

Quantum holographic surface anomalies

Nadav Drukker^{*1}, Omar Shahpo^{†1}, and Maxime Trépanier^{‡1,2}

¹*Department of Mathematics, King's College London,
London, WC2R 2LS, United Kingdom*

²*Department of Physics, Technion, Haifa, 32000, Israel*

Abstract

Expectation values of surface operators suffer from logarithmic divergences reflecting a conformal anomaly. In a holographic setting, where surface operators can be computed by a minimal surface in AdS , the leading contribution to the anomaly comes from a divergence in the classical action (or area) of the minimal surface. We study the subleading correction to it due to quantum fluctuations of the minimal surface. In the same way that the divergence in the area does not require a global solution but only a near-boundary analysis, the same holds for the quantum corrections. We study the asymptotic form of the fluctuation determinant and show how to use the heat kernel to calculate the quantum anomaly. In the case of M2-branes describing surface operators in the $\mathcal{N} = (2, 0)$ theory in 6d, our calculation of the one-loop determinant reproduces expressions for the anomaly that have been found by less direct methods.

^{*}nadav.drukker@gmail.com

[†]omar.shahpo@kcl.ac.uk

[‡]trepanier.maxime@gmail.com

Contents

1	Introduction	1
2	Asymptotic $AdS_3 \subset AdS_7$	3
2.1	Asymptotic AdS_7 geometry	3
2.2	Asymptotic AdS_3 brane geometry	5
3	Determinants on asymptotically AdS_3	7
3.1	Heat kernel asymptotics	7
3.2	The bosonic determinant	11
3.3	Spinor bundles	14
4	Conclusions	17
A	Quadratic M2-brane action	19
A.1	Bosonic fluctuations	19
A.2	Fermionic fluctuations	20
B	Convoluting the fermionic heat kernels	22

1 Introduction

Surface operators are considered rather exotic observables in quantum field theories, except perhaps in the context of three dimensional theories where they appear as boundaries and interfaces, see e.g. [1–3] for early work. The reason is probably that they are harder to define and compute than local and line operators. And yet, if one bothers to look for them, they are ubiquitous: they appear already in the simplest examples of theories with scalar fields [4–9], in the context of entanglement entropy in four dimensions [10, 11], in 4d supersymmetric gauge theories [12–14], in six dimensional theories [15–17] and more. A simple way to engineer them is by adding extra two dimensional degrees of freedom on a surface and coupling them in some way to the bulk [18].

Once we define a surface operator we are faced with the task of computing its expectation value (or more generally correlation functions). This seems like a daunting problem, as it depends on the shape of the surface. In conformal field theories the situation is dramatically simplified, as surface operators are described by three *anomaly coefficients* [19]. Those are similar to dimensions of local operators and central charges for bulk theories. Calculating the expectation value of a surface operator is akin to computing a two-point function of local operators

$$\langle O(x)O(y) \rangle = \frac{c}{|x - y|^{2\Delta_O}}, \quad (1.1)$$

where Δ_O is the dimension of the operator. c is a normalisation constant that in most cases is scheme dependent, with exceptions for specific operators like the energy momentum tensor or conserved currents. The two point function is not invariant under conformal transformations, but transforms in a well defined way determined by Δ_O . In perturbative calculations of the dimension, the quantum corrections $\Delta_O = \Delta_O^{\text{cl}} + \delta\Delta_O$ appear multiplying a logarithm of a cutoff, like

$$\log \langle O(x)O(y) \rangle \sim 2\delta\Delta_O \log \epsilon. \quad (1.2)$$

We include the log on the left-hand side to account for the exponentiation of the irreducible diagrams.

The anomaly coefficients of surface operators play a similar role. They appear in the expectation value of the surface operator multiplied by particular conformally covariant local densities. They govern how the observable transforms under conformal transformations and in explicit calculations are the prefactors of logarithmic divergences. For a surface operator along the submanifold $\bar{\Sigma}$ with induced metric \bar{g} , its expectation value is given by

$$\langle O_{\bar{\Sigma}} \rangle = c r^{-\int_{\bar{\Sigma}} d^2\tau \sqrt{\bar{g}} \mathcal{A}}, \quad (1.3)$$

where \mathcal{A} is known as the anomaly density. Here r is some length-scale, similar to $|x - y|$ in (1.1) and c a scheme dependent factor. The bars are used to indicate quantities defined in the field theory or the boundary of AdS and to distinguish from quantities in the bulk.

In practice, what we would normally find is

$$\log \langle O_{\bar{\Sigma}} \rangle \sim \int_{\bar{\Sigma}} d^2\tau \sqrt{\bar{g}} \mathcal{A} \log \epsilon. \quad (1.4)$$

The form of \mathcal{A} is constrained by the Wess-Zumino conditions to be a linear combination of independent conformally invariant densities on a surface [20–22, 19]¹

$$\mathcal{A}_{\bar{\Sigma}} = \frac{1}{4\pi} \left[a_1 \bar{\mathcal{R}} + a_2 (\bar{H}^2 + 4 \text{tr} \bar{P}) + b \text{tr} \bar{W} \right], \quad (1.5)$$

where $\bar{\mathcal{R}}$ is the Ricci scalar of the surface, $\bar{H}^{\mu'}$ is the mean curvature of the surface, \bar{P} is the pullback of the Schouten tensor and \bar{W} is the pullback of the Weyl tensor to the surface. The geometric dependence is given by these three local densities and their prefactors a_1 , a_2 and b are the anomaly coefficients. They are characteristic of the operator, and should be thought of as part of the CFT data, just like Δ_O .

Given a definition of a surface operator, we should then aim to determine these three numbers. One approach is to choose three geometries for which $\{\bar{\mathcal{R}}, H^2 + 4 \text{tr} P, \text{tr} \bar{W}\}$ are linearly independent and determine the coefficients from the three examples. But given that \mathcal{A} is made of local quantities, determining it by a local calculation should also be possible.

This was realised beautifully by Graham and Witten for the surface operators in the six dimensional $\mathcal{N} = (2, 0)$ theory [24]. For the A_N theory at large N , the surface operators are given by M2-branes that end on $\bar{\Sigma}$ on the boundary of AdS_7 (or asymptotically locally AdS_7). While full classical M2-brane solutions are known in only a handful of examples [25–30], what Graham and Witten showed is that it is possible to solve for the near-boundary embedding and that this solution is enough in order to calculate the anomaly coefficients at large N .

The finite N corrections were found only much later and by indirect methods. They were first conjectured in [31] based on the holographic description of the 1/2 BPS plane in terms of bubbling geometries [32] and the calculation of entanglement entropy [33, 34]. The conjecture for a_1 is confirmed by b -extremisation [35] and the coefficient a_2 can be calculated using the superconformal index and the chiral algebra sector of the $\mathcal{N} = (2, 0)$ theory [36–38]. The coefficient b was conjectured to vanish in [39] and later proven in [40] using supersymmetry. The linear combination $a_1 + 2a_2$ was determined by direct calculation of the holographic dual

¹There is a second basis of conformal invariants commonly used, where $H^2 + 4 \text{tr} P$ is replaced by the traceless part of the second fundamental form squared, $\text{tr} \bar{\mathbb{I}}^2$. The relation between that basis and (1.5) can be found in [23].

of the 1/2 BPS spherical surface. First to leading order in N using the classical M2-brane [41] and to first subleading order in N by evaluating 1-loop determinants [42].

The purpose of this paper is to rederive these results by a direct extension of the asymptotic analysis of Graham and Witten. The same philosophy still holds: it is enough to know the near boundary geometry of the brane to extract the near boundary quadratic fluctuation action and the near boundary fluctuations are enough to determine the anomaly coefficients. We emphasize that the conformal anomalies obtained this way are associated with IR divergences near the boundary of AdS_3 —these are distinct from the well-studied Seeley coefficients capturing UV divergences of determinants (see e.g. [43]).

At the technical level we study the determinant of differential operators on asymptotic AdS_3 space. For pure AdS_3 , this can be easily extracted from the heat kernel for scalars and spinors which is known exactly [44–46]. For the purpose of deriving the anomaly, we require the first subleading correction of asymptotic AdS_3 .

To do that we employ perturbation theory, as was used for AdS_2 in a somewhat different context in [47]. The AdS_3 heat kernel times a step function serves as a Green’s function for the heat equation, allowing for a systematic perturbative expansion. The first correction involves a convolution of two heat kernels with an extra local differential operator between them. Evaluating this operator and using the properties of the heat kernel results in expressions related to the geometric invariants in (1.5) and a direct convolution of two heat kernels, which is easy to perform, see Section 3.1.

This formalism is valid for evaluating determinants in any context of asymptotic AdS_3 space. For the specific problem of the anomaly of the surface operators of the $\mathcal{N} = (2, 0)$ theory we need the quadratic action for M2-branes with asymptotically AdS_3 geometry inside asymptotically AdS_7 . This is derived in Appendix A without restriction to this geometry, but rather the quadratic fluctuation action around an arbitrary classical solution. This can be seen as an auxiliary result of this paper.

Applying our heat kernel technology to these differential operators, we evaluate their determinants for both bosonic and fermionic modes and extract the anomaly coefficients

$$a_1 = \frac{1}{2} + \mathcal{O}(1/N), \quad a_2 = -N + \frac{1}{2} + \mathcal{O}(1/N), \quad b = \mathcal{O}(1/N). \quad (1.6)$$

These expressions agree with the existing literature [31, 33–40, 42].

2 Asymptotic $AdS_3 \subset AdS_7$

In this section we set up the asymptotic AdS geometries that play a role in the calculation. We first review the work of Graham and Witten [24] finding a near-boundary classical brane solution. We then implement a change of coordinates to bring also the induced metric on the brane to Fefferman-Graham form [48], which simplifies the calculation of the determinants in the next section.

2.1 Asymptotic AdS_7 geometry

Following [24, 23], we look at a bulk geometry asymptotic to $AdS_7 \times S^4$. We choose y as the coordinate normal to the boundary, such that the asymptotic form of the metric is [24, 23]

$$G = \frac{R^2}{y^2} \left(dy^2 + \bar{G} + y^2 \bar{G}^{(1)} \right) + \frac{R^2}{4} G_{S^4} + \mathcal{O}(y^2). \quad (2.1)$$

\bar{G} is the metric on the boundary of space and $\bar{G}^{(1)}$ is fixed by the (super)gravity equations to be the Schouten tensor of \bar{G}

$$\bar{G}_{MN}^{(1)} = -\bar{P}_{MN}(\bar{G}). \quad (2.2)$$

Here we use MN for coordinates on all of (asymptotically) AdS_7 .

We study the embedding of a 3d M2-brane with world-volume Σ into this geometry, where the brane ends along a 2d surface $\bar{\Sigma}$ in the 6d boundary of asymptotically locally AdS_7 . We use coordinates τ^a with $a = 1, 2$ on the 2d surface and σ^μ with $\mu = 1, 2, 3$ and $\sigma^3 = y$ on the M2-brane world-volume. The asymptotically locally AdS_7 space is parametrised by coordinates x^M . We take three to be $x^\mu = \sigma^\mu$, and we require that the remaining coordinates $x^{\mu'}(\sigma)$ are orthogonal to $\bar{\Sigma}$ at the boundary. Finally we parametrise the S^4 by z^i with $i = 1, \dots, 4$ and we restrict to classical solutions localised at a point on S^4 $z^i(\sigma) = 0$, representing the north pole of S^4 .

The bosonic part of the M2-brane action [49] is the volume form of the induced metric and the pullback of the three-form A_3

$$S_{M2} = T_{M2} \int_{\Sigma} (\text{vol}_{\Sigma} + iA_3). \quad (2.3)$$

Here A_3 is the potential for the flux

$$F_4 = dA_3 = \frac{3}{8}R^3 \text{vol}_{S^4} + \mathcal{O}(y^2), \quad (2.4)$$

where vol_{S^4} is the volume of the unit sphere, as in (2.1). As mentioned above, we assume the surface is localised at a point in S^4 , so the pullback of A_3 in (2.3) vanishes.

The equations of motion for the brane are those of a minimal surface, which can be elegantly expressed as the vanishing of the mean curvature vector H^M . Recalling some definitions, the second fundamental form is

$$\mathbb{I}_{\mu\nu}^M = (\partial_{\mu}\partial_{\nu}x^P + \partial_{\mu}x^Q\partial_{\nu}x^R\Gamma_{QR}^P)(\delta_P^M - \partial^{\rho}x^M\partial_{\rho}x^N G_{NP}), \quad (2.5)$$

and using the inverse of the induced metric $g_{\mu\nu}$ we get the mean curvature vector as

$$H^M = g^{\mu\nu}\mathbb{I}_{\mu\nu}^M. \quad (2.6)$$

In fact, if the coordinates $x^{\mu'}$ are orthogonal to the brane not just at the boundary but everywhere, then the second fundamental form simplifies and the only nonzero components are

$$\mathbb{I}_{\mu\nu}^{\mu'} = -\frac{1}{2}G^{\mu'\nu'}\partial_{\nu'}G_{\mu\nu}\Big|_{\Sigma}. \quad (2.7)$$

We can express $H^{\mu'}$ in a Fefferman-Graham expansion as power series in y in (2.1). If $x^{\mu'}$ were constant, then it would be the same as the mean curvature on the boundary surface $\bar{\Sigma}$, so $H^{\mu'} = \bar{H}^{\mu'}$ (or strictly speaking, the pullback of $\bar{H}^{\mu'}$, since these two objects are in different bundles). If $x^{\mu'}$ is not a constant, we find to lowest nontrivial order in y [23]

$$H^{\mu'} = \bar{H}^{\mu'} + y^3\partial_y(y^{-3}\partial_y x^{\mu'}) + \mathcal{O}(y^2). \quad (2.8)$$

Imposing the equations of motion sets this to zero and fixes $x^{\mu'} = \bar{H}^{\mu'}y^2/4$, as found in [24].

Before imposing the equations of motion and keeping only terms of order $\mathcal{O}(y^0)$, the induced metric on the world-volume is

$$\begin{aligned} g_{yy} &= \frac{R^2}{y^2} \left(1 + \partial_y x^{\mu'} \partial_y x^{\nu'} \bar{g}_{\mu'\nu'} \right), \\ g_{ab} &= \frac{R^2}{y^2} \left(\bar{g}_{ab} - y^2 \bar{P}_{ab} - 2 \bar{\mathbb{I}}_{ab}^{\mu'} x^{\nu'} \bar{g}_{\mu'\nu'} \right), \\ g_{ay} &= 0, \end{aligned} \tag{2.9}$$

Here \bar{g}_{ab} is the metric on $\bar{\Sigma}$ and \bar{P}_{ab} is the pullback of the bulk Schouten tensor to the brane, and likewise for the second fundamental form. We also use $g_{\mu'\nu'} = G_{\mu'\nu'}|_{\Sigma}$ for the metric evaluated on the brane, and $\bar{g}_{\mu'\nu'} = G_{\mu'\nu'}|_{\bar{\Sigma}}$ for its value at the boundary of AdS .

Plugging in the solution to the asymptotic equations, $x^{\mu'} = \bar{H}^{\mu'} y^2/4$, gives the induced metric

$$\begin{aligned} g_{yy} &= \frac{R^2}{y^2} \left(1 + \frac{y^2}{4} \bar{H}^2 \right), \\ g_{ab} &= \frac{R^2}{y^2} \left(\bar{g}_{ab} - \bar{P}_{ab} y^2 - \frac{y^2}{2} \bar{\mathbb{I}}_{ab}^{\mu'} \bar{H}^{\nu'} \bar{g}_{\mu'\nu'} \right), \\ g_{ay} &= 0. \end{aligned} \tag{2.10}$$

The classical action (expanding the integrand to order y^{-1}) is now

$$\begin{aligned} S_{\text{classical}} &= T_{\text{M2}} \int_{\Sigma} d^3 \sigma \frac{R^3}{y^3} \sqrt{\bar{g} \left(1 + \frac{y^2}{4} \bar{H}^2 \right) \left(1 - y^2 \text{tr} \bar{P} - \frac{y^2}{2} \bar{H}^2 \right)} \\ &= T_{\text{M2}} R^3 \int_0^{\infty} \frac{dy}{y^3} \int_{\bar{\Sigma}} d^2 \tau \sqrt{\bar{g}} \left(1 - \frac{y^2}{8} \bar{H}^2 - \frac{y^2}{2} \text{tr} \bar{P} \right). \end{aligned} \tag{2.11}$$

Using $T_{\text{M2}} R^3 = 2N/\pi$, we get a quadratic and logarithmic divergences

$$S_{\text{classical}} = \frac{N}{\pi \epsilon^2} \text{vol}(\bar{\Sigma}) + \frac{N}{4\pi} \int_{\bar{\Sigma}} d^2 \tau \sqrt{\bar{g}} \left(\bar{H}^2 + 4 \text{tr} \bar{P} \right) \log \epsilon. \tag{2.12}$$

The quadratic divergence can be cancelled by an appropriate Legendre transform [50, 51]. Evaluating $\exp[-S_{\text{classical}}]$ then gives a power of $1/\epsilon$ which should be minus the anomaly (1.3), so we can identify the leading result at large N for the anomaly coefficients, namely $a_1 = b = 0$ and $a_2 = -N$ as in (1.6) [24].

2.2 Asymptotic AdS_3 brane geometry

The induced metric (2.10) is perfectly fine in order to plug into the action and evaluate the classical anomalies. The coordinates and metric have some issues that make them less than ideal for the quantum calculation in Section 3. First, the induced metric is not in Fefferman-Graham form, so the asymptotic AdS_3 structure needed in there is not manifest. Second, the quadratic fluctuation action derived in Appendix A assumes coordinates tangent and normal to the brane. The fact that $x^{\mu'}$ depends on y means that they are not orthogonal.

Both of those issues can be resolved with a change of coordinates to z and $u^{\mu'}$ defined via

$$\begin{aligned} x^{\mu'} &= u^{\mu'} + \frac{y^2}{4} \bar{H}^{\mu'}, \\ y &= z \exp \left(-\frac{1}{2} x^{\mu'} \bar{H}_{\mu'} - \frac{z^2}{16} \bar{H}^2 \right). \end{aligned} \tag{2.13}$$

The coordinates τ^a remain untouched for now. The choice of $u^{\mu'}$ is such that it vanishes on the classical solution (2.8) and then the definition of z makes the metric block diagonal and simplifies the induced metric. Plugging this into (2.1), we find to order z^0

$$\begin{aligned} \frac{G}{R^2} = & \frac{dz^2}{z^2} + \frac{e^{u^{\rho'} \bar{H}_{\rho'}}}{z^2} \left[\left(\bar{G}_{ab} d\tau^a d\tau^b + \bar{G}_{\mu'\nu'} du^{\mu'} du^{\nu'} \right) \left(1 + \frac{z^2}{8} \bar{H}^2 \right) + 2\bar{G}_{a\mu'} d\tau^a du^{\mu'} \right] \\ & - \frac{1}{2} \bar{H}_{\mu'} \bar{H}_{\nu'} du^{\mu'} du^{\nu'} + \bar{G}_{a\mu'} \left(\frac{dz d\tau^a}{z} \bar{H}^{\mu'} - \frac{d\tau^a du^{\nu'}}{2} \bar{H}^{\mu'} \bar{H}_{\nu'} \right) - \bar{P} + \frac{G_{S^4}}{4}. \end{aligned} \quad (2.14)$$

Like (2.1), this metric is also in Fefferman-Graham form. Indeed, the Fefferman-Graham expansion is unique only given a boundary metric \bar{G} . The terms in the square bracket in the first line of (2.14) (excluding $z^2 \bar{H}^2/8$) are exactly \bar{G} , so we see that we conformally transformed the boundary metric by $\exp[u^{\rho'} \bar{H}_{\rho'}]$. Thus (2.14) is the Fefferman-Graham expansion for a different choice of boundary metric in the same conformal class as \bar{G} .

All the terms of order z^0 (with the exception of the S^4 part) are minus the Schouten tensor for the conformally transformed boundary metric. Of course, given that now $u^{\mu'} = 0$ on the classical solution, the mean curvature for the surface $\bar{\Sigma}$ in this conformally transformed metric vanishes.

We can find the induced metric by setting $u^{\mu'} = 0$, where also $\bar{g}_{a\mu'} = 0$, resulting in

$$\frac{g}{R^2} = \frac{dz^2}{z^2} + \frac{d\tau^a d\tau^b}{z^2} \left(\bar{g}_{ab} - z^2 \bar{\Pi}_{ab} \right). \quad (2.15)$$

where we defined

$$\bar{\Pi}_{ab} = -\frac{1}{8} \bar{H}^2 \bar{g}_{ab} + \frac{1}{2} \bar{\Pi}_{ab}^{\nu'} \bar{H}^{\mu'} \bar{g}_{\mu'\nu'} + \bar{P}_{ab}, \quad \text{tr } \bar{\Pi} = \frac{1}{4} \bar{H}^2 + \text{tr } \bar{P}. \quad (2.16)$$

The induced metric (2.15) is now also in Fefferman-Graham form for an asymptotically locally AdS_3 with boundary metric \bar{g} on $\bar{\Sigma}$. Note that the correction $z^2 \bar{\Pi}$ is not the Schouten tensor of \bar{g} , as in the bulk case (2.2), since we do not impose an Einstein equation on the world-volume, but rather the minimal surface equations.

In fact, we could have started the analysis from this statement

Given a surface $\bar{\Sigma} \subset \bar{M}_6$ and a conformal class on \bar{M}_6 , one can choose a metric in that class such that $\bar{\Sigma}$ is minimal, so $\bar{H}^{\mu'} = 0$. The Graham-Witten solution then is $u^{\mu'} = 0$ and the induced metric (2.10) is automatically in Fefferman-Graham form.²

Note that the conformal transformation used in (2.13), (2.14) is defined only locally, but that shouldn't matter for our calculation of a local anomaly density.

In practical term, this other approach would set $\bar{H}^{\mu'} = 0$ in all calculations, so $\bar{\Pi}_{ab} = \bar{P}_{ab}$. We could also work in this setting and replace $\text{tr } \bar{P} \rightarrow \bar{H}^2/4 + \text{tr } \bar{P}$ in the final expressions, as this is the combination that appears in the anomaly (1.5). We did perform the calculation keeping nonzero $\bar{H}^{\mu'}$ just to make sure that we do not make any mistakes.

²The discussion above makes this statement manifest and despite asking several experts, we could not find a reference that discusses this in any detail.

3 Determinants on asymptotically AdS_3

Having reviewed the classical M2-brane solution and presented a convenient form for the metric with explicit asymptotically AdS_3 induced metric on the brane, we now develop the tools to evaluate determinants of differential operators on such submanifolds. In turn, we apply this to the semiclassical M2-brane, where the action for quadratic fluctuations is evaluated in Appendix A.

In the absence of a full classical solution one cannot expect to be able to fully evaluate the determinant, but we are interested only in the logarithmically divergent terms that contribute to the anomaly and arise from IR divergences near the boundary of AdS . We rely on the heat kernel method to evaluate the determinants and apply them only in the near boundary region to first nontrivial order beyond pure AdS_3 .

The determinant of a differential operator L can be, in principle, calculated from the heat kernel $K(t; \sigma, \sigma_0)$ satisfying

$$(L_\sigma + \partial_t)K(t; \sigma, \sigma_0) = 0, \quad \lim_{t \rightarrow 0} K(t; \sigma, \sigma_0) = \frac{1}{\sqrt{g}} \delta^{(3)}(\sigma - \sigma_0). \quad (3.1)$$

The subscript σ here is meant to emphasize that the differential operator acts on the point σ . If one can solve the heat kernel equation, the determinant of the operator is then obtained as

$$\log \det(L) = - \int_\Sigma \text{vol}_\Sigma \int_0^\infty \frac{dt}{t} \lim_{\sigma \rightarrow \sigma_0} K(t; \sigma, \sigma_0). \quad (3.2)$$

For more details on the heat kernel and its applications, see the review [43].

We proceed now to study heat kernels on asymptotically AdS_3 for the massive scalar laplacian, for vector bundles and for spinors. In the case of the M2-brane that we are interested in, there are four massless scalars from the fluctuations in S^4 , four scalars for fluctuations in AdS_7 , which see the nontrivial geometry of the normal bundle of the M2-brane so should be treated as taking value in an $SO(4)$ -vector bundle, and finally there are 16 fermi fields. See Appendix A for the derivation of the quadratic action.

3.1 Heat kernel asymptotics

In practice, solving the heat kernel equation for an arbitrary kinetic operator L on an arbitrary manifold is impossible. Fortunately, for the purpose of extracting anomaly coefficients, we only need the behavior of the differential operators near the boundary of AdS .

We start with a massive scalar laplacian $L = -\Delta + M^2$ on the asymptotically locally AdS_3 world-volume given in (2.15) and further simplify the calculation by choosing Riemann normal coordinates for the metric \bar{g} about a particular point (corresponding to $\tau_0 = (0, 0)$ and at fixed z_0), such that

$$\begin{aligned} g_{zz} &= \frac{R^2}{z^2}, \\ g_{az} &= 0, \\ g_{ab} &= \frac{R^2}{z^2} \delta_{ab} - R^2 \left(\frac{1}{3z^2} \bar{\mathcal{R}}_{abcd} \tau^c \tau^d + \bar{\Pi}_{ab} \right). \end{aligned} \quad (3.3)$$

For simplicity we set the AdS radius $R = 1$ in the following. We proceed by considering the parenthesis on the last line as a perturbation about the AdS_3 geometry. More precisely, we treat (τ^a, z) as homogeneous coordinates and expand in a power series in them, so $g_{\mu\nu} = g_{\mu\nu}^{(-2)} + g_{\mu\nu}^{(0)} + \dots$ with the index indicating the degree and

$$\begin{aligned} g_{\mu\nu}^{(-2)} d\sigma^\mu d\sigma^\nu &= \frac{1}{z^2} (dz^2 + \delta_{ab} d\tau^a d\tau^b), \\ g_{\mu\nu}^{(0)} d\sigma^\mu d\sigma^\nu &= - \left(\frac{1}{3} \bar{\mathcal{R}}_{abcd} \frac{\tau^c \tau^d}{z^2} + \bar{\Pi}_{ab} \right) d\tau^a d\tau^b. \end{aligned} \quad (3.4)$$

Likewise, the determinant of the metric is expanded as

$$\begin{aligned} \sqrt{g}^{(-3)} &= \frac{1}{z^3}, \\ \sqrt{g}^{(-1)} &= -\frac{1}{z} \left(\frac{1}{6} \bar{\mathcal{R}}_{cd} \frac{\tau^c \tau^d}{z^2} + \frac{1}{2} \text{tr} \bar{\Pi} \right). \end{aligned} \quad (3.5)$$

We now expand the kinetic operator $L = -\Delta + M^2$ in the same way as

$$L = L^{(0)} + L^{(2)} + \dots \quad (3.6)$$

For a massive field, we assume an expansion of the mass term $M^2 = \mathcal{M}^{(0)} + \mathcal{M}^{(2)} + \dots$, and the kinetic operator takes the form

$$\begin{aligned} L^{(0)} &= -\frac{1}{\sqrt{g}^{(-3)}} \partial_\mu \sqrt{g}^{(-3)} g^{(2)\mu\nu} \partial_\nu + \mathcal{M}^{(0)}, \\ L^{(2)} &= \frac{\sqrt{g}^{(-1)}}{(\sqrt{g}^{(-3)})^2} \partial_\mu \sqrt{g}^{(-3)} g^{(2)\mu\nu} \partial_\nu - \frac{1}{\sqrt{g}^{(-3)}} \partial_\mu \left(\sqrt{g}^{(-1)} g^{(2)\mu\nu} + \sqrt{g}^{(-3)} g^{(4)\mu\nu} \right) \partial_\nu + \mathcal{M}^{(2)}, \end{aligned} \quad (3.7)$$

where of course $g^{(4)\mu\nu} = -g^{(2)\mu\rho} g_{\rho\sigma}^{(0)} g^{(2)\sigma\nu}$.

Plugging in the metric (3.4) yields

$$\begin{aligned} L^{(0)} &= -z^3 \partial_z \left(\frac{1}{z} \partial_z \right) - z^2 \delta^{ab} \partial_a \partial_b + \mathcal{M}^{(0)}, \\ L^{(2)} &= \text{tr} \bar{\Pi} z^3 \partial_z + \frac{2z^2}{3} \bar{\mathcal{R}}_{ab} \tau^a \partial_b - \left(\frac{z^2}{3} \bar{\mathcal{R}}_{abcd} \tau^c \tau^d + z^4 \bar{\Pi}_{ab} \right) \partial_a \partial_b + \mathcal{M}^{(2)}. \end{aligned} \quad (3.8)$$

We emphasize that the kinetic operator L acts on the point σ , and is evaluated at that point. The quantities $(\mathcal{M}, \bar{\Pi}, \dots)$ appearing in $L^{(2)}$ are all evaluated at σ_0 .

The heat kernel itself can also be expanded as

$$K(t; \sigma, \sigma_0) = K^{(0)}(t; \sigma, \sigma_0) + K^{(2)}(t; \sigma, \sigma_0) + \dots \quad (3.9)$$

To calculate the anomaly we only need those first two terms. $K^{(0)}$ is of homogeneous degree zero, so the volume integral in (3.2) has a quadratic divergence. It may also contribute logarithmic divergences from the subleading terms in the volume form $\sqrt{g}^{(-1)}$ (3.5). $K^{(2)}$ is quadratic, so like the y^2 terms in the classical action (2.11), gives logarithmic divergences contributing to the conformal anomaly.

The heat kernel equation (3.1) then reduces to the recursive relations

$$(L^{(0)} + \partial_t)K^{(0)} = 0, \quad (L^{(0)} + \partial_t)K^{(2)} = -L^{(2)}K^{(0)}. \quad (3.10)$$

The first equation is the AdS_3 heat kernel equation, its solution is given by [44] (also [45,46])

$$K^{(0)}(t; \sigma, \sigma_0) = \frac{1}{(4\pi t)^{3/2}} \frac{\rho}{\sinh \rho} \exp\left(-\left(1 + \mathcal{M}^{(0)}\right)t - \frac{\rho^2}{4t}\right), \quad (3.11)$$

where ρ is the geodesic distance in AdS_3 between σ and σ_0

$$\rho = \text{arccosh}\left(\frac{z^2 + z_0^2 + |\tau - \tau_0|^2}{2zz_0}\right). \quad (3.12)$$

The equation for $K^{(2)}$ can be solved by observing that $K^{(0)}(t' - t; \sigma, \sigma_0)\Theta(t' - t)$ (with Θ the Heaviside step function) is a Green's function for $L^{(0)} + \partial_t$, i.e.

$$(L^{(0)} + \partial_t)K^{(0)}(t' - t; \sigma, \sigma_0)\Theta(t' - t) = \frac{1}{\sqrt{g^{(-3)}}}\delta^{(3)}(\sigma - \sigma_0)\delta(t' - t). \quad (3.13)$$

This allows us to express $K^{(2)}$ as the integral

$$\begin{aligned} K^{(2)}(t; \sigma, \sigma_0) &= -\int_0^t dt' \int_{\Sigma} d^3\sigma' \sqrt{g^{(-3)}} K^{(0)}(t - t'; \sigma, \sigma') L_{\sigma'}^{(2)} K^{(0)}(t'; \sigma', \sigma_0) \\ &\quad - \frac{\sqrt{g^{(-1)}}}{\sqrt{g^{(-3)}}} K^{(0)}(t; \sigma, \sigma_0), \end{aligned} \quad (3.14)$$

where the second line ensures that

$$\lim_{t \rightarrow 0} K^{(2)}(t; \sigma, \sigma_0) = -\frac{\sqrt{g^{(-1)}}}{(\sqrt{g^{(-3)}})^2} \delta^{(3)}(\sigma - \sigma_0), \quad (3.15)$$

so that the boundary conditions at $t = 0$ (3.1) are compatible with the subleading terms in the measure \sqrt{g} (3.5).

To evaluate the determinant we need (3.2) the coincident point limit of the heat kernel $K(t; \sigma_0, \sigma_0)$, also known as the trace of the heat kernel. For $K^{(0)}$, we have the explicit expression (3.11) from which we get

$$\lim_{\sigma \rightarrow \sigma_0} K^{(0)}(t; \sigma, \sigma_0) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\left(1 + \mathcal{M}^{(0)}\right)t\right). \quad (3.16)$$

For $K^{(2)}$ we can use the integral expression (3.14) to evaluate the coincident limit. $K^{(0)}$ depends on the coordinates only through the geodesic distance $\rho(\sigma', \sigma_0)$, so it is natural to change coordinates for AdS_3 to

$$ds^2 = \left(d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)\right). \quad (3.17)$$

The change of variables is given by

$$z = \frac{z_0}{\cosh \rho - \sinh \rho \cos \theta}, \quad \tau_a = \frac{z_0 \sinh \rho \sin \theta e_a}{\cosh \rho - \sinh \rho \cos \theta}, \quad (3.18)$$

where $e_a = (\cos \phi, \sin \phi)_a$.

We now can express $L^{(2)}$ (3.8) in these coordinates, but since we act with it on $K^{(0)}$, we only need to keep derivatives with respect to ρ . Those are

$$L^{(2)} f(\rho) = -z_0^2 \left[\frac{\bar{\mathcal{R}} \sinh \rho \sin^2 \theta}{6(\cosh \rho - \sinh \rho \cos \theta)^3} \partial_\rho + \text{tr} \bar{\Pi} \frac{\coth \rho}{(\cosh \rho - \sinh \rho \cos \theta)^2} \partial_\rho + \bar{\Pi}_{ab} \frac{\sin^2 \theta e_a e_b}{(\cosh \rho - \sinh \rho \cos \theta)^4} (\partial_\rho^2 - \coth \rho \partial_\rho) \right] f(\rho) + \mathcal{M}^{(2)} f(\rho). \quad (3.19)$$

The integral in (3.14) is

$$- \int_0^t dt' \int_\Sigma d\rho d\theta d\phi \sinh^2 \rho \sin \theta K^{(0)}(t - t'; \rho) L^{(2)} K^{(0)}(t'; \rho). \quad (3.20)$$

The ϕ integral is easy to do using

$$\int_0^{2\pi} d\phi e_a e_b = \pi \delta_{ab}. \quad (3.21)$$

The θ integral can be done as well to get

$$-4\pi \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho K^{(0)}(t - t'; \rho) \times \left[z_0^2 \left(-\frac{1}{3} \text{tr} \bar{\Pi} (\partial_\rho^2 + 2 \coth \rho \partial_\rho) + \frac{\bar{\mathcal{R}} \cosh \rho \sinh \rho - \rho}{6 \sinh^2 \rho} \partial_\rho \right) + \langle \mathcal{M}^{(2)} \rangle \right] K^{(0)}(t'; \rho). \quad (3.22)$$

$\langle \mathcal{M}^{(2)} \rangle$ denotes the average of $\mathcal{M}^{(2)}$ over the spherical coordinates. We can simplify further the coefficient of $\text{tr} \bar{\Pi}$ by using the heat kernel equation (3.10), which in these coordinates is

$$(-\partial_\rho^2 - 2 \coth \rho \partial_\rho + \mathcal{M}^{(0)} + \partial_t') K^{(0)}(t'; \rho) = 0, \quad (3.23)$$

which removes both first and second order ρ derivatives and we obtain

$$4\pi \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho K^{(0)}(t - t'; \rho) \times \left[\frac{z_0^2}{3} \text{tr} \bar{\Pi} (\partial_{t'} + \mathcal{M}^{(0)}) - \frac{z_0^2 \bar{\mathcal{R}} \cosh \rho \sinh \rho - \rho}{6 \sinh^2 \rho} \partial_\rho - \langle \mathcal{M}^{(2)} \rangle \right] K^{(0)}(t'; \rho). \quad (3.24)$$

The remaining integrals can be evaluated without much difficulty after plugging in the expression for $K^{(0)}$ in (3.11). A more elegant way to evaluate them is to use partial integration to reduce them to a convolution of heat kernels. Consider first the term proportional to $\text{tr} \bar{\Pi}$. We can take out the t' derivative as

$$\begin{aligned} & \frac{4\pi z_0^2}{3} \text{tr} \bar{\Pi} \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho K^{(0)}(t - t'; \rho) (\partial_{t'} + \mathcal{M}^{(0)}) K^{(0)}(t'; \rho) \\ &= \frac{4\pi z_0^2}{3} \text{tr} \bar{\Pi} \int_0^t dt' (\partial_{t'} + \partial_t + \mathcal{M}^{(0)}) \int_0^\infty d\rho \sinh^2 \rho K^{(0)}(t - t'; \rho) K^{(0)}(t'; \rho). \end{aligned} \quad (3.25)$$

Now the ρ integral is the convolution of two heat kernels (up to the factor 4π from the spherical integral), giving simply $K^{(0)}(t; \sigma_0, \sigma_0)$. The t' dependence completely drops out and the integral over it gives t , leaving

$$\frac{z_0^2}{3} \text{tr} \bar{\Pi} t (\partial_t + \mathcal{M}^{(0)}) K^{(0)}(t; \sigma_0, \sigma_0). \quad (3.26)$$

Second, for the term proportional to $\bar{\mathcal{R}}$, with an exchange of $t' \rightarrow t - t'$ the derivative can be moved to the other $K^{(0)}$, so we can write it as

$$\begin{aligned} & -\frac{\pi z_0^2}{3} \bar{\mathcal{R}} \int_0^t dt' \int_0^\infty d\rho (\cosh \rho \sinh \rho - \rho) \partial_\rho \left(K^{(0)}(t - t'; \rho) K^{(0)}(t'; \rho) \right) \\ & = \frac{2\pi z_0^2}{3} \bar{\mathcal{R}} \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho K^{(0)}(t - t'; \rho) K^{(0)}(t'; \rho) = \frac{z_0^2}{6} \bar{\mathcal{R}} t K^{(0)}(t; \sigma_0, \sigma_0). \end{aligned} \quad (3.27)$$

The last term in (3.24) is that proportional to $\mathcal{M}^{(2)}$, and assuming $\langle \mathcal{M}^{(2)} \rangle$ doesn't depend on ρ (as we show in the case of interest below), it is directly the convolution of the two heat kernels.

Adding (3.26), (3.27) and the contribution from $\mathcal{M}^{(2)}$ and plugging the value of the heat kernel at coincident points (3.16) gives

$$\left[-\frac{z_0^2}{3} \text{tr} \bar{\Pi} \left(\frac{3}{2t} + 1 \right) + \frac{z_0^2}{6} \bar{\mathcal{R}} + \langle \mathcal{M}^{(2)} \rangle \right] \frac{\exp \left(-(1 + \mathcal{M}^{(0)})t \right)}{(4\pi)^{3/2} t^{1/2}}. \quad (3.28)$$

Finally, adding the term which corrects the boundary conditions from the second line of (3.14), it exactly cancels the $3/2t$ term and we get

$$K^{(2)}(t; \sigma_0, \sigma_0) = \left[\frac{z_0^2}{6} (\bar{\mathcal{R}} - 2 \text{tr} \bar{\Pi}) - \langle \mathcal{M}^{(2)} \rangle \right] \frac{\exp \left(-(1 + \mathcal{M}^{(0)})t \right)}{(4\pi)^{3/2} t^{1/2}}. \quad (3.29)$$

Note the appearance of the combination $\bar{\mathcal{R}} - 2 \text{tr} \bar{\Pi}$. The correction to the AdS_3 heat kernel coming from geometric terms is expected to be proportional to this combination (up to perhaps also a contribution from terms involving the Weyl tensor), because of the following consistency check. Take the surface to be a sphere $\bar{\Sigma} = S^2$, with corresponding classical geometry AdS_3 [26]. In that case the heat kernel is exactly $K^{(0)}$, so we should find $K^{(2)} = 0$. Indeed, although both $\bar{\mathcal{R}} = 2$ and $\text{tr} \bar{\Pi} = 1$ are nonzero for the sphere, the combination $\bar{\mathcal{R}} - 2 \text{tr} \bar{\Pi}$ vanishes.

For the evaluation of the determinant (3.2), we need the t integral of the trace of the heat kernel, (3.16) and (3.29), which are simple gamma function integrals. Reinstating powers of R we get

$$\int_0^\infty \frac{dt}{t} K^{(0)}(t; \sigma_0, \sigma_0) = \frac{(1 + \mathcal{M}^{(0)} R^2)^{\frac{3}{2}}}{6\pi R^3}, \quad (3.30)$$

$$\int_0^\infty \frac{dt}{t} K^{(2)}(t; \sigma_0, \sigma_0) = \frac{\sqrt{1 + \mathcal{M}^{(0)} R^2}}{4\pi R} \left[\frac{z_0^2}{6R^2} (2 \text{tr} \bar{\Pi} - \bar{\mathcal{R}}) + \langle \mathcal{M}^{(2)} \rangle \right]. \quad (3.31)$$

3.2 The bosonic determinant

We can use the results above to evaluate the determinant of bosonic fluctuations of M2-branes describing surface operators. The quadratic action is derived in Appendix A and consists of eight bosonic and eight fermionic modes. The bosonic modes are in turn split into the four modes related to the S^4 part of the geometry and four directions transverse to the M2-brane world-volume within AdS_7 .

The first four bosonic modes are massless scalars, and their kinetic operator is simply $L = -\Delta$, see (A.6). The path integral over these fluctuations contributes $-\frac{4}{2} \log \det(-\Delta)$ to

the log of the partition function, which can be evaluated using (3.2). The integrand contains both a cubic and a simple pole in z . Performing the z integral, the cubic pole leads to a quadratic divergence in the cutoff, which we drop (it can be removed by a local counterterm). The simple pole leads to $\log \epsilon$ divergence, and leaves an integral over $\bar{\Sigma}$ as in the expression for the anomaly (1.4). The contribution of these four modes to the anomaly density is therefore

$$\begin{aligned} \mathcal{A}^{S^4} &= -\frac{4}{2} \operatorname{res}_{z_0 \rightarrow 0} \left[\int_0^\infty \frac{dt}{t} \frac{\sqrt{g}^{(-3)} + \sqrt{g}^{(-1)} + \dots}{\sqrt{g}} \left(K^{(0)}(t; \sigma_0, \sigma_0) + K^{(2)}(t; \sigma_0, \sigma_0) + \dots \right) \right] \\ &= -2 \operatorname{res}_{z_0 \rightarrow 0} \left[\left(-\frac{1}{2z_0} \operatorname{tr} \bar{\Pi} \right) \int_0^\infty \frac{dt}{t} K^{(0)}(t; \sigma_0, \sigma_0) + \frac{1}{z_0^3} \int_0^\infty \frac{dt}{t} K^{(2)}(t; \sigma_0, \sigma_0) + \dots \right] \\ &= \frac{1}{4\pi} \frac{1}{3} \bar{\mathcal{R}}. \end{aligned} \tag{3.32}$$

The second line has the metric in Riemann normal coordinates (3.5), and in going to the last line we use the expressions for the integrated heat kernel (3.30) and (3.31).

The second set of four bosonic modes parametrise the normal bundle $N\Sigma$ to the world-volume of the M2-brane. The kinetic operator derived in (A.6) acts on the fiber bundle (with fiber $V = \mathbb{R}^4$), and the path integral over the fluctuations contributes a factor $-\frac{1}{2} \log \det_V(-\Delta + M^2)$ to the log of the partition function. Evaluating this determinant requires a slight generalisation of the formalism for the scalars.

The heat kernel is still defined by (3.1), with the initial condition multiplied on the right hand side by the identity matrix $\mathbf{1}_V$, and the determinant of L has the additional trace

$$\log \det_V(L) = - \int_\Sigma d^3\sigma \sqrt{g} \int_0^\infty \frac{dt}{t} \operatorname{tr}_V K(t; \sigma, \sigma). \tag{3.33}$$

The class of differential operators we need to consider are laplacians on the normal bundle along with a mass term. They take the form

$$L = -\frac{1}{\sqrt{g}} D_\mu (\sqrt{g} g^{\mu\nu} D_\nu) + M^2, \tag{3.34}$$

with D_μ the covariant derivative acting on fluctuations $\zeta^{m'}$

$$D_\mu \zeta^{m'} \equiv \partial_\mu \zeta^{m'} + \omega_\mu^{m'n'} \zeta^{n'}, \tag{3.35}$$

where $\omega_\mu^{m'n'}$ is the spin connection on the normal bundle. To evaluate it, we pick a vielbein for the metric G (2.14). We then expand it at $u = 0$ in (z, τ^a) as

$$E = E^{(-1)} + E^{(1)} + \dots, \tag{3.36}$$

where $E^{(-1)}$ is a vielbein for the AdS_7 metric, for instance

$$E^{(-1)m} = \frac{1}{z} \delta_\mu^m d\sigma^\mu, \quad E^{(-1)m'} = \frac{1}{z} \bar{E}_{\mu'}^{m'} du^{\mu'}. \tag{3.37}$$

The relevant spin connection is defined by

$$\omega_\mu^{m'n'} = E_{\mu'}^{m'} \left(\partial_\mu E^{\mu'n'} + \Gamma_{\mu\rho'}^{\mu'} E^{\rho'n'} \right). \tag{3.38}$$

Importantly, this vanishes using the leading order vielbein $E^{(-1)}$, so $\omega^{(-1)} = 0$. If we expand L (3.34), to leading order we recover four copies of the scalar laplacian on AdS_3 (3.8). The mass matrix $\mathcal{M}_{m'n'}^{(0)}$ is diagonal with eigenvalues 3 [42]. Likewise $K^{(0)}$ is a diagonal matrix with entries as in (3.11). Had $\omega^{(-1)}$ not vanished, we would have had to solve for the vector heat kernel, in analogy with the spinors in the next section. Instead, the nontrivial bundle only affects $L^{(2)}$ and $K^{(2)}$.

The first subleading correction is $L^{(2)}$ as in (3.7) with the additional contribution from the spin connection

$$-\frac{1}{\sqrt{g^{(-3)}}} \partial_\mu \left(\sqrt{g^{(-3)}} g^{(2)\mu\nu} \right) \omega_\mu^{(1)m'n'} - 2g^{(2)\mu\nu} \omega_\mu^{(1)m'n'} \partial_\nu. \quad (3.39)$$

Here $\omega^{(1)}$ is the subleading correction to the spin connection. $K^{(2)}$ is then given by (3.14), and we are interested in the trace of the heat kernel $\text{tr}_V K^{(2)}$. Using that $\mathcal{M}^{(0)}$ is a diagonal matrix (and so is $K^{(0)}$), while $\omega^{(1)}$ is antisymmetric, we see that these additional contributions simply drop out of the trace

$$\text{tr}_V K^{(0)} \omega^{(1)} K^{(0)} = 0. \quad (3.40)$$

We therefore do not even need to write down explicit expressions for the corrections $E^{(1)}$ and $\omega_\mu^{(1)m'n'}$. Instead, $K^{(2)}$ is a straightforward generalisation of the scalar case, where the only nontrivial matrix structure is in $\mathcal{M}^{(2)}$. The result is the same as (3.29)

$$\text{tr}_V K^{(2)}(t; \sigma_0, \sigma_0) = \text{tr}_V \left[\left(\frac{z_0^2}{6} (\bar{\mathcal{R}} - 2 \text{tr} \bar{\Pi}) - \langle \mathcal{M}^{(2)} \rangle \right) \frac{\exp\left(- (1 + \mathcal{M}^{(0)})t\right)}{(4\pi)^{3/2} t^{1/2}} \right]. \quad (3.41)$$

For the M2-brane, the mass matrix M^2 is a function over the world-volume Σ derived in (A.6). Its trace over the four transverse AdS_7 modes is

$$M_{m'm'}^2 = -g^{\mu\nu} G^{\mu'\nu'} R_{\mu'\mu\nu'\nu} - g^{\mu\nu} g^{\rho\sigma} G_{\mu'\nu'} \tilde{\Pi}_{\mu\rho}^{\mu'} \tilde{\Pi}_{\sigma\nu}^{\nu'}, \quad (3.42)$$

where here $R_{\mu'\mu\nu'\nu}$ is the bulk Riemann tensor (as opposed to the world-volume Riemann tensor \mathcal{R}). In order to read $\mathcal{M}^{(0)}$ and $\mathcal{M}^{(2)}$ we rewrite M^2 as follows. Recall that μ, ν are the M2-brane directions and μ', ν' the transverse ones, so we can write

$$G^{\mu'\nu'} R_{\mu'\mu\nu'\nu} = R_{\mu\nu} - g^{\rho\sigma} R_{\mu\rho\nu\sigma}. \quad (3.43)$$

The Ricci tensor is fixed by the bulk equations of motion

$$R_{\mu\nu} = -6G_{\mu\nu} = -6g_{\mu\nu}, \quad (3.44)$$

and for the components of the Riemann tensor, we relate them to the world-volume curvature \mathcal{R} via the Gauss-Codazzi equation

$$R_{\mu\rho\nu\sigma} = \mathcal{R}_{\mu\rho\nu\sigma} + 2\Pi_{\mu[\sigma}^{\mu'} \Pi_{\nu]\rho}^{\nu'} G_{\mu'\nu'}. \quad (3.45)$$

Together we then find

$$M_{m'm'}^2 = 18 + \mathcal{R}, \quad (3.46)$$

which is now expressed solely in terms of quantities depending on the induced metric g (3.3). It is easy to show that the world-volume Ricci scalar has expansion $\mathcal{R} = -6 + \mathcal{R}^{(2)} + \dots$ with

$$\mathcal{R}^{(2)} = \frac{z_0^2}{(\cosh \rho - \cos \theta \sinh \rho)^2} (\bar{\mathcal{R}} - 2 \operatorname{tr} \bar{\Pi}). \quad (3.47)$$

The leading trace of the mass matrix is $\operatorname{tr} \mathcal{M}^{(0)} = 12$, as expected for four modes of mass-squared 3. The subleading correction is

$$\operatorname{tr} \mathcal{M}^{(2)} = \mathcal{R}^{(2)} = \frac{z_0^2}{(\cosh \rho - \cos \theta \sinh \rho)^2} (\bar{\mathcal{R}} - 2 \operatorname{tr} \bar{\Pi}). \quad (3.48)$$

Integrating over S^2 ,

$$\langle \operatorname{tr} \mathcal{M}^{(2)} \rangle = \frac{1}{4\pi} \int d\theta d\phi \sin \theta \operatorname{tr} \mathcal{M}^{(2)} = z_0^2 (\bar{\mathcal{R}} - 2 \operatorname{tr} \bar{\Pi}). \quad (3.49)$$

As we anticipated in (3.27), this does not depend on ρ .

To get the contribution to the anomaly, we again take the t integral of the trace of the heat kernel (3.30), (3.31), and look for the residue at $z_0 \rightarrow 0$ as in (3.32). We get

$$\begin{aligned} \mathcal{A}^{N\Sigma} &= -\frac{1}{2} \operatorname{res}_{z_0 \rightarrow 0} \left[\left(-\frac{1}{2z_0} \operatorname{tr} \bar{\Pi} \right) \int_0^\infty \frac{dt}{t} \operatorname{tr}_V K^{(0)}(t; \sigma_0, \sigma_0) + \frac{1}{z_0^3} \int_0^\infty \frac{dt}{t} \operatorname{tr}_V K^{(2)}(t; \sigma_0, \sigma_0) + \dots \right] \\ &= \frac{1}{4\pi} \left(-\frac{1}{3} \bar{\mathcal{R}} + 6 \operatorname{tr} \bar{\Pi} \right). \end{aligned} \quad (3.50)$$

3.3 Spinor bundles

A similar analysis can be performed for spinors. The Dirac operator in euclidean signature is $\mathcal{D} - M$ and it acts on the spinor bundle S with fiber \mathbb{C}^8 corresponding to the (chiral) spinor representations of $SO(3)$, $SO(4)_N$ for normal directions and $SO(4)_S$ for S^4 directions. We are interested in the determinant of its square

$$\det \left(-\mathcal{D}^2 + M^2 \right) \equiv \det (L_F). \quad (3.51)$$

The square of the Dirac operator is related to the spinor laplacian via a generalisation of the Lichnerowicz formula (see e.g. [52]). Using γ_m and $\rho_{m'}$ for the gamma matrices of $SO(3)$ and $SO(4)_N$ respectively (so $\{\gamma_m, \gamma_n\} = 2\delta_{mn}$ and similarly for ρ) and $\gamma_{mn} = \frac{1}{2} [\gamma_m, \gamma_n]$ for the antisymmetrised product, it reads

$$\mathcal{D}^2 = \Delta_{1/2} - \frac{\mathcal{R}}{4} + \frac{1}{8} \gamma_{mn} \rho_{m'n'} R_{mnm'n'}. \quad (3.52)$$

$\Delta_{1/2}$ is spinor laplacian given in terms of the spinor covariant derivative as

$$\Delta_{1/2} = \frac{1}{\sqrt{g}} D_\mu (\sqrt{g} g^{\mu\nu} D_\nu), \quad D_\mu \psi = \left(\partial_\mu - \frac{1}{4} \gamma_{mn} \omega_\mu^{mn} - \frac{1}{4} \rho_{m'n'} \omega_\mu^{m'n'} \right) \psi, \quad (3.53)$$

and R is the curvature 2-form of the normal bundle

$$R_{\mu\nu}^{m'n'} = 2\partial_{[\mu}\omega_{\nu]}^{m'n'} + 2\omega_{[\mu}^{m'p'}\omega_{\nu]}^{p'n'} . \quad (3.54)$$

Note that we assume here again that the embedding in S^4 is trivial, so we do not see its spin connection.

To proceed, we again look for a near- AdS_3 expansion of the operator $L_F = -\not{D}^2 + M^2$ and the corresponding heat kernel. We also restrict to constant $M^2 = \mathcal{M}^{(0)}$. We then require the generalisation of Section 3.1 to the case of spinor bundles on AdS_3 .

To leading order, the metric (3.4) is AdS_3 and the differential operator $L^{(0)}$ is the AdS_3 spinor laplacian. The spinor heat kernel derived in [53] (see also [54, 55]) transforms nontrivially as a bispinor at points σ and σ_0 . It takes the factorised form

$$K^{(0)}(t; \sigma, \sigma_0) = U(\sigma, \sigma_0)k_F(t; \rho) , \quad (3.55)$$

where U encodes the bispinor transformations and k_F is a function of t and the geodesic distance ρ between σ and σ_0 only. On a curved background, U can be obtained by parallel transporting a spinor along the geodesic connecting σ and σ_0

$$U(\sigma, \sigma_0) = \mathcal{P} \exp \int_{\sigma_0}^{\sigma} dx^{\mu} \frac{1}{2} \omega_{\mu}^{mn} \gamma_{mn} . \quad (3.56)$$

To write explicit expressions, we use the coordinates in (3.17) with σ_0 the origin of AdS_3 and the vielbein

$$\begin{aligned} e^1 &= \sin \theta \cos \phi d\rho + \sinh \rho (\cos \theta \cos \phi d\theta - \sin \theta \sin \phi d\phi) , \\ e^2 &= \sin \theta \sin \phi d\rho + \sinh \rho (\cos \theta \sin \phi d\theta + \sin \theta \cos \phi d\phi) , \\ e^3 &= \cos \theta d\rho - \sinh \rho \sin \theta d\theta . \end{aligned} \quad (3.57)$$

This frame is diagonal in projective coordinates and the corresponding spin connection is

$$\begin{aligned} \omega^{12} &= -2 \sinh^2 \frac{\rho}{2} \sin^2 \theta d\phi , \\ \omega^{13} &= 2 \sinh^2 \frac{\rho}{2} (\cos \phi d\theta - \cos \theta \sin \theta \sin \phi d\phi) , \\ \omega^{23} &= 2 \sinh^2 \frac{\rho}{2} (\sin \phi d\theta + \cos \theta \sin \theta \cos \phi d\phi) . \end{aligned} \quad (3.58)$$

Now the geodesic connecting the origin to a point $\sigma = (\rho, \theta, \phi)$ is along the vector ∂_{ρ} . Since the spin connection along that direction is trivial $\omega_{\rho}^{mn} = 0$, this frame is known as a parallel frame and the matrix U (3.56) reduces to the identity matrix³

$$U(\sigma, \sigma_0) = \mathbf{1}_8 . \quad (3.59)$$

Using this frame, the heat kernel equation reduces to

$$(L_F^{(0)} + \partial_t)K^{(0)}(t; \sigma, \sigma_0) = \left[-\partial_{\rho}^2 - 2 \coth \rho \partial_{\rho} + \frac{1}{2} \tanh^2 \frac{\rho}{2} - \frac{3}{2} + \mathcal{M}^{(0)} + \partial_t \right] k_F(t; \rho) = 0 , \quad (3.60)$$

³Note that also for the frame $e = (d\rho, \sinh \rho d\theta, \sinh \rho \sin \theta d\phi)$ the spin connection component $\omega_{\rho}^{mn} = 0$, so U is independent of ρ . However, the spin connection does not vanish at $\rho = 0$, which introduces angular dependence into U . It can still be computed, as was done in the case of AdS_2 in [56].

where $\frac{1}{2} \tanh^2(\rho/2)$ is the leading order contribution of $g^{\mu\nu} \omega_\mu^{mn} \omega_\nu^{mn}/8$ and $-3/2 = \mathcal{R}^{(0)}/4$ is the scalar curvature of AdS_3 . If we define

$$k_F(t; \rho) = -\frac{1}{4\pi \sinh(\rho/2)} \partial_\rho \left(\frac{h(t; \rho)}{\cosh(\rho/2)} \right), \quad (3.61)$$

then h satisfies the one-dimensional heat kernel equation

$$\left[-\partial_\rho^2 + \mathcal{M}^{(0)} + \partial_t \right] h(t; \rho) = 0. \quad (3.62)$$

A solution to this equation is

$$h(t; \rho) = \frac{\exp\left(-\mathcal{M}^{(0)}t - \rho^2/4t\right)}{\sqrt{4\pi t}}, \quad (3.63)$$

and its derivative

$$k_F(t; \rho) = \frac{\rho + t \tanh(\rho/2)}{\sinh \rho} \frac{\exp\left(-\mathcal{M}^{(0)}t - \rho^2/4t\right)}{(4\pi t)^{3/2}}, \quad (3.64)$$

satisfies the initial condition (3.1) [53].

To find the subleading correction to the heat kernel $K^{(2)}$, we can again use (3.14), which extends to the spinor case. One obvious difference is that the differential operator (3.52) contains extra terms compared to the scalar laplacian

$$L_F^{(2)} = \left[L^{(2)} + \frac{1}{8} g^{(4)\mu\nu} \omega_\mu^{(-1)mn} \omega_\nu^{(-1)mn} + \frac{1}{4} g^{(2)\mu\nu} \omega_\mu^{(1)mn} \omega_\nu^{(-1)mn} + \frac{\mathcal{R}^{(2)}}{4} \right] \mathbf{1}_8 + \dots \quad (3.65)$$

where the ellipses involve terms with single gamma matrices which vanish upon taking the trace over the spinor bundle. One should also worry about the matrix U . As stated, in our frame $U(\sigma', \sigma_0) = \mathbf{1}$, but the same is not generally true for $U(\sigma, \sigma')$. As we only require $K^{(2)}(\sigma_0, \sigma_0)$, the second heat kernel comes with $U(\sigma_0, \sigma')$, which is also the identity.

The correction to the metric in (3.65) is given in (3.4) and that of the scalar curvature is (3.47). To obtain that of the spin connection, use that the vielbein for the asymptotically AdS_3 metric (3.4) is

$$e_\mu^m = e_\mu^{(-1)m} + \frac{1}{2} g_{\mu\nu}^{(0)} e_m^{(1)\nu} + \dots \quad (3.66)$$

with $e_\mu^{(-1)m}$ the leading vielbein defined in (3.57) and $e_m^{(1)\mu}$ its inverse. Note that $g^{(0)}$ can be written in the ρ, θ, ϕ coordinates using the change of variables (3.18).

The expression for $K^{(2)}$ is then as in (3.14). Explicitly,

$$-\int_0^t dt' \int_\Sigma d\rho d\theta d\phi \sinh^2 \rho \sin \theta k_F(t-t'; \sigma_0, \sigma') \text{tr}_S L_F^{(2)} k_F(t'; \sigma', \sigma_0). \quad (3.67)$$

We then need to do the spherical integral, as in (3.22). For the scalar laplacian, we get the same as above and for the other terms in (3.65) we find

$$\frac{1}{8} \int \sin \theta d\theta d\phi g^{(4)\mu\nu} \omega_\mu^{(-1)mn} \omega_\nu^{(-1)mn} = \frac{2\pi}{3} \bar{\Pi} \tanh^2 \frac{\rho}{2} + \frac{\pi}{6} \bar{\mathcal{R}} (\rho \cosh \rho - \sinh \rho) \frac{\sinh(\rho/2)}{\cosh^3(\rho/2)}, \quad (3.68)$$

$$\frac{1}{4} \int \sin \theta d\theta d\phi g^{(2)\mu\nu} \omega_\mu^{(1)mn} \omega_\nu^{(1)mn} = -\frac{\pi}{3} \bar{\mathcal{R}} \rho \tanh \frac{\rho}{2}, \quad (3.69)$$

$$\frac{1}{4} \int \sin \theta d\theta d\phi \mathcal{R}^{(2)} = \pi (\bar{\mathcal{R}} - 2\bar{\Pi}). \quad (3.70)$$

Assembling the results, we obtain

$$\begin{aligned}
& -4\pi \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho k_F(t-t'; \rho) \left[\frac{z_0^2}{3} \text{tr} \bar{\Pi} \left(-\partial_\rho^2 - 2 \coth \rho \partial_\rho + \frac{1}{2} \tanh^2 \frac{\rho}{2} - \frac{3}{2} \right) \right. \\
& \quad \left. + z_0^2 \bar{\mathcal{R}} \left(\frac{1}{6} \frac{\cosh \rho \sinh \rho - \rho}{\sinh^2 \rho} \partial_\rho + \frac{1}{4} - \frac{\rho + \sinh \rho \frac{\sinh(\rho/2)}{\cosh^3(\rho/2)}}{24} \right) \right] k_F(t'; \rho) \text{tr}_S(\mathbf{1}).
\end{aligned} \tag{3.71}$$

This integral is evaluated explicitly in Appendix B. Adding (B.3) and (B.7) together with the boundary term appearing in the second line of (3.14), we finally get the trace of the heat kernel

$$\text{tr}_S K^{(2)}(t; \sigma_0, \sigma_0) = -\frac{z_0^2}{12} (\bar{\mathcal{R}} - 2 \text{tr} \bar{\Pi}) \frac{e^{-\mathcal{M}^{(0)} t}}{(4\pi)^{3/2} t^{1/2}} \text{tr}_S(\mathbf{1}). \tag{3.72}$$

Its t integral is easy to evaluate and gives (reinstating powers of R)

$$\int_0^\infty \frac{dt}{t} \text{tr}_S K^{(0)}(t; \sigma_0, \sigma_0) = \frac{\sqrt{\mathcal{M}^{(0)}}}{24\pi R^2} (4\mathcal{M}^{(0)} R^2 - 3) \text{tr}_S(\mathbf{1}), \tag{3.73}$$

$$\int_0^\infty \frac{dt}{t} \text{tr}_S K^{(2)}(t; \sigma_0, \sigma_0) = \frac{z_0^2}{48\pi R^2} (\bar{\mathcal{R}} - 2 \text{tr} \bar{\Pi}) \sqrt{\mathcal{M}^{(0)}} \text{tr}_S(\mathbf{1}). \tag{3.74}$$

Getting from this to the contribution of the fermions to the anomaly of the surface operator is immediate. The Dirac operator governing the quadratic fermionic fluctuations for an M2-brane is derived in Appendix A, and from equation (A.17) we can read $\mathcal{M}^{(0)} = 9/4$.

Evaluating the path integral yields the pfaffian of the Dirac operator, or $\frac{1}{2} \log \det_S(-\mathcal{D}^2 + 9/4)$. The contribution of the determinant to the anomaly is obtained from the pole of the z integral as in the bosonic case (3.32). Using (3.73), (3.74), and $\text{tr}_S(\mathbf{1}) = 8$ we get

$$\begin{aligned}
\mathcal{A}^F &= \frac{1}{2} \text{res}_{z_0 \rightarrow 0} \left[\left(-\frac{1}{2z_0} \text{tr} \bar{\Pi} \right) \int_0^\infty \frac{dt}{t} \text{tr}_S K^{(0)}(t; \sigma_0, \sigma_0) + \frac{1}{z_0^3} \int_0^\infty \frac{dt}{t} \text{tr}_S K^{(2)}(t; \sigma_0, \sigma_0) + \dots \right] \\
&= \frac{1}{4\pi} \left(\frac{1}{2} \bar{\mathcal{R}} - 4 \text{tr} \bar{\Pi} \right).
\end{aligned} \tag{3.75}$$

4 Conclusions

Combining the results of the massless scalars on S^4 (3.32), the normal bundle inside AdS_7 (3.50), the fermions (3.75), and replacing $\text{tr} \bar{\Pi} = H^2/4 + \text{tr} \bar{P}$ (2.16), we find

$$\frac{1}{2} \log \frac{\det_S(-\mathcal{D}^2 + 9/(4R^2))}{\det^4(-\Delta) \det_V(-\Delta_{m'n'} + M_{m'n'}^2)} = \frac{\log \epsilon}{4\pi} \int_{\bar{\Sigma}} \text{vol}_{\bar{\Sigma}} \left[\frac{1}{2} \bar{\mathcal{R}} + \frac{1}{2} (\bar{H}^2 + 4 \text{tr} \bar{P}) \right] + \dots \tag{4.1}$$

where the \dots stand for quadratic divergences in ϵ and finite terms. The integrand takes the form of an anomaly density (1.5) and from it we read the order N^0 corrections to the anomaly coefficients, respectively $1/2$, $1/2$ and 0 for $\bar{\mathcal{R}}$, $\bar{H}^2 + 4 \text{tr} \bar{P}$, $\text{tr} \bar{W}$, giving (1.6). This provides a first-principles evaluation of these anomaly coefficients previously derived in [31, 33–40, 42].

While the values of the anomaly coefficients are already known, the tools we develop are new. In particular, we obtain explicit expressions for the trace of the heat kernel associated to

massive scalars (3.31) and spinors (3.74) in asymptotically AdS_3 spaces. Using these results we are able to extend the classical near boundary asymptotic analysis of Graham-Witten [24] to the quantum level.

Along the way we also obtain the quadratic fluctuation action about any classical M2-brane solution in $AdS_7 \times S^4$, which is presented in Appendix A. These expressions reproduce known results when specifying the appropriate classical solution, in particular we recover the spectrum of fluctuations of the parallel planes [57] and sphere [42]. In these cases and the other few examples of known classical M2-brane solution [27–29], our results provide the determinants capturing the 1-loop fluctuations around the classical solutions. One may now try to evaluate these determinants exactly and go beyond the analysis of divergences that lead to the anomalies studied here.

Our work also serves to enlarge the scope of recent progress on the semiclassical quantisation of M2-branes. This has been done in several contexts like Wilson loops and instanton corrections to the free energy of ABJM theory [58–62] and reproduced results from supersymmetric localisation. In most of those cases the classical M2-brane solutions are BPS and their spectrum can be solved exactly and summed up using ζ function regularisation. Here the brane does not need to preserve supersymmetry.

One limitation of our calculation is that we assume that the classical surface is located at a single point in S^4 , which is not the most general setting (and in particular not sufficient for most of the cases studied in [63, 28, 29]). When the classical M2-brane is extended in S^4 , there is an extra anomaly coefficient denoted c in [23]—if the embedding into S^4 is represented by a unit vector $n^i(\tau^a)$, then c multiplies $(\partial n^i)^2$. This term can be thought more generally as arising from the coupling of a surface operator to a global symmetry. It was proven in [40] that in the $\mathcal{N} = (2, 0)$ theory $c = -a_2$, so our analysis here establishes that it is $c = N - 1/2 + \mathcal{O}(1/N)$. Nonetheless, it would be instructive to rederive it from first principles, by generalising the analysis in this paper to generic embeddings in $AdS_7 \times S^4$.

Finally, our analysis suggests many further directions. Perhaps the most straightforward is the calculation of anomaly coefficients for the non-supersymmetric surface operator of the 6d $\mathcal{N} = (2, 0)$ theory [42]. This is the surface operator analog of the usual Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills without scalar coupling [64, 65], and in holography its scalar fluctuations on S^4 obey Neumann instead of Dirichlet boundary conditions; we hope to report on it soon. It should also be possible to calculate anomaly coefficients for surface operators in large representations using a probe M5-brane as in [66, 51] and its fluctuations. More generally, a similar analysis should also be possible for other settings where the asymptotic analysis proved useful, for instance in the calculation of entanglement entropy [10, 67, 11], the study of surface operators in 4d [12, 68] and that of codimension-2 observables in 6d [69, 70].

Acknowledgements

We are grateful to C.R. Graham, F. Hübner, L. Pando Zayas, G. Papadopoulos, R. Sinha, A. Tseytlin and E. Witten for helpful discussions, and especially A. Tseytlin his many helpful comments on a preliminary version of this paper. ND would like to thank EPFL, CERN and DESY for their hospitality in the course of this work. ND’s research is supported by the Science Technology & Facilities council under the grants ST/P000258/1 and ST/X000753/1. OS’s research is funded by the Engineering & Physical Sciences Research Council under grant number EP/W524025/1. MT gratefully acknowledges the support from the Institute for Theoretical and Mathematical Physics (ITMP, Moscow) where this project began, and

the Perimeter Institute where part of this project was realised. MT's research is funded by the Engineering & Physical Sciences Research Council under the grant EP/W522429/1.

A Quadratic M2-brane action

In this appendix we derive the action for quadratic fluctuations of M2-branes in $AdS_7 \times S^4$. This is used in Section 3 where the determinant of the kinetic operators are evaluated. The spectrum of fluctuations of specific M2-brane classical solutions in different backgrounds have been previously calculated in [57–59, 61, 62]. The calculation follows closely that of the fluctuations of semiclassical strings pioneered in [71], elaborated on in [72] and generalised to other branes in [73].

For the purpose of this paper it would be enough to study the asymptotic form of the M2-brane quadratic fluctuation action, but it turns out to be simple enough to derive it for any classical solution. In this appendix we therefore abandon the near boundary metric and use one near the classical brane solution. Indeed, the brane does not need to have a boundary and can be a bulk instanton, as those studied in [59, 61].

As in Section 2, we take σ^μ as the three world-volume coordinates parametrising Σ and for the remaining eight directions, we take $u^{\mu'}$ to be normal to Σ and z^i to parametrise S^4 , such that the classical solution is at $u^{\mu'} = 0$ and $z^i = 0$. We ignore global issues and write the metric near the brane as

$$ds^2 = G_{\mu\nu}(\sigma, u)d\sigma^\mu d\sigma^\nu + 2G_{\mu\mu'}(\sigma, u)d\sigma^\mu du^{\mu'} + G_{\mu'\nu'}(\sigma, u)du^{\mu'} du^{\nu'} + \frac{dz^i dz^j}{(1 + z^j z^j)^2}, \quad (\text{A.1})$$

with $G_{\mu\nu}(\sigma, 0) = g_{\mu\nu}(\sigma)$ the induced metric on Σ and $G_{\mu\mu'}(\sigma, 0) = 0$. Note that the metric in (2.14) satisfies all the same properties, so realises this in the near-boundary regime.

We proceed to analyse the bosonic and fermionic fields on the M2-brane world-volume in the geometry (A.1) in turn.

A.1 Bosonic fluctuations

We consider fluctuations $u^{\mu'}, z^i$ around a classical solution described by world-volume coordinates x^μ as in the metric (A.1). The M2-brane action (2.3) contains two terms, the volume form and a term coupling to the pullback of the three-form A_3 . Because we restrict to classical solutions localised at a point on S^4 , there are no tangent vectors along S^4 and the pullback of A_3 is at least cubic in the fluctuations, and we can neglect it. The quadratic terms arising from the volume form is obtained in [73]⁴ (see also [74])

$$\frac{T_{\text{M2}}}{2} \int_{\Sigma} \text{vol}_{\Sigma} \left[\partial_{\mu} u^{\mu'} \partial_{\nu} u^{\nu'} g^{\mu\nu} g_{\mu'\nu'} + \partial_{\mu} z^i \partial_{\nu} z^j g^{\mu\nu} - \left(R_{\mu'\mu\nu\nu'} + g_{\mu'\rho'} g_{\nu'\sigma'} \tilde{\Pi}_{\mu\rho}^{\rho'} \tilde{\Pi}_{\sigma\nu}^{\sigma'} g^{\rho\sigma} \right) g^{\mu\nu} u^{\mu'} u^{\nu'} \right], \quad (\text{A.2})$$

with $\tilde{\Pi}$ the traceless part of the second fundamental form and R the Riemann tensor.

To put the kinetic term in canonical form, we introduce the vielbeine $E_{m'}^{\mu'}(\sigma)$ and parametrise u as

$$u^{\mu'} = E_{m'}^{\mu'} \zeta^{m'}. \quad (\text{A.3})$$

⁴But correcting a factor of 2 in [73].

The derivatives are then expressed as

$$\partial_\mu u^{\mu'} = E_{m'}^{\mu'} D_\mu \zeta^{m'}, \quad D_\mu \zeta^{m'} \equiv \partial_\mu \zeta^{m'} + \omega_\mu^{m'n'} \zeta^{n'}, \quad (\text{A.4})$$

with ω the spin connection on the normal bundle. With these substitutions the quadratic action becomes

$$\frac{T_{\text{M2}}}{2} \int_\Sigma \text{vol}_\Sigma \left[D_\mu \zeta^{m'} D_\nu \zeta^{m'} g^{\mu\nu} - \left(E_{m'}^{\mu'} E_{n'}^{\nu'} R_{\mu'\nu'\rho\sigma} + E_{\mu'm'} E_{\nu'n'} \tilde{\Pi}_{\mu\rho}^{\mu'} \tilde{\Pi}_{\sigma\nu}^{\nu'} g^{\rho\sigma} \right) g^{\mu\nu} \zeta^{m'} \zeta^{n'} \right]. \quad (\text{A.5})$$

Finally, to bring this to the form of a differential operator, we integrate the kinetic term by parts. This yields a boundary term, and assuming it vanishes, we get [73]

$$\begin{aligned} S_{\text{fluc}}^{(\text{bos})} &= \frac{T_{\text{M2}}}{2} \int_\Sigma \text{vol}_\Sigma \zeta^{m'} \left[-\Delta_{m'n'} + M_{m'n'}^2 \right] \zeta^{n'} - z^i \Delta z^i, \\ M_{m'n'}^2 &= -g^{\mu\nu} \left(E_{m'}^{\mu'} E_{n'}^{\nu'} R_{\mu'\nu'\rho\sigma} + E_{\mu'm'} E_{\nu'n'} \tilde{\Pi}_{\mu\rho}^{\mu'} \tilde{\Pi}_{\sigma\nu}^{\nu'} g^{\rho\sigma} \right), \end{aligned} \quad (\text{A.6})$$

with Δ the usual laplacian and $\Delta_{m'n'}$ the vector laplacian acting on $\zeta^{m'}$ (the covariant derivative is defined in (A.4))

$$\Delta_{m'n'} \zeta^{n'} = \frac{1}{\sqrt{g}} [D_\mu (\sqrt{g} g^{\mu\nu} D_\nu) \zeta]_{m'}. \quad (\text{A.7})$$

A.2 Fermionic fluctuations

The fermionic part of the M2-brane action is obtained in [49] and is expressed in terms of an 11d spinor $\Psi(\sigma)$ (whose indices we suppress). On $AdS_7 \times S^4$ Ψ is in the spinor representation of $\mathfrak{so}(1,6)$ and $\mathfrak{so}(4)$. To quadratic order and working in lorentzian signature, it reads [75, 76] (see also [77])

$$T_{\text{M2}} \int_\Sigma d^3\sigma \sqrt{-g} \left[\bar{\Psi} \left(g^{\rho\mu} \Gamma_\mu - \frac{1}{2} \varepsilon^{\mu\nu\rho} \Gamma_{\mu\nu} \right) \left(D_\rho + \frac{1}{288} \left(\Gamma_\rho^{NPQR} - 8\delta_\rho^{[N} \Gamma^{PQR]} \right) F_{NPQR} \right) \Psi \right]. \quad (\text{A.8})$$

Here we use M, N, \dots for coordinates on the full 11d space. Γ^M are the 11d gamma matrices and we use the shorthand for antisymmetrised indices $\Gamma_{\mu\nu} \equiv \Gamma_{[\mu} \Gamma_{\nu]}$ (with the appropriate 1/2). The spinor covariant derivative is

$$D_\rho \Psi = \partial_\rho \Psi - \frac{1}{4} \omega_\rho^{MN} \Gamma_{MN} \Psi, \quad (\text{A.9})$$

where to avoid introducing extra notations, we use curved space indices on the spin connection and gamma matrices.

As mentioned previously, we restrict to M2-branes located at a point on S^4 , so the term proportional to $\delta_\rho^{[N} \Gamma^{PQR]}$ vanishes and the covariant derivative (A.9) includes only the AdS_7 spin connection. We can also simplify the remaining flux term by noting that Γ_ρ commutes with Γ^{NPQR} and using the explicit expression for the flux (2.4) we get

$$\frac{1}{288} \Gamma_\rho \Gamma^{NPQR} F_{NPQR} = \frac{1}{2R} \Gamma_\rho \Gamma_{789\ddagger}. \quad (\text{A.10})$$

To proceed, we can pick a frame where $E_{\mu}^{m'} = E_{\mu'}^m = 0$. So the gamma matrices used above can now be expressed in terms of those with flat space indices as $\Gamma_{\mu} = E_{\mu}^m \Gamma_m$ and $\Gamma_{\mu'} = E_{\mu'}^{m'} \Gamma_{m'}$. We can further specify the frame by requiring that the flat space gamma matrices $\Gamma_m, \Gamma_{m'}$ and Γ_i are given by⁵

$$\Gamma_m = \gamma_m \otimes \rho_* \otimes \tau_*, \quad \Gamma_{m'} = \mathbb{1}_2 \otimes \rho_{m'} \otimes \tau_*, \quad \Gamma_i = \mathbb{1}_2 \otimes \mathbb{1}_4 \otimes \tau_i, \quad (\text{A.11})$$

with $\gamma_m, \rho_{m'}$ and τ_i respectively the gamma matrices for $SO(1, 2), SO(4)_N$ and $SO(4)_S$, and $\rho_* \equiv \rho_1 \rho_2 \rho_3 \rho_4, \tau_* \equiv \tau_1 \tau_2 \tau_3 \tau_4$. Note that $\rho_*^2 = \tau_*^2 = 1$. With this basis the lagrangian becomes

$$\begin{aligned} \bar{\Psi} \left(\gamma^p \rho_* \tau_* - \frac{1}{2} \gamma_{mn} \varepsilon^{mnp} \right) E_p^{\mu} \left(\partial_{\mu} \psi - \frac{1}{4} \gamma_{qr} \omega_{\mu}^{qr} - \frac{1}{2} \gamma_q \rho_* \rho_{r'} \omega_{\mu}^{qr'} - \frac{1}{4} \rho_{q'r'} \omega_{\mu}^{q'r'} \right) \Psi \\ + \frac{3}{R} \bar{\Psi} (1 - \gamma_{012} \rho_* \tau_*) \tau_* \Psi. \end{aligned} \quad (\text{A.12})$$

ω_{μ}^{qr} and $\omega_{\mu}^{q'r'}$ are respectively the spin connections on the world-volume and on the normal bundle. $\omega_{\mu}^{q'r'}$ can be expressed as

$$\omega_{\mu}^{q'r'} = E^{\sigma q} E_{\rho'}^{r'} \Gamma_{\mu\sigma}^{\rho'} = -E^{\sigma q} E_{\rho'}^{r'} \mathbf{\Gamma}_{\mu\sigma}^{\rho'}, \quad (\text{A.13})$$

where in the first equality we used the definition of the spin connection in terms of a frame (analogous to (3.38)), and in the second we used that the second fundamental form (2.7) is also the Christoffel symbol for the metric G .

With some gamma matrix algebra and using that $\gamma_{012} = 1$, this can be rewritten as

$$\begin{aligned} -\bar{\Psi} (1 - \rho_* \tau_*) \gamma^{\mu} \left(\partial_{\mu} - \frac{1}{4} \gamma_{mn} \omega_{\mu}^{mn} - \frac{1}{4} \rho_{m'n'} \omega_{\mu}^{m'n'} \right) \Psi + \frac{3}{2R} \bar{\Psi} (1 - \rho_* \tau_*) \tau_* \Psi \\ - \frac{1}{2} \bar{\Psi} (1 - \rho_* \tau_*) \rho_* \rho_{m'} \Psi H^{m'}, \end{aligned} \quad (\text{A.14})$$

and recall that the mean curvature vector H vanishes because of the equations of motion. Finally, the M2-brane action has κ -symmetry, and we can fix a gauge by imposing the condition

$$\left(1 + \frac{1}{6} E_{\mu}^m E_{\nu}^n E_{\rho}^p \Gamma_{mnp} \varepsilon^{\mu\nu\rho} \right) \Psi = (1 + \rho_* \tau_*) \Psi = 0. \quad (\text{A.15})$$

This condition is solved by decomposing $\Psi = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$ in terms of spinors ψ_{\pm} with chirality

$$\tau_* \psi_{\pm} = -\rho_* \psi_{\pm} = \pm \psi_{\pm}. \quad (\text{A.16})$$

We then obtain the gauge-fixed quadratic action

$$\begin{aligned} S_{\text{fluc}}^{(\text{fer})} = -T_{\text{M2}} \int_{\Sigma} d^3 \sigma \sqrt{-g} \left[\bar{\psi}_+ \left(\not{D} + \frac{3}{2R} \right) \psi_+ + \bar{\psi}_- \left(\not{D} - \frac{3}{2R} \right) \psi_- \right], \\ \not{D} \equiv \gamma^{\mu} D_{\mu} = \gamma^{\mu} \left(\partial_{\mu} - \frac{1}{4} \gamma_{mn} \omega_{\mu}^{mn} - \frac{1}{4} \rho_{m'n'} \omega_{\mu}^{m'n'} \right). \end{aligned} \quad (\text{A.17})$$

⁵In any frame the gamma matrices are related to these by a similarity transformation $\Gamma_M \rightarrow S^{-1} \Gamma_M S = \Lambda_M^N \Gamma_N$. The (local) Lorentz transformation Λ can be absorbed by a change of frame to ensure that the gamma matrices are in the desired basis.

We can read the relevant determinant for the analysis of section 3 from this action. After Wick rotation to euclidean signature, the path integral over quadratic fluctuations gives the pfaffian

$$\text{Pf} \left(\not{D} + \frac{3}{2R} \right) \text{Pf} \left(\not{D} - \frac{3}{2R} \right). \quad (\text{A.18})$$

Using the identity

$$\text{Pf} \left(\not{D} \pm \frac{3}{2R} \right)^2 = \det \left(\not{D} \pm \frac{3}{2R} \right) = \det^{1/2} \left(-\not{D}^2 + \frac{9}{4R^2} \right), \quad (\text{A.19})$$

we can write

$$\text{Pf} \left(\not{D} + \frac{3}{2R} \right) \text{Pf} \left(\not{D} - \frac{3}{2R} \right) = \det^{1/2} \left(-\not{D}^2 + \frac{9}{4R^2} \right), \quad (\text{A.20})$$

up to a possible overall sign ambiguity in the path integral.

We note that our gauge-fixed action (A.17) and the determinant (A.20) agree with existing results for fermionic fluctuations derived for specific geometries: they reproduce that of the parallel planes [57] and the Dirac operator for AdS_3 [42].

B Convoluting the fermionic heat kernels

Here we evaluate the integral (3.71) (without the factor of $\text{tr}_S(\mathbb{1})$)

$$\begin{aligned} & -4\pi \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho k_F(t-t'; \rho) \left[\frac{z_0^2}{3} \text{tr} \bar{\Pi} \left(-\partial_\rho^2 - 2 \coth \rho \partial_\rho + \frac{1}{2} \tanh^2 \frac{\rho}{2} - \frac{3}{2} \right) \right. \\ & \left. + z_0^2 \bar{\mathcal{R}} \left(\frac{1}{6} \frac{\cosh \rho \sinh \rho - \rho}{\sinh^2 \rho} \partial_\rho + \frac{1}{4} - \frac{\rho + \sinh \rho}{24} \frac{\sinh(\rho/2)}{\cosh^3(\rho/2)} \right) \right] k_F(t'; \rho). \end{aligned} \quad (\text{B.1})$$

The term multiplying $\bar{\Pi}$ is related to the heat kernel equation (3.60). Then integrating by parts the first term on the second line we get

$$\begin{aligned} & 4\pi \int_0^t dt' \int_0^\infty d\rho \sinh^2 \rho k_F(t-t'; \rho) \\ & \times \left[\frac{z_0^2}{3} \text{tr} \bar{\Pi} (\partial_t + \mathcal{M}^{(0)}) k_F(t'; \rho) - z_0^2 \bar{\mathcal{R}} \frac{2 - \rho \tanh(\rho/2)}{24 \cosh^2(\rho/2)} k_F(t'; \rho) \right]. \end{aligned} \quad (\text{B.2})$$

For the first term, we can apply the same analysis as in the scalar case (3.26), yielding

$$\frac{z_0^2}{3} \text{tr} \bar{\Pi} t (\partial_t + \mathcal{M}^{(0)}) k_F(t; 0). \quad (\text{B.3})$$

While k_F does not satisfy a general convolution, as it is not a heat kernel in 3d, this calculation still works because $U = 1$ for the radial motion from 0 to ρ and back to 0. It would be nice to try to repeat the calculation in this appendix using proper convolution, relying on the full fermionic heat kernel including angular dependence.

For the term proportional to $\bar{\mathcal{R}}$, we can use the explicit expression for the AdS_3 spinor heat kernel to evaluate the integral. Plugging in k_F (3.64), we get

$$-\frac{z_0^2 \bar{\mathcal{R}} e^{-t\mathcal{M}^{(0)}}}{384\pi^2} \int_0^\infty d\rho \frac{2 - \rho \tanh(\rho/2)}{\cosh^2(\rho/2)} \int_0^t dt' \left(\rho \left(\rho + t \tanh \frac{\rho}{2} \right) \frac{e^{-t\rho^2/4t'(t-t')}}{(t'(t-t'))^{3/2}} + \tanh^2 \frac{\rho}{2} \frac{e^{-t\rho^2/4t'(t-t')}}{(t'(t-t'))^{1/2}} \right). \quad (\text{B.4})$$

This can be evaluated using a small trick. We integrate the term with $(t'(t-t'))^{-1/2}$ by parts with respect to ρ and note that the boundary term evaluates to zero, giving

$$-\frac{z_0^2 \bar{\mathcal{R}} e^{-t\mathcal{M}^{(0)}}}{1536\pi^2} \int_0^\infty d\rho \frac{\rho(2\rho(t+2) - (2\rho^2 - 3t) \sinh \rho + \rho(4-t) \cosh \rho)}{\cosh^4(\rho/2)} \int_0^t dt' \frac{e^{-t\rho^2/4t'(t-t')}}{(t'(t-t'))^{3/2}}. \quad (\text{B.5})$$

Making a change of variables to $\tau = (t - 2t')^2/4t'(t - t')$, the integral over τ is simply a gamma function and gives

$$\frac{z_0^2 \bar{\mathcal{R}} e^{-t\mathcal{M}^{(0)}}}{384(\pi t)^{3/2}} \int_0^\infty d\rho \frac{(2\rho^2 - 3t) \sinh \rho - 2\rho(t+2) + \rho(t-4) \cosh \rho}{\cosh^4(\rho/2)} e^{-(\rho^2/t)}. \quad (\text{B.6})$$

Finally the ρ integral can be done straightforwardly and yields

$$-\frac{z_0^2 \bar{\mathcal{R}}}{96\pi^{3/2}} \frac{e^{-\mathcal{M}^{(0)}t}}{\sqrt{t}}. \quad (\text{B.7})$$

References

- [1] A. J. Bray and M. A. Moore, “Critical behaviour of semi-infinite systems,” *J. Phys. A* **10** no. 11, (1977) 1927.
- [2] K. Ohno and Y. Okabe, “The $1/N$ expansion for the extraordinary transition of semi-infinite system,” *Prog. Th. Phys.* **72** no. 4, (1984) 736–745.
- [3] J. L. Cardy, “Conformal invariance and surface critical behavior,” *Nuclear Physics B* **240** no. 4, (1984) 514–532.
- [4] M. A. Metlitski, “Boundary criticality of the $O(N)$ model in $d = 3$ critically revisited,” *SciPost Phys.* **12** no. 4, (2022) 131, [arXiv:2009.05119](#).
- [5] D. Rodriguez-Gomez, “A scaling limit for line and surface defects,” *JHEP* **06** (2022) 071, [arXiv:2202.03471](#).
- [6] T. Shachar, R. Sinha, and M. Smolkin, “RG flows on two-dimensional spherical defects,” [arXiv:2212.08081](#).
- [7] M. Trépanier, “Surface defects in the $O(N)$ model,” *JHEP* **09** (2023) 074, [arXiv:2305.10486](#).
- [8] S. Giombi and B. Liu, “Notes on a surface defect in the $O(N)$ model,” [arXiv:2305.11402](#).
- [9] A. Raviv-Moshe and S. Zhong, “Phases of surface defects in scalar field theories,” *JHEP* **08** (2023) 143, [arXiv:2305.11370](#).
- [10] S. N. Solodukhin, “Entanglement entropy, conformal invariance and extrinsic geometry,” *Phys. Lett. B* **665** (2008) 305–309, [arXiv:0802.3117](#).

- [11] L. Bianchi, M. Meineri, R. C. Myers, and M. Smolkin, “Rényi entropy and conformal defects,” *JHEP* **07** (2016) 076, [arXiv:1511.06713](#).
- [12] S. Gukov and E. Witten, “Gauge theory, ramification, and the geometric Langlands program,” [hep-th/0612073](#).
- [13] S. Gukov and E. Witten, “Rigid surface operators,” *Adv. Theor. Math. Phys.* **14** no. 1, (2010) 87–178, [arXiv:0804.1561](#).
- [14] J. Gomis and B. Le Floch, “M2-brane surface operators and gauge theory dualities in Toda,” *JHEP* **04** (2016) 183, [arXiv:1407.1852](#).
- [15] E. Witten, “Some comments on string dynamics,” in *STRINGS 95: Future Perspectives in String Theory*, pp. 501–523. 7, 1995. [hep-th/9507121](#).
- [16] A. Strominger, “Open p -branes,” *Phys. Lett. B* **383** (1996) 44–47, [hep-th/9512059](#).
- [17] O. J. Ganor, “Six-dimensional tensionless strings in the large N limit,” *Nucl. Phys. B* **489** (1997) 95–121, [hep-th/9605201](#).
- [18] D. Gaiotto, “Surface operators in $\mathcal{N} = 2$ 4d gauge theories,” *JHEP* **11** (2012) 090, [arXiv:0911.1316](#).
- [19] A. Schwimmer and S. Theisen, “Entanglement entropy, trace anomalies and holography,” *Nucl. Phys. B* **801** (2008) 1–24, [arXiv:0802.1017](#).
- [20] J. Wess and B. Zumino, “Consequences of anomalous Ward identities,” *Phys. Lett. B* **37** (1971) 95–97.
- [21] S. Deser and A. Schwimmer, “Geometric classification of conformal anomalies in arbitrary dimensions,” *Phys. Lett. B* **309** (1993) 279–284, [hep-th/9302047](#).
- [22] N. Boulanger, “General solutions of the Wess-Zumino consistency condition for the Weyl anomalies,” *JHEP* **07** (2007) 069, [arXiv:0704.2472](#).
- [23] N. Drukker, M. Probst, and M. Trépanier, “Surface operators in the 6d $\mathcal{N} = (2, 0)$ theory,” *J. Phys. A* **53** no. 36, (2020) 365401, [arXiv:2003.12372](#).
- [24] C. R. Graham and E. Witten, “Conformal anomaly of submanifold observables in AdS/CFT correspondence,” *Nucl. Phys.* **B546** (1999) 52–64, [hep-th/9901021](#).
- [25] J. M. Maldacena, “Wilson loops in large N field theories,” *Phys. Rev. Lett.* **80** (1998) 4859–4862, [hep-th/9803002](#).
- [26] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large N limit,” *Phys. Rev. D* **59** (1999) 105023, [hep-th/9809188](#).
- [27] M. Mezei, S. S. Pufu, and Y. Wang, “Chern-Simons theory from M5-branes and calibrated M2-branes,” *JHEP* **08** (2019) 165, [arXiv:1812.07572](#).
- [28] N. Drukker and M. Trépanier, “M2-doughnuts,” *JHEP* **02** (2022) 071, [arXiv:2111.09385](#).
- [29] N. Drukker and M. Trépanier, “Ironing out the crease,” *JHEP* **08** (2022) 193, [arXiv:2204.12627](#).
- [30] N. Drukker and M. Trépanier, “BPS surface operators and calibrations,” *J. Phys. A* **56** no. 17, (2023) 175403, [arXiv:2210.07251](#).
- [31] K. Jensen, A. O’Bannon, B. Robinson, and R. Rodgers, “From the Weyl anomaly to entropy of two-dimensional boundaries and defects,” *Phys. Rev. Lett.* **122** no. 24, (2019) 241602, [arXiv:1812.08745](#).

- [32] S. A. Gentle, M. Gutperle, and C. Marasinou, “Entanglement entropy of Wilson surfaces from bubbling geometries in M-theory,” *JHEP* **08** (2015) 019, [arXiv:1506.00052](#).
- [33] R. Rodgers, “Holographic entanglement entropy from probe M-theory branes,” *JHEP* **03** (2019) 092, [arXiv:1811.12375](#).
- [34] J. Estes, D. Krym, A. O’Bannon, B. Robinson, and R. Rodgers, “Wilson surface central charge from holographic entanglement entropy,” *JHEP* **05** (2019) 032, [arXiv:1812.00923](#).
- [35] Y. Wang, “Surface defect, anomalies and b -extremization,” *JHEP* **11** (2021) 122, [arXiv:2012.06574](#).
- [36] M. Bullimore and H.-C. Kim, “The superconformal index of the (2,0) theory with defects,” *JHEP* **05** (2015) 048, [arXiv:1412.3872](#).
- [37] A. Chalabi, A. O’Bannon, B. Robinson, and J. Sisti, “Central charges of 2d superconformal defects,” *JHEP* **05** (2020) 095, [arXiv:2003.02857](#).
- [38] C. Meneghelli and M. Trépanier, “Bootstrapping string dynamics in the 6d $\mathcal{N} = (2, 0)$ theories,” *JHEP* **07** (2023) 165, [arXiv:2212.05020](#).
- [39] L. Bianchi and M. Lemos, “Superconformal surfaces in four dimensions,” *JHEP* **06** (2020) 056, [arXiv:1911.05082](#).
- [40] N. Drukker, M. Probst, and M. Trépanier, “Defect CFT techniques in the 6d $\mathcal{N} = (2, 0)$ theory,” *JHEP* **03** (2021) 261, [arXiv:2009.10732](#).
- [41] R. Corrado, B. Florea, and R. McNees, “Correlation functions of operators and Wilson surfaces in the $d = 6$, (0,2) theory in the large N limit,” *Phys. Rev. D* **60** (1999) 085011, [hep-th/9902153](#).
- [42] N. Drukker, S. Giombi, A. A. Tseytlin, and X. Zhou, “Defect CFT in the 6d (2, 0) theory from M2 brane dynamics in $AdS_7 \times S^4$,” *JHEP* **07** (2020) 101, [arXiv:2004.04562](#).
- [43] D. V. Vassilevich, “Heat kernel expansion: user’s manual,” *Phys. Rept.* **388** (2003) 279–360, [hep-th/0306138](#).
- [44] R. Camporesi, “Harmonic analysis and propagators on homogeneous spaces,” *Phys. Rept.* **196** (1990) 1–134.
- [45] S. Giombi, A. Maloney, and X. Yin, “One-loop partition functions of 3D gravity,” *JHEP* **08** (2008) 007, [arXiv:0804.1773](#).
- [46] S. Giombi and I. R. Klebanov, “One loop tests of higher spin AdS/CFT,” *JHEP* **12** (2013) 068, [arXiv:1308.2337](#).
- [47] V. Forini, A. A. Tseytlin, and E. Vescovi, “Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in $AdS_5 \times S^5$,” *JHEP* **03** (2017) 003, [arXiv:1702.02164](#).
- [48] C. Fefferman, “Conformal invariants,” *“Elie Cartan et les mathématiques d’aujourd’hui,” Astérisque, hors série* (1985) 95–116.
- [49] E. Bergshoeff, E. Sezgin, and P. K. Townsend, “Supermembranes and eleven-dimensional supergravity,” *Phys. Lett.* **B189** (1987) 75–78.
- [50] N. Drukker, D. J. Gross, and H. Ooguri, “Wilson loops and minimal surfaces,” *Phys. Rev. D* **60** (1999) 125006, [hep-th/9904191](#).
- [51] H. Mori and S. Yamaguchi, “M5-branes and Wilson surfaces in AdS_7/CFT_6 correspondence,” *Phys. Rev. D* **90** no. 2, (2014) 026005, [1404.0930](#).

- [52] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry and the Dirac Operators*, pp. 77–165. Princeton University Press, 1989.
<http://www.jstor.org/stable/j.ctt1bpmb28.7>.
- [53] R. Camporesi, “The spinor heat kernel in maximally symmetric spaces,” *Commun. Math. Phys.* **148** (1992) 283–308.
- [54] R. Camporesi and A. Higuchi, “On the eigen functions of the Dirac operator on spheres and real hyperbolic spaces,” *J. Geom. Phys.* **20** (1996) 1–18, [gr-qc/9505009](https://arxiv.org/abs/gr-qc/9505009).
- [55] J. R. David, M. R. Gaberdiel, and R. Gopakumar, “The heat kernel on AdS_3 and its applications,” *JHEP* **04** (2010) 125, [arXiv:0911.5085](https://arxiv.org/abs/0911.5085).
- [56] R. Bergamin and A. A. Tseytlin, “Heat kernels on cone of AdS_2 and k -wound circular Wilson loop in $AdS_5 \times S^5$ superstring,” *J. Phys. A* **49** no. 14, (2016) 14LT01, [arXiv:1510.06894](https://arxiv.org/abs/1510.06894).
- [57] Förste, Stefan, “Membrany corrections to the string anti-string potential in M5-brane theory,” *JHEP* **05** (1999) 002, [hep-th/9902068](https://arxiv.org/abs/hep-th/9902068).
- [58] S. Giombi and A. A. Tseytlin, “Wilson loops at large N and the quantum M2-brane,” *Phys. Rev. Lett.* **130** no. 20, (2023) 201601, [arXiv:2303.15207](https://arxiv.org/abs/2303.15207).
- [59] M. Beccaria, S. Giombi, and A. A. Tseytlin, “Instanton contributions to the ABJM free energy from quantum M2 branes,” *JHEP* **10** (2023) 029, [arXiv:2307.14112](https://arxiv.org/abs/2307.14112).
- [60] F. K. Seibold and A. A. Tseytlin, “S-matrix on effective string and compactified membrane,” *J. Phys. A* **56** no. 48, (2023) 485401, [arXiv:2308.12189](https://arxiv.org/abs/2308.12189).
- [61] M. Beccaria, S. Giombi, and A. A. Tseytlin, “(2,0) theory on $S^5 \times S^1$ and quantum M2 branes,” [arXiv:2309.10786](https://arxiv.org/abs/2309.10786).
- [62] N. Drukker and O. Shahpo. To appear.
- [63] N. Drukker and M. Trépanier, “Observations on BPS observables in 6d,” *J. Phys. A* **54** no. 20, (2021) 20, [arXiv:2012.11087](https://arxiv.org/abs/2012.11087).
- [64] J. Polchinski and J. Sully, “Wilson loop renormalization group flows,” *JHEP* **10** (2011) 059, [arXiv:1104.5077](https://arxiv.org/abs/1104.5077).
- [65] M. Beccaria, S. Giombi, and A. Tseytlin, “Non-supersymmetric Wilson loop in $\mathcal{N} = 4$ SYM and defect 1d CFT,” *JHEP* **03** (2018) 131, [arXiv:1712.06874](https://arxiv.org/abs/1712.06874).
- [66] B. Chen, W. He, J.-B. Wu, and L. Zhang, “M5-branes and Wilson surfaces,” *JHEP* **08** (2007) 067, [arXiv:0707.3978](https://arxiv.org/abs/0707.3978).
- [67] A. Lewkowycz and E. Perlmutter, “Universality in the geometric dependence of Rényi entropy,” *JHEP* **01** (2015) 080, [arXiv:1407.8171](https://arxiv.org/abs/1407.8171).
- [68] N. Drukker, J. Gomis, and S. Matsuura, “Probing $\mathcal{N} = 4$ SYM with surface operators,” *JHEP* **10** (2008) 048, [arXiv:0805.4199](https://arxiv.org/abs/0805.4199).
- [69] A. Chalabi, C. P. Herzog, A. O’Bannon, B. Robinson, and J. Sisti, “Weyl anomalies of four dimensional conformal boundaries and defects,” *JHEP* **02** (2022) 166, [arXiv:2111.14713](https://arxiv.org/abs/2111.14713).
- [70] P. Capuzzo, J. Estes, B. Robinson, and B. Suzzoni, “Holographic Weyl anomalies for 4d defects in 6d SCFTs,” [arXiv:2310.17447](https://arxiv.org/abs/2310.17447).
- [71] N. Drukker, D. J. Gross, and A. A. Tseytlin, “Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function,” *JHEP* **04** (2000) 021, [hep-th/0001204](https://arxiv.org/abs/hep-th/0001204).
- [72] V. Forini, V. G. M. Puletti, L. Griguolo, D. Seminara, and E. Vescovi, “Remarks on the geometrical properties of semiclassically quantized strings,” *J. Phys. A* **48** no. 47, (2015) 475401, [arXiv:1507.01883](https://arxiv.org/abs/1507.01883).

- [73] R. de León Ardón, “Semiclassical p -branes in hyperbolic space,” *Class. Quant. Grav.* **37** no. 23, (2020) 237001, [arXiv:2007.03591](#).
- [74] G. Goon, S. Melville, and J. Noller, “Quantum corrections to generic branes: DBI, NLSM, and more,” *JHEP* **01** (2021) 159, [arXiv:2010.05913](#).
- [75] B. de Wit, K. Peeters, and J. Plefka, “Superspace geometry for supermembrane backgrounds,” *Nucl. Phys. B* **532** (1998) 99–123, [hep-th/9803209](#).
- [76] B. de Wit, K. Peeters, J. Plefka, and A. Sevrin, “The M-theory two-brane in $AdS_4 \times S^7$ and $AdS_7 \times S^4$,” *Phys. Lett. B* **443** (1998) 153–158, [hep-th/9808052](#).
- [77] M. Sakaguchi, H. Shin, and K. Yoshida, “Semiclassical analysis of M2-brane in $AdS_4 \times S^7/Z_k$,” *JHEP* **12** (2010) 012, [arXiv:1007.3354](#).