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An extension of the Liouville theorem for Fourier multipliers to sub-exponentially growing solutions

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Abstract. We study the equation $m(D)f = 0$ in a large class of sub-exponentially growing functions. Under appropriate restrictions on $m \in C(\mathbb{R}^n)$, we show that every such solution can be analytically continued to a sub-exponentially growing entire function on $\mathbb{C}^n$ if and only if $m(\xi) \neq 0$ for $\xi \neq 0$.

1. Introduction

The classical Liouville theorem for the Laplace operator $\Delta := \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ on $\mathbb{R}^n$ says that every bounded (polynomially bounded) solution of the equation $\Delta f = 0$ is in fact constant (is a polynomial). Recently, similar results have been obtained for solutions of more general equations of the form $m(D)f = 0$, where $m(D) := \mathcal{F}^{-1}m(\xi)\mathcal{F}$, and

$$\mathcal{F}\phi(\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}u(\xi)\,d\xi$$

are the Fourier and the inverse Fourier transforms (see [**1**], [**2**], [**3**], [**11**], and the references therein). Namely, it was shown that, under appropriate restrictions on $m \in C(\mathbb{R}^n)$, the implication

$$f \text{ is bounded (polynomially bounded) and } m(D)f = 0 \implies f \text{ is constant (is a polynomial)}$$

holds if and only if $m(\xi) \neq 0$ for $\xi \neq 0$. Much of this research has been motivated by applications to infinitesimal generators of Lévy processes.

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In this paper, we deal with solutions of $m(D)f = 0$ that can grow faster than any polynomial. Of course, one cannot expect such solutions to have simple structure, not even in the case of $\Delta f = 0$ in $\mathbb{R}^2$ (see, e.g., [**21**, Ch. I, §2]). We consider sub-exponentially growing solutions whose growth is controlled by a submultiplicative function (see (1)) satisfying the Beurling-Domar condition (3), and show that, under appropriate restrictions on $m \in C(\mathbb{R}^n)$, every such solution admits analytic continuation to a sub-exponentially growing entire function on $\mathbb{C}^n$ if and only if $m(\xi) \neq 0$ for $\xi \neq 0$ (see Corollary 4.5). Results of this type have been obtained for solutions of partial differential equations with constant coefficients by A. Kaneko and G.E. Šilov (see [**16**], [**17**], [**26**], [**7**, Ch. 10, Sect. 2, Theorem 2], and Section 5 below).

Keeping in mind applications to infinitesimal generators of Lévy processes, we do not assume that m is the Fourier transform of a distribution with compact support, so our setting is different from that in, e.g., [**6**], [**15**, Ch. XVI].

The paper is organized as follows. In Chapter 2, we consider submultiplicative functions satisfuing the Beurling-Domar condition and, for every such function g, introduce an auxiliary function S_g (see (14), (15)), which appears in our main estimates. Chapter 3 contains weighted L^p estimates for entire functions on $\mathbb{C}^n$, which are a key ingredient in the proof of our main results in Chapter 4. Another key ingredient is the Tauberian theorem 4.1, which is similar to [**3**, Theorem 7] and [**23**, Theorem 9.3]. The main difference is that the function f in Theorem 4.1 is not assumed to be polynomially bounded, and hence it might not be a tempered distribution. So, we avoid using the Fourier transform $\widehat{f} = \mathcal{F}f$ and its support (and non-quasianalytic type ultradistributions). Although we are mainly interested in the case $m(\xi) \neq 0$ for $\xi \neq 0$, we also prove a Liouville type result for m with compact zero set $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\}$ (see Theorem 4.4). Finally, we discuss in Section 5 A. Kaneko's Liouville type results for partial differential equations with constant coefficients ([**16**], [**17**]), which show that the Beurling-Domar condition is in a sense optimal in our setting.

2. Submultiplicative functions and the Beurling-Domar condition

Let $g : \mathbb{R}^n \to (0, \infty)$ be a locally bounded, measurable *submultiplicative* function, i.e. a locally bounded measurable function satisfying

$$g(x + y) \leq Cg(x)g(y) \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

where the constant $C \in [1, \infty)$ does not depend on x and y. Without loss of generality, we will always assume that $g \geq 1$, as otherwise one can replace g with $g + 1$. Also, replacing g with Cg, one can assume that

$$g(x + y) \leq g(x)g(y) \quad \text{for all} \quad x, y \in \mathbb{R}^n. \tag{1}$$

A locally bounded submultiplicative function is exponentially bounded, i.e.

$$|g(x)| \leq Ce^{a|x|} \tag{2}$$

for suitable constants $C, a > 0$ (see [**24**, Section 25] or [**13**, Ch. VII]).

We will say that g satisfies the *Beurling-Domar* condition if

$$\sum_{l=1}^{\infty} \frac{\log g(lx)}{l^2} < \infty \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{3}$$

If g satisfies the Beurling-Domar condition, then it also satisfies the Gelfand-Raikov-Shilov condition

$$\lim_{l\to\infty} g(lx)^{1/l} = 1 \quad \text{for all} \quad x \in \mathbb{R}^n,$$

while $g(x) = e^{|x|/\log(e+|x|)}$ satisfies the latter but not the former (see [**9**]). It is also easy to see that $g(x) = e^{|x|/\log^\gamma(e+|x|)}$ satisfies the Beurling-Domar condition if and only if $\gamma > 1$. The function

$$g(x) = e^{a|x|^b}(1+|x|)^s(\log(e+|x|))^t$$

satisfies the Beurling-Domar condition for any $a, s, t \geq 0$ and $b \in [0, 1)$ (see [**9**]).

Lemma 2.1. *Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition* (3)*. Then for every $\varepsilon > 0$, there exists $R_\varepsilon \in (0, \infty)$ such that*

$$\int_{R_\varepsilon}^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau < \varepsilon \quad \textit{for all} \quad x \in \mathbb{S}^{n-1} := \{y \in \mathbb{R}^n : \ |y| = 1\}. \tag{4}$$

Proof. Since $g \geq 1$ is locally bounded,

$$0 \leq M := \sup_{|y|\leq 1} \log g(y) < \infty. \tag{5}$$

Take any $x \in \mathbb{S}^{n-1}$. It follows from (1) that

$$\log g((l+1)x) - M \leq \log g(\tau x) \leq \log g(lx) + M \quad \text{for all} \quad \tau \in [l, l+1].$$

Hence

$$\sum_{l=L}^{\infty} \frac{\log g((l+1)x) - M}{(l+1)^2} \leq \sum_{l=L}^{\infty} \int_l^{l+1} \frac{\log g(\tau x)}{\tau^2}\, d\tau \leq \sum_{l=L}^{\infty} \frac{\log g(lx) + M}{l^2}$$

$$\Longrightarrow \sum_{l=L+1}^{\infty} \frac{\log g(lx)}{l^2} - \frac{M}{L} \leq \int_L^{\infty} \frac{\log g(\tau x)}{\tau^2}\, d\tau \leq \sum_{l=L}^{\infty} \frac{\log g(lx)}{l^2} + \frac{M}{L-1} \tag{6}$$

for $L \in \mathbb{N}$.

Let

$$\mathbf{e}_j := (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0), \ j = 1, \dots, n, \qquad \mathbf{e}_0 := \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right), \tag{7}$$

$$Q := \left\{ y = (y_1, \dots, y_n) \in \mathbb{R}^n : \ \frac{1}{2\sqrt{n}} < y_j < \frac{2}{\sqrt{n}}, \ j = 1, \dots, n \right\}.$$

For every $x \in \mathbb{S}^{n-1}$ there exists an orthogonal matrix $A_x \in O(n)$ such that $x = A_x \mathbf{e}_0$. Hence $\{AQ\}_{A\in O(n)}$ is an open cover of $\mathbb{S}^{n-1}$. Let $\{A_k Q\}_{k=1,\ldots,K}$ be a finite subcover. Take an arbitrary $\varepsilon > 0$. It follows from (3) and (6) that there exists $R_\varepsilon > 0$ for which

$$\int_{\frac{R_\varepsilon}{2\sqrt{n}}}^\infty \frac{\log g(\tau A_k \mathbf{e}_j)}{\tau^2}\, d\tau < \frac{\varepsilon}{2\sqrt{n}}, \quad k = 1, \ldots, K,\ j = 1, \ldots, n.$$

For any $x \in \mathbb{S}^{n-1}$, there exist $k = 1, \ldots, K$ and $a_j \in \left(\frac{1}{2\sqrt{n}}, \frac{2}{\sqrt{n}}\right)$, $j = 1, \ldots, n$ such that

$$x = \sum_{j=1}^n a_j A_k \mathbf{e}_j.$$

Using (1), one gets

$$\int_{R_\varepsilon}^\infty \frac{\log g(\tau x)}{\tau^2}\, d\tau \le \sum_{j=1}^n \int_{R_\varepsilon}^\infty \frac{\log g(\tau a_j A_k \mathbf{e}_j)}{\tau^2}\, d\tau = \sum_{j=1}^n a_j \int_{a_j R_\varepsilon}^\infty \frac{\log g(r A_k \mathbf{e}_j)}{r^2}\, dr$$
$$\le \sum_{j=1}^n \frac{2}{\sqrt{n}} \int_{\frac{R_\varepsilon}{2\sqrt{n}}}^\infty \frac{\log g(r A_k \mathbf{e}_j)}{r^2}\, dr < \sum_{j=1}^n \frac{2}{\sqrt{n}} \cdot \frac{\varepsilon}{2\sqrt{n}} = n\,\frac{\varepsilon}{n} = \varepsilon.$$

□

Let

$$I_{g,x}(r) := \int_{\max\{r,1\}}^\infty \frac{\log g(\tau x)}{\tau^2}\, d\tau < \infty,$$
$$J_{g,x}(r) := \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x)\, d\tau < \infty,$$
$$S_{g,x}(r) := \frac{1}{\pi} \int_{-\infty}^\infty \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2}\, d\tau \quad r \ge 0,\ x \in \mathbb{S}^{n-1}.$$

One has, for $r > 1$ and any $\beta \in (0,1)$,

$$\begin{aligned} J_{g,x}(r) &= \frac{1}{r^2}\int_0^r \log g(\tau x)\, d\tau = \frac{1}{r^2}\int_0^1 \log g(\tau x)\, d\tau \\ &\quad + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{r^{2\beta}}\, d\tau + \int_{r^\beta}^r \frac{\log g(\tau x)}{r^2}\, d\tau \\ &\le \frac{M}{r^2} + \frac{1}{r^{2(1-\beta)}} \int_1^{r^\beta} \frac{\log g(\tau x)}{\tau^2}\, d\tau + \int_{r^\beta}^r \frac{\log g(\tau x)}{\tau^2}\, d\tau \\ &\le \frac{M}{r^2} + \frac{I_{g,x}(1)}{r^{2(1-\beta)}} + I_{g,x}(r^\beta) \end{aligned} \tag{8}$$

(see (5)). Further, if $r > 1$, then

$$\pi S_{g,x}(r) = \int_0^\infty \frac{\log g(\tau x)}{\tau^2 + r^2}\, d\tau + \int_0^\infty \frac{\log g(-\tau x)}{\tau^2 + r^2}\, d\tau$$

$$\begin{aligned}
&\le \int_0^r \frac{\log g(\tau x)}{r^2}\,d\tau + \int_r^\infty \frac{\log g(\tau x)}{\tau^2}\,d\tau \\
&+ \int_0^r \frac{\log g(-\tau x)}{r^2}\,d\tau + \int_r^\infty \frac{\log g(-\tau x)}{\tau^2}\,d\tau \\
&= I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r), \qquad (9)\\
\pi S_{g,x}(r) &\ge \int_0^r \frac{\log g(\tau x)}{2r^2}\,d\tau + \int_r^\infty \frac{\log g(\tau x)}{2\tau^2}\,d\tau \\
&+ \int_0^r \frac{\log g(-\tau x)}{2r^2}\,d\tau + \int_r^\infty \frac{\log g(-\tau x)}{2\tau^2}\,d\tau \\
&= \frac{1}{2}\left(I_{g,x}(r) + J_{g,x}(r) + I_{g,-x}(r) + J_{g,-x}(r)\right). \qquad (10)
\end{aligned}$$

Since g is locally bounded, it follows from Lemma 2.1 that I_g defined by

$$I_g(r) := \sup_{x\in\mathbb{S}^{n-1}} I_{g,x}(r) = \sup_{x\in\mathbb{S}^{n-1}} \int_{\max\{r,1\}}^\infty \frac{\log g(\tau x)}{\tau^2}\,d\tau < \infty, \qquad (11)$$

is a decreasing function such that

$$I_g(r) \to 0 \quad \text{as} \quad r \to \infty. \qquad (12)$$

Let

$$J_g(r) := \sup_{x\in\mathbb{S}^{n-1}} J_{g,x}(r) = \sup_{x\in\mathbb{S}^{n-1}} \frac{1}{\max\{r,1\}^2} \int_0^r \log g(\tau x)\,d\tau, \qquad (13)$$

$$S_g(r) := \sup_{x\in\mathbb{S}^{n-1}} S_{g,x}(r) = \sup_{x\in\mathbb{S}^{n-1}} \frac{1}{\pi} \int_{-\infty}^\infty \frac{\log g(\tau x)}{\tau^2 + \max\{r,1\}^2}\,d\tau. \qquad (14)$$

Then

$$\begin{aligned}
&J_g(r) \le \frac{M}{r^2} + \frac{I_g(1)}{r^{2(1-\beta)}} + I_g(r^\beta),\\
&\frac{1}{2\pi}\max\{I_g(r), J_g(r)\} \le S_g(r) \le \frac{2}{\pi}\left(I_g(r) + J_g(r)\right)
\end{aligned}$$

(see (8), (9), (10)). So, $J_g(r) \to 0$, and

$$S_g(r) \to 0 \quad \text{as} \quad r \to \infty \qquad (15)$$

(see (12)). It is clear that

$$S_g(r) = S_g(1) \text{ for } r \in [0,1], \quad \text{and} \quad S_g \text{ is a decreasing function.} \qquad (16)$$

Examples.

1) If $g(x) = (1+|x|)^s$, $s \ge 0$, then

$$\begin{aligned}
S_g(r) &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{s\log(1+|\tau|)}{\tau^2+r^2}\,d\tau = \frac{s}{\pi r}\int_{-\infty}^\infty \frac{\log(1+r|\lambda|)}{\lambda^2+1}\,d\lambda\\
&\le \frac{s}{\pi r}\int_{-\infty}^\infty \frac{\log(1+|\lambda|)}{\lambda^2+1}\,d\lambda + \frac{s\log(1+r)}{\pi r}\int_{-\infty}^\infty \frac{1}{\lambda^2+1}\,d\lambda
\end{aligned}$$

$$= \frac{c_1 s}{r} + \frac{s\log(1+r)}{r}, \quad r \ge 1, \tag{17}$$

where

$$c_1 := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log(1+|\lambda|)}{\lambda^2+1}\, d\lambda < \infty.$$

2) If $g(x) = (\log(e+|x|))^t$, $t \ge 0$, then using the obvious inequality

$$u + v \le 2uv, \qquad u, v \ge 1,$$

one gets

$$\begin{aligned} S_g(r) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t\log\log(e+|\tau|)}{\tau^2+r^2}\, d\tau = \frac{t}{\pi r}\int_{-\infty}^{\infty} \frac{\log\log(e+r|\lambda|)}{\lambda^2+1}\, d\lambda \\ &\le \frac{t}{\pi r}\int_{-\infty}^{\infty} \frac{\log\big(\log(e+|\lambda|)+\log(e+r)\big)}{\lambda^2+1}\, d\lambda \\ &\le \frac{t}{\pi r}\int_{-\infty}^{\infty} \frac{\log\big(2\log(e+|\lambda|)\big)}{\lambda^2+1}\, d\lambda + \frac{t\log\log(e+r)}{\pi r}\int_{-\infty}^{\infty} \frac{1}{\lambda^2+1}\, d\lambda \\ &= \frac{c_2 t}{r} + \frac{t\log\log(e+r)}{r}, \quad r \ge 1, \end{aligned} \tag{18}$$

where

$$c_2 := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log\big(2\log(e+|\lambda|)\big)}{\lambda^2+1}\, d\lambda < \infty.$$

3) If $g(x) = e^{a|x|^b}$, $a \ge 0$, $b \in [0,1)$, then

$$\begin{aligned} S_g(r) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a|\tau|^b}{\tau^2+r^2}\, d\tau = \frac{ar^{b-1}}{\pi}\int_{-\infty}^{\infty} \frac{|\lambda|^b}{\lambda^2+1}\, d\lambda = \frac{2ar^{b-1}}{\pi}\int_0^{\infty} \frac{t^b}{t^2+1}\, dt \\ &= \frac{ar^{b-1}}{\pi}\int_0^{\infty} \frac{s^{\frac{b-1}{2}}}{s+1}\, ds = \frac{ar^{b-1}}{\sin\left(\frac{1-b}{2}\pi\right)}, \quad r \ge 1 \end{aligned} \tag{19}$$

(see, e.g., [**4**, Ch. V, Example 2.12]).

4) Finally, let $g(x) = e^{|x|/\log^\gamma(e+|x|)}$, $\gamma > 1$. Since

$$\frac{\tau(e+\tau)}{\tau^2+r^2} = \frac{1+\frac{e}{\tau}}{1+\frac{r^2}{\tau^2}} \le 1 + \frac{e}{\tau} \le 1 + \frac{e}{r} \quad \text{for} \quad \tau \ge r,$$

then for any $\beta \in (0,1)$,

$$\begin{aligned} S_g(r) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|}{(\tau^2+r^2)\log^\gamma(e+|\tau|)}\, d\tau = \frac{2}{\pi}\int_0^{\infty} \frac{\tau}{(\tau^2+r^2)\log^\gamma(e+\tau)}\, d\tau \\ &= \frac{2}{\pi}\left(\int_0^{r^\beta} + \int_{r^\beta}^{r} + \int_r^{\infty}\right) \frac{\tau}{(\tau^2+r^2)\log^\gamma(e+\tau)}\, d\tau \\ &\le \frac{2}{\pi}\int_0^{r^\beta} \frac{\tau}{\tau^2+r^2}\, d\tau + \frac{2}{\pi\log^\gamma(e+r^\beta)}\int_{r^\beta}^{r} \frac{\tau}{\tau^2+r^2}\, d\tau \\ &\quad + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\int_r^{\infty} \frac{1}{(e+\tau)\log^\gamma(e+\tau)}\, d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\log(\tau^2+r^2)\Big|_0^{r^\beta} + \frac{1}{\pi\log^\gamma(e+r^\beta)}\log(\tau^2+r^2)\Big|_{r^\beta}^{r} \\
&\quad + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{1-\gamma}\log^{1-\gamma}(e+\tau)\Big|_r^\infty \\
&\le \frac{1}{\pi}\log(1+r^{2(\beta-1)}) + \frac{\log 2}{\pi\log^\gamma(e+r^\beta)} + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{\gamma-1}\log^{1-\gamma}(e+r) \\
&\le \frac{r^{2(\beta-1)}}{\pi} + \frac{\log 2}{\pi\log^\gamma(e+r^\beta)} + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{\gamma-1}\log^{1-\gamma}(e+r), \quad r\ge 1.
\end{aligned}
$$

Since

$$\lim_{r\to\infty}\frac{r^{2(\beta-1)}+(\log 2)\log^{-\gamma}(e+r^\beta)}{\log^{-\gamma}(e+r)} = \frac{\log 2}{\beta^\gamma} \quad \text{for all} \quad \beta\in(0,1),$$

one gets, upon taking $\beta\in\left((\log 2)^{1/\gamma},1\right)$, the following estimate

$$S_g(r) \le \frac{\log^{-\gamma}(e+r)}{\pi} + \frac{2}{\pi}\left(1+\frac{e}{r}\right)\frac{1}{\gamma-1}\log^{1-\gamma}(e+r) \tag{20}$$

for sufficiently large r.

3. Estimates for entire functions

Let $1\le p\le\infty$ and let $\omega:\mathbb{R}^n\to[0,\infty)$ be a measurable function such that $\omega>0$ Lebesgue almost everywhere. We set

$$
\begin{aligned}
&\|f\|_{L^p_\omega} := \|\omega f\|_{L^p} \quad \text{and} \\
&L^p_\omega(\mathbb{R}^n) := \left\{f:\mathbb{R}^n\to\mathbb{C} \mid f \text{ measurable}, \ \|f\|_{L^p_\omega}<\infty\right\}.
\end{aligned} \tag{21}
$$

Lemma 3.1. *Let $g:\mathbb{R}^n\to[1,\infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition* (3)*. Let φ be a measurable function such that for almost every $x'=(x_2,\dots,x_n)\in\mathbb{R}^{n-1}$, $\varphi(z_1,x')$ is analytic in z_1 for $\operatorname{Im} z_1>0$ and continuous up to $\mathbb{R}$. Suppose also that $\log|\varphi(z_1,x')| = O(|z_1|)$ for $|z_1|$ large, $\operatorname{Im} z_1\ge 0$, and that the restriction of φ to $\mathbb{R}^n$ belongs to $L^p_{g^{\pm1}}(\mathbb{R}^n)$, $1\le p\le\infty$. Finally, suppose that*

$$k_\varphi := \operatorname*{ess\,sup}_{x'\in\mathbb{R}^{n-1}}\left(\limsup_{0<y_1\to\infty}\frac{\log|\varphi(iy_1,x')|}{y_1}\right)<\infty. \tag{22}$$

Then

$$\|\varphi(\cdot+iy_1,\cdot)\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)} \le C_g e^{(k_\varphi+S_g(y_1))y_1}\|\varphi\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)}, \quad y_1>0 \tag{23}$$

(see (14), (15)*), where the constant $C_g<\infty$ depends only on g.*

Proof. Let $a^+ := \max\{a,0\}$ for $a\in\mathbb{R}$. It follows from (1) that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\log^+\left(g^{\mp1}(t,x')\right)}{1+t^2}\,dt \le \int_{-\infty}^{\infty}\frac{\log\left(g(t,x')\right)}{1+t^2}\,dt \\
&\le \int_{-\infty}^{\infty}\frac{\log(g(t,0))+\log(g(0,x'))}{1+t^2}\,dt \le \pi\left((S_g(1)+\log(g(0,x'))\right)<+\infty.
\end{aligned}
$$

Since $g^{\pm 1}\varphi \in L^p(\mathbb{R}^n)$, Fubini's theorem implies that

$$g^{\pm 1}(\cdot, x')\varphi(\cdot, x') \in L^p(\mathbb{R})$$

for almost all $x' \in \mathbb{R}^{n-1}$. For such $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\log^+ |\varphi(t, x')|}{1+t^2}\, dt \\ &\quad \le \int_{-\infty}^{\infty} \frac{\log^+ \left(g^{\pm 1}(t, x')|\varphi(t, x')|\right)}{1+t^2}\, dt + \int_{-\infty}^{\infty} \frac{\log^+ \left(g^{\mp 1}(t, x')\right)}{1+t^2}\, dt < \infty.\end{aligned}$$

Then

$$\log|\varphi(x_1 + iy_1, x')| \le k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\varphi(t, x')|}{(t-x_1)^2 + y_1^2}\, dt, \quad x_1 \in \mathbb{R},\ y_1 > 0$$

([**19**, Ch. III, G, 2], see also [**21**, Ch. V, Theorems 5 and 7]).

Applying (1) again, one gets

$$\begin{aligned}&\log g(x) \le \log g(t, x') + \log g(x_1 - t, 0),\\ &\log g(t, x') \le \log g(x) + \log g(t - x_1, 0) \quad \text{for all} \quad x = (x_1, x') \in \mathbb{R}^n,\ t \in \mathbb{R}.\end{aligned}$$

The latter inequality can be rewritten as follows

$$\log g^{-1}(x) \le \log g^{-1}(t, x') + \log g(t - x_1, 0).$$

Hence

$$\log g^{\pm 1}(x) \le \log g^{\pm 1}(t, x') + \log g(\pm(x_1 - t), 0) \quad \text{for all} \quad x = (x_1, x') \in \mathbb{R}^n,\ t \in \mathbb{R},$$

and

$$\begin{aligned}&\log\left(|\varphi(x_1 + iy_1, x')| g^{\pm 1}(x)\right) \le k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\varphi(t, x')|}{(t-x_1)^2 + y_1^2}\, dt + \log g^{\pm 1}(x)\\ &= k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\varphi(t, x')| + \log g^{\pm 1}(x)}{(t-x_1)^2 + y_1^2}\, dt\\ &\le k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log\left(|\varphi(t, x')| g^{\pm 1}(t, x')\right)}{(t-x_1)^2 + y_1^2}\, dt + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log g(\pm(x_1 - t), 0)}{(t-x_1)^2 + y_1^2}\, dt\\ &= k_\varphi y_1 + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log\left(|\varphi(t, x')| g^{\pm 1}(t, x')\right)}{(t-x_1)^2 + y_1^2}\, dt + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2}\, d\tau.\end{aligned}$$

If $y_1 \in [0, 1]$, then

$$\begin{aligned}\frac{y_1}{\pi}\int_0^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2}\, d\tau &\le M\,\frac{y_1}{\pi}\int_0^1 \frac{1}{\tau^2 + y_1^2}\, d\tau + \frac{y_1}{\pi}\int_1^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2}\, d\tau\\ &\le M\,\frac{y_1}{\pi}\int_{\mathbb{R}} \frac{1}{\tau^2 + y_1^2}\, d\tau + \frac{1}{\pi}\int_1^{\infty} \frac{\log g(\tau, 0)}{\tau^2}\, d\tau \le M + \frac{I_g(1)}{\pi}. \qquad (24)\end{aligned}$$

It follows from (14) that for $y_1 > 1$,

$$\frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log g(\tau, 0)}{\tau^2 + y_1^2}\, d\tau \le y_1 S_g(y_1).$$

So,

$$\log\left(|\varphi(x_1+iy_1,x')|g^{\pm1}(x)\right) \le c_g + (k_\varphi + S_g(y_1))\,y_1 \\ + \frac{y_1}{\pi}\int_{-\infty}^{\infty} \frac{\log\left(|\varphi(t,x')|g^{\pm1}(t,x')\right)}{(t-x_1)^2+y_1^2}\,dt,$$

where $c_g := M + \frac{I_g(1)}{\pi}$. Using Jensen's inequality, one gets

$$|\varphi(x_1+iy_1,x')|g^{\pm1}(x) \le C_g e^{(k_\varphi+S_g(y_1))y_1}\,\frac{y_1}{\pi}\int_{-\infty}^{\infty}\frac{|\varphi(t,x')|g^{\pm1}(t,x')}{(t-x_1)^2+y_1^2}\,dt,$$

where

$$C_g := e^{M+\frac{I_g(1)}{\pi}}. \tag{25}$$

Estimate (23) now follows from Young's convolution inequality and (21). □

Remark 3.2. Let $n=1$, $g:\mathbb{R}\to[1,\infty)$ be a Hölder continuous submultiplicative function satisfying the Beurling-Domar condition, and let

$$w(x+iy) := \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log g(t)}{(t-x)^2+y^2}\,dt \\ + \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-t}{(t-x)^2+y^2}+\frac{t}{t^2+1}\right)\log g(t)\,dt, \quad x\in\mathbb{R},\ y>0.$$

Then $\varphi(z) := e^{w(z)}$ is analytic in z for $\operatorname{Im} z>0$ and continuous up to $\mathbb{R}$,

$$|\varphi(x)| = e^{\operatorname{Re}(w(x))} = e^{\log g(x)} = g(x), \quad x\in\mathbb{R}$$

(see, e.g., [**8**, Ch. III, §1]), and

$$|\varphi(iy)| = e^{\operatorname{Re}(w(iy))} = \exp\left(\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log g(t)}{t^2+y^2}\,d\right) = e^{S_g(y)y}, \quad y>0.$$

So,

$$k_\varphi = \limsup_{0<y\to\infty}\frac{\log|\varphi(iy)|}{y} = \limsup_{y\to\infty} S_g(y) = 0$$

(see (15)), and

$$\|\varphi(\cdot+iy)\|_{L^\infty_{g^{-1}}(\mathbb{R})} \ge \frac{|\varphi(iy)|}{g(0)} \ge |\varphi(iy)| = e^{S_g(y)y} = e^{S_g(y)y}\|1\|_{L^\infty(\mathbb{R})} \\ = e^{S_g(y)y}\|g^{-1}\varphi\|_{L^\infty(\mathbb{R})} = e^{S_g(y)y}\|\varphi\|_{L^\infty_{g^{-1}}(\mathbb{R})},$$

which shows that the factor $e^{S_g(y_1)y_1}$ in the right-hand side of (23) is optimal in this case.

Clearly,

$$S_{\breve g} = S_g, \quad C_{\breve g} = C_g, \tag{26}$$

where $\breve g(x) := g(Ax)$ and $A\in O(n)$ is an arbitrary orthogonal matrix (see (14), (25) and (5)).

THEOREM 3.3. *Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition* (3). *Let $\varphi : \mathbb{C}^n \to \mathbb{C}$ be an entire function such that $\log|\varphi(z)| = O(|z|)$ for $|z|$ large, $z \in \mathbb{C}^n$, and that the restriction of φ to $\mathbb{R}^n$ belongs to $L^p_{g^{\pm 1}}(\mathbb{R}^n)$, $1 \le p \le \infty$. Then for every multi-index $\alpha \in \mathbb{Z}^n_+$,*

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)} \le C_\alpha e^{(\kappa_\varphi(y/|y|)+S_g(|y|))|y|} \|\varphi\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{27}$$

where

$$\kappa_\varphi(\omega) := \sup_{x\in\mathbb{R}^n} \left(\limsup_{0<t\to\infty} \frac{\log|\varphi(x+it\omega)|}{t} \right) < \infty, \quad \omega \in \mathbb{S}^{n-1}, \tag{28}$$

and the constant $C_\alpha \in (0, \infty)$ depends only on α and g.

PROOF. (Cf. the proof of Lemma 9.29 in [**20**].) Take any $y \in \mathbb{R}^n \setminus \{0\}$. There exist an orthogonal matrix $A \in O(n)$ such that $A\mathbf{e}_1 = \omega := y/|y|$ (see (7)). Let $\breve{\varphi}(z) := \varphi(Az)$, $z \in \mathbb{C}^n$, and $\breve{g}(x) := g(Ax)$, $x \in \mathbb{R}^n$. Then $\breve{\varphi} : \mathbb{C}^n \to \mathbb{C}$ is an entire function, and one can apply to it Lemma 3.1 with $\breve{g}$ in place of g (see (26)).

For any $x \in \mathbb{R}^n$, one has $\varphi(x+iy) = \breve{\varphi}(\tilde{x} + i|y|\mathbf{e}_1) = \breve{\varphi}(\tilde{x}_1 + i|y|, \tilde{x}_2, \dots, \tilde{x}_n)$, where $\tilde{x} := A^{-1}x$. Hence

$$\begin{aligned}\|\varphi(\cdot + iy)\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)} &= \|\breve{\varphi}(\cdot + i|y|, \cdot)\|_{L^p_{\breve{g}^{\pm1}}(\mathbb{R}^n)} \le C_{\breve{g}} e^{\left(k_{\breve{\varphi}}+S_{\breve{g}}(|y|)\right)|y|} \|\breve{\varphi}\|_{L^p_{\breve{g}^{\pm1}}(\mathbb{R}^n)} \\ &\le C_g e^{(\kappa_\varphi(y/|y|)+S_g(|y|))|y|} \|\breve{\varphi}\|_{L^p_{\breve{g}^{\pm1}}(\mathbb{R}^n)} = C_g e^{(\kappa_\varphi(y/|y|)+S_g(|y|))|y|} \|\varphi\|_{L^p_{g^{\pm1}}(\mathbb{R}^n)}\end{aligned}$$

(see (26)), which proves (27) for $\alpha = 0$ and $y \ne 0$. This estimate is trivial for $\alpha = 0$ and $y = 0$.

Iterating the standard Cauchy integral formula for one complex variable, one gets

$$\varphi(\zeta) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)} \left(\prod_{k=1}^n e^{i\theta_k} \right) d\theta_1 \cdots d\theta_n,$$
$$\zeta \in \Delta(z) := \{\eta \in \mathbb{C}^n : \ |\eta_k - z_k| < 1, \ k = 1, \dots, n\}, \ z \in \mathbb{C}^n$$

(cf. [**20**, Ch. 1, §1]), which implies

$$\partial^\alpha \varphi(\zeta) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n (z_k + e^{i\theta_k} - \zeta_k)^{\alpha_k+1}} \left(\prod_{k=1}^n e^{i\theta_k} \right) d\theta_1 \cdots d\theta_n.$$

Hence

$$\partial^\alpha \varphi(z) = \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})}{\prod_{k=1}^n e^{i\alpha_k\theta_k}} \, d\theta_1 \cdots d\theta_n,$$

and

$$|\partial^\alpha \varphi(z)| \le \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left|\varphi(z_1 + e^{i\theta_1}, \dots, z_n + e^{i\theta_n})\right| d\theta_1 \cdots d\theta_n. \tag{29}$$

Since $g \geq 1$ is locally bounded,

$$1 \leq M_1 := \sup_{|s_k| \leq 1,\, k=1,\dots,n} g(s) < \infty.$$

Then it follows from (1) that

$$g^{\pm 1}(x_1 - \cos\theta_1, \dots, x_n - \cos\theta_n) \leq M_1 g^{\pm 1}(x). \tag{30}$$

According to the conditions of the theorem, there exists a constant $c_\varphi \in (0, \infty)$ such that $\log|\varphi(\zeta)| \leq c_\varphi |\zeta|$ for $|\zeta|$ large. Then $\kappa_\varphi(\omega) \leq c_\varphi$ (see (28)). Let $\varphi_y := \varphi(\cdot + iy)$, $y = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$. Then, similarly to the above inequality, $\kappa_{\varphi_y}(\omega) \leq c_\varphi$. Applying (27) with $\alpha = 0$ to the function φ_y in place of φ and using (16), (30), one derives from (29)

$$\begin{aligned}
&\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \\
&\leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left\|\varphi(\cdot + iy_1 + e^{i\theta_1}, \dots, \cdot + iy_n + e^{i\theta_n})\right\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \, d\theta_1 \cdots d\theta_n \\
&\leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 \left\|\varphi(\cdot + iy_1 + i\sin\theta_1, \dots, \cdot + iy_n + i\sin\theta_n)\right\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \, d\theta_1 \cdots d\theta_n \\
&\leq \frac{\alpha!}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \, d\theta_1 \cdots d\theta_n \\
&= \alpha! M_1 C_0 e^{(c_\varphi + S_g(1))\sqrt{n}} \|\varphi(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}.
\end{aligned}$$

Applying (27) with $\alpha = 0$ again, one gets

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \leq \alpha! M_1 C_0^2 e^{(c_\varphi + S_g(1))\sqrt{n}} e^{(\kappa_\varphi(y/|y|) + S_g(|y|))|y|} \|\varphi\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}.$$

□

Corollary 3.4. *Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition* (3)*. Let $\varphi : \mathbb{C}^n \to \mathbb{C}$ be an entire function such that $\log|\varphi(z)| = O(|z|)$ for $|z|$ large, $z \in \mathbb{C}^n$, and that the restriction of φ to $\mathbb{R}^n$ belongs to $L^p_{g^{\pm 1}}(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then for every multi-index $\alpha \in \mathbb{Z}^n_+$ and every $\varepsilon > 0$,*

$$\|(\partial^\alpha \varphi)(\cdot + iy)\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)} \leq C_{\alpha,\varepsilon} e^{(\kappa_\varphi(y/|y|) + \varepsilon)|y|} \|\varphi\|_{L^p_{g^{\pm 1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{31}$$

where κ_φ is defined by (28)*, and the constant $C_{\alpha,\varepsilon} \in (0, \infty)$ depends only on α, ε, and g.*

Proof. It follows from (15) that for every $\varepsilon > 0$, there exists c_ε such that

$$S_g(|y|)|y| \leq c_\varepsilon + \varepsilon|y| \quad \text{for all} \quad y \in \mathbb{R}^n.$$

Hence (27) implies (31). □

4. Main results

We will use the notation $\widetilde{g}(x) := g(-x)$, $x \in \mathbb{R}^n$.

Taking $y - x$ in place of y in (1) and rearranging, one gets

$$\frac{1}{g(x)} \le \frac{g(y-x)}{g(y)}\,. \tag{32}$$

Using this inequality, one can easily show that $f * u \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ for every $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ and $u \in L^1_{\widetilde{g}}(\mathbb{R}^n)$. The Fubini-Tonelli theorem implies that

$$f * (v * u) = (f * v) * u \quad \text{for all} \quad f \in L^\infty_{g^{-1}}(\mathbb{R}^n) \quad \text{and} \quad v, u \in L^1_{\widetilde{g}}(\mathbb{R}^n). \tag{33}$$

Theorem 4.1. *Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition* (3)*, $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$, and Y be a linear subspace of $L^1_{\widetilde{g}}(\mathbb{R}^n)$ such that*

$$f * v = 0 \quad \textit{for every } v \in Y. \tag{34}$$

Suppose the set

$$Z(Y) := \bigcap_{v \in Y} \{\xi \in \mathbb{R}^n \mid \widehat{v}(\xi) = 0\} \tag{35}$$

is bounded, and $u \in L^1_{\widetilde{g}}(\mathbb{R}^n)$ is such that $\widehat{u} = 1$ in a neighbourhood of $Z(Y)$. Then $f = f * u$.

Proof. It is sufficient to show that

$$\langle f, h\rangle = \langle f * u, h\rangle \quad \text{for every } h \in L^1_g(\mathbb{R}^n). \tag{36}$$

Since the set of functions h with compactly supported Fourier transforms $\widehat{h}$ is dense in $L^1_g(\mathbb{R}^n)$ (see [**5**, Theorems 1.52 and 2.11]), it is sufficient to prove (36) for such h. Further,

$$\langle f, h\rangle = \left(f * \widetilde{h}\right)(0).$$

So, it is sufficient to show that

$$f * w = f * u * w \tag{37}$$

for every $w \in L^1_{\widetilde{g}}(\mathbb{R}^n)$ with compactly supported Fourier transform $\widehat{w}$. Take any such w and choose $R > 0$ such that the support of $\widehat{w}$ lies in $B_R := \{\xi \in \mathbb{R}^n : \ |\xi| \le R\}$. It is clear that $\widetilde{g}$ satisfies the Beurling-Domar condition. Then there exists $u_R \in L^1_{\widetilde{g}}(\mathbb{R}^n)$ such that $0 \le \widehat{u_R} \le 1$, $\widehat{u_R}(\xi) = 1$ for $|\xi| \le R$, and $\widehat{u_R}(\xi) = 0$ for $|\xi| \ge R + 1$ (see [**5**, Lemma 1.24]).

Let V be an open neighbourhood of $Z(Y)$ such that $\widehat{u} = 1$ in V. Similarly to the above, there exists $u_0 \in L^1_{\widetilde{g}}(\mathbb{R}^n)$ such that $0 \le \widehat{u_0} \le 1$, $\widehat{u_0} = 1$ in a neighbourhood $V_0 \subset V$ of $Z(Y)$, and $\widehat{u_0} = 0$ outside V (see [**5**, Lemma 1.24]).

Since Y is a linear subspace, for every $\eta \in B_{R+1} \setminus V_0 \subset \mathbb{R}^n \setminus Z(Y)$, there exists $v_\eta \in Y$ such that $\widehat{v_\eta}(\eta) = 1$. Since $v_\eta \in L^1(\mathbb{R}^n)$, $\widehat{v_\eta}$ is continuous, and there is a neighbourhood V_η of η such that $|\widehat{v_\eta}(\xi) - 1| < 1/2$ for all $\xi \in V_\eta$.

Similarly to the above, there exists $u_\eta \in L^1_{\tilde{g}}(\mathbb{R}^n)$ such that $\mathrm{Re}\,(\widehat{v_\eta}\widehat{u_\eta}) \ge 0$, and $\mathrm{Re}\,(\widehat{v_\eta}\widehat{u_\eta}) > \frac{1}{2}$ in a neighbourhood $V^0_\eta \subset V_\eta$ of η.

Since $B_{R+1} \setminus V_0$ is compact, its open cover $\{V^0_\eta\}_{\eta \in B_{R+1}\setminus V_0}$ has a finite subcover. So, there exist functions $v_j \in Y$ and $u_j \in L^1_{\tilde{g}}(\mathbb{R}^n)$, $j = 1, \dots, N$ such that

$$\mathrm{Re}\,(\sigma) > \frac{1}{2}, \quad \text{where} \quad \sigma := \widehat{u_0} + \sum_{j=1}^{N} \widehat{v_j}\widehat{u_j} + 1 - \widehat{u_R}.$$

Then there exists $\upsilon \in L^1_{\tilde{g}}(\mathbb{R}^n)$ such that $\widehat{\upsilon} = 1/\sigma$ (see [**5**, Theorem 1.53]).

Since $\widehat{u_0}(1 - \widehat{u}) = 0$ and $(1 - \widehat{u_R})\,\widehat{w} = 0$, one has

$$\begin{aligned}&\left(\widehat{u} + \sum_{j=1}^{N} \widehat{v_j}\widehat{u_j}\widehat{\upsilon}\,(1-\widehat{u})\right)\widehat{w} = \big(\widehat{u} + (\sigma - (\widehat{u_0} + 1 - \widehat{u_R}))\,\widehat{\upsilon}\,(1-\widehat{u})\,\big)\widehat{w}\\ &= \big(\widehat{u} + (1-\widehat{u}) - (\widehat{u_0} + 1 - \widehat{u_R})\widehat{\upsilon}\,(1-\widehat{u})\,\big)\widehat{w} = \big(1 - (1-\widehat{u_R})\widehat{\upsilon}\,(1-\widehat{u})\,\big)\widehat{w}\\ &= \widehat{w} - (1-\widehat{u_R})\widehat{w}\widehat{\upsilon}\,(1-\widehat{u}) = \widehat{w}.\end{aligned}$$

It now follows from (33) and (34) that

$$\begin{aligned} f * w &= f * \left(u + \sum_{j=1}^{N} v_j * u_j * (\upsilon - \upsilon * u)\right) * w\\ &= f * u * w + f * \left(\sum_{j=1}^{N} v_j * u_j * (\upsilon - \upsilon * u)\right) * w\\ &= f * u * w + \sum_{j=1}^{N} (f * v_j) * u_j * (\upsilon - \upsilon * u) * w = f * u * w.\end{aligned}$$

□

Corollary 4.2. *If $Z(Y) = \emptyset$ in Theorem 4.1, then $f = 0$.*

Proof. It is sufficient to show that

$$f * w = 0$$

for every $w \in L^1_{\tilde{g}}(\mathbb{R}^n)$ with compactly supported Fourier transform $\widehat{w}$ (see the beginning of the proof of Theorem 4.1). Take $u \in L^1_{\tilde{g}}(\mathbb{R}^n)$ is such that $\widehat{u} = 1$ in an open set, and the support of $\widehat{u}$ does not intersect that of $\widehat{w}$. The latter condition implies that $u * w = 0$. Since $Z(Y) = \emptyset$, it follows from Theorem 4.1 that $f = f * u$. Hence,

$$f * w = (f * u) * w = f * (u * w) = f * 0 = 0$$

(see (33)). □

For a bounded set $E \subset \mathbb{R}^n$, let $\operatorname{conv}(E)$ denote its closed convex hull and H_E denote its support function:

$$H_E(y) := \sup_{\xi\in E} y\cdot\xi = \sup_{\xi\in\operatorname{conv}(E)} y\cdot\xi, \quad y\in\mathbb{R}^n.$$

Clearly, H_E is positively homogeneous and convex:

$$H_E(\tau y) = \tau H_E(y), \quad H_E(y+x) \le H_E(y) + H_E(x) \quad \text{for all} \quad y,x\in\mathbb{R}^n,\ \tau\ge 0.$$

For every positively homogeneous convex function H,

$$K := \{\xi\in\mathbb{R}^n |\ y\cdot\xi \le H(y) \text{ for all } y\in\mathbb{R}^n\} \tag{38}$$

is the unique convex compact set such that $H_K = H$ (see, e.g., [**14**, Theorem 4.3.2]).

Theorem 4.3. *Let g, f, and Y satisfy the conditions of Theorem 4.1, and let*

$$\mathcal{H}_Y(y) := H_{Z(Y)}(-y) = \sup_{\xi\in Z(Y)} (-y)\cdot\xi = -\inf_{\xi\in Z(Y)} y\cdot\xi, \quad y\in\mathbb{R}^n. \tag{39}$$

Then f admits analytic continuation to an entire function $f : \mathbb{C}^n \to \mathbb{C}$ such that for every multi-index $\alpha\in\mathbb{Z}^n_+$,

$$\|(\partial^\alpha f)(\cdot+iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \le C_\alpha e^{\mathcal{H}_Y(y)+S_g(|y|)|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y\in\mathbb{R}^n \tag{40}$$

(see (14), (15)*), where the constant $C_\alpha\in(0,\infty)$ depends only on α and g.*

Proof. Take any $\varepsilon > 0$. There exists $u\in L^1_{\widetilde{g}}(\mathbb{R}^n)$ such that $\widehat{u} = 1$ in a neighbourhood of $Z(Y)$, and $\widehat{u} = 0$ outside the $\frac{\varepsilon}{2}$-neighbourhood of $Z(Y)$ (see [**5**, Lemma 1.24]). It follows from the Paley–Wiener–Schwartz theorem (see, e.g., [**14**, Theorem 7.3.1]) that $u = \mathcal{F}^{-1}\widehat{u}$ admits analytic continuation to an entire function $u : \mathbb{C}^n \to \mathbb{C}$ satisfying the estimate

$$|u(x+iy)| \le c_\varepsilon e^{\mathcal{H}_Y(y)+\varepsilon|y|/2} \quad \text{for all} \quad x,y\in\mathbb{R}^n$$

with some constant $c_\varepsilon\in(0,\infty)$. So, u satisfies the conditions of Corollary 3.4 with $\widetilde{g}$ in place of g, and

$$\|u(\cdot+iy)\|_{L^1_{\widetilde{g}}(\mathbb{R}^n)} \le C_{0,\varepsilon/2}\, e^{\mathcal{H}_Y(y)+\varepsilon|y|}\|u\|_{L^1_{\widetilde{g}}(\mathbb{R}^n)}, \quad y\in\mathbb{R}^n. \tag{41}$$

Since

$$f(x) = \int_{\mathbb{R}^n} u(x-s)f(s)\,ds$$

(see Theorem 4.1), f admits analytic continuation

$$f(x+iy) := \int_{\mathbb{R}^n} u(x+iy-s)f(s)\,ds$$

(see Corollary 3.4), and

$$\begin{aligned}\|f(\cdot+iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} &\le \|u(\cdot+iy)\|_{L^1_{\widetilde{g}}(\mathbb{R}^n)}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}\\ &\le C_{0,\varepsilon/2}\, e^{\mathcal{H}_Y(y)+\varepsilon|y|}\|u\|_{L^1_{\widetilde{g}}(\mathbb{R}^n)}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} =: M_\varepsilon e^{\mathcal{H}_Y(y)+\varepsilon|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}\end{aligned}$$

(see (32)). Since

$$\frac{|f(x+iy)|}{g(x)} \le M_\varepsilon e^{\mathcal{H}_Y(y)+\varepsilon|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)},$$

one has $\log|f(x+iy)| = O(|x+iy|)$ for $|x+iy|$ large (see (2)), and

$$\begin{aligned}\limsup_{0<t\to\infty}\frac{\log|f(x+it\omega)|}{t} &\le \limsup_{0<t\to\infty}\frac{\log\left(M_\varepsilon g(x)\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}\right)+t\mathcal{H}_Y(\omega)+\varepsilon t}{t}\\ &= \mathcal{H}_Y(\omega)+\varepsilon.\end{aligned}$$

Hence,

$$\kappa_f(\omega) := \sup_{x\in\mathbb{R}^n}\left(\limsup_{0<t\to\infty}\frac{\log|f(x+it\omega)|}{t}\right) \le \mathcal{H}_Y(\omega)+\varepsilon$$

for every $\varepsilon>0$, i.e.

$$\kappa_f(\omega) \le \mathcal{H}_Y(\omega).$$

So, (40) follows from Theorem 3.3. □

THEOREM 4.4. *Let $g:\mathbb{R}^n\to[1,\infty)$ be a locally bounded, measurable sub-multiplicative function satisfying the Beurling-Domar condition* (3) *and let $m\in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n)\ni\phi\mapsto\widetilde{m}(D)\phi := \mathcal{F}^{-1}(\widetilde{m}\widehat{\phi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_g(\mathbb{R}^n)$. Suppose $f\in L^\infty_{g^{-1}}(\mathbb{R}^n)$ is such that $m(D)f=0$ as a distribution, i.e.

$$\langle f,\widetilde{m}(D)\phi\rangle = 0 \quad \textit{for all } \phi\in C_c^\infty(\mathbb{R}^n). \tag{42}$$

If $K:=\{\eta\in\mathbb{R}^n \mid m(\eta)=0\}$ is compact, then f admits analytic continuation to an entire function $f:\mathbb{C}^n\to\mathbb{C}$ such that for every multi-index $\alpha\in\mathbb{Z}^n_+$,

$$\|(\partial^\alpha f)(\cdot+iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \le C_\alpha e^{H(y)+S_g(|y|)|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y\in\mathbb{R}^n \tag{43}$$

(see (14), (15)*), where where $H(y):=H_K(-y)$, and the constant $C_\alpha\in(0,\infty)$ depends only on α and g.*

Conversely, if every $f\in L^\infty(\mathbb{R}^n)$ satisfying (42) *admits analytic continuation to an entire function $f:\mathbb{C}^n\to\mathbb{C}$ such that*

$$\|f(\cdot+iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \le M_\varepsilon e^{H(y)+\varepsilon|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y\in\mathbb{R}^n, \tag{44}$$

holds for every $\varepsilon>0$ with a constant $M_\varepsilon\in(0,\infty)$ that depends only on ε, m, and g, then $\{\eta\in\mathbb{R}^n \mid m(\eta)=0\}\subseteq K$, where K is the unique convex compact set such that $H_K(y)=H(-y)$ (cf. (38)*).*

PROOF. Let

$$(T_\upsilon\phi)(x) := \phi(x-\upsilon), \quad x,\upsilon\in\mathbb{R}^n.$$

Since $T_\upsilon\phi\in C_c^\infty(\mathbb{R}^n)$ for every $\phi\in C_c^\infty(\mathbb{R}^n)$ and all $\upsilon\in\mathbb{R}^n$, it follows from (42) that

$$\left(f*\widetilde{\widetilde{m}(D)\phi}\right)(\upsilon) = \langle f, T_\upsilon\widetilde{m}(D)\phi\rangle = \langle f,\widetilde{m}(D)\,(T_\upsilon\phi)\rangle = 0 \quad \text{for all } \upsilon\in\mathbb{R}^n.$$

Hence

$$f * \widetilde{\widetilde{m}(D)\phi} = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

It is easy to see that

$$\bigcap_{\phi \in C_c^\infty(\mathbb{R}^n)} \left\{\eta \in \mathbb{R}^n \mid \widehat{\widetilde{\widetilde{m}(D)\phi}}(\eta) = 0\right\} = \bigcap_{\phi \in C_c^\infty(\mathbb{R}^n)} \left\{\eta \in \mathbb{R}^n \mid \widehat{\widetilde{m}(D)\phi}(-\eta) = 0\right\}$$

$$= \bigcap_{\phi \in C_c^\infty(\mathbb{R}^n)} \left\{\eta \in \mathbb{R}^n \mid m(\eta)\widehat{\phi}(-\eta) = 0\right\} = \{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = K.$$

Applying Theorem 4.3 with

$$Y := \left\{\widetilde{\widetilde{m}(D)\phi} \mid \phi \in C_c^\infty(\mathbb{R}^n)\right\} \subset L^1_{\widetilde{g}}(\mathbb{R}^n)$$

and $Z(Y) = K$, one gets (43).

For the converse direction, we assume the contrary, i.e. that the zero-set $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\}$ contains some $\gamma \notin K$ (see (38)). Then there exists $y_0 \in \mathbb{R}^n \setminus \{0\}$ such that $y_0 \cdot \gamma > H_K(y_0) = H(-y_0)$. It is easy to see that $f(x) := e^{ix\cdot\gamma}$ satisfies $m(D)e^{ix\cdot\gamma} = e^{ix\cdot\gamma}m(\gamma) = 0$ for all $x \in \mathbb{R}^n$. Take $\varepsilon < (y_0 \cdot \gamma - H(-y_0))/|y_0|$. Clearly, $f \in L^\infty(\mathbb{R}^n)$, and

$$\frac{\|f(\cdot - i\tau y_0)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}}{e^{H(-\tau y_0)+\varepsilon|\tau y_0|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}} = \frac{e^{\tau(y_0\cdot\gamma)}}{e^{\tau(H(-y_0)+\varepsilon|y_0|)}}$$

$$= e^{\tau(y_0\cdot\gamma - H(-y_0)-\varepsilon|y_0|)} \to \infty \quad \text{as} \quad \tau \to \infty.$$

So, f does not satisfy (44). □

Corollary 4.5. *Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function satisfying the Beurling-Domar condition* (3) *and let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator*

$$C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \widetilde{m}(D)\phi := \mathcal{F}^{-1}(\widetilde{m}\widehat{\phi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_g(\mathbb{R}^n)$. Suppose $f \in L^\infty_{g^{-1}}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, i.e. (42) *holds. If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \{0\}$, then f admits analytic continuation to an entire function $f : \mathbb{C}^n \to \mathbb{C}$ such that for every multi-index $\alpha \in \mathbb{Z}^n_+$,*

$$\|(\partial^\alpha f)(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \le C_\alpha e^{S_g(|y|)|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{45}$$

where the constant $C_\alpha \in (0, \infty)$ depends only on α and g. If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$, then $f = 0$.

Conversely, if every $f \in L^\infty(\mathbb{R}^n)$ satisfying (42) *admits analytic continuation to an entire function $f : \mathbb{C}^n \to \mathbb{C}$ such that*

$$\|f(\cdot + iy)\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)} \le M_\varepsilon e^{\varepsilon|y|}\|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^n)}, \quad y \in \mathbb{R}^n, \tag{46}$$

holds for every $\varepsilon > 0$ with a constant $M_\varepsilon \in (0, \infty)$ that depends only on ε, m, and g, then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subseteq \{0\}$.

PROOF. The only part that does not follow immediately from Theorem 4.4 is that $f = 0$ in the case $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} = \emptyset$. In this case, one can take the same Y as in the proof of Theorem 4.4, note that $Z(Y) = \emptyset$ and apply Corollary 4.2 to conclude that $f = 0$. (It is instructive to compare this result to [**17**, Proposition 2.2].) □

REMARK 4.6. The condition that $\widetilde{m}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ to $L_g^1(\mathbb{R}^n)$ is satisfied if m is a linear combination of terms of the form ab, where $a = F\mu$, μ is a finite complex Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \widetilde{g}(y)\, |\mu|(dy) < \infty,$$

and b is the Fourier transform of a compactly supported distribution. Indeed, it is easy to see that $\widetilde{b}(D)$ maps $C_c^\infty(\mathbb{R}^n)$ into itself, while the convolution operator $\phi \mapsto \widetilde{\mu} * \phi$ maps $C_c^\infty(\mathbb{R}^n)$ to $L_g^1(\mathbb{R}^n)$.

REMARK 4.7. We are mostly interested in super-polynomially growing weights here as polynomially growing ones have been dealt with in our previous paper [**3**]. Nevertheless, it is instructive to look at the behaviour of the factor $e^{S_g(|y|)|y|}$ for typical super-polynomially, polynomially, and sub-polynomially growing weights.

It follows from (20) that if $g(x) = e^{|x|/\log^\gamma(e+|x|)}$, $\gamma > 1$, then there exists a constant C_γ such that

$$\begin{aligned} e^{S_g(|y|)|y|} &\le C_\gamma e^{\frac{1}{\pi}|y|\log^{-\gamma}(e+|y|)\left(1+\frac{2}{\gamma-1}\log(e+|y|)\right)} \\ &= C_\gamma \left(e^{|y|/\log^\gamma(e+|y|)}\right)^{\frac{1}{\pi}\left(1+\frac{2}{\gamma-1}\log(e+|y|)\right)} = C_\gamma (g(y))^{\frac{1}{\pi}\left(1+\frac{2}{\gamma-1}\log(e+|y|)\right)}. \end{aligned}$$

Similarly, if $g(x) = e^{a|x|^b}$, $a \ge 0$, $b \in [0,1)$, then (19) implies

$$e^{S_g(|y|)|y|} = e^{a|y|^b\left(\sin\left(\frac{1-b}{2}\pi\right)\right)^{-1}} = (g(y))^{\left(\sin\left(\frac{1-b}{2}\pi\right)\right)^{-1}}. \tag{47}$$

If $g(x) = (1+|x|)^s$, $s \ge 0$, then (17) implies

$$e^{S_g(|y|)|y|} \le e^{c_1 s + s\log(1+|y|)} = C_s(1+|y|)^s = C_s\, g(y). \tag{48}$$

Finally, if $g(x) = (\log(e+|x|))^t$, $t \ge 0$, then (18) implies

$$e^{S_g(|y|)|y|} \le e^{c_2 t + t\log\log(e+|y|)} = C_t(\log(e+|y|))^t = C_t\, g(y).$$

REMARK 4.8. If g is polynomially bounded in Corollary 4.5, then it follows from (45) and (48) that f is a polynomially bounded entire function on $\mathbb{C}^n$ and hence a polynomial (see, e.g., [**20**, Corollary 1.7]). The fact that f is a polynomial in this case was established in [**3**] and [**11**].

REMARK 4.9. Let $n = 2$, $g(x) := (1+|x|)^k$, $k \in \mathbb{N}$, $f(x_1, x_2) := (x_1 + ix_2)^k$ (or $f(x_1, x_2) := (x_1 + ix_2)^k + (x_1 - ix_2)^k$ if one prefers to have a real-valued f). Then $f \in L^\infty_{g^{-1}}(\mathbb{R}^2)$, $\Delta f = 0$, $f(x + iy_1\mathbf{e}_1) = (x_1 + iy_1 + ix_2)^k$ for any $y_1 \in \mathbb{R}$ (see (7)), and

$$\frac{\|f(\cdot + iy_1\mathbf{e}_1)\|_{L^\infty_{g^{-1}}(\mathbb{R}^2)}}{g(y_1\mathbf{e}_1)} \ge \frac{|y_1|^k}{(1+|y_1|)^k} \to 1 = \|f\|_{L^\infty_{g^{-1}}(\mathbb{R}^2)} \quad \text{as} \quad |y_1| \to \infty.$$

So, the factor $e^{S_g(|y|)|y|} \le C_k\, g(y)$ (see (48)) is optimal in (45) in this case.

The case $g(x) = e^{a|x|^b}$, $a > 0$, $b \in [0, 1)$ is perhaps more interesting. Let us take $b = \frac{1}{2}$. Then it follows from (47) that $e^{S_g(|y|)|y|} = (g(y))^{\sqrt{2}}$. Let us show that one cannot replace this factor in (45) with $(g(y))^{\sqrt{2}\,(1-\varepsilon)}$, $\varepsilon > 0$. Take any $\varepsilon > 0$. Since

$$\sqrt[4]{1+\tau^2}\,\cos\left(\frac{1}{2}\arctan\frac{1}{\tau}\right) \to \frac{1}{\sqrt{2}} \quad \text{as} \quad \tau \to 0,\ \tau > 0,$$

there exists $\tau_\varepsilon > 0$ such that

$$\sqrt[4]{1+\tau_\varepsilon^2}\,\cos\left(\frac{1}{2}\arctan\frac{1}{\tau_\varepsilon}\right) \le \frac{1+\varepsilon}{\sqrt{2}}\,.$$

Let us estimate $\mathrm{Re}\,\sqrt{x_1 + i\kappa x_2}$, where $x = (x_1, x_2) \in \mathbb{R}^2$, $\kappa > 0$ is a constant to be chosen later, and $\sqrt{\cdot}$ is the branch of the square root that is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and positive on $(0, +\infty)$. If $x_1 \ge \tau_\epsilon \kappa |x_2|$, then

$$\begin{aligned}\mathrm{Re}\,\sqrt{x_1 + i\kappa x_2} &\le \left|\sqrt{x_1 + i\kappa x_2}\right| = \sqrt[4]{x_1^2 + \kappa^2 x_2^2} \le \sqrt[4]{\left(1 + \frac{1}{\tau_\varepsilon^2}\right) x_1^2} \\ &\le \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4} \sqrt{x_1} \le \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4} \sqrt{|x|}\,.\end{aligned}$$

If $0 < x_1 < \tau_\epsilon \kappa |x_2|$, then

$$\begin{aligned}&\mathrm{Re}\,\sqrt{x_1 + i\kappa x_2} \le \left|\sqrt{x_1 + i\kappa x_2}\right| \cos\left(\frac{1}{2}\arctan\frac{\kappa|x_2|}{x_1}\right) \\ &\le \left|\sqrt{\tau_\epsilon \kappa |x_2| + i\kappa x_2}\right| \cos\left(\frac{1}{2}\arctan\frac{1}{\tau_\varepsilon}\right) \\ &= \kappa^{1/2}|x_2|^{1/2}\,\sqrt[4]{1+\tau_\varepsilon^2}\,\cos\left(\frac{1}{2}\arctan\frac{1}{\tau_\varepsilon}\right) \le \frac{1+\varepsilon}{\sqrt{2}}\kappa^{1/2}|x|^{1/2}.\end{aligned}$$

Now, take $\kappa_\varepsilon \ge 1$ such that

$$\frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2} \ge \left(1 + \frac{1}{\tau_\varepsilon^2}\right)^{1/4}.$$

Then

$$\mathrm{Re}\,\sqrt{x_1 + i\kappa_\varepsilon x_2} \le \frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}|x|^{1/2} \tag{49}$$

for $x_1 > 0$. If $x_1 \le 0$, then the argument of $\sqrt{x_1 + i\kappa_\varepsilon x_2}$ belongs to $\pm[\pi/4, \pi/2]$, depending on the sign of x_2. Hence

$$\mathrm{Re}\,\sqrt{x_1 + i\kappa_\varepsilon x_2} \le \left|\sqrt{x_1 + i\kappa_\varepsilon x_2}\right| \cos\frac{\pi}{4} \le \frac{1}{\sqrt{2}}\kappa_\varepsilon^{1/2}|x|^{1/2},$$

and (49) holds for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Since the Taylor series of $\cos w$ contains only even powers of w, $\cos(i\sqrt{z})$ is an analytic function of $z \in \mathbb{C}$. So, $\cos(i\sqrt{x_1 + ix_2})$ is a harmonic function of

$x = (x_1, x_2) \in \mathbb{R}^2$. Hence $f(x_1, x_2) := \cos(i\sqrt{x_1 + i\kappa_\varepsilon x_2})$ is a solution of the elliptic partial differential equation

$$\left(\partial_{x_1}^2 + \frac{1}{\kappa_\varepsilon^2}\partial_{x_2}^2\right) f(x_1, x_2) = 0.$$

It follows from (49) that

$$|f(x_1, x_2)| \le \frac{1}{2}\left(1 + e^{\operatorname{Re}\sqrt{x_1 + i\kappa_\varepsilon x_2}}\right) \le e^{\frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}|x|^{1/2}}.$$

So, $f \in L^\infty_{g^{-1}}(\mathbb{R}^2)$, where $g(x) = e^{a|x|^{1/2}}$ with $a = \frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}$. Clearly, the analytic continuation of f to $\mathbb{C}^2$ is given by the formula

$$f(x_1 + iy_1, x_2 + iy_2) = \cos\left(i\sqrt{x_1 + iy_1 + i\kappa_\varepsilon(x_2 + iy_2)}\right).$$

Then (see (7))

$$\begin{aligned}\frac{\|f(\cdot + iy_2\mathbf{e}_2)\|_{L^\infty_{g^{-1}}(\mathbb{R}^2)}}{(g(y_2\mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} &\ge \frac{|f(0 + iy_2\mathbf{e}_2)|}{g(0)(g(y_2\mathbf{e}_2))^{\sqrt{2}(1-\varepsilon)}} = \frac{|\cos(i\sqrt{-\kappa_\varepsilon y_2})|}{e^{\sqrt{2}(1-\varepsilon)\frac{1+\varepsilon}{\sqrt{2}}\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} \\ &\ge \frac{e^{\kappa_\varepsilon^{1/2}|y_2|^{1/2}}}{2e^{(1-\varepsilon^2)\kappa_\varepsilon^{1/2}|y_2|^{1/2}}} = \frac{e^{\varepsilon^2\kappa_\varepsilon^{1/2}|y_2|^{1/2}}}{2} \to \infty \quad \text{as} \quad y_2 \to -\infty.\end{aligned}$$

5. Concluding remarks

Corollary 4.5 shows that sub-exponentially growing solutions of $m(D)f = 0$ admit analytic continuation to entire functions on $\mathbb{C}^n$. It is well known that no growth restrictions are necessary in the case when $m(D)$ is an elliptic partial differential operator with constant coefficients, and every solution of $m(D)f = 0$ in $\mathbb{R}^n$ admits analytic continuation to an entire function on $\mathbb{C}^n$ (see [**22**], [**6**]).

Remark 5.1. The latter result has a local version similar to Hayman's theorem on harmonic functions (see [**12**, Theorem 1]) : for every elliptic partial differential operator $m(D)$ with constant coefficients there exists a constant $c_m \in (0, 1)$ such that every solution of $m(D)f = 0$ in the ball $\{x \in \mathbb{R}^n : |x| < R\}$ of any radius $R > 0$ admits continuation to an analytic function in the ball $\{x \in \mathbb{C}^n : |x| < c_m R\}$. Indeed, let $m_0(D) = \sum_{|\alpha|=N} a_\alpha D^\alpha$ be the principal part of $m(D) = \sum_{|\alpha|\le N} a_\alpha D^\alpha$. There exists $C_m > 0$ such that

$$\sum_{|\alpha|=N} a_\alpha(a + ib)^\alpha = 0, \quad a, b \in \mathbb{R}^n \qquad \Longrightarrow \qquad |a| \ge C_m|b|$$

(see, e.g., [**25**, §7]). Then the same argument as in the proof of [**18**, Corollary 8.2] shows that f admits continuation to an analytic function in the ball $\left\{x \in \mathbb{C}^n : |x| < (1 + C_m^{-2})^{-1/2}R\right\}$. Note that in the case of the Laplacian, one can take $C_m = 1$ and $c_m = (1 + C_m^{-2})^{-1/2} = \frac{1}{\sqrt{2}}$, which is the optimal constant for harmonic functions (see [**12**]).

Let us return to equations in $\mathbb{R}^n$. Below, $m(\xi)$ will always denote a polynomial with $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} \subseteq \{0\}$. For non-elliptic partial differential operators $m(D)$, one needs to place growth restrictions on solutions of $m(D)f = 0$ to make sure that they admit analytic continuation to entire functions on $\mathbb{C}^n$.

We say that a function f defined on $\mathbb{R}^n$ (or $\mathbb{C}^n$) is of *infra-exponential* growth if for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(z)| \le C_\varepsilon e^{\varepsilon|z|} \quad \text{for all} \quad z \in \mathbb{R}^n \ (z \in \mathbb{C}^n).$$

Let $\mu : [0, \infty) \to [0, \infty)$ be an increasing to infinity function such that

$$\mu(t) \le At + B, \quad t \ge 0$$

for some $A, B > 0$, and

$$\int_1^\infty \frac{\mu(t)}{t^2}\, dt < \infty. \tag{50}$$

Suppose $\{\xi \in \mathbb{R}^n \mid m(\xi) = 0\} = \{0\}$. Then, under additional restrictions on μ, every solution f of $m(D)f = 0$ that has growth $O(e^{\varepsilon\mu(|x|)})$ for every $\varepsilon > 0$ admits analytic continuation to an entire function of infra-exponential growth on $\mathbb{C}^n$ (see [**17**]). It is easy to see that (50) is equivalent to the Beurling-Domar condition (3) for $g(x) := e^{\mu(|x|)}$.

One cannot replace $O(e^{\varepsilon\mu(|x|)})$ with $O(e^{\varepsilon|x|})$ in the above result without placing a restriction on the complex zeros of m. If there exists $\delta > 0$ such that $m(\zeta)$ has no complex zeros in

$$|\mathrm{Im}\,\zeta| < \delta, \quad |\mathrm{Re}\,\zeta| > \delta^{-1}, \tag{51}$$

then every solution of $m(D)f = 0$ that, together with its partial derivatives up to the order of $m(D)$, is of infra-exponential growth on $\mathbb{R}^n$, admits analytic continuation to an entire function of infra-exponential growth on $\mathbb{C}^n$ (see [**16**], [**17**]).

On the other hand, if for every $\delta > 0$, (51) contains complex zeros of $m(\zeta)$, then $m(D)f = 0$ has a solution in C^∞ all of whose derivatives are of infra-exponential growth, but which is not entire infra-exponential in $\mathbb{C}^n$. The proof of the latter result in [**16**], [**17**] is not constructive, and the author writes: "Unfortunately we cannot present concrete examples of such" solutions. However, it is not difficult to construct, for any $\varepsilon > 0$, a solution in C^∞ all of whose derivatives have growth $O(e^{\varepsilon|x|})$, but which is not real-analytic. Indeed, according to the assumption, there exist complex zeros

$$\zeta_k = \xi_k + i\eta_k, \quad \xi_k, \eta_k \in \mathbb{R}^n, \qquad k \in \mathbb{N}$$

of $m(\zeta)$ such that

$$|\eta_k| < k^{-1}, \quad |\xi_k| > k. \tag{52}$$

Choosing a subsequence, we can assume that $\omega_k := |\xi_k|^{-1}\xi_k$ converge to a point $\omega_0 \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ as $k \to \infty$, and that $|\omega_k - \omega_0| < 1$ for all

$k \in \mathbb{N}$. Then

$$\omega_k \cdot \omega_0 = \frac{|\omega_k|^2 + |\omega_0|^2 - |\omega_k - \omega_0|^2}{2} > \frac{1+1-1}{2} = \frac{1}{2}, \quad k \in \mathbb{N}. \tag{53}$$

Consider

$$f(x) := \sum_{k>\varepsilon^{-1}} \frac{e^{i\zeta_k \cdot x}}{e^{|\xi_k|^{1/2}}} = \sum_{k>\varepsilon^{-1}} \frac{e^{i\xi_k \cdot x - \eta_k \cdot x}}{e^{|\xi_k|^{1/2}}}, \quad x \in \mathbb{R}^n. \tag{54}$$

Then, for every multi-index α,

$$|\partial^\alpha f(x)| = \left| \sum_{k>\varepsilon^{-1}} \frac{(i\zeta_k)^\alpha e^{i\zeta_k \cdot x}}{e^{|\xi_k|^{1/2}}} \right| \le \sum_{k>\varepsilon^{-1}} \frac{(|\xi_k|+1)^{|\alpha|} e^{|\eta_k||x|}}{e^{|\xi_k|^{1/2}}}$$

$$\le e^{\varepsilon|x|} \sum_{k>\varepsilon^{-1}} \frac{(|\xi_k|+1)^{|\alpha|}}{e^{|\xi_k|^{1/2}}} =: C_\alpha e^{\varepsilon|x|}, \quad x \in \mathbb{R}^n$$

(see (52)). Further,

$$m(D)f(x) = \sum_{k>\varepsilon^{-1}} \frac{m(\zeta_k) e^{i\zeta_k \cdot x}}{e^{|\xi_k|^{1/2}}} = 0.$$

On the other hand, f is not real-analytic. Before we prove this, note that formally putting $x - it\omega_0$, $t > 0$ in place of x in the right-hand side of (54), one gets a divergent series. Indeed, its terms can be estimated as follows

$$\left| \frac{e^{i\xi_k \cdot x + t\xi_k \cdot \omega_0 - \eta_k \cdot x + it\eta_k \cdot \omega_0}}{e^{|\xi_k|^{1/2}}} \right| = \frac{e^{t|\xi_k|\omega_k \cdot \omega_0 - \eta_k \cdot x}}{e^{|\xi_k|^{1/2}}} \ge e^{-\varepsilon|x|} \frac{e^{t|\xi_k|/2}}{e^{|\xi_k|^{1/2}}} \to \infty$$

as $k \to \infty$ (see (52), (53)).

For any $j > \varepsilon^{-1}$, there exists $\ell_j \in \mathbb{N}$ such that

$$\ell_j \le |\xi_j|^{1/2} < \ell_j + 1. \tag{55}$$

It is clear that $\ell_j \to \infty$ as $j \to \infty$ (see (52)). Note that

$$|\arg(\omega_0 \cdot \zeta_k)| \le \frac{|\omega_0 \cdot \eta_k|}{|\omega_0 \cdot \xi_k|} \le \frac{2}{k|\xi_k|}.$$

If $|\xi_k| \ge \frac{6\ell_j}{\pi k}$, then

$$|\arg(\omega_0 \cdot \zeta_k)^{\ell_j}| \le \frac{2\ell_j}{k|\xi_k|} \le \frac{\pi}{3},$$

and

$$\operatorname{Re}(\omega_0 \cdot \zeta_k)^{\ell_j} \ge \frac{1}{2} |\omega_0 \cdot \zeta_k|^{\ell_j} \ge \frac{1}{2^{\ell_j+1}} |\xi_k|^{\ell_j}.$$

Clearly, $|\xi_j| \ge \frac{6\ell_j}{\pi j}$ for sufficiently large j (see (55)). Hence, one has the following estimate for the directional derivative ∂_{ω_0}

$$\left| \left((-i\partial_{\omega_0})^{\ell_j} f \right)(0) \right| \ge \sum_{k>\varepsilon^{-1}} \frac{\operatorname{Re}(\omega_0 \cdot \zeta_k)^{\ell_j}}{e^{|\xi_k|^{1/2}}}$$

$$\geq -\sum_{k>\varepsilon^{-1},\ |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{|\zeta_k|^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \sum_{k>\varepsilon^{-1},\ |\xi_k|\geq\frac{6\ell_j}{\pi k}} \frac{|\xi_k|^{\ell_j}}{2^{\ell_j+1}e^{|\xi_k|^{1/2}}}$$

$$\geq -\sum_{k>\varepsilon^{-1},\ |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{\left(|\xi_k|+\frac{1}{k}\right)^{\ell_j}}{e^{|\xi_k|^{1/2}}} + \frac{|\xi_j|^{\ell_j}}{2^{\ell_j+1}e^{|\xi_j|^{1/2}}}$$

$$\geq -\sum_{k>\varepsilon^{-1},\ |\xi_k|<\frac{6\ell_j}{\pi k}} \frac{1}{e^{|\xi_k|^{1/2}}}\left(\frac{10\ell_j}{\pi k}\right)^{\ell_j} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1}e^{(\ell_j^2+1)^{1/2}}}$$

$$\geq -(10\ell_j)^{\ell_j}\sum_{k=1}^{\infty}\frac{1}{e^{|\xi_k|^{1/2}}k^2} + \frac{\ell_j^{2\ell_j}}{2^{\ell_j+1}e^{\ell_j+1}} = -C(10\ell_j)^{\ell_j} + (2e)^{-(\ell_j+1)}\ell_j^{2\ell_j}.$$

Hence

$$\left|\left((-i\partial_{\omega_0})^{\ell_j}f\right)(0)\right| \geq \ell_j^{\frac{3}{2}\ell_j}$$

for all sufficiently large j, which means that f is not real-analytic in a neighbourhood of 0.

The operator $m(D)$ in the previous example is not hypoelliptic. If $m(D)$ is hypoelliptic, then every solution of $m(D)f = 0$, such that $|f(x)| \leq Ae^{a|x|}$, $x \in \mathbb{R}^n$, for some constants $A, a > 0$, admits analytic continuation to an entire function of order one on $\mathbb{C}^n$ (see [**10**, §4, Corollary 2]). For elliptic operators, this result can be strengthened: every solution of $m(D)f = 0$, such that $|f(x)| \leq Ae^{a|x|^\beta}$, $x \in \mathbb{R}^n$, for $\beta \geq 1$ and some constants $A, a > 0$, admits analytic continuation to an entire function of order β on $\mathbb{C}^n$ (see [**10**, §4, Corollary 3]). Let us show that for every $\beta > 1$ there exists a semi-elliptic operator $m(D)$ (see [**15**, Theorem 11.1.11]) and a C^∞ solution of $m(D)f = 0$, all of whose derivatives have growth $O(e^{a|x|^\beta})$, but which does not admit analytic continuation to an entire function on $\mathbb{C}^n$.

A simple example of such a semi-elliptic operator is $\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2}$ with $\ell \in \mathbb{N}$ satisfying $1 + \frac{1}{2\ell} \leq \beta$, i.e. $\ell \geq \frac{1}{2(\beta-1)}$.

Let

$$f(x_1, x_2) := \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}x_1 + kx_2}}{e^{k^{2\ell+1}}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

If $x_2 > 0$, then the function $t \mapsto tx_2 - t^{2\ell+1}$ achieves maximum at $t = \left(\frac{x_2}{2\ell+1}\right)^{\frac{1}{2\ell}}$, and this maximum is equal to

$$2\ell\left(\frac{1}{2\ell+1}\right)^{1+\frac{1}{2\ell}} x_2^{1+\frac{1}{2\ell}} =: c_\ell\, x_2^{1+\frac{1}{2\ell}}.$$

Hence, for every multi-index α,

$$|\partial^\alpha f(x_1, x_2)| \leq \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{kx_2 - k^{2\ell+1}}$$

$$= \sum_{k=1}^{\left[x_2^{\frac{1}{2\ell}}\right]+1} k^{(2\ell+1)|\alpha|} e^{kx_2 - k^{2\ell+1}} + \sum_{k=\left[x_2^{\frac{1}{2\ell}}\right]+2}^{\infty} k^{(2\ell+1)|\alpha|} e^{k\left(x_2 - k^{2\ell}\right)}$$

$$\leq \left(\left[x_2^{\frac{1}{2\ell}}\right] + 1\right)^{(2\ell+1)|\alpha|+1} e^{c_\ell\, x_2^{1+\frac{1}{2\ell}}} + \sum_{k=1}^{\infty} k^{(2\ell+1)|\alpha|} e^{-k}$$

$$\leq 2^{(2\ell+1)|\alpha|+1} \left(x_2^{2|\alpha|+1} + 1\right) e^{c_\ell\, x_2^{1+\frac{1}{2\ell}}} + c_{\ell,\alpha} \leq C_{\ell,\alpha} e^{(c_\ell+1)x_2^{1+\frac{1}{2\ell}}}.$$

If $x_2 \leq 0$, then

$$|\partial^\alpha f(x_1, x_2)| \leq \sum_{k=1}^{\infty} \frac{k^{(2\ell+1)|\alpha|}}{e^{k^{2\ell+1}}} < \sum_{j=1}^{\infty} \frac{j^{|\alpha|}}{e^j} =: C_\alpha < \infty.$$

So, $f \in C^\infty(\mathbb{R}^2)$, and $\partial^\alpha f(x_1, x_2) = O\left(e^{(c_\ell+1)|x_2|^{1+\frac{1}{2\ell}}}\right) = O\left(e^{(c_\ell+1)|x|^{1+\frac{1}{2\ell}}}\right)$. It is easy to see that $\left(\partial_{x_1}^2 + \partial_{x_2}^{4\ell+2}\right) f(x_1, x_2) = 0$.

The function f admits analytic continuation to the set

$$\Pi_1 := \left\{(z_1, z_2) \in \mathbb{C}^2 | \ \mathrm{Im}\, z_1 < 1\right\}.$$

Indeed, let

$$f(z_1, z_2) = f(x_1 + iy_1, x_2 + iy_2) = \sum_{k=1}^{\infty} \frac{e^{-ik^{2\ell+1}(x_1+iy_1)+k(x_2+iy_2)}}{e^{k^{2\ell+1}}}$$

$$= \sum_{k=1}^{\infty} e^{i\left(ky_2 - k^{2\ell+1}x_1\right)} e^{k^{2\ell+1}(y_1-1)+kx_2}.$$

It is easy to see that the last series is uniformly convergent on compact subsets of Π_1. So, f admits analytic continuation to Π_1. On the other hand, $f(iy_1, 0) \to \infty$ as $y_1 \to 1-0$. Indeed,

$$f(iy_1, 0) = \sum_{k=1}^{\infty} e^{k^{2\ell+1}(y_1-1)}.$$

Take any $N \in \mathbb{N}$. If $y_1 > 1 - N^{-(2\ell+1)}$, then

$$f(iy_1, 0) > \sum_{k=1}^{\infty} e^{-k^{2\ell+1}N^{-(2\ell+1)}} > \sum_{k=1}^{N} e^{-k^{2\ell+1}N^{-(2\ell+1)}} \geq \sum_{k=1}^{N} e^{-1} = \frac{N}{e}.$$

So, $f(iy_1, 0) \to \infty$ as $y_1 \to 1-0$.

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