) + \mathbb{T}^{\mathbf{6}}(v)Q(v) + \mathbb{T}^{\bar{\mathbf{4}}}(v-i/2)Q(v-i) + \mathbb{T}^{\bar{\mathbf{1}}}(v-i)Q(v-2i) = 0 \ . \tag{8.5.1}$$

As we discussed above the transfer matrices have a number of trivial factors. In order to remove these fixed, non-dynamical factors, we perform the following *gauge* transformation of the Q-function

$$Q(v) = q(v)\frac{e^{\pi(J+1)v}\Gamma(-iv)\xi^{2i(J+1)v}\Gamma(iv+1)^{-2J-1}}{\Gamma\left(-iv-\frac{1}{2}\right)\Gamma(iv+2)} \ , \tag{8.5.2}$$

[7] Notice that this is actually the *dual* Baxter TQ relation presented in chapter 6, with the main difference being an overall shift in the transfer matrices.

which brings (8.5.1) to a simpler and more symmetric form:

$$\begin{aligned}\frac{P^{\mathbf{6}}_{2J+2}(v^2)}{v^{2J+3}}q(v) &= -(v+i)^{2J+1}q(v+2i) - \frac{v+\frac{i}{2}}{v(v+i)}P^{\mathbf{4}}_{J+1}\left((v+\tfrac{i}{2})^2\right)q(v+i) \\ &\quad -(v-i)^{2J+1}q(v-2i) - \frac{v-\frac{i}{2}}{v(v-i)}P^{\bar{\mathbf{4}}}_{J+1}\left((v-\tfrac{i}{2})^2\right)q(v-i)\,.\end{aligned} \tag{8.5.3}$$

As a test of this equation we can compare with the case $J=0$, studied as a ladder limit of QSC in $\mathcal{N}=4$ SYM. For $J=0$, by plugging in the explicit form of the polynomials (8.4.31) into (8.5.3), we obtain:

$$\begin{aligned}&q(v)\left(\frac{2\left(8\hat{g}^2v^2\cos(\varphi)+8\hat{g}^4+v^4(\cos(2\varphi)+2)\right)}{v^3}-\frac{4\Delta^2\sin^2(\varphi)}{v}\right)\\ &+\frac{2(2v-i)q(v-i)\left(2\hat{g}^2+v(v-i)\cos(\varphi)\right)}{v(v-i)}+\frac{2(2v+i)q(v+i)\left(2\hat{g}^2+v(v+i)\cos(\varphi)\right)}{v(v+i)}\\ &+(v-i)q(v-2i)+(v+i)q(v+2i)=0\,.\end{aligned} \tag{8.5.4}$$

This is the same as what was found in [25, 157] for a cusped Wilson line in the ladders limit, as expected (see detailed comparison in Appendix F.3).

The equation (8.5.3) for general J is one of our main result. As we show in section 8.6 it lets us evaluate numerically the spectrum.

8.5.1 Large v asymptotic of Q-functions

For the numerical evaluation, which we describe in the next section, it is important to have the large v asymptotics under control. As the leading and partially subleading coefficients in the polynomials P are known from (8.4.34) and (8.4.35), we can deduce that the 4 linearly independent solutions of the equation (8.5.3) should have the following large v asymptotic expansion:

$$\begin{aligned}q_1 &= e^{+\phi v}v^{+\Delta-S-J}\left(1+\frac{c_{1,1}}{v}+\dots\right)\,,\\ q_2 &= e^{-\phi v}v^{+\Delta+S-J}\left(1+\frac{c_{2,1}}{v}+\dots\right)\,,\\ q_3 &= e^{+\phi v}v^{-\Delta+S-J}\left(1+\frac{c_{3,1}}{v}+\dots\right)\,,\\ q_4 &= e^{-\phi v}v^{-\Delta-S-J}\left(1+\frac{c_{4,1}}{v}+\dots\right)\,.\end{aligned} \tag{8.5.5}$$

where $\phi=\pi-\varphi$. The above asymptotics suggest the following relation to the QSC Q-functions of [126]:

$$q_i(v)\sim\frac{Q_i(v)}{v^{J+1/2}}\,, \tag{8.5.6}$$

which is similar to the relations found in the fishnet model [33]. Subleading coefficients in $1/v$ can be found systematically in terms of the coefficients of the polynomials P, i.e. a_i, b_i

and c_i, by plugging the expansion (8.5.5) into (8.5.3). In order to fix the coefficients of the polynomials P one has to use the gluing (or quantisation) condition, which we describe in the next section.

8.6 Numerical solution

After having established the key properties of the Baxter equation we can solve them numerically and fix the remaining coefficients a_i, b_i and c_i. The method we implement is essentially the one of [158] which was adopted and simplified to the current type of problems in [33, 93, 121, 157]. The 4th order finite difference equation (8.5.3) has 4 linearly independent solutions with the asymptotic (8.5.5). The way to find them numerically is first finding the asymptotic solution at large v, where (8.5.3) reduces to a linear problem for the asymptotic expansion coefficients. The truncated asymptotic series gives a very good approximation at sufficiently large $|\mathrm{Im}\, v|$. In order to bring $\mathrm{Im}\, v$ to a finite value, we can simply use (8.5.3) itself, as it allows to find $q(v)$ in terms of $q(v+in)$, $n=1,\dots,4$ (or $q(v-in)$, $n=1,\dots,4$). Using (8.5.3) as a recursion relation, we can gradually decrease $|\mathrm{Im}\, v|$. By doing this there are two options: starting from $+i\infty$ or from $-i\infty$. Correspondingly, we will find 4 analytic solutions in the upper-half plane, $q_i^{\downarrow}$, and other 4 analytic in the lower-half plane, $q_i^{\uparrow}$. Since the Baxter equation is a fourth order equation, we can have only four independent solutions, meaning that the $q_i^{\uparrow}$ and $q_i^{\downarrow}$ should be related by a linear transformation. We should therefore have:

$$q_i^{\uparrow}(v) = \Omega_i^{\ j}(v) q_j^{\downarrow}(v), \quad \Omega_i^{\ j}(v+i) = \Omega_i^{\ j}(v)\,, \tag{8.6.1}$$

where

$$\Omega_i^{\ j}(v) = \frac{\epsilon^{j\, j_1\, j_2\, j_3}}{3!} \frac{\det_{n=0,\dots,3}\left\{q_i^{\uparrow}(v-in), q_{j_1}^{\downarrow}(v-in), q_{j_2}^{\downarrow}(v-in), q_{j_3}^{\downarrow}(v-in)\right\}}{\det_{n=0,\dots,3}\left\{q_1^{\downarrow}(v-in), q_2^{\downarrow}(v-in), q_3^{\downarrow}(v-in), q_4^{\downarrow}(v-in)\right\}}\,. \tag{8.6.2}$$

$\Omega_i^{\ j}(v)$ is an i-periodic function which can have poles at $v=in$ of order no higher than $q_i(v)$'s themselves. From the Baxter equation (8.5.3) it is easy to see that $q_i(v)$ only has poles at $v=in$ of maximal order $2J+2$, which implies that $\Omega_i^{\ j}(v)$ is a trigonometric rational function of the form:

$$\Omega_i^{\ j}(v) = \frac{\sum_{n=0}^{2J+2} C^{(n)}{}_i^{\ j} e^{2\pi n u}}{(1-e^{2\pi u})^{2J+2}}\,, \tag{8.6.3}$$

The quantisation condition can be obtained by comparing with the QSC description of the cusped Wilson line [126], where one defines an antisymmetric matrix ω_{ik}, related to $\Omega_i^{\ j}$ in the following way:

$$\omega_{ik} = \Omega_i^{\ j}\Gamma_{jk}\,, \tag{8.6.4}$$

where the so-called *gluing* matrix is:

$$\Gamma_{jk} = \begin{pmatrix} 0 & \gamma_1 \sinh(2\pi v) & 0 & \gamma_3 \\ \gamma_2 \sinh(2\pi v) & 0 & \gamma_4 & 0 \\ 0 & \gamma_5 & 0 & 0 \\ \gamma_6 & 0 & 0 & 0 \end{pmatrix}, \tag{8.6.5}$$

where γ_i are some constants. All we need to know, from QSC, is that ω in (8.6.4) is anti-symmetric i.e. $\Omega_i^j \Gamma_{jk} = -\Omega_k^j \Gamma_{ji}$, which, in particular, implies:

$$\Omega_{41} = \Omega_{32} = 0\,. \tag{8.6.6}$$

As each component of $\Omega(u)$ is a nontrivial function parametrised in terms of $2J+3$ constants $C_{ij}^{(n)}$, imposing (8.6.6) is usually sufficient to fix $4J+4$ unknown constants, contained in the Baxter equations.

Tests By applying the numerical method we studied the spectrum for $J=1$ and $J=2$ cases. For $J=1$ we also found a large number of excited states (see figure 8.6), corresponding to additional insertions of Φ_2 and Φ_1 fields at the cusp, as discussed in [25]. We tested our results against the weak coupling result of [125], which in our notations reads

$$\Delta = J + \hat{g}^{2J+2} \frac{(-1)^J 2^{4J+3} \pi^{2J+1} \csc(\varphi) B_{2J+1}\left(\frac{\varphi}{2\pi}\right)}{\Gamma(2J+2)} + \mathcal{O}(\hat{g}^{4J+4})\,, \tag{8.6.7}$$

which agreed with high precision (of more than 15 digits) with our numerical data for $J=0$, $J=1$ and $J=2$. For example for $J=2$ and $\varphi = 2\pi/3$ we get the following fit for the numerical data on figure 8.7:

$$\Delta = 2 - 124.08839542210\hat{g}^6 + 23271.513371517\hat{g}^{12} + \dots \tag{8.6.8}$$

in agreement with (8.6.7), which for $J=2$ gives

$$2 + \hat{g}^6 \frac{8}{45}\varphi\left(3\varphi^4 - 15\pi\varphi^3 + 20\pi^2\varphi^2 - 8\pi^4\right)\csc\varphi = 2 - \hat{g}^6 \frac{512\pi^5}{729\sqrt{3}}\,. \tag{8.6.9}$$

The states Z^J for the cases $J=1$ and $J=2$ do not behave classically at large ξ, i.e. Δ decreases faster than linear. Like in [33] we expect the classical regime to describe the highly excited states.

8.7 Summary

In this chapter we showed that the cusped Wilson-Maldacena loop with insertions of J orthogonal scalars is dual to an open integrable spin chain. We computed the transfer matrices of this model, and obtained a Baxter TQ equation which can be solved numerically for any J, finding the spectrum of dimensions Δ non-perturbatively (for $J = 0, 1, 2$). This lets us find the Q-functions of the system, which could be used as a building block for future calculations, such as the three-point correlation functions using Functional SoV. Some hints that this approach will work in this setting can be found in [25], while some initial progress for the cousin Fishnet theory has been done in [92].

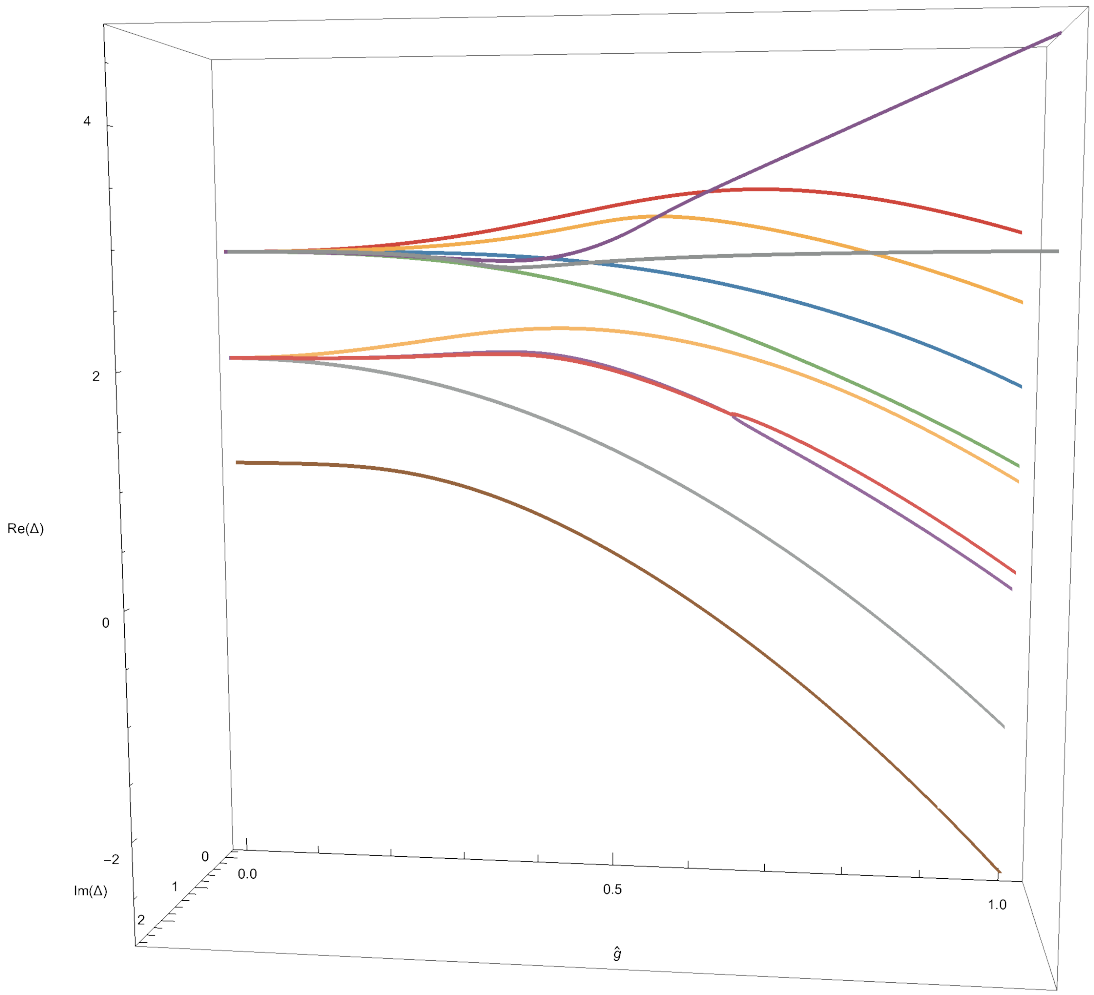


Figure 8.6: Numerical spectrum with excited states for $J = 1$ and $S = 0$. The lowest curve (starting at $\Delta = 1$ at zero coupling) corresponds to the case with a single insertion of Z at the cusp. The curves which begin at higher integers at zero coupling correspond to excited states of the solution of the Baxter equation, which correspond to additional insertions of Φ_2 and Φ_1 at the cusp (see [25] for some explicit examples). Whereas for the ground state the dimension Δ is real, excited states could appear in complex conjugate pairs.

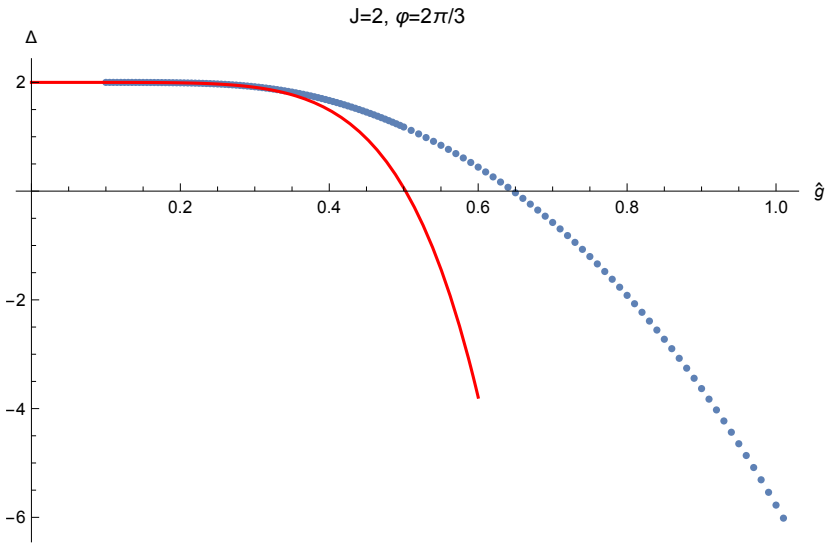


Figure 8.7: Numerical data (dots) for the ground state of length-2 chain with $\varphi = 2\pi/3$. Red solid line shows the Lüscher formula prediction of [125].

Chapter 9

Conclusions and Outlooks

In chapters 5 and 6 of this thesis, we developed the Functional Separation of Variables. For periodic rational spin chains, we presented the Character Projection (CP) technique to compute a complete basis of observables, the Principal Operators for the spin chain in terms of Baxter Q-functions. We then showed that FSoV + CP let us reconstruct from first principles the Sklyanin's B and C operators, whose form for rank $N \geqslant 4$ spin chains was only conjectured before (the rank $N = 3$ case was treated in [75]). We also derived the SoV basis and the SoV basis matrix elements of all Principal Operators, which form a complete set of observables: from their knowledge, as we explained, it is possible to extract any correlation function for rational integrable spin chains.

For open rational spin chains, based on the twisted Yangian $Y^+(2N)$, we also developed the FSoV approach, culminating in the functional orthogonality relations and the scalar product between Bethe states. While the latter is a trivial relation[1], it is practically immediate to update it to obtain diagonal matrix elements of various operators, using the perturbation theory techniques explained in chapter 5.

In chapter 7, we proved from first principles that the cusped Maldacena-Wilson line in $\mathcal{N} = 4$ SYM with orthogonal scalar insertions at the cusp, is integrable in the ladders limit. We derived the holographic dual of this observable, the open Fishchain. Solving the Baxter TQ relation for this open rational spin chain allowed us to find the non-perturbative values of the scaling dimension of the cusped line with orthogonal scalar insertions, for any value of the coupling constant and cusp angle. It also provides an ideal playground for the application of FSoV to a concrete model in an important Quantum Field Theory such as $\mathcal{N} = 4$ SYM.

9.1 Future directions

FSoV We believe that the formulas presented for form-factors in this thesis extend immediately to the q-deformed high-rank XXZ case [77] after simple modification as is already

[1] Recall that Bethe states are orthogonal.

the case in the gl_2 setting [159], and it would be interesting to check directly, allowing one to extend the recent rank 2 results [160] and to study high-rank correlators at zero temperature along the lines of [86]. From our results it is also possible to extract form-factors of local operators using the quantum inverse transform [85]. From here there are many interesting directions to pursue. For example, these can be used in the computation of current operators [161] which have numerous applications. It would also be very interesting to compare with the results of [162], where certain mean-values related to current operators are shown to factorise, and to understand such results from an SoV perspective.

Another interesting direction would be to construct the quantum version of the classical A operator of Sklyanin seen at the beginning of chapter 4, which is expected to act as a raising / lowering operator on the SoV bases, as some combination of principal operators as was done here for the B and C operators. This would solve an important puzzle as it is known that Sklyanin's quantum construction is singular for the highest-weight representations we consider here, see for example the discussion in [70].

It would be very interesting to develop the FSoV formalism and the approach to correlators developed in this thesis for spin chains based on different algebras. The Q-system for models with orthogonal symmetry has attracted huge attention recently [163, 164] and will likely play a large role in the SoV approach to correlators in conformal fishnet theories in $D \neq 4$ [165, 166].

The SoV construction for models with open boundary conditions still needs quite a lot of work. First of all, we have not developed yet a Character Projection technique, nor we know a companion "boundary" frame in which the SoV basis is independent of the boundary matrices eigenvalues. This in turn does not let us define Principal Operators, nor a way to compute off-diagonal form factors. Furthermore, the B and C operators have not been built yet. An operatorial approach for open spin chains has recently been studied [86, 99, 167–171], but this is for a different twisted Yangian, the reflection algebra. We used some insights from this work, such as the procedure to build the SoV basis, but it would be interesting to develop it fully for our open spin chain. Finally, it would be interesting to extend FSoV to *any* open rational spin chain, which would mean extending it to all the known twisted Yangians [95]. This would let us study any so-called integrable boundary state within the SoV formalism. First steps were already taken in [92, 172, 173] and recently this problem received increased interest [174–183].

Open Fishchain Since we have the spectrum of the model under control, the natural next step is to compute correlation functions. In [25, 184], the first steps towards this were taken, where the authors calculated the three point functions of three cusped Wilson lines in the ladders limit. Remarkably, they observed that the structure constants can be expressed as overlaps between some states via formulas that are highly reminiscent of FSoV. We have given some preliminary results in this thesis, but it would be important to extend them to the general case. This would naturally lead to the calculation of structure constants for cusped Wilson lines in the ladders limit.

It would also be interesting to get away from the ladders/fishnet limit both in our set-up

and in the Fishnet theory. Such an exploration could give some clues as to how to develop a first principles holographic derivation for the full theory. Some progress for the Fishnet theory has been made in [185], where a more complicated version of the Fishnet theory, which includes fermions and gauge fields, has been proven to be fully dual to a periodic integrable spin chain.

Another direction of exploration would be to try to expand our construction to Wilson loops in the ABJM theory [186] where there is already some evidence of integrability [187] and further, it admits treatment from a defect CFT point of view [188, 189]. Here too, a fishnet limit exists with Feynman graphs which look like a triangular lattice [190] and one can envisage the definition and study of a similar CFT wavefunction like the one we studied in this thesis.

Appendix A

Technical Details for Chapter 3

In this Appendix, we will put some more technical derivations of the subjects treated in chapter 3.

A.1 Derivation of eigenstates of the Heisenberg XXX spin chain

Our goal is to solve the spectral problem:

$$t(u)|\Psi\rangle = \tau(u)|\Psi\rangle \tag{A.1.1}$$

We start by considering the monodromy matrix (3.4.5). We will write it as acting as a matrix on the auxiliary space $\mathcal{H}_{aux} = \mathbb{C}^2$, with entries being operators in the full physical space $\mathcal{H}_{phys} = \bigotimes_{i=1}^{L} \mathcal{H}_i$:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{A.1.2}$$

The transfer matrix in this notation is simply $t(u) = A(u) + D(u)$.

We now need to compute the operators in the entries of (A.1.2). Recall that T is defined as a product of Lax operators (3.4.8), whose action on a physical spin up state (tensored with a generic state $|\Phi\rangle$ on the auxiliary space) is:

$$L_i(u)|\uparrow\rangle_i \otimes |\Phi\rangle = \begin{pmatrix} (u+i/2)|\uparrow\rangle & i|\downarrow\rangle \\ 0 & (u-i/2)|\uparrow\rangle \end{pmatrix} |\Phi\rangle. \tag{A.1.3}$$

Thus, if we choose as the vacuum state for the physical space a state with all spin ups,

$|\Omega\rangle = \bigotimes_{i=1}^{L} |\uparrow\rangle_i$, the action of the monodromy matrix on it will be given by:

$$T(u)|\Omega\rangle \otimes |\Phi\rangle = \begin{pmatrix} (u+i/2)|\uparrow\rangle_L & i|\downarrow\rangle_L \\ 0 & (u-i/2)|\uparrow\rangle_L \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} (u+i/2)|\uparrow\rangle_1 & i|\downarrow\rangle_1 \\ 0 & (u-i/2)|\uparrow\rangle_1 \end{pmatrix} |\Phi\rangle \tag{A.1.4}$$
$$= \begin{pmatrix} (u+i/2)^L|\Omega\rangle & |\dots\rangle \\ 0 & (u-i/2)^L|\Omega\rangle \end{pmatrix} |\Phi\rangle\,,$$

where $|\dots\rangle$ is a value we do not need to write explicitly. Comparing with (A.1.2), we see that:

$$A(u)|\Omega\rangle = (u+i/2)^L|\Omega\rangle, \qquad D(u)|\Omega\rangle = (u-i/2)^L|\Omega\rangle, \tag{A.1.5}$$

$$C(u)|\Omega\rangle = 0, \qquad B(u)|\Omega\rangle = |\dots\rangle\,. \tag{A.1.6}$$

So now we know that the vacuum state $|\Omega\rangle$ is an eigenvector for $t(u)$:

$$t(u)|\Omega\rangle = \left((u+i/2)^L + (u-i/2)^L\right)|\Omega\rangle\,. \tag{A.1.7}$$

We now need to build the other eigenstates. The fact that $C(u)|\Omega\rangle = 0$ suggests us to treat $C(u)$ as a lowering operator. Therefore, we may think of $B(u)$ as a raising operator, with eigenstates of $T(u)$ being of the form:

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_i)|\Omega\rangle\,. \tag{A.1.8}$$

To prove this, we need to check the commutation relations between the various operators, and in particular we need that $[B(u), t(u)] \sim t(u)$. Luckily, all the algebra we need is contained into the RTT relation (2.2.11). In particular, by using the matrix forms for T and R and comparing the entries, we obtain the following relations:

$$\begin{aligned} A(u)B(v) &= \frac{u-v-i}{u-v}B(v)A(u) + \frac{i}{u-v}B(u)A(v)\,, \\ D(u)B(v) &= \frac{u-v+i}{u-v}B(v)D(u) - \frac{i}{u-v}B(u)D(v)\,, \\ B(u)B(v) &= B(v)B(u)\,. \end{aligned} \tag{A.1.9}$$

It is quite simple from here to prove that the state with one excitation, $B(v)|\Omega\rangle$, is an eigenstate of $t(u) = A(u) + D(u)$. In fact, we can easily compute $A(u)B(v)|\Omega\rangle$ and $D(u)B(v)|\Omega\rangle$, obtaining:

$$A(u)B(v)|\Omega\rangle = \frac{u-v-i}{u-v}(u+i/2)^L B(v)|\Omega\rangle + \frac{i}{u-v}(v+i/2)^L B(u)|\Omega\rangle\,, \tag{A.1.10}$$

$$D(u)B(v)|\Omega\rangle = \frac{u-v+i}{u-v}\,(u-i/2)^L\,B(v)|\Omega\rangle - \frac{i}{u-v}\,(v-i/2)^L\,B(u)|\Omega\rangle\,. \tag{A.1.11}$$

Therefore, we have that:

$$t(u)B(v)|\Omega\rangle = \left(\frac{(u-v-i)(u+i/2)^L + (u-v+i)(u-i/2)^L}{u-v}\right) B(v)|\Omega\rangle + \quad \text{(A.1.12)}$$

$$+ \left(\frac{i}{u-v}\left((v+i/2)^L - (v-i/2)^L\right)\right) B(u)|\Omega\rangle\,. \quad \text{(A.1.13)}$$

So, under the condition that the second row cancels in the above equation, i.e. that:

$$\left(\frac{v+i/2}{v-i/2}\right)^L = 1\,, \quad \text{(A.1.14)}$$

we get that $B(v)|\Omega\rangle$ is an eigenstate of the transfer matrix $t(u)$ with eigenvalue:

$$\tau(u) = \left(\frac{(u-v-i)(u+i/2)^L + (u-v+i)(u-i/2)^L}{u-v}\right)\,. \quad \text{(A.1.15)}$$

With a little more work, it is possible to generalize the above argument to states created by applying M creation operators B evaluated at different spectral parameters u_k, obtaining that they again are eigenstates of $t(u)$ with eigenvalues:

$$\tau(u) = \prod_{j=1}^{M} \frac{u-u_j-i}{u-u_j}(u+i/2)^L + (u-i/2)^L \prod_{j=1}^{M} \frac{u-u_j+i}{u-u_j}\,, \quad \text{(A.1.16)}$$

provided that the following *Bethe equations* hold:

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = \prod_{j\neq k}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}\,. \quad \text{(A.1.17)}$$

The solutions of the Bethe equations u_k are known as *Bethe roots*.

Of course, the cancellation of the unwanted terms given by the Bethe Equations (A.1.17) is an ansatz, but there are many other ways to see that this is indeed correct.

A.2 Derivation of eigenstates for the $Y(gl_3)$ spin chain

We have seen that the transfer matrix of the $Y(gl_3)$ spin chain in the fundamental representation is:

$$t(u) = \mathrm{tr}\, T(z) = A(z) + D_{11}(z) + D_{22}(z) = A(z) + \mathrm{tr}\mathbb{D}(z) \quad \text{(A.2.1)}$$

We want to check what are the conditions for which the states built from the two operators B_1, B_2 as:

$$|\Psi_M(\bar{u})\rangle = \prod_{i=1}^{M}\prod_{\beta_i=1}^{2} B_{\beta_i}(u_i)|0\rangle \quad \text{(A.2.2)}$$

are eigenstates of $t(u)$.

It turns out that the ansatz (A.2.2) does not work; one needs to consider linear combinations:

$$|\Psi_M(\bar{u})\rangle = \sum_{\beta_1 \dots \beta_M = 1}^{2} B_{\beta_1}(u_1) \dots B_{\beta_M}(u_M) F_{\beta_1 \dots \beta_M} |0\rangle\,, \tag{A.2.3}$$

where $F_{\beta_1 \dots \beta_M}$ are some complex coefficients. This notation can be simplified using the vectors $\mathbb{B}(u) = \begin{pmatrix} B_1(u) \\ B_2(u) \end{pmatrix}$. If $\mathbb{B}_1 \dots \mathbb{B}_M$ is the tensor product of M such vectors, then equation (A.2.3) can be written compactly as:

$$|\Psi_M(\bar{u})\rangle = \prod_{i=1}^{M} \mathbb{B}_i(u_i) \mathbb{F}(\bar{u}) |0\rangle\,, \tag{A.2.4}$$

where now $\mathbb{F}$ is a vector in the tensor product of M copies of $\mathbb{C}^2$. We now need to compute the action of (A.2.1) on these vectors.

Commutator with tr$\mathbb{D}$ Using the RTT relation for $Y(gl_3)$, it is possible to prove that (we indicate the trace on the 2×2 matrix $\mathbb{D}$ with the subscript 0):

$$\mathrm{tr}_0 \mathbb{D}_0(z) |\Psi_M(\bar{u})\rangle = \mathrm{tr}_0 \prod_{i=1}^{M} \mathbb{B}_i(u_i) \mathbb{D}_0 \mathcal{T}_0^{(M)}(z) \mathbb{F}(\bar{u}) |0\rangle + \text{unwanted terms} \tag{A.2.5}$$

where $\mathcal{T}_0^{(M)}(z) \equiv r_{0M}(z, u_M) \dots r_{01}(z, u_1)$ is a gl_2 monodromy matrix built via the gl_2 R-matrix r and the unwanted terms all contain at least one of the $\mathbb{B}$ evaluated at z. For now we will leave the unwanted terms alone.

The first term in the RHS of (A.2.5) is almost of the form we want; we just need to impose that $\mathbb{F}(\bar{u})$ is an eigenvector of the gl_2 transfer matrix:

$$\mathrm{tr}_0 \mathcal{T}_0^{(M)}(z) = \begin{pmatrix} \mathcal{A}^M(z) & \mathcal{B}^M(z) \\ \mathcal{C}^M(z) & \mathcal{D}^M(z) \end{pmatrix}. \tag{A.2.6}$$

We know exactly how to get them - via the Algebraic Bethe Ansatz for $Y(gl_2)$. Therefore, we impose that:

$$\mathbb{F}(\bar{u}) = \prod_{i=1}^{N} \mathcal{B}^M(v_i) |\Omega^M\rangle\,, \tag{A.2.7}$$

where $|\Omega^M\rangle$ is the vacuum vector for (A.2.6), and the *auxiliary* Bethe roots $v_1 \dots v_N$ satisfy the *auxiliary* Bethe equations:

$$\prod_{l=1}^{M} \frac{v_j - u_l + i}{v_j - u_l} = \prod_{k \neq j}^{N} \frac{v_j - v_k + i}{v_k - v_j + i}, \quad j = 1 \dots N \tag{A.2.8}$$

Under these conditions, we get that:

$$\mathrm{tr}\mathbb{D}(z)|\Psi_{M,N}(\bar{u},\bar{v})\rangle = \tau_D(z,\bar{u},\bar{v})|\Psi_{M,N}(\bar{u},\bar{v})\rangle + \text{unwanted terms} \tag{A.2.9}$$

Commutator with A Using again the RTT relations for gl_3, it is possible to show that:

$$A(z)|\Psi_{M,N}(\bar{u},\bar{v})\rangle = \tau_A(z,\bar{u},\bar{v})|\Psi_{M,N}(\bar{u},\bar{v})\rangle + \text{unwanted terms} \tag{A.2.10}$$

Combining, we see that:

$$t(z)|\Psi_{M,N}(\bar{u},\bar{v})\rangle = \tau(z,\bar{u},\bar{v})|\Psi_{M,N}(\bar{u},\bar{v})\rangle + \text{unwanted terms} \tag{A.2.11}$$

Then it turns out that the overall coefficient of the unwanted terms can be set to zero provided that $\bar{u} = (u_1 \dots u_M)$ satisfy the Bethe equations:

$$(u_k + i/2)^L = \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_j - u_k + i} \prod_{l=1}^{L} \frac{v_l - u_k + i}{v_l - u_k}, \quad k = 1 \dots M\,. \tag{A.2.12}$$

A.3 Fusion for transfer matrices

Fusion is a generalisation of the procedure used to build the quantum minors of section 3.7.1.

In particular, fusion takes m copies of the monodromy matrix (3.1.4), and computes their projection on the tensor product of m auxiliary spaces $\bigotimes_{i=1}^{m} \mathbb{C}^N$, opportunely symmetrised and/or antisymmetrised. This symmetrisation of auxiliary spaces can be described in terms of the Young tableaux appearing in the representation theory of the gl_N Lie algebra.

Young diagrams and Young tableaux

A *Young diagram* is a two-dimensional array of boxes, arranged in left-justified rows with row lenghts that are in non-increasing order.

The number of boxes in each row is defined as $\lambda = (\lambda_1, \lambda_2 \dots \lambda_N)$, where N is the number of rows and $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N$, and specifies completely the Young diagram[1].

A *Young Tableau* is any set of numbers that we put in the boxes of a Young diagram.

To index the boxes in a Young diagram, we will use Cartesian coordinates (s, a). Both s and a start from 0 at the top-left box; s increases moving to the right along columns, while a increases moving down along rows.

The Young Tableaux we use to fill Young diagrams for fusion are simply expressed in terms of these Cartesian coordinates; to the box of coordinates (s, a) we assign the number:

$$c_j = a - s\,, \tag{A.3.1}$$

where the order of the boxes j is given in figure A.1.

[1]This notation is used because Young tableaux are in one-to-one correspondence with highest weight representations of gl_N, which are determined uniquely by their weights λ_i.

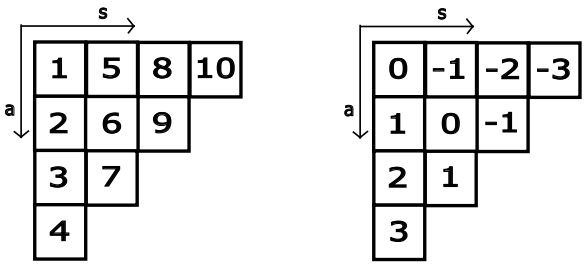


Figure A.1: Left: Column-ordering of boxes on the Young diagram λ. Right: $c_j = a - s$ associated with each box j.

Fused monodromy matrices

We now describe how to use Young tableaux in the fusion procedure. We start by defining $T^\lambda(u)$ as the monodromy matrix whose auxiliary space is $P^\lambda \left(\bigotimes_{i=1}^{|\lambda|} \mathbb{C}^N\right)$, where P^λ is the *Young symmetriser* associated to the Young diagram λ.

To obtain $T^\lambda(u)$ explicitly, we just need to generalise the results of the previous section; in fact, the antisymmetriser A_m we have used there is just the Young symmetriser associated to a Young diagram with a single vertical column of length m.

Any Young symmetriser P^λ with total number of boxes $|\lambda| = m$ can be obtained from the generalised R-matrix (3.7.1) using the following spectral parameters:

$$u_j = u + ic_j, \;\; j = 1 \dots m \tag{A.3.2}$$

where c_j are the numbers in the Young tableaux defined in (A.3.1). Then the fused monodromy matrix is:

$$T^\lambda(u) \equiv P^\lambda \left(T_1(u + ic_1) \dots T_{|\lambda|}(u + ic_{|\lambda|})\right) = T_{|\lambda|}(u + ic_{|\lambda|}) \dots T_1(u + ic_1) P^\lambda \tag{A.3.3}$$

The fused monodromy matrices form representation of $Y(gl_N)$, and the coproduct for them is given by (3.7.8).

Fused transfer matrices as generators of conserved charges

The fused monodromy matrices satisfy the generalised RTT relation:

$$R_{12}^{\lambda\lambda}(u - v) T_1^\lambda(u) T_2^\lambda(v) = T_2^\lambda(v) T_1^\lambda(u) R_{12}^{\lambda\lambda}(u - v) \tag{A.3.4}$$

where $R_{12}^{\lambda\lambda}$ is the fused R-matrix:

$$R_{AB}^{\lambda\lambda}(u) = P_A^\lambda P_B^\lambda \left(R_{a_1,b_1}(u + ic_1) \dots R_{a_{|\lambda|},b_{|\lambda|}}(u + ic_{|\lambda|})\right) \tag{A.3.5}$$

By taking the trace on the auxiliary spaces $1, 2$ in (A.3.4), it is easy to see that:

$$\left[t^\lambda(u), t^\lambda(v)\right] = 0, \;\; \forall \lambda \tag{A.3.6}$$

where $t^\lambda(u) \equiv \mathrm{tr}(T^\lambda(u))$ is the *fused transfer matrix* corresponding to the Young tableau λ. Therefore, the fused transfer matrices commute with themselves at different values of the spectral parameters, for each Young tableau λ.

This property can be generalised to the case where we have two different fused transfer matrices associated to different Young tableaux λ, μ. To prove this, we define the corresponding fused R-matrix as:

$$R^{\lambda\mu}(u,\theta) = P^\mu P^\lambda \left(\prod_{j=1}^{|\lambda|} \prod_{k=1}^{|\mu|} R_{a_j k}\left(u + ic_j^\lambda - ic_k^\mu\right) \right) \tag{A.3.7}$$

Then the fused monodromy matrices corresponding to any two Young diagrams λ, μ satisfy:

$$R_{12}^{\lambda\mu}(u-v) T_1^\lambda(u) T_2^\mu(v) = T_2^\mu(v) T_1^\lambda(u) R_{12}^{\lambda\mu}(u-v) \tag{A.3.8}$$

Taking the trace on auxiliary spaces $1, 2$, we conclude that:

$$\left[t^\lambda(u), t^\mu(v)\right] = 0, \ \ \forall \lambda, \mu \tag{A.3.9}$$

Hence, all fused transfer matrices commute with each other, forming a large class of commuting operators. The fundamental transfer matrix $t(u)$ corresponds to the Young diagram formed by a single box, and is part of this family. Furthermore, the fused antisymmetric transfer matrices t_a $a = 1 \dots N$ defined in section 3.7.1 are also part of this family, since t_a can be built via fusion using a Young diagram with a single column of a boxes.

A.3.1 Hirota and CBR equations

The T-system is composed of the fused transfer matrices we have just defined and two functional relations between them, which take the form of finite difference equations.

The first one is the *Hirota equation*. It is a relation between transfer matrices corresponding to *rectangular* Young diagrams, which are completely specified by their number of rows and columns. Therefore, we will denote λ_{rect} as (a, s), where a is the number of rows and s is the number of columns.

The Hirota equation reads:

$$t_{a,s} t_{a,s}^{[-2]} = t_{a+1,s}^{[-2]} t_{a-1,s} + t_{a,s+1} t_{a,s-1}^{[-2]} \tag{A.3.10}$$

where $t_{0,i} = 1, \ t_{a+N,s} \sim t_{a,s}$ and $t_{a,i} = 0, \ \forall a < 0$.

It is possible to use this relation repeatedly to obtain all rectangular transfer matrices $t_{a,s}$ from the subset of N elements $t_{a,1}, \ a = 1 \dots N$, which correspond to the totally antisymmetric transfer matrices (3.7.14). This is one of the reasons why quantum minors play a fundamental role in the T-system.

In fact, one can start by fixing $s = 1$ in (A.3.10), and read off $t_{a,2}$ for any a in terms of $t_{a+1,1}, t_{a,1}, t_{a-1,1}$. Then fixing $s = 2$ one can obtain $t_{a,3}$ in terms of $t_{a+1,2}, t_{a,2}, t_{a-1,2}, t_{a,1}$. This procedure can be then iterated for any s.

The Hirota equation admits the following gauge symmetry:

$$t_{(a,s)}(u) \to g^{i,j}_{(a,s)} t_{(a,s)}(u), \;\; i,j = 1,2 \tag{A.3.11}$$

where $g^{i,j}_{(a,s)}$ are 4 independent analytic functions of u who depend in a fixed way on the constants a, s. We can use the transformation (A.3.11) to impose a specific normalisation on the transfer matrices corresponding to rectangular Young diagrams.

The second relation between fused transfer matrices is the CBR formula:

$$t_\lambda(u) = \det_{1 \leqslant k,j \leqslant \lambda_1} t_{\lambda'_j + k - j, 1}(u - i(k-1)) \tag{A.3.12}$$

where λ'_j denotes the number of boxes in the j-th column of the Young diagram λ.

Appendix B

Technical Details for Chapter 5

B.1 Gelfand-Tsetlin patterns

A Gelfand-Tsetlin pattern is a collection of numbers that labels uniquely a state of a spin chain based on the Yangian $Y(gl_N)$[1]. This procedure is based on the chain of subalgebras $Y(gl_1) \subset Y(gl_2) \subset \dots Y(gl_{N-1}) \subset Y(gl_N)$.

We can take as a starting point the fact that gl_k has a gl_{k-1} subalgebra naturally identified with the subset of generators $E_{ij},\ i,j = 1 \dots k-1$, and this subalgebra can be decomposed into a direct sum of irreps. of gl_{k-1}:

$$R(E_{ij})^{\lambda}|_{k-1} = \bigoplus_{\mu} R(F)^{\mu} \tag{B.1.1}$$

where the LHS is the restriction of the irrep. R, of weights λ, of gl_k to its gl_{k-1} subalgebra, and on the RHS F are the generators of gl_{k-1} in a irrep. with weights μ.

This decomposition is unique and each μ can appear at most once, and furthermore it is subject to the branching rules:

$$\lambda_j \geqslant \mu_j \geqslant \lambda_{j+1} \tag{B.1.2}$$

Considering now the chain of subalgebras $gl_1 \subset gl_2 \subset \dots gl_{N-1} \subset gl_N$ and using the decomposition (B.1.1), we can start from gl_N and pick up a single μ irrep. of gl_{N-1} from the RHS of the decomposition (B.1.1), subject to the branching rules (B.1.2); then, we can decompose this gl_{N-1} irrep into a sum of gl_{N-2} irreps, pick one of them and continue until we reach a single gl_1 irrep. All the possible highest weights of these chains of irreps. of

[1]Technically one has to assume non-degenerateness of the spin chain; thanks to twisting, this is always true in the setting we consider.

subalgebras form the Gelfand-Tsetlin patterns for gl_N:

$$\begin{array}{ccccccc} \lambda_{N,1} & & \lambda_{N,2} & \dots & \lambda_{N,N} \\ & \lambda_{N-1,1} & \dots & \lambda_{N-1,N-1} & \\ & & \dots & & \\ & & \lambda_{1,1} & & \end{array} \tag{B.1.3}$$

where $\lambda_{k,i}$ is the $i-th$ weight of the irrep. of gl_k, subject to branching rules:

$$\lambda_{k,j} \geqslant \lambda_{k-1,j-1} \geqslant \lambda_{k,j+1} \tag{B.1.4}$$

The Gelfand-Tsetlin patterns can immediately be lifted to the Yangian. In fact, a similar chain of subalgebras is present in $Y(gl_N)$, and the Gelfand-Tsetlyn patterns for the $Y(gl_N)$ are built with the same philosophy as the ones for gl_N. The main difference is that now we will have a GT pattern for each site of the spin chain, that we will label via α[2]:

$$\begin{array}{ccccccc} \lambda^{\alpha}_{N,1} & & \lambda^{\alpha}_{N,2} & \dots & \lambda^{\alpha}_{N,N} \\ & \lambda^{\alpha}_{N-1,1} & \dots & \lambda^{\alpha}_{N-1,N-1} & \\ & & \dots & & \\ & & \lambda^{\alpha}_{1,1} & & \end{array} \tag{B.1.5}$$

subject to branching rules:

$$\lambda^{\alpha}_{k,j} \geqslant \lambda^{\alpha}_{k-1,j-1} \geqslant \lambda^{\alpha}_{k,j+1},\ \forall \alpha = 1 \dots L\,. \tag{B.1.6}$$

We find it convenient to label the gl_N weights as $\nu_i,\ i = 1 \dots N$ and introduce for the other weights in the GT pattern the notation $\mu^{\alpha}_{k,j} \equiv \lambda^{\alpha}_{n-k+j-1,j}$. In this notation the GT pattern becomes:

$$\begin{array}{ccccccccccccc} \nu^{\alpha}_1 & & \nu^{\alpha}_2 & & \nu^{\alpha}_3 & \dots & \nu^{\alpha}_{N-2} & & \nu^{\alpha}_{N-1} & & & & \nu^{\alpha}_N \\ & \mu^{\alpha}_{1,1} & & \mu^{\alpha}_{2,2} & & \dots & \mu^{\alpha}_{N-2,N-2} & & & & \mu^{\alpha}_{N-1,N-1} & & \\ & & \mu^{\alpha}_{2,1} & & \mu^{\alpha}_{3,2} & \dots & & & \mu^{\alpha}_{N-1,N-2} & & & & \\ & & & \mu^{\alpha}_{3,1} & & \dots & \mu^{\alpha}_{N-1,N-3} & & & & & & \\ & & & & & \dots & & & & & & & \\ & & & & & \mu^{\alpha}_{N-1,1} & & & & & & & \end{array} \tag{B.1.7}$$

[2]In principle there can be a different representation of $Y(gl_N)$ at each site. We will always assume that every site is in the same representation.

$\mu_k^\alpha = (\mu_{k,1}^\alpha \dots \mu_{k,k}^\alpha)$ is said the be the k-th dual diagonal of the GT pattern, since it is placed on the k-th bottom left to top right diagonal of (B.1.7). The dual diagonals are the natural objects labelling the separated variables.

Appendix C

Technical Details for chapter 6

C.1 Mapping $\langle\Psi_A|\hat{O}|\Psi_B\rangle$ to $\langle \mathsf{y}|\hat{O}|\mathsf{x}\rangle$

Our goal in this section is to prove the relation (5.3.30) which we repeat here for convenience

$$\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big|L_3;\mathbf{u}_3\Big]_{\Psi}=\sum_{\mathsf{xy}}\tilde{\Psi}_B(\mathsf{x})\Psi_A(\mathsf{y})\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big|L_3;\mathbf{u}_3\Big]_{\mathsf{xy}} \tag{C.1.1}$$

where we use the notation

$$\begin{aligned}&\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big|L_3;\mathbf{u}_3\Big]_{\Psi}=\frac{1}{\mathcal{N}}\times \\ &\Big[\Big\{\frac{\Delta_{\mathbf{u}_0\cup w}}{\Delta_{\mathbf{u}_0}}w^jD^3\Big\}_{j=0}^{L_0-1},\Big\{\frac{\Delta_{\mathbf{u}_1\cup w}}{\Delta_{\mathbf{u}_1}}w^jD^1\Big\}_{j=0}^{L_1-1},\Big\{\frac{\Delta_{\mathbf{u}_2\cup w}}{\Delta_{\mathbf{u}_2}}w^jD^{-1}\Big\}_{j=0}^{L_2-1},\Big\{\frac{\Delta_{\mathbf{u}_3\cup w}}{\Delta_{\mathbf{u}_3}}w^jD^{-3}\Big\}_{j=0}^{L_3-1}\Big],\end{aligned} \tag{C.1.2}$$

and

$$\Big[L_0;\mathbf{u}_0\Big|L_1;\mathbf{u}_1\Big|L_2;\mathbf{u}_2\Big|L_3;\mathbf{u}_3\Big]_{\mathsf{xy}}:=\frac{s_{\mathbf{L}}}{\Delta_\theta^2}\sum_k\mathrm{sign}(\sigma)\prod_{\alpha,a}\frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}}\prod_b\frac{\Delta_{\mathbf{u}_b\cup\mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}}\Big|_{\sigma_{a,\alpha}=k_{a,\alpha}-m_{\alpha,a}+a}. \tag{C.1.3}$$

Our starting point is the l.h.s. of (C.1.1). By explicitly writing each entry of the matrix we can pull out the measure factors $\mu_\alpha(\omega_{\alpha,a})$ and Q-functions $\tilde{Q}_1^B$ associated to the state $|\tilde{\Psi}_B\rangle$, as the finite-difference operators in the determinant do not act on them. Hence we obtain

$$\int t(\{w_{\alpha,a}\})\prod_{\alpha,a}\tilde{Q}_1^B(w_{\alpha,a})\mu_\alpha(w_{\alpha,a})\mathrm{d}w_{\alpha,a} \tag{C.1.4}$$

where

$$t(\{w_{\alpha,a}\})=\det_{(\alpha,a),(b,\beta)}f_b(w_{\alpha,a})w_{\alpha,a}^{\beta-1}Q_{1,1+a}\left(w_{\alpha,a}+\frac{i}{2}(3-2b)\right), \tag{C.1.5}$$

and

$$f_b(w)=\frac{\Delta_{\mathbf{u}_b\cup w}}{\Delta_{\mathbf{u}_b}}. \tag{C.1.6}$$

Let us note the range of indices in the above determinant formula. (α, a) takes values in the set

$$\{(1,1),(1,2),(2,1),\ldots,(L,2)\} \tag{C.1.7}$$

whereas (b,β) takes values in the set

$$\{(0,1),\ldots,(0,L_0),(1,1),\ldots,(1,L_1),\ldots,(3,L_3)\}\,. \tag{C.1.8}$$

Note that, in order to simplify this derivation, this notation is in contrast to the one used in the main text, where the rows of the determinant were labelled by (a,α) instead of (α,a). At the end we will convert back to the original ordering.

In [74] a determinant relation was used to extract the SoV matrix elements for the measure, which in our notation corresponds to the case $L_0 = L_3 = 0$ and $L_1 = L_2 = L$. For the general case we have the following updated determinant relation, valid for any two tensors $H_{a,\alpha,\beta}$ and $G_{a,\alpha,b}$, which reads

$$\det_{(\alpha,a),(b,\beta)} H_{a,\alpha,\beta}G_{a,\alpha,b} = \sum_{\sigma}(-1)^{|\sigma|}\left(\prod_b \det_{(\alpha,a)\in\sigma^{-1}(b),\beta_b} H_{a,\alpha,\beta_b}\right)\prod_{a,\alpha} G_{a,\alpha,\sigma_{a,\alpha}} \tag{C.1.9}$$

which is easy to derive. Here, σ is a permutation of

$$\{\underbrace{0,\ldots,0}_{L_0},\ldots,\underbrace{3,\ldots,3}_{L_3}\} \tag{C.1.10}$$

with $\sigma_{\alpha,a}$ denoting the number at position $a + (N-1)(\alpha-1)$ and

$$\sigma^{-1}(b) = \{(\alpha,a) : \sigma_{a,\alpha} = b\}\,. \tag{C.1.11}$$

We have $\beta_b \in \{1,\ldots,L_b\}$ and finally $|\sigma|$ denotes the number of elementary permutations needed to bring the set $\bigcup_b \sigma^{-1}(b)$ to the canonical ordering (C.1.7).

We now apply (C.1.9) to (C.1.4) by identifying

$$H_{a,\alpha,\beta} = w_{\alpha,a}^{\beta-1}, \quad G_{a,\alpha,b} = f_b(w_{\alpha,a})Q_{1,1+a}\left(w_{\alpha,a} + \frac{i}{2}(3-2b)\right)\,. \tag{C.1.12}$$

Notice that $\det_{(\alpha,a)\in\sigma^{-1}(b),\beta_b} H_{a,\alpha,\beta_b} = (-1)^{\frac{L_b}{2}(L_b-1)}\Delta_b$ where Δ_b denotes the Vandermonde determinant built out of $w_{\alpha,a}$ for which $\sigma_{\alpha,a} = b$, that is

$$\Delta_b := \prod_{(\alpha,a)<(\alpha',a')} (w_{\alpha,a} - w_{\alpha',a'}) \tag{C.1.13}$$

where $<$ is to be understood in lexicographical ordering as explained above. The result then reads

$$t(\{w_{\alpha,a}\} = s'_{\mathbf{L}}\sum_{\sigma}(-1)^{\sigma|}\prod_b \Delta_b \prod_{\alpha,a} f_{\sigma_{a,\alpha}}(w_{\alpha,a})Q_{1,1+a}(w_{\alpha,a} + \tfrac{i}{2} + is_{a,\alpha}) \tag{C.1.14}$$

where we have defined $s_{\alpha,a} = 1 - \sigma_{\alpha,a}$ and

$$s'_{\mathbf{L}} := \prod_b (-1)^{\frac{L_b}{2}(L_b-1)} \tag{C.1.15}$$

Using the explicit form of $f_b(w)$, which is (C.1.6), it is easy to verify that

$$\prod_b \Delta_b \prod_{\alpha,a} f_{\sigma_{a,\alpha}}(w_{\alpha,a}) = \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \tag{C.1.16}$$

and hence we obtain

$$t(\{w_{\alpha,a}\}) = s'_{\mathbf{L}} \sum_\sigma (-1)^{|\sigma|} \prod_b (-1)^{\frac{L_b}{2}(L_b-1)} \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \prod_{\alpha,a} Q_{1,1+a}(w_{\alpha,a} + \tfrac{i}{2} + i s_{\alpha,a}) \,. \tag{C.1.17}$$

We now symmetrise[1] over the integration variables $w_{\alpha,1}$ and $w_{\alpha,2}$. The only factor in (C.1.4) not invariant under this operation is $t(\{w_{\alpha,a}\})$, so symmetrising it gives

$$\operatorname*{sym}_{w_{\alpha,1} \leftrightarrow w_{\alpha,2}} t(\{w_{\alpha,a}\}) = \frac{s'_{\mathbf{L}}}{2^L} \sum_\sigma (-1)^{|\sigma|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \prod_\alpha F_\alpha^{s_{\alpha,1} s_{\alpha,2}} \,, \tag{C.1.18}$$

where

$$F_\alpha^{s_{\alpha,1} s_{\alpha,2}} = \det_{1 \leqslant a,a' \leqslant 2} Q_{1,1+a}(w_{\alpha,a'} + \tfrac{i}{2} + i s_{\alpha,a'}) \,. \tag{C.1.19}$$

We will now derive this relation. We introduce the expression

$$h_{\sigma,\alpha} := (-1)^{|\sigma|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} Q_{1,2}(w_{\alpha,1} + \tfrac{i}{2} + i s_{\alpha,1}) Q_{1,3}(w_{\alpha,2} + \tfrac{i}{2} + i s_{\alpha,2}) \,. \tag{C.1.20}$$

Consider the interchange of $w_{\alpha,1}$ and $w_{\alpha,2}$. This produces

$$h_{\sigma,\alpha} \to (-1)^{|\sigma|} \prod_b \frac{\bar{\Delta}_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} Q_{1,2}(w_{\alpha,2} + \tfrac{i}{2} + i s_{\alpha,1}) Q_{1,3}(w_{\alpha,1} + \tfrac{i}{2} + i s_{\alpha,2}) \,. \tag{C.1.21}$$

where $\bar{\Delta}$ denotes that we have interchanged $w_{\alpha,1}$ and $w_{\alpha,2}$ inside the Vandermonde determinant.

There are two possible types of σ. Either $\sigma_{\alpha,1} = \sigma_{\alpha,2} = c$ for some $c \in \{0,1,2,3\}$ or not. First suppose $\sigma_{\alpha,1} = \sigma_{\alpha,2} = c$. Then $s_{\alpha,1} = s_{\alpha,2}$ and, since $(\alpha,1)$ and $(\alpha,2)$ are adjacent to each other in the properly ordered set (C.1.7) we have

$$\bar{\Delta}_{\mathbf{u}_c \cup w_{\sigma^{-1}(c)}} = -\Delta_{\mathbf{u}_c \cup w_{\sigma^{-1}(c)}}, \quad \bar{\Delta}_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}} = \Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}, \; b \neq c \,. \tag{C.1.22}$$

[1] To avoid confusion, for any function $f(u,v)$ we define the symmetrisation of f over u and v as being $\operatorname*{sym}_{u \leftrightarrow v} f(u,v) = \frac{1}{2}\left(f(u,v) + f(v,u)\right)$, as used in Sections 5.2 and 5.3 of [74].

Hence, after exchanging $w_{\alpha,1}$ and $w_{\alpha,2}$ for such a σ we obtain

$$h_{\sigma,\alpha} \to -(-1)^{|\sigma|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} Q_{1,2}(w_{\alpha,2} + \tfrac{i}{2} + is_{\alpha,2}) Q_{1,3}(w_{\alpha,1} + \tfrac{i}{2} + is_{\alpha,1}) \,. \quad \text{(C.1.23)}$$

Hence for such a σ, after symmetrising over $w_{\alpha,1}$ and $w_{\alpha,2}$ we obtain

$$\operatorname*{sym}_{w_{\alpha,1} \leftrightarrow w_{\alpha,2}} h_{\sigma,\alpha} = \frac{1}{2}(-1)^{|\sigma|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} F_\alpha^{s_{\alpha,1} s_{\alpha,2}} \,. \quad \text{(C.1.24)}$$

We now consider the case of σ such that $\sigma_{\alpha,1} = c_1$, $\sigma_{\alpha,2} = c_2$ with $c_1 \neq c_2$. Note that there is another (unique) permutation $\tilde{\sigma}$ with $\tilde{\sigma}_{\alpha,1} = c_2$, $\tilde{\sigma}_{\alpha,2} = c_1$ and $\tilde{\sigma}_{\alpha',a'} = \sigma_{\alpha',a'}$ for all other pairs (α', a'). Clearly since these two permutations are equivalent up to interchanging a single pair we have $(-1)^{|\sigma|} = -(-1)^{|\tilde{\sigma}|}$. Denote $\tilde{s}_{\alpha,a} = 1 - \tilde{\sigma}_{\alpha,a}$. Then it immediately follows that under exchanging $w_{\alpha,1}$ and $w_{\alpha,2}$ we have

$$h_{\sigma,\alpha} \to -(-1)^{|\tilde{\sigma}|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\tilde{\sigma}^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} Q_{1,2}(w_{\alpha,2} + \tfrac{i}{2} + i\tilde{s}_{\alpha,2}) Q_{1,3}(w_{\alpha,2} + \tfrac{i}{2} + i\tilde{s}_{\alpha,1}) \,. \quad \text{(C.1.25)}$$

Hence, after symmetrisation we have

$$\operatorname*{sym}_{w_{\alpha,1} \leftrightarrow w_{\alpha,2}} (h_{\sigma,\alpha} + h_{\tilde{\sigma},\alpha}) = \frac{1}{2} \left((-1)^{|\sigma|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup w_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} F_\alpha^{s_{\alpha,1} s_{\alpha,2}} + \sigma \leftrightarrow \tilde{\sigma} \right) . \quad \text{(C.1.26)}$$

Of course, the conclusion is unchanged if $h_{\sigma,\alpha}$ is multiplied by any function independent of (α, a) and hence (C.1.18) immediately follows by sequentially symmetrising over $(\alpha, 1)$ and $(\alpha, 2)$ for $\alpha = 1, 2, \ldots, L$.

We now put (C.1.18) under the integration (C.1.4) and compute the integral by residues, closing the contour in the upper-half plane. This produces a sum over poles at the locations $w_{\alpha,a} = \mathsf{x}_{\alpha,a} = \theta_\alpha + i(\mathbf{s} + n_{\alpha,a})$, with $n_{\alpha,a}$ ranging over all non-negative integers. If all $n_{\alpha,a}$ are distinct for a fixed α we can use the symmetry of the integrand to remove a factor of 2 for each α and restrict the summation to $n_{\alpha,1} \geqslant n_{\alpha,2}$. If some $n_{\alpha,a}$ coincide for a fixed α then removing the 2^L factor will result in an overcounting which we must compensate for, by introducing the factor M_α.

As a result, we obtain

$$\begin{aligned} &\sum_{n_{\alpha,1} \geqslant n_{\alpha,2} \geqslant 0} \prod_\alpha \frac{1}{M_\alpha} \prod_{\alpha,a} \tilde{Q}_1^B(\mathsf{x}_{\alpha,a}) \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \\ &\qquad s'_{\mathbf{L}} \sum_\sigma (-1)^{|\sigma|} \prod_b \frac{\Delta_{\mathbf{u}_b \cup \mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \prod_\alpha \det_{1 \leqslant a,a' \leqslant 2} Q_{1,a+1}(\mathsf{x}_{\alpha,a'} + \tfrac{i}{2} + is_{\alpha,a}) \,. \end{aligned} \quad \text{(C.1.27)}$$

We now compare with the general expression (C.1.1). We see that in order for a term

$$\Big[L_0; \mathbf{u}_0 \Big| L_1; \mathbf{u}_1 \Big| L_2; \mathbf{u}_2 \Big| L_3; \mathbf{u}_3 \Big]_{\mathsf{xy}} \quad \text{(C.1.28)}$$

in the summand of the r.h.s. to be non-zero it must be possible to write each $\mathsf{y}_{\alpha,a} = \theta_\alpha + i(\mathbf{s} + m_{\alpha,a} + 1 - a)$, for each fixed α, as

$$\mathsf{y}_{\alpha,a} = \mathsf{x}_{\alpha,\rho_a^\alpha} + i s_{\alpha,\rho_a^\alpha},, \quad a = 1, 2 \tag{C.1.29}$$

where ρ^α is a permutation of $\{1, 2\}$ and $\rho_a^\alpha := \rho^\alpha(a)$ and hence we require

$$m_{\alpha,a} + 1 - a = n_{\alpha,\rho_a^\alpha} + s_{\alpha,\rho_a^\alpha} . \tag{C.1.30}$$

Since each of the numbers $m_{\alpha,a} + 1 - a$ must be distinct, as otherwise the determinant built from $Q_{1,1+a}$ will vanish, there is a unique permutation ρ^α (if such a permutation exists) for which (C.1.30) holds. If such a permutation does not exist then the matrix element (C.1.28) vanishes. The permutation ρ^α amounts to sorting the set $\{n_{\alpha,1} + s_{\alpha,1}, n_{\alpha,2} + s_{\alpha,2}\}$ and so we should keep track of the sign of this permutation. Hence, for a fixed permutation σ we read off the following contribution to (C.1.28)

$$s'_{\mathbf{L}}(-1)^{\frac{L}{2}(L-1)(N-1)} \frac{(-1)^{|\sigma|}}{\Delta_\theta^2} \prod_\alpha \frac{(-1)^{|\rho^\alpha|}}{M_\alpha} \prod_{\alpha,a} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \prod_b (-1)^{\frac{L_b}{2}(L_b-1)} \frac{\Delta_{\mathbf{u}_b \cup \mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \Big|_{m_{\alpha,a} = n_{\alpha,\rho_a^\alpha} - \sigma_{\alpha,\rho_a^\alpha} + a} \tag{C.1.31}$$

where we have included the corresponding normalisation factor $\mathcal{N}$. Finally, in order to determine (C.1.28) we note that for a given set of $\mathsf{x}_{\alpha,a}$ and $\mathsf{y}_{\alpha,a}$ there can be many different σ for which the relation (C.1.30) holds and we must sum over all such σ in (C.1.31) in order to obtain (C.1.28). When there is a degeneracy in $n_{\alpha,a}$ for fixed α there are multiple σ that give the same result. Their number is exactly M_α, so we can simplify the expression by only summing over k inequivalent permutations of $n_{\alpha,a}$ within each α. We denote such permutations $\mathrm{perm}_\alpha n$ and hence obtain

$$\langle \mathsf{y}|\hat{O}|\mathsf{x}\rangle = s_{\mathbf{L}}(-1)^{\frac{L}{2}(L-1)(N-1)} \sum_k \frac{(-1)^{|\sigma|}}{\Delta_\theta^2} \prod_{\alpha,a} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \prod_b (-1)^{\frac{L_b}{2}(L_b-1)} \frac{\Delta_{\mathbf{u}_b \cup \mathsf{x}_{\sigma^{-1}(b)}}}{\Delta_{\mathbf{u}_b}} \Big|_{\sigma_{\alpha,a} = k_{\alpha,a} - m_{\alpha,a} + a} . \tag{C.1.32}$$

Moving back to the original ordering of the rows of the determinant introduces a sign $(-1)^{\frac{L}{4}(N^2-3N+2)}$ which combines with $s'_{\mathbf{L}}$ to produce $s_{\mathbf{L}}$ given by

$$s_{\mathbf{L}} := (-1)^{\frac{LN}{4}(L-1)(N-1) + \sum_{n=0}^N L_n} . \tag{C.1.33}$$

Finally, the above argument is rigourous in the finite-dimensional setting. To pass to the infinite-dimensional case we notice that the matrix elements are block diagonal with each block having finite size. The spin $\mathbf{s}$ enters each block as a universal polynomial pre-factor. Then, each block is fixed by analysing a finite-number of finite-dimensional representations and the matrix elements can be analytically continued to values of $\mathbf{s}$ not being negative half-integers. So the matrix elements we found are valid in the infinite-dimensional case as well for generic $\mathbf{s}$. Note that since the SoV basis vectors are polynomial in the spin $\mathbf{s}$ our formula for the SoV matrix elements of the principal operators are valid for all values of $\mathbf{s}$.

This completes the derivation. The gl_N case is identical up to extending the range of indices $\{1,2\}$ to $\{1,\ldots,N-1\}$ but can be carried out in exactly the same way as was demonstrated for the measure in [74]. Note that for a function $f(w_{\alpha,1},\ldots,w_{\alpha,N-1})$ we define the symmetrisation over $w_{\alpha,1},\ldots,w_{\alpha,N-1}$ as

$$\frac{1}{(N-1)!}\sum_{p\in\sigma_{N-1}} f(w_{\alpha,p(1)},\ldots,w_{\alpha,p(N-1)}) \tag{C.1.34}$$

where σ_{N-1} denotes the permutation group on $N-1$ letters.

Appendix D

Technical details for Chapter 5

D.1 Alternative derivation

In this appendix we present an alternative derivation of (5.3.15) which avoids using Cramer's rule and hence avoids expressing the integral of motion eigenvalues $I^A_{b',\beta'}$ as a ratio with a potentially vanishing denominator.

Our starting point is the following trivial equality

$$[(b',\beta') \to \mathcal{O}^\dagger_A] = 0\,. \tag{D.1.1}$$

We then expand out $\mathcal{O}^\dagger_A$

$$\mathcal{O}^\dagger_A = \sum_{b,\beta}(-1)^b I^A_{b,\beta} w^{\beta-1} D^{N-2b} + \sum_{r=0}^{N} \chi_r \mathcal{O}^\dagger_{(r)} \tag{D.1.2}$$

and notice a number of cancellations. Indeed, in the sum

$$\sum_{b,\beta}(-1)^b I^A_{b,\beta}[(b',\beta') \to w^{\beta-1} D^{N-2b}] \tag{D.1.3}$$

only a single term will survive and it is precisely $(-1)^{b'} I^A_{b',\beta'}[(b',\beta') \to w^{\beta'-1} D^{N-2b'}]$. This is a result of the anti-symmetry of the determinant as all other terms in the sum (D.1.3) will produce two identical columns in the determinant and hence vanish.

As such we obtain the relation

$$(-1)^b[(b',\beta') \to w^{\beta'-1} D^{N-2b'}] I^A_{b',\beta'} = -\sum_{r=0}^{N}[(b',\beta') \to \mathcal{O}^\dagger_{(r)}] \tag{D.1.4}$$

and see that the coefficient of $I^A_{b',\beta'}$ is precisely $(-1)^{b'}\mathcal{N}\langle\Psi_A|\tilde{\Psi}_B\rangle$. From here on the derivation proceeds exactly as in section 5.3. Since $I^A_{b',\beta'}$ is the eigenvalue of the integral of motion

$\hat{I}_{b',\beta'}$ on the state $\langle\Psi_A|$ we can replace $\langle\Psi_A|\tilde{\Psi}_B\rangle I^A_{b',\beta'}$ with $\langle\Psi_A|\hat{I}_{b',\beta'}|\tilde{\Psi}_B\rangle$. Then, we expand $\hat{I}_{b',\beta'}$ into a sum over characters χ_r to obtain

$$\sum_{r=0}^{N}\chi_r\langle\Psi_A|\hat{I}_{b',\beta'}|\tilde{\Psi}_B\rangle = \frac{(-1)^{b'+1}}{\mathcal{N}}\sum_{r=0}^{N}[(b',\beta')\to\mathcal{O}^\dagger_{(r)}]\,. \tag{D.1.5}$$

Finally, we equate the coefficients of the characters χ_r on both sides (character projection), which was justified in section 5.3, and obtain the result

$$\langle\Psi_A|\hat{I}^{(r)}_{b',\beta'}|\tilde{\Psi}_B\rangle = \frac{(-1)^{b'+1}}{\mathcal{N}}[(b',\beta')\to\mathcal{O}^\dagger_{(r)}]\,. \tag{D.1.6}$$

The derivation presented here is valid for any transfer matrix eigenstate $\langle\Psi_A|$ and any factorisable state $|\tilde{\Psi}_B\rangle$. These classes of states are enough to completely constrain the SoV matrix elements $\langle\mathsf{y}|\hat{I}^{(r)}_{b',\beta'}|\mathsf{x}\rangle$ (as is proven in Appendix C.1) and hence the formula (D.1.6) is valid for any two factorisable states $\langle\Psi_A|$ and $|\tilde{\Psi}_B\rangle$.

D.2 SoV basis

In this section we will demonstrate that knowledge of the structure of the SoV basis and the FSoV approach allows one to derive the form of Sklyanin's **B** operator.

We start by defining the SoV ground states $|0\rangle$ and $\langle 0|$ which correspond to the constant polynomial 1. These states satisfy the following properties

$$T_{j1}(\theta_\alpha+i\mathbf{s})|0\rangle = 0 = T_{1j}(\theta_\alpha+i\mathbf{s})|0\rangle = 0,\quad j=1,\dots,N,\quad \alpha=1,\dots,L\,. \tag{D.2.1}$$

We can now follow the logic of [70] and build vectors by action of transfer matrices on $\langle 0|$ and $|0\rangle$. The key idea of [70] is that if such vectors form a basis then it is automatically an SoV basis since the transfer matrix wave functions will factorise. We will choose the following set of transfer matrices

$$\mathbb{T}^*_\mu(u) := \det_{1\leqslant j,k\leqslant\mu_1}\mathbb{T}_{\mu'_j-j+k,1}\left(u-\frac{i}{2}\left(\mu'_1-\mu_1-\mu'_j+j+k-1\right)\right) \tag{D.2.2}$$

where $\mathbb{T}_{a,1}(u)$ are the transfer matrices in anti-symmetric representations and μ denotes an integer partition (Young diagram)

$$\mu=(\mu_1,\dots,\mu_{N-1},0) \tag{D.2.3}$$

and μ'_j denotes the height of the j-th column of the Young diagram. The states $|\mathsf{y}\rangle$ are then constructed as

$$|\mathsf{y}\rangle\propto\prod_{\alpha=1}^{L}\mathbb{T}^*_{\mu^\alpha}\left(\theta_\alpha+i\mathbf{s}+\frac{i}{2}\left(\mu^\alpha_1-\mu^{\alpha,\prime}_1\right)\right)|0\rangle \tag{D.2.4}$$

and we label the constructed states by the L Young diagrams μ^α, $\alpha=1,\dots,L$.

We also construct a set of left vectors

$$\langle \mathsf{x}| \propto \langle 0| \prod_{\alpha=1}^{L} \prod_{j=1}^{N-1} \mathbb{T}_{N-1,s_j^\alpha} \left(\theta_\alpha + i\mathbf{s} - \frac{i}{2}(N - s_j^\alpha - 1) \right) \tag{D.2.5}$$

where $(N-1,s)$ denotes the Young diagram of height $N-1$ and width s, that is $\mu = (\underbrace{s,\dots,s}_{N-1},0)$ and the corresponding transfer matrix is defined by the Cherednik-Bazhanov-Reshetikhin (CBR) formula

$$\mathbb{T}_\mu(u) = \det_{1\leqslant j,k\leqslant \mu_1} \mathbb{T}_{\mu'_j - j + k,1} \left(u - \frac{i}{2}\left(\mu'_1 - \mu_1 - \mu'_j + j + k - 1\right) \right) . \tag{D.2.6}$$

We now note two key properties of the constructed set of vectors. First, they are linearly independent. This was proven in [73] and the argument relies on the fact that the twist matrix (4.4.4) can be deformed slightly with $N-1$ parameters $w_1,\dots,w_{N-1}$. Then, in the limit where all w_i are sent sequentially to infinity the constructed set of vectors reduce to eigenvectors of the so-called Gelfand-Tsetlin algebra [73], a key object in representation theory. Furthermore, the Gelfand-Tsetlin algebra has non-degenerate spectrum and it was shown in [73] that a basis of eigenvectors are given by (D.2.5) and (D.2.4) in the above-described limit. Hence, the vectors (D.2.5) and (D.2.4) form a basis, and the transfer matrix wave functions are guaranteed to factorise.

The next key property is that the constructed set of vectors are independent of the twist eigenvalues. This follows from the fact that all transfer matrices in our chosen reference frame have the structure

$$\mathbb{T}^*_\mu(u) = \mathbb{T}^{*,0}_\mu(u) + \sum_{r=0}^{N} \cdots \times \chi_r T_{r1}(u) \tag{D.2.7}$$

and

$$\mathbb{T}_{N-1,s}(u) = \mathbb{T}^0_{N-1,s}(u) + \sum_{r=0}^{N} \chi_r T_{1,r}(u) \times \dots \tag{D.2.8}$$

where $\mathbb{T}^0_\mu(u)$ denotes a part which is independent of the twist eigenvalues. The property (D.2.1) then ensures that the twist-dependent part of the transfer matrices never contributes, see [72–74].

We now exploit known relations for transfer matrices in terms of Baxter Q-functions. The transfer matrix eigenvalues admit the form

$$\langle \Psi|\mathbb{T}^*_{\mu^\alpha}\left(\theta_\alpha + i\mathbf{s} + \frac{i}{2}\left(\mu_1^\alpha - \mu_1^{\alpha,\prime}\right) \right) \propto \frac{\det_{1\leqslant a,a'\leqslant N-1} Q^{a+1}\left(\mathsf{y}_{\alpha,a'} + \frac{i}{2}(N-2) \right)}{\det_{1\leqslant a,a'\leqslant N-1} Q^{a+1}\left(\theta_\alpha + i\mathbf{s} + \frac{i}{2}(N-2k) \right)} \langle \Psi| \tag{D.2.9}$$

with $\mathsf{y}_{\alpha,a} = \theta_\alpha + i(\mathbf{s} + \mu_a^\alpha + 1 - a)$ and

$$\prod_{\alpha=1}^{L}\prod_{a=1}^{N-1}\mathbb{T}_{N-1,s_a^\alpha}\left(\theta_\alpha + i\mathbf{s} - \frac{i}{2}(N - s_j^\alpha - 1)\right)|\Psi\rangle \propto \prod_{\alpha=1}^{L}\prod_{a=1}^{N-1}\frac{Q_1(\theta_\alpha + i\mathbf{s} + is_a^\alpha)}{Q_1(\mathsf{x}_{\alpha,a})}|\Psi\rangle \quad \text{(D.2.10)}$$

where $\mathsf{x}_{\alpha,a} = \theta_\alpha + i(\mathbf{s} + s_a^\alpha)$.

We can now write down the wave functions. By normalising $\langle\Psi|$ and $|\Psi\rangle$ appropriately we have

$$\langle\Psi|\mathsf{y}\rangle = \prod_{\alpha=1}^{L}\det_{1\leqslant a,a'\leqslant N-1} Q^{j+1}\left(\mathsf{y}_{\alpha,a} + \frac{i}{2}(N-2)\right). \quad \text{(D.2.11)}$$

Similarly, we have

$$\langle\mathsf{x}|\Psi\rangle = \prod_{\alpha=1}^{L}\prod_{a=1}^{N-1} Q_1(\mathsf{x}_{\alpha,a}) \quad \text{(D.2.12)}$$

Since the proposed sets of vectors form a basis we can write the scalar product between two transfer matrix eigenstates as

$$\langle\Psi_A|\Psi_B\rangle = \sum_{\mathsf{x},\mathsf{y}}\Psi_B(\mathsf{x})\mathcal{M}_{\mathsf{y},\mathsf{x}}\Psi_A(\mathsf{y})\,. \quad \text{(D.2.13)}$$

We now turn to the FSoV construction which allows us to extract the measure (6.7.15) in the two SoV bases. This is just a special case of the formula (5.3.30). Since the SoV bases are independent of twist, the character projection trick is valid and all of the techniques developed earlier in sections 5.3 and 5.4 can be carried out. In particular, we can compute correlation functions of multi-insertions of principal operators. Following the logic of section 5.4.3 we see that there is a distinguished operator diagonalised in the basis $|\mathsf{x}\rangle$ which then must also be diagonalised in the basis $\langle\mathsf{x}|$ defined in (D.2.5). Hence, we have obtained Sklyanin's $\mathbf{B}$ operator, and the basis diagonalising it, starting solely from the FSoV approach and the knowledge of the SoV basis.

D.3 Existence of basis of Bethe algebra eigenstates

In this appendix we will prove the existence of the decomposition used in (5.3.12) which states that one can write a resolution of the identity as

$$1 = \sum_A |\Psi_A\rangle\langle\bar{\Psi}_A| \quad \text{(D.3.1)}$$

where each $|\Psi_A\rangle$ is a joint eigenvector of the transfer matrices $t_1(u)\,, t_2(u)$, $\langle\bar{\Psi}_A|$ is defined by the property

$$\langle\bar{\Psi}_A|\Psi_B\rangle = \delta_{AB} \quad \text{(D.3.2)}$$

and the index A in the sum labels all transfer matrix eigenstates.

Outline of proof The proof is very similar in spirit to our proof in the paper [74] showing that eigenstates of the SoV B operator form a basis of the representation space. An outline of the steps we will take are as follows. We will begin by decomposing the representation space $\mathcal{H}$ into a direct sum of finite-dimensional spaces $\mathcal{H}_k$, $k \in \mathbb{Z}_{\geqslant 0}$ from which the identity operator inherits the decomposition

$$1 = \sum_{k \geqslant 0} 1_k \tag{D.3.3}$$

where 1_k denotes the restriction of the identity operator to $\mathcal{H}_k$. We will then prove that each $\mathcal{H}_k$ admits a basis of transfer matrix eigenstates and as a result we can write

$$1_k = \sum_{A_k} |\Psi_{A_k}\rangle\langle\bar{\Psi}_{A_k}| \tag{D.3.4}$$

where the sum is over a finite number of certain transfer matrix eigenstates enumerated by A_k to be specified later. Hence, the decomposition (D.3.1) holds with

$$1 = \sum_{k \geqslant 0} \sum_{A_k} |\Psi_{A_k}\rangle\langle\bar{\Psi}_{A_k}| \,. \tag{D.3.5}$$

Proof The representation space of the gl_3 spin chain is the space of polynomials in x_α, y_α, $\alpha = 1, \ldots, L$. By definition, a vector in this space is a finite linear combination of monomials

$$\prod_{\alpha=1}^{L} x_\alpha^{n_\alpha} y_\alpha^{m_\alpha}, \quad n_\alpha, m_\alpha \in \mathbb{Z}_{\geqslant 0} \,. \tag{D.3.6}$$

In order to perform $\mathsf{SL}(3)$ (group)-valued linear transformations the representation space must be extended from polynomials to analytic functions which can be represented as power series in the above variables convergent in some neighbourhood of the origin, see [74] for a discussion and examples. Hence, in order to show that a collection of vectors form a basis of the representation space we must show that any such analytic function can be written as an infinite series in those vectors with finite coefficients.

Note that there are legitimate questions about convergence of such infinite series and the existence of the corresponding finite coefficients. Indeed even if one manages to construct a basis for the original space of polynomials it does not necessarily extend to a basis of the space of analytic functions.

Let us introduce the following operator $\mathcal{E} = -\mathcal{E}_{11} - L\,\mathbf{s}\times 1$ of the global Cartan subalgebra of $U(\mathfrak{gl}(3))$, see (5.6.2) for the definition of $\mathcal{E}_{11}$. A direct calculation yields

$$\mathcal{E} \prod_{\alpha=1}^{L} x_\alpha^{n_\alpha} y_\alpha^{m_\alpha} = \left(\sum_{\alpha=1}^{L} n_\alpha + m_\alpha \right) \prod_{\alpha=1}^{L} x_\alpha^{n_\alpha} y_\alpha^{m_\alpha} \,. \tag{D.3.7}$$

Hence, we see that the spectrum of this operator is bounded from below and furthermore each eigenvalue is non-negative. Hence the representation space $\mathcal{H}$ decomposes into a direct

sum of eigenspaces $\mathcal{H}_k$ corresponding to the eigenvalue k

$$\mathcal{H} = \bigoplus_{k\geqslant 0} \mathcal{H}_k, \quad \mathcal{H}_k := \{v \in \mathcal{H} \,:\, \mathcal{E}v = k\, v\}\,. \tag{D.3.8}$$

Clearly, each $\mathcal{H}_k$ is finite-dimensional as there are only finitely many ways to write a given non-negative integer k as a sum of non-negative integers n_α and m_α. Clearly the decomposition (D.3.8) is valid for the space of polynomials. However, since the space of regular at the origin analytic functions admit a Taylor expansion into the variables x_α, y_α the decomposition (D.3.8) is also valid for this enlarged space. To complete the proof it remains to show that each $\mathcal{H}_k$ admits a basis of transfer matrix eigenstates. To this end it suffices to show that they have a distinct set of eigenvalues in each subspace the number of which matches the dimension of that space. Since transfer matrix eigenvalues are algebraic functions of the twist parameters it suffices to prove there are sufficiently many for some special value of the twist to prove that it is true generically.

We will proceed as follows. Consider the transfer matrices with diagonal twist $g = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. As a result the transfer matrices commute with $\mathcal{E}$ and preserve the subspaces $\mathcal{H}_k$. We consider the singular twist limit $\lambda_1 \gg \lambda_2 \gg \lambda_3$. In this limit $t_1(u)$ and $t_2(u)$ reduce to the generators $\mathsf{GT}_1(u)$ and $\mathsf{GT}_2(u)$ of the Gelfand-Tsetlin subalgebra of the Yangian whose properties are well-understood [43]. Let $\sigma \in \mathbb{C}$ be a generic parameter and consider the combination

$$t(u) = \frac{1}{\lambda_1} t_1(u) + \frac{\sigma}{\lambda_1\lambda_2} t_2(u)\,. \tag{D.3.9}$$

In the singular twist limit $t(u) \to t^{\mathsf{GT}}(u) := \mathsf{GT}_1(u) + \sigma\, \mathsf{GT}_2(u)$. We will prove that this operator has distinct eigenstates in each Cartan subspace, and hence so does $t(u)$ and hence so do the family $t_1(u)$ and $t_2(u)$.

The eigenvectors of $t^{\mathsf{GT}}(u)$ are well-understood for finite-dimensional representations where $\mathbf{s} \in \{0, -\frac{1}{2}, -1, \dots\}$. In particular, they are all polynomial functions of the spin $\mathbf{s}$ with each eigenvalue also being a polynomial in $\mathbf{s}$. What is not obvious is that each eigenvector remains an eigenvectors when analytically continued to non half-integer values of $\mathbf{s}$. Let us consider the expression

$$t^{\mathsf{GT}}(u)|\Psi_{\mathbf{s}}\rangle - \tau^{\mathsf{GT}}(u)|\Psi_{\mathbf{s}}\rangle \tag{D.3.10}$$

where $|\Psi_{\mathbf{s}}\rangle$ is an eigenvectors of $t^{\mathsf{GT}}(u)$ for $\mathbf{s} \in \{0, -\frac{1}{2}, -1, \dots\}$. For such values of $\mathbf{s}$ this expression equates to zero. However, for generic $\mathbf{s}$ the operator $t^{\mathsf{GT}}(u)$ is a differential operator with coefficients which are polynomial in $\mathbf{s}$ and hence the action of $t^{\mathsf{GT}}(u)$ on $|\Psi_{\mathbf{s}}\rangle$ results in a vector which is polynomial in $\mathbf{s}$ and hence (D.3.10) vanishes for all values of $\mathbf{s}$. Hence, $|\Psi_{\mathbf{s}}\rangle$ is an eigenvector for all $\mathbf{s}$ and is non-zero for generic $\mathbf{s}$ and $t^{\mathsf{GT}}(u)$ has distinct eigenvalue for each such eigenvector. Since each $\mathcal{H}_k$ can be obtained by considering enough finite-dimensional representations with sufficiently large $-\mathbf{s}$ we can promote a basis of eigenvectors of $\mathcal{H}_k$ for $\mathbf{s} \in \{0, -\frac{1}{2}, \dots\}$ to a basis of eigenvectors for generic $\mathbf{s}$. This completes the proof.

Appendix E

Technical details for Chapter 6

E.1 Parity of the open transfer matrix

We will now prove that, up to a trivial prefactor, the open transfer matrix $\mathbb{T}(u)$ built from the Lax operators and boundary matrices is even in the spectral parameter.

This is quite simple to do once we establish a few identities.

Unitarity of R^t Just like the R-matrix, the R^t matrix is also unitary:

$$S^{t\,i\,m}_{\quad n\,l}(2u)S^{t\,n\,k}_{\quad j\,m}(-2u) = \delta^i_j\delta^k_l\,, \tag{E.1.1}$$

where we are summing over repeated indices and S is the normalised rational R-matrix introduced in (6.3.1).

KR identities We also need special identities between boundary matrices and R-matrices, first found in [28]. They are given by:

$$S^{t\,i\,n}_{\quad m\,j}(2u)K^-_{mn} = (K^t_-)_{ij}\,, \tag{E.1.2}$$

$$K^+_{mn}S^{t\,j\,n}_{\quad m\,i}(-2u) = \frac{2u+i}{-2u+i}(K^t_+)_{ij}\,. \tag{E.1.3}$$

In particular, the first can be immediately upgraded to an identity for the boundary monodromy matrix $U_{ij}(u)$:

$$S^{t\,i\,n}_{\quad m\,j}(2u)U_{\,mn}(u) = (U^t)_{ij}(-u)\,. \tag{E.1.4}$$

Using these identities, we see that:

$$\tilde{\mathbb{T}}(u) = K^+_{il}U_{li}(u) = K^+_{jk}S^{t\,n\,k}_{\quad j\,m}(-2u)S^{t\,m\,i}_{\quad l\,n}(2u)U_{il}(u) = \tag{E.1.5}$$

$$= \tfrac{2u+i}{-2u+i}(K^t_+)_{nm}(U^t)_{mn}(-u) = \tfrac{2u+i}{-2u+i}\tilde{\mathbb{T}}(-u)\,, \tag{E.1.6}$$

where in the last passage we have used the fact that trace is invariant under transposition and that $V^2 = 1$. Therefore, the quantity $\mathbb{T}(u) \equiv \frac{\tilde{\mathbb{T}}(u)}{u+i/2}$ is even in u.

Appendix F

Technical details for Chapter 8

F.1 Proof of Poisson-commutativity of $\mathbb{T}(u)$

In this appendix, we prove that the classical transfer matrix (8.3.14) forms a family of mutually Poisson-commuting functions for any value of the spectral parameter and for any J. We start by the $J = 0$ case. We have that:

$$\{\mathbb{T}(u), \mathbb{T}(v)\} = \{\mathbb{K}_{ab}(u), \mathbb{K}_{\alpha\beta}(v)\}\bar{\mathbb{K}}^{ba}(u)\bar{\mathbb{K}}^{\beta\alpha}(v) + \mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\{\bar{\mathbb{K}}^{ba}(u), \bar{\mathbb{K}}^{\beta\alpha}(v)\}\,. \quad \text{(F.1.1)}$$

Using (8.3.18) and its analog for $\bar{\mathbb{K}}$:

$$\begin{aligned}\{\mathbb{T}(u), \mathbb{T}(v)\} = &- \frac{\bar{\mathbb{K}}^{ba}(u)\bar{\mathbb{K}}^{\beta\alpha}(v)}{\xi(u+v)}\left[\mathbb{K}_{\beta b}(u)\mathbb{K}_{\alpha a}(v) - \mathbb{K}_{b\beta}(v)\mathbb{K}_{a\alpha}(u)\right] + \\ &+ \frac{\bar{\mathbb{K}}^{ba}(u)\bar{\mathbb{K}}^{\beta\alpha}(v)}{\xi(u-v)}\left[\mathbb{K}_{a\beta}(u)\mathbb{K}_{\alpha b}(v) - \mathbb{K}_{a\beta}(v)\mathbb{K}_{\alpha b}(u)\right] - \\ &- \frac{\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)}{\xi(u+v)}\left[\bar{\mathbb{K}}^{\alpha a}(u)\bar{\mathbb{K}}^{\beta b}(v) - \bar{\mathbb{K}}^{a\alpha}(v)\bar{\mathbb{K}}^{b\beta}(u)\right] + \\ &+ \frac{\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)}{\xi(u-v)}\left[\bar{\mathbb{K}}^{b\alpha}(u)\bar{\mathbb{K}}^{\beta a}(v) - \bar{\mathbb{K}}^{b\alpha}(v)\bar{\mathbb{K}}^{\beta a}(u)\right].\end{aligned} \quad \text{(F.1.2)}$$

By relabelling indices appropriately, all terms cancel as expected.

For the rest of this section, we will use the shorthand notation:

$$(\mathbb{L})^a_{\ b}(u) \equiv (\mathbb{L}_1(u).\mathbb{L}_2(u)\ldots\mathbb{L}_J(u))^a_{\ b}\,, \qquad (\bar{\mathbb{L}})_a^{\ b}(u) \equiv (\mathbb{L}_{-J}(u).\mathbb{L}_{-(J-1)}(u)\ldots\mathbb{L}_{-1}(u))_a^{\ b}\,. \quad \text{(F.1.3)}$$

It is easy to see that these matrices follow the same Poisson brackets as individual $\mathbb{L}$-

matrices. Thus, for general J we have that:

$$
\begin{aligned}
\{\mathbb{T}(u),\mathbb{T}(v)\} =&\{\mathbb{K}_{ab}(u),\mathbb{K}_{\alpha\beta}(v)\}(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)+\\
&\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)\{\bar{\mathbb{K}}^{dc}(u),\bar{\mathbb{K}}^{\delta\gamma}(v)\}(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)+\\
&+\text{(Poisson brackets between }\mathbb{L})=\\
&=(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)\\
&\Bigg\{-\frac{1}{\xi(u+v)}\Bigg[\mathbb{K}_{\beta b}(u)\mathbb{K}_{\alpha a}(v)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)-\mathbb{K}_{b\beta}(v)\mathbb{K}_{a\alpha}(u)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)+\\
&+\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{\gamma c}(u)\bar{\mathbb{K}}^{\delta d}(v)-\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{c\gamma}(v)\bar{\mathbb{K}}^{d\delta}(u)\Bigg]+\\
&+\frac{1}{\xi(u-v)}\Bigg[\mathbb{K}_{a\beta}(u)\mathbb{K}_{\alpha b}(v)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)-\mathbb{K}_{a\beta}(v)\mathbb{K}_{\alpha b}(u)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)+\\
&+\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{d\gamma}(u)\bar{\mathbb{K}}^{\delta c}(v)-\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{d\gamma}(v)\bar{\mathbb{K}}^{\delta c}(u)\Bigg]\Bigg\}+\\
&+\text{(Poisson brackets between }\mathbb{L})\,.
\end{aligned}
\tag{F.1.4}
$$

We now relabel indices in order to collect the boundary reflection matrices as:

$$
\begin{aligned}
\{\mathbb{T}(u),\mathbb{T}(v)\} =&\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)\\
&\Bigg\{-\frac{1}{\xi(u+v)}\Bigg[(\bar{\mathbb{L}})_{c}^{\ \beta}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{a}_{\ \delta}(v)-\\
&-(\bar{\mathbb{L}})_{\delta}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{\beta}_{\ c}(v)+\\
&+(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{d}^{\ \alpha}(v)(\mathbb{L})^{b}_{\ \gamma}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)-\\
&-(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ b}(v)(\mathbb{L})^{\alpha}_{\ d}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)\Bigg]+\\
&+\frac{1}{\xi(u-v)}\Bigg[(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)(\mathbb{L})^{\beta}_{\ d}(u)(\mathbb{L})^{b}_{\ \delta}(v)-\\
&-(\bar{\mathbb{L}})_{c}^{\ \alpha}(u)(\bar{\mathbb{L}})_{\gamma}^{\ a}(v)(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)+\\
&+(\bar{\mathbb{L}})_{\gamma}^{\ a}(u)(\bar{\mathbb{L}})_{c}^{\ \alpha}(v)(\mathbb{L})^{b}_{\ d}(u)(\mathbb{L})^{\beta}_{\ \delta}(v)-\\
&-(\bar{\mathbb{L}})_{c}^{\ a}(u)(\bar{\mathbb{L}})_{\gamma}^{\ \alpha}(v)(\mathbb{L})^{b}_{\ \delta}(u)(\mathbb{L})^{\beta}_{\ d}(v)\Bigg]\Bigg\}+\\
&+\text{(Poisson brackets between }\mathbb{L})\,.
\end{aligned}
\tag{F.1.5}
$$

The Poisson Brackets between $\mathbb{L}$-matrices give:

$$\begin{aligned}\{\mathbb{T}(u),\mathbb{T}(v)\} =&(\text{Poisson brackets between } \mathbb{K})+\\ &+\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)\\ &\Bigg[\{(\bar{\mathbb{L}})_c^{\ a}(u),(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)\}(\mathbb{L})^b_{\ d}(u)(\mathbb{L})^\beta_{\ \delta}(v)+\\ &+(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)\{(\bar{\mathbb{L}})_c^{\ a}(u),(\mathbb{L})^\beta_{\ \delta}(v)\}(\mathbb{L})^b_{\ d}(u)+\\ &+(\bar{\mathbb{L}})_c^{\ a}(u)(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)\{(\mathbb{L})^b_{\ d}(u),(\mathbb{L})^\beta_{\ \delta}(v)\}+\\ &+(\bar{\mathbb{L}})_c^{\ a}(u)\{(\mathbb{L})^b_{\ d}(u),(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)\}(\mathbb{L})^\beta_{\ \delta}(v)\Bigg].\end{aligned}\tag{F.1.6}$$

Using the Poisson brackets:

$$\xi\,\{(\mathbb{L}_{-n})_a^{\ b}(u),(\mathbb{L}_{-m})_c^{\ d}(v)\}=\frac{(\mathbb{L}_{-n})_a^{\ d}(u)(\mathbb{L}_{-n})_c^{\ b}(v)-(\mathbb{L}_{-n})_c^{\ b}(u)(\mathbb{L}_{-n})_a^{\ d}(v)}{u-v}\,\delta_{nm}\,,\tag{F.1.7}$$

$$\xi\,\{(\mathbb{L}_{-n})_a^{\ b}(u),(\mathbb{L}_m)^c_{\ d}(v)\}=\frac{(\mathbb{L}_{-n})_a^{\ c}(u)(\mathbb{L}_n)^b_{\ d}(v)-(\mathbb{L}_{-n})_d^{\ b}(u)(\mathbb{L}_n)^c_{\ a}(v)}{u+v}\delta_{m\,n}\,,\tag{F.1.8}$$

we get:

$$\begin{aligned}\{\mathbb{T}(u),\mathbb{T}(v)\} =&(\text{Poisson brackets between } \mathbb{K})+\\ &+\mathbb{K}_{ab}(u)\mathbb{K}_{\alpha\beta}(v)\bar{\mathbb{K}}^{dc}(u)\bar{\mathbb{K}}^{\delta\gamma}(v)\\ &\Bigg\{\frac{1}{\xi(u-v)}\Bigg[(\bar{\mathbb{L}})_c^{\ \alpha}(u)(\bar{\mathbb{L}})_\gamma^{\ a}(v)(\mathbb{L})^b_{\ d}(u)(\mathbb{L})^\beta_{\ \delta}(v)-\\ &-(\bar{\mathbb{L}})_\gamma^{\ a}(u)(\bar{\mathbb{L}})_c^{\ \alpha}(v)(\mathbb{L})^b_{\ d}(u)(\mathbb{L})^\beta_{\ \delta}(v)+\\ &+(\bar{\mathbb{L}})_c^{\ a}(u)(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)(\mathbb{L})^b_{\ \delta}(u)(\mathbb{L})^\beta_{\ d}(v)-\\ &-(\bar{\mathbb{L}})_c^{\ a}(u)(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)(\mathbb{L})^\beta_{\ d}(u)(\mathbb{L})^b_{\ \delta}(v)\Bigg]+\\ &+\frac{1}{\xi(u+v)}\Bigg[(\bar{\mathbb{L}})_c^{\ \beta}(u)(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)(\mathbb{L})^b_{\ d}(u)(\mathbb{L})^a_{\ \delta}(v)-\\ &-(\bar{\mathbb{L}})_\delta^{\ a}(u)(\bar{\mathbb{L}})_\gamma^{\ \alpha}(v)(\mathbb{L})^b_{\ d}(u)(\mathbb{L})^\beta_{\ c}(v)-\\ &-(\bar{\mathbb{L}})_c^{\ a}(u)(\bar{\mathbb{L}})_\gamma^{\ b}(v)(\mathbb{L})^\alpha_{\ d}(u)(\mathbb{L})^\beta_{\ \delta}(v)+\\ &+(\bar{\mathbb{L}})_c^{\ a}(u)(\bar{\mathbb{L}})_d^{\ \alpha}(v)(\mathbb{L})^b_{\ \gamma}(u)(\mathbb{L})^\beta_{\ \delta}(v)\Bigg]\Bigg\}.\end{aligned}\tag{F.1.9}$$

It is easy to verify that the terms from the Poisson Brackets of $\mathbb{L}$-matrices cancel exactly the ones from the Poisson brackets of $\mathbb{K}$-matrices. Therefore, the transfer matrices form a family of functions in convolution between themselves:

$$\{\mathbb{T}(u),\mathbb{T}(v)\}=0\,.\tag{F.1.10}$$

F.2 Parity of quantum transfer matrices

In this appendix we will prove explicitly the parity of the quantum transfer matrices in all the antisymmetric representations of the auxiliary space.

Parity of $\mathbb{T}^4$

We need to evaluate:

$$\hat{\mathbb{T}}^{\mathbf{4}}(-u) = \mathrm{Tr}(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(u).\hat{\mathbb{K}}^{\mathbf{4}}(-u).\hat{\mathbb{L}}^{\mathbf{4}}_1(-u).\hat{\mathbb{L}}^{\mathbf{4}}_2(-u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(-u).G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}}(-u).G^{\mathbf{4}\,T})\,. \tag{F.2.1}$$

Transposing inside of the trace:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&\mathrm{Tr}(G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T}.\hat{\mathbb{L}}^{\mathbf{4}\,T}_J(-u).\hat{\mathbb{L}}^{\mathbf{4}\,T}_{J-1}(-u)\dots\hat{\mathbb{L}}^{\mathbf{4}\,T}_1(-u).\\ &.\hat{\mathbb{K}}^{\mathbf{4}\,T}(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}\,T}_1(u).\hat{\bar{L}}^{\mathbf{4}\,T}_2(u)\dots\hat{\bar{L}}^{\mathbf{4}\,T}_J(u))\,.\end{aligned} \tag{F.2.2}$$

Now since $\hat{\mathbb{L}}^{\mathbf{4}\,T}_j(-u) = -\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_j(-u)$ and $\hat{\bar{\mathbb{L}}}^{\mathbf{4}\,T}_j(u) = -\hat{\mathbb{L}}^{\mathbf{4}}_1(u)$ we get:

$$\hat{\mathbb{T}}^{\mathbf{4}}(-u) = \mathrm{Tr}(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(-u).\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).\hat{\mathbb{L}}^{\mathbf{4}}_1(u).\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u).G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T})\,. \tag{F.2.3}$$

We can now insert a pair of $\bar{S}$-matrices near the $\hat{\mathbb{K}}$-operator using the unitarity condition $\bar{S}(2u)\bar{S}(-2u) = I$ and then commute $\bar{S}(2u)$ through the $\hat{\mathbb{L}}$-operators using the Yang-Baxter equation, obtaining:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&\mathrm{Tr}(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(-u).\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).\bar{S}(-2u).\\ &.\hat{\mathbb{L}}^{\mathbf{4}}_1(u).\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u).G^{\mathbf{4}}.\bar{S}(2u).\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T})\,.\end{aligned} \tag{F.2.4}$$

Using the following identities:

$$\bar{\alpha}(2u)\bar{R}^{c\,d}_{a\,b}(2u)\hat{\mathbb{K}}^{\mathbf{4}}_{cd}(-u) = \hat{\mathbb{K}}^{\mathbf{4}}_{ba}(u)\,, \tag{F.2.5}$$

$$\bar{\alpha}(-2u)\bar{R}^{a\,b}_{d\,c}(-2u)(\hat{\bar{\mathbb{K}}}^{\mathbf{4}})^{dc}(-u) = (\hat{\bar{\mathbb{K}}}^{\mathbf{4}})^{ba}(u)\,, \tag{F.2.6}$$

which in matrix notation are:

$$\hat{\mathbb{K}}^{\mathbf{4}\,T}(-u).\bar{S}(2u) = \hat{\mathbb{K}}^{\mathbf{4}}(u)\,, \tag{F.2.7}$$

$$\bar{S}(-2u).\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u) = \hat{\bar{\mathbb{K}}}^{\mathbf{4}}(u)\,, \tag{F.2.8}$$

we obtain that $\hat{\mathbb{T}}^{\mathbf{4}}(-u) = \hat{\mathbb{T}}^{\mathbf{4}}(u)$, thus seeing that $\hat{\mathbb{T}}^4$ is even for any J.

We will now give a more detailed proof of the last passage above. We will use the RTT relations (substituting $R(-u)$ with $\bar{S}(u)$), the unitarity condition $S(u)S(-u) = 1$ and

the identities (F.2.5) and (F.2.6). Starting from (F.2.3) we insert an identity and we use unitarity to get:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(-u))_a{}^{\gamma}\bar{S}^{\delta\ \alpha}_{\zeta\ \eta}(2u)\bar{S}^{\zeta\ \eta}_{\gamma\ \beta}(-2u)\hat{\mathbb{K}}^{\mathbf{4}}_{\alpha\delta}(-u)\\ &(\hat{\mathbb{L}}^{\mathbf{4}}_1(u).\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u))^{\beta}_{\ h}(G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T})^{ha}\,.\end{aligned} \tag{F.2.9}$$

We can now use (F.2.5) to get:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(-u))_a{}^{\gamma}\bar{S}^{\zeta\ \eta}_{\gamma\ \beta}(-2u)\hat{\mathbb{K}}^{\mathbf{4}}_{\zeta\eta}(u)\\ &(\hat{\mathbb{L}}^{\mathbf{4}}_1(u).\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u))^{\beta}_{\ h}(G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T})^{ha}\,.\end{aligned} \tag{F.2.10}$$

We now use the YBE (substituting $R(-u)$ with $\bar{S}(u)$) to commute the remaining S-matrix through all the $\hat{\mathbb{L}}_i$ and $\hat{\bar{\mathbb{L}}}_i$, obtaining for $i=1$:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_2(-u))_a{}^{\omega}\,\bar{S}^{\gamma\ \beta}_{\epsilon\ \omega}(-2u)\hat{\mathbb{L}}^{\mathbf{4}\ \eta}_{1\ \gamma}(u)\hat{\mathbb{L}}^{\mathbf{4}}_{1\ \beta}{}^{\zeta}(-u)\hat{\mathbb{K}}^{\mathbf{4}}_{\zeta\eta}(u)\\ &(\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u))^{\epsilon}_{\ h}(G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T})^{ha}\,.\end{aligned} \tag{F.2.11}$$

Continuing this process for $\forall i=2...J$ we obtain:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(-u).\hat{\mathbb{K}}^{\mathbf{4}}(u).\hat{\mathbb{L}}^{\mathbf{4}}_1(u).\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u))_{\epsilon\,\omega}\\ &\bar{S}^{\epsilon\ \omega}_{h\ a}(-2u)(G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}\,T}(-u).G^{\mathbf{4}\,T})^{ha}\,.\end{aligned} \tag{F.2.12}$$

Finally we use (F.2.6) and get:

$$\begin{aligned}\hat{\mathbb{T}}^{\mathbf{4}}(-u) =&(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_J(-u).\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J-1}(-u)\dots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_1(-u).\hat{\mathbb{K}}^{\mathbf{4}}(u).\hat{\mathbb{L}}^{\mathbf{4}}_1(u).\hat{\mathbb{L}}^{\mathbf{4}}_2(u)\dots\hat{\mathbb{L}}^{\mathbf{4}}_J(u))_{\epsilon\,\omega}\\ &(G^{\mathbf{4}}.\hat{\bar{\mathbb{K}}}^{\mathbf{4}}(u).G^{\mathbf{4}\,T})^{\omega\,\epsilon}=\hat{\mathbb{T}}^{\mathbf{4}}(u)\,.\end{aligned} \tag{F.2.13}$$

Parity of $\mathbb{T}^{\mathbf{6}}$

Remembering that $\bar{\mathbb{L}}^{\mathbf{6}}_i(u)=\mathbb{L}^{\mathbf{6}}_i(-u)$, we write:

$$\hat{\mathbb{T}}^{\mathbf{6}}(u)=\mathrm{Tr}\left(\hat{\mathbb{L}}^{\mathbf{6}}_J(u)\dots\hat{\mathbb{L}}^{\mathbf{6}}_1(u).\hat{\mathbb{K}}^{\mathbf{6}}(u).\hat{\mathbb{L}}^{\mathbf{6}}_1(u)\dots\hat{\mathbb{L}}^{\mathbf{6}}_J(u).G^{\mathbf{6}}.\hat{\bar{\mathbb{K}}}^{\mathbf{6}}(u).G^{6\,T}\right)\,. \tag{F.2.14}$$

Hence we have that:

$$\hat{\mathbb{T}}^{\mathbf{6}}(-u)=\mathrm{Tr}\left(\hat{\mathbb{L}}^{\mathbf{6}}_J(-u)\dots\hat{\mathbb{L}}^{\mathbf{6}}_1(-u).\hat{\mathbb{K}}^{\mathbf{6}}(-u).\hat{\mathbb{L}}^{\mathbf{6}}_1(-u)\dots\hat{\mathbb{L}}^{\mathbf{6}}_J(-u).G^{\mathbf{6}}.\hat{\bar{\mathbb{K}}}^{\mathbf{6}}(-u).G^{\mathbf{6}\,T}\right)\,. \tag{F.2.15}$$

Taking a transpose inside the trace and noticing from definition (8.4.13) that $\hat{\mathbb{L}}^{\mathbf{6}\,T}_i(-u)=\hat{\mathbb{L}}^{\mathbf{6}}_i(u)$:

$$\hat{\mathbb{T}}^{\mathbf{6}}(-u)=\mathrm{Tr}\left(\hat{\mathbb{L}}^{\mathbf{6}}_J(u)\dots\hat{\mathbb{L}}^{\mathbf{6}}_1(u).\hat{\mathbb{K}}^{6\,T}(-u).\hat{\mathbb{L}}^{\mathbf{6}}_1(u)\dots\hat{\mathbb{L}}^{\mathbf{6}}_J(u).G^{\mathbf{6}}.\hat{\bar{\mathbb{K}}}^{\mathbf{6}\,T}(-u).G^{\mathbf{6}\,T}\right)\,. \tag{F.2.16}$$

We now need identities analogous to (F.2.5) for **6** irrep. First, we need the $\bar{S}^{\mathbf{6}}$-matrix. Making the ansatz that it is formed by all compatible indices structures, we can fix the relative coefficients by requiring that it satisfies the Yang-Baxter equation:

$$\hat{\mathbb{L}}^{\mathbf{6}\,B}{}_{E}(u)\hat{\bar{\mathbb{L}}}^{\mathbf{6}}{}_{F}{}^{D}(-v)\bar{S}^{\mathbf{6}E\,F}{}_{A\,C}(u+v) = \bar{S}^{\mathbf{6}B\,D}{}_{F\,E}(u+v)\hat{\bar{\mathbb{L}}}^{\mathbf{6}}{}_{C}{}^{E}(-v)\hat{\mathbb{L}}^{\mathbf{6}\,F}{}_{A}(u)\,. \tag{F.2.17}$$

We get that:

$$\bar{S}^{\mathbf{6}\,A}{}_{B}{}^{C}{}_{D}(u) = c(u)\left(\delta^{A}_{B}\,\delta^{C}_{D} - \frac{i}{\xi\,u}\delta^{A}_{D}\,\delta^{C}_{B} + \frac{i}{\xi\left(u-\frac{2i}{\xi}\right)}\eta^{AC}\,\eta_{BD}\right). \tag{F.2.18}$$

The overall coefficient $c(u)$ is fixed by unitarity, $\bar{S}^{\mathbf{6}\,A}{}_{B}{}^{C}{}_{D}(u)\bar{S}^{\mathbf{6}\,B}{}_{E}{}^{D}{}_{F}(-u) = \delta^{A}_{E}\,\delta^{C}_{F}$, as:

$$c(u) = \frac{u\left(u-\frac{2i}{\xi}\right)}{\frac{2}{\xi^2}+u\left(u+\frac{i}{\xi}\right)}\,. \tag{F.2.19}$$

The identities we need are:

$$\hat{\mathbb{K}}^{\mathbf{6}\,T}(-u).\bar{S}^{\mathbf{6}}(2u) = \hat{\mathbb{K}}^{\mathbf{6}}(u)\,, \tag{F.2.20}$$

$$\bar{S}^{\mathbf{6}}(-2u).\hat{\bar{\mathbb{K}}}^{\mathbf{6}\,T}(-u) = \hat{\bar{\mathbb{K}}}^{\mathbf{6}}(u)\,. \tag{F.2.21}$$

Hence, inserting into (F.2.16) a pair of $\bar{S}$-matrices via unitarity and repeating the passages of the section above, it is easy to prove that:

$$\hat{\mathbb{T}}^{\mathbf{6}}(-u) = \hat{\mathbb{T}}^{\mathbf{6}}(u)\,. \tag{F.2.22}$$

Parity of $\mathbb{T}^{\bar{\mathbf{4}}}$

Using the definitions of section 8.4.3 one can rewrite $\hat{\mathbb{T}}^{\bar{\mathbf{4}}}(u)$ in terms of $\hat{\mathbb{L}}^{\mathbf{4}}$-operators and $\hat{\mathbb{K}}^{4}$-operators as:

$$\hat{\mathbb{T}}^{\bar{\mathbf{4}}}(u) = \hat{\beta}(u)\,\mathrm{Tr}(\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{1}(u)\ldots\hat{\bar{\mathbb{L}}}^{\mathbf{4}}_{J}(u).(G^{\mathbf{4}})^{-T}.\hat{\mathbb{K}}^{\mathbf{4}}(-u).(G^{\mathbf{4}})^{-1}.\hat{\mathbb{L}}^{\mathbf{4}}_{J}(-u)\ldots\hat{\mathbb{L}}^{\mathbf{4}}_{1}(-u).\hat{\bar{\mathbb{K}}}^{\mathbf{4}}(-u))\,. \tag{F.2.23}$$

Where $\hat{\beta}(u)$ is an operator which is an even polynomial in u, composed by all the prefactors appearing in the definitions of the $\bar{\mathbf{4}}$ operators. Following the same passages used for the parity of $\mathbb{T}^{\mathbf{4}}$, it is then easy to prove that:

$$\mathbb{T}^{\bar{\mathbf{4}}}(-u) = \mathbb{T}^{\bar{\mathbf{4}}}(u)\,. \tag{F.2.24}$$

Parity of $\mathbb{T}^{\bar{\mathbf{1}}}$

From the definitions of section 8.4.3, it is evident that the $\mathbb{L}^{\bar{\mathbf{1}}}$-operators are even polynomials in u. Also, since the $\mathbb{K}^{\bar{\mathbf{1}}}$-operators are proportional to the identity operator, we can move them together through all $\mathbb{L}^{\bar{\mathbf{1}}}$. Their product is:

$$\mathbb{K}^{\bar{\mathbf{1}}}(u)\bar{\mathbb{K}}^{\bar{\mathbf{1}}}(u) = \left(u^2 + \frac{4}{\xi^2}\right)\left(u^2 + \frac{1}{\xi^2}\right)^2 u^2 \,. \tag{F.2.25}$$

Hence, $\mathbb{T}^{\bar{\mathbf{1}}}(u)$ is even as it is a product of even functions.

F.3 Explicit form of transfer matrices

$J = 0$ case

The polynomials P that enter the transfer matrices for the $J = 0$ case in (8.4.31) are

$$\begin{aligned} P^{\mathbf{4}} = P^{\bar{\mathbf{4}}} &= 4\,\cos\varphi\, v^2 - 8\,h + \cos\varphi\,, \\ P^{\mathbf{6}} &= (2\cos 2\varphi + 4)v^4 - (4\,\Delta^2 \sin^2\varphi + 16\,h\,\cos\varphi)v^2 + 16\,h^2\,, \end{aligned} \tag{F.3.1}$$

where $h = -\hat{g}^2\, B^{-1}$. The above expressions lead to the Baxter equation (8.5.4).

In order to make a comparison with [25, 157] we introduce the notation for the final difference operators $\hat{O}_{\pm}$

$$\hat{O}_{\pm}q \equiv q(u)\left(4\hat{g}^2 - 2u^2\cos(\phi) \pm 2\Delta u \sin(\phi)\right) + u^2 q(u-i) + u^2 q(u+i)\,. \tag{F.3.2}$$

The second order equations used in [25, 157] was of the form $\hat{O}_{\pm}q = 0$. At the same time the forth order equation (8.5.4) can we written as

$$-\frac{1}{u^2}\hat{O}_+\frac{1}{u}\hat{O}_- q = -\frac{1}{u^2}\hat{O}_-\frac{1}{u}\hat{O}_+ q = 0\,. \tag{F.3.3}$$

We see that the four independent solutions of the two second order equations $\hat{O}_{\pm}q = 0$ are the 4 solutions of (8.5.4), which indeed demonstrates their equivalence.

$J = 1$ case

For the $J = 1$ case we explicitly built all 3 transfer matrices as differential operators acting on the CFT wavefunction of 6 variables $s, t, \vec{x}_1$. We verified the general analytic properties outlined in the text in section 8.4.3. Furthermore, we found some additional relations

between the coefficients as shown below

$$\begin{aligned} P^{\mathbf{4}} &= 4\cos\varphi\, v^4 + a_1\, v^2 + \frac{a_1}{4} + 8h - \frac{1}{4}\cos\varphi\,, \\ P^{\mathbf{6}} &= (2\cos 2\varphi + 4)\, v^8 + 2\Big[\cos\varphi\,\frac{a_1+c_1}{2} + \cos 2\varphi\,\Delta^2 - (\Delta^2+1) \\ &\quad - 2S^2\sin^2\varphi\Big]\, v^6 + b_2\, v^4 + b_3\, v^2 + 16\, h^2\,, \\ P^{\bar{\mathbf{4}}} &= 4\cos\varphi\, v^4 + c_1\, v^2 + \frac{c_1}{4} + 8h - \frac{1}{4}\cos\varphi\,. \end{aligned} \tag{F.3.4}$$

Here, $h = -\hat{g}^4 B^{-1} \simeq -\hat{g}^4$ so we can see that the relation (8.4.26) does hold indeed. The coefficients a_1, c_1, b_2, b_3 and h are complicated differential operators whose explicit form can be computed if needed. There are no further simple relations we found between them except for $c_1 - a_1 = 8\, i\, S\, \Delta\, \sin\varphi$, agreeing with (8.4.35). This implies that under spin flipping $S \to -S$, $\mathbb{T}^{\mathbf{4}}(u)$ interchanges with $\mathbb{T}^{\bar{\mathbf{4}}}(u)$ up to the trivial explicit prefactor. We see that in total we have 6 independent commuting operators $a_1, b_2, b_3, h, S, \Delta$, which equates the number of degrees of freedom in $J = 1$ case.

The limit of straight line $\varphi \to \pi$ is especially interesting as 1D Conformal symmetry gets restored. The space naturally decomposes into a $1D$ line and the $3D$ space orthogonal to it. The corresponding symmetry is thus $SO(3) \times SO(2,1)$, and its representations are parametrised by the spin S of $SO(3)$ and the conformal weight Δ of $SO(2,1)$. Fixing Δ and S removes two variables in our CFT wavefunction out of 6. Furthermore, we can restrict ourselves to Highest Weight states w.r.t. to both subgroups, which imposes on the wave function 2 more conditions $K\psi = 0$ and $S^+\psi = 0$, which can be used to further reduce the number of variables from 4 to 2. In this reduced system B^{-1} and b_3 remain two non-trivial differential operators, whereas all others can be expressed explicitly in terms of Δ and S. In particular, a_1 becomes $2(P_1 K_1 + \Delta^2 - \Delta + 1)$, where K_μ is the special conformal transformation generator and P_μ is the generator of translations, and thus simplifies considerably for the primary operators in the 1D defect CFT, for which by definition $K_1 = 0$.

In the simplified case $S = 0$ we get the following relations

$$\begin{aligned} P^{\mathbf{4}} &= -4v^4 + \left(2\Delta^2 - 2\Delta + 2\right) v^2 + \frac{1}{4}\left(2\Delta^2 - 2\Delta + 3\right) - 8\hat{g}^4\,, \\ P^{\mathbf{6}} &= +6v^8 - (4(\Delta-1)\Delta + 6)v^6 + v^4\left((\Delta-1)^2\Delta^2 + 16\hat{g}^4\right) + b_3 v^2 + 16\hat{g}^8\,, \\ P^{\bar{\mathbf{4}}} &= -4v^4 + \left(2\Delta^2 - 2\Delta + 2\right) v^2 + \frac{1}{4}\left(2\Delta^2 - 2\Delta + 3\right) - 8\hat{g}^4\,, \end{aligned} \tag{F.3.5}$$

so there are only two non-trivial functions $\Delta(g)$ and $b_3(g)$, which can only be deduced numerically.

F.4 Generalisation: addition of impurities

In order to develop an SoV construction for this spin chain, it would be good to introduce impurities like in [69]. This is done by introducing a dependence on some parameters

$\{\theta_i\}$, $i = 1 \dots J$ in the rapidities of the bulk particles. To preserve parity in the argument u of the T-operators, the correct choice (up to a normalisation of the θ_i) amounts to:

$$\hat{\mathbb{T}}^{\lambda}_{\theta}(u) = \mathrm{Tr}\left(\hat{\mathbb{L}}^{t}_{J}{}^{\lambda}(-u-\theta_J)\dots\hat{\mathbb{L}}^{t}_{1}{}^{\lambda}(-u-\theta_1).\hat{\mathbb{K}}^{\lambda}(u).\hat{\mathbb{L}}^{\lambda}_{1}(u-\theta_1)\dots\hat{\mathbb{L}}^{\lambda}_{J}(u-\theta_J).G^{\lambda}.\hat{\bar{\mathbb{K}}}^{\lambda}(u).G^{\lambda\,t}\right) , \tag{F.4.1}$$

where $\lambda = \{\mathbf{4}, \bar{\mathbf{4}}, \mathbf{6}, \bar{\mathbf{1}}\}$. These transfer matrices form a family of mutually commuting operators: this was verified explicitly up to the case $J = 1$. However, they do not commute with the original Hamiltonian H: this is expected, as introducing impurities changes the physical system and thus the Hamiltonian as well.

The next step is to introduce the polynomials P^{λ}_{k} and to write the Baxter equation. In this case, the polynomials will acquire a $\{\theta_i\}$ dependence. Moreover, the prefactors appearing in equations (8.4.32) will also be modified. We obtain that:

$$\begin{aligned}
&\mathbb{T}^{\mathbf{1}}_{\theta}(v,\{\zeta_i\}) = 1 \,,\\
&\mathbb{T}^{\mathbf{4}}_{\theta}(v,\{\zeta_i\}) \equiv \frac{P^{\mathbf{4}}_{J+1}(v^2,\{\zeta_i\})}{\xi^{2J+2}} \,,\\
&\mathbb{T}^{\mathbf{6}}_{\theta}(v,\{\zeta_i\}) \equiv A(2v)\frac{v^2+1}{v^2}\frac{P^{\mathbf{6}}_{2J+2}(v^2,\{\zeta_i\})}{\xi^{4J+4}} \,,\\
&\mathbb{T}^{\bar{\mathbf{4}}}_{\theta}(v,\{\zeta_i\}) = A(2v)A(2v+i)A(2v-i)\frac{(v^2+\frac{9}{4})(v^2+\frac{1}{4})\prod_{i=1}^{J}\left(\zeta_i^2+\left(v^2-\zeta_i^2+\frac{1}{4}\right)^2\right)P^{\bar{\mathbf{4}}}_{J+1}(v^2,\{\zeta_i\})}{\xi^{6J+6}} \,,\\
&\mathbb{T}^{\bar{\mathbf{1}}}_{\theta}(v,\{\zeta_i\}) = A^2(2v)A(2v+i)A(2v-i)A(2v+2i)A(2v-2i)\\
&\qquad\qquad\frac{(v^2+4)(v^2+1)^2v^2\prod_{i=1}^{J}\left(\left(v^2-\zeta_i^2\right)^2\left(4\zeta_i^2+\left(v^2-\zeta_i^2+1\right)^2\right)\right)}{\xi^{8J+8}} \,,
\end{aligned} \tag{F.4.2}$$

where $v \equiv u\,\xi$ and $\zeta_i \equiv \theta_i\,\xi$. We can now rewrite the Baxter equation (5.2.2). Defining $\zeta_0 \equiv 0$, $\zeta_{-i} \equiv -\zeta_i$ and identifying:

$$Q(v) \to \frac{\Gamma(-iv)\exp(\pi(J+1)v)\,q(v)\,\xi^{2iv(J+1)}}{\Gamma\left(-iv-\frac{1}{2}\right)\Gamma(iv+2)}\prod_{i=-J}^{J}\left(\Gamma(i(v+\zeta_i)+1)^{-1}\right) \,, \tag{F.4.3}$$

we obtain:

$$\begin{aligned}
\frac{P^{\mathbf{6}}_{2J+2}(v^2)}{v^2\prod_{i=-J}^{J}(v-\zeta_i)}q(v) \;=\; &-\prod_{i=-J}^{J}(v+i-\zeta_i)\,q(v+2i) - \frac{v+\frac{i}{2}}{v(v+i)}P^{\mathbf{4}}_{J+1}\left((v+\tfrac{i}{2})^2\right)q(v+i)\\
&-\prod_{i=-J}^{J}(v-i-\zeta_i)\,q(v-2i) - \frac{v-\frac{i}{2}}{v(v-i)}P^{\bar{\mathbf{4}}}_{J+1}\left((v-\tfrac{i}{2})^2\right)q(v-i) \,.
\end{aligned}$$

A similar construction with inhomogeneities in the closed fishchain is introduced in [92].

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