Trivial and Non-trivial Defect Conformal Manifolds

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A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics
King's College London

Abstract

It has been known for many years that there exist families of superconformal field theories (SCFTs) connected by exactly marginal deformations. Such families are called "conformal manifolds". In the presence of boundaries or defects, we can study the analogue construction, defect conformal manifolds. Just as exactly marginal operators parameterise the conformal manifold, the corresponding operators on conformal defects allow for their marginal deformations.

In this thesis, we consider two kinds of defect exactly marginal operators. One is "trivial" that arises from global symmetry breaking. When a defect breaks a global symmetry, there is a contact term in the conservation equation with defect exactly marginal operators. The resulting defect conformal manifold is the symmetry breaking coset and its Zamolodchikov metric is expressed as the 2-point function of the exactly marginal operators. As the Riemann tensor on the conformal manifold can be expressed as an integrated 4-point function of the marginal operators, we find an exact relation to the curvature of the coset space. We confirm this relation against previously obtained 4-point functions for insertions into the 1/2 BPS Wilson loop in $\mathcal{N} = 4$ super Yang Mills, the 1/2 BPS surface operator of the 6d $\mathcal{N} = (2, 0)$ theory and 1/2 BPS Wilson loops in ABJM theory. We also construct the 1/3 BPS loops in ABJM and examine the relation there.

However, defect conformal manifolds do not require broken symmetries. One natural setting is in 3d, where line operators have multiple marginal couplings. We constructed many new moduli spaces of both conformal and non-conformal BPS Wilson loops in $\mathcal{N} = 4$ quiver Chern-Simons-matter theory on S^3, connected by continuous supersymmetric deformations. In the case of conformal BPS loops, the deformations play the role of defect exactly marginal operators which generate the "nontrivial" conformal manifolds. With the same method, we also address a longstanding question of whether ABJM theory has 1/3 BPS Wilson loop operators, where such loops are made of a large supermatrix combining two 1/2 BPS Wilson loops.

Acknowledgements

Many days later, as I face my examiners, Dr. Anatoly Konechny and Dr. Madalena Lemos, I am to remember that distant afternoon when I firstly entered the office of my supervisor Nadav Drukker, and he said, "Clay and students are two malleable materials to form beautiful things from." Words cannot express my gratitude to him for all the time and efforts into shaping me that he has dedicated in the past four years and will dedicate to me in the future. Carrying the torch lightened by his impressive insights, I get to catch a glimpse of the elegance of mathematical universes.

I would like to extend my sincere thanks to all of my collaborators for their great contributions to our joint projects. They are Bartomeu Fiol, Malte Probst, Georgios Sakkas, Marcia Tenser and Diego Trancanelli. I have benefited a lot from their excellent expertise and overwhelming passions.

I would also like to thank my second supervisor, Yang-Hui He, for his kind support, his hospitality, and his encouragement to me to explore the dark side of the moon.

Special thanks goes to my academic brothers, Maxime Trépanier, Malte Probst and Georgios Sakkas, who have been extremely patient with all of my questions and always welcome for either academic discussions or daily chats. In particular, I want to thank Maxime for detailed comments and thoughtful suggestions on the draft of this thesis.

I am lucky to have the pleasure of spending time with my fellow PhD students and researchers in the theoretical physics group at King's along the way. All the shining moments they have brought into my memories are much appreciated.

My gratitude also goes to my beautiful friend Danqian Cao, who is light of my life, my soul twin, the other star of my binary system.

I am deeply indebted to my parents for all of their support, understanding and everlasting love. They let me go for almost four whole years and have always been backing me whatever happened. I am truly grateful.

I would be remiss in not mentioning the support by CSC grant No. 201906340174 for providing me with the scholarship that allowed me to complete this thesis.

Publications

This thesis is based on the five publications. Sections 3 to 7 follow closely in order

[1] N. Drukker, Z. Kong, and G. Sakkas, "Broken global symmetries and defect conformal manifolds," *Phys. Rev. Lett.* **129** no. 20, (2022) 201603, arXiv:2203.17157.

[2] N. Drukker, Z. Kong, M. Probst, M. Tenser, and D. Trancanelli, "Conformal and non-conformal hyperloop deformations of the 1/2 BPS circle," *JHEP* **08** (2022) 165, arXiv:2206.07390.

[3] N. Drukker, Z. Kong, M. Probst, M. Tenser, and D. Trancanelli, "Classifying BPS bosonic Wilson loops in 3d $\mathcal{N} = 4$ Chern-Simons-matter theories," *JHEP* **11** (2022) 163, arXiv:2210.03758.

[4] Z. Kong, "A network of hyperloops," *JHEP* **06** (2023) 111, arXiv:2212.09418.

[5] N. Drukker and Z. Kong, "1/3 BPS loops and defect CFTs in ABJM theory," *JHEP* **06** (2023) 137, arXiv:2212.03886.

with appendices.

The results of the following publication are not included.

[6] B. Fiol and Z. Kong, "The planar limit of integrated 4-point functions," *JHEP* **07** (2023) 100, arXiv:2303.09572.

Contents

1 Introduction

Quantum Field Theory (QFT) remains of great interest in theoretical physics after almost one century since its inception. A central problem is to classify the QFTs and understand the relations between them. The concept of "space of quantum field theories" which was introduced by Wilson, Friedan and others [8–12] in the 1970's, plays an important role in organizing and classifying QFTs, and allows us to tell when two theories are related by finite variations of the coupling or by Renormalization Group (RG) flows.

In broad terms, one may classify quantum field theories according to their spacetime dimensions, as well as the presence of any possible continuous symmetries. Given the prominence of the Poincaré group in relativistic quantum field theories, the candidates of extended symmetries turn out to be surprisingly limited. In 1967, Coleman and Mandula [13] proved that the Poincaré group can only be combined with internal continuous symmetries in a trivial way, i.e. as a direct product. In the search for nontrivial extensions of Poincaré group, conformal symmetry and supersymmetry have attracted wide interests. Theories enhanced with either conformal or supersymmetries can be constructed in arbitrary spacetime dimensions and have been studied abundantly in the literature[1].

However, after more than fifty years of effort, the complete classification of QFTs is still an unsolved mystery, except for a few special subclasses of theories. One successful example is the two-dimensional Conformal Field Theories[2] (CFTs) [16–18] with central charge $c < 1$, where the combined constraints are given by their unitarity and one loop modular invariance [19–22]. See [23, 24] for an overview of this classification.

For a long time the best existing classification of QFTs was based on perturbation theory [25]. However, we may distrust a perturbation theory because of the large bare coupling, or the relevant coupling that becomes large at low energy. Even more, there are many QFTs for which the perturbation theory is not a good description, since the associated Lagrangian descriptions are often non-existent or just useless. To that end, conformal field theory provides a powerful tool to classify and characterize QFTs, since a UV-complete QFT can always be thought of as a RG flow between CFTs[3] that govern the critical behaviours near the UV and IR fixed points, see (1.1). One famous example is the Quantum Chromodynamics (QCD) with a free fixed point in the UV [26, 27] and a trivial[4] fixed point in the IR [28]. In other words, the classification of QFTs can be addressed by identifying the CFTs that are

[1]The early history of supersymmetry and a guide to the original literatures can be found in [14], and a canonical reading for conformal field theory is [15].

[2]Conformal field theory is a special class of QFTs that is invariant under conformal transformations, i.e. rotations, translations, dilation and the so-called special conformal transformations.

[3]It is generic to have a CFT in the IR. Though having a CFT in the UV is not essential, it is very useful to assume one.

[4]Here by "trivial" we mean there is not enough energy to produce any excitations in the theory, so that the theory is empty, or, trivial.

connected to them through RG flows, and their deformations. The main advantage of this approach is that it requires no perturbative Langrangian descriptions of the QFTs but just the non-perturbative definition of CFTs.

$$\text{QFT}\begin{cases}\text{CFT}_{\text{UV}}\\ \quad\downarrow \text{RG flow}\\ \text{CFT}_{\text{IR}}\end{cases} \tag{1.1}$$

Occasionally, a conformal field theory is not isolated but parameterized continuously as a member of a family. The set of possible values for these parameters often has the structure of a manifold, known as the conformal manifold. It occurs when the theory possesses one or more marginal couplings λ^i, where two well known examples are the compactification radius R of a compact scalar in 2d [29], and the complexified gauge coupling $\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ of 4d $\mathcal{N} = 4$ super Yang-Mills theory [30]. One can interpret these couplings as coordinates on the manifold. More precisely, for a CFT located at point $\mathcal{P}$ on the conformal manifold $\mathcal{M}$, the deformation by an exactly marginal potential

$$\mathcal{W} = \lambda^i \mathcal{O}_i \tag{1.2}$$

takes it to some nearby CFT, $\mathcal{P}'$, where $\mathcal{O}_i$ are the exactly marginal operators.

Especially, when the points along the conformal manifold describe genuinely different theories, rather than being related by a relabelling of operators, the resulting conformal manifold is called "non-trivial" [31, 32]. Instead, if the points on the conformal manifold are related to each other through a group generator and thus the associated deformations do not really change the CFT, the resulting manifold is "trivial". The most general exactly marginal deformation one can write out is an arbitrary linear combinations of the above two types.

Generally, for a given theory there are two approaches to determine the conformal manifold [33]. The first one is based on a direct evaluation of the β-functions requiring their vanishing [34], which is used to show that conformal manifolds are common in 4d $\mathcal{N} = 1$ supersymmetric gauge theories [35]. While the other approach [36] relies on group theory and proves that when a given superconformal field theory is equipped with a global continuous (non-R) symmetry group G', the conformal manifold is determined by the quotient of the space of couplings by the complexified symmetry group

$$\mathcal{M} = \{\lambda^i\}/G'_{\mathbb{C}}\,. \tag{1.3}$$

We have to point out that the existence of conformal manifolds is not guaranteed, and in fact their existence imposes non-trivial constraints on the CFT data [32, 37–39]. There are many 2d CFTs where exactly marginal couplings occur, but for the dimension $D \geq 2$, conformal manifolds are much less common and so far all known examples of conformal manifolds enjoy some degree of supersymmetry. As shown in [40], the superconformal symmetry allows for the existence of marginal couplings only in superconformal field theories (SCFT) with $\mathcal{N} = 1$ or 2 supersymmetry in 3d and $\mathcal{N} = 1$, 2, or 4 supersymmetry in 4d. In even higher dimensions, not any 5d or 6d SCFTs with exactly marginal couplings exist.[5] There are some independent arguments for the existence of non-supersymmetric conformal manifolds, relying on conformal perturbation theory, discussed in [32, 37]. Recently some new holographic evidence for the existence of nonsupersymmetric conformal manifolds have been found in [42], but to our knowledge, we still do not know any explicit examples in this case so far.

In this thesis, we mainly concern defect conformal field theories (dCFTs) that are reviewed in section 2.3. Luckily, in the presence of defects, we can study analogue construction to conformal manifolds, where the same technical restrictions do not apply any more. Much like exactly marginal bulk[6] operators, there may be defect exactly marginal operators and they lead to defect conformal manifolds. In analogy to Goldstone's theorem [43, 44], such marginal defect operators are guaranteed to exist when the defect breaks a global symmetry including R-symmetry. Thus unlike bulk marginal operators, exactly marginal defect operators are ubiquitous.

While conformal defects and their deformations play an important role both in condensed matter physics and in string theory (for example the truly marginal boundary deformations of $c = 1$ theories in [45]), this point of view remains relatively unexplored. When a defect $\mathcal{D}$ breaks a global symmetry G to G', defect exactly marginal operators $\mathbb{O}_i$ naturally emerge from contact terms in the conservation equation (3.2). We find that the resulting defect conformal manifold contains the symmetry breaking coset

$$\mathcal{M} = G/G' \,, \tag{1.4}$$

To see the coset structure, one may think of $\mathcal{M}$ as the space of the defects produced by the actions of all the group elements g of G on $\mathcal{D}$, which remains the same when $g \in G'$. In other words, the space of the defects at least contains the coset G/G'. A natural metric defined locally by the two point function of the defect exactly marginal operators [46] is consistent with the coset structure.

Our method allows defect conformal manifold to exist in very general dCFTs, even including the non-supersymmetric ones, such as the critical $O(N)$ model [8] with defects studied in [47–56]. Two symmetry breaking cosets $O(N)/(O(p) \times O(q))$ and $U(N)/(U(p) \times U(q))$ in two free theories, the free scalar and spinor, in boundary conformal field theories (bCFTs) have been already confirmed in [57]. We have to clarify that the associated defect conformal manifolds generated in this way can be absorbed by a field redefinition, in other words,

[5] There are no interacting SCFTs in $d > 6$ [41].

[6] We use the word "bulk" for the spacetime of the field theory away from the defect.

the points on such manifolds can be related to each other through the action of the broken generators of the group [57]. Consequently, we name them as the "trivial" manifolds.

Examples of "non-trivial" defect conformal manifold are also known. A distinguishing feature of three-dimensional supersymmetric conformal field theories is the vast moduli spaces of BPS line operators including multiple marginal couplings [2,3,58–70]. We present a full exploration of connected components of the moduli spaces of both conformal and non-conformal BPS Wilson loops in $\mathcal{N}=4$ quiver Chern-Simons-matter theory on S^3 along a great circle, where the loops preserving the same supercharges are connected by continuous supersymmetric deformations (6.2). We apply a similar philosophy to previous papers [58, 59], but employ as the starting point of the deformation arbitrary supersymmetric Wilson loops in the theory, and then choose a preserved supercharge and look for BPS deformations built out of the matter fields in the proper representations.[7] Our construction guarantees that all the entries of the superconnection have classical dimension 1, which is a necessary but not sufficient condition for the conformality of the loops. As a result, we exhausted all the connected moduli spaces of BPS Wilson loops in 3d $\mathcal{N}=4$ quiver Chern-Simons-matter theory, except for the possible isolated components that share no overlapping supercharges with any bosonic loops.

Concentrating on the conformal Wilson loops, there is a branch of 1/4 BPS loops interpolating between the bosonic and 1/2 BPS loops whose conformalities are known [59], because of the one-dimensional conformal algebra generated by the supercharges these loops preserve. Besides, we find a new candidate, which is another branch of 1/4 BPS loops that interpolate between two 1/2 BPS loops, without any intersection with the first branch except for the 1/2 BPS points. Unlike the first branch, the supercharges preserved by the new loops do not generate the conformal algebra, but are an outer automorphism of it. In other words, the new loops are classically conformal invariants. As we cannot rely on supersymmetry to guarantee conformality, it would be extremely interesting to examine them at the quantum level and verify whether they are the truly conformal invariants. Besides, the geometric properties of the non-trivial defect conformal manifolds are still unexplored and we will discuss them in more details in the outlook section 8.

This thesis collects a number of results, organised as follows. In section 2, we start with a quick review of the backgrounds, including the field contents and superconformal algebras of two main superconformal field theories that will be used throughout the remainder: 3d $\mathcal{N}=4$ super Chern-Simons theories on S^3 and ABJM theory in flat space. We also present standard constructions relating to conformal manifolds and defect operators. In section 3, we introduce the defect conformal manifold and show that when a defect breaks a global symmetry, the resulting defect conformal manifold is the symmetry breaking coset. We find exact relations to the curvature of the coset space, and confirm this relation against

[7] In most of the cases, we consider the theories contain both hypermultiplets and twisted hypermultiplets, seeing Figure 1.

previously obtained 4-point functions for insertions into the 1/2 BPS Wilson loop in $\mathcal{N} = 4$ SYM and 3d $\mathcal{N} = 6$ theory and the 1/2 BPS surface operator of the 6d $\mathcal{N} = (2, 0)$ theory. In sections 4, 5 and 6, we construct many new large classes of BPS Wilson loops in three-dimensional $\mathcal{N} = 4$ quiver Chern-Simons-matter theory on S^3 including both bosonic and fermionic ones, preserving one to six supercharges. In section 7, we address a longstanding question of whether ABJM theory has Wilson loop operators preserving eight supercharges (so 1/3 BPS) and present such Wilson loops made of a large supermatrix combining two 1/2 BPS Wilson loops. We also construct the defect conformal manifold arising from marginal defect operators. Finally, we end with a discussion of possible extensions of this work, especially how to apply the techniques we employ in trivial defect conformal manifolds to the non-trivial cases.

2 Background

2.1 Superconformal field theories

One of the major reasons that we study supersymmetric field theories is that, both conformality and supersymmetry provide powerful constraints that help to calculate a lot of interesting quantities. Theories enhanced with either conformal symmetries or supersymmetries generally exist and have been studied in great detail [14, 15]. However, if we require a theory to preserve both symmetries, it proves to be much more restrictive requirement and as a consequence the space of such theories are rather more limited. Still, there are many examples of supersymmetric conformal field theories are known in 2d and 4d, though much less was known in 3d for a long time.

It was pointed out in [71] that the supersymmetric Chern-Simons theories in three dimensions give rise to a natural class of conformal theories. Pure Chern-Simons theory with $U(N)$ gauge symmetry has a real Lagrangian in the Euclidean signature that is proportional to

$$\mathcal{L}_{CS} = \mathrm{tr}\left[\epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho)\right] , \tag{2.1}$$

where $\mu, \nu, \rho = 1, 2, 3$. For simplicity here and throughout we use A_μ to denote the gauge fields $A^a_\mu t^a$, where the Hermitian matrices t^a are generators of the Lie algebra of the gauge group in the adjoint representation. Such theories are topological by themselves, but may be coupled to other matter fields. The number of supersymmetries of superconformal Chern-Simons theories has a natural division between $\mathcal{N} \leq 3$ and $\mathcal{N} > 3$ theories. For the theories with $\mathcal{N} \leq 3$, the constructions are rather straight forward, see [71–76]. When $\mathcal{N} > 3$, the most symmetrical choice according to the AdS/CFT correspondence [77] is the model proposed by Bagger, Lambert [78–80] and Gustavsson [81, 82], which can be regarded as a special class of Chern-Simons-matter theories with $\mathcal{N} = 8$ supersymmetry and $OSp(8|4)$ superconformal symmetry.

The $\mathcal{N} = 4$ supersymmetric Chern-Simons-matter models were fistly proposed by Gaiotto and Witten in [83] with $OSp(4|4)$ superconformal symmetry, then generalized in [84–86]. In the same year, Aharony, Bergman, Jafferis and Maldacena constructed a supersymmetric Chern-Simons-matter model [87] with gauge group $U(N) \times U(N)$ and proved that it has explicitly $\mathcal{N} = 6$ supersymmetry, which is now known as ABJM theory. Very soon later, Aharony, Bergman and Jafferis generalized it with the same matter content and interactions, but with gauge group $U(N_1) \times U(N_2)$ where $N_1 \neq N_2$ [88], sometimes referred to as ABJ theory. In this thesis we make no distinction between the names and call these two cases ABJ(M) or ABJM theory.

In the following we give a brief presentation of $\mathcal{N} = 4$ super Chern-Simons-matter theories and ABJM theory about their field contents and superalgebras, based on [83–88]. These theories are known to have intricate spectrums of line operators.

2.1.1 $\mathcal{N} = 4$ Chern-Simons-matter theories on S^3

$\mathcal{N} = 4$ Chern-Simons-matter theories can be represented in terms of either circular or linear quiver diagrams [83–85, 89]. For the most part we focus on a node labeled by I with gauge field A_I and its adjacent node with A_{I+1}, but in Section 4.4 we also consider more nodes. The edges of the diagram represent hypermultiplets and twisted hypermultiplets. The hypermultiplet $(q_I^a, \psi_{I\dot{a}})$ couples to A_I and A_{I+1}, while the twisted hypermultiplet $(\tilde{q}_{I-1\,\dot{a}}, \tilde{\psi}_{I-1}^a)$ couples to A_I and A_{I-1}, and so on in an alternate fashion. The field content is summarized in the quiver diagram of Figure 1, where the solid lines between nodes represent the matter fields.

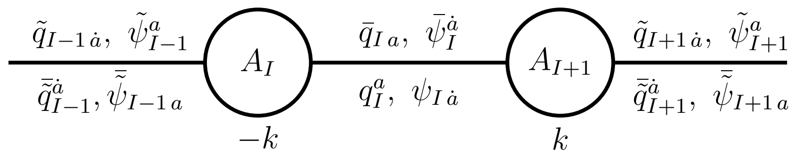

Figure 1: The quiver and field content of the $\mathcal{N} = 4$ theory.

As shown in [59], the (twisted) hypermultiplets can be decomposed into pairs of chiral multiplets. Figure 2 explicitly displays the (anti-)chiral scalars in this decomposition. Chiral fields are consistently indicated with solid arrows, and anti-chiral fields with dashed arrows. The arrow orientation corresponds to the field's representation. For example, the fields q_I^2 is in $(\mathbf{N}_I, \bar{\mathbf{N}}_{I+1})$ of $U(N_I) \times U(N_{I+1})$ and q_I^1 is in the conjugate representation $(\bar{\mathbf{N}}_I, \mathbf{N}_{I+1})$.

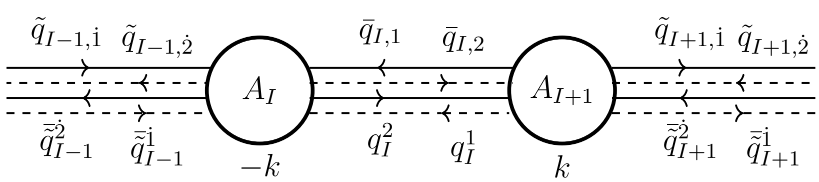

Figure 2: The decomposition of the $\mathcal{N} = 4$ matter multiplets into pairs of chiral multiplets.

The scalar fields in the hypermultiplet have indices $a, b = 1, 2$ and are doublets of the $SU(2)_L$ R-symmetry. The fermions with indices $\dot{a}, \dot{b} = \dot{1}, \dot{2}$ are charged instead under $SU(2)_R$. This is reversed in the twisted hypermultiplets. These indices are raised and lowered using the appropriate epsilon symbols: $v^a = \epsilon^{ab} v_b$ and $v_a = \epsilon_{ab} v^b$ with $\epsilon^{12} = \epsilon_{21} = 1$, and similarly for the dotted indices. Together with the conformal group, they form the bosonic subalgebra $\mathfrak{so}(1, 4) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ of the superalgebra $\mathfrak{osp}(4|4)$ of 3d $\mathcal{N} = 4$ superconformal field theories.

Although S^3 and $\mathbb{R}^3$ are conformal to each other, and therefore they share the same algebras, in the later sections 4, 5 and 6 we focus on the case of a three-sphere of radius r

embedded in $\mathbb{R}^4$ that is parameterised by 4d coordinates x^m, with $m, n = 1, 2, 3, 4$. More explicitly, the algebra of geometric symmetries is spanned by M_{mn}, the generators of S^3 isometry algebra $\mathfrak{so}(4)$, and the those of conformal maps T_m

$$M_{mn} = x_m\partial_n - x_n\partial_m, \qquad T_m = r\partial_m - \frac{1}{r}x_m x^n\partial_n\,, \tag{2.2}$$

with commutators

$$[M_{mn}, M_{rs}] = \delta_{m[s}M_{r]n} - \delta_{n[s}M_{r]m}\,, \quad [M_{mn}, T_r] = -\delta_{r[m}T_{n]}\,, \quad [T_m, T_n] = M_{mn}\,. \tag{2.3}$$

It is convenient to introduce capital indices $M, N = 0, 1, 2, 3, 4$ and to define $M_{0m} = T_m$, then the above equations can be summarised by

$$[M_{MN}, M_{RS}] = \eta_{M[S}M_{R]N} - \eta_{N[S}M_{R]M} \tag{2.4}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1, 1)$.

The R-symmetry algebra is spanned by J_i and $\bar{J}_i$

$$[J_i, J_j] = 2i\epsilon_{ijk}J_k\,, \qquad [\bar{J}_i, \bar{J}_j] = 2i\epsilon_{ijk}\bar{J}_k\,, \tag{2.5}$$

which are two independent sets of $\mathfrak{su}(2)$ generators. To be consistent with our previous conventions, we define

$$J^{ab} = (\sigma^i)^{ab}J_i \tag{2.6}$$

and similarly for $\bar{J}^{\dot{a}\dot{b}}$, with commutators

$$[J^{ab}, J^{cd}] = -\epsilon^{c(a}J^{b)d} - \epsilon^{d(a}J^{b)c}\,, \qquad [\bar{J}^{\dot{a}\dot{b}}, \bar{J}^{\dot{c}\dot{d}}] \;= -\epsilon^{\dot{c}(\dot{a}}\bar{J}^{\dot{b})\dot{d}} - \epsilon^{\dot{d}(\dot{a}}\bar{J}^{\dot{b})\dot{c}}\,. \tag{2.7}$$

In addition, there are 16 supercharges $Q^{\dot{a}a}_\alpha$ in the theory, where besides the R-symmetry indices $a, \dot{a}$, the spinor indices α represent that those supercharges transform as spinors of the conformal group $\mathfrak{so}(1, 4)$. Their anticommutators are given by

$$\{Q^{\dot{a}a}_\alpha, Q^{\dot{b}b}_\beta\} = \epsilon^{\dot{a}\dot{b}}\epsilon^{ab}(\Gamma^{MN}C^{-1})_{\alpha\beta}M_{MN} + \epsilon^{\dot{a}\dot{b}}C^{-1}_{\alpha\beta}J^{ab} + \epsilon^{ab}C^{-1}_{\alpha\beta}\bar{J}^{\dot{a}\dot{b}}\,, \tag{2.8}$$

where Γ_M are the 5d gamma matrices and C are the 5d charge conjugation matrices. Finally, the mixed commutators are

$$[M_{MN}, Q^{\dot{a}a}_\alpha] = -\frac{1}{2}(\Gamma_{MN})_\alpha^\beta Q^{\dot{a}a}_\beta\,, \tag{2.9}$$

and

$$[J^{bc}, Q^{\dot{a}a}_\alpha] = \epsilon^{ba}Q^{\dot{a}c}_\alpha + \epsilon^{ca}Q^{\dot{a}b}_\alpha\,, \qquad [\bar{J}^{\dot{b}\dot{c}}, Q^{\dot{a}a}_\alpha] = \epsilon^{\dot{b}\dot{a}}Q^{\dot{c}a}_\alpha + \epsilon^{\dot{c}\dot{a}}Q^{\dot{b}a}_\alpha\,. \tag{2.10}$$

They are actually the $\mathcal{N} = 4$ superconformal subset of the ABJM ($\mathcal{N} = 6$) superalgebras.

To write down the Wilson loops and the supersymmetry variations, it is useful to define moment maps and currents, following [63, 59]

$$\begin{aligned}
\mu_I{}^a{}_b &= q_I^a \bar{q}_{I\,b} - \frac{1}{2}\delta_b^a q_I^c \bar{q}_{I\,c}\,, & j_I^{a\dot{b}} &= q_I^a \bar{\psi}_I^{\dot{b}} - \epsilon^{ac}\epsilon^{\dot{b}\dot{c}}\psi_{I\,\dot{c}}\bar{q}_{I\,c}\,, \\
\tilde{\mu}_I{}^{\dot{a}}{}_{\dot{b}} &= \bar{\tilde{q}}_{I-1}^{\dot{a}} \tilde{q}_{I-1\,\dot{b}} - \frac{1}{2}\delta_{\dot{b}}^{\dot{a}} \bar{\tilde{q}}_{I-1}^{\dot{c}} \tilde{q}_{I-1\,\dot{c}}\,, & \tilde{j}_I^{\dot{b}a} &= \bar{\tilde{q}}_{I-1}^{\dot{b}} \tilde{\psi}_{I-1}^a - \epsilon^{\dot{b}\dot{c}}\epsilon^{ac}\bar{\tilde{\psi}}_{I-1\,c}\tilde{q}_{I-1\,\dot{c}}\,, \\
\nu_I &= q_I^a \bar{q}_{I\,a}\,, & \tilde{\nu}_I &= \bar{\tilde{q}}_{I-1}^{\dot{a}} \tilde{q}_{I-1\,\dot{a}}\,.
\end{aligned} \tag{2.11}$$

These are bilinears of the matter fields and transform in the adjoint representation of $U(N_I)$. Note that other bilinears of the same matter fields can transform in the adjoint of $U(N_{I\pm1})$. For example, $\nu_{I+1} = \bar{q}_{Ia} q_I^a$ is built out of the same fields as ν_I, but it transforms in the adjoint of $U(N_{I+1})$ because of the reversed order.

As stated in the Introduction, we define the theory on S^3 and the hyperloops we construct are supported along the equator of this sphere. The corresponding on-shell $\mathcal{N} = 4$ supersymmetry transformations were derived in [59] by relying on the decomposition of $\mathcal{N} = 4$ to $\mathcal{N} = 2$ theories and the transformation rules of the latter in [89, 90]. They are

$$\begin{aligned}
\delta A_{\mu\,I} &= \frac{i}{k}\xi_{a\dot{b}}\gamma_\mu (j_I^{a\dot{b}} - \tilde{j}_I^{\dot{b}a})\,, \\
\delta q_I^a &= \xi^{a\dot{b}}\psi_{I\,\dot{b}}\,, \qquad\qquad \delta\bar{q}_{I\,a} = \xi_{a\dot{b}}\bar{\psi}_I^{\dot{b}}\,, \\
\delta\tilde{q}_{I-1\,\dot{b}} &= -\xi_{a\dot{b}}\tilde{\psi}_{I-1}^a\,, \qquad\qquad \delta\bar{\tilde{q}}_{I-1}^{\dot{b}} = -\xi^{a\dot{b}}\bar{\tilde{\psi}}_{I-1\,a}\,, \\
\delta\psi_{I\,\dot{a}} &= i\gamma^\mu \xi_{b\dot{a}} D_\mu q_I^b + i\zeta_{b\dot{a}} q_I^b - \frac{i}{k}\xi_{b\dot{a}}(\nu_I q_I^b - q_I^b \nu_{I+1}) + \frac{2i}{k}\xi_{b\dot{c}}\left(\tilde{\mu}_I{}^{\dot{c}}{}_{\dot{a}} q_I^b - q_I^b \tilde{\mu}_{I+1\dot{a}}{}^{\dot{c}}\right), \\
\delta\bar{\psi}_I^{\dot{a}} &= i\gamma^\mu \xi^{b\dot{a}} D_\mu \bar{q}_{I\,b} + i\zeta^{b\dot{a}}\bar{q}_{I\,b} - \frac{i}{k}\xi^{b\dot{a}}(\bar{q}_{I\,b}\nu_I - \nu_{I+1}\bar{q}_{I\,b}) + \frac{2i}{k}\xi^{b\dot{c}}\left(\bar{q}_{I\,b}\tilde{\mu}_I{}^{\dot{a}}{}_{\dot{c}} - \tilde{\mu}_{I+1\dot{c}}{}^{\dot{a}}\bar{q}_{I\,b}\right), \\
\delta\tilde{\psi}_{I-1}^a &= -i\gamma^\mu \xi^{a\dot{b}} D_\mu \tilde{q}_{I-1\,\dot{b}} - i\zeta^{a\dot{b}}\tilde{q}_{I-1\,\dot{b}} + \frac{i}{k}\xi^{a\dot{b}}(\tilde{q}_{I-1\,\dot{b}}\tilde{\nu}_I - \tilde{\nu}_{I-1}\tilde{q}_{I-1\,\dot{b}}) \\
&\quad - \frac{2i}{k}\xi^{b\dot{c}}\left(\tilde{q}_{I-1\,\dot{c}}\mu_I{}^a{}_b - \mu_{I-1b}{}^a\tilde{q}_{I-1\,\dot{c}}\right), \\
\delta\bar{\tilde{\psi}}_{I-1\,a} &= -i\gamma^\mu \xi_{a\dot{b}} D_\mu \bar{\tilde{q}}_{I-1}^{\dot{b}} - i\zeta_{a\dot{b}}\bar{\tilde{q}}_{I-1}^{\dot{b}} + \frac{i}{k}\xi_{a\dot{b}}(\tilde{\nu}_I \bar{\tilde{q}}_{I-1}^{\dot{b}} - \bar{\tilde{q}}_{I-1}^{\dot{b}}\tilde{\nu}_{I-1}) \\
&\quad - \frac{2i}{k}\xi_{b\dot{c}}\left(\mu_I{}^b{}_a \bar{\tilde{q}}_{I-1}^{\dot{c}} - \bar{\tilde{q}}_{I-1}^{\dot{c}}\mu_{I-1a}{}^b\right),
\end{aligned} \tag{2.12}$$

where $\xi_{a\dot{b}}$ are the Killing spinors and $\zeta_{a\dot{b}} = \frac{1}{3}\gamma^\mu \nabla_\mu \xi_{a\dot{b}}$. The covariant derivative acts as, for instance, $D_\mu q_I^a = \partial_\mu q_I^a - iA_{\mu,I} q_I^a + i q_I^a A_{\mu,I}$.

Specifically, each supersymmetry parameter $\xi^{a\dot{b}}$ is a linear combination of four (conformal) Killing spinors on S^3 denoted $\{\xi^l, \xi^{\bar{l}}, \xi^r, \xi^{\bar{r}}\}$, *i.e.*

$$\xi_\alpha^{a\dot{b}} = \xi_\imath^{a\dot{b}}\xi_\alpha^\imath + \xi_{\bar{\imath}}^{a\dot{b}}\xi_\alpha^{\bar{\imath}}\,, \tag{2.13}$$

where $\imath = l, r$ and $\bar{\imath} = \bar{l}, \bar{r}$ label doublets of the $SO(2,1)$ conformal symmetry along the circle. All together they form a quartet of the $SO(4,1)$ symmetry of S^3. The index $\alpha = \pm$ is the spinor index.

The Killing spinors obey

$$\nabla_\mu \xi^{l,\bar{l}} = \frac{i}{2}\gamma_\mu \xi^{l,\bar{l}}\,, \qquad \nabla_\mu \xi^{r,\bar{r}} = -\frac{i}{2}\gamma_\mu \xi^{r,\bar{r}}\,. \tag{2.14}$$

Along the circle we take $\gamma_\varphi = \sigma_3$ and the Killing spinors reduce to [91]

$$\xi^l_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi^{\bar{l}}_\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \xi^r_\alpha = \begin{pmatrix} e^{-i\varphi} \\ 0 \end{pmatrix}, \qquad \xi^{\bar{r}}_\alpha = \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix}, \tag{2.15}$$

whence one finds $\zeta^{l,\bar{l}}_{a\dot{b}} = \frac{i}{2}\xi^{l,\bar{l}}_{a\dot{b}}$ and $\zeta^{r,\bar{r}}_{a\dot{b}} = -\frac{i}{2}\xi^{r,\bar{r}}_{a\dot{b}}$.

We work in Euclidean signature and take the gamma-matrices, $(\gamma^\mu)_\alpha{}^\beta$, to be given by the Pauli matrices. As usual, the spinor contractions are such that

$$\xi_1\xi_2 \equiv \xi_1^\alpha \xi_{2,\alpha} = +\xi_2\xi_1\,, \qquad \xi_1\gamma^\mu\xi_2 \equiv \xi_1^\alpha (\gamma^\mu)_\alpha{}^\beta \xi_{2,\beta} = -\xi_2\gamma^\mu\xi_1\,. \tag{2.16}$$

It follows that the Killing spinors in (2.15) satisfy $\xi^{\bar{l}}\xi^l = \xi^l\xi^{\bar{l}} = 1$ and $\xi^{\bar{l}}\gamma^\mu\xi^l = -\xi^l\gamma^\mu\xi^{\bar{l}} = \delta^\mu_\varphi$, and similarly for the contractions involving ξ^r and $\xi^{\bar{r}}$.

2.1.2 ABJM theory

The ABJM theory [87] is a conformal field theory in three dimensions with $\mathcal{N} = 6$ supersymmetry, or a total of 12 real supercharges Q^{IJ}_α and 12 real superconformal charges S^{IJ}_α, to be discussed below. The field content are the following: Two gauge fields $A^{(1)}_\mu$ and $A^{(2)}_\mu$ belonging respectively to the adjoint representation of $U(N_1)_k$ and $U(N_2)_{-k}$, four complex scalars C_I (or $\bar{C}^I$) and four complex fermions $\bar{\psi}^I$ (ψ_I) that transform under $(\mathbf{N}_1, \bar{\mathbf{N}}_2)$ $((\bar{\mathbf{N}}_1, \mathbf{N}_2))$ of the gauge group $U(N_1)_k \times U(N_2)_{-k}$. They can be summarised in the quiver diagram of Figure 3[8].

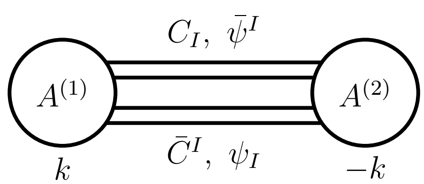

Figure 3: The quiver and field content of the ABJM theory.

The symmetry algebra of ABJM theory in flat space includes the conformal group $\mathfrak{so}(4,1)$, the R-symmetry $\mathfrak{su}(4)$ and the supersymmetry generators forming the $\mathfrak{osp}(6|4)$ superalgebra. The first is comprised of the Lorentz generators $M_{\mu\nu}$, translations P_μ, special conformal transformations K_μ and dilation D. The R-symmetry generators can be written as $J_I{}^J$ and

[8] Here we use the double lines to indicate that the amount of matter fields in ABJM is twice of those in $\mathcal{N} = 4$ theories, cross-ref to figure 1.

we write them in a redundant notation allowing $I, J = 1, \ldots, 4$, with the constraint $J_I{}^I = 0$. The supercharges are Q^{IJ}_α and S^{IJ}_α and satisfy the reality constraint $Q^{IJ}_\alpha = \frac{1}{2}\epsilon^{IJKL}\bar{Q}_{KL\alpha}$ and $S^{IJ}_\alpha = \frac{1}{2}\epsilon^{IJKL}\bar{S}_{KL\alpha}/2$.

The nonzero commutators in the conformal algebra are [92, 93]

$$\begin{aligned} [P_\mu, K_\nu] &= 2\delta_{\mu\nu}D + 2M_{\mu\nu}\,, & [D, P_\mu] &= P_\mu\,, & [D, K_\mu] &= -K_\mu\,, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \delta_{\mu[\sigma}M_{\rho]\nu} - \delta_{\nu[\sigma}M_{\rho]\mu}\,, & [P_\mu, M_{\nu\rho}] &= \delta_{\mu[\nu}P_{\rho]}\,, & [K_\mu, M_{\nu\rho}] &= \delta_{\mu[\nu}K_{\rho]}\,. \end{aligned} \tag{2.17}$$

For the R-symmetry generators

$$[J_I{}^J, J_K{}^L] = \delta^L_I J_K{}^J - \delta^J_K J_I{}^L\,. \tag{2.18}$$

The anticommutators of the fermionic generators are

$$\begin{aligned} \{Q^{IJ}_\alpha, Q^{KL\beta}\} &= 2\epsilon^{IJKL}(\gamma^\mu)_\alpha{}^\beta P_\mu\,, \qquad \{S^{IJ}_\alpha, S^{KL\beta}\} = 2\epsilon^{IJKL}(\gamma^\mu)_\alpha{}^\beta K_\mu\,, \\ \{Q^{IJ}_\alpha, S^{KL\beta}\} &= \epsilon^{IJKL}(\gamma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + 2\delta^\beta_\alpha\left(\epsilon^{IJKL}D - \epsilon^{NJKL}J_N{}^I - \epsilon^{INKL}J_N{}^J\right)\,, \end{aligned} \tag{2.19}$$

Finally, the mixed commutators are

$$\begin{aligned} [D, Q^{IJ}_\alpha] &= \frac{1}{2}Q^{IJ}_\alpha\,, & [D, S^{IJ}_\alpha] &= -\frac{1}{2}S^{IJ}_\alpha\,, \\ [M_{\mu\nu}, Q^{IJ}_\alpha] &= -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q^{IJ}_\beta\,, & [M_{\mu\nu}, S^{IJ}_\alpha] &= -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta S^{IJ}_\beta\,, \\ [K_\mu, Q^{IJ}_\alpha] &= (\gamma_\mu)_\alpha{}^\beta S^{IJ}_\beta\,, & [P_\mu, S^{IJ}_\alpha] &= (\gamma_\mu)_\alpha{}^\beta Q^{IJ}_\beta\,, \\ [J_I{}^J, Q^{KL}_\alpha] &= \delta^K_I Q^{JL}_\alpha + \delta^L_I Q^{KJ}_\alpha - \frac{1}{2}\delta^J_I Q^{KL}_\alpha\,, & [J_I{}^J, S^{KL}_\alpha] &= \delta^K_I S^{JL}_\alpha + \delta^L_I S^{KJ}_\alpha - \frac{1}{2}\delta^J_I S^{KL}_\alpha\,. \end{aligned} \tag{2.20}$$

Since supersymmetry plays an essential role in section 7, it is also necessary for us to find out the supersymmetry transformations in ABJM. Based on the conventions in [94] with $\epsilon^{+-} = \epsilon_{-+} = 1$ and some factors of 2 to be compatible with the algebra in Appendix 7.C and with [95], the variations of the fields are

$$\begin{aligned} Q^{IJ}_\alpha C_K &= \delta^I_K\bar{\psi}^J_\alpha - \delta^J_K\bar{\psi}^I_\alpha\,, \\ Q^{IJ}_\alpha \bar{C}^K &= -\epsilon^{IJKL}\epsilon_{\alpha\beta}\psi^\beta_L\,, \\ Q^{IJ}_\alpha \psi^\beta_K &= -2\delta^I_K\left(i(\gamma^\mu)_\alpha{}^\beta D_\mu\bar{C}^J + 2\alpha\bar{\alpha}\delta^\beta_\alpha\bar{C}^{[J}C_L\bar{C}^{L]}\right) \\ &\quad + 2\delta^J_K\left(i(\gamma^\mu)_\alpha{}^\beta D_\mu\bar{C}^I + 2\alpha\bar{\alpha}\delta^\beta_\alpha\bar{C}^{[I}C_L\bar{C}^{L]}\right) - 8\alpha\bar{\alpha}\delta^\beta_\alpha\bar{C}^{[I}C_K\bar{C}^{J]}\,, \\ Q^{IJ}_\alpha \bar{\psi}^K_\beta &= 2\epsilon^{IJKL}\left(i\epsilon_{\alpha\gamma}(\gamma^\mu)_\beta{}^\gamma D_\mu C_L + 2\alpha\bar{\alpha}\epsilon_{\alpha\beta}C_{[L}\bar{C}^M C_{M]}\right) \\ &\quad + 4\alpha\bar{\alpha}\epsilon^{IJLM}\epsilon_{\alpha\beta}C_{[L}\bar{C}^K C_{M]}\,, \\ Q^{IJ}_\alpha A^{(1)}_\mu &= -\alpha\bar{\alpha}\epsilon^{IJKL}\epsilon_{\alpha\gamma}(\gamma_\mu)_\beta{}^\gamma C_K\psi^\beta_L - 2\alpha\bar{\alpha}(\gamma^\mu)_\alpha{}^\beta\bar{\psi}^{[I}_\beta\bar{C}^{J]}\,, \\ Q^{IJ}_\alpha A^{(2)}_\mu &= \alpha\bar{\alpha}\epsilon^{IJKL}\epsilon_{\alpha\gamma}(\gamma_\mu)_\beta{}^\gamma\psi^\beta_K C_L + 2\alpha\bar{\alpha}(\gamma_\mu)_\alpha{}^\beta\bar{C}^{[I}\bar{\psi}^{J]}_\beta\,. \end{aligned} \tag{2.21}$$

The anti-symmetrisation symbol is normalised with a factor of 1/2.

2.2 Conformal manifolds

Amongst all operators of a conformal field theory, exactly marginal operators hold a special place, as they allow for continuous deformations of the theory, forming a space of CFTs known as the conformal manifold. In a D dimensional CFT, marginal operators $\mathcal{O}_i$ have scaling dimension D. If the CFT has an action, the deformations can be written as

$$S \to S + \int \lambda^i \mathcal{O}_i \, d^D x \,, \tag{2.22}$$

where the parameters λ^i are local coordinates on the conformal manifold $\mathcal{M}$. In the absence of the action, correlation functions of any operators ϕ_a in the deformed theory are then written simply with the extra insertion of the exponential of the integral in (2.22)

$$\left\langle \phi_{a_1} \cdots \phi_{a_n} \right\rangle_{\lambda^i} = \left\langle e^{-\int \lambda^i \mathcal{O}_i \, d^D x} \phi_{a_1} \cdots \phi_{a_n} \right\rangle_0 \,, \tag{2.23}$$

with subscript 0 indicating the undeformed theory.

Such a manifold admits a natural Riemannian structure given by the Zamolodchikov metric [46]

$$g_{ij} = \langle \mathcal{O}_i(0) \mathcal{O}_j(\infty) \rangle \,, \tag{2.24}$$

where $\mathcal{O}(\infty) = \lim_{x\to\infty} x^{2D} \mathcal{O}(x)$. Note that we cannot put $g_{ij} = \delta_{ij}$ globally, even though on the right hand of (2.24) the two-point functions are indeed always proportional to δ_{ij}. Instead, it has to be regarded as being defined at a local point p on $\mathcal{M}$.

Differentiating the metric with respect to λ (2.23) gives the integrated three- and four-point functions

$$\begin{aligned} \partial_k g_{ij} &= \int d^D x \langle \mathcal{O}_i(0) \mathcal{O}_j(\infty) \mathcal{O}_k(x) \rangle \\ \partial_l \partial_k g_{ij} &= \int d^D x_1 \int d^D x_2 \langle \mathcal{O}_i(0) \mathcal{O}_j(\infty) \mathcal{O}_k(x_1) \mathcal{O}_l(x_2) \rangle_c \end{aligned} \tag{2.25}$$

where $\langle \mathcal{O} \cdots \mathcal{O} \rangle_c$ denotes the connected 3-(4-)point functions, because with a particular choice of regularization: the hard-sphere cutoff and minimal subtraction of the divergences (explained below (2.30)), the disconnected contribution to the integral vanishes [96, 97]. Assuming that the three-point functions vanish when the points are non-coincident, more explicitly, they can be expressed in terms of the connections on the manifold $\mathcal{M}$ as [97]

$$\langle \mathcal{O}_i(0) \mathcal{O}_j(\infty) \mathcal{O}_k(x) \rangle = \Gamma^l_{ik} g_{lj} \delta^D(x) + \Gamma^l_{jk} g_{li} \delta^D(x - \infty) \tag{2.26}$$

By integrating this result over x, we actually recover the following equation in Riemannian geometry

$$\partial_k g_{ij} = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li} \,. \tag{2.27}$$

Since minimal subtraction means that there are no finite counterterms allowed, the first derivatives of the metric have to vanish. So that the expression for the Riemannian curvature tensor can be written as

$$R_{ijkl} = \frac{1}{2}(\partial_k \partial_j g_{li} - \partial_k \partial_i g_{jl} - \partial_l \partial_j g_{ki} + \partial_l \partial_i g_{jk}) \tag{2.28}$$

After plugging in all the terms (2.25), the curvature tensor consists of a sum of double integrals of four-point functions. However, noticing that the 4-point functions depend at non-coincident points only on the cross ratio(s) $\chi(\bar{\chi})$, one can perform one of the two integrals explicitly and reduce the formula to just a single integral. This was done in [96] with careful treatment of the integration domain and possible singularities giving the 1d expression

$$\begin{aligned} R_{ijkl} &= -\mathrm{RV} \int_{-\infty}^{+\infty} d\eta \log|\eta| \Big[\langle \mathcal{O}_i(1) \mathcal{O}_j(\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(0) \rangle_c \\ &\qquad\qquad + \langle \mathcal{O}_i(0) \mathcal{O}_j(1-\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(1) \rangle_c \Big] \\ &= -\lim_{\epsilon \to 0} \Bigg[\int\limits_{\substack{\epsilon<|\eta|<\epsilon^{-1} \\ \epsilon<|1-\eta|}} d\eta \log|\eta| \Big[\langle \mathcal{O}_i(1) \mathcal{O}_j(\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(0) \rangle_c \\ &\qquad\qquad + \langle \mathcal{O}_i(0) \mathcal{O}_j(1-\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(1) \rangle_c + \Delta R_{ijkl}(\epsilon) \Big] . \end{aligned} \tag{2.29}$$

And in 2d

$$\begin{aligned} R_{ijkl} &= -\, 2\pi \, \mathrm{RV} \int d^2\eta \log|\eta| \langle \mathcal{O}_i(1) \mathcal{O}_j(\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(0) \rangle_c \\ &= -\, 2\pi \lim_{\epsilon \to 0} \Bigg[\int\limits_{\substack{\epsilon<|\eta|<\epsilon^{-1} \\ \epsilon<|1-\eta|}} d^2\eta \log|\eta| \langle \mathcal{O}_i(1) \mathcal{O}_j(\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(0) \rangle_c + \Delta R_{ijkl}(\epsilon) \Bigg] . \end{aligned} \tag{2.30}$$

where we fix three points in the 4-point functions located at respectively $0, 1$ and ∞, and leave the last point at η so that the cross ratios are just functions of η. The symbol RV represents a particular regularisation prescription for regularizing and subtracting the divergences—a hard-sphere (point-splitting) cutoff, i.e. the integral removing the region around $\eta = 0$, $\eta = 1$ and $\eta = \infty$ as expressed in the second line, and $\Delta R_{ijkl}(\epsilon)$ is a counterterm removing residual power-law divergences. See [96] for the full details.

In arbitrary dimension $D \geq 2$, the integration measure are obtained in [97] with the help of inversion formula [98, 99]

$$R_{ijkl} \propto -\int d^2\eta \left| \frac{\eta - \bar{\eta}}{2i} \right|^{D-2} \log|\eta| \, \langle \mathcal{O}_i(1) \mathcal{O}_j(\eta) \mathcal{O}_k(\infty) \mathcal{O}_l(0) \rangle_c \, , \tag{2.31}$$

with the proportionality constant given by the volumes of the (D-1)- and (D-2)-spheres.

2.3 Defect CFTs

The inclusion of defects much extends the contents of conformal field theories, thus known as extended CFTs or defect CFTs [100–103]. An important feature of the conformal defects is that they are very similar to the conformal field theories in the absence of defects.

Generally, there are two ways to define a defect [104], as:

- insertions of integrals of defect densities $\mathcal{L}$

$$\mathcal{D} = e^{i\int_\Sigma \mathcal{L}}\,, \tag{2.32}$$

  with Σ the world volume of the defect. This form is often referred to as the order-type defect. This is the case that we mainly focus on in the later sections. A canonical example is the Wilson loop [105]

$$W = \mathrm{Tr}\,\mathcal{P}\exp\left(i\int \mathcal{L}\,ds\right)\,, \tag{2.33}$$

  where $\mathcal{L}$ is the connection composed of gauge fields (as well as matter fields), which we will explain later in details. $\mathcal{P}$ denotes the path ordering and s is the affine parameter along the contour of Wilson loops. In principle it can be arbitrary lines or loops, but since here we are interested in conformal defects, it has to be an infinite straight line or a circle. With some specific $\mathcal{L}$, for instance (2.39), (2.40), W also preserves additional symmetries such as the supersymmetry.

- boundary conditions along Σ in the path integral

$$\langle\mathcal{D}\cdots\rangle = \int [d\phi]|_{\phi(\Sigma)=\phi_0}(e^{iS[\phi]}\cdots)\,, \tag{2.34}$$

  where ϕ is the bulk elementary field. This form is known as the disorder-type defect, such as the 't Hooft loops [106]. More general defects such as Wilson-'t Hooft lines [107] can be defined by superimposing both types of defects.

Just like any CFT, the fundamental observables in a dCFT are the correlation functions of local operators ϕ inserted into the defect

$$\langle\!\langle\phi(x_1)\cdots\phi(x_n)\rangle\!\rangle = \frac{1}{\langle\mathcal{D}\rangle}\langle\mathcal{D}[\phi(x_1)\cdots\phi(x_n)]\rangle\,. \tag{2.35}$$

We use the double brace notation to represent the correlation function in the dCFT normalized by the VEV of the defect without insertions.

Every defect has a distinguished operator known as the displacement operator, which captures the breaking of translation invariance by the defect. It can be seen in the divergence of the energy momentum tensor

$$\partial_\mu T^{\mu n}(x) = \mathbb{D}^n(x_\parallel)\delta^{D-d}(x_\perp)\,, \tag{2.36}$$

where $x_\parallel$ represent the directions along the defect and $x_\perp$ the transverse ones, and $n = 1, \cdots d$ is an index in the $x_\perp$ directions. As $T^{\mu\nu}$ has dimension D, the displacement operator has dimension $d+1$. Another important consequence of this equation is that the normalisation of $\mathbb{D}$ is fixed by the normalisation of the $T^{\mu\nu}$ and therefore the factor $C_\mathbb{D}$ given by the two-point function

$$\langle\!\langle \mathbb{D}^n(x_\parallel)\, \mathbb{D}^m(0) \rangle\!\rangle = \frac{C_\mathbb{D} \delta^{nm}}{x_\parallel^{2d+2}}\,, \tag{2.37}$$

is determined. It is interesting that in some cases the defect exactly marginal operators $\mathbb{O}_i$ are also unavoidable. Such operators play a crucial role in this thesis and we will discuss them at length in the next section 3.

Among all the conformal defects, there are some reasons to focus on Wilson loops. They can be defined in any gauge theory and particularly in the case of pure Chern-Simons theories, they are the principle observables. Even in a Chern-Simons theory including additional matter fields, Wilson loops are still very natural observables.

The standard Wilson loops is constructed by the gauge fields purely

$$\mathcal{L} = A_\mu \dot{x}^\mu, \tag{2.38}$$

along either a circle or an infinite straight line. Such loops are usually not supersymmetric, which can be checked through direct calculations. In $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetric Chern-Simons-matter theories, the BPS Wilson loops were firstly constructed by Gaiotto and Yin [76] in analogy to the 1/2 BPS Wilson loops (3.9) in 4d $\mathcal{N} = 4$ super Yang-Mills theory [108, 109], by including only gauge fields and scalar fields, thus we call them "bosonic loops". Such loops are 1/2 BPS in $\mathcal{N} = 2$ theories and 1/3 BPS in $\mathcal{N} = 3$ theories. Then the idea is generalized to ABJM theory [60–62], to find

$$\mathcal{L} = A_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_J^I C_I \bar{C}^J\,, \tag{2.39}$$

where M_J^I is a matrix whose properties will be determined by supersymmetry. The maximal supersymmetry of such loops turns out to be 1/6 BPS. However, AdS/CFT correspondence suggests that there should be 1/2 BPS loops in ABJM theory, which was still absent. After about one year, finally Drukker and Trancanelli [110] proposed a way to construct the 1/2 BPS loops, by introducing the superconnection $\mathcal{L}$ containing the fermionic fields

$$\mathcal{L} = \begin{pmatrix} A_\mu^{(1)} \dot{x}^\mu + \alpha\bar{\alpha} |\dot{x}| M_J^I C_I \bar{C}^J & i\bar{\alpha}|\dot{x}|\bar{\psi}_+^1 \\ -i\alpha|\dot{x}|\psi_1^+ & A_\mu^{(2)} \dot{x}^\mu + \alpha\bar{\alpha}|\dot{x}| M_J^I \bar{C}^J C_I \end{pmatrix}, \tag{2.40}$$

with $\alpha\bar{\alpha} = -2\pi i/k$ and $M = \mathrm{diag}(-1, 1, 1, 1)$. For $U(N_1) \times U(N_2)$ ABJM theory, the superconnection $\mathcal{L}$ is an $U(N_1|N_2)$ supermatrix. In the following several years, more and more new BPS loops have been found in ABJM theory as well as $\mathcal{N} = 4$ supersymmetric Chern-Simons-matter theories [59, 63–69, 111].

In [58], a systematic formalism is uncovered to reorganize moduli space of BPS Wilson loops. The idea is to start with particular block-diagonal combinations of bosonic connections annihilated by a supercharge $\mathcal{Q}$ and look for their BPS deformations generated by the same supercharge. The resulting operator is still supersymmetric by construction, and is defined in terms of a superconnection containing the fermionic fields, which is something typical of supersymmetric Chern-Simons theories. We develop this formalism in the later sections 4, 5, 6 and 7.

3 Broken global symmetries and defect conformal manifolds

This section is based on [1] with minor edits.

3.1 Introduction and summary

Theories with conformal boundaries or defects are ubiquitous and play an important role both in condensed matter physics and in string theory. They form a defect CFT (dCFT) involving operators on and off the defect. A relatively unexplored topic (notable exceptions are [32, 45, 112–114]) are marginal deformations of dCFTs by defect operators.

For a defect of dimension d, exactly marginal defect operators $\mathbb{O}_i$ have scaling dimension d and the correlation function of defect operators ϕ in the deformed theory can be expressed as

$$\langle\!\langle \phi\phi' \dots \rangle\!\rangle_{\zeta^i} = \langle\!\langle e^{-\int \zeta^i \mathbb{O}_i \, d^d x} \phi\phi' \dots \rangle\!\rangle_0 \,. \tag{3.1}$$

where ζ^i are local coordinates on the defect conformal manifold and the double bracket notation represents the correlation function in the dCFT normalized by the expectation value of the defect without insertions.

If the theory has a global symmetry G with current $J^{\mu a}$, broken by the defect to G', its conservation equation is modified to

$$\partial_\mu J^{\mu a} = \mathbb{O}_i(x_\parallel)\delta^{ia}\delta^{D-d}(x_\perp) \,, \tag{3.2}$$

where i is an index for the broken generators, $x_\parallel$ the directions along the defect and $x_\perp$ the transverse ones.

In a theory in D dimensions, $J^{\mu a}$ has dimension $D-1$. Therefore $\mathbb{O}_i$ has dimension d, so in the undeformed theory

$$\langle\!\langle \mathbb{O}_i(x_\parallel)\mathbb{O}_j(0) \rangle\!\rangle = \frac{C_{\mathbb{O}}\delta_{ij}}{|x_\parallel|^{2d}} \,. \tag{3.3}$$

This leads naturally to consider a defect conformal manifold, and with the usual rescaling of the operator at infinity, it has the Zamolodchikov metric [46]

$$g_{ij} = \langle\!\langle \mathbb{O}_i(\infty)\mathbb{O}_j(0) \rangle\!\rangle = C_{\mathbb{O}}\delta_{ij} \,, \qquad \mathbb{O}_i(\infty) \equiv \lim_{x_\parallel \to \infty} |x_\parallel|^{2d}\mathbb{O}_i(x_\parallel) \,. \tag{3.4}$$

While the metric is locally flat, if the theory has R-symmetry group G_R, broken by the defect to G'_R, the full defect conformal manifold is G_R/G'_R. Furthermore, the size of this manifold is set by $C_{\mathbb{O}}$.

It is no surprise that an object that breaks global symmetries transforms nontrivially under the broken symmetries and is parametrised by this coset. Still, this point of view allows us to find non-trivial identities on integrated correlators. The defect analog of (2.23) is

$$\langle\!\langle \phi_{a_1} \cdots \phi_{a_n} \rangle\!\rangle_{\lambda^i} = \langle\!\langle e^{-\int \lambda^i \mathbb{O}_i \, d^d x} \phi_{a_1} \cdots \phi_{a_n} \rangle\!\rangle_0 \,. \tag{3.5}$$

In particular for a pair of $\phi = \mathbb{O}_i$ we have

$$\langle\!\langle \mathbb{O}_i \mathbb{O}_j \rangle\!\rangle_{\lambda^i} = \langle\!\langle e^{-\int \lambda^i \mathbb{O}_i \, d^d x} \mathbb{O}_i \mathbb{O}_j \rangle\!\rangle_0 \,. \tag{3.6}$$

This is the extension of the local Zamolodchikov metric (3.4) beyond the flat space approximation and the derivatives with respect to λ^i give the Riemann tensor. Indeed, as in [115], one finds

$$R_{ijkl} = \int d^d x_1 d^d x_2 \Big[\langle\!\langle \mathbb{O}_j(x_1) \mathbb{O}_k(x_2) \mathbb{O}_i(0) \mathbb{O}_l(\infty) \rangle\!\rangle_c - \langle\!\langle \mathbb{O}_j(0) \mathbb{O}_k(x_2) \mathbb{O}_i(x_1) \mathbb{O}_l(\infty) \rangle\!\rangle_c \Big] \,, \tag{3.7}$$

where $\langle\!\langle \ldots \rangle\!\rangle_c$ implies the connected correlator, as stressed for example in [96]. This integral is $2d$ dimensional, but it can be reduced to an integral over cross-ratios [96]. See equations (2.29), (2.30) below.

Given that the manifold is G_R/G'_R, there is no mystery in the metric. Indeed if it is a maximally symmetric space, the Riemann tensor is determined by the Ricci scalar R as

$$R_{ijkl} = \frac{R}{p(p-1)} (g_{ik} g_{jl} - g_{il} g_{jk}) \,. \tag{3.8}$$

where p is the dimension of the conformal manifold. If we know the exact value of $C_{\mathbb{O}}$, then we know the exact form of the curvature and equating the last two equations gives a non-trivial relation on 4-point function, which is one of our main results.

In the remainder of this paper we apply this idea to two defects. In Section 3.2 we consider the 1d dCFT of 1/2 BPS Wilson loops in $\mathcal{N} = 4$ SYM. Then in Section 3.4 we look at the case of surface operators in the 6d $\mathcal{N} = (2, 0)$ theory. Some details of the calculations are presented in appendices.

3.2 Maldacena-Wilson loops

We start by looking at the case of the 1/2 BPS Wilson loop in $\mathcal{N} = 4$ SYM in 4d along the Euclidean time direction

$$W = \operatorname{Tr} \mathcal{P} e^{\int (iA_0 + \Phi_6) dt} \,. \tag{3.9}$$

Another 1/2 BPS loop is the circle, and there are some subtle differences between the two [116, 117], but of our purposes here the differences are immaterial and one finds the same results with either the circle or the line.

3.2.1 The Wilson loop dCFT

The defect CFT point of view on the Wilson loop was developed in [118–123]. Defect operators are any adjoint valued word inserted along the Wilson loop and their dimensions can be calculated using integrability [118, 124–126].

The lowest dimension insertions are the six scalar fields Φ_I. The one already in the Wilson loop, Φ_6, has an anomalous dimension studied in [122, 127–129]. In fact this scalar is the marginally irrelevant operator at the bottom of the renormalisation group flow from the non-BPS Wilson loop with no scalar coupling [117, 130, 131].

The remaining five scalars are marginal and in fact are $\mathbb{O}_i$, the superpartners of the displacement operator. Note that deforming the loop by a finite $\lambda^i\Phi_i$ in the exponent gives a non-BPS loop with a non-vanishing beta function. The exactly marginal deformation is a rotation of scalar field Φ_6

$$\Phi_6 \to \Phi_6\sqrt{1-|\lambda|^2} + \lambda^i\Phi_i\,, \tag{3.10}$$

so for finite deformations, the operator Φ_i includes the appropriate subtraction of Φ_6 to account for that.

The two point function of Φ_i is indeed as in (3.4) with C_Φ twice the bremsstrahlung function B given in terms of the expectation value of the circular Wilson loop [119, 126, 132, 133]

$$C_\Phi = 2B = \frac{1}{\pi^2}\lambda\partial_\lambda \log\left\langle W_\circ\right\rangle, \qquad W_\circ = \frac{1}{N}L^1_{N-1}(-\lambda/4N)e^{\lambda/8N}\,, \tag{3.11}$$

where here λ is the Yang-Mills coupling, so the bulk marginal coupling. Explicitly in the planar limit

$$C_\Phi = \begin{cases} \dfrac{\lambda}{8\pi^2} - \dfrac{\lambda^2}{192\pi^2} + \dfrac{\lambda^3}{3072\pi^2} - \dfrac{\lambda^4}{46080\pi^2} + O(\lambda^5)\,, & \lambda \ll 1\,, \\ \dfrac{\sqrt{\lambda}}{2\pi^2} - \dfrac{3}{4\pi^2} + \dfrac{3}{16\pi^2\sqrt{\lambda}} + \dfrac{3}{16\pi^2\lambda} + O(\lambda^{-3/2})\,, & \lambda \gg 1\,. \end{cases} \tag{3.12}$$

Let us present the general form of the four point functions of the scalars. We define

$$\Phi(x,t) = t^i\Phi_i(x)\,, \tag{3.13}$$

where t^i are auxiliary five vectors introduced to contract the R-symmetry indices. It is convenient to write the four point functions as

$$\langle\!\langle \Phi(x_1,t_1)\Phi(x_2,t_2)\Phi(x_3,t_3)\Phi(x_4,t_4)\rangle\!\rangle = \frac{t_{12}t_{34}}{x_{12}^2x_{34}^2}\mathcal{G}(\chi;\zeta_1,\zeta_2)\,, \tag{3.14}$$

where $t_{12} = t_1\cdot t_2$ and the cross-ratios χ, ζ_1, ζ_2 are defined via

$$\chi = \frac{x_{12}x_{34}}{x_{13}x_{24}}\,, \qquad \zeta_1\zeta_2 = \frac{t_{12}t_{34}}{t_{13}t_{24}}\,, \qquad (1-\zeta_1)(1-\zeta_2) = \frac{t_{14}t_{23}}{t_{13}t_{24}}\,. \tag{3.15}$$

The dynamical part of the correlator is encoded in $\mathcal{G}(\chi;\zeta_1,\zeta_2)$ where it is symmetric under $\zeta_1 \leftrightarrow \zeta_2$ and it satisfies the Ward identities [123]

$$\left(\frac{\partial \mathcal{G}}{\partial \zeta_1} + \frac{1}{2}\frac{\partial \mathcal{G}}{\partial \chi}\right)\Big|_{\chi=\zeta_1} = \left(\frac{\partial \mathcal{G}}{\partial \zeta_2} + \frac{1}{2}\frac{\partial \mathcal{G}}{\partial \chi}\right)\Big|_{\chi=\zeta_2} = 0\,. \tag{3.16}$$

Moreover, the dependence of $\mathcal{G}(\chi;\zeta_1,\zeta_2)$ on ζ_1,ζ_2 is constrained by the fact that it has to be a polynomial of degree four in the t^i. For a detailed discussion on this, see [123].

The superconformal Ward identities (3.16) were solved in [123,134] in the elegant expression[9]

$$\mathcal{G}(\chi;\zeta_1,\zeta_2) = C_\Phi^2\left(\mathbb{F}\mathfrak{X} + \mathbb{D}f(\chi)\right). \tag{3.17}$$

where $\mathbb{F}$ does not depend on χ, ζ_1, ζ_2 and can be determined from the topological sector of the correlators, which occurs for the choice $\chi = \zeta_1 = \zeta_2$ [123,135,136].

$\mathfrak{X}$ in (3.17) is one of two superconformal cross-ratios defined in [123] as follows

$$\mathfrak{X} = \frac{\chi^2}{\zeta_1\zeta_2}\,, \qquad \bar{\mathfrak{X}} = \frac{(1-\chi)^2}{(1-\zeta_1)(1-\zeta_2)}\,. \tag{3.18}$$

$\mathbb{D}$ is a differential operator given by

$$\mathbb{D} = (2\chi^{-1} - \zeta_1^{-1} - \zeta_2^{-1}) - \chi^2(\zeta_1^{-1} - \chi^{-1})(\zeta_2^{-1} - \chi^{-1})\frac{\partial}{\partial \chi}\,. \tag{3.19}$$

Crossing symmetry ($x_1 \leftrightarrow x_3$) is manifested on $\mathcal{G}(\chi;\zeta_1,\zeta_2)$ by

$$\bar{\mathfrak{X}}\mathcal{G}(\chi;\zeta_1,\zeta_2) = \mathfrak{X}\mathcal{G}(1-\chi;1-\zeta_1,1-\zeta_2)\,, \tag{3.20}$$

and on $f(\chi)$ as

$$(1-\chi)^2 f(\chi) + \chi^2 f(1-\chi) = 0\,. \tag{3.21}$$

The $\zeta_{1,2}$ dependence in $\mathcal{G}(\chi;\zeta_1,\zeta_2)$ comes purely from $\mathfrak{X}$ (3.18) and $\mathbb{D}$ (3.19). We can therefore also decompose it as

$$\mathcal{G}(\chi;\zeta_1,\zeta_2) = g_2(\chi)(\zeta_1\zeta_2)^{-1} + g_1(\chi)(\zeta_1^{-1} + \zeta_2^{-1}) + g_0(\chi)\,, \tag{3.22}$$

where

$$g_2 = C_\Phi^2\left(\chi^2\mathbb{F} - \chi^2\frac{\partial f}{\partial \chi}\right), \qquad g_1 = C_\Phi^2\left(-f + \chi\frac{\partial f}{\partial \chi}\right), \qquad g_0 = C_\Phi^2\left(\frac{2f}{\chi} - \frac{\partial f}{\partial \chi}\right). \tag{3.23}$$

From the crossing symmetry equation of $\mathcal{G}(\chi;\zeta_1,\zeta_2)$ (3.20), we can also find the properties for $g_{0,1,2}$, which are

$$\begin{aligned}
\chi^2 g_2(1-\chi) &= (1-\chi)^2\left(g_2(\chi) + 2g_1(\chi) + g_0(\chi)\right),\\
\chi^2 g_1(1-\chi) &= -(1-\chi)^2\left(g_1(\chi) + g_0(\chi)\right),\\
\chi^2 g_0(1-\chi) &= (1-\chi)^2 g_0(\chi)\,.
\end{aligned} \tag{3.24}$$

[9] The factor of C_Φ^2 in (3.17) is not in [123,134] because they use operators normalised to the identity.

To get the four point function of scalar insertions, we need to differentiate with respect to the t's, c.f. (3.14)

$$\begin{aligned}\langle\!\langle \Phi_i(x_1)\Phi_j(x_2)\Phi_k(x_3)\Phi_l(x_4)\rangle\!\rangle &= \frac{\partial}{\partial t_1^i}\frac{\partial}{\partial t_2^j}\frac{\partial}{\partial t_3^k}\frac{\partial}{\partial t_4^l}\left(\frac{t_{12}t_{34}}{x_{12}^2x_{34}^2}\mathcal{G}(\chi;\zeta_1,\zeta_2)\right) \\ &= \frac{1}{x_{12}^2x_{34}^2}\big(g_2(\chi)\delta_{ik}\delta_{jl} + g_1(\chi)(\delta_{ik}\delta_{jl}+\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk}) + g_0(\chi)\delta_{ij}\delta_{kl}\big)\,.\end{aligned} \tag{3.25}$$

3.2.2 Sum rules for Wilson loop insertions

Clearly the dCFT of the 1/2 BPS Wilson loop has a defect conformal manifold with geometry S^5 of radius $\sqrt{C_\Phi}$ (3.11). The curvature (3.8) is then

$$R_{ijkl} = C_\Phi(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,, \qquad R = \frac{20}{C_\Phi}\,. \tag{3.26}$$

We would now like to identify this with the expression for the curvature (3.7). In the case of a 1d defect there is a subtlety, as the order of the insertions is meaningful. Taking the 4-point function $\langle\!\langle \Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0)\rangle\!\rangle$ for example, it can be regarded as shorthand for

$$\langle\!\langle \Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0)\rangle\!\rangle_c = \begin{cases} \langle\!\langle \Phi_j(\eta)\Phi_l(0)\Phi_i(1)\Phi_k(\infty)\rangle\!\rangle\,, & \text{for } \eta<0\,, \\ \langle\!\langle \Phi_l(0)\Phi_j(\eta)\Phi_i(1)\Phi_k(\infty)\rangle\!\rangle\,, & \text{for } 0<\eta<1\,, \\ \langle\!\langle \Phi_l(0)\Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\rangle\!\rangle\,, & \text{for } \eta>1\,. \end{cases} \tag{3.27}$$

Taking the order of insertions into consideration as shown in (3.27), the curvature tensor (2.29) is actually the sum of six terms. In each term we make use of the expression (3.25) for the 4-point functions, and conformal symmetry such that the cross ration χ is in the domain $\chi \in (0,1)$. Eventually we find

$$\begin{aligned}R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\bigg(&\int_0^1 \frac{d\chi}{\chi^2}\log\chi\big(g_2(\chi)-2g_1(\chi)-2g_0(\chi)\big) \\ &+\int_0^1\frac{d\chi}{\chi^2}\log(1-\chi)\big(g_2(\chi)+4g_1(\chi)+g_0(\chi)\big)\bigg)\,.\end{aligned} \tag{3.28}$$

We now change the variable in the second integration from χ to $1-\chi$ and then apply the crossing relations for $g_{0,1,2}$ (3.24), to find that the second integral is exactly equal to the first one. Therefore, the curvature tensor becomes

$$R_{ijkl} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\int_0^1\frac{d\chi}{\chi^2}\log\chi\big(g_2(\chi)-2g_1(\chi)-2g_0(\chi)\big)\,. \tag{3.29}$$

Comparing with (3.22), the integrand can be written as

$$\frac{\log\chi}{\chi^2}\mathcal{G}(\chi;\zeta_1^*,\zeta_2^*)\,, \qquad \zeta_{1,2}^* = -1\pm\sqrt{3}\,, \tag{3.30}$$

such that $\zeta_1^* + \zeta_2^* = \zeta_1^*\zeta_2^* = -2$.

To perform the integral, we plug in the expressions for $g_{0,1,2}$ in terms of $\mathbb{F}$ and f (3.23) to find

$$R_{ijkl} = 2C_\Phi^2(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \int_0^1 d\chi \log\chi \left(\mathbb{F} + 2\left(\frac{1}{\chi^2} - \frac{2}{\chi^3}\right) f(\chi) - \left(1 + \frac{2}{\chi} - \frac{2}{\chi^2}\right) \frac{\partial f(\chi)}{\partial \chi} \right). \tag{3.31}$$

The tensor structure is as expected for a maximally symmetric space, and after contracting the indices with the inverse of the metric (3.4), the Ricci scalar is

$$R = 40 \int_0^1 d\chi \log\chi \left(\mathbb{F} + 2\left(\frac{1}{\chi^2} - \frac{2}{\chi^3}\right) f(\chi) - \left(1 + \frac{2}{\chi} - \frac{2}{\chi^2}\right) \frac{\partial f(\chi)}{\partial \chi} \right). \tag{3.32}$$

In Appendix 3.A we further simplify this to

$$R = 40 \int_0^1 d\chi \left(-\left(1 + \frac{1}{\chi}\right)\mathbb{F} + \left(1 - \frac{2}{\chi^3}\right) f(\chi) \right). \tag{3.33}$$

This integral with $R = 20/C_\Phi$ (3.26) can also be deduced from the integral identities in [7], as shown in Appendix 3.B.

3.2.3 Comparison to explicit 4-point functions

The 4-point function of Φ_i insertions was calculated at strong coupling by explicit world-sheet Witten diagrams in [122] and extended up to three loop order in [134] based on the formalism in [120, 123].

Representing the 4-point function as in (3.17), $\mathbb{F}$ is given by the following series at strong coupling [134]

$$\mathbb{F} = -\frac{3}{\sqrt{\lambda}} + \frac{45}{8}\frac{1}{\lambda^{\frac{3}{2}}} + \frac{45}{4}\frac{1}{\lambda^2} + O(\lambda^{-\frac{5}{2}}) \tag{3.34}$$

Likewise $f(\chi)$ is expanded in a power series

$$f(\chi) = \sum_{n=1}^{\infty} \lambda^{-\frac{n}{2}} f^{(n)}(\chi)\,, \tag{3.35}$$

where

$$f^{(1)}(\chi) = -(1-\chi^2)\log(1-\chi) + \frac{\chi^3(2-\chi)}{(1-\chi)^2}\log(\chi) - \frac{\chi(1-2\chi)}{1-\chi}\,. \tag{3.36}$$

$f^{(n)}$ for $n = 2, 3, 4$ are of transcendentality $\leq n$ and are successively more complicated. They can be found in [134] and are not repeated here.

At these orders, the integrand in (3.33) involves terms of the form

$$r(x)\log^a x \log^b(1-x)\,\mathrm{Li}_n(y(x))\,, \qquad y(x) \in \left\{x,\, 1-x,\, \frac{x}{x-1}\right\}\,, \tag{3.37}$$

Where $r(x)$ is a rational function with poles at $x = 0$ and $x = 1$. They can all be evaluated recursively by integration by parts. Doing the integrals and combining back into a power series we find

$$2\int_0^1 d\chi \left(-\left(1+\frac{1}{\chi}\right)\mathbb{F} + \left(1-\frac{2}{\chi^3}\right)f(\chi)\right) = \frac{2\pi^2}{\sqrt{\lambda}} + \frac{3\pi^2}{\lambda} + \frac{15\pi^2}{4\lambda^{\frac{3}{2}}} + \frac{15\pi^2}{4\lambda^2} + O(\lambda^{-\frac{5}{2}})\,. \tag{3.38}$$

This exactly agrees with the large λ expansion of $1/C_\Phi$, whose inverse is in (3.12).

Expressions for the 4-point function at weak coupling where given in [137, 138, 7]. As [7] checked their integral identities against those results, and (3.33) can be related to those (see Appendix 3.B), it is clearly also satisfied.

3.3 1/2 BPS loops in ABJM theory

Another line defect is the 1/2 BPS Wilson loop (2.40) of the ABJM theory [110]. The 4-point functions of defect exactly marginal operators are studied in [95] at the tree level order in the strong coupling.

3.3.1 The Wilson loop dcft

Now the marginal operators are chiral and have the structure of a supermatrix

$$S \to S + \lambda^i \int \mathbb{O}_i dx + \bar{\lambda}^{\bar{i}} \int \bar{\mathbb{O}}_{\bar{i}} dx\,. \tag{3.39}$$

with $i = 1, 2, 3$ and their complex pairs. The two point function of the exactly marginal operators are

$$g_{i\bar{j}} = \langle\!\langle \mathbb{O}_i(0)\bar{\mathbb{O}}_{\bar{j}}(1)\rangle\!\rangle = 4B_{1/2}\delta_{ij}\,, \tag{3.40}$$

where $B_{1/2} = \sqrt{2\lambda}/4\pi + \dots$ is the bremsstrahlung function for these operators [139–141].

For the 4-point function we need to distinguish two orderings [95]

$$\begin{aligned} \langle\!\langle \mathbb{O}_i(x_1)\bar{\mathbb{O}}_{\bar{j}}(x_2)\mathbb{O}_k(x_3)\bar{\mathbb{O}}_{\bar{l}}(x_4)\rangle\!\rangle &= \frac{g_{i\bar{j}}g_{k\bar{l}}F_1(z) - g_{i\bar{l}}g_{k\bar{j}}F_2(z)}{x_{12}^2x_{34}^2}\,, \\ \langle\!\langle \mathbb{O}_i(x_1)\bar{\mathbb{O}}_{\bar{j}}(x_2)\bar{\mathbb{O}}_{\bar{k}}(x_3)\mathbb{O}_l(x_4)\rangle\!\rangle &= \frac{g_{i\bar{j}}g_{l\bar{k}}H_1(\chi) - g_{i\bar{k}}g_{l\bar{j}}H_2(\chi)}{x_{12}^2x_{34}^2}\,. \end{aligned} \tag{3.41}$$

Here χ is the cross-ratio defined in (3.15), and the other one

$$z = \frac{x_{12}x_{34}}{x_{14}x_{32}} \tag{3.42}$$

is related to χ by $z = \frac{\chi}{\chi-1}$. With the ordering $x_1 < x_2 < x_3 < x_4$, we always have $z < 0$ and $0 < \chi < 1$. Other orderings can be determined by conformal invariance. The functions F_i

and H_i are expressed in terms of functions $h(\chi)$ and $f(z)$ as

$$\begin{aligned} F_1(z) &= f(z) - zf'(z) + z^2 f''(z), \quad F_2(z) = z^2 f'(z) + z^3 f''(z) \\ H_1(\chi) &= h(\chi) - \chi h'(\chi) + \chi^2 h''(\chi), \quad H_2(\chi) = \chi^2 h'(\chi) + \chi^3 h''(\chi). \end{aligned} \tag{3.43}$$

However, for reasons of simplicity we will define the following

$$\begin{aligned} K_1(\chi) &= F_1\Big(\frac{\chi}{\chi - 1}\Big) \\ K_2(\chi) &= F_2\Big(\frac{\chi}{\chi - 1}\Big). \end{aligned} \tag{3.44}$$

In [95] the crossing equation was also given and it simply reads

$$f(z) = -zf(1/z). \tag{3.45}$$

Using now equation (3.45) and substituting in $F_2(z)$ of (3.43), we find the following relation between $F_1(z)$ and $F_2(z)$

$$F_1(z) = -z^2 F_2(1/z), \tag{3.46}$$

or in terms of the new $K_1(\chi), K_2(\chi)$ functions

$$(1-\chi)^2 K_1(\chi) = -\chi^2 K_2(1-\chi). \tag{3.47}$$

Using this, we derive some useful identities in Appendix 3.C that are used to simplify our calculations in the next sections.

3.3.2 Sum rules for Wilson loop insertions

The metric on the dCFT is read off from the two-point functions

$$g_{i\bar{j}} = \langle\!\langle \mathbb{O}_i(\infty)\bar{\mathbb{O}}_{\bar{j}}(0)\rangle\!\rangle = 2C_\Phi \delta_{ij}, \quad \mathbb{O}_i(\infty) \equiv \lim_{t\to\infty} x^2 \mathbb{O}_i(x). \tag{3.48}$$

Applying conjugation, since $g_{i\bar{j}}$ is real, $g_{\bar{i}j} = g_{i\bar{j}} = 2C_\Phi\delta_{ij}$. Besides, we also have $g_{ij} = g_{\bar{i}\bar{j}} = 0$.

Now, we wish to find the Ricci scalar of the conformal manifold by simple geometrical arguments. First, the conformal manifold is the coset space $SU(4)/SU(3) \simeq \mathbb{CP}^3$, the only unkwown from this point of view is the radius of the $\mathbb{CP}^3$. But, from (3.48) and the fact that a simple calculation yields at normal coordinates

$$g_{i\bar{j}} = \frac{1}{2}\delta_{ij}, \tag{3.49}$$

after comparsion of (3.49) with (3.48) we see that the radius is $r = 2\sqrt{C_\Phi}$ and the Ricci scalar reads

$$R = \frac{4p(p+1)}{4C_\Phi}. \tag{3.50}$$

By specifying $p = 3$ for our case, we have

$$R = \frac{12}{C_\Phi}\,. \tag{3.51}$$

We would now like to identify this with the expression for the curvature (3.7), for this, we define the curvature tensor as usual (in normal coordinates)

$$R_{IJKL} = \frac{1}{2}(\partial_J\partial_K g_{IL} - \partial_I\partial_K g_{JL} - \partial_J\partial_L g_{IK} + \partial_I\partial_L g_{JK}), \tag{3.52}$$

where the capital letters I, J run in $\{i, \bar{i}\}$ and $\{j, \bar{j}\}$ and the second derivatives of the metric are

$$\partial_J\partial_K g_{IL} = \iint dx_1 dx_2 \langle\!\langle \mathcal{O}_J(x_1)\mathcal{O}_K(x_2)\mathcal{O}_I(0)\mathcal{O}_L(\infty)\rangle\!\rangle_c \tag{3.53}$$

where $\mathcal{O}_I = \{\mathbb{O}_i, \bar{\mathbb{O}}_i\}$. By the definition (3.52) and curvatute formula of integrated correlators in [96] or (2.29), we can easily see that the curvature tensor enjoys the skew symmetry $R_{(i\bar{j})k\bar{l}} = R_{i\bar{j}(k\bar{l})} = 0$ and interchange symmetry $R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}}$ just like a Kahler curvature tensor. However, a Kahler curvature tensor also has the properties

$$R_{ij\bar{k}\bar{l}} = 0, \qquad R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} \tag{3.54}$$

which are obvious through the definition (3.52) but not through the one given in [96]. However, we will show in the following, by explicit calculation, that these properties are indeed satisfied when the integrated correlator formula (2.29) is used .

Written in terms of four-point functions, the curvature tensor is (2.29) [96]. Firstly, we want to check that the component $R_{ij\bar{k}\bar{l}} = 0$ is true, where

$$\begin{aligned} R_{ij\bar{k}\bar{l}} = -\mathrm{RV}\int_{-\infty}^{+\infty} d\eta \log|\eta| \Big[&\langle\!\langle \mathbb{O}_i(1)\mathbb{O}_j(\eta)\bar{\mathbb{O}}_k(\infty)\bar{\mathbb{O}}_l(0)\rangle\!\rangle_c \\ &+ \langle\!\langle \mathbb{O}_i(0)\mathbb{O}_j(1-\eta)\bar{\mathbb{O}}_k(\infty)\bar{\mathbb{O}}_l(1)\rangle\!\rangle_c \Big]\,. \end{aligned} \tag{3.55}$$

Plugging in the expressions for the four-point functions (3.41), it turns to

$$R_{ij\bar{k}\bar{l}} = -4C_\Phi^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,\mathrm{Int}_1 \tag{3.56}$$

where

$$\mathrm{Int}_1 = \int_0^1 \frac{d\chi}{\chi^2}\left[\log\left(\frac{\chi}{\chi-1}\right)(K_1(\chi) + K_2(\chi)) + 2\log\chi(H_1(\chi) + H_2(\chi))\right]. \tag{3.57}$$

By calculating the integral explicitly using the tree level expressions (3.64) we obtain that $\mathrm{Int}_1^{(1)} = 0$ [10], verifying at least at leading order that $R_{ij\bar{k}\bar{l}}$ vanishes. However, since $R_{ij\bar{k}\bar{l}} = 0$

[10] $\mathrm{Int}_1 = \sum_{n=0}^{\infty} \epsilon^n \mathrm{Int}_1^{(n)}$ where $\mathrm{Int}_1^{(n)}$ means that n-th order expressions for $h(\chi), f(z)$ and hence H_1, H_2, F_1 and F_2 are used. (See section (3.3.3) for details on ϵ)

must hold at all orders and not just tree level, we postulate that $\text{Int}_1 = 0$ holds at all orders, which leads then asymptotics constraints on H_1, H_2, F_1 and F_2, see Appendix 3.C for more details.

More importantly, the only non-zero independent component of curvature tensor (2.29) reads

$$R_{i\bar{j}\bar{k}l} = -\text{RV}\int_{-\infty}^{+\infty} d\eta \log|\eta| \Big[\langle\!\langle \mathbb{O}_i(1)\bar{\mathbb{O}}_j(\eta)\bar{\mathbb{O}}_k(\infty)\mathbb{O}_l(0)\rangle\!\rangle_c + \langle\!\langle \mathbb{O}_i(0)\bar{\mathbb{O}}_j(1-\eta)\bar{\mathbb{O}}_k(\infty)\mathbb{O}_l(1)\rangle\!\rangle_c \Big] . \tag{3.58}$$

Plugging in (3.41), we have

$$R_{i\bar{j}\bar{k}l} = 4C_\Phi^2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})\text{Int}_2 \tag{3.59}$$

where we have applied $\text{Int}_1 = 0$ and

$$\text{Int}_2 = \int_0^1 \frac{d\chi}{\chi^2}\left[2\log(1-\chi)H_1(\chi) - 2\log\left(\frac{1-\chi}{\chi}\right)H_2(\chi) + \log\chi K_2(\chi) - \log(1-\chi)K_1(\chi)\right] . \tag{3.60}$$

Using now

$$R = 2g^{i\bar{k}}g^{\bar{j}l}R_{i\bar{j}\bar{k}l} \tag{3.61}$$

and finally equation (3.48) We arrive at the simplified expression for the Ricci scalar that reads as follows

$$R = 24\int_0^1 \frac{d\chi}{\chi^2}\left[2\log(1-\chi)H_1(\chi) - 2\log\left(\frac{1-\chi}{\chi}\right)H_2(\chi) + 2\log\chi K_2(\chi)\right] . \tag{3.62}$$

Lastly, using the constraints derived in Appendix 3.C we find

$$R = 48\int_0^1 \frac{d\chi}{\chi^2}\Big[\log(1-\chi)\Big(H_1(\chi) - H_2(\chi)\Big) + \log\chi K_2(\chi)\Big] . \tag{3.63}$$

3.3.3 Comparison to explicit 4-point functions

The 4-point functions of $\{\mathbb{O}, \bar{\mathbb{O}}\}$ were calculated at strong coupling by explicit worldsheet Witten diagrams and bootsrap methods in [95] at tree level.

Representing the 4-point functions as in (3.41), the functions $f(z)$ and $h(\chi)$ read at tree level

$$\begin{aligned} f^{(1)}(z) &= \epsilon\left[-\frac{(1-z)^3}{z}\log(1-z) + z(3-z)\log|z| + z - 1\right] \\ h^{(1)}(\chi) &= \epsilon\left[-\frac{(1-\chi)^3}{\chi}\log(1-\chi) + \chi(3-\chi)\log(\chi) + \chi - 1\right] \end{aligned} \tag{3.64}$$

and hence the functions $H_1(\chi), H_2(\chi), K_1(\chi), K_2(\chi)$ read at tree level as

$$H_1^{(1)}(\chi) = -4 + \chi + (3 - \frac{4}{\chi} + \chi^2)\log(1-\chi) - \chi^2 \log\chi \tag{3.65}$$

$$H_2^{(1)}(\chi) = (-1 - 3\chi^2 + 4\chi^3)\log(1-\chi) + \chi(-1 + 4\chi + (3-4\chi)\chi\log\chi) \tag{3.66}$$

$$K_1^{(1)}(\chi) = \frac{(-4 + 9\chi - 6\chi^2)\log(1-\chi) - \chi\Big(4 - 7\chi + 3\chi^2 + \chi^2\log\left(\frac{\chi}{1-\chi}\right)\Big)}{(\chi-1)^2\chi} \tag{3.67}$$

$$K_2^{(1)}(\chi) = \frac{(-1 + 3\chi - 6\chi^2)\log(1-\chi) - \chi\Big(1 + 2\chi - 3\chi^2 + \chi(3+\chi)\log\left(\frac{\chi}{1-\chi}\right)\Big)}{(\chi-1)^3} \tag{3.68}$$

where ϵ is a small parameter, whose precise relation with the string tension cannot be predicted by symmetry considerations, but can be fixed by the calculation with Witten diagrams in AdS space that $\epsilon = 1/(2\pi\sqrt{2\lambda})$ [95]. The expression of $h(\chi)$ is essentially replacing f with h and z with χ in $f(z)$, neglecting the imaginary part of the logarithm. Because $z < 0$ and $0 < \chi < 1$, $f(z)$ and $h(\chi)$ are both real.

Using now (3.64) to evaluate the integral (3.63) we obtain

$$R = 48\pi^2\epsilon = \frac{24\pi}{\sqrt{2\lambda}} \tag{3.69}$$

at leading order, which matches exactly with the large λ expansion of $\frac{12}{C_\Phi}$, again at the specified leading order. The expansion reads as follows

$$\frac{12}{C_\Phi} = \frac{24\pi}{\sqrt{2\lambda}} + \frac{12}{\lambda} + O(\lambda^{-\frac{3}{2}}). \tag{3.70}$$

Another example is the 1/3 BPS Wilson loops in ABJM theory that we introduce later in section 7.5, where the symmetry breaking coset is $SU(4)/S(U(2)\times U(1)\times U(1))$ which is a 10 dimensional manifold (7.50).

3.4 Surface operators in 6d

The 6d $\mathcal{N} = (2,0)$ theory has 1/2 BPS surface operators [142] with the geometry of the plane or the sphere. In the absence of a Lagrangian description, we cannot write an expression like (3.9). Yet many properties of the surface operators are known: they carry a representation of the A_{N-1} algebra of the theory [143–145] and we focus on the fundamental representation, described by an M2-brane in $AdS_7 \times S^4$ [108]. Their symmetry algebra is $\mathfrak{osp}(4^*|2)^2$, the anomaly coefficients have been evaluated and properties of their defect CFT were also studied.

3.4.1 The surface dCFT

The defect CFT approach to surface operators was developed in [146]. Again the displacement operator is in a multiplet, whose bosonic operators are the displacement itself and the scalar associated to breaking of $SO(5)$ R-symmetry

$$\begin{aligned}\partial_\mu T^{\mu m}(x_\parallel, x_\perp)\, V &= V[\mathbb{D}^m(x_\parallel)]\delta^{(4)}(x_\perp)\,,\\ \partial_\mu j^{\mu i5}(x_\parallel, x_\perp)\, V &= V[\mathbb{O}^i(x_\parallel)]\delta^{(4)}(x_\perp)\,.\end{aligned} \tag{3.71}$$

We write here explicitly the surface operator V that leads to the symmetry breaking and on the right hand side the operator with the appropriate insertion.

Their two point functions take the form

$$\langle\!\langle \mathbb{D}^m(x_\parallel)\mathbb{D}^n(0)\rangle\!\rangle = \frac{C_{\mathbb{D}}\delta^{mn}}{|x_\parallel|^6}\,, \qquad \langle\!\langle \mathbb{O}_i(x_\parallel)\mathbb{O}_j(0)\rangle\!\rangle = \frac{C_{\mathbb{O}}\delta_{ij}}{|x_\parallel|^4}\,. \tag{3.72}$$

As shown in [146], the normalisation constants $C_{\mathbb{D}}$ and $C_{\mathbb{O}}$ are related to each other and to the anomaly coefficients c and a_2 [147, 148] by

$$C_{\mathbb{O}} = \frac{1}{16}C_{\mathbb{D}} = \frac{c}{\pi^2} = -\frac{a_2}{\pi^2} = \frac{1}{\pi^2}\left(N - \frac{1}{2} - \frac{1}{2N}\right). \tag{3.73}$$

The value of the anomaly coefficients were fully determined in [149–154].

In order to write the 4-point function of $\mathbb{O}_i$, we define as in the case of the Wilson loop dCFT (3.13), the operator $\mathbb{O}(\vec{x};t) = t^i\mathbb{O}_i(\vec{x})$ where t^i are constant 4-vectors. We then give their 4-point function as

$$\langle\!\langle \mathbb{O}(\vec{x}_1;t_1)\mathbb{O}(\vec{x}_2;t_2)\mathbb{O}(\vec{x}_3;t_3)\mathbb{O}(\vec{x}_4;t_4)\rangle\!\rangle = \frac{t_{12}t_{34}}{\vec{x}_{12}^4\vec{x}_{34}^4}\mathcal{G}(\chi,\bar{\chi};\alpha,\bar{\alpha})\,. \tag{3.74}$$

As in Section 3.2, $t_{ij} \equiv t_i \cdot t_j$ and now the cross-ratios are χ, $\bar{\chi}$, α and $\bar{\alpha}$ are now (c.f. (3.15))

$$\begin{aligned} U &= \frac{\vec{x}_{12}^2\vec{x}_{34}^2}{\vec{x}_{13}^2\vec{x}_{24}^2} = \chi\bar{\chi}\,, \qquad & V &= \frac{\vec{x}_{14}^2\vec{x}_{23}^2}{\vec{x}_{13}^2\vec{x}_{24}^2} = (1-\chi)(1-\bar{\chi})\,,\\ \sigma &= \frac{t_{13}t_{24}}{t_{12}t_{34}} = \alpha\bar{\alpha}\,, & \tau &= \frac{t_{14}t_{23}}{t_{12}t_{34}} = (1-\alpha)(1-\bar{\alpha})\,,\end{aligned} \tag{3.75}$$

Since the correlator is not sensitive to the order of the four $\mathbb{O}$'s, it should be invariant under the exchanges of any two $\mathbb{O}$. The simple crossing equation arises for $\mathcal{G}$, arises from the $1 \leftrightarrow 3$ exchange

$$\mathcal{G}\left(1-\chi, 1-\bar{\chi}; \frac{\alpha}{\alpha-1}, \frac{\bar{\alpha}}{\bar{\alpha}-1}\right) = \frac{|1-\chi|^4}{|\chi|^4(1-\alpha)(1-\bar{\alpha})}\mathcal{G}(\chi,\bar{\chi};\alpha,\bar{\alpha})\,. \tag{3.76}$$

This is the analogue of (3.20).

3.4.2 Sum rules for surface insertions

The expression for the curvature tensor of the Zamolodchikov metric in terms of the 4-point function is as usual (3.7). In this case that is a pair of two dimensional integrals. This expression was reduced in [96] to the expression (2.30) with a single integral over the complex cross ratio. We further simplify the integral by splitting the integration domain into three parts

$$\begin{aligned}\mathcal{R}_1 &= \{\eta \,|\, \epsilon < |\eta| < 1-\epsilon\}\,,\\ \mathcal{R}_2 &= \{\eta \,|\, \epsilon < |1-\eta|\} \cap \{\eta \,|\, 1-\epsilon < |\eta| < 1+\epsilon\}\,,\\ \mathcal{R}_3 &= \{\eta \,|\, 1+\epsilon < |\eta| < \epsilon^{-1}\}\,,\end{aligned} \tag{3.77}$$

In the region $\mathcal{R}_2$ we checked that the explicit expressions (3.84), (3.85) below are maximised along $|\eta - 1| = \epsilon$ and is bound by

$$\sup_{\eta\in\mathcal{R}_2} \langle\!\langle \mathbb{O}_i(1)\mathbb{O}_j(\eta)\mathbb{O}_k(\infty)\mathbb{O}_l(0)\rangle\!\rangle = O(\log\epsilon)\,. \tag{3.78}$$

The integral in (2.30) over $\mathcal{R}_2$ then clearly vanishes without any counterterms.

For the region $\mathcal{R}_3$ we consider the conformal transformation

$$0 \to 0\,, \qquad \eta \to 1\,, \qquad 1 \to \frac{1}{\eta}\,, \qquad \infty \to \infty\,. \tag{3.79}$$

Clearly for $\eta \in \mathcal{R}_3$ we have $1/\eta \in \mathcal{R}_1$, and that the conformal transformation of the 4-point function of marginal operators cancels the $1/|\eta|^4$ Jacobian, thus the integral in $\mathcal{R}_3$ becomes

$$-\int_{\mathcal{R}_3} d^2\eta \log|\eta| \langle\!\langle \mathbb{O}_i(1)\mathbb{O}_j(\eta)\mathbb{O}_k(\infty)\mathbb{O}_l(0)\rangle\!\rangle_c = \int_{\mathcal{R}_1} d^2\eta \log|\eta| \langle\!\langle \mathbb{O}_i(\eta)\mathbb{O}_j(1)\mathbb{O}_k(\infty)\mathbb{O}_l(0)\rangle\!\rangle_c \tag{3.80}$$

Finally we find an expression for the curfacture tensor over $\mathcal{R}_1$ alone and (2.30) becomes

$$\begin{aligned}R_{ijkl} = -2\pi \lim_{\epsilon\to 0}\Bigg[\int\limits_{\epsilon<|\eta|<1-\epsilon} d^2\eta \log|\eta| \Big[&\langle\!\langle \mathbb{O}_l(0)\mathbb{O}_j(\eta)\mathbb{O}_i(1)\mathbb{O}_k(\infty)\rangle\!\rangle_c \\ &- \langle\!\langle \mathbb{O}_l(0)\mathbb{O}_i(\eta)\mathbb{O}_j(1)\mathbb{O}_k(\infty)\rangle\!\rangle_c\Big] + \Delta R_{ijkl}(\epsilon)\Bigg]\,.\end{aligned} \tag{3.81}$$

In our case of marginal operators, it turns out that $\Delta R_{ijkl}(\epsilon) \to 0$, but one still has to be careful about the exact domain of integration. In this expression η is equal to the cross-ratio χ as defined in (3.75).

3.4.3 Comparison to explicit 4-point functions

We now are ready to evaluate (3.81) for large N and match it to the curvature as deduced from the breaking of R-symmetry by the surface operator, where the analogue of (3.26) is

$$R_{ijkl} = C_{\mathbb{O}}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,, \qquad R = \frac{12}{C_{\mathbb{O}}}\,. \tag{3.82}$$

The 4-point function was calculated at leading order at large N from the M2-brane with the geometry of AdS_3 in AdS_7 [155]. The expression for the 4-point function (3.74) can be divided into two parts

$$\mathcal{G}_{\text{tree}} = \mathcal{G}_1 + \mathcal{G}_2 \tag{3.83}$$

where

$$\begin{aligned}\mathcal{G}_1 = -\frac{6N}{\pi^4}U^2\Big((U-1-V)\bar{D}_{3333} - U\bar{D}_{3322} + \bar{D}_{2222}\big] \\ + \sigma\big[(1-U-V)\bar{D}_{3333} - \bar{D}_{3232} + \bar{D}_{2222}\big] + \tau\big[(V-1-U)\bar{D}_{3333} - \bar{D}_{3223} + \bar{D}_{2222}\big]\Big)\end{aligned} \tag{3.84}$$

and

$$\mathcal{G}_2 = -\frac{9N}{2\pi^4}U^2(\chi-\bar{\chi})(\alpha-\bar{\alpha})\bar{D}_{3333}\,. \tag{3.85}$$

The definition of the $\bar{D}$ functions is given in appendix 3.D. Note that the expressions here are 16 times smaller than in [155] because of a factor of 2 difference in the normalisation of the scalar operators $\mathbb{O}_i$ compared to the S^4 coordinates y_i in [155].

The 4-point correlator is given by the t_n^i derivatives of (3.74)

$$\langle\!\langle \mathbb{O}_i(\vec{x}_1)\mathbb{O}_j(\vec{x}_2)\mathbb{O}_k(\vec{x}_3)\mathbb{O}_l(\vec{x}_4)\rangle\!\rangle = \frac{\partial}{\partial t_1^i}\frac{\partial}{\partial t_2^j}\frac{\partial}{\partial t_3^k}\frac{\partial}{\partial t_4^l}\langle\!\langle \mathbb{O}(\vec{x}_1;t_1)\mathbb{O}(\vec{x}_2;t_2)\mathbb{O}(\vec{x}_3;t_3)\mathbb{O}(\vec{x}_4;t_4)\rangle\!\rangle \tag{3.86}$$

$\mathcal{G}_2$ is parity odd and does not contribute to the curvature tensor. This is easiest to see by changing the integration variables from χ, $\bar{\chi}$ to U, V, which gives

$$d^2\chi = \frac{1}{|\chi-\bar{\chi}|}\,dU\,dV\,. \tag{3.87}$$

identifying $U = |\chi^2|$ and combining with the measure, we have

$$\frac{\log|\chi|}{U^2}\mathcal{G}_2\,d^2\chi = -\frac{9N}{2\pi^4}\log|\chi|\,\text{sign}(\text{Im}\,\chi)(\alpha-\bar{\alpha})\bar{D}_{3333}\,dU\,dV\,. \tag{3.88}$$

Differentiation with respect to t^i, leaves this expression odd under $\chi \to \bar{\chi}$ and since the integration domain $\mathcal{R}_1$ is symmetric, the integral vanishes.

Ignoring the contribution of $\mathcal{G}_2$, we write the contribution of $\mathcal{G}_1$ to the 4-point function as

$$\begin{aligned}\langle\!\langle \mathbb{O}_l(0)\mathbb{O}_j(\chi)\mathbb{O}_i(1)\mathbb{O}_k(\infty)\rangle\!\rangle_{c,\mathcal{G}_1} = -\frac{6N}{\pi^4}\Big(&\delta_{ik}\delta_{jl}[(U-1-V)\bar{D}_{3333} - U\bar{D}_{3322} + \bar{D}_{2222}] \\ &+\delta_{il}\delta_{jk}[(1-U-V)\bar{D}_{3333} - \bar{D}_{3232} + \bar{D}_{2222}] \\ &+\delta_{ij}\delta_{kl}[(V-1-U)\bar{D}_{3333} - \bar{D}_{3223} + \bar{D}_{2222}]\Big)\,.\end{aligned} \tag{3.89}$$

The expression of $\langle\!\langle \mathbb{O}_l(0)\mathbb{O}_i(\chi)\mathbb{O}_j(1)\mathbb{O}_k(\infty)\rangle\!\rangle_c$ is similar just with i, j exchanged. So the final expression of curvature tensor is

$$R_{ijkl} = \frac{12N}{\pi^3}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\int\limits_{|\chi|<1} d^2\chi\log|\chi|\,\big[(2|\chi|^2-2)\bar{D}_{3333} - |\chi|^2\bar{D}_{3322} + \bar{D}_{3232}\big]\,. \tag{3.90}$$

By numerical integration, we confirm that

$$R_{ijkl} = \frac{N}{\pi^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,, \tag{3.91}$$

in agreement with (3.8) with Ricci scalar $R = 12/C_{\mathbb{O}}$, and the large N limit of $C_{\mathbb{O}} \sim N/\pi^2$ (3.73).

3.5 Discussion

In this paper we used the breaking of the global R-symmetry by defects to realise defect conformal manifolds. As the defect conformal manifold is the coset arising from symmetry breaking, its geometry is determined up to an overall scale. The marginal operators for these deformations are superpartners of the displacement operator and an appropriate integral of their 4-point function (3.7) gives the Riemann curvature of the conformal manifold [115].

Fixing the scale requires knowledge of the normalisation of the marginal operator, which is a natural observable in the defect CFT. With that, one finds an exact identity for the integrated 4-point functions. We studied those in the case of the Maldacena-Wilson line dCFT and for surface operators in the 6d $(2,0)$ theory. The exact identities that we wrote were based on a simplification of the integral due to Friedan and Konechny [96], and some further simplifications and are given in (3.33) and (3.90).

We verified those identities against the previously computed 4-point functions in [122, 123, 155, 134] and found perfect agreement.

Similar constraints can be found for higher point functions (see e.g. [138]). The fully integrated correlators are again derivatives of the Zamolodchikov metric, and therefore fixed by the geometry of the manifold.

The identity we derived for the Maldacena-Wilson loop is related to the two identities noted in [7] where it was used as part of their numerical bootstrap studies and also compared to new analytic results. It would be interesting to see if these integral identities can be used in analytic bootstrap calculations to derive results at higher loops and in other systems. Further integral identities were identified in [156–158].

In the two cases that we studied, the defect conformal manifolds are symmetric spaces, S^5 and S^4, so have just this single scale, giving one integral identity. Defects that break the R-symmetry in more interesting ways will give more interesting metrics, have a variety of marginal operators with different 2-point functions and one could find integral constraints for different components of the Riemann tensor.

The same analysis can be applied to Wilson loops and surface operators in higher dimensional representations, where some results for the bremsstralung function and anomaly coefficient c are known [159, 149–154]. This can be compared to explicit holographic computations in terms of D3, D5 and M5-branes [160–165]. In the case of the Wilson loop the explicit calculation was carried out in [166], where the result was proportional to the same function of the cross-ratios (3.36) as in the case of the fundamental string.

A natural next avenue would be to examine line operators in 3d supersymmetric theories, which have a much richer spectrum (see [58,59] for an overview and recent results). Expressions for the bremsstrahlung function were found in [139–141], but their interpretation is not clear, as it is not positive definite. Beyond that, there are many other supersymmetric theories with defects for which these techniques can be applied.

It would be interesting to study defect conformal manifolds that do not arise from broken symmetries. Hopefully exactly marginal defect operators are not as rare or hard to find as bulk marginal operators. It is also natural to look at systems with both defect and bulk marginal operators to construct richer structures. Some work in that direction is in [114] and it would be interesting to see if simple theories like mixed dimensional QED [167] and generalisations thereof admit such deformations.

Acknowledgements

We are indebted to S. Giombi, N. Gromov, Z. Komargodski, C. Meneghelli, M. Probst, A. Stergiou, M. Trépanier and G. Watts for invaluable discussions. ND's research is supported by the Science Technology & Facilities council under the grants ST/T000759/1 and ST/P000258/1. GS is funded by the Science Technology & Facilities council under the grant ST/W507556/1. The work of ZK was supported by China Scholarship Council under CSC No. 201906340174.

3.A Simplifying the integral of $f(\chi)$

To simplify the expression for the Ricci scalar (3.32)

$$\frac{R}{40} = \int_0^1 d\chi \log\chi \left(\mathbb{F} + 2\left(\frac{1}{\chi^2} - \frac{2}{\chi^3}\right) f(\chi) - \left(1 + \frac{2}{\chi} - \frac{2}{\chi^2}\right)\frac{\partial f(\chi)}{\partial\chi}\right), \tag{3.92}$$

we integrate the first term, giving $-\mathbb{F}$ and integrate the last term by parts, to find

$$\frac{R}{40} = -\mathbb{F} - \left[\left(1 + \frac{2}{\chi} - \frac{2}{\chi^2}\right) f(\chi)\log\chi\right]_0^1 + \int_0^1 \frac{d\chi}{\chi}\left(1 + \frac{2}{\chi} - \frac{2}{\chi^2}\right) f(\chi)\,. \tag{3.93}$$

Noticing the boundary behaviour of $f(\chi)$

$$\begin{aligned} \chi \to 0\,, \quad & f(\chi) \sim -\frac{\mathbb{F}}{2}\chi^2\,, \\ \chi \to 1\,, \quad & f(\chi) \sim \frac{\mathbb{F}}{2}\,, \end{aligned} \tag{3.94}$$

we can now use (3.94) to evaluate the boundary term in (3.93)

$$-\left(1+\frac{2}{\chi}-\frac{2}{\chi^2}\right)f(\chi)\log\chi\Big|_{\epsilon}^{1-\epsilon}=\mathbb{F}\log\epsilon\,. \tag{3.95}$$

The crossing symmetry equation (3.21) for $f(\chi)$ leads to the integral identities

$$\int_{\epsilon}^{1-\epsilon}d\chi\frac{f(\chi)}{\chi^2}=0\,,\qquad\int_{\epsilon}^{1-\epsilon}d\chi\frac{f(\chi)}{\chi}=\int_{\epsilon}^{1-\epsilon}d\chi f(\chi)\,. \tag{3.96}$$

This allows us to further simplify (3.93) to

$$\begin{aligned}R&=40\lim_{\epsilon\to0}\left(-\mathbb{F}+\mathbb{F}\log\epsilon+\int_{\epsilon}^{1-\epsilon}d\chi f(\chi)\left(1-\frac{2}{\chi^3}\right)\right)\\&=40\int_0^1d\chi\left(-\left(1+\frac{1}{\chi}\right)\mathbb{F}+\left(1-\frac{2}{\chi^3}\right)f(\chi)\right).\end{aligned} \tag{3.97}$$

3.B Relation to integral identities of [7]

In this appendix we show that the two integral identities in [7] encode the identity (3.32), and thus our results prove one of the identities stated there.

The 4-point function in [7] is expressed in terms of a function δG, which in the notations of Section 3.2 is

$$\delta G(\chi)=\chi^2\mathbb{F}-\left(1-\frac{2}{\chi}\right)f(\chi)-(\chi^2-\chi+1)\frac{\partial f(\chi)}{\partial\chi}\,. \tag{3.98}$$

The two integral constraints noticed in [7] are

$$\int_0^1d\chi\,\frac{\delta G(\chi)}{\chi^2}(1+\log\chi)=\frac{3\mathbb{C}-B}{8B^2}\,,\qquad\int_0^1d\chi\,\frac{f(\chi)}{\chi}=\frac{\mathbb{C}}{4B^2}+\mathbb{F}\,. \tag{3.99}$$

$B=\mathbb{C}_\Phi/2$ is the bremsstrahlung function and $\mathbb{C}$ is a function defined in [7] and since does not appear in this paper, we can take a linear combination to eliminate it

$$\int_0^1d\chi\left(3\frac{f(\chi)}{\chi}-2\frac{\delta G(\chi)}{\chi^2}(1+\log\chi)\right)=\frac{1}{4B}+3\mathbb{F}\,. \tag{3.100}$$

Noting that $\delta G(\chi)/\chi^2$ can be written as

$$\frac{\delta G(\chi)}{\chi^2}=\mathbb{F}-\partial_\chi\left(\left(1-\frac{1}{\chi}+\frac{1}{\chi^2}\right)f(\chi)\right). \tag{3.101}$$

Using (3.94), the left hand side of (3.100) is

$$2\mathbb{F}+\mathbb{F}\log\epsilon+\int_0^1d\chi\left(3\frac{f(\chi)}{\chi}-2\mathbb{F}(1+\log\chi)-2\left(\frac{1}{\chi}-\frac{1}{\chi^2}+\frac{1}{\chi^3}\right)f(\chi)\right). \tag{3.102}$$

Then using (3.96), this is

$$\int_0^1 d\chi \left(\left(2 - \frac{1}{\chi}\right) \mathbb{F} + \left(1 - \frac{2}{\chi^3}\right) f(\chi) \right), \tag{3.103}$$

and finally using our expression for the Ricci tensor (3.33), this is

$$3\mathbb{F} + \frac{R}{40} = 3\mathbb{F} + \frac{1}{2\mathbb{C}_\Phi} = 3\mathbb{F} + \frac{1}{4B}, \tag{3.104}$$

proving (3.100).

3.C Integral identities for H_i, K_i

To simplify the integrals in (3.57) and (3.60) we note that the crossing relation (3.47) implies

$$\int_0^1 \frac{d\chi}{\chi^2} \log\chi \, K_i = -\int_0^1 \frac{d\chi}{\chi^2} \log(1-\chi) \, K_{3-i}, \tag{3.105}$$

which yields

$$\mathrm{Int}_1 = 2\int_0^1 \frac{d\chi}{\chi^2} \Big(\log\left(\frac{\chi}{1-\chi}\right) F_1 + \log\chi H_1 \Big). \tag{3.106}$$

Using (3.43), this is a total derivative. Furthermore, assuming the asymptotics

$$\begin{aligned} h(\chi) &\sim \begin{cases} a_0\chi + a_1\chi\log\chi\,, & \chi \to 0\,, \\ a_2(\chi - 1)\,, & \chi \to 1\,, \end{cases} \\ f(z) &\sim \begin{cases} b_0 z + b_1 z \log|z|\,, & z \to 0\,, \\ b_2 + b_3 \log|z|\,, & z \to -\infty\,, \end{cases} \end{aligned} \tag{3.107}$$

we find

$$\int_0^1 \frac{d\chi}{\chi^2} \log\chi \, H_2 = 0 \tag{3.108}$$

and

$$\mathrm{Int}_1 = 2a_0 - 2b_0\,. \tag{3.109}$$

This indeed vanishes in the perturbative analytic bootstrap, where $a_0 = b_0 = -b_2$ as a consequence of the crossing of h and a braiding relation to f [95].

By using (3.105), (3.107) and (3.108) we can also simplify (3.60) to

$$\mathrm{Int}_2 = 2\int_0^1 \frac{d\chi}{\chi^2} \log(1-\chi)(H_1 - H_2 - K_1)\,. \tag{3.110}$$

3.D $\bar{D}$-functions

We collect here the recursive definition of $\bar{D}$-functions in (3.84), (3.85). These expressions can be used to obtain the explicit form of correlators as functions of the cross ratios U, V or χ, $\bar{\chi}$ in (3.75). See [168] for more details.

The simplest $\bar{D}$-function is $\bar{D}_{1111}$, which is just the scalar one-loop box diagram in four dimensions

$$\bar{D}_{1111} = \frac{1}{\chi - \bar{\chi}} \left[\log(\chi\bar{\chi}) \log\left(\frac{1-\chi}{1-\bar{\chi}}\right) + 2\,\mathrm{Li}_2(\chi) - 2\,\mathrm{Li}_2(\bar{\chi}) \right] \tag{3.111}$$

To obtain $\bar{D}$-functions with higher weights, we can use the following differential operators

$$\begin{aligned}
\bar{D}_{\Delta_1+1,\Delta_2+1,\Delta_3,\Delta_4} &= -\partial_U \,, \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \,, \\
\bar{D}_{\Delta_1,\Delta_2,\Delta_3+1,\Delta_4+1} &= (\Delta_3 + \Delta_4 - \Sigma - U\partial_U)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \,, \\
\bar{D}_{\Delta_1,\Delta_2+1,\Delta_3+1,\Delta_4} &= -\partial_V \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \,, \\
\bar{D}_{\Delta_1+1,\Delta_2,\Delta_3,\Delta_4+1} &= (\Delta_1 + \Delta_4 - \Sigma - V\partial_V)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \,, \\
\bar{D}_{\Delta_1,\Delta_2+1,\Delta_3,\Delta_4+1} &= (\Delta_2 + U\partial_U + V\partial_V)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \,, \\
\bar{D}_{\Delta_1+1,\Delta_2,\Delta_3+1,\Delta_4} &= (\Sigma - \Delta_4 + U\partial_U + V\partial_V)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \,,
\end{aligned} \tag{3.112}$$

where $\Sigma = (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)/2$.

4 Conformal and non-conformal hyperloop deformations of the 1/2 BPS circle

This section is based on [2] with minor edits.

4.1 Introduction and summary

A distinguishing feature of three-dimensional supersymmetric conformal field theories are the vast moduli spaces of BPS line operators annihilated by some supercharges. For operators that are conformal, this was understood from an algebraic point of view in [104], but many examples of conformally invariant circular line operators, including continuous families of them, were found before, see for example [59, 63–69, 111] and [58] for a review.

In the absence of an approach allowing for a full classification, we continue here to develop and employ constructive methods of identifying BPS Wilson loop operators called *hyperloops*, finding a plethora of new observables, some of which are conformally invariant and some of which are not, greatly enlarging the known moduli spaces.

The theories we study are $\mathcal{N} = 4$ supersymmetric Chern-Simons-matter with either linear or circular quiver structure, characterized by the coupling of the gauge multiplet to hypermultiplets and twisted hypermultiplets [83–85, 89]. The 2-node circular quiver has $\mathcal{N} = 6$ supersymmetry and is the ABJ(M) theory [87, 88], so most of what we say applies there as well. For concreteness, we consider theories on S^3 and focus on operators supported along a great circle.[11]

In a recent paper [59], some of us already studied Wilson loops in this same setting. Those hyperloops were written as deformations of bosonic Wilson loops that preserve 2 or 4 supercharges (so they are 1/8 or 1/4 BPS). Starting with particular block-diagonal combinations of bosonic connections $\mathcal{L}_{\text{bos}}$ annihilated by a supercharge Q, it was found that one can deform them as follows

$$\mathcal{L}_{\text{bos}} \to \mathcal{L} = \mathcal{L}_{\text{bos}} - iQG + G^2, \tag{4.1}$$

where G is a matrix constructed out of bosonic fields in the hypermultiplets. The resulting operator is still supersymmetric, by construction, and is defined in terms of a superconnection containing the fermionic fields, which is something typical of supersymmetric Chern-Simons theories [110]. Another peculiarity of three-dimensional theories is that the Q variation of $\mathcal{L}$ does not vanish *per se*, as it happens in the four-dimensional counterpart of these objects, but it is instead a total covariant derivative, so the entire Wilson loop, which is a gauge invariant object, is still annihilated by Q.

[11] Of course, it would be interesting to consider other contours, such as latitudes, or generic curves on an $S^2 \subset S^3$, along the lines of what has been done in [169–171] for $\mathcal{N} = 4$ super Yang-Mills in four dimensions and in [172] for the ABJ(M) theory.

In the current work we apply a similar philosophy to [59], but we employ as the starting point of the deformation the 1/2 BPS Wilson loop found in [63] (see also [65]), rather than a bosonic loop:

$$\mathcal{L}_{1/2} \to \mathcal{L} = \mathcal{L}_{1/2} + \text{deformation}, \tag{4.2}$$

with the details of the deformation given after (4.23) below. The 1/2 BPS loop is also a particular deformation of the bosonic loop as in (4.1), so our current construction includes all of those found previously.

Moreover, unlike the construction in [59], where a single choice of supercharge based on the original Wilson loops was employed, here we consider any supercharge annihilating the 1/2 BPS loop, so any linear combination of a basis of 8 supercharges. In particular, in cases when the supercharge Q has an appropriate kernel, we find infinite-dimensional moduli spaces, since (roughly speaking) we can insert any of the operators in the kernel any number of times at any point along the loop.

This new procedure allows us to uncover new families of supersymmetric line operators. For example, we have discovered:

- Previously unrecognized bosonic loops preserving 2 and 3 supercharges, which are therefore 1/8 and 3/16 BPS, in addition to the known ones preserving 2 or 4 supercharges, see Section 4.5.1.
- New 1/8 and 1/4 BPS loops that do not share supercharges with any known bosonic Wilson loops, so could not have been found by relying on (4.1). Of particular note is a subclass of these loops, which depends on one parameter (after fixing 4 supercharges), for which the variation of the superconnection under conformal transformations of the circle is a total derivative, see Section 4.5.3.

  This forms a new class of previously unrecognized line operators that are classically conformally invariant. Unlike the 1/2 BPS or 1/4 BPS bosonic loops, the one-dimensional conformal algebra is not generated by the supercharges that they preserve, but is an outer automorphism of it. As we cannot rely on supersymmetry to guarantee conformality, it would be extremely interesting to examine them at the quantum level and verify whether they are truly conformally invariant.

There are various natural directions that could be pursued starting from these results. The most obvious one is to try to compute the expectation value of these operators, using localization for example. This typically starts with determining to which cohomological class the various operators belong. In previous examples [59] based on (4.1), as well as in the original papers [111, 63], it was found that the bosonic operators and their fermionic deformations are cohomologically equivalent. In this context we know however that this does not hold, as we find loops, such as the latitudes, that are known to have different expectation values from the 1/2 BPS circle [173–176]. This of course makes these new classes of operators even more interesting.

The next natural question is about the holographic duals. While the holographic duals of 1/2 BPS loops in some $\mathcal{N}=4$ Chern-Simons-matter theories have been identified [65,63,177], the question of what is dual to less supersymmetric (and/or higher representation) operators has not been addressed yet.[12]

Finally, it would be interesting to study the moduli spaces of conformal loops as defect conformal manifolds and analyze the defect conformal field theory they define, along the lines of what has been done for the ABJ(M) theory in [95] and see also [1]. For non-conformal loops it would be interesting to understand their renormalisation group flows [130, 117].

This paper is organised as follows. In the next section we present the notation for the theories and the supersymmetry variations of the fields. In Section 4.2 we present the simplest 1/2 BPS Wilson loop in these theories, which is the starting point of the deformations. The bulk of the calculations is in Sections 4.3 and 4.4, focusing respectively on loops involving only two nodes of the quiver and those involving more, respectively. For the benefit of the casual reader we collect the main results and present a detailed analysis of special interesting examples in Section 4.5. Some details are presented in the appendices.

4.2 The 1/2 BPS Wilson loop and its symmetries

The starting point of the deformation (4.2) considered in this paper is a particular 1/2 BPS loop of the theory. As shown originally for the ABJ(M) theory in [110] and for $\mathcal{N}=4$ theories in [63] (see also [65]), such a Wilson loop must couple to at least two vector fields, as well as to the matter fields charged under them. We take the loop built around the I and $I+1$ nodes of the particular form

$$W_{1/2} = \mathrm{sTr}\,\mathcal{P}\exp i\oint \mathcal{L}_{1/2}\,d\varphi\,, \qquad \mathcal{L}_{1/2} = \begin{pmatrix} \mathcal{A}_I & -i\bar{\alpha}\psi_{I\dot{1}-} \\ i\alpha\bar{\psi}^{\dot{1}}_{I+} & \mathcal{A}_{I+1}-\frac{1}{2} \end{pmatrix}, \tag{4.3}$$

with

$$\mathcal{A}_I = A_{\varphi,I} + \frac{i}{k}\big(\nu_I - \tilde{\mu}_I{}^{\dot{1}}{}_{\dot{1}} + \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}\big)\,, \qquad \mathcal{A}_{I+1} = A_{\varphi,I+1} + \frac{i}{k}\big(\nu_{I+1} - \tilde{\mu}_{I+1\dot{1}}{}^{\dot{1}} + \tilde{\mu}_{I+1\dot{2}}{}^{\dot{2}}\big)\,. \tag{4.4}$$

The constants α and $\bar{\alpha}$ (which are not complex conjugate to each other) satisfy $\alpha\bar{\alpha}=2i/k$ and the Wilson loop does not depend on their actual value, so we could fix them to be equal, but we leave them instead as a constant gauge parameter. We could allow for them to depend on φ at the expense of a $U(1)$ gauge transformation at the bottom right entry: $\mathcal{A}_{I+1}-\frac{1}{2} \to \mathcal{A}_{I+1}-\frac{1}{2}-i\alpha^{-1}\partial_\varphi\alpha$. The origin of the shift $-1/2$ in the connection (and the resulting appearance of the supertrace if compared with the original definition in [63] in terms of the trace) is explained in [58].

[12] A first examination of a possible moduli space of 1/6 BPS loops in ABJ(M) theory was done in [178,179].

As we verify below, the eight supercharges preserved by this loop are

$$Q_{\imath}^{\dot{2}a+}, \qquad Q_{\bar{\imath}}^{\dot{1}a-}. \tag{4.5}$$

The spinor indices $\alpha = \pm$ are taken upstairs, to contract with the downstairs indices of the Killing spinors in (2.15). To relate to the notation in (2.12), we can represent the supersymmetry transformation as $\delta = -\xi^{\imath}_{a\dot{b}\alpha} Q^{\dot{b}a\alpha}_{\imath} - \xi^{\bar{\imath}}_{a\dot{b}\alpha} Q^{\dot{b}a\alpha}_{\bar{\imath}}$.

Looking at the form of the Killing spinors along the circle (2.15), one can write a general superposition of the preserved supercharges (4.5) as

$$Q = \eta^{\imath}_a Q_{\imath}^{\dot{2}a+} + \bar{\eta}^{\imath}_a (\sigma^1)_{\imath}{}^{\bar{\imath}} Q_{\bar{\imath}}^{\dot{1}a-} = \eta^{\imath}_a \bar{v}_{\imath} Q^{\dot{2}a+} + \bar{\eta}^{\imath}_a v_{\imath} Q^{\dot{1}a-}, \tag{4.6}$$

with Grassmann-even parameters $\eta^{\imath}_a$, $\bar{\eta}^{\imath}_a$ (which, again, are not complex conjugate) and auxiliary $SO(2,1)$ spinors

$$v_{\imath} = \begin{pmatrix} e^{+i\varphi} \\ 1 \end{pmatrix}_{\imath}, \qquad \bar{v}_{\imath} = \begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix}_{\imath}. \tag{4.7}$$

The supersymmetry variations generated by a supercharge parameterised in such fashion can then be computed by reading off

$$\xi_{a\dot{1}} = \begin{pmatrix} (\eta\bar{v})_a \\ 0 \end{pmatrix}, \qquad \xi_{a\dot{2}} = \begin{pmatrix} 0 \\ (\bar{\eta}v)_a \end{pmatrix}. \tag{4.8}$$

In the right-most expression in (4.6), $Q^{\dot{a}a}$ acts in the same way as $Q^{\dot{a}a}_{l}$, that is without the extra phases $e^{\pm i\varphi}$, which have been absorbed in the definition of $v_{\imath}$ and $\bar{v}_{\imath}$. There are four $\eta^{\imath}_a$ and four $\bar{\eta}^{\imath}_a$ parameters, but the supercharges are identified up to rescalings, so the space of real supercharges is in fact $\mathbb{RP}^7$.

As noted already in [63], there exists another Wilson loop with the same gauge fields and preserving the exact same symmetries, but coupling instead to other fields in the hypermultiplets. This other operator has the superconnection

$$\mathcal{L}'_{1/2} = \begin{pmatrix} \mathcal{A}_I & -i\bar{\alpha}\psi_{I\dot{2}+} \\ i\alpha\bar{\psi}^{\dot{2}}_{I-} & \mathcal{A}_{I+1} + \frac{1}{2} \end{pmatrix}, \tag{4.9}$$

with the opposite sign for the ν's compared to the ones appearing in (4.4). All the moduli spaces that we find include in them also this operator as a special point of enhanced supersymmetry. It is then just a matter of choice to do the analysis around (4.3), rather than around this one.

Before examining in detail the supersymmetries preserved by the loop defined in (4.3), let us compute its bosonic symmetries. Our notation and further details on the algebra can be found in Appendix 4.A. Firstly, notice that the superconnection (4.3) contains only singlets of the $\mathfrak{su}(2)_L$ R-symmetry, which is clearly preserved. The bosonic part of $\mathcal{L}_{1/2}$ is also annihilated by transverse rotations $T_\perp$, but it acts on the fermions by the Pauli matrix σ_3, see (4.147). Since spinor indices appear in $\mathcal{L}_{1/2}$ accompanied by opposite R-symmetry indices, we can cancel the action of $T_\perp$ by an appropriate multiple of the $\bar{R}_3$ generator of the unbroken $\mathfrak{u}(1)_R$ R-symmetry, and, indeed, the combination $L_\perp \equiv -i\left(T_\perp + i\bar{R}_3/2\right)$ annihilates $\mathcal{L}_{1/2}$. As for the action of the conformal generators J_0 and $J_\pm$ on the 1/2 BPS loop, using the expressions (4.145) and (4.146)

$$iJ_0\,\mathcal{L}_{1/2} = \frac{d\mathcal{L}_{1/2}}{d\varphi} - \frac{\partial\mathcal{L}_{1/2}}{\partial\varphi} - [\sigma_3, \mathcal{L}_{1/2}]\,. \tag{4.10}$$

Since the $\mathcal{L}_{1/2}$ does not contain any explicit φ-dependence, we may bring this into the form[13]

$$iJ_0\,\mathcal{L}_{1/2} = \mathcal{D}_\varphi^{\mathcal{L}_{1/2}}\left(\mathcal{L}_{1/2} + \sigma_3\right). \tag{4.11}$$

Total covariant derivatives can be integrated away, so this guarantees invariance of the 1/2 BPS loop under J_0. Similar arguments show that $J_\pm$ are preserved as well. Finally, note that while acting on $\mathcal{L}_{1/2}$ with $T_\perp$ (or equivalently $\bar{R}_3$) gives a non-zero result, it still takes the form of a covariant derivative

$$T_\perp\mathcal{L}_{1/2} \propto [\sigma_3, \mathcal{L}_{1/2}] = \mathcal{D}_\varphi^{\mathcal{L}_{1/2}}\sigma_3\,. \tag{4.12}$$

Consequently, $\bar{R}_3$ and $T_\perp$ are preserved separately.

We now proceed to evaluate the action of the supercharge Q in (4.6) on the superconnection $\mathcal{L}_{1/2}$ (4.3) and to verify that it is equal to a total derivative. This also introduces a lot of the notation required in the rest of the paper.

First, to write the action of Q on the hypermultiplet fields it is useful to define rotated scalar fields

$$r^1 \equiv (\eta\bar{v})_a q^a\,, \quad r^2 \equiv (\bar{\eta}v)_a q^a\,, \quad \bar{r}_1 \equiv \epsilon^{ab}(\bar{\eta}v)_a\bar{q}_b\,, \quad \bar{r}_2 \equiv -\epsilon^{ab}(\eta\bar{v})_a\bar{q}_b\,, \tag{4.13}$$

where $(\eta\bar{v})_a = \eta_a^\imath\bar{v}_\imath$ and likewise for $(\bar{\eta}v)_a$. Now

$$Qr^1 = -\Pi\psi_{\dot{2}+}\,, \quad Qr^2 = \Pi\psi_{\dot{1}-}\,, \quad Q\bar{r}_1 = \Pi\bar{\psi}_-^{\dot{2}}\,, \quad Q\bar{r}_2 = -\Pi\bar{\psi}_+^{\dot{1}}\,, \tag{4.14}$$

where the $\pm$ subscripts are spinor indices and

$$\Pi \equiv \epsilon^{ab}(\bar{\eta}v)_a(\eta\bar{v})_b \tag{4.15}$$

is a quantity that plays a central role in our analysis.

[13]The precise definition of the covariant derivative $\mathcal{D}_\varphi^{\mathcal{L}_{1/2}}$ is in Appendix 4.B.

It is not too hard to show, using (2.12), that the second variation of the rotated scalars is

$$\begin{aligned} Q^2 r^1 &= \Pi\left(i(\eta\bar{v})_a\partial_\varphi q^a - \frac{1}{2}(\eta\sigma^3\bar{v})_a q^a + \mathcal{A}_I r^1 - \frac{2i}{k}\nu_I r^1 - r^1\mathcal{A}_{I+1} + \frac{2i}{k} r^1\nu_{I+1}\right), \\ Q^2 r^2 &= \Pi\left(i(\bar{\eta}v)_a\partial_\varphi q^a - \frac{1}{2}(\bar{\eta}\sigma^3 v)_a q^a + \mathcal{A}_I r^2 - r^2\mathcal{A}_{I+1}\right). \end{aligned} \tag{4.16}$$

Now, using

$$2i\partial_\varphi(\eta\bar{v})_a = (\eta\bar{v})_a - (\eta\sigma^3\bar{v})_a\,, \qquad -2i\partial_\varphi(\bar{\eta}v)_a = (\bar{\eta}v)_a + (\bar{\eta}\sigma^3 v)_a\,, \tag{4.17}$$

and

$$r^1\bar{r}_1 + r^2\bar{r}_2 = \Pi(q^1\bar{q}_1 + q^2\bar{q}_2) = \Pi\nu\,, \tag{4.18}$$

these second variations can be written as

$$\begin{aligned} Q^2 r^1 &= \Pi\left(i\partial_\varphi r^1 - \frac{1}{2} r^1 + \mathcal{A}_I r^1 - r^1\mathcal{A}_{I+1}\right) - \frac{2i}{k}(r^2\bar{r}_2 r^1 - r^1\bar{r}_2 r^2)\,, \\ Q^2 r^2 &= \Pi\left(i\partial_\varphi r^2 + \frac{1}{2} r^2 + \mathcal{A}_I r^2 - r^2\mathcal{A}_{I+1}\right). \end{aligned} \tag{4.19}$$

Likewise, the anti-chiral components have double variations given by

$$\begin{aligned} Q^2 \bar{r}_1 &= \Pi\left(i\partial_\varphi \bar{r}_1 + \frac{1}{2}\bar{r}_1 + \mathcal{A}_{I+1}\bar{r}_1 - \bar{r}_1\mathcal{A}_I\right) - \frac{2i}{k}(\bar{r}_2 r^2\bar{r}_1 - \bar{r}_1 r^2\bar{r}_2)\,, \\ Q^2 \bar{r}_2 &= \Pi\left(i\partial_\varphi \bar{r}_2 - \frac{1}{2}\bar{r}_2 + \mathcal{A}_{I+1}\bar{r}_2 - \bar{r}_2\mathcal{A}_I\right). \end{aligned} \tag{4.20}$$

It is now straightforward to check that, when $\Pi \neq 0$, the off-diagonal entries in $\mathcal{L}_{1/2}$ are equal to $-i\Pi^{-1}QH$, with

$$H = \begin{pmatrix} 0 & \bar{\alpha} r^2 \\ \alpha\bar{r}_2 & 0 \end{pmatrix}. \tag{4.21}$$

One can combine this with the results above to find that the supersymmetry variation of the 1/2 BPS connection is

$$Q\mathcal{L}_{1/2} = \mathcal{D}_\varphi^{\mathcal{L}_{1/2}} H\,. \tag{4.22}$$

The covariant derivative used here includes a commutator with the diagonal part of $\mathcal{L}_{1/2}$ and an anticommutator with the off-diagonal part, as explained in detail in Appendix 4.B.

For the purpose of this calculation it was not needed to evaluate the action of Q^2 on r^1, but only on r^2. The former is included here as it is of relevance for the rest of the paper. Also, if one wanted to repeat the calculation for the other 1/2 BPS loop in (4.9), one would need to replace r^2 and $\bar{r}_2$ in H in (4.21) with r^1 and $\bar{r}_1$.

4.3 Two-node hyperloops

Here we systematically study continuous deformations of the $\mathcal{L}_{1/2}$ in (4.3) preserving the supercharge Q defined in (4.6). Again, the strategy is not to find a superconnection which is strictly annihilated by Q, but that rather transforms as a total covariant derivative, precisely as $\mathcal{L}_{1/2}$ in (4.22) above. For the moment, we focus on the case in which the hyperloop couples to only two nodes of the quiver of the theory, but in the next section we generalize this to longer quivers.[14]

Following [59], we take a deformation of the form

$$\mathcal{L} = \mathcal{L}_{1/2} + F + B + C\,, \tag{4.23}$$

where F is off-diagonal and Grassmann-odd, B is a diagonal bilinear of the scalar fields and C is annihilated by Q. This is the most general form consistent with the gauge group representations, the supermatrix structure and with all dimensions being equal to one. BPS non-conformal loops with higher dimension insertions are also possible, but are not considered here.

The condition $QC = 0$ distinguishes two cases: when the supercharge annhilates some of the matter fields and when it does not. Nontrivial solutions include any BPS bosonic loop where the supersymmetry variation should be simply zero, rather than a total derivative. We exclude that case at the moment, because for a compact gauge group the coefficient of the gauge field in the Wilson loop is the identity (or more precisely i). As the gauge field already appears in the appropriate form in $\mathcal{L}_{1/2}$, we should not allow for extra gauge field terms in C. An exception to this would arise if B also has gauge fields, a possibility discussed in Appendix 4.C.

The other possibility is that Q annihilates fields from the hypermultiplet. Note that the action of Q on the scalars in (4.14) is always proportional to the bilinear of the parameters $\eta^\imath_a$ and $\bar{\eta}^\imath_a$ that we called Π. When Π is identically zero, we see that Q has a nontrivial kernel (in this case $r^1 \propto r^2$, so they do not form a basis of the scalar fields). One has therefore to distinguish the cases when Π does not vanish (or has isolated zeros) and the case when $\Pi = 0$, studied later in Section 4.3.2.

4.3.1 Deformations with $\Pi \neq 0$

Starting from the ansatz (4.23), we want to determine the most general F, B and C giving BPS loops, under the assumption that $\Pi \neq 0$.

The simplest term to address is C. The only solutions to $QC = 0$ which is at most bilinear in the fields and excluding the gauge field is $C = \mathrm{diag}(c_I, c_{I+1})$, a numerical matrix not containing the fields. Note that we set the radius R of S^3 to 1, otherwise this should

[14]The representation of the hyperloops in terms of quiver diagrams, which may include some or all of the nodes and edges of the original quiver defining the gauge theory, is explained in detail in [59, 69].

scale with $1/R$ on dimensional grounds. The term proportional to the identity is completely trivial, so we remove it and take $C = \mathrm{diag}(0, c)$.

Moving on to F, in order for QF to involve a derivative in the φ direction, F is restricted to have the fermions in (4.14). Therefore, if $\Pi \neq 0$, one can take

$$F = -iQG\,, \qquad G = \begin{pmatrix} 0 & \bar{b}_a r^a \\ b^a \bar{r}_a & 0 \end{pmatrix}. \tag{4.24}$$

Here the b^a, $\bar{b}_a$ parameters may be functions of φ.

In terms of G, we can combine (4.19) and (4.20) into

$$-iQ^2 G = \partial_\varphi(\Pi G) - i[\mathcal{L}^B_{1/2}, \Pi G] + i[H^2, G] - \Pi\hat{\mathcal{G}}\,, \tag{4.25}$$

with the remainder

$$\Pi\hat{\mathcal{G}} = \begin{pmatrix} 0 & \partial_\varphi(\Pi\bar{b}_a)r^a - i\Pi\bar{b}_1 r^1 \\ \partial_\varphi(\Pi b^a)\bar{r}_a + i\Pi b^1 \bar{r}_1 & 0 \end{pmatrix}. \tag{4.26}$$

To evaluate the supersymmetry variation, it is sometimes useful to split the connection into the diagonal (bosonic) and off-diagonal (fermionic) part: $\mathcal{L} = \mathcal{L}^B + \mathcal{L}^F$, and likewise for $\mathcal{L}_{1/2}$. One can then write

$$\begin{aligned} Q\mathcal{L} &= Q\mathcal{L}_{1/2} - iQ^2 G + QB \\ &= \mathcal{D}^{\mathcal{L}_{1/2}} H + \partial_\varphi(\Pi G) - i[\mathcal{L}^B_{1/2}, \Pi G] + i[H^2, G] + QB - \Pi\hat{\mathcal{G}} \\ &= \mathcal{D}^{\mathcal{L}_{1/2}}(H + \Pi G) - i\{\mathcal{L}^F_{1/2}, \Pi G\} + i[H^2, G] + QB - \Pi\hat{\mathcal{G}} \\ &= \mathcal{D}^{\mathcal{L}}(H + \Pi G) + i[B, H + \Pi G] + i[C, H + \Pi G] + i[H^2, G] - \Pi\hat{\mathcal{G}} \\ &\quad - \{QG, H + \Pi G\} - \{QH, G\} + QB\,. \end{aligned} \tag{4.27}$$

The terms on the last line are all diagonal and vanish by setting $B = \{G, H\} + \Pi G^2$. With this form for B, also the second and fourth terms on the previous line (which are cubic in the scalar fields) vanish. Another way to write these equations is in terms of the variations of the extra terms in $\mathcal{L}$ in (4.23)

$$\begin{aligned} QB &= i\{F, H\} + \{\mathcal{L}^F_{1/2} + F, \Delta H\}\,, \\ QF &= \partial_\varphi \Delta H - i[\mathcal{L}^B_{1/2}, \Delta H] - i[B + C, H + \Delta H]\,. \end{aligned} \tag{4.28}$$

We see that this is indeed satisfied with $\Delta H = \Pi G$.

The deformed connection can then be written as

$$\mathcal{L} = \mathcal{L}_{1/2} - iQG + \{G, H\} + \Pi G^2 + C\,, \tag{4.29}$$

and it is a total derivative if we further impose that the remainders in the last equality of (4.27) cancel

$$i[C, H + \Pi G] - \Pi\hat{\mathcal{G}} = 0\,. \tag{4.30}$$

These are four differential equations for b^a and $\bar{b}_a$

$$\begin{aligned}
\partial_\varphi(\Pi b^1) - i(c-1)\Pi b^1 &= 0\,, \\
\partial_\varphi(\Pi b^2) - ic(\alpha + \Pi b^2) &= 0\,, \\
\partial_\varphi(\Pi \bar{b}_1) + i(c-1)\Pi \bar{b}_1 &= 0\,, \\
\partial_\varphi(\Pi \bar{b}_2) + ic(\bar{\alpha} + \Pi \bar{b}_2) &= 0\,.
\end{aligned} \tag{4.31}$$

Taking $\hat{c}(\varphi)$ to be the primitive of c, the general solution can be written as

$$\Pi b^1 = e^{-i\varphi + i\hat{c}}\beta^1\,, \quad \Pi b^2 = e^{i\hat{c}}\beta^2 - \alpha\,, \quad \Pi \bar{b}_1 = e^{i\varphi - i\hat{c}}\bar{\beta}_1\,, \quad \Pi \bar{b}_2 = e^{-i\hat{c}}\bar{\beta}_2 - \bar{\alpha}\,, \tag{4.32}$$

with constant β^1, β^2, $\bar{\beta}_1$, $\bar{\beta}_2$.

There is a lot of freedom in choosing c. It can in principle be an arbitrary function of φ, but this is a gauge symmetry, which is absorbed in $\mathcal{A}_{I+1}$. We can always fix to the same gauge as in (4.3) by setting $c = 0$. Note that in generic gauges, when $\hat{c}$ is not periodic, the parameters b and $\bar{b}$ are also not periodic (as it was in the original paper [110]).

In the gauge $c = 0$, the deformed connection (4.29) is

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + M_a{}^b r^a \bar{r}_b - \frac{i}{k}(\tilde{\mu}_I{}^{\dot 1}{}_{\dot 1} - \tilde{\mu}_I{}^{\dot 2}{}_{\dot 2}) & -i\bar{\beta}_2\psi_{I\dot{1}-} + ie^{i\varphi}\bar{\beta}_1\psi_{I\dot{2}+} \\ i\beta^2\bar{\psi}^{\dot 1}_{I+} - ie^{-i\varphi}\beta^1\bar{\psi}^{\dot 2}_{I-} & A_{\varphi,I+1} + M_a{}^b \bar{r}_b r^a - \frac{i}{k}(\tilde{\mu}_{I+1\,\dot 1}{}^{\dot 1} - \tilde{\mu}_{I+1\,\dot 2}{}^{\dot 2}) - \frac{1}{2} \end{pmatrix}, \tag{4.33}$$

where

$$M = \Pi^{-1}\begin{pmatrix} \bar{\beta}_1\beta^1 + \frac{i}{k} & e^{i\varphi}\bar{\beta}_1\beta^2 \\ e^{-i\varphi}\bar{\beta}_2\beta^1 & \bar{\beta}_2\beta^2 - \frac{i}{k} \end{pmatrix}. \tag{4.34}$$

After fixing a supercharge Q, the possible space of hyperloops it generates can be represented by the matrix M in (4.34). It is given by 4 complex parameters β^a and $\bar{\beta}_a$, modded out by a C^* action, which is the conifold. This is the same type of moduli space found in [58, 59].

Note that the effect of the shift of β^2 and $\bar{\beta}_2$ by α and $\bar{\alpha}$ in (4.32) means that the "origin of β space", which is the tip of the conifold, is a bosonic loop. We can thus view all the hyperloops that we find here as deformations around some bosonic loop by some supercharge that it preserves. This is similar to the structure in [59], but here we have far more general bosonic loops (see Section 4.5.1) and choose any of the supercharges that they preserve.

Specific examples of hyperloops of this type are presented in Section 4.5.2. Their symmetry algebras are also studied there, as well as a closer inspection of the connection between them and the hyperloops of [59].

The condition $\Pi \neq 0$ from an algebraic point of view

The conditions on Π being zero or not can be interpreted from an algebraic point of view. To do that, let us start by looking at the square of the supercharge (4.6), which using (4.150) reads

$$Q^2 = -\Pi_- J_- - \Pi_0 J_0 + \Pi_+ J_+ - \lambda L_\perp - \frac{1}{2}\lambda_{ab} R^{ab}\,, \tag{4.35}$$

with $\Pi_\pm$ and Π_0 the Fourier coefficients of Π, defined through

$$\Pi \equiv \Pi_- e^{-i\varphi} + \Pi_0 + \Pi_+ e^{+i\varphi}\,, \tag{4.36}$$

and

$$\lambda_{ab} \equiv \epsilon_{\imath\jmath}\bar{\eta}^{\imath}_a \eta^{\jmath}_b\,, \qquad \lambda \equiv \epsilon^{ab}\lambda_{ab}\,. \tag{4.37}$$

As mentioned in Section 4.2, J_0 and $J_\pm$ are the generators of the conformal group along the circle, R^{ab} are $\mathfrak{su}(2)_L$ generators, and $L_\perp$ is a combination of rotation orthogonal to the circle and the unbroken $\mathfrak{u}(1)_R$ (see Appendix 4.A for further details).

As manifest from (4.35), Q^2 generically generates $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{L_\perp}$, which is the algebra preserved by the 1/2 BPS Wilson loop. When $\Pi \neq 0$ the conformal generators are part of this preserved algebra (at least in part). It is now possible to consider subcases of the condition $\Pi \neq 0$ in which one progressively decouples some of the generators on the right hand side of (4.35). This imposes conditions on the parameters η and $\bar{\eta}$, which we derive below and which are going to be useful in Section 4.5, where we construct specific examples.

We start by considering cases in which the $\mathfrak{su}(2)_L$ is "turned off". In order for the contribution of R^{ab} to Q^2 to vanish, one must require that λ_{ab} in (4.37) be antisymmetric. This implies that

$$\lambda_{11} = \epsilon_{\imath\jmath}\bar{\eta}^{\imath}_1 \eta^{\jmath}_1 = 0\,, \tag{4.38}$$

which allows to deduce $\bar{\eta}^{\imath}_1 \propto \eta^{\imath}_1$, and similarly for λ_{22} and $\bar{\eta}^{\imath}_2, \eta^{\imath}_2$. We may then factorize these parameter in terms of some other quantities carrying a single index, as follows (bars do not indicate complex conjugation, as usual)

$$\begin{aligned} \bar{\eta}^{\imath}_1 &= \bar{w}_1 s^{\imath}\,, \qquad & \eta^{\imath}_1 &= w_1 s^{\imath}\,, \\ \bar{\eta}^{\imath}_2 &= \bar{w}_2 t^{\imath}\,, \qquad & \eta^{\imath}_2 &= w_2 t^{\imath}. \end{aligned} \tag{4.39}$$

It remains to impose

$$\lambda_{12} + \lambda_{21} = (\epsilon_{\imath\jmath} s^{\imath} t^{\jmath})(\epsilon^{ab}\bar{w}_a w_b) = 0\,, \tag{4.40}$$

which can be achieved by setting either $s^{\imath} \propto t^{\imath}$ or $\bar{w}_a \propto w_a$. As a consequence, the remaining parameters that determine Q^2 are given by

$$\lambda = (\epsilon_{\imath\jmath} s^{\imath} t^{\jmath})(\bar{w}_1 w_2 + \bar{w}_2 w_1)\,, \qquad \Pi = \left(s^l e^{+i\varphi/2} + s^r e^{-i\varphi/2}\right)^2 \epsilon^{ab}\bar{w}_a w_b\,. \tag{4.41}$$

In order to avoid that $\Pi = 0$, we must ensure $\epsilon^{ab}\bar{w}_a w_b \neq 0$, which implies $\epsilon_{\imath\jmath} s^{\imath} t^{\jmath} = 0$. In particular, the contribution of $L_\perp$ vanishes automatically. In other words, $Q^2 \in \mathfrak{so}(2,1)$.

More restrictive cases can be easily constructed by considering special choices of s^l, s^r. In particular, setting $s^r = 0$ gives $Q^2 \propto J_+$ and similarly $s^l = 0$ gives $Q^2 \propto J_-$.

Next, one could maintain the $\mathfrak{su}(2)_L$ and set instead individual Fourier coefficients of Π to zero, looking, for example, to the case $Q^2 \in \mathfrak{u}(1)_{J_0} \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{L_\perp}$. The contributions of $J_\pm$ to Q^2 vanish if and only if

$$\epsilon^{ab}\eta^l_a\bar{\eta}^l_b = 0\,, \qquad \epsilon^{ab}\eta^r_a\bar{\eta}^r_b = 0\,, \tag{4.42}$$

namely if the η's are linearly dependent

$$\begin{aligned} \eta^l_a &= t^l w_a\,, \qquad & \eta^r_a &= t^r z_a\,, \\ \bar{\eta}^l_a &= \bar{t}^l w_a\,, \qquad & \bar{\eta}^r_a &= \bar{t}^r z_a\,. \end{aligned} \tag{4.43}$$

Without loss of generality, one can take z, w to be normalized, finding the corresponding parameters

$$\begin{aligned} \Pi &= (\epsilon^{ab} z_a w_b)\,(\epsilon_{\imath\jmath}\bar{t}^\imath t^\jmath)\,, \\ \lambda_{ab} &= (\epsilon_{\imath\jmath}\bar{t}^\imath t^\jmath) z_{(a} w_{b)} + \frac{1}{2}\bar{t}^\imath \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\imath\jmath} t^\jmath (\epsilon^{cd} z_c w_d)\epsilon_{ab}\,. \end{aligned} \tag{4.44}$$

One could go on and, for example, turn off Π_- and Π_0 by imposing

$$0 = \epsilon^{ab}\bar{\eta}^r_a\eta^r_b\,, \qquad 0 = \epsilon^{ab}\bar{\eta}^r_a\eta^l_b + \epsilon^{ab}\bar{\eta}^l_a\eta^r_b\,, \tag{4.45}$$

which is achieved by taking

$$\eta^r_a = s w_a, \qquad \bar{\eta}^r_a = \bar{s} w_a\,, \tag{4.46}$$

and yields

$$\lambda_{ab} = \bar{s}\left(\eta^l_a w_b - w_a \eta^l_b\right) - t w_a w_b\,. \tag{4.47}$$

The specific cases considered above do not form an exhaustive classification of supercharges with $\Pi \neq 0$, but have been selected because they are of interest in the study of some loops, like the bosonic loops in Section 4.5.1. Supercharges whose squares are a linear combination of both $\mathfrak{su}(2)_L$ and conformal generators can nonetheless be easily constructed.

4.3.2 Deformations with $\Pi = 0$

The analysis above gives Wilson loops rather similar to those already studied in [59] (though far more general). As seen, it requires that the function Π be non-zero. Now we turn to look at the interesting case when

$$\Pi = (\bar{\eta}v)_1(\eta\bar{v})_2 - (\bar{\eta}v)_2(\eta\bar{v})_1 = 0\,, \tag{4.48}$$

and define

$$\xi = \frac{(\eta\bar{v})_1}{(\bar{\eta}v)_1} = \frac{(\eta\bar{v})_2}{(\bar{\eta}v)_2}\,, \tag{4.49}$$

thus $\xi(\varphi) \in \mathbb{C} \cup \{\infty\}$.

This case is subtle because the supercharge Q in (4.6) annihilates the rotated scalars (4.14) and, furthermore, the pairs of rotated fields are not linearly independent

$$r^1 = \xi r^2 \,, \qquad \bar{r}_2 = -\xi \bar{r}_1 \,. \tag{4.50}$$

For convenience we define (assuming $(\bar{\eta}v)_1 \neq 0$)

$$r^{\parallel} = r^2 \,, \qquad \bar{r}_{\parallel} = -\bar{r}_1 \,, \tag{4.51}$$

and an orthogonal pair which are not annihilated by Q

$$r^{\perp} = (\bar{\eta}v)_2 q^1 - (\bar{\eta}v)_1 q^2 \,, \qquad \bar{r}_{\perp} = (\bar{\eta}v)_1 \bar{q}_1 + (\bar{\eta}v)_2 \bar{q}_2 \,. \tag{4.52}$$

We then find that

$$\begin{aligned}
Qr^{\perp} &= \Lambda(\xi\psi_{\dot{1}-} + \psi_{\dot{2}+}) \,, \qquad Q\bar{r}_{\perp} = -\Lambda(\bar{\psi}^{\dot{1}}_{+} - \xi\bar{\psi}^{\dot{2}}_{-}) \,, \\
Q^2 r^{\perp} &= -\Lambda\left((i\partial_\varphi \xi - \xi) r^{\parallel} - \frac{2i}{k}\xi(\nu_I r^{\parallel} - r^{\parallel}\nu_{I+1})\right) , \\
Q^2 \bar{r}_{\perp} &= \Lambda\left((i\partial_\varphi \xi - \xi)\bar{r}_{\parallel} + \frac{2i}{k}\xi(\nu_{I+1}\bar{r}_{\parallel} - \bar{r}_{\parallel}\nu_I)\right) ,
\end{aligned} \tag{4.53}$$

where

$$\Lambda \equiv (\bar{\eta}v)_1^2 + (\bar{\eta}v)_2^2 \,, \tag{4.54}$$

and similarly to (4.18)

$$r^{\parallel}\bar{r}_{\perp} + r^{\perp}\bar{r}_{\parallel} = \Lambda\nu_I \,, \qquad \bar{r}_{\perp}r^{\parallel} + \bar{r}_{\parallel}r^{\perp} = \Lambda\nu_{I+1} \,. \tag{4.55}$$

We can apply now the same formalism as in the $\Pi \neq 0$ case and take

$$\mathcal{L} = \mathcal{L}_{1/2} - iQG + \{G, H\} + C \,, \qquad QC = 0 \,. \tag{4.56}$$

H is the same as above, see (4.21), which in the new notations becomes

$$H = \begin{pmatrix} 0 & \bar{\alpha} r^{\parallel} \\ \alpha\xi\bar{r}_{\parallel} & 0 \end{pmatrix} . \tag{4.57}$$

In G we include only $r^{\perp}$ and $\bar{r}_{\perp}$ and C may contain scalar bilinears as well as the numerical factors discussed before

$$G = \begin{pmatrix} 0 & \bar{\beta}_{\perp} r^{\perp} \\ \beta^{\perp}\bar{r}_{\perp} & 0 \end{pmatrix} , \qquad C = \begin{pmatrix} \bar{\beta}_{\parallel} r^{\parallel}\bar{r}_{\parallel} & 0 \\ 0 & \beta^{\parallel}\bar{r}_{\parallel} r^{\parallel} + c \end{pmatrix} . \tag{4.58}$$

QG gives a single linear combination of the fermions $\xi\psi_{\dot{1}-}+\psi_{\dot{2}+}$. In Appendix 4.C we explore the possibility of adding another combinations of the fermions, but find that this can only be done in the case of $\xi = 0$, presented in Section 4.3.2 below.

Going back to the deformation (4.48), one can get $Q\mathcal{L} = \mathcal{D}_\varphi^{\mathcal{L}} H$, provided that

$$Q^2 G = [G, H^2] + [C, H] \,. \tag{4.59}$$

Unlike the $\Pi \neq 0$ case, here H remains the same regardless of the deformation.

Besides, one can check that the cubic terms inside Q^2G cancel $[G, H^2] + [C, H]$ provided $\beta^\parallel = \bar{\beta}_\parallel$. The remaining equations for the terms linear in the scalars are

$$\Lambda\bar{\beta}_\perp \partial_\varphi(e^{i\varphi}\xi) = -ie^{i\varphi}c\bar{\alpha} \,, \qquad \Lambda\beta^\perp \partial_\varphi(e^{i\varphi}\xi) = -ie^{i\varphi}\xi c\alpha \,, \tag{4.60}$$

which are simple algebraic relations on $\beta^\perp$, $\bar{\beta}_\perp$ and c.

In the generic case, we can have

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + M_a{}^b r^a \bar{r}_b - \frac{i}{k}(\tilde{\mu}_I{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}) & -i(\bar{\alpha} + \xi\Lambda\bar{\beta}_\perp)\psi_{\dot{1}-} - i\Lambda\bar{\beta}_\perp\psi_{\dot{2}+} \\ i(\alpha + \Lambda\beta^\perp)\bar{\psi}_+^{\dot{1}} - i\xi\Lambda\beta^\perp\bar{\psi}_-^{\dot{2}} & A_{\varphi,I+1} + M_a{}^b \bar{r}_b r^a - \frac{i}{k}(\tilde{\mu}_{I+1\dot{1}}{}^{\dot{1}} - \tilde{\mu}_{I+1\dot{2}}{}^{\dot{2}}) + c - \frac{1}{2} \end{pmatrix}, \tag{4.61}$$

where

$$M_a{}^b = \begin{pmatrix} 0 & \frac{i}{k\Lambda} + \xi\alpha\bar{\beta}_\perp \\ \frac{i}{k\Lambda} + \bar{\alpha}\beta^\perp & \beta^\parallel \end{pmatrix}, \tag{4.62}$$

with $a, b = \perp, \parallel$. Plugging in the solutions of (4.60), the resulting loops generically preserve only one supercharge. However, at some special points we find supersymmetry enhancement. In fact, we find some very interesting subclasses of those loops, which are analyzed in detail in Section 4.5.3.

The special cases: $\xi = 0$ and $\xi = \infty$

Two further degenerations of the $\Pi = 0$ supercharges are when ξ in (4.49) vanishes or is infinite. Both cases are equivalent under the replacement of η with $\bar{\eta}$ (or $Q_{\imath}^{\dot{2}a+}$ and $Q_{\bar{\imath}}^{\dot{1}a-}$) and for simplicity we focus on $\xi = 0$. This means that the supercharge Q is comprised of only the four supercharges $Q_{\bar{\imath}}^{\dot{1}a-}$ and is nilpotent $Q^2 = 0$.

In all cases when $\Pi = 0$, we have two scalar fields $r^\parallel$ and $\bar{r}_\parallel$ in (4.51) that are annihilated by Q. For $\xi = 0$, as can be seen from (4.53), there are also two fermionic field in the hypermultiplet annihilated by Q. Those are $\psi_{\dot{2}+}$ and $\bar{\psi}_+^{\dot{1}}$ and we can therefore insert any distribution of these fields in the hyperloop while still preserving supersymmetry.

As the bottom left entry in $\mathcal{L}_{1/2}$ is comprised of $\bar{\psi}_+^{\dot{1}}$, see (4.3), the matrix H appearing in the variation of $\mathcal{L}_{1/2}$ is upper-triangular, as can indeed be read off from (4.57). To construct the deformed loops we take G as in (4.58) and add the extra fermionic fields to C.

Alternatively, they can also be added as extra terms into F beyond QG

$$G = \begin{pmatrix} 0 & \bar{\beta}_\perp r^\perp \\ \beta^\perp \bar{r}_\perp & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \bar{\beta}_\| r^\| \bar{r}_\| & \bar{\delta}\psi_{\dot{2}+} \\ \delta \bar{\psi}^{\dot{1}}_+ & \beta^\| \bar{r}_\| r^\| + c \end{pmatrix}. \tag{4.63}$$

Plugging G and C into $Q\mathcal{L} = \mathcal{D}^{\mathcal{L}}_\varphi H$, one gets again the same condition that appeared in (4.59), which can be solved by

$$\delta = c = 0\,, \qquad \bar{\beta}_\| = \beta^\|\,. \tag{4.64}$$

This gives the superconnection

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + M_a{}^b r^a \bar{r}_b - \frac{i}{k}(\tilde{\mu}_I{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}) & -i\bar{\alpha}\psi_{\dot{1}-} + (\bar{\delta} - i\Lambda\bar{\beta}_\perp)\psi_{\dot{2}+} \\ i(\alpha + \Lambda\beta^\perp)\bar{\psi}^{\dot{1}}_+ & A_{\varphi,I+1} + M_a{}^b \bar{r}_b r^a - \frac{i}{k}(\tilde{\mu}_{I+1}{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_{I+1}{}^{\dot{2}}{}_{\dot{2}}) - \frac{1}{2} \end{pmatrix}, \tag{4.65}$$

where $M_a{}^b$ is the same as (4.62) with $\xi = 0$. Note that $\bar{\delta}$ and $\bar{\beta}_\perp$ appear only as the combination $\bar{\delta} - i\Lambda\bar{\beta}_\perp$, so we can eliminate any one of them.

The same answer is found from a different approach in Appendix 4.C, where extra fermionic fields are added in F.

The condition $\Pi = 0$ from an algebraic point of view

As done for $\Pi \neq 0$ in Section 4.3.1, one can consider the condition $\Pi = 0$ from an algebraic point of view. Here we give a complete classification of all possible subcases. From the discussion around (4.42), with $Q^2 \in \mathfrak{u}(1)_{J_0} \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{L_\perp}$, the conditions on $\bar{\eta}^\imath_a, \eta^\imath_a$ for Π to vanish are easily derived, since one only needs to enforce $\Pi_0 = 0$, so that $Q^2 \in \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{L_\perp}$. By (4.44), there are two possibilities: either $\epsilon^{ab} z_a w_b = 0$ which implies $\lambda_{ab} = \lambda_{ba}$ and $Q^2 \in \mathfrak{su}(2)_L$, or $\epsilon_{\imath\jmath}\bar{t}^\imath t^\jmath = 0$, which implies $\lambda_{ab} = -\lambda_{ba}$ and $Q^2 \in \mathfrak{u}(1)_{L_\perp}$.

In the former case, $Q^2 \in \mathfrak{su}(2)_L$, one can let $z_a = w_a$ without loss of generality, leading to

$$Q^2 \propto w_a w_b R^{ab}\,. \tag{4.66}$$

The functions ξ and Λ are given by

$$\xi = \frac{t^l + e^{-i\varphi} t^r}{e^{+i\varphi}\bar{t}^l + \bar{t}^r}\,, \qquad \Lambda = (e^{+i\varphi}\bar{t}^l + \bar{t}^r)^2\,. \tag{4.67}$$

In the case $Q^2 \in \mathfrak{u}(1)_{L_\perp}$, one may write instead $t^\imath = t\, s^\imath, \bar{t}^\imath = \bar{t}\, s^\imath$, leading to

$$Q^2 \propto \epsilon^{ab} z_a w_b\, L_\perp\,, \tag{4.68}$$

as well as to

$$\xi = \frac{t e^{-i\varphi}}{\bar{t}}\,, \qquad \Lambda = \bar{t}^2 e^{i\varphi}\Big(e^{i\varphi}(s^l)^2 + e^{-i\varphi}(s^r)^2 + 2 s^l s^r (\epsilon^{ab} z_a w_b)\Big)\,. \tag{4.69}$$

Finally, when both of the conditions above are met the supercharge becomes nilpotent, $Q^2 = 0$. The parameters are of the form

$$\eta_a^\imath = a\rho^\imath w_a \,, \qquad \bar{\eta}_a^\imath = \bar{a}\rho^\imath w_a \,. \tag{4.70}$$

This factorisation is expected since each term in (4.35) antisymmetrises over either $\imath, \jmath$ or a, b (or both). The functions ξ and Λ take the simple form

$$\xi = \frac{ae^{-i\varphi}}{\bar{a}} \,, \qquad \Lambda = \bar{a}^2(e^{i\varphi}\rho^l + \rho^r)^2 \,. \tag{4.71}$$

Note that the function ξ provides a handy way of distinguishing these cases. Concretely, $\partial_\varphi\left(e^{i\varphi}\xi\right) = 0$ if and only if $Q^2 \in \mathfrak{u}(1)_{L_\perp}$. ξ vanishes identically if and only if Q is composed entirely of barred supercharges.

4.4 Longer quivers and twisted hypers

All the constructions in Section 4.3 involve only two nodes of the quiver. Here we turn to hyperloops coupling to more nodes. As a guiding example and starting point of the deformation, we consider the 1/2 BPS loop on two pairs of nodes, with undeformed superconnection given by

$$\mathcal{L}_{1/2} = \begin{pmatrix} \mathcal{A}_I & -i\bar{\alpha}_I\psi_{I,\mathrm{i}-} & 0 & 0 \\ i\alpha_I\bar{\psi}^{\mathrm{i}}_{I,+} & \mathcal{A}_{I+1} - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{I+2} - c & -i\bar{\alpha}_{I+2}\psi_{I+2,\mathrm{i}-} \\ 0 & 0 & i\alpha_{I+2}\bar{\psi}^{\mathrm{i}}_{I+2,+} & \mathcal{A}_{I+3} - c - \frac{1}{2} \end{pmatrix} . \tag{4.72}$$

We introduce a constant shift c between the two pairs of nodes representing the effect of a $U(N_{I+1})$ gauge freedom. In this block-diagonal form, there is no restriction on c. The resulting Wilson loop is well defined with constant α_{I+2} and $\bar{\alpha}_{I+2}$ satisfying $\alpha_{I+2}\bar{\alpha}_{I+2} = 2i/k$. We find (the supertrace sums lines with signs $+, -, +, -$)

$$W = \mathrm{sTr}\,\mathcal{P}\exp i\oint \mathcal{L}\,d\varphi = W_{(I,I+1)} + \exp\left(-i\oint c\,d\varphi\right) W_{(I+2,I+3)} \,. \tag{4.73}$$

Clearly with this block-diagonal structure, we can take any linear combination of the two Wilson loops. Adding deformations by the hypermultiplets keeps the block-diagonal structure, so again it works with any c. As already noted in [59], deformations by twisted hypermultiplets with $\tilde{q}^{\dot{a}}_{I+1}$ are more subtle and fix c.

The Wilson loop based on (4.72) still satisfies $Q\mathcal{L}_{1/2} = \mathcal{D}_\varphi^{\mathcal{L}_{1/2}} H$, this time with

$$H = \begin{pmatrix} 0 & \bar{\alpha}_I r_I^2 & 0 & 0 \\ \alpha_I \bar{r}_{I\,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\alpha}_{I+2} r_{I+2}^2 \\ 0 & 0 & \alpha_{I+2}\bar{r}_{I+2\,2} & 0 \end{pmatrix} . \tag{4.74}$$

It is now natural to rotate the fermions from the twisted hypermultiplets

$$\tilde{\rho}^1_- = -(\eta\bar{v})_a\tilde{\psi}^a_- \,, \quad \tilde{\rho}^2_+ = (\bar{\eta}v)_a\tilde{\psi}^a_+ \,, \quad \bar{\tilde{\rho}}_{1+} = \epsilon^{ab}(\bar{\eta}v)_a\bar{\tilde{\psi}}_{b+} \,, \quad \bar{\tilde{\rho}}_{2-} = \epsilon^{ab}(\eta\bar{v})_a\bar{\tilde{\psi}}_{b-} \,, \tag{4.75}$$

such that the supersymmetry transformations are

$$Q\tilde{q}_{\dot{1}} = \tilde{\rho}^2_+ \,, \quad Q\tilde{q}_{\dot{2}} = \tilde{\rho}^1_- \,, \quad Q\bar{\tilde{q}}^{\dot{1}} = \bar{\tilde{\rho}}_{2-} \,, \quad Q\bar{\tilde{q}}^{\dot{2}} = \bar{\tilde{\rho}}_{1+} \,. \tag{4.76}$$

The double variations are then

$$\begin{aligned}
Q^2\tilde{q}_{\dot{1}} &= \Pi\big(i\partial_\varphi\tilde{q}_{\dot{1}} + \mathcal{A}_{I+1}\tilde{q}_{\dot{1}} - \tilde{q}_{\dot{1}}\mathcal{A}_{I+2}\big) - \frac{2i}{k}(\bar{r}_2r^2\tilde{q}_{\dot{1}} - \tilde{q}_{\dot{1}}r^2\bar{r}_2) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}v)_a(\eta\sigma^3\bar{v})_b\tilde{q}_{\dot{1}} \,, \\
Q^2\tilde{q}_{\dot{2}} &= \Pi\big(i\partial_\varphi\tilde{q}_{\dot{2}} + \mathcal{A}_{I+1}\tilde{q}_{\dot{2}} - \tilde{q}_{\dot{2}}\mathcal{A}_{I+2}\big) - \frac{2i}{k}(\bar{r}_2r^2\tilde{q}_{\dot{2}} - \tilde{q}_{\dot{2}}r^2\bar{r}_2) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\sigma^3v)_a(\eta\bar{v})_b\tilde{q}_{\dot{2}} \,, \\
Q^2\bar{\tilde{q}}^{\dot{1}} &= \Pi\big(i\partial_\varphi\bar{\tilde{q}}^{\dot{1}} + \mathcal{A}_{I+2}\bar{\tilde{q}}^{\dot{1}} - \bar{\tilde{q}}^{\dot{1}}\mathcal{A}_{I+1}\big) - \frac{2i}{k}(r^2\bar{r}_2\bar{\tilde{q}}^{\dot{1}} - \bar{\tilde{q}}^{\dot{1}}\bar{r}_2r^2) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\sigma^3v)_a(\eta\bar{v})_b\bar{\tilde{q}}^{\dot{1}} \,, \\
Q^2\bar{\tilde{q}}^{\dot{2}} &= \Pi\big(i\partial_\varphi\bar{\tilde{q}}^{\dot{2}} + \mathcal{A}_{I+2}\bar{\tilde{q}}^{\dot{2}} - \bar{\tilde{q}}^{\dot{2}}\mathcal{A}_{I+1}\big) - \frac{2i}{k}(r^2\bar{r}_2\bar{\tilde{q}}^{\dot{2}} - \bar{\tilde{q}}^{\dot{2}}\bar{r}_2r^2) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}v)_a(\eta\sigma^3v)_b\bar{\tilde{q}}^{\dot{2}} \,.
\end{aligned} \tag{4.77}$$

Using (4.17), the linear terms above can rewritten as

$$\epsilon^{ab}(\bar{\eta}v)_a(\eta\sigma^3\bar{v})_b = -i\partial_\varphi\Pi - \lambda \,, \qquad \epsilon^{ab}(\bar{\eta}\sigma^3v)_a(\eta\bar{v})_b = -i\partial_\varphi\Pi + \lambda \,, \tag{4.78}$$

such that the double variations become

$$\begin{aligned}
Q^2\tilde{q}_{\dot{1}} &= \Pi\left(i\partial_\varphi\tilde{q}_{\dot{1}} - \frac{1}{2}\tilde{q}_{\dot{1}} + \Gamma\tilde{q}_{\dot{1}} + \mathcal{A}_{I+1}\tilde{q}_{\dot{1}} - \tilde{q}_{\dot{1}}\mathcal{A}_{I+2}\right) - \frac{2i}{k}(\bar{r}_2r^2\tilde{q}_{\dot{1}} - \tilde{q}_{\dot{1}}r^2\bar{r}_2) \,, \\
Q^2\tilde{q}_{\dot{2}} &= \Pi\left(i\partial_\varphi\tilde{q}_{\dot{2}} + \frac{1}{2}\tilde{q}_{\dot{2}} + \bar{\Gamma}\tilde{q}_{\dot{2}} + \mathcal{A}_{I+1}\tilde{q}_{\dot{2}} - \tilde{q}_{\dot{2}}\mathcal{A}_{I+2}\right) - \frac{2i}{k}(\bar{r}_2r^2\tilde{q}_{\dot{2}} - \tilde{q}_{\dot{2}}r^2\bar{r}_2) \,, \\
Q^2\bar{\tilde{q}}^{\dot{1}} &= \Pi\left(i\partial_\varphi\bar{\tilde{q}}^{\dot{1}} + \frac{1}{2}\bar{\tilde{q}}^{\dot{1}} + \bar{\Gamma}\bar{\tilde{q}}^{\dot{1}} + \mathcal{A}_{I+2}\bar{\tilde{q}}^{\dot{1}} - \bar{\tilde{q}}^{\dot{1}}\mathcal{A}_{I+1}\right) - \frac{2i}{k}(r^2\bar{r}_2\bar{\tilde{q}}^{\dot{1}} - \bar{\tilde{q}}^{\dot{1}}\bar{r}_2r^2) \,, \\
Q^2\bar{\tilde{q}}^{\dot{2}} &= \Pi\left(i\partial_\varphi\bar{\tilde{q}}^{\dot{2}} - \frac{1}{2}\bar{\tilde{q}}^{\dot{2}} + \Gamma\bar{\tilde{q}}^{\dot{2}} + \mathcal{A}_{I+2}\bar{\tilde{q}}^{\dot{2}} - \bar{\tilde{q}}^{\dot{2}}\mathcal{A}_{I+1}\right) - \frac{2i}{k}(r^2\bar{r}_2\bar{\tilde{q}}^{\dot{2}} - \bar{\tilde{q}}^{\dot{2}}\bar{r}_2r^2) \,,
\end{aligned} \tag{4.79}$$

where for latter convenience we introduce

$$\Gamma = \frac{1}{2}\left(i\partial_\varphi\ln\Pi + \frac{\lambda}{\Pi} + 1\right), \qquad \bar{\Gamma} = \frac{1}{2}\left(i\partial_\varphi\ln\Pi - \frac{\lambda}{\Pi} - 1\right). \tag{4.80}$$

4.4.1 Deformations with $\Pi \neq 0$

We now proceed to deform the loop (4.72) as in (4.23). We take G to be of the form

$$G = \begin{pmatrix} 0 & \bar{b}_{Ia}r^a_I & 0 & 0 \\ b^a_I\bar{r}_{Ia} & 0 & \bar{d}^{\dot{1}}_{I+1}\tilde{q}_{I+1\,\dot{1}} & 0 \\ 0 & d_{I+1\,\dot{1}}\bar{\tilde{q}}^{\dot{1}}_{I+1} & 0 & \bar{b}_{I+2\,a}r^a_{I+2} \\ 0 & 0 & b^a_{I+2}\bar{r}_{I+2\,a} & 0 \end{pmatrix}. \tag{4.81}$$

We allow a coupling to all the scalars in the hypermultiplets, but in the twisted hypers we restrict to $\tilde{q}_{I+1\,\dot{\mathrm{i}}}$ and $\bar{\tilde{q}}^{\dot{\mathrm{i}}}_{I+1}$. The second pair of scalar fields is examined below.

Using (4.79), the analogue of (4.25) adapted for a longer quiver is

$$-iQ^2G = \partial_\varphi(\Pi G) - i[\mathcal{L}^B_{1/2}, \Pi G] + i[H^2, G] - \Pi\hat{\mathcal{G}}\,, \tag{4.82}$$

with

$$\begin{aligned}\Pi\hat{\mathcal{G}} = &\begin{pmatrix} 0 & \partial_\varphi(\Pi\bar{b}_{I\,a})r^a_I & 0 & 0 \\ \partial_\varphi(\Pi b^a_I)\bar{r}_{I\,a} & 0 & \partial_\varphi(\Pi\bar{d}^{\dot{\mathrm{i}}})\tilde{q}_{I+1\,\dot{\mathrm{i}}} & 0 \\ 0 & \partial_\varphi(\Pi d_{\dot{\mathrm{i}}})\bar{\tilde{q}}^{\dot{\mathrm{i}}}_{I+1} & 0 & \partial_\varphi(\Pi\bar{b}_{I+2\,a})r^a_{I+2} \\ 0 & 0 & \partial_\varphi(\Pi b^a_{I+2})\bar{r}_{I+2\,a} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -i\Pi\bar{b}_{I\,1}r^1_I & 0 & 0 \\ i\Pi b^1_I\bar{r}_{I\,1} & 0 & -i\Pi(c-\Gamma)\bar{d}^{\dot{\mathrm{i}}}\tilde{q}_{I+1\,\dot{\mathrm{i}}} & 0 \\ 0 & i\Pi(c+\bar{\Gamma})d_{\dot{\mathrm{i}}}\bar{\tilde{q}}^{\dot{\mathrm{i}}}_{I+1} & 0 & -i\Pi\bar{b}_{I+2\,1}r^1_{I+2} \\ 0 & 0 & i\Pi b^1_{I+2}\bar{r}_{I+2\,1} & 0 \end{pmatrix}.\end{aligned} \tag{4.83}$$

Proceeding as before, the analogue of (4.27) sets $B = \{G, H\} + \Pi G^2$ and supersymmetry invariance of $\mathcal{L}$ now requires solving

$$i[C, H + \Pi G] - \Pi\hat{\mathcal{G}} = 0\,, \qquad C = \mathrm{diag}(c_I, c_{I+1}, c_{I+2}, c_{I+3})\,. \tag{4.84}$$

We recover two copies of the equations (4.31), now for b_I, $\bar{b}_I$, b_{I+2}, and $\bar{b}_{I+2}$. In addition, using $\Gamma + \bar{\Gamma} = i\partial_\varphi \ln\Pi$, we find the two following equations for $d_{I+1\,\dot{\mathrm{i}}}$ and $\bar{d}^{\dot{\mathrm{i}}}_{I+1}$

$$\begin{aligned}\partial_\varphi(\bar{d}^{\dot{\mathrm{i}}}_{I+1}) - i(c_{I+1} - c_{I+2} + c + \bar{\Gamma})\bar{d}^{\dot{\mathrm{i}}}_{I+1} = 0\,, \\ \partial_\varphi(\Pi d_{I+1\,\dot{\mathrm{i}}}) + i(c_{I+1} - c_{I+2} + c + \bar{\Gamma})\Pi d_{I+1\,\dot{\mathrm{i}}} = 0\,.\end{aligned} \tag{4.85}$$

Note that these involve not only the numerical factors arising from C but also the relative shift c that was left arbitrary in (4.72). In particular, we can make use of this gauge freedom to make the convenient choice $c_{I+1} = c_{I+2} = 0$ and then with $c = -\bar{\Gamma}$, the equations above are solved by

$$\bar{d}^{\dot{\mathrm{i}}}_{I+1} = \bar{\delta}^{\dot{\mathrm{i}}}_{I+1}\,, \qquad d_{I+1\,\dot{\mathrm{i}}} = \frac{\delta_{I+1\,\dot{\mathrm{i}}}}{\Pi}\,, \tag{4.86}$$

with constant δ's. Other gauges are possible, but they are completely equivalent to this one.

One can write the explicit expression for $\mathcal{L}$ using (4.29). Two points to note are that in addition to the diagonal bosonic terms and first off-diagonal fermionic terms, there are also off-off-diagonal bosonic terms that contain the bilinears $\tilde{q}_{I+1\,\dot{\mathrm{i}}}r^a_{I+2}$ and $\bar{\tilde{q}}^{\dot{\mathrm{i}}}_{I+1}\bar{r}_{Ia}$. Also, the diagonal terms in the central nodes now include the modification of the bilinears of the scalars in the twisted hypermultiplets via

$$\widetilde{M}^{\dot{a}}{}_{\dot{b}}\tilde{q}_{I+1\,\dot{a}}\bar{\tilde{q}}^{\dot{b}}_{I+1}\,, \qquad \widetilde{M} = \begin{pmatrix} -i/k + \bar{\delta}^{\dot{\mathrm{i}}}_{I+1}\delta_{I+1\,\dot{\mathrm{i}}} & 0 \\ 0 & i/k \end{pmatrix}. \tag{4.87}$$

Instead of writing the full complicated 4×4 form of the general $\mathcal{L}$, we look at some special cases in Section 4.5.4.

To couple $\mathcal{L}$ to $\tilde{q}_{I+1\,\dot{2}}$ and $\bar{\tilde{q}}^{\dot{2}}_{I+1}$, we take instead

$$G=\begin{pmatrix} 0 & \bar{b}_{Ia}r^a_I & 0 & 0\\ b^a_I\bar{r}_{Ia} & 0 & \bar{d}^{\dot{2}}_{I+1}\tilde{q}_{I+1\,\dot{2}} & 0\\ 0 & d_{I+1\,\dot{2}}\bar{\tilde{q}}^{\dot{2}}_{I+1} & 0 & \bar{b}_{I+2\,a}r^a_{I+2}\\ 0 & 0 & b^a_{I+2}\bar{r}_{I+2\,a} & 0\end{pmatrix}, \tag{4.88}$$

then (4.82) holds with

$$\begin{aligned}\Pi\hat{\mathcal{G}}=&\begin{pmatrix} 0 & \partial_\varphi(\Pi\bar{b}_{I\,a})r^a_I & 0 & 0\\ \partial_\varphi(\Pi b^a_I)\bar{r}_{I\,a} & 0 & \partial_\varphi(\Pi\bar{d}^{\dot{2}}_{I+1})\tilde{q}_{I+1\,\dot{2}} & 0\\ 0 & \partial_\varphi(\Pi d_{I+1\,\dot{2}})\bar{\tilde{q}}^{\dot{2}}_{I+1} & 0 & \partial_\varphi(\Pi\bar{b}_{I+2\,a})r^a_{I+2}\\ 0 & 0 & \partial_\varphi(\Pi b^a_{I+2})\bar{r}_{I+2\,a} & 0\end{pmatrix}\\ &+\begin{pmatrix} 0 & -i\Pi\bar{b}_{I\,1}r^1_I & 0 & 0\\ i\Pi b^1_I\bar{r}_{I\,1} & 0 & -i\Pi(c-\bar{\Gamma}-1)\bar{d}^{\dot{2}}_{I+1}\tilde{q}_{I+1\,\dot{2}} & 0\\ 0 & i\Pi(c+\Gamma-1)d_{I+1\,\dot{2}}\bar{\tilde{q}}^{\dot{2}}_{I+1} & 0 & -i\Pi\bar{b}_{I+2\,1}r^1_{I+2}\\ 0 & 0 & i\Pi b^1_{I+2}\bar{r}_{I+2\,1} & 0\end{pmatrix}.\end{aligned} \tag{4.89}$$

This time, (4.84) gives two equations for $d_{I+1\,\dot{2}}$ and $\bar{d}^{\dot{2}}_{I+1}$

$$\begin{aligned}\partial_\varphi(\bar{d}^{\dot{2}}_{I+1})-i(c_{I+1}-c_{I+2}+c+\Gamma-1)\Pi\bar{d}^{\dot{2}}_{I+1}&=0\,,\\ \partial_\varphi(\Pi d_{I+1\,\dot{2}})+i(c_{I+1}-c_{I+2}+c+\Gamma-1)d_{I+1\,\dot{2}}&=0\,.\end{aligned} \tag{4.90}$$

In this case the convenient gauge is $c_{I+1}=c_{I+2}=0$ where these equations are solved with $c=-\Gamma+1$ and

$$\bar{d}^{\dot{2}}_{I+1}=\bar{\delta}^{\dot{2}}_{I+1}\,,\qquad d_{I+1\,\dot{2}}=\frac{\delta_{I+1\,\dot{2}}}{\Pi}, \tag{4.91}$$

with constant δ's. Now $\widetilde{M}$ is given by

$$\widetilde{M}=\begin{pmatrix}-i/k & 0\\ 0 & i/k+\bar{\delta}^{\dot{2}}_{I+1}\delta_{I+1\,\dot{2}}\end{pmatrix}. \tag{4.92}$$

Notice that we performed the analysis separately for the two pairs of scalars in the twisted hypermultiplets and the resulting expressions required different conditions on c, namely $c=-\bar{\Gamma}$ and $c=-\Gamma+1$. To allow $\mathcal{L}$ to couple to all scalars of the twisted hypermultiplet at the same time, these need to be related by a gauge transformation, requiring

$$\hat{c}(\varphi)=-\int_0^\varphi\big(\bar{\Gamma}-\Gamma+1\big)\,d\varphi'=\int_0^\varphi\frac{\lambda}{\Pi}\,d\varphi'\,, \tag{4.93}$$

to be single valued. Thus, if

$$e^{i\hat{c}(2\pi)} = \exp i \oint \frac{\lambda}{\Pi}\, d\varphi = 1\,, \tag{4.94}$$

is satisfied, $\mathcal{L}$ may couple to all twisted scalars, otherwise it may couple either to the pair $\tilde{q}_{\dot{1}}, \bar{\tilde{q}}^{\dot{1}}$ or to $\tilde{q}_{\dot{2}}, \bar{\tilde{q}}^{\dot{2}}$.

To be concrete, if we choose the gauge $c_I = c_{I+1} = c_{I+2} = c_{I+3} = 0$ and $c = -\bar{\Gamma}$, a G including all twisted scalars is then composed from (4.81) and the gauge transformed version of (4.88), giving

$$G = \begin{pmatrix} 0 & \bar{b}_{Ia} r_I^a & 0 & 0 \\ b_I^a \bar{r}_{Ia} & 0 & \bar{d}^{\dot{1}}_{I+1} \tilde{q}_{I+1\,\dot{1}} + e^{i\hat{c}(\varphi)} \bar{d}^{\dot{2}}_{I+1} \tilde{q}_{I+1\,\dot{2}} & 0 \\ 0 & d_{I+1\,\dot{1}} \bar{\tilde{q}}^{\dot{1}}_{I+1} + e^{-i\hat{c}(\varphi)} d_{I+1\,\dot{2}} \bar{\tilde{q}}^{\dot{2}}_{I+1} & 0 & \bar{b}_{I+2\,a} r^a_{I+2} \\ 0 & 0 & b^a_{I+2} \bar{r}_{I+2\,a} & 0 \end{pmatrix}. \tag{4.95}$$

The construction then follows as before. Differential equations for $\bar{b}_{Ia}$, b_I^a, $\bar{b}_{I+2\,a}$, b^a_{I+2} and for $d_{I+1\,\dot{1}}$, $\bar{d}^{\dot{1}}_{I+1}$ are as in (4.31) and (4.85) and are solved by (4.32) and (4.86). As for $d_{I+1\,\dot{2}}, \bar{d}^{\dot{2}}_{I+1}$, we find the equivalent to (4.90) in the $c = -\bar{\Gamma}$ gauge

$$\begin{aligned} \partial_\varphi(\Pi e^{i\hat{c}(\varphi)} \bar{d}^{\dot{2}}_{I+1}) - i\lambda e^{i\hat{c}(\varphi)} \bar{d}^{\dot{2}}_{I+1} = 0\,, \\ \partial_\varphi(\Pi e^{-i\hat{c}(\varphi)} d_{I+1\,\dot{2}}) + i\lambda e^{-i\hat{c}(\varphi)} d_{I+1\,\dot{2}} = 0\,, \end{aligned} \tag{4.96}$$

which is still solved by (4.91).

We found therefore the form of $\mathcal{L}$ coupling to both twisted scalars, under the condition (4.94). Now $\widetilde{M}$ is given by

$$\widetilde{M} = \begin{pmatrix} -i/k + \bar{\delta}^{\dot{1}}_{I+1} \delta_{I+1\,\dot{1}} & e^{-i\hat{c}(\varphi)} \bar{\delta}^{\dot{1}}_{I+1} \delta_{I+1\,\dot{2}} \\ e^{i\hat{c}(\varphi)} \bar{\delta}^{\dot{2}}_{I+1} \delta_{I+1\,\dot{1}} & i/k + \bar{\delta}^{\dot{2}}_{I+1} \delta_{I+1\,\dot{2}} \end{pmatrix}. \tag{4.97}$$

A special case of this construction was already carried out in [59]. In the parameterization of that paper, $\Pi = 1$ and $\lambda = \cos\theta$, with θ the so-called "latitude" angle. It was then possible to include all scalar fields in G for $\theta = 0$ (see equation (4.9) of [59]). The analog of the obstruction (4.94) arose there for $\theta \neq 0$ (see the comment below (5.15) of [59]). The reasoning for that is precisely the fact that $e^{i\hat{c}(\varphi)} = e^{i\varphi\cos\theta}$ considered there is not single valued.

4.4.2 Deformations with $\Pi = 0$

Generalizing Section 4.3.2 to allow for twisted hypers, we start again with the 1/2 BPS loop with four nodes in (4.72). H is the same as in (4.74), now written generalizing (4.57) to

$$H = \begin{pmatrix} 0 & \bar{\alpha}_I r_I^{\parallel} & 0 & 0 \\ -\alpha_I \xi \bar{r}_{I\,\parallel} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\alpha}_{I+2} r_{I+2}^{\parallel} \\ 0 & 0 & -\alpha_{I+2}\xi \bar{r}_{I+2\,\parallel} & 0 \end{pmatrix} . \tag{4.98}$$

As before, the fact that $\Pi = 0$ implies that the variation of the deformed loop is still a covariant derivative of H regardless of the deformation. Since H does not include twisted scalars, we do not expect the relative shift between the two pairs of nodes (c in (4.72)) to be fixed by the requirement that the deformed loop is supersymmetric. Below we see that this is indeed the case.

The $\Pi = 0$ version of the double transformations (4.77) is

$$\begin{aligned} Q^2 \tilde{q}_{\mathrm{1}} &= \frac{2i}{k}\xi(\bar{r}_{\parallel} r^{\parallel} \tilde{q}_{\mathrm{1}} - \tilde{q}_{\mathrm{1}} r^{\parallel} \bar{r}_{\parallel}) + \frac{\lambda}{2}\tilde{q}_{\mathrm{1}} \,, \\ Q^2 \tilde{q}_{\dot{2}} &= \frac{2i}{k}\xi(\bar{r}_{\parallel} r^{\parallel} \tilde{q}_{\dot{2}} - \tilde{q}_{\dot{2}} r^{\parallel} \bar{r}_{\parallel}) - \frac{\lambda}{2}\tilde{q}_{\dot{2}} \,, \\ Q^2 \bar{\tilde{q}}^{\mathrm{1}} &= \frac{2i}{k}\xi(r^{\parallel} \bar{r}_{\parallel} \bar{\tilde{q}}^{\mathrm{1}} - \bar{\tilde{q}}^{\mathrm{1}} \bar{r}_{\parallel} r^{\parallel}) - \frac{\lambda}{2}\bar{\tilde{q}}^{\mathrm{1}} \,, \\ Q^2 \bar{\tilde{q}}^{\dot{2}} &= \frac{2i}{k}\xi(r^{\parallel} \bar{r}_{\parallel} \bar{\tilde{q}}^{\dot{2}} - \bar{\tilde{q}}^{\dot{2}} \bar{r}_{\parallel} r^{\parallel}) + \frac{\lambda}{2}\bar{\tilde{q}}^{\dot{2}} \,. \end{aligned} \tag{4.99}$$

The building blocks are then the 4×4 versions of G and C (we set $c_I = c_{I+2} = 0$ for convenience)

$$\begin{aligned} G &= \begin{pmatrix} 0 & \bar{\beta}_{I\,\perp} r_I^{\perp} & 0 & 0 \\ \beta_I^{\perp} \bar{r}_{I\,\perp} & 0 & \bar{d}^{\dot{a}} \tilde{q}_{I+1\,\dot{a}} & 0 \\ 0 & d_{\dot{a}} \bar{\tilde{q}}_{I+1}^{\dot{a}} & 0 & \bar{\beta}_{I+2\,\perp} r_{I+2}^{\perp} \\ 0 & 0 & \beta_{I+2}^{\perp} \bar{r}_{I+2\perp} & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} \bar{\beta}_{I\,\parallel} r_I^{\parallel} \bar{r}_{I\,\parallel} & 0 & 0 & 0 \\ 0 & \beta_I^{\parallel} \bar{r}_{I\,\parallel} r_I^{\parallel} + c_{I+1} & 0 & 0 \\ 0 & 0 & \bar{\beta}_{I+2\,\parallel} r_{I+2}^{\parallel} \bar{r}_{I+2\,\parallel} & 0 \\ 0 & 0 & 0 & \beta_{I+2}^{\parallel} \bar{r}_{I+2\,\parallel} r_{I+2}^{\parallel} + c_{I+3} \end{pmatrix}. \end{aligned} \tag{4.100}$$

With these in hand, the superconnection $\mathcal{L} = \mathcal{L}_{1/2} - iQG + \{G, H\} + C$ is supersymmetric provided that the same condition as in (4.59) is obeyed.

The analysis for the β parameters follows as in the 2-node case. Cubic terms on the fields cancel for $\bar{\beta}_{I\,\parallel} = \beta_I^{\parallel}$ and $\bar{\beta}_{I+2\,\parallel} = \beta_{I+2}^{\parallel}$. Linear terms are such that we find, in addition to (4.60), its $I+2$-node version

$$\Lambda \bar{\beta}_{I+2\,\perp} \partial_\varphi (e^{i\varphi}\xi) = -ie^{i\varphi} c_{I+3} \bar{\alpha}_{I+2} \,, \qquad \Lambda \beta_{I+2}^{\perp} \partial_\varphi (e^{i\varphi}\xi) = ie^{i\varphi} c_{I+3} \alpha_{I+2} \xi \,. \tag{4.101}$$

For the central block containing the d parameters, one realizes that the cubic term in the double variations (4.99) is exactly equal to $[G, H^2]$. There is no contribution related to d from $[C, H]$, so one is left with the linear terms arising from $Q^2 G$

$$\begin{pmatrix} \ddots & & & \\ & 0 & \frac{\lambda}{2}(\bar{d}^{\dot{1}}\tilde{q}_{\dot{1}} - \bar{d}^{\dot{2}}\tilde{q}_{\dot{2}}) & \\ & -\frac{\lambda}{2}(d_{\dot{1}}\bar{\tilde{q}}^{\dot{1}} - d_{\dot{2}}\bar{\tilde{q}}^{\dot{2}}) & 0 & \\ & & & \ddots \end{pmatrix} = 0\,. \tag{4.102}$$

Solutions with nonvanishing d parameters and a non block-diagonal structure are only possible for supercharges with

$$\lambda = 0\,. \tag{4.103}$$

In this case there are no constraints on $\bar{d}^{\dot{a}}$ and $d_{\dot{a}}$, and they can be arbitrary functions. At the level of the algebra, see Section 4.3.2, this means that loops in this section are constructed from Q's that square only to $\mathfrak{su}(2)_L$ generators R^{ab}.

Note that as anticipated the derivation above does not set restrictions on the relative shift c appearing in (4.72), in contrast to the $\Pi \neq 0$ case. We examine some special cases of the resulting operators in Section 4.5.5.

The special cases: $\xi = 0$ and $\xi = \infty$

The analysis of $\xi = 0$ and $\xi = \infty$ follows in analogy with Section 4.3.2. As before, both cases are equivalent under the replacement of η and $\bar{\eta}$, so we focus only on the $\xi = 0$ case.

Here, since we are considering longer quivers coupling to twisted hypermultiplets, we need to include in G not only $r^{\perp}$, $\bar{r}_{\perp}$ but also the twisted scalars that are not annihilated by Q. From (4.76) we see that these are $\tilde{q}_{\dot{1}}$ and $\bar{\tilde{q}}^{\dot{2}}$, so we have

$$G = \begin{pmatrix} 0 & \bar{\beta}_{I\perp} r_I^{\perp} & 0 & 0 \\ \beta_I^{\perp} & 0 & \bar{d}^{\dot{1}}\tilde{q}_{I+1\,\dot{1}} & 0 \\ 0 & d_{\dot{2}}\bar{\tilde{q}}^{\dot{2}}_{I+1} & 0 & \bar{\beta}_{I+2} r^{\perp}_{I+2} \\ 0 & 0 & \beta^{\perp}_{I+2}\bar{r}_{I+2\perp} & 0 \end{pmatrix}. \tag{4.104}$$

Conversely, the fields $\tilde{q}_{\dot{2}}$ and $\bar{\tilde{q}}^{\dot{1}}$ are annihilated by Q and are included in the matrix C. In addition to them, we should also include $\tilde{\rho}_+^2$ and $\bar{\tilde{\rho}}_{1+}$, which are the linear combination of fermionic fields from the twisted hypermultiplet that are annihilated by Q. Thus, we have (setting again c_I and c_{I+2} to zero for convenience)

$$C = \begin{pmatrix} \bar{K}_I & \bar{\delta}_I \psi_{I\dot{2}+} & \gamma_1 r_I^{\parallel}\tilde{q}_{I+1\,\dot{2}} & 0 \\ \delta_I \bar{\psi}^{\dot{1}}_{I+} & K_I + c_{I+1} & \bar{\delta}_{I+1}\tilde{\rho}^2_{I+1,+} & \gamma_2 \tilde{q}_{I+1\,\dot{2}} r^{\parallel}_{I+2} \\ \gamma_3 \bar{\tilde{q}}^{\dot{1}}_{I+1}\bar{r}_{I\parallel} & \delta_{I+1}\bar{\tilde{\rho}}_{I+1,1+} & \bar{K}_{I+2} & \bar{\delta}_{I+2}\psi_{I+2\,\dot{2}+} \\ 0 & \gamma_4 \bar{r}_{I+2\parallel}\bar{\tilde{q}}^{\dot{1}}_{I+1} & \delta_{I+2}\bar{\psi}^{\dot{1}}_{I+2,+} & K_{I+2} + c_{I+3} \end{pmatrix}, \tag{4.105}$$

with $K_I \equiv \beta_I^{\|} \bar{r}_{I\|} r_I^{\|} + \tau_{I+1} \tilde{q}_{I+1\,\dot{2}} \bar{\tilde{q}}_{I+1}^{\dot{1}}$ and $\bar{K}_I \equiv \bar{\beta}_{I\|} r_I^{\|} \bar{r}_{I\|} + \tau_I \bar{\tilde{q}}_{I-1}^{\dot{1}} \tilde{q}_{I-1\,\dot{2}}$.

As before, the superconnection $\mathcal{L} = \mathcal{L}_{1/2} - iQG + \{G, H\} + C$ is supersymmetric provided that (4.59) is obeyed. This is solved by

$$\bar{\beta}_{I\|} = \beta_I^{\|}\,, \qquad \bar{\beta}_{I+2\|} = \beta_{I+2}^{\|}\,, \qquad \gamma_1 \bar{\alpha}_{I+2} = \gamma_2 \bar{\alpha}_I\,, \tag{4.106}$$

and by setting the remaining parameters in C to zero, except for $\bar{\delta}_I$, $\bar{\delta}_{I+2}$ and δ_{I+1}, which are left arbitrary. We write down the resulting operator at the end of Section 4.5.5.

4.5 Special cases

Having carried out the systematic construction of BPS hyperloops described above, we turn now to some special examples of the constructions. This includes making contact with previously described operators and identifying new ones. Our emphasis is on operators preserving more than one supercharge.

4.5.1 Single node bosonic loops

We start with the simplest possible BPS Wilson loops in three-dimensional Chern-Simons-matter theories, those involving only a single node and $\mathcal{L}$ is a 1×1 block with only the gauge field and bilinears of the scalars. The first such bosonic loops were constructed by Gaiotto and Yin in off-shell $\mathcal{N} = 2$ language in [76]. Analogues of them in ABJ(M) theory were described in [60–62] and that description carries over also to $\mathcal{N} = 4$ theories. Such loops preserve at most four supercharges. The other previously identified family of bosonic loops are the "bosonic latitude" loops of [172, 176, 59], which preserve a pair of supercharges.

To get such loops in our setting we may decouple the nodes by simply setting $\beta^1 = \beta^2 = \bar{\beta}_1 = \bar{\beta}_2 = 0$ in the analysis of Section 4.3 for the case $\Pi \neq 0$ (we comment below on the case $\Pi = 0$). This eliminates all the fermions in the superconnection $\mathcal{L}$, which becomes block-diagonal with a connection in the I-th block taking the form

$$\mathcal{A} = A_\varphi + \frac{i}{k}\Pi^{-1}(r^1 \bar{r}_1 - r^2 \bar{r}_2) - \frac{i}{k}(\tilde{\mu}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}^{\dot{2}}{}_{\dot{2}})\,, \tag{4.107}$$

It is easy to show that these loops preserve at least two supercharges. Consider in fact the supercharge Q' gotten by the replacement $\bar{\eta}_a^\imath \to -\bar{\eta}_a^\imath$ in (4.6)

$$Q' = \eta_a^\imath Q_\imath^{\dot{2}a+} - \bar{\eta}_a^\imath (\sigma^1)_\imath{}^{\bar{\imath}} Q_{\bar{\imath}}^{\dot{1}a-}\,. \tag{4.108}$$

Under this change of sign, $\Pi \to -\Pi$, $r^2 \to -r^2$ and $\bar{r}_1 \to -\bar{r}_1$, such that (4.107) is left invariant. Note that because $\Pi \neq 0$, Q is the sum of barred and unbarred supercharges and by the above argument these must be preserved separately.

Alternatively, this can be seen by investigating the bosonic symmetries. In particular, note that the transverse rotation $T_\perp$ keeps the loop fixed pointwise, and therefore acts trivially on the scalars as well as on the parallel component of the gauge field, the only fields in the bosonic loop. Closure of the symmetry algebra then implies that, in addition to Q, the supercharge $[T_\perp, Q]$ is preserved by the loop. From (4.147) we see that this generates Q', so we come to the same conclusion as above (an analogous argument can be made using the generator $\bar{R}_3$).

A useful way to write the connection (4.107) is in terms of the moment maps $\mu^a{}_b$ as

$$\mathcal{A} = A_\varphi + \frac{i}{k}\frac{1}{(\chi-\bar{\chi})}\left((\chi+\bar{\chi})(\mu^1{}_1-\mu^2{}_2)+2\mu^2{}_1-2\chi\bar{\chi}\,\mu^1{}_2\right)-\frac{i}{k}(\tilde{\mu}^{\dot{1}}{}_{\dot{1}}-\tilde{\mu}^{\dot{2}}{}_{\dot{2}})\,, \tag{4.109}$$

with

$$\chi = \frac{(\eta\bar{v})_1}{(\eta\bar{v})_2}\,, \qquad \bar{\chi} = \frac{(\bar{\eta}v)_1}{(\bar{\eta}v)_2}\,, \tag{4.110}$$

which are generally linear fractional transformations of $e^{i\varphi}$ (4.7) (and as usual, they are not conjugates).

The most degenerate case is when both χ and $\bar{\chi}$ have no φ dependence. This requires the numerators and denominators to be proportional to each-other, spanning a two dimensional space of η's and likewise $\bar{\eta}$'s. This implies that the loop preserves 4 supercharges and having no φ dependence, it also preserves the $SO(2,1)$ conformal group. To recover the Gaiotto-Yin Loop [76] we take $\chi = 1/\bar{\chi} \to \infty$. Other values of χ, $\bar{\chi}$ are related by the action of the complexification of the broken $SU(2)_L$ symmetry.

When χ is a constant and $\bar{\chi}$ depends on φ (or vice versa), there is only partial degeneracy, and the loops preserve three supercharges, or are 3/16 BPS. Such loops have not been previously discussed in the literature.

When both χ and $\bar{\chi}$ depend on φ, the loops preserve a pair of supercharges. A simple example is when they are just monomials, for example $\chi = -\tan(\theta/2)e^{-i\varphi}$ and $\bar{\chi} = \cot(\theta/2)e^{-i\varphi}$. The connection takes the form

$$\mathcal{A} = A_\varphi - \frac{i}{k}\left(\cos\theta(\mu^1{}_1-\mu^2{}_2)+\sin\theta\, e^{-i\varphi}\mu^1{}_2+\sin\theta\, e^{i\varphi}\mu^2{}_1\right)-\frac{i}{k}(\tilde{\mu}^{\dot{1}}{}_{\dot{1}}-\tilde{\mu}^{\dot{2}}{}_{\dot{2}})\,. \tag{4.111}$$

These are the latitude loops found in [172] and studied in [176, 59]. As the φ dependence breaks conformal invariance, acting with the (complexified) conformal group $SL_2(\mathbb{C})$ on the loop above generates many other loops, including those where χ and $\bar{\chi}$ are proper rational functions and not mere monomials.

There are yet more peculiar bosonic loops that preserve two supercharges, but are not similar to the latitude loops. Representatives of those have

$$\chi = e^{-i\varphi} + \nu\,, \qquad \bar{\chi} = e^{-i\varphi} - \nu\,, \tag{4.112}$$

with an arbitrary parameter ν.

Despite all the machinery in the previous sections, the analysis of the most general BPS bosonic loop requires yet further techniques, so those will be explored in the next section

5. That exploration will also relax the condition in this paper that the loops arise from continuous deformations of the 1/2 BPS loop, which could give rise to further BPS bosonic loops.

4.5.2 Two-node hyperloops with $\Pi \neq 0$

Let us look now at some special examples of the hyperloops with two nodes constructed in Section 4.3.1. Examining (4.34), the most symmetric possibility is that M is proportional to the identity, restoring $SU(2)_L$ symmetry. There are two such solutions. The first with $\beta^1 = \bar\beta_1 = 0$ and $\beta^2\bar\beta_2 = 2i/k$, which is just the original 1/2 BPS loop in (4.3). The second has $\beta^2 = \bar\beta_2 = 0$ and $\beta^1\bar\beta_1 = -2i/k$, which is the second 1/2 BPS loop with the same symmetries in (4.9) (albeit written in a different gauge).

A less symmetric case is when M is diagonal, but not necessarily proportional to the identity, so when $\bar\beta_1\beta^2 = \bar\beta_2\beta^1 = 0$. If $\beta^1 = \beta^2 = 0$ or $\bar\beta_1 = \bar\beta_2 = 0$, the connection becomes upper or lower triangular, respectively. As discussed in [58, 59, 69], the resulting loops are effectively the same as if all the $\beta^a = \bar\beta_a = 0$, since they are all identical as quantum operators. The interesting case is then when $\bar\beta_1 = \beta^1 = 0$ or $\bar\beta_2 = \beta^2 = 0$. Taking the former as an example, we find

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + M_a{}^b r^a \bar r_b - \frac{i}{k}(\tilde\mu_I{}^{\dot 1}{}_{\dot 1} - \tilde\mu_I{}^{\dot 2}{}_{\dot 2}) & -i\bar\beta_2\psi_{I\dot 1-} \\ i\beta^2\bar\psi^{\dot 1}_{I+} & A_{\varphi,I+1} + M^b{}_a \bar r_b r^a - \frac{i}{k}(\tilde\mu_{I+1}{}_{\dot 1}{}^{\dot 1} - \tilde\mu_{I+1}{}_{\dot 2}{}^{\dot 2}) - \frac{1}{2} \end{pmatrix}, \tag{4.113}$$

with

$$M = \Pi^{-1}\begin{pmatrix} \frac{i}{k} & 0 \\ 0 & \bar\beta_2\beta^2 - \frac{i}{k} \end{pmatrix}. \tag{4.114}$$

In addition to the supercharge Q, these hyperloops preserve a supercharge Q' arising from same $\eta^\imath_a$ but with $\bar\eta^\imath_a \to -\bar\eta^\imath_a$. The argument is identical to the case of the bosonic loops presented in Section 4.5.1. In this case we see that the fermionic terms are unchanged if we keep the same β's and $M \to -M$, so the diagonal entries $M_1{}^1 r^1 \bar r_1$ and $M_2{}^2 r^2 \bar r_2$ are also left invariant. The requirement that M is diagonal guarantees, therefore, that the loop is also invariant under Q' and is 1/8 BPS.

Thus, for any choice of Q with $\Pi \neq 0$, if we restrict the parameters such that $\beta^1 = \bar\beta_1 = 0$, we find a family of 1/8 BPS hyperloops parametrized by β^2 and $\bar\beta_2$. However, as we can conjugate $\mathcal{L}$ by a constant matrix

$$\mathcal{L} \to \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} \mathcal{L} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \tag{4.115}$$

this gauge transformation eliminates one of the parameters, and we end up with a one (complex) dimensional moduli space.

This is very similar to the discussion in [59], but it is much more general, as it works with any of the supercharges Q in (4.6) with $\Pi \neq 0$. To make contact with the constructions in [59] we can look at the moduli space of 1/4 BPS hyperloops studied there, which are all deformations of the usual bosonic Gaiotto-Yin loops [76]. Those loops preserve a one-dimensional conformal group, under which the supercharges are charged. Looking at the algebra (4.35) and requiring only conformal transformations in the square of the supercharge imposes $\epsilon_{\imath\jmath}\,(\bar{\eta}^{\imath}_a\eta^{\jmath}_b + \bar{\eta}^{\imath}_b\eta^{\jmath}_a) = 0$. To realize this, we choose two vectors $\bar{w}_a$ and w_a (as usual, bar does not indicate complex conjugation). For an arbitrary vector $s^{\imath}$, define parameters $\bar{\eta}, \eta$ as

$$\eta^{\imath}_a = w_a s^{\imath}\,, \qquad \bar{\eta}^{\imath}_a = \bar{w}_a s^{\imath}\,. \tag{4.116}$$

The resulting supercharges are all linear combinations of

$$w_a Q^{\dot{2}a}_{\imath}\,, \qquad \bar{w}_a Q^{\dot{1}a}_{\bar{\imath}}\,, \tag{4.117}$$

whose anticommutators generate the bosonic algebra $\mathfrak{so}(2,1)\oplus\mathfrak{u}(1)$, where the $\mathfrak{u}(1)$ summand is generated by $L_\perp + \frac{1}{2}w_a\bar{w}_b R^{ab}$ (see Section 4.3.1 for details).

In [59] the vector w_a was δ^2_a and $\bar{w}_a$ was δ^1_a. Other choices can be achieved by an $SU(2)_L$ rotation. What was more restrictive there is that only a single choice of Q (or $s^{\imath}$) was used. As long as we turn on only the parameters as in (4.114), we preserve all the supercharges in (4.117), so any choice (with $\Pi \neq 0$) is equivalent. When turning on more β parameters, we find different moduli spaces, depending on the exact choice of Q. Our analysis here therefore generalizes also this simple case of deformations of the 1/4 BPS bosonic loop.

As discussed in Section 4.5.1, there are several new bosonic loops generated by our construction that are not related to those in [59]. Clearly their deformations with $\beta \neq 0$ are also new.

4.5.3 Two-node hyperloops with $\Pi = 0$

This case is presented in Section 4.3.2, where it is shown that the general deformation is of the form (4.56) with G and C as in (4.58), subject to the constraints that $\bar{\beta}_\parallel = \beta^\parallel$ and the conditions on $\bar{\beta}_\perp$, $\beta^\perp$ and c in (4.60). The resulting expression for $\mathcal{L}$ is then in (4.61).

A simple way to find loops with enhanced supersymmetry is when the superconnection is invariant under $\mathfrak{su}(2)_L$, which arises when $M_a{}^b r^a\bar{r}_b \propto \nu_I$. Looking at the expression for ν in (4.55) and M in (4.62), we see that one needs to impose

$$\beta^\parallel = 0\,, \qquad \xi\alpha\bar{\beta}_\perp = \bar{\alpha}\beta^\perp\,. \tag{4.118}$$

These equations are consistent with (4.60)[15], combining all the parameters to a single periodic

[15](4.60) is also solved with $\xi = \xi_0 e^{-i\varphi}$, with a constant $\xi_0 \neq 0$, arbitrary $\beta^\perp$, $\bar{\beta}_\perp$, $\beta^\parallel = \bar{\beta}_\parallel$ and $c = 0$.

function $\gamma = 1 - ik\Lambda\bar{\alpha}\beta^{\perp}$ appearing in the superconnection as

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + \frac{i}{k}\gamma\nu_I - \frac{i}{k}(\tilde{\mu}_I{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}) & -\frac{i\bar{\alpha}}{2}(\gamma+1)\psi_{\dot{1}-} - \frac{i\bar{\alpha}}{2}(\gamma-1)\xi^{-1}\psi_{\dot{2}+} \\ \frac{i\alpha}{2}(\gamma+1)\bar{\psi}_+^{\dot{1}} - \frac{i\alpha}{2}(\gamma-1)\xi\bar{\psi}_-^{\dot{2}} & A_{\varphi,I+1} + \frac{i}{k}\gamma\nu_{I+1} - \frac{i}{k}(\tilde{\mu}_{I+1}{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_{I+1}{}^{\dot{2}}{}_{\dot{2}}) + c - \frac{1}{2} \end{pmatrix}, \tag{4.119}$$

and $c = i\frac{\gamma-1}{2}\partial_\varphi \log(\xi e^{i\varphi})$.

The degree of supersymmetry enhancement depends on the choice of supercharge Q. Specifically, following Section 4.3.2, we distinguish three cases.

1/8 BPS loops

First, suppose $0 \neq Q^2 \in \mathfrak{su}(2)_L$. Putting together (4.43) and (4.46), one sees that the parameters η, $\bar{\eta}$ may be cast into the form

$$\eta_a^{\imath} = t^{\imath}\, w_a\,, \qquad \bar{\eta}_a^{\imath} = \bar{t}^{\imath}\, w_a\,, \tag{4.120}$$

with some vector $w_a \neq 0$ and $\epsilon_{\imath\jmath}t^{\imath}\bar{t}^{\jmath} \neq 0$. Acting on the resulting supercharge with $\mathfrak{su}(2)_L$, we find that, regardless of the choice of w_a, the loop preserves the two supercharges (with a convenient normalization)

$$Q_1 = \frac{1}{\sqrt{\epsilon_{\imath\jmath}t^{\imath}\bar{t}^{\jmath}}}\left(t^{\imath}\, Q_{\imath}^{\dot{2}1} + \bar{t}^{\imath}\,(\sigma_1)_{\imath}{}^{\bar{\jmath}}\, Q_{\bar{\jmath}}^{\dot{1}1}\right), \qquad Q_2 = \frac{1}{\sqrt{\epsilon_{\imath\jmath}t^{\imath}\bar{t}^{\jmath}}}\left(t^{\imath}\, Q_{\imath}^{\dot{2}2} + \bar{t}^{\imath}\,(\sigma_1)_{\imath}{}^{\bar{\jmath}}\, Q_{\bar{\jmath}}^{\dot{1}2}\right). \tag{4.121}$$

Using (4.150) it is easy to verify that their anticommutators generate $\mathfrak{su}(2)_L$

$$\{Q_1, Q_1\} = \frac{1}{2}R_+\,, \qquad \{Q_1, Q_2\} = -R_3\,, \qquad \{Q_2, Q_2\} = -\frac{1}{2}R_-\,. \tag{4.122}$$

1/4 BPS loops and conformal loops

Another case is when the supercharge satisfies $0 \neq Q^2 \in \mathfrak{u}(1)_{L_\perp}$. In this case, as derived in (4.69), we have $\xi = \xi_0 e^{-i\varphi}$, which immediately implies $c = 0$. As discussed in Section 4.3.2, the parameters of Q take the form

$$\begin{aligned} \eta_a^l &= ts^l w_a\,, & \eta_a^r &= ts^r z_a\,, \\ \bar{\eta}_a^l &= \bar{t}s^l w_a\,, & \bar{\eta}_a^r &= \bar{t}s^r z_a\,, \end{aligned} \tag{4.123}$$

where both $\epsilon^{ab}w_a z_b \neq 0$ and $s^l s^r \neq 0$. Knowing that the loop is invariant under $\mathfrak{su}(2)_L$ we can project (4.123) to those terms involving either only w_a or only z_a. Acting then with raising and lowering operators projects further to the two components $a = 1, 2$, removing the dependence on w_a and z_a altogether, and leaving us with four supercharges

$$\begin{aligned} Q_1 &= tQ_r^{\dot{2}1} + \bar{t}Q_{\bar{l}}^{\dot{1}1}\,, & Q_2 &= tQ_l^{\dot{2}1} + \bar{t}Q_{\bar{r}}^{\dot{1}1}\,, \\ Q_3 &= tQ_r^{\dot{2}2} + \bar{t}Q_{\bar{l}}^{\dot{1}2}\,, & Q_4 &= tQ_l^{\dot{2}2} + \bar{t}Q_{\bar{r}}^{\dot{1}2}\,. \end{aligned} \tag{4.124}$$

Examining these, we see that they form doublets of $\mathfrak{so}(2,1)$ (exchanging l and r).

The algebra generated by these supercharges is very simple, with the only non-vanishing anticommutators

$$\{Q_1, Q_4\} = -2t\bar{t}L_\perp\,, \qquad \{Q_2, Q_3\} = 2t\bar{t}L_\perp\,. \tag{4.125}$$

Note that the bosonic part of this 1/4 BPS algebra is just $\mathfrak{u}(1)_{L_\perp}$, while $\mathfrak{su}(2)_L$ and the one-dimensional conformal algebra $\mathfrak{so}(2,1)$ act as outer automorphisms.

We noted that the superconnection (4.119) is invariant under $\mathfrak{su}(2)_L$. It is interesting to check whether it is also invariant under $\mathfrak{so}(2,1)$. This clearly requires γ to be a constant, as otherwise $\mathcal{L}$ is not invariant even under rotations. Considering then a general conformal generator $J = a_+J_+ + a_0J_0 + a_-J_-$ and using (4.145)-(4.146), one finds that the conformal transformation of $\mathcal{L}$ in (4.119) is a total derivative

$$J\mathcal{L} = \mathcal{D}^{\mathcal{L}}_\varphi(a\mathcal{L} + H), \tag{4.126}$$

with

$$a = a_+e^{i\varphi} - ia_0 + a_-e^{-i\varphi}, \qquad H = \begin{pmatrix} 0 & 0 \\ 0 & a/2 \end{pmatrix}. \tag{4.127}$$

The resulting Wilson loops are then invariant under $\mathfrak{so}(2,1) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_{L_\perp}$, providing a previously unidentified family of conformal 1/4 BPS loops.

Note that the argument here is classical and as the superalgebra (4.125) does not include the conformal generators, we cannot be sure that it is not spoiled by quantum corrections.

Further 1/8 BPS loops

The last example arising from (4.119) are loops with nilpotent Q. Since this case lies at the intersection of the previous two, we have to impose all the conditions discussed above. For the parameters, we have

$$\eta^\imath_a = a\rho^\imath w_a\,, \qquad \bar{\eta}^\imath_a = \bar{a}\rho^\imath w_a\,. \tag{4.128}$$

They give a pair of nilpotent supercharges

$$\begin{aligned} Q_1 &= a\rho^\imath Q^{\dot{2}1}_\imath + \bar{a}\rho^\imath(\sigma_1)_\imath{}^{\bar{\jmath}}Q^{\dot{2}1}_{\bar{\jmath}}\,, \\ Q_2 &= a\rho^\imath Q^{\dot{2}2}_\imath + \bar{a}\rho^\imath(\sigma_1)_\imath{}^{\bar{\jmath}}Q^{\dot{2}2}_{\bar{\jmath}}\,, \end{aligned} \tag{4.129}$$

whose anticommutator vanishes as well.

Another family of loops with enhanced supersymmetry arises if, instead of $\mathfrak{su}(2)_L$ symmetry (as in (4.119)), we demand conformal invariance from the beginning. Generalising the discussion in Section 4.5.3, we impose the equation (4.126) directly on the superconnection (4.61). The off-diagonal components of this matrix equation are satisfied, as in

Section 4.5.3, as long as $\xi = \xi_0 e^{-i\varphi}$ and $c = 0$, which identically solves the supersymmetry conditions (4.60). Additionally, if we redefine

$$\beta^{\perp} = \frac{\alpha}{2\Lambda}(\gamma - 1), \qquad \bar{\beta}_{\perp} = \frac{\bar{\alpha}}{2\Lambda\xi}(\bar{\gamma} - 1), \qquad \beta^{\parallel} = \frac{i}{k\Lambda}\gamma^{\parallel}, \tag{4.130}$$

then we need to impose that γ, $\bar{\gamma}$ and $\gamma^{\parallel}$ are constants.

The expression for the superconnection (4.61) then becomes

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + M_a{}^b r^a \bar{r}_b - \frac{i}{k}(\tilde{\mu}_I{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}) & -\frac{i\bar{\alpha}}{2}(\bar{\gamma}+1)\psi_{\dot{1}-} - \frac{i\bar{\alpha}}{2}(\bar{\gamma}-1)\xi^{-1}\psi_{\dot{2}+} \\ \frac{i\alpha}{2}(\gamma+1)\bar{\psi}^{\dot{1}}_{+} - \frac{i\alpha}{2}(\gamma-1)\xi\bar{\psi}^{\dot{2}}_{-} & A_{\varphi,I+1} + M_a{}^b \bar{r}_b r^a - \frac{i}{k}(\tilde{\mu}_{I+1}{}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}_{I+1}{}^{\dot{2}}{}_{\dot{2}}) - \frac{1}{2} \end{pmatrix}, \tag{4.131}$$

with the couplings to the rotated scalars (4.62) given by

$$M_a{}^b = \frac{i}{k\Lambda}\begin{pmatrix} 0 & \bar{\gamma} \\ \gamma & \gamma^{\parallel} \end{pmatrix}. \tag{4.132}$$

The remaining check is whether the diagonal part of equation (4.126) is satisfied, which imposes that the couplings to the unrotated scalars $q^a, \bar{q}_a$ are constant. This can be arranged in two ways. Firstly, by (4.55) we can set $\bar{\gamma} = \gamma, \gamma^{\parallel} = 0$ to obtain scalar terms proportional to ν_I, ν_{I+1} without any explicit φ dependence. These loops are just the conformal 1/4 BPS loops described in the previous section.

Alternatively, constant scalar couplings can be obtained for arbitrary $\gamma, \bar{\gamma}, \gamma^{\parallel}$ by demanding instead $\epsilon^{ab}\bar{\eta}^l_a\bar{\eta}^r_b = 0$ or, equivalently, $Q^2 = 0$. In order to derive the symmetries preserved by these loops, we parametrise the supercharge using (4.70) and act on it with the conformal generators. This process generates another supercharge, so in total we have

$$Q_1 = w_a\left(aQ_l^{\dot{2}a} + \bar{a}Q_{\bar{r}}^{\dot{1}a}\right), \qquad Q_2 = w_a\left(aQ_r^{\dot{2}a} + \bar{a}Q_{\bar{l}}^{\dot{1}a}\right). \tag{4.133}$$

Both these supercharges are nilpotent and their anticommutator vanishes. By construction, $\mathfrak{so}(2,1)$ acts on the algebra as an outer automorphism.

There is yet another example of supersymmetry enhancement without $\mathfrak{su}(2)_L$ symmetry, but with invariance under $T_\perp$ (but not $L_\perp$ in (4.149)). Recalling that $T_\perp$ acts diagonally and separates barred from unbarred supercharges, it is easily seen that the commutator $Q' = [T_\perp, Q]$ is linearly independent of Q, provided Q comprises both barred and unbarred supercharges (so $\xi \neq 0, \infty$). To see which loops are invariant under Q', we note that keeping $\eta^{\imath}_a$ and changing $\bar{\eta}^{\imath}_a \to -\bar{\eta}^{\imath}_a$ leaves $\Pi = 0$, likewise Λ is unmodified, and $\xi \to -\xi$. Noticing that (4.61) contains terms proportional to both $\Lambda\beta^\perp$ and $\xi\Lambda\beta^\perp$, we have to set $\beta^\perp = 0$ and similarly for $\bar{\beta}_\perp$, which by (4.60) also fixes $c = 0$. The resulting superconnection is

$$\mathcal{L} = \mathcal{L}_{1/2} + \begin{pmatrix} \beta^{\parallel} r^{\parallel}\bar{r}_{\parallel} & 0 \\ 0 & \beta^{\parallel}\bar{r}_{\parallel} r^{\parallel} \end{pmatrix}, \tag{4.134}$$

where $\beta^{\parallel}$ can be an arbitrary periodic function of φ. One can check that generically the supersymmetry is not enhanced further.

The special cases: $\xi = 0$ and $\xi = \infty$

When $\xi = 0$, the superconnection of loops are the same as (4.61) with $\xi = c = 0$ and $\beta^\perp$, $\bar{\beta}_\perp$ and $\beta^\parallel$ free. If we want to study the $\mathfrak{su}(2)_L$ enhanced points, we should impose $\beta^\perp = \beta^\parallel = 0$ and get the loops

$$\mathcal{L} = \mathcal{L}_{1/2} + \begin{pmatrix} 0 & -i\Lambda\bar{\beta}_\perp \psi_{\dot{2}+} \\ 0 & 0 \end{pmatrix} . \tag{4.135}$$

The case $\xi = \infty$ is similar with a term on the lower left corner.

In all of these examples the free parameters $\beta^\parallel$, $\beta^\perp$ (and in the last case also $\bar{\beta}_\perp$) are any periodic functions of φ. The reason is most transparent with regards to $\beta^\parallel$, as Q annihilates $r^\parallel \bar{r}_\parallel$ and we can insert any density of them along the loop.

In Sections 4.5.1, 4.5.2 and 4.5.3 above, we noted multiple examples of hyperloops that in addition to Q preserve also Q' with $\bar{\eta}^{\imath}_a \to -\bar{\eta}^{\imath}_a$. They clearly also preserve $Q \pm Q'$, which are supercharges with $\Pi = 0$ and $\xi = 0$ and $\xi = \infty$.

4.5.4 Hyperloops with twisted hypers and $\Pi \neq 0$

To couple our hyperloops to the twisted hypermultiplets, the starting point in Section 4.4 is a 4×4 superconnection (4.72) which takes a block-diagonal form and is deformed with parameters β and δ. Here we focus on special examples of these loops. As a first step, we set all the β's to zero. In the absence of the δ terms, this would give a diagonal connection with only bosonic fields.

With $\beta = 0$ and $\delta \neq 0$, we find instead a block-diagonal form, with a 2×2 block involving the nodes $I+1$ and $I+2$, and two decoupled nodes I and $I+3$. We ignore in the following the decoupled nodes and concentrate only on the remaining 2×2 block. Note that often the decoupled nodes do not preserve the symmetries of the central block. This can be remedied in the setting of a circular quiver.

In the case of a deformation with $\delta_{I+1\,\mathrm{i}}$ and $\bar{\delta}^{\mathrm{i}}_{I+1}$, the central block takes the form

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I+1} + M^b{}_a \bar{r}_{I\,b} r^a_I + \widetilde{M}^{\dot{a}}{}_{\dot{b}} \tilde{q}_{I+1\,\dot{a}} \bar{\tilde{q}}^{\dot{b}}_{I+1} - \frac{1}{2} & -i\bar{\delta}^{\mathrm{i}}_{I+1} \tilde{\rho}^2_{I+1\,+} \\ -i\Pi^{-1} \delta_{I+1\,\mathrm{i}} \bar{\tilde{\rho}}_{I+1\,2-} & A_{\varphi,I+2} + M_a{}^b r^a_{I+2} \bar{r}_{I+2\,b} + \widetilde{M}_{\dot{b}}{}^{\dot{a}} \bar{\tilde{q}}^{\dot{b}}_{I+1} \tilde{q}_{I+1\,\dot{a}} + \bar{\Gamma} \end{pmatrix} \tag{4.136}$$

with (see (4.107))

$$M = \frac{i}{k} \Pi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \widetilde{M} = \begin{pmatrix} -i/k + \bar{\delta}^{\mathrm{i}}_{I+1} \delta_{I+1\,\mathrm{i}} & 0 \\ 0 & i/k \end{pmatrix} . \tag{4.137}$$

This structure is the analog of the two-node quiver with a coupling to a single pair of scalars in the hypermultiplets as in (4.114). Just as in that example, these loops have enhanced supersymmetry with the second supercharge Q' given by exchanging $\bar{\eta}^{\imath}_a \to -\bar{\eta}^{\imath}_a$. So all these loops are at least 1/8 BPS.

Further supersymmetry enhancement arises in the schemes explained in Section 4.5.1 leading to operators that can preserve either 3 or 4 supercharges. Even further supersymmetry enhancement arises by setting $\delta_{I+1\,\dot{1}}\bar{\delta}^{\dot{1}}_{I+1} = 2i/k$, as then the loop enjoys $\mathfrak{su}(2)_R$ symmetry. In this case, the analysis of the previous paragraph is extended to supercharges with $\dot{1} \leftrightarrow \dot{2}$ and we have a doubling of the amount of preserved supersymmetry. Note that because of the $\dot{1} \leftrightarrow \dot{2}$ exchange, these supercharges are not preserved by the original 1/2 BPS loop, see (4.5). The 1/4 BPS loop becomes the 1/2 BPS operator coupling to the single pair of scalars $\tilde{q}_{\dot{1}}$, $\bar{\tilde{q}}^{\dot{1}}$ from the twisted hypermultiplet. The 1/8 BPS operator becomes 1/4 BPS and for the particular parameterization

$$\bar{\eta}^r_1 = \eta^l_2 = \cos\frac{\theta}{2}\,, \qquad \bar{\eta}^l_2 = -\eta^r_1 = \sin\frac{\theta}{2}\,, \tag{4.138}$$

we recover the "fermionic latitude" loops constructed first in ABJM theory in [172] and generalized to $\mathcal{N} = 4$ theories in [59], see also [176]. The 3/16 BPS operator becomes 3/8 BPS.

Completely analog constructions arise with $\delta_{I+1\,\dot{1}} = \bar{\delta}^{\dot{1}}_{I+1} = 0$ and nonzero couplings $\delta_{I+1\,\dot{2}}$ and $\bar{\delta}^{\dot{2}}_{I+1}$. The most symmetric loop of this class is the second 1/2 BPS loop coupling instead to the pair of scalars $\tilde{q}_{\dot{2}}$, $\bar{\tilde{q}}^{\dot{2}}$. The cases with all four δ parameters non-vanishing is allowed, as long as (4.94) is satisfied. The analysis follows as before, but $\mathfrak{su}(2)_R$ symmetry is preserved only when we restrict to a single pair of δ.

4.5.5 Hyperloops with twisted hypers and $\Pi = 0$

These operators are considered in Section 4.4.2, where we find supersymmetric loops built out of the G and C in (4.100). In particular, the β parameters that couple to scalars from the untwisted hypermultiplet satisfy the same constraints as in the 2-node case, while the couplings to the twisted scalars, $\bar{d}^{\dot{a}}$ and $d_{\dot{a}}$, are arbitrary periodic functions as long as $\lambda = 0$.

Denoting the superconnection in (4.61) as $\mathcal{L}_{\Pi=0}$, the expression we find for $\mathcal{L}$ is

$$\mathcal{L} = \begin{pmatrix} \multicolumn{2}{c}{\mathcal{L}_{\Pi=0}} & \begin{matrix}\bar{\alpha}_I\bar{d}^{\dot{a}}r^{\parallel}_I\tilde{q}_{I+1\,\dot{a}} \\ \bar{d}^{\dot{1}}\tilde{\rho}^2_{I+1,+} + \bar{d}^{\dot{2}}\tilde{\rho}^1_{I+1,-}\end{matrix} & \begin{matrix}0 \\ \bar{d}^{\dot{a}}\bar{\alpha}_{I+2}\tilde{q}_{I+1\,\dot{a}}r^{\parallel}_{I+2}\end{matrix} \\ \begin{matrix}-d_{\dot{a}}\alpha_I\xi\bar{\tilde{q}}^{\dot{a}}_{I+1}\bar{r}_{I\parallel} \\ 0\end{matrix} & \begin{matrix}d_{\dot{1}}\bar{\tilde{\rho}}_{I+1,2-} + d_{\dot{2}}\bar{\tilde{\rho}}_{I+1,1+} \\ -\alpha_{I+2}d_{\dot{a}}\xi\bar{r}_{I+2\parallel}\bar{\tilde{q}}^{\dot{a}}_{I+1}\end{matrix} & \multicolumn{2}{c}{\mathcal{L}_{\Pi=0}} \end{pmatrix}. \tag{4.139}$$

Note that the coupling to the twisted scalar bilinears is unchanged and the $\widetilde{M}$ in the central nodes does not receive contributions from the d's. In general, these loops preserve a single supercharge.

One special case is similar to the 1/4 BPS hyperloop of Section 4.5.3, when ξ (4.49) is of the form $\xi_0 e^{-i\varphi}$ with constant ξ_0. This can arise with either $\xi_0 = \eta^r/\bar{\eta}^r$ or $\xi_0 = \eta^l/\bar{\eta}^l$ leading to a two fold degeneracy. This is a symmetry of the superconnection (4.139) when

$$\bar{d}^{\dot{1}} = d_{\dot{2}} = \frac{1}{(\eta\bar{v})_1}\,, \quad \bar{d}^{\dot{2}} = d_{\dot{1}} = \frac{1}{(\bar{\eta}v)_1}\,, \tag{4.140}$$

and $(\bar{\eta}v)_1 = (\bar{\eta}v)_2$, $(\eta\bar{v})_1 = (\eta\bar{v})_2$. The resulting hyperloop preserves two supercharges and, as before, $\mathfrak{so}(2,1)$ acts as an outer automorphism on the preserved superalgebra. Unlike the 2-nodes case in (4.119), there is no way to restore $\mathfrak{su}(2)_L$ symmetry and find further supersymmetry enhancement.

The special cases: $\xi = 0$ and $\xi = \infty$

In Section 4.4.2 the analysis of the case of $\xi = 0$ is extended to include the twisted hypermultiplets. Denoting the superconnection in (4.65) as $\mathcal{L}_{\xi=0}$, the extension to include twisted hypermultiplets gives

$$\mathcal{L} = \begin{pmatrix} \multicolumn{2}{c}{\mathcal{L}_{\xi=0}} & r_I^{\|}(\gamma_1\tilde{q}_{I+1\,\dot{2}} + \bar{\alpha}_I\bar{d}^{\dot{1}}\tilde{q}_{I+1\,\dot{1}}) & 0 \\ & & -i\bar{d}^{\dot{1}}\tilde{\rho}^2_{I+1,+} & \frac{\bar{\alpha}_{I+2}}{\bar{\alpha}_I}(\gamma_1\tilde{q}_{I+1\,\dot{2}} + \bar{\alpha}_I\bar{d}^{\dot{1}}\tilde{q}_{I+1\,\dot{1}})r^{\|}_{I+2} \\ 0 & (\delta_{I+1} - id_{\dot{2}})\bar{\tilde{\rho}}_{I+1,1+} & \multicolumn{2}{c}{\mathcal{L}_{\xi=0}} \\ 0 & 0 & & \end{pmatrix}. \tag{4.141}$$

Note that, as δ_{I+1} and $d_{\dot{2}}$ appear only through the linear combination $\delta_{I+1} - id_{\dot{2}}$, we can eliminate one of them. Supersymmetry enhancement relying on manifest $\mathfrak{su}(2)_L$ symmetry happens only by setting to zero off-block-diagonal parameters, in which case we simply recover two decoupled copies of (4.135).

Acknowledgements

We would like to acknowledge fruitful discussions with L. Griguolo, E. Pomoni and Y. Wang. DT is grateful to the Galileo Galilei Institute for hospitality in the final stages of this work. ND is supported by the Science Technology & Facilities Council under the grants ST/T000759/1 and ST/P000258/1. The work of ZK is supported by CSC grant No. 201906340174. MT acknowledges the support of the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq). DT is supported in part by the INFN grant *Gauge and String Theory (GAST)* and would like to thank FAPESP's partial support through the grants 2016/01343-7 and 2019/21281-4.

4.A Symmetries of the 1/2 BPS Wilson loop

We start by recalling that the symmetries of an $\mathcal{N} = 4$ superconformal theory on S^3 form an $\mathfrak{osp}(4|4) \cong D(2,2)$ superalgebra, with the bosonic symmetries $\mathfrak{so}(4,1) \oplus \mathfrak{so}(4)$. These are, respectively, the three-dimensional conformal algebra and the R-symmetry algebra. The latter is conveniently thought of as $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. The 16 supercharges transform as conformal spinors under $\mathfrak{so}(4,1)$ and in the fundamental representations of both R-symmetry $\mathfrak{su}(2)$'s.

The circular 1/2 BPS loop breaks part of these symmetries. Specifically, of the conformal generators, it preserves only the one-dimensional conformal algebra along the contour of the loop and the rotations in the plane perpendicular to it

$$\mathfrak{so}(2,1) \oplus \mathfrak{u}(1)_\perp \,. \tag{4.142}$$

$\mathfrak{su}(2)_L$ is preserved by the loop, whereas $\mathfrak{su}(2)_R$ is broken to $\mathfrak{u}(1)_R$.[16]

We denote the conformal generators along the circle by J_0 and $J_\pm$, with nonvanishing commutators

$$[J_0, J_\pm] = \pm J_\pm \,, \qquad [J_+, J_-] = 2J_0 \,. \tag{4.143}$$

Parametrising the circle by the angular coordinate φ, these generators can be represented by differential operators

$$J_0 = -i\partial_\varphi, \qquad J_\pm = e^{\pm i\varphi}\partial_\varphi \,. \tag{4.144}$$

The action on the fields can be obtained by evaluating the usual conformal transformations on the circle. Suppressing R-symmetry indices, we find for the bosonic fields involved in our Wilson loops

$$\begin{aligned} J_0 A_\varphi &= -i\partial_\varphi A_\varphi \,, & J_\pm A_\varphi &= e^{\pm i\varphi}\left(\partial_\varphi \pm i\right) A_\varphi \,, \\ J_0 q &= -i\partial_\varphi q \,, & J_\pm q &= e^{\pm i\varphi}(\partial_\varphi \pm i/2) q \,, \\ J_0 \bar{q} &= -i\partial_\varphi \bar{q} \,, & J_\pm \bar{q} &= e^{\pm i\varphi}(\partial_\varphi \pm i/2)\bar{q} \,. \end{aligned} \tag{4.145}$$

The second term in the action of $J_\pm$ picks up the scaling dimension of the respective fields. Similarly, for the fermions

$$\begin{aligned} J_0 \psi &= -i\left(\partial_\varphi + i\sigma_3/2\right)\psi \,, & J_\pm \psi &= e^{\pm i\varphi}(\partial_\varphi \pm i + i\sigma_3/2)\psi \,, \\ J_0 \bar{\psi} &= -i\left(\partial_\varphi + i\sigma_3/2\right)\bar{\psi} \,, & J_\pm \bar{\psi} &= e^{\pm i\varphi}(\partial_\varphi \pm i + i\sigma_3/2)\bar{\psi} \,. \end{aligned} \tag{4.146}$$

We denote by $T_\perp$ the generator of rotations $\mathfrak{u}(1)_\perp$ in the orthogonal plane to the contour, which commutes with all other preserved conformal generators. The normalization of $T_\perp$ is fixed such that

$$\begin{aligned} T_\perp \psi &= \frac{i}{2}\sigma_3 \psi \,, & [T_\perp, Q^{\dot{a}a}_\imath] &= \frac{i}{2} Q^{\dot{a}a}_\imath \,, \\ T_\perp \bar{\psi} &= \frac{i}{2}\sigma_3 \bar{\psi} \,, & [T_\perp, Q^{\dot{a}a}_{\bar{\imath}}] &= -\frac{i}{2} Q^{\dot{a}a}_{\bar{\imath}} \,. \end{aligned} \tag{4.147}$$

[16] Of course, the choice of which of the R-symmetry factors is broken and which one is preserved is a matter of which 1/2 BPS loop one considers, as explained in [63].

The generators of $\mathfrak{su}(2)_L$ are $R_\pm, R_3$, with commutation relations

$$[R_3, R_\pm] = \pm R_\pm\,, \qquad [R_+, R_-] = 2R_3\,. \tag{4.148}$$

As mentioned above, these symmetries are preserved by the loop. We distinguish $\mathfrak{su}(2)_R$ with bars: $\bar{R}_\pm, \bar{R}_3$. Only $\bar{R}_3$ is preserved by the loop. It is also useful to defined the twisted generator

$$L_\perp \equiv -i\left(T_\perp + \frac{i}{2}\bar{R}_3\right), \tag{4.149}$$

which mixes the rotations in the perpendicular plane in $\mathfrak{u}(1)_\perp$ with the R-symmetry rotations in $\mathfrak{u}(1)_R$ [104].

The supercharges preserved by the loop are given in (4.5) and anticommute to

$$\begin{aligned}
&\{Q_l^{\dot{2}a}, Q_{\bar{l}}^{\dot{1}b}\} = \epsilon^{ab}\left(J_0 + L_\perp\right) + R^{ab}\,, \quad \{Q_l^{\dot{2}a}, Q_{\bar{r}}^{\dot{1}b}\} = \epsilon^{ab} J_+\,,\\
&\{Q_r^{\dot{2}a}, Q_{\bar{l}}^{\dot{1}b}\} = -\epsilon^{ab} J_-\,,\\
&\{Q_r^{\dot{2}a}, Q_{\bar{r}}^{\dot{1}b}\} = \epsilon^{ab}\left(J_0 - L_\perp\right) - R^{ab}\,.
\end{aligned} \tag{4.150}$$

Here, we have contracted the $\mathfrak{su}(2)_L$ generators with the Pauli matrices in the usual fashion and raised one index by ϵ^{ab} (with $\epsilon^{12} = 1$), such that

$$R^{ab} = \begin{pmatrix} R_+ & -R_3 \\ -R_3 & -R_- \end{pmatrix}. \tag{4.151}$$

In order to fully specify the superalgebra, one computes the commutators of bosonic and fermionic generators using the super-Jacobi identities. Explicitly, we find that the residual conformal generators act on the supercharges as follows

$$\begin{aligned}
J_+ \begin{pmatrix} Q_l \\ Q_r \end{pmatrix} &= \begin{pmatrix} 0 \\ -Q_l \end{pmatrix}, & J_+ \begin{pmatrix} Q_{\bar{l}} \\ Q_{\bar{r}} \end{pmatrix} &= \begin{pmatrix} -Q_{\bar{r}} \\ 0 \end{pmatrix},\\
J_- \begin{pmatrix} Q_l \\ Q_r \end{pmatrix} &= \begin{pmatrix} -Q_r \\ 0 \end{pmatrix}, & J_- \begin{pmatrix} Q_{\bar{l}} \\ Q_{\bar{r}} \end{pmatrix} &= \begin{pmatrix} 0 \\ -Q_{\bar{l}} \end{pmatrix},\\
J_0 \begin{pmatrix} Q_l \\ Q_r \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} Q_l \\ -Q_r \end{pmatrix}, & J_0 \begin{pmatrix} Q_{\bar{l}} \\ Q_{\bar{r}} \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} -Q_{\bar{l}} \\ Q_{\bar{r}} \end{pmatrix},\\
T_\perp \begin{pmatrix} Q_l \\ Q_r \end{pmatrix} &= \begin{pmatrix} Q_l \\ Q_r \end{pmatrix}, & T_\perp \begin{pmatrix} Q_{\bar{l}} \\ Q_{\bar{r}} \end{pmatrix} &= -\begin{pmatrix} Q_{\bar{l}} \\ Q_{\bar{r}} \end{pmatrix}.
\end{aligned} \tag{4.152}$$

These (anti-)commutators together with the bosonic structure outlined above define the Lie superalgebra $\mathfrak{sl}(2|2)$. As is easily checked, $L_\perp$ commutes with all supercharges as well as all bosonic generators. Indeed, $\mathfrak{sl}(2|2)$ is a central extension of the classical Lie superalgebra $A(1,1)$ by $\mathfrak{u}(1)$, so this structure is expected [180].

4.B The covariant derivative

Here we explain what it concretely means when a supersymmetry transformation on a superconnection $\mathcal{L}$ acts as a total covariant derivative, as in (4.22)

$$Q\mathcal{L} = \mathcal{D}^{\mathcal{L}}_{\varphi} H\,. \tag{4.153}$$

Consider the open Wilson loop (we shall worry about taking the supertrace later)

$$W_{2\pi,0} = \mathcal{P}\exp i\int_0^{2\pi} d\varphi\,\mathcal{L}\,, \tag{4.154}$$

and act with Q on the loop. It is crucial that the superconnection $\mathcal{L} = \mathcal{L}^B + \mathcal{L}^F$ is an even supermatrix, *i.e.* a matrix whose diagonal entries $\mathcal{L}^B$ are exclusively bosonic and whose off-diagonal entries $\mathcal{L}^F$ are exclusively fermionic, and likewise for the Wilson loop. Commuting Q through a product of two such superconnections $\mathcal{L}_1$ and $\mathcal{L}_2$, one gets $Q(\mathcal{L}_2\mathcal{L}_1) = Q(\mathcal{L}_2)\mathcal{L}_1 + \sigma_3\mathcal{L}_2\sigma_3 Q\mathcal{L}_1$, where the Pauli matrix is introduced to flip the sign of the odd part of $\mathcal{L}_1$.

Acting with Q on $W_{2\pi,0}$, one needs to apply the Leibniz rule, as Q can act on any $\mathcal{L}(\varphi)$. Keeping track of the sign changes, one finds

$$QW_{2\pi,0} = i\sigma_3\int_0^{2\pi} d\varphi\,W_{2\pi,\varphi}\left(\sigma_3 Q\mathcal{L}(\varphi)\right)W_{\varphi,0}\,. \tag{4.155}$$

Now let us assume it exists an $H(\varphi)$, such that $Q\mathcal{L} = \sigma_3\mathcal{D}^{\mathcal{L}}_{\varphi}(\sigma_3 H(\varphi))$. Then, by the standard relations for Wilson loops, one finds

$$QW_{2\pi,0} = i\sigma_3\int_0^{2\pi} d\varphi\,W_{2\pi,\varphi}\mathcal{D}^{\mathcal{L}}_{\varphi}(\sigma_3 H(\varphi))W_{\varphi,0} = iH(2\pi)W_{2\pi,0} - i\sigma_3 W_{2\pi,0}\sigma_3 H(0)\,. \tag{4.156}$$

Assuming $H(\varphi)$ to be periodic and taking the supertrace, one gets

$$QW = i\,\mathrm{sTr}(H(0)W_{2\pi,0} - \sigma_3 W_{2\pi,0}\sigma_3 H(0)) = i\,\mathrm{Tr}([\sigma_3 H(0), W_{2\pi,0}]) = 0\,. \tag{4.157}$$

This implies that the covariant derivative that should appear in the supersymmetry transformations is

$$Q\mathcal{L} = \sigma_3\mathcal{D}^{\mathcal{L}}_{\varphi}(\sigma_3 H) = \partial_\varphi H - i[\mathcal{L}_{\mathrm{bos}}, H] + i\{\mathcal{L}_{\mathrm{fer}}, H\}\,. \tag{4.158}$$

In the main text we write this as $\mathcal{D}^{\mathcal{L}}_{\varphi}H$, but we really mean the expression above with the anticommutator of the fermionic part of the superconnection.

If one prefers working instead with bosonic variations, one can introduce a Grassmann parameter ξ and write $\delta = \xi Q$. The analogous supersymmetry condition reads

$$\delta\mathcal{L} = \mathcal{D}^{\mathcal{L}}_{\varphi}(\xi H)\,. \tag{4.159}$$

4.C Extra fermionic terms

In this appendix we examine the possibility to add extra fermionic terms to the F in the superconnection, beyond the term $-iQG$ in (4.56). This term arises in the case of $\Pi = 0$ in Section 4.3.2, where G includes only two scalar fields (4.58) and, consequently, QG has only two linear combinations of the fermions (4.53). To generalize it, we take an extra term related to the fermions in the original 1/2 BPS connection

$$F = -iQG + (D-1)\mathcal{L}^F_{1/2}\,. \tag{4.160}$$

Here $D = \mathrm{diag}(\bar{d}, d)$.

The result of the analysis below is that such addition is only possible for $\xi = 0$ or $\xi = \infty$, and those cases are already treated in Section 4.3.2. So this appendix leads to no further hyperloops beyond those described in the main text.

Taking (4.160) and using the same equations for the variations QB and QF in (4.28), one gets $\Delta H = (D-1)H$, because there is no derivative term in Q^2G. Plugging everything known into (4.28) yields

$$\begin{aligned} -iQ^2G &= (\partial_\varphi D)H - i[B + C, DH]\,,\\ QB &= \{QG, DH\} + i(\det D - 1)\{\mathcal{L}^F_{1/2}, H\}\,. \end{aligned} \tag{4.161}$$

Focusing on the second equation for now and using $QH = 0$ and $Q\mathcal{L}^B_{1/2} = i\{\mathcal{L}^F_{1/2}, H\}$, one gets

$$QB = Q\{DH, G\} + (\det D - 1)Q\mathcal{L}^B_{1/2}\,, \tag{4.162}$$

which is simply solved by

$$B = \{DH, G\} + (\det D - 1)\mathcal{L}^B_{1/2}\,. \tag{4.163}$$

Extra terms annihilated by Q are included in C.

The case of $\det D = 1$ is simply a gauge transformation, changing α and $\bar{\alpha}$. So we are left with examining the case $\det D \neq 1$. This results in B having a term proportional to $\mathcal{L}^B_{1/2}$, which includes the gauge fields. Since the gauge fields cannot appear in a Wilson loop with an arbitrary prefactor (they should have prefactor i), one needs to cancel part of this term with factors of the gauge field in C. This amounts to finding a connection annihilated by Q, which one can assume to be purely bosonic: $\mathcal{L}^{B\prime} = \mathrm{diag}(\mathcal{A}'_I, \mathcal{A}'_{I+1} - 1/2)$. We take

$$\mathcal{A}'_I = A_\varphi - \frac{i}{k}(M_a{}^b r^a \bar{r}_b + \tilde{\mu}^{\dot{1}}_{\dot{1}} - \tilde{\mu}^{\dot{2}}_{\dot{2}})\,, \tag{4.164}$$

where $a, b \in \{\|, \perp\}$, and the task is now to find the coefficient matrix $M_a{}^b$. Using

$$\begin{aligned} &r^\|(\bar{\psi}^{\dot{1}}_+ + \xi\bar{\psi}^{\dot{2}}_-) - (\xi\psi_{\dot{1}-} - \psi_{\dot{2}+})\bar{r}_\| \\ &\quad= \Lambda(M_\|{}^\perp r^\| + M_\perp{}^\perp r^\perp)(\bar{\psi}^{\dot{1}}_+ - \xi\bar{\psi}^{\dot{2}}_-) - \Lambda(\xi\psi_{\dot{1}-} + \psi_{\dot{2}+})(M_\perp{}^\| \bar{r}_\| + M_\perp{}^\perp \bar{r}_\perp)\,, \end{aligned} \tag{4.165}$$

and imposing $Q\mathcal{A}'_I = 0$ results in

$$Q\left(A_\varphi - \frac{i}{k}(-\nu_I + \tilde{\mu}_{\mathrm{i}}^{\mathrm{i}} - \tilde{\mu}_{\dot{2}}^{\dot{2}})\right) = -\frac{2i}{k}(\xi\psi_{\mathrm{i}-}\bar{r}_\parallel + r^\parallel\bar{\psi}_+^{\mathrm{i}}), \tag{4.166}$$

which is solved by

$$\xi = M_\perp{}^\perp = 0, \quad M_\parallel{}^\perp = -M_\perp{}^\parallel = 1/\Lambda. \tag{4.167}$$

We see that indeed this works only for $\xi = 0$ and therefore it falls under the cases already analyzed in Section 4.3.2.

To compare with the analysis in Section 4.3.2, we note that for $\xi = 0$ there are many specific features, such as $Q^2G = H^2 = 0$. We can also check that $\mathcal{L}^B_{1/2} - \mathcal{L}^B$ commutes with DH and the only remaining supersymmetry conditions is

$$\partial_\varphi \bar{d} = -ic\bar{d}. \tag{4.168}$$

Including the bosonic loop

$$\mathcal{A}'_I = A_\varphi - \frac{i}{k\Lambda}(\Lambda M_\parallel{}^\parallel r^\parallel \bar{r}_\parallel + r^\parallel\bar{r}_\perp - \bar{r}^\perp\bar{r}_\parallel) - \frac{i}{k}(\tilde{\mu}_{\mathrm{i}}^{\mathrm{i}} - \tilde{\mu}_{\dot{2}}^{\dot{2}}), \tag{4.169}$$

and the analogous expression for $\mathcal{A}'_{I+1}$ in C with prefactors $1 - \det D$ and combining all the terms, one finally gets the superconnection

$$\mathcal{L} = \begin{pmatrix} A_{\varphi,I} + M_a{}^b r^a \bar{r}_b - \frac{i}{k}(\tilde{\mu}_{I\,\mathrm{i}}^{\;\mathrm{i}} - \tilde{\mu}_{I\,\dot{2}}^{\;\dot{2}}) & -i\bar{\alpha}\bar{d}\psi_{\mathrm{i}-} - i\Lambda\bar{\beta}_\perp\psi_{\dot{2}+} \\ i(\alpha d + \Lambda\beta^\perp)\bar{\psi}_+^{\mathrm{i}} & A_{\varphi,I+1} + M_a{}^b\bar{r}_b r^a - \frac{i}{k}(\tilde{\mu}_{I+1\mathrm{i}}^{\quad\mathrm{i}} - \tilde{\mu}_{I+1\dot{2}}^{\quad\dot{2}}) + c - \frac{1}{2} \end{pmatrix}, \tag{4.170}$$

with $c = i\partial_\varphi \log \bar{d}$ and

$$M_a{}^b = \begin{pmatrix} 0 & \frac{i}{k\Lambda} \\ \beta^\perp\bar{d}\bar{\alpha} + (2\bar{d}d - 1)\frac{i}{k\Lambda} & \beta^\parallel \end{pmatrix}, \tag{4.171}$$

where $M_\parallel{}^\parallel$ has been absorbed into $\beta^\parallel$, since both of them are free parameters. One can further absorb $\bar{d}$ into $\bar{\alpha}$ and $\bar{\beta}_\perp$, which sets $c = 0$ and replaces $\alpha \to \alpha\bar{d}$ and $\beta^\perp \to \beta^\perp\bar{d}$. Then, with $\hat{\beta}^\perp = (\alpha(d\bar{d} - 1) + \Lambda\beta^\perp\bar{d})/\Lambda$ the bottom left entry in $\mathcal{L}$ becomes $i(\alpha + \Lambda\hat{\beta}^\perp)\bar{\psi}_+^{\mathrm{i}}$ and the bottom left entry in $M_a{}^b$ becomes $\bar{\alpha}\hat{\beta}^\perp + i/k\Lambda$.

This eliminates the parameters d and $\bar{d}$ from $\mathcal{L}$, so they are completely redundant. Furthermore, we see that these loops are exactly those found directly in the $\xi = 0$ case in (4.65) in Section 4.3.2.

5 Classifying BPS bosonic Wilson loops in 3d $\mathcal{N}=4$ Chern-Simons-matter theories

This section is based on [3] with minor edits.

5.1 Introduction and conclusions

Over the past few years, more and more examples of supersymmetry preserving (BPS) line operators have been found [2, 59–69, 111] in Chern-Simons-matter theories like ABJM [87]. For a relatively recent introduction to the topic, see [58]. While many papers discuss the bosonic 1/6 BPS loop and fermionic 1/2 BPS loop, there are in fact many more Wilson loops including rich moduli spaces of 1/6 BPS loops with fermionic fields and Wilson loops preserving fewer supercharges.

Following work on $\mathcal{N}=2$ theories [69], the recent papers [59, 2] started to methodically treat the space of BPS Wilson loops in the context of theories with $\mathcal{N}=4$ supersymmetry, called there "hyperloops". In the course of writing the last paper in the series, we realised that even the case of Wilson loops with a single gauge field and no fermi fields is very rich and includes previously unnoticed operators preserving 1, 2 and 3 supercharges. As the classification of those was beyond the scope of that paper and requires many different tools, it is the topic of this paper.

A generic $\mathcal{N}=4$ Chern-Simons-matter theory has vector multiplets, hypermultiplets as well as twisted hypers which can be organised graphically in either a circular or linear quiver diagram [83–85, 89]. Restricting to bosonic fields, those are the vector fields A_μ and the bi-fundamental scalars in the hypermultiplet, q^a, $\bar{q}_a$ and in the twisted hypermultiplets, $\tilde{q}_{\dot{a}}$, $\bar{\tilde{q}}^{\dot{a}}$.

The scalar fields in the hypermultiplet have undotted indices and are doublets of the $SU(2)_L$ R-symmetry. The fermions $\psi_{\dot{b}}$, $\bar{\psi}^{\dot{a}}$ with dotted indices are charged instead under $SU(2)_R$. This is reversed for the twisted hypermultiplets. We use the usual epsilon symbols to raise and lower indices: $v^a = \epsilon^{ab} v_b$ and $v_a = \epsilon_{ab} v^b$ with $\epsilon^{12} = \epsilon_{21} = 1$, and likewise for the dotted indices.

We consider the theory on S^3 with the loops supported along the equator of this sphere with coordinate φ. The theories are conformal and this setup allows for conformal line operators preserving $SO(2,1) \times SO(2) \subset SO(4,1)$, but the loops we study are not necessarily conformal, as is discussed below. Still, we restrict the connection $\mathcal{L}$ to have canonical dimension one, so it does not have dimensionful couplings. With this restriction, $\mathcal{L}$ is comprised of the gauge field and bilinear of the scalars. This leads to the natural ansatz

$$W = \operatorname{Tr} \mathcal{P} \exp \oint i\mathcal{L}\, d\varphi\,, \qquad \mathcal{L} = A_\varphi + \frac{i}{k} q^a M_a{}^b \bar{q}_b + \frac{i}{k} \bar{\tilde{q}}^{\dot{a}} \widetilde{M}_{\dot{a}}{}^{\dot{b}} \tilde{q}_{\dot{b}}\,. \tag{5.1}$$

M and $\widetilde{M}$—the couplings of the scalar bilinears—are the main protagonists of this paper. We allow for them to have explicit φ-dependence, which breaks rotational symmetry and therefore also the conformal symmetry along the circle.

The scalar fields may be charged under a flavour group or in fact other gauge groups. The ansatz above assumes that they form singlets of those groups, since the supersymmetry variation (5.3) below requires a cancellation between the variation of the gauge fields that are not charged under these other groups and that of the scalar bilinears.

More general Wilson loops in these theories include also fermi fields and a connection that is naturally extended to a supermatrix with multiple gauge fields, fermions and scalar bilinears. The restriction to bosonic structures still allows the scalar bilinears. Those appearing in (5.1) are in the adjoint of the gauge group (plus the singlet), but in a quiver theory one could also construct bilinears of scalars from different multiplets $q^a\tilde{q}_{\dot{a}}$ which transform in the bifundamental of next to nearest neighbours in the quiver. Wilson loops without fermions but with these couplings are BPS only when the supercharge annihilates these bilinears. As this is a rather trivial additional constraint on our general analysis of (5.1), we do not discuss this possibility further.

With the increasingly rich and intricate structure of BPS loops in 3d theories, it is worth mentioning some of the possible applications of such operators. First, BPS protected quantities serve as a rich laboratory for refining the tools of quantum field theory, for example Seiberg-Witten theory [181, 182] or AdS/CFT [183]. The circular BPS Wilson loop, in particular, is amenable to exact calculations [184–189, 176] and the rich spectrum of BPS Wilson loops in 4d [169–171, 190, 191, 186] allows for further exact results in quantities such as the bremsstrahlung function [119, 133] and its 3d generalisations [140, 141, 172, 192–195]. Some of these loops or close analogues of them arise in our analysis and we hope that the new examples we uncover here will play a similar role in future work.

With this analysis of the bosonic loops complete, it is evident that the story of Wilson loops involving fermi fields and more than one gauge field is still richer than all those already identified in [59, 63–69, 2]. This is explored in [4].

5.2 General analysis

We start by looking at the variation of (5.1) under a generic supercharge. This leads to a set of conditions on the supercharges that can preserve such a loop. These are then used in the subsequent section to reconstruct the loop operator invariant under the possible supercharges.

$\mathcal{N} = 4$ Chern-Simons-matter theories were constructed in [83–85, 89] and the supersymmetry transformations in flat space were presented there. Those were adapted to S^3 in [59], relying also on the decomposition to $\mathcal{N} = 2$ theories and the transformation rules in [89, 90].

Suppressing spinor indices, the variations of the bosonic fields are (2.12)

$$\begin{aligned}
\mathcal{Q}A_\mu &= \frac{i}{k}\xi_{a\dot{b}}\gamma_\mu(q^a\bar{\psi}^{\dot{b}} - \epsilon^{ac}\epsilon^{\dot{b}\dot{c}}\psi_{\dot{c}}\bar{q}_c - \bar{\tilde{q}}^{\dot{b}}\tilde{\psi}^a + \epsilon^{\dot{b}\dot{c}}\epsilon^{ac}\bar{\tilde{\psi}}_c\tilde{q}_{\dot{c}})\,,\\
\mathcal{Q}q^a &= \xi^{a\dot{b}}\psi_{\dot{b}}\,, \qquad \mathcal{Q}\bar{q}_a = \xi_{a\dot{b}}\bar{\psi}^{\dot{b}}\,,\\
\mathcal{Q}\tilde{q}_{\dot{b}} &= -\xi_{a\dot{b}}\tilde{\psi}^a\,, \qquad \mathcal{Q}\bar{\tilde{q}}^{\dot{b}} = -\xi^{a\dot{b}}\bar{\tilde{\psi}}_a\,,
\end{aligned} \tag{5.2}$$

This is all in Euclidean signature and in the frame outlined below $\gamma_\varphi = \sigma_3$.

The supersymmetry variation of the connection (5.1) is then

$$\begin{aligned}
\mathcal{Q}\mathcal{L} = &\frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha + M_a{}^b\xi^\alpha_{b\dot{a}}\right)q^a\bar{\psi}^{\dot{a}}_\alpha - \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha - \xi^\alpha_{b\dot{a}}\epsilon^{bc}M_c{}^d\epsilon_{da}\right)\psi^{\dot{a}}_\alpha\bar{q}^a\\
&- \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha + \widetilde{M}_{\dot{a}}{}^{\dot{b}}\xi^\alpha_{a\dot{b}}\right)\bar{\tilde{q}}^{\dot{a}}\tilde{\psi}^a_\alpha + \frac{i}{k}\left(\xi^\beta_{a\dot{a}}(\sigma_3)_\beta{}^\alpha - \xi^\alpha_{a\dot{b}}\epsilon^{\dot{b}\dot{c}}\widetilde{M}_{\dot{c}}{}^{\dot{d}}\epsilon_{\dot{d}\dot{a}}\right)\bar{\tilde{\psi}}^a_\alpha\tilde{q}^{\dot{a}}\,.
\end{aligned} \tag{5.3}$$

The supercharge $\mathcal{Q}$ is a linear combination of the 16 supercharges $Q^{a\dot{a}}_l$, $Q^{a\dot{a}}_r$, $Q^{a\dot{a}}_{\bar{l}}$ and $Q^{a\dot{a}}_{\bar{r}}$ given by ($\imath$ takes values l, r and likewise $\bar{\imath}$)

$$\mathcal{Q} = \eta^\imath_{a\dot{a}}Q^{a\dot{a}}_\imath + \bar{\eta}^\imath_{a\dot{a}}(\sigma_1)_\imath{}^{\bar{\jmath}}Q^{a\dot{a}}_{\bar{\jmath}}\,. \tag{5.4}$$

The interpolating σ_1 guarantees that $\eta^\imath_{a\dot{a}}, \bar{\eta}^\imath_{a\dot{a}}$ transform in the same representation of the conformal group, see Appendix 5.A for details. The right-hand-side of (5.3) is expressed in terms of $\xi^\alpha_{a\dot{a}}$, which package together the η parameters and the four Killing spinors $\xi^\imath_\alpha$, $\xi^{\bar{\imath}}_\alpha$

$$\xi^\alpha_{a\dot{a}} = \eta^\imath_{a\dot{a}}\xi^\alpha_\imath + \bar{\eta}^\imath_{a\dot{a}}(\sigma_1)_\imath{}^{\bar{\imath}}\xi^\alpha_{\bar{\imath}}\,. \tag{5.5}$$

Note that the Killing spinors appear here with raised spinor and lowered $\imath$, $\bar{\imath}$ indices compared to (2.15).

Let us recall some facts about the Killing spinors. They obey the equations

$$\nabla_\mu\xi^{l,\bar{l}} = \frac{i}{2}\gamma_\mu\xi^{l,\bar{l}}\,, \qquad \nabla_\mu\xi^{r,\bar{r}} = -\frac{i}{2}\gamma_\mu\xi^{r,\bar{r}}\,. \tag{5.6}$$

Specifically, following [187, 91], we can use the Lie group structure of S^3 to construct a left-invariant dreibein e_i with spin connection $\omega_{ij} = \epsilon_{ijk}e^k$. Then the spin connection in the first equation cancels the right-hand-side and the solutions are simply constant spinors. For the great circle along the $\mu = \varphi$ direction, those can be chosen as eigenstates of σ_3. In this setup, the second equation reads

$$\partial_\varphi\xi^{r,\bar{r}} = -i\sigma_3\xi^{r,\bar{r}}\,. \tag{5.7}$$

Clearly they can again be chosen along the circle to be eigenvectors of σ_3 such that we find

$$\xi^l_\alpha = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \xi^{\bar{l}}_\alpha = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \xi^r_\alpha = \begin{pmatrix}e^{-i\varphi}\\0\end{pmatrix}, \qquad \xi^{\bar{r}}_\alpha = \begin{pmatrix}0\\e^{i\varphi}\end{pmatrix}. \tag{5.8}$$

In particular note that the unbarred spinors have only + components and the barred ones only −. This makes the indices redundant and allows us to eliminate some of them as already done in (5.4). With those expressions for the Killing spinors, (5.5) becomes

$$\xi^{+}_{a\dot{a}} = \bar{\eta}^{l}_{a\dot{a}} - \bar{\eta}^{r}_{a\dot{a}} e^{i\varphi}, \qquad \xi^{-}_{a\dot{a}} = \eta^{l}_{a\dot{a}} e^{-i\varphi} - \eta^{r}_{a\dot{a}}. \tag{5.9}$$

For the supersymmetry variation of the connection to vanish, all four terms in (5.3) must vanish. A Wilson loop can also be invariant under a symmetry when the variation of the connection is an appropriate covariant derivative [58], but as there are no derivatives on the right-hand-side, this is not the case here. Multiplying from the left the second term in (5.3) by M and the fourth by $\widetilde{M}$, and both by σ_3 from the right, we find

$$\begin{aligned} \xi^{\beta}_{a\dot{a}}(\sigma_3)_{\beta}{}^{\alpha} + M_a{}^{b}\xi^{\alpha}_{b\dot{a}} &= 0\,, & \xi^{\beta}_{a\dot{a}}(\sigma_3)_{\beta}{}^{\alpha} + \widetilde{M}_{\dot{a}}{}^{\dot{b}}\xi^{\alpha}_{a\dot{b}} &= 0\,, \\ M_a{}^{b}\xi^{\alpha}_{b\dot{a}} - \det(M)\xi^{\beta}_{a\dot{a}}(\sigma_3)_{\beta}{}^{\alpha} &= 0\,, & \widetilde{M}_{\dot{a}}{}^{\dot{b}}\xi^{\alpha}_{a\dot{b}} - \det(\widetilde{M})\xi^{\beta}_{a\dot{a}}(\sigma_3)_{\beta}{}^{\alpha} &= 0\,. \end{aligned} \tag{5.10}$$

Comparing the two lines, this can only be solved by all $\xi^{\alpha}_{a\dot{a}} = 0$ (which means no supersymmetry) or by

$$\det M = \det \widetilde{M} = -1\,. \tag{5.11}$$

The supersymmetry parameters then have to satisfy the eigenvector equations

$$M_a{}^{b}\xi^{\pm}_{b\dot{a}} = \mp\xi^{\pm}_{a\dot{a}}\,, \qquad \widetilde{M}_{\dot{a}}{}^{\dot{b}}\xi^{\pm}_{a\dot{b}} = \mp\xi^{\pm}_{a\dot{a}}\,. \tag{5.12}$$

If this is solved by any nonzero $\xi^{+}_{a\dot{a}}$, both M and $\widetilde{M}$ must have an eigenvalue −1, and if it is solved by $\xi^{-}_{a\dot{a}}$, one eigenvalue of both M, $\widetilde{M}$ must be 1. From (5.11) we see that regardless, M and $\widetilde{M}$ have both the eigenvalues 1 and −1.

The simplest possibility is of course when $M = \widetilde{M} = \mathrm{diag}(1, -1)$. Plugging them back into (5.1), we find the Gaiotto-Yin loop [76] with four preserved supercharges $Q^{\dot{1}1}_{l,r}, Q^{\dot{2}2}_{\bar{l},\bar{r}}$

$$\mathcal{L} = A_\varphi + \frac{i}{k}(q^1\bar{q}_1 - q^2\bar{q}_2 + \bar{\tilde{q}}^{\dot{1}}\tilde{q}_{\dot{1}} - \bar{\tilde{q}}^{\dot{2}}\tilde{q}_{\dot{2}})\,. \tag{5.13}$$

This is indeed the most symmetric and supersymmetric bosonic loop, discussed in more details in Section 5.3.1. Up to rotations to other matrices with eigenvalues 1 and −1, this is the only possibility with constant matrices. So in fact the main focus of this work are the cases when M or $\widetilde{M}$ have φ dependence. For example,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}, \tag{5.14}$$

also have eigenvalues 1 and −1, but are not constant. Those are "latitude" bosonic loops [172,176,59], preserving two supercharges and are discussed in detail in Section 5.3.3. Loops with different values of θ are not related by symmetry and are truly different (and their expectation values are also different [196]). We find below one further inequivalent example in the same class as well as a few new classes of Wilson loops preserving two supercharges.

To study all the possible supersymmetric bosonic loops systematically, we now proceed to look for the most general configurations of M and $\widetilde{M}$ which allow for nonzero solutions to (5.12). Since any 2×2 matrix with two distinct eigenvalues is uniquely determined by its eigenvectors, it is sufficient to determine those.

Note that in the two equations in (5.12) there is a free parameter. For example

$$\xi^-_{a\dot{a}} = M_a{}^b \xi^-_{b\dot{a}}\,, \qquad \dot{a} = \dot{1}, \dot{2}\,. \tag{5.15}$$

In other words, both $\xi^-_{a\dot{1}}$ and $\xi^-_{a\dot{2}}$ are eigenvectors with the eigenvalue 1, so are linearly dependent. In particular, if we view $\xi^-_{a\dot{a}}$ as a 2×2 matrix, its determinant must vanish. This already incorporates the second equation in (5.12) and likewise for $\xi^+_{a\dot{a}}$, giving

$$\det{}_{a\dot{a}}(\xi^-_{a\dot{a}}) = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\xi^-_{a\dot{a}}\xi^-_{b\dot{b}} = 0\,, \qquad \det{}_{a\dot{a}}(\xi^+_{a\dot{a}}) = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\xi^+_{a\dot{a}}\xi^+_{b\dot{b}} = 0\,. \tag{5.16}$$

Being of rank ≤ 1, we can clearly write $\xi^\pm_{a\dot{a}}$ as the outer product of two vectors. But recall that these are linear combinations of two Killing spinors

$$\xi^-_{a\dot{a}} = \eta^\imath_{a\dot{a}}\xi^-_\imath\,. \tag{5.17}$$

The expression for the determinant translates to

$$2\det{}_{a\dot{a}}(\xi^-_{a\dot{a}}) = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\left((\xi^-_l)^2\eta^l_{a\dot{a}}\eta^l_{b\dot{b}} + 2\xi^-_l\xi^-_r\eta^l_{a\dot{a}}\eta^r_{b\dot{b}} + (\xi^-_r)^2\eta^r_{a\dot{a}}\eta^r_{b\dot{b}}\right). \tag{5.18}$$

Since $\xi^-_\imath$ are different functions (as are their squares), (2.15), this vanishes only if the three terms vanish separately

$$\det{}_{a\dot{a}}(\eta^l_{a\dot{a}}) = \det{}_{a\dot{a}}(\eta^r_{a\dot{a}}) = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\eta^l_{a\dot{a}}\eta^r_{b\dot{b}} = 0\,. \tag{5.19}$$

The first two equations allow us to represent η as a the product of two vectors (no sum implied on the right-hand-side)

$$\eta^\imath_{a\dot{a}} = w^\imath_a z^\imath_{\dot{a}}\,, \tag{5.20}$$

where $w^\imath_a$ and $z^\imath_{\dot{a}}$ are constants and the remaining condition is

$$(\epsilon^{ab}w^l_a w^r_b)(\epsilon^{\dot{a}\dot{b}}z^l_{\dot{a}}z^r_{\dot{b}}) = \det{}_{a\imath}(w^\imath_a)\det{}_{\dot{a}\jmath}(z^\jmath_{\dot{a}}) = \det{}_{a\dot{a}}\left(\sum_\imath w^\imath_a z^\imath_{\dot{a}}\right) = 0\,. \tag{5.21}$$

All the same arguments carry over to $\xi^+_{a\dot{a}}$, expressing

$$\bar{\eta}^\imath_{a\dot{a}} = \bar{w}^\imath_a \bar{z}^\imath_{\dot{a}}\,, \tag{5.22}$$

subject to the constraint

$$(\epsilon^{ab}\bar{w}^l_a\bar{w}^r_b)(\epsilon^{\dot{a}\dot{b}}\bar{z}^l_{\dot{a}}\bar{z}^r_{\dot{b}}) = \det{}_{a\dot{a}}\left(\sum_{\imath}\bar{w}^{\imath}_a\bar{z}^{\imath}_{\dot{a}}\right) = 0\,. \tag{5.23}$$

We may define two matrices of Killing spinors $\Xi^- = \mathrm{diag}(\xi^-_l, \xi^-_r)$ and $\Xi^+ = \mathrm{diag}(\xi^+_{\bar{r}}, \xi^+_{\bar{l}})$, where the indices are properly raised and lowered with respect to (2.15). Also, it is natural to incorporate the action of σ_1, as in (5.5), such that they both have two unbarred $\imath$ indices and we can now combine (5.20), (5.22) to write $\xi^{\pm}_{a\dot{a}}$ as

$$\xi^-_{a\dot{a}} = w^{\imath}_a\Xi^-_{\imath\jmath}z^{\jmath}_{\dot{a}}\,, \qquad \xi^+_{a\dot{a}} = \bar{w}^{\imath}_a\Xi^+_{\imath\jmath}\bar{z}^{\jmath}_{\dot{a}}\,. \tag{5.24}$$

Going back to (5.21), there is a nonzero $\xi^-_{a\dot{a}}$ giving a preserved supersymmetry if either $\det_{a\imath}(w^{\imath}_a)$ or $\det_{\dot{a}\jmath}(z^{\jmath}_{\dot{a}})$ vanish. To enumerate the different possibilities:

1. $w^{\imath}_a = w_a y^{\imath}$, but can immediately absorb $y^{\imath}$ in $z^{\imath}_{\dot{a}}$, such that $\eta^{\imath}_{a\dot{a}} = w_a z^{\imath}_{\dot{a}}$.
2. $z^{\imath}_{\dot{a}}$ factors, so $\eta^{\imath}_{a\dot{a}} = w^{\imath}_a z_{\dot{a}}$.
3. If both determinants vanish independently, we have $\eta^{\imath}_{a\dot{a}} = y^{\imath}w_a z_{\dot{a}}$.

In the first two cases all the components of $\eta^{\imath}_{a\dot{a}}$ are related, representing a single preserved supercharge. In the last case the factorization to $y^{\imath}$ means that we have two independent solutions with $\eta^l_{a\dot{a}} = w_a z_{\dot{a}}$ and another one with $\eta^r_{a\dot{a}} = w_a z_{\dot{a}}$.

All the same analysis carries over to $\xi^+_{a\dot{a}}$ allowing to find zero, one or two independent solutions.

Lastly, because $\xi^-_{a\dot{a}}$ and $\xi^+_{a\dot{a}}$ have different eigenvalues, they cannot be proportional to each-other, so for any pair of nonzero eigenvectors

$$\epsilon^{ab}\xi^-_{a\dot{a}}\xi^+_{b\dot{b}} \neq 0\,, \qquad \epsilon^{\dot{a}\dot{b}}\xi^-_{a\dot{a}}\xi^+_{b\dot{b}} \neq 0\,. \tag{5.25}$$

When both $\xi^-_{a\dot{a}}$ and $\xi^+_{a\dot{a}}$ factorise, this translates into conditions such as $\epsilon^{ab}w_a\bar{w}_b \neq 0$.

5.3 Representative examples

We are now ready to write down examples of Wilson loops preserving varying number of supercharges. In this section we list the cases up to actions of symmetries explained in Appendix 5.A. We also outline the proof that the examples we present are indeed representatives of every orbit and demonstrate some details of the other elements in the orbit.

The relevant symmetries are the R-symmetry group $SU(2)_L \times SU(2)_R$ (and its complexification) acting on the indices a and $\dot{a}$ respectively and the conformal group $SL(2,\mathbb{R})$ (and its complexification), acting on functions of φ (and on the indices l, r, $\bar{l}$, $\bar{r}$). In addition the supersymmetry equations (5.3) have two discrete transformations relating different solutions:

- The exchange $M \leftrightarrow \widetilde{M}$ or equivalently $SU(2)_L \leftrightarrow SU(2)_R$ relates cases with factorised w and/or $\bar{w}$ to cases with factorised z and/or $\bar{z}$.
- The simultaneous change of sign of M and $\widetilde{M}$ exchanges the 1 and -1 eigenvalues in (5.12). Since those are matched to the spinor index and then via (2.15) to the exchange $l, r \leftrightarrow \bar{l}, \bar{r}$. This has the effect of relating cases with factorised w and/or z to cases with factorised $\bar{w}$ and/or $\bar{z}$.

The symmetry actions are explained in more detail in Appendix 5.A.

If we know two eigenvectors of M with the different eigenvalues, say $\xi^-_{a\dot{1}}$ and $\xi^+_{a\dot{2}}$, we can easily reconstruct M as

$$M_a{}^b = \begin{pmatrix} \xi^-_{a\dot{1}} & \xi^+_{a\dot{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \xi^-_{b\dot{1}} & \xi^+_{b\dot{2}} \end{pmatrix}^{-1} = \frac{1}{\epsilon^{ab}\xi^-_{a\dot{1}}\xi^+_{b\dot{2}}} \begin{pmatrix} \xi^-_{1\dot{1}}\xi^+_{2\dot{2}} + \xi^-_{2\dot{1}}\xi^+_{1\dot{2}} & -2\xi^-_{1\dot{1}}\xi^+_{1\dot{2}} \\ 2\xi^-_{2\dot{1}}\xi^+_{2\dot{2}} & -\xi^-_{1\dot{1}}\xi^+_{2\dot{2}} - \xi^-_{2\dot{1}}\xi^+_{1\dot{2}} \end{pmatrix}, \tag{5.26}$$

and likewise $\widetilde{M}$. In this way we can find all BPS Wilson loops by going over all possible preserved supercharges. If the loop preserves both $\xi^-_{a\dot{1}}$ and $\xi^-_{a\dot{2}}$ (and say one $\xi^+_{a\dot{1}}$), we would get the same expression from any linear combination of the two ξ^-. The cases with single ξ^- or ξ^+ arise from not fully factorised $\eta^\pm$, which is different from a linear combination of factorised ones.

In the case when the loop preserves only one or two ξ^- and no ξ^+, we can plug into the formula above any vector ξ^+ which is not proportional to the eigenvector ξ^- to get an appropriate M. As ξ^+ is any vector not necessarily made of the Killing spinors, it can have arbitrary dependence on φ, and in turn so does M. See Section 5.3.3 and Section 5.3.4 below.

5.3.1 1/4 BPS loops

The most supersymmetric bosonic loops preserve two components of $\xi^-_{a\dot{a}}$ and two of $\xi^+_{a\dot{a}}$. They are given by $\eta^\imath_{a\dot{a}} = y^\imath w_a z_{\dot{a}}$ and $\bar{\eta}^\imath_{a\dot{a}} = \bar{y}^\imath \bar{w}_a \bar{z}_{\dot{a}}$. We can choose $w_a = \delta^1_a$, $z_{\dot{a}} = \delta^{\dot{1}}_{\dot{a}}$, $\bar{w}_a = \delta^2_a$, $\bar{z}_{\dot{a}} = \delta^{\dot{2}}_{\dot{a}}$ such that the four independent supercharges have the nonzero parameters

$$\eta^l_{1\dot{1}}, \qquad \eta^r_{1\dot{1}}, \qquad \bar{\eta}^l_{2\dot{2}}, \qquad \bar{\eta}^r_{2\dot{2}}. \tag{5.27}$$

Clearly they are eigenvector with eigenvalues ± 1 of

$$M = \widetilde{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.28}$$

which is also what one gets from (5.26), matching (5.13).

This is the $\mathcal{N} = 4$ avatar of the Wilson loops first constructed by Gaiotto and Yin [76] and then rediscovered as the 1/6 BPS loops of ABJM theory [60–62]. These M and $\widetilde{M}$ break $SU(2)_L \times SU(2)_R$ to $U(1)^2$. Conjugation of M by elements of the complexified $SL(2, \mathbb{C})_L$

can produce any other constant matrix with eigenvalues ± 1, and likewise for $\widetilde{M}$. So this is the orbit of the broken symmetry group.

Looking at it from the point of view of the parameters w_a, $\bar{w}_a$, $z_{\dot{a}}$, $\bar{z}_{\dot{a}}$, $y^\imath$ and $\bar{y}^\imath$, starting with any other factorised choice, we can use two $SL(2,\mathbb{C})_{L,R}$ actions to set $w_a = \delta_a^1$ and $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$ as in the above example. This on its own does not fix the barred parameters. But as they cannot be linearly dependent on the unbarred ones (5.25), we can use the Gram-Schmidt process to produce $\bar{w}_a \propto \delta_a^2$, $\bar{z}_{\dot{a}} \propto \delta_{\dot{a}}^{\dot{2}}$, and then can rescale them to reproduce (5.27).

The superalgebra generated by the four preserved supercharges in (5.27) includes the $SO(2,1)_C$ conformal group of the circle, guaranteeing that these loops are conformal [195,2]. In fact, of all the loops discussed in this paper, these are the only ones that are conformal and all the other ones have explicit φ dependence in M and/or $\widetilde{M}$.

5.3.2 3/16 BPS loops

This case arises when three of w, z, $\bar{w}$, $\bar{z}$ factorise while the remaining one doesn't. Let us focus on the case when $\bar{z}$ does not factorise. As above, we take $w_a = \delta_a^1$, $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$, $\bar{w}_a = \delta_a^2$ and $\bar{z}_{\dot{a}}^\imath = \delta_{\dot{a}}^\imath$ (with $l \simeq \dot{1}$).

The preserved supercharges are then linear combinations of

$$Q_l^{1\dot{1}}\,, \qquad Q_r^{1\dot{1}}\,, \qquad Q_{\bar{r}}^{2\dot{1}} + Q_{\bar{l}}^{2\dot{2}}\,, \tag{5.29}$$

and the connection is as in (5.1) with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 & 2e^{-i\varphi} \\ 0 & -1 \end{pmatrix}. \tag{5.30}$$

This is a new example of a Wilson loop with an exotic number of preserved supercharges. Unlike the 1/4 BPS loops, this is not conformally invariant, as is evident by the explicit φ dependence in $\widetilde{M}$. Acting with the broken $SU(2)_L \times SU(2)_R \times SO(2,1)_C$ (possibly complexified) gives a rich orbit of further examples. Note that the action of the complexified conformal group $SL(2,\mathbb{C})$ can transform $e^{i\varphi}$ to any fractional linear function. Other examples include the replacement of $M \leftrightarrow \widetilde{M}$ as well as changing their signs. Those correspond to a choice of a different parameter among w, z, $\bar{w}$, $\bar{z}$ that doesn't factorise. In particular the example with lower triangular $\widetilde{M}$ is related by symmetry to the case where M and $\widetilde{M}$ have opposite signs.

To prove that there is a single orbit of the group and (5.30) is indeed a representative, note that any unfactorised $\bar{z}$ is a rank 2 matrix which can be brought into the form above with the action of either $SL(2,\mathbb{C})_R$ or $SL(2,\mathbb{C})_C$. It is still invariant under conjugation by any identical element of the two groups, which means we have $SL(2,\mathbb{C})_R$ freedom to set $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$. Then the $SL(2,\mathbb{C})_L$ action can bring $\bar{w}$ into the desired form. This procedure leaves w as a vector not parallel to $\bar{w}$, so we can choose a linear combination producing $w_a = \delta_a^1$.

Supersymmetry enhancement can be easily seen by exploiting the action of $SL(2,\mathbb{C})_C$, which allows us to apply arbitrary constant rescalings to $e^{i\varphi}$. In the limit where the phase in (5.30) vanishes, supersymmetry is enhanced to 1/4 BPS.

5.3.3 1/8 BPS loops

There are of course $\binom{4}{2} = 6$ different pairs out of w, z, $\bar{w}$, $\bar{z}$ to factorise, but they are pairwise related by extra symmetries. We discuss the three inequivalent classes below.

Factorised w and $\bar{w}$

In this case we can take representatives with $w_a = \delta_a^1$, $\bar{w}_a = \delta_a^2$ and $z_{\dot{a}}^{\imath} = \delta_{\dot{a}}^{\imath}$. To get to this form from an arbitrary unfactorised $z_{\dot{a}}^{\imath}$, we can either act from the left with $SL(2,\mathbb{C})_R$ or from the right with $SL(2,\mathbb{C})_C$. This form is still invariant under conjugation by the same elements of the two groups, so that is the remaining freedom we have to act on $\bar{z}$, as well as overall rescaling, which is immaterial. By rescaling we can make $\det_{\dot{a}\imath} \bar{z} = 1$ (since it has rank 2) and then by conjugation bring it to Jordan normal form

$$\bar{z}_\lambda = \begin{pmatrix} \bar{z}_{\dot{1}}^l & \bar{z}_{\dot{1}}^r \\ \bar{z}_{\dot{2}}^l & \bar{z}_{\dot{2}}^r \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \bar{z}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{5.31}$$

Plugging the diagonal case into (5.26) gives

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \frac{1}{\lambda - \lambda^{-1}} \begin{pmatrix} -\lambda - \lambda^{-1} & -2\lambda e^{-i\varphi} \\ 2\lambda^{-1} e^{i\varphi} & \lambda + \lambda^{-1} \end{pmatrix}. \tag{5.32}$$

These loops are in fact related to the "bosonic latitude loops" (5.14) of [172, 176, 59]. To see that, we take $\cos\theta = -(\lambda + \lambda^{-1})/(\lambda - \lambda^{-1})$, such that

$$\widetilde{M}_\theta = \begin{pmatrix} \cos\theta & i\lambda e^{-i\varphi} \sin\theta \\ -i\lambda^{-1} e^{i\varphi} \sin\theta & -\cos\theta\,. \end{pmatrix}. \tag{5.33}$$

Conformal symmetry acts on $e^{\pm i\varphi}$ as Möbius transformations, which in particular includes the rescaling that eliminates $i\lambda$ from the matrix above, reproducing (5.14). This can also be realised by conjugating $\widetilde{M}$ by

$$\begin{pmatrix} 1/\sqrt{i\lambda} & 0 \\ 0 & \sqrt{i\lambda} \end{pmatrix}, \tag{5.34}$$

which is an $SL(2,\mathbb{C})_R$ transformation.

In terms of z and $\bar{z}$, this $SL(2,\mathbb{C})_R$ acts on them from the left giving

$$z \to \begin{pmatrix} 1/\sqrt{i\lambda} & 0 \\ 0 & \sqrt{i\lambda} \end{pmatrix}, \qquad \bar{z}_\lambda \to \begin{pmatrix} -i\sqrt{i\lambda} & 0 \\ 0 & i/\sqrt{i\lambda} \end{pmatrix}. \tag{5.35}$$

Inserting the resulting $\xi^{\pm}_{a\dot{a}}$ into (5.26) and taking the above relation between λ and θ produces the same result (5.14). Since the original and new z and $\bar{z}$ are all diagonal, the exact same result can be achieved by right multiplication, which is an $SL(2,\mathbb{C})_C$ transformation. The fact that we can act with either of the groups indicates that these loops are invariant under a particular combination of the two group actions, which is a known symmetry of the latitude [196, 2].

Turning to the non-diagonal case in (5.31). This is a new example, which to our knowledge has not been previously described. We find the matrices

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} -1 + 2e^{-i\varphi} & -2e^{-i\varphi} + 2e^{-2i\varphi} \\ -2 & 1 - 2e^{-i\varphi} \end{pmatrix}. \tag{5.36}$$

Another representative of this orbit is given by

$$z = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \bar{z} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \tag{5.37}$$

which leads to

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 - 2e^{2i\varphi} & 2e^{i\varphi}\left(1 + e^{i\varphi}\right) \\ 2e^{i\varphi}\left(1 - e^{i\varphi}\right) & -1 + 2e^{2i\varphi} \end{pmatrix}. \tag{5.38}$$

Instead of constructing the detailed map between the two cases, as we did for the latitudes, note that z and $\bar{z}$ offer a simple way to identify the orbit. Since z, $\bar{z}$ both transform in the fundamental of $SL(2,\mathbb{C})_R$ and the antifundamental of $SL(2,\mathbb{C})_C$. The matrix $z^{-1}\bar{z}$ transforms in the adjoint of $SL(2,\mathbb{C})_C$ and is invariant under the other group. Conversely $\bar{z}z^{-1}$ is in the adjoint of $SL(2,\mathbb{C})_R$ and invariant under the other group. The eigenvalues of these matrices are then invariant under both groups, but these two matrices have the same eigenvalues. Since any nonzero rescaling is immaterial, we can always set them to λ and $1/\lambda$ and compare with (5.31).

Clearly for the latitudes in (5.35), we reproduce the same eigenvalues as $\bar{z}_\lambda$ in (5.31). Likewise, the Jordan form of those for (5.37) is the same as z'.

The case with factorized z and $\bar{z}$ gives similar loops under the exchange $M \leftrightarrow \widetilde{M}$.

To see how to get supersymmetry enhancement to the previous examples, the case of enhancement to 4 supercharges in Section 5.3.1 is obvious, by taking $\theta \to 0$ in (5.14).

To get to the 3/16 BPS case in Section 5.3.2, we need to take $\lambda \to \infty$ in (5.32). In terms of θ, this is a double scaling limit by first using a complexified conformal transformation that scales $e^{i\varphi} \to 2e^{i\varphi}/\theta$ and $e^{-i\varphi} \to \theta\, e^{-i\varphi}/2$ and then take $\theta \to 0$. The expression in (5.14) then clearly becomes the transpose of (5.30), which is another example of a 3/16 BPS loop.

The nondiagonalisable case admits similar limits. Rescaling $e^{-i\varphi}$ in (5.36) allows us to tune out the φ dependent terms entirely, which brings us back to the 1/4 BPS case. To hit a 3/16 BPS orbit instead, we first rescale the phases by x^2, and then conjugate $\widetilde{M}$ with $\mathrm{diag}(x, x^{-1})$, which shifts one factor of x^2 from the bottom left to the top right. The limit $x \to 0$ then removes all but one phase, and we recover the 3/16 BPS loop (5.30), as before.

Factorised w and $\bar{z}$

As an illustration of this case, we can take $w_a = \delta_a^1$, $\bar{z}_{\dot{a}} = \delta_{\dot{a}}^{\dot{2}}$, $\bar{w}_a^\imath = \delta_a^1 \delta_l^\imath - \delta_a^2 \delta_r^\imath$ and $z_{\dot{a}}^\imath = \delta_{\dot{a}}^\imath$ to get

$$M = \begin{pmatrix} 1 & -2e^{-i\varphi} \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 & 0 \\ -2e^{i\varphi} & -1 \end{pmatrix}. \tag{5.39}$$

Unlike the previous example, here we could use the symmetry to choose a unique representative, so there is only one conjugacy class (the argument follows the same logic as in the previous examples). To our knowledge, such loops have not been previously described. The action of the conformal group on these loops produces more loops with fractional linear functions in both M and $\widetilde{M}$.

The case when $\bar{w}$ and z are instead factorised is related again by $M \leftrightarrow \widetilde{M}$.

The loop (5.39) admits 3/16 BPS limits. Conjugation with $\mathrm{diag}(x, x^{-1}) \in SL(2,\mathbb{C})_L$ and taking $x \to \infty$ allows us to tune out the phase in M and the same can be done with $\widetilde{M}$.

Factorised w and z

When w and z both factorise, we have two preserved ξ^- supercharges and no ξ^+ ones. We choose representative supercharges with $w_a = \delta_a^1$ and $z_{\dot{a}} = \delta_{\dot{a}}^{\dot{1}}$. This does not completely fix M and $\widetilde{M}$, as (5.26) requires also to specify ξ^+. From the above information alone, we find

$$M_1{}^1 = \widetilde{M}_1{}^1 = 1\,, \qquad M_2{}^1 = \widetilde{M}_2{}^1 = 0\,. \tag{5.40}$$

Then, since $\det M = \det \widetilde{M} = -1$, we get $M_2{}^2 = \widetilde{M}_2{}^2 = -1$, which leaves $M_1{}^2$ and $\widetilde{M}_1{}^2$ as completely arbitrary periodic functions of φ. We denote them $2\mu(\varphi)$ and $2\tilde{\mu}(\varphi)$ respectively to get

$$M = \begin{pmatrix} 1 & 2\mu(\varphi) \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} 1 & 2\tilde{\mu}(\varphi) \\ 0 & -1 \end{pmatrix}. \tag{5.41}$$

Alternatively, we can arrive at the same result by choosing the second eigenvectors for M and $\widetilde{M}$ as

$$\begin{pmatrix} \mu(\varphi) \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\mu}(\varphi) \\ -1 \end{pmatrix}. \tag{5.42}$$

Of course, if $\mu(\varphi)$ or $\tilde{\mu}(\varphi)$ are constants or $e^{\pm i\varphi}$, these eigenvectors are in fact Killing spinors and the loop will have enhanced supersymmetry to 3/16 or 1/4 and will match the forms in Section 5.3.2 or Section 5.3.1 up to symmetry action.

To understand the reason for this freedom of arbitrary functions in M and $\widetilde{M}$, it is instructive to reconstruct the general supercharge preserved by these loops, that is (5.4)

$$\mathcal{Q} = \eta_{1\dot{1}}^\imath Q_\imath^{1\dot{1}}\,. \tag{5.43}$$

Examining the supersymmetry variations (2.12), we clearly see that $\mathcal{Q}(q^1 \bar{q}_2) = \mathcal{Q}(\bar{\tilde{q}}^{\dot{1}} \tilde{q}_{\dot{2}}) = 0$. So this Wilson loop is simply the 1/4 BPS Wilson loop of Section 5.3.1 with arbitrary insertions of these bilinears that are chiral under this pair of supercharges.

5.3.4 1/16 BPS loops

Among the four possible cases, we choose the one with factorised w, so in particular no ξ^+ supercharges. If we choose $w_a = \delta_a^1$ and $z_{\dot a}^{\imath} = \delta_{\dot a}^{\imath}$, then for M things are similar to the last case, where we get ${M_1}^1 = 1$, ${M_2}^1 = 0$, and by $\det M = 1$ we have the other two entries ${M_1}^2 = 2\mu(\varphi)$, ${M_2}^2 = -1$. The second matrix, $\widetilde{M}$, is different. Its entries satisfy the two equations

$$
{\widetilde{M}_{\dot 1}}{}^{\dot 1} e^{-i\varphi} - {\widetilde{M}_{\dot 1}}{}^{\dot 2} = e^{-i\varphi}\,, \qquad {\widetilde{M}_{\dot 2}}{}^{\dot 1} e^{-i\varphi} - {\widetilde{M}_{\dot 2}}{}^{\dot 2} = -1\,, \tag{5.44}
$$

and there is the extra condition $\det \widetilde{M} = -1$, $\operatorname{tr} \widetilde{M} = 0$. It therefore still has one completely free parameter and we take ${\widetilde{M}_{\dot 1}}{}^{\dot 1} = \tilde\mu(\varphi)$. Then

$$
M = \begin{pmatrix} 1 & 2\mu(\varphi) \\ 0 & -1 \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} \tilde\mu(\varphi) & e^{-i\varphi}(\tilde\mu(\varphi) - 1) \\ -e^{i\varphi}(\tilde\mu(\varphi) + 1) & -\tilde\mu(\varphi) \end{pmatrix}. \tag{5.45}
$$

5.4 Theories without twisted hypers

In the discussion so far, we assumed that the theory contains both hypermultiplets and twisted hypermultiplets, but this is not necessary for $\mathcal{N} = 4$ supersymmetry. The discussion follows through if the theory has only hypermultiplets. The supersymmetry variations are as in (2.12) except that we should remove all the twisted hypermultiplet fields.

The ansatz for the Wilson loops is as in (5.1), but without the term involving $\widetilde{M}$. The theory has the same 16 supercharges and Killing spinors, but the supersymmetry conditions impose far fewer constraints. Specifically, only the left equation in (5.12) remains, meaning that there is no constraint on the dotted indices of the supercharges.

In terms of the conditions on η and $\bar\eta$, the z and $\bar z$ parameters are always factorised, so equation (5.20) becomes

$$
\eta^{\imath}_{a\dot a} = w^{\imath}_a z_{\dot a}\,, \tag{5.46}
$$

and the requirement for supersymmetry is that $\det_{a\imath} w^{\imath}_a = 0$ or $\det_{a\imath} \bar w^{\imath}_a = 0$, meaning one of them also factorises.

The completely factorised case is as in Section 5.3.1, with

$$
M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.47}
$$

Now this loop preserves 8 supercharges, so is 1/2 BPS.

When only w factorises but not $\bar w$ we have loops similar to the construction in Section 5.3.2

$$
M = \begin{pmatrix} 1 & 2e^{-i\varphi} \\ 0 & -1 \end{pmatrix}, \tag{5.48}
$$

which are now 1/4 BPS. Likewise, when only $\bar w$ factorises we can get a similar M with the phase in the bottom left.

In a linear quiver theory with two nodes, bifundamental hypermultiplets and no twisted hypermultiplets, there are then two bosonic 1/2 BPS loops, one at each of the nodes as well as fermionic 1/2 BPS loops. The bosonic loops break $SU(2)_L$ and preserve $SU(2)_R$. The 1/2 BPS fermionic loops break $SU(2)_R$ and preserve $SU(2)_L$.

Acknowledgements

We would like to acknowledge fruitful discussions with E. Pomoni. ND is supported by STFC under the grants ST/T000759/1 and ST/P000258/1 and by the National Science Foundation under Grant No. NSF PHY-1748958. The work of ZK is supported by CSC grant No. 201906340174. ND would like to thank École Polytechnique Fédérale de Lausanne, the Simons Center for Geometry and Physics, Stony Brook and the KITP, Santa Barbara for their hospitality in the course of this work. ZK would like to thank DT and the University of Modena and Reggio Emilia as well as T. Fiol and the University of Barcelona for their hospitality.

5.A Symmetries

The theories studied here have $SO(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ R-symmetry and $SO(4,1)$ conformal symmetry, which are packaged together into an $OSp(4|4)$ supergroup. The geometry of the circle breaks the conformal group to $SO(2,1)_C \times SO(2)$, and the particular choice of line operator can break the symmetry further. However, the bosonic loops all are invariant under the transverse $SO(2)$, so what we focus on is the action of $SU(2)_L \times SU(2)_R \times SO(2,1)_C$ on the loops constructed in the body of the paper.

To be precise, we are not imposing any reality or hermiticity conditions, so we allow for the action of the complexified group $SL(2,\mathbb{C})_L \times SL(2,\mathbb{C})_R \times SL(2,\mathbb{C})_C$. The scalars q_a are in the fundamental of $SU(2)_L$ and $\bar{q}^a$ in the anti-fundamental. The scalars from the twisted hyper are charged under $SU(2)_R$. Clearly the matrices M and $\widetilde{M}$ are in the adjoint of $SU(2)_L$ and $SU(2)_R$ respectively. The action on the supercharges as well as on $\eta, \bar{\eta}$ and their decomposition into the parameters w, z, y, $\bar{w}$, $\bar{z}$ and $\bar{y}$ can be read off from their index structure.

The action of the conformal group is more involved. The parameters $\eta^{\imath}_{a\dot{a}}$ and $\bar{\eta}^{\imath}_{a\dot{a}}$ are doublets, with the indices $\imath$ interchanged under the action of the conformal generators $J_\pm$ satisfying the algebra

$$[J_0, J_\pm] = \pm J_\pm \,, \qquad [J_+, J_-] = 2J_0 \,. \tag{5.49}$$

Under a finite conformal transformation $U \in SL(2,\mathbb{C})_C$, they transform as

$$\eta^{\imath}_{a\dot{a}} \longmapsto \eta^{\jmath}_{a\dot{a}} \left(U^T\right)_{\jmath}^{\ \imath} , \qquad \bar{\eta}^{\imath}_{a\dot{a}} \longmapsto \bar{\eta}^{\jmath}_{a\dot{a}} \left(U^T\right)_{\jmath}^{\ \imath} . \tag{5.50}$$

The action on M and $\widetilde{M}$ is set by their φ dependence. The 1/4 BPS loops in Section 5.3.1 are all invariant. In all the other cases we know that the conformal group acts on the unit circle via Möbius transformations

$$U:\ e^{i\varphi} \longmapsto \frac{de^{i\varphi}-b}{-ce^{i\varphi}+a}\,, \qquad U=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad-bc=1\,. \tag{5.51}$$

This determines the transformations of all the Wilson loops discussed in Section 5.3.

6 A net of hyperloops

This section is based on [4] with minor edits.

6.1 Introduction and conclusions

In three-dimensional supersymmetric conformal field theories there are vast moduli spaces of BPS Wilson loops. In addition to the bosonic loops that couple to only one gauge field and bilinear of the scalars [76], most BPS Wilson loops have the superconnections $\mathcal{L}$ comprised of at least two vector fields as well as the matter fields in figure 1

$$W = \mathrm{sTr}\,\mathcal{P}\exp i\oint \mathcal{L}d\varphi\,. \tag{6.1}$$

More and more such examples have been found in the past years [2, 3, 58–70]. For a recent review about what was known at the time on this topic, see [58].

To discover new BPS Wilson loops, instead of relying on complicated ansatze, there is a more efficient algorithm constructed in the series of "Hyperloop" papers [59, 2, 3]

Algorithm 1

1. *Pick a BPS Wilson loop of the theory with (super)connection $\mathcal{L}$.*
2. *Choose a supercharge it preserves, $\mathcal{Q}$.*
3. *Look for deformations that still preserve that supercharge.*

The hyperloops refer to the supersymmetric Wilson loops, especially those coupling to the fermionic fields in $\mathcal{N} = 4$ supersymmetric Chern-Simons-matter theories with either linear or circular quiver structure [83–86, 89] on S^3 along a great circle. However, this algorithm can be easily applied to other supersymmtric theories such as ABJM[17] Step 3 of this algorithm centers on the formula

$$\mathcal{L} \to \mathcal{L} - i\mathcal{Q}G + \{H, G\} + \Pi\tilde{\Pi}G^2 + C\,, \tag{6.2}$$

where $\Pi, \tilde{\Pi}$ (4.15), (6.29)are parameters related to the supercharge $\mathcal{Q}$, and H is determined by the relaxed supersymmetry condition [18]

$$\mathcal{Q}\mathcal{L} = \mathcal{D}_\varphi^{\mathcal{L}} H\,, \tag{6.3}$$

and G, C are some other supermatrices that will be explained in detail later (6.34), (6.63), (6.62).

[17] Actually this algorithm is firstly proposed by [58] in ABJM theory.

[18] The definition of the covariant derivative is consistent with [2] that $\mathcal{D}_\varphi^{\mathcal{L}} H = \partial_\varphi H - i[L_{\mathrm{bos}}, H] + i\{L_{\mathrm{fer}}, H\}$, where L_{bos}, L_{fer} are the bosonic and fermionic part of the supermatrix $\mathcal{L}$ respectively.

The first Hyperloop paper [59] studied the cases with bosonic loops as the starting points, and deformed with certain linear combinations of supercharge $\mathcal{Q}$. The resulting moduli spaces are composed of hyperloops preserving the same supercharge. The second Hyperloop paper [2] focused on the deformations of 1/2 BPS loops with arbitrary linear combination of the supercharges. Since there are eight supercharges preserved by a 1/2 BPS loop, in this way, we discovered an eight-dimensional space, where each point corresponds to one moduli space generated by a fixed supercharge. These moduli spaces intersect where hyperloops preserve more than one supercharge, for example loops with $SU(2)$ R-symmetry enhancements.

We dub the connected components of moduli spaces as the "network". In this paper, instead of a fixed starting point, we allow $\mathcal{L}$ to travel along the network, and look for all possible moduli spaces produced by the algorithm. At each step of the itinerary we run the whole algorithm 1 to obtain the consequent moduli spaces with (6.2), and the starting point $\mathcal{L}$ and the supercharge[19] $\mathcal{Q}$ are chosen as below

i) Starting with the 1/2 BPS loops and arbitrary non-nilpotent supercharges[20] preserved by it, the resulting moduli spaces are found in [2],

ii) Starting with the bosonic loops[21] [3] and arbitrary supercharges preserved by them. In the resulting moduli spaces, there are some special points that receive supersymmetry enhancements, especially those preserving $SU(2)$ R-symmetry (thus being 1/2, 3/8 and 1/4 BPS).

iii) Starting with the $SU(2)$ enhanced points and any preserved supercharge including both nilpotent and non-nilpotent ones.

Note that in step ii) we actually employ a further trick. The 1/2 BPS loops we start with are built around the adjacent I and $I{+}1$ nodes, when it comes to the bosonic loops, since they contain no terms that are linear in the matter fields, one may decouple the nodes and rebuild a superconnection around the $I{+}1$ and $I{+}2$ ones, which are the starting point of the second deformation. All of the new hyperloops constructed here are in this setting. Alternatively, one can also take the superconnections in larger supermatrices that are built around I, $I{+}1$, $I{+}2$ nodes and study BPS deformations in such cases, like in the section 5 of [2].

In fact, we can choose any supersymmetric loop in the network as the new starting point for the deformation, which will always take the same form as (6.2) with the replacement of the corresponding $\Pi', \tilde{\Pi}'$ and $\mathcal{Q}', H'$. However, the resulting moduli spaces will be subspaces of those generated in i), ii) and iii). In other words, the network is closed under the algorithm. Compared to the previous Hyperloop papers, the difference in the implementation of algorithm 1 is summarised in the flowcharts 4

[19] In this paper, we do not consider $\mathcal{Q}$ combined with only supercharges in the same chirality.

[20] The $\Pi \neq 0$ supercharges in [2].

[21] There are some subtleties that we will explain later around (6.11).

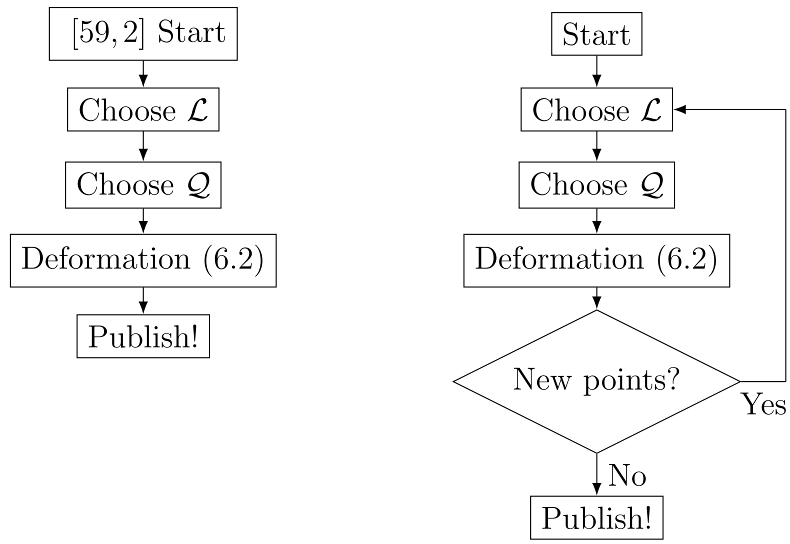

Figure 4: Strategies used in [59,2] and this paper. "New points" in the conditional block refers to those with supersymmetry enhancements.

Let us review some of the resulting network. One special kind of loops is the bosonic loops. Their full classification in $\mathcal{N}=4$ Chern-Simons-matter theory is given in the third Hyperloop paper [3], where in the notation employed here, all of them can be summarized as[22]

$$\mathcal{A}^{\text{bos}} = \mathcal{A}_\varphi + \frac{i}{k\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2) - \frac{i}{k\tilde{\Pi}}(\bar{\tilde{r}}^{\dot{1}}\tilde{r}_{\dot{1}} - \bar{\tilde{r}}^{\dot{2}}\tilde{r}_{\dot{2}})\,, \tag{6.4}$$

where $r, \tilde{r}$ (6.13), (6.28) are the rotated scalar fields. Because of the full classification of bosonic loops, in step ii) we exhaust all the hyperloops (6.21) with $SU(2)$ R-symmetry enhancements in the diagonal entries. By setting them as the new starting point in step iii) and perform the supersymmetric deformations, we find all the moduli spaces of hyperloops that are connected to theirs.

All the hyperloops in this network can be classified into two types. One is what we call $\Pi \neq 0$ (or $\tilde{\Pi} \neq 0$) loops that can be deformed from all the possible bosonic loops and arbitrary preserved supercharge $\mathcal{Q}$. The other one is what we call $\Pi = 0$ (or $\tilde{\Pi} = 0$)[23] loops, first found in [2], where the preserved supercharges $\mathcal{Q}$ are nilpotent and some scalars are annihilated by them. They are produced in step i) and iii) and do not live in the moduli spaces generated by the deformation from bosonic loops ii). In particular, in this case we find other loops that preserves $SU(2)_L$ (or $SU(2)_R$) R-symmetries.

Then a question naturally comes up, whether these moduli spaces are really complete? Although we have exhausted all the connected moduli spaces of hyperloops through our

[22] Except for two special cases given in section 3.3.3 and 3.4 in [3] where the preserved supercharges are composed entirely of either barred or unbarred ones.

[23] Π and $\tilde{\Pi}$ do not vanish at the same time.

construction, we are still unable to give a definite answer since there might be some isolated components in the complete moduli space. It would be very interesting to look for such BPS Wilson loops. In particular, a longstanding question was whether 1/3 BPS Wilson loops exist in ABJM theory. This was answered in the recent paper [5], but when we set the 1/3 BPS loop as the starting point and played our algorithm, we did not find more 1/3 BPS loops up to global gauge symmetries. Therefore, if other 1/3 BPS loops really exist, they probably live in the isolated points.

Another question is about Wilson loops with superconnections in larger supermatrices, especially those coupling to repetitive gauge fields [69]. We attempted to set the starting point as two copies of 1/2 BPS loops (6.5), in which case the gauge symmetry is $S(GL(2,\mathbb{C})\times GL(2,\mathbb{C}))$, where S denotes that the center of $GL(4,\mathbb{C})$ is excluded. The outcome is very similar to [2] and does not include new conformal hyperloops in the moduli spaces, expect for the ones that are direct sums of two conformal loops in 2×2 supermatrices [2] up to the gauge transformation, so we skip the presentation of this in the paper.

Other future directions are very similar to those mentioned in [2]. One is to compute the expectation value of all the hyperloops in the network, among which cases of Gaiotto-Yin loops and their fermionic deformations are studied in [63, 2, 111] and the "latitude" loops are in [173–176, 196]. Another direction is about the holographic duals of hyperloops [65, 63, 177–179, 95, 197]. Besides, since all the hyperloops newly found in this paper are not conformal, they do not contribute to the defect conformal manifolds [1, 32, 45, 112–114]. However, it would be interesting to understand their renormalization group flows [130, 117, 70].

This paper is organised as follows. In the next section we present notations for the theories and supersymmetry variations of the fields, as well as details of the bosonic loops (6.4) and the hyperloops with $SU(2)_L$ and $SU(2)_R$ supersymmetry enhancements, which by the amount of preserved supercharges can be classified into 1/4, 3/8 and 1/2 BPS loops. In Sections 6.3 and 6.4 we discuss the moduli spaces produced by deformations from 1/4 and 3/8 BPS hyperloops[24] respectively. The supersymmetry transformation rules are collected in section 2.1.1.

6.2 $\mathcal{N}=4$ Chern-Simons-matter theories on unit S^3 and hyperloops

Our setting is $\mathcal{N}=4$ Chern-Simons-matter theories, with either a circular or a linear quiver diagram [83–85, 89]. In the quiver diagram 1, the edges represent hypermultiplets $(q_I^a, \psi_{I\dot{a}})$ and the twisted hypermultiplet $(\tilde{q}_{I-1\,\dot{a}}, \tilde{\psi}_{I-1}^a)$, and so on in an alternate fashion.

The fields with indices $a, b = 1, 2$ are doublets of the $SU(2)_L$ R-symmetry, while those with indices $\dot{a}, \dot{b} = \dot{1}, \dot{2}$ are of $SU(2)_R$ R-symmetry. These indices are raised and lowered

[24] The deformation from 1/2 BPS hyperloops have been studied in [2].

using the appropriate epsilon symbols: $v^a = \epsilon^{ab} v_b$ and $v_a = \epsilon_{ab} v^b$ with $\epsilon^{12} = \epsilon_{21} = 1$, and similarly for the dotted indices.

Our first example of a hyperloop in the theory is the 1/2 BPS loop [110]. We take a 1/2 BPS Wilson loop built around the I and $I+1$ nodes, whose superconnection is in the $GL(N_I|N_{I+1})$ supermatrix preserving $SU(2)_L$ symmetry

$$\mathcal{L}_{I,1/2} = \begin{pmatrix} \mathcal{A}_I & -i\bar{\alpha}\psi_{I\mathrm{i}-} \\ i\alpha\bar{\psi}^{\mathrm{i}}_{I+} & \mathcal{A}_{I+1} - \frac{1}{2} \end{pmatrix}, \tag{6.5}$$

where

$$\mathcal{A}_I = A_{\varphi,I} + \frac{i}{k}(\nu_I - \tilde{\mu}_I{}^{\mathrm{i}}{}_{\mathrm{i}} + \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}), \quad \mathcal{A}_{I+1} = A_{\varphi,I+1} + \frac{i}{k}(\nu_{I+1} - \tilde{\mu}_{I+1\mathrm{i}}{}^{\mathrm{i}} + \tilde{\mu}_{I+1\dot{2}}{}^{\dot{2}}), \tag{6.6}$$

and the constants α and $\bar{\alpha}$ (which are not complex conjugate to each other) satisfy $\alpha\bar{\alpha} = 2i/k$. The Wilson loop does not depend on their actual value, since the loops are invariant under the (constant) gauge transformation [69]

$$\mathcal{L}_{I,1/2} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix} \mathcal{L}_{I,1/2} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \tag{6.7}$$

the equivalence relation $(\bar{\alpha}, \alpha) \sim (\bar{\alpha}x, \alpha/x)$ gives the global gauge symmetry $\mathbb{C}^\star$, so the moduli space is just one point. We could also allow for $\alpha, \bar{\alpha}$ to depend on φ at the expense of a $U(1)$ gauge transformation at the bottom right entry

$$\mathcal{A}_{I+1} - \frac{1}{2} \to \mathcal{A}_{I+1} - \frac{1}{2} - i\alpha^{-1}\partial_\varphi \alpha\,. \tag{6.8}$$

This shift in the connection can be compared with another 1/2 BPS loop, which includes the same gauge fields and preserves the exact same symmetries [63], but couples instead to other fields in the hypermultiplets

$$\mathcal{L}'_{I,1/2} = \begin{pmatrix} \mathcal{A}'_I & i\bar{\alpha}\psi_{I\dot{2}+} \\ i\alpha\bar{\psi}^{\dot{2}}_{I-} & \mathcal{A}'_{I+1} + \frac{1}{2} \end{pmatrix}, \tag{6.9}$$

where $\mathcal{A}'_I = A_{\varphi,I} - \frac{i}{k}(\nu_I + \tilde{\mu}_I{}^{\mathrm{i}}{}_{\mathrm{i}} - \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}})$. The shift $+1/2$ can be transformed to $-1/2$ by bringing in extra phases to $\bar{\alpha}', \alpha'$. These two loops preserve the same eight supercharges, where we can write a general superposition of them as

$$\mathcal{Q}_{1/2} = \eta^{\imath}_a Q^{\dot{2}a+}_{\imath} + \bar{\eta}^{\imath}_a (\sigma^1)^{\bar{\imath}}_{\imath} Q^{\mathrm{i}a-}_{\bar{\imath}}\,. \tag{6.10}$$

Analogous to [2], we start with either of the two 1/2 BPS loops and play the algorithm with (6.10). Among all the resulting moduli spaces, in particular we pay our attention to the bosonic loops where the nodes get decoupled

$$\mathcal{A}^{\mathrm{bos}} = A_\varphi + \frac{i}{k\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2) - \frac{i}{k}(\tilde{\mu}^{\mathrm{i}}{}_{\mathrm{i}} - \tilde{\mu}^{\dot{2}}{}_{\dot{2}}), \tag{6.11}$$

with

$$\Pi \equiv \epsilon^{ab}(\bar{\eta}v)_a(\eta\bar{v})_b\,. \tag{6.12}$$

The rotated scalar fields $r, \bar{r}$ are defined as

$$r^1 \equiv (\eta\bar{v})_a q^a\,, \quad r^2 \equiv (\bar{\eta}v)_a q^a\,, \quad \bar{r}_1 \equiv \epsilon^{ab}(\bar{\eta}v)_a\bar{q}_b\,, \quad \bar{r}_2 \equiv -\epsilon^{ab}(\eta\bar{v})_a\bar{q}_b\,, \tag{6.13}$$

where $(\eta\bar{v})_a = \eta_a^\imath \bar{v}_\imath$ and likewise for $(\bar{\eta}v)_a$, with the auxiliary vectors

$$v_\imath = \begin{pmatrix} e^{+i\varphi} \\ 1 \end{pmatrix}_\imath\,, \quad \bar{v}_\imath = \begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix}_\imath\,. \tag{6.14}$$

The bosonic loops (6.11) preserve (at least) two supercharges $\eta_a^\imath Q_\imath^{\dot{2}a+}$ and $\bar{\eta}_a^\imath(\sigma_1)_\imath^{\bar{\imath}} Q_{\bar{\imath}}^{\dot{1}a-}$. As discussed in [3], we can further classify them into the following cases:

- When both $\eta_a^\imath$ and $\bar{\eta}_a^\imath$ factorise, i.e. $\eta_a^\imath = y^\imath w_a$, $\bar{\eta}_a^\imath = \bar{y}^\imath \bar{w}_a$, (6.11) preserves 4 supercharges and thus is 1/4 BPS. In particular, with the choice $w_a = \delta_a^2$ and $\bar{w}_a = \delta_a^1$, we recover the Gaiotto-Yin loop [76]

$$\mathcal{A}_{\text{GY}}^{\text{bos}} = A_\varphi - \frac{i}{k}(q^1\bar{q}_1 - q^2\bar{q}_2) - \frac{i}{k}(\bar{\tilde{q}}^{\dot{1}}\tilde{q}_{\dot{1}} - \bar{\tilde{q}}^{\dot{2}}\tilde{q}_{\dot{2}})\,. \tag{6.15}$$

- When one of $\eta_a^\imath, \bar{\eta}_a^\imath$ factorises while the other does not, we get a 3/16 BPS bosonic loop [2,3]. For instance, when $\bar{\eta}_1^l = \bar{\eta}_2^r = \eta_2^l = 1$ and other $\eta, \bar{\eta}$'s vanish, it leads to the loop

$$\mathcal{A}_{3/16}^{\text{bos}} = A_\varphi - \frac{i}{k}(q^1\bar{q}_1 + 2e^{-i\varphi}q^2\bar{q}_1 - q^2\bar{q}_2) - \frac{i}{k}(\bar{\tilde{q}}^{\dot{1}}\tilde{q}_{\dot{1}} - \bar{\tilde{q}}^{\dot{2}}\tilde{q}_{\dot{2}})\,. \tag{6.16}$$

- When neither $\eta_a^\imath$ nor $\bar{\eta}_a^\imath$ factorises, the resulting loop is 1/8 BPS. For example, when

$$\bar{\eta}_1^r = \eta_2^l = \cos\frac{\theta}{2}, \quad \bar{\eta}_2^l = -\eta_1^r = \sin\frac{\theta}{2}\,, \tag{6.17}$$

  and other $\eta, \bar{\eta}$'s vanish, the resulting loops are the "bosonic latitude loops [89, 59, 172]

$$\mathcal{A}_{\text{LA}}^{\text{bos}} = A_\varphi + \frac{i}{k}M_{LA}q^a\bar{q}_b - \frac{i}{k}(\tilde{\mu}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}^{\dot{2}}{}_{\dot{2}}), \quad M_{LA} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}\,. \tag{6.18}$$

  And when

$$\begin{pmatrix} \bar{\eta}_1^l & \bar{\eta}_1^r \\ \bar{\eta}_2^l & \bar{\eta}_2^r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \eta_1^l & \eta_1^r \\ \eta_2^l & \eta_2^r \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{6.19}$$

  the resulting loops are the non-trivial Jordan normal form bosonic loops in section 3.3.1 of [3]

$$\mathcal{A}_{\text{JN}}^{\text{bos}} = A_\varphi + \frac{i}{k}M_{JN}q^a\bar{q}_b - \frac{i}{k}(\tilde{\mu}^{\dot{1}}{}_{\dot{1}} - \tilde{\mu}^{\dot{2}}{}_{\dot{2}}), \quad M_{JN} = \begin{pmatrix} 1+2e^{i\varphi} & -2(e^{i\varphi}+e^{2i\varphi}) \\ 2 & -(1+2e^{i\varphi}) \end{pmatrix}. \tag{6.20}$$

Note that not all the bosonic loops found in [3] can be represented as (6.11). There are some other 1/8 BPS loops in section 3.3.2 of [3] that will be presented later in (6.86).

Furthermore, from the deformation of the bosonic loops (6.11) we can get the general fermionic hyperloops in $GL(U_{I+1}|U_{I+2})$ supermatrices that preserves $SU(2)_R$ symmetry

$$\mathcal{L} = \begin{pmatrix} \tilde{\mathcal{A}}_{I+1} - \frac{1}{2} & -i\bar{\delta}^{\text{i}}\tilde{\rho}^2_+ \\ -i\frac{\delta_{\text{i}}}{\Pi}\bar{\tilde{\rho}}_{2-} & \tilde{\mathcal{A}}_{I+2} + \bar{\Gamma} \end{pmatrix}, \tag{6.21}$$

where $\Pi \neq 0$ and

$$\tilde{\mathcal{A}} = A_\varphi + \frac{i}{k\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2) + \frac{i}{k}\tilde{\nu}, \quad \bar{\Gamma} = \frac{1}{2}\left(i\partial_\varphi \log\Pi - \frac{\lambda}{\Pi} - 1\right), \tag{6.22}$$

with

$$\lambda = \epsilon^{ab}\epsilon_{\imath\jmath}\bar{\eta}^\imath_a\eta^\jmath_b \tag{6.23}$$

satisfying

$$\epsilon^{ab}(\bar{\eta}v)_a(\eta\sigma^3\bar{v})_b = -i\partial_\varphi\Pi - \lambda, \quad \epsilon^{ab}(\bar{\eta}\sigma^3 v)_a(\eta\bar{v})_b = -i\partial_\varphi\Pi + \lambda\,. \tag{6.24}$$

The rotated fermions $\bar{\rho}^2_+, \bar{\tilde{\rho}}_{2-}$ are defined as follows, forming a set of complete bases of fermionic fields together with $\bar{\rho}^1_-, \bar{\tilde{\rho}}_{1+}$

$$\tilde{\rho}^1_- = -(\eta\bar{v})_a\tilde{\psi}^a_-, \quad \tilde{\rho}^2_+ = (\bar{\eta}v)_a\tilde{\psi}^a_+, \quad \bar{\tilde{\rho}}_{1+} = \epsilon^{ab}(\bar{\eta}v)_a\bar{\tilde{\psi}}_{b+}, \quad \bar{\tilde{\rho}}_{2-} = \epsilon^{ab}(\eta\bar{v})_a\bar{\tilde{\psi}}_{b-}\,. \tag{6.25}$$

Similar to the parametrization in 1/2 BPS Wilson loops, the constants $\bar{\delta}^{\text{i}}$ and δ_{i} (which are again not complex conjugate) satisfy $\delta_{\text{i}}\bar{\delta}^{\text{i}} = \frac{2i}{k}$. They are also allowed to depend on φ with proper transformation of the shift, whose origin $\bar{\Gamma}$ in the superconnection is obtained in [2] to satisfy the relaxed supersymmtry condition (6.3). Depending on the factorisation of $\eta^\imath_a, \bar{\eta}^\imath_a$, the supersymmetries of the hyperloops in (6.21) could vary from 1/4, 3/8 to 1/2 BPS. Analogous to the other 1/2 BPS loops (6.9), we can also construct the other hyperloops with the same gauge fields that preserve exactly the same symmetries as (6.21), with the replacements $\tilde{\rho}^2_+ \to \tilde{\rho}^{\text{i}}_-$, $\bar{\tilde{\rho}}_{2-} \to -\bar{\tilde{\rho}}_{1+}$, $\bar{\Gamma} \to \bar{\Gamma} + \lambda/\Pi$ and the opposite sign for the $\tilde{\nu}$'s, compared to the ones appearing in (6.22).

In the following sections 6.3 and 6.4, we study 1/4 and 3/8[25] BPS (6.21) respectively and their supersymmetric deformations.

6.3 1/4 BPS hyperloops

We notice that with the special choice of parameters in (6.17), the hyperloops in (6.21) become the 1/4 BPS "fermionic latitude" loops [90, 176, 59], and the supercharges preserved

[25] We skip discussions about the 1/2 BPS case, because in principle it should be the same as the two-node hyperloops in [2] with exchange of hypermultiplets and twisted ones.

by them are not the subset of those of any 1/2 BPS loop. In other words, they are not in the moduli spaces of hyperloops obtained in [2], so we can take them as the new starting point of supersymmetric deformation to discover new moduli spaces of hyperloops.

However, since the latitude loop is just a special case of the general 1/4 BPS hyperloops in (6.21) given by unfactorized $\eta_a^\imath$ and $\bar{\eta}_a^\imath$, we prefer to take the later ones as the starting point, in which case the preserved four supercharges are

$$\eta_a^\imath Q_\imath^{\dot{1}a+}, \quad \eta_a^\imath Q_\imath^{\dot{2}a+}, \quad \bar{\eta}_a^\imath(\sigma^1)_\imath^{\bar{\imath}} Q_{\bar{\imath}}^{\dot{1}a-}, \quad \bar{\eta}_a^\imath(\sigma^1)_\imath^{\bar{\imath}} Q_{\bar{\imath}}^{\dot{2}a-} \,. \tag{6.26}$$

These supercharges make up the bases of a more general $\mathcal{Q}$ with constant coefficients $w_{\dot{b}}$ and $\bar{w}_{\dot{b}}$ that carry $SU(2)_R$ indices

$$\mathcal{Q}_{1/4} = w_{\dot{b}}\eta_a^\imath Q_\imath^{\dot{b}a+} + \bar{w}_{\dot{b}}\bar{\eta}_a^\imath(\sigma^1)_\imath^{\bar{\imath}} Q_{\bar{\imath}}^{\dot{b}a-} \,, \tag{6.27}$$

where the subscript 1/4 is written explicitly here to distinguish with the 3/8 BPS case in the next section.

We now proceed to evaluate the action of this supercharge $\mathcal{Q}_{1/4}$ on the superconnection $\mathcal{L}_{1/4}$ in (6.21) and to look for $H_{1/4}$ given in the total derivative (6.3). To do so we define the rotated twisted scalar fields

$$\tilde{r}_{\dot{1}} = -\epsilon^{\dot{a}\dot{b}} w_{\dot{a}}\tilde{q}_{\dot{b}}, \quad \tilde{r}_{\dot{2}} = \epsilon^{\dot{a}\dot{b}}\bar{w}_{\dot{a}}\tilde{q}_{\dot{b}}, \quad \bar{\tilde{r}}^{\dot{1}} = \bar{w}_{\dot{a}}\bar{\tilde{q}}^{\dot{a}}, \quad \bar{\tilde{r}}^{\dot{2}} = w_{\dot{a}}\bar{\tilde{q}}^{\dot{a}} \,. \tag{6.28}$$

It is useful to introduce a new parameter analogous to Π (6.11)

$$\tilde{\Pi} = \epsilon^{\dot{a}\dot{b}}\bar{w}_{\dot{a}} w_{\dot{b}} \,, \tag{6.29}$$

such that the supersymmetry variations of the rotated twisted scalar fields can be written in terms of it as

$$\mathcal{Q}_{1/4}\tilde{r}_{\dot{1}} = \tilde{\Pi}\tilde{\rho}_+^2, \quad \mathcal{Q}_{1/4}\tilde{r}_{\dot{2}} = \tilde{\Pi}\tilde{\rho}_-^1, \quad \mathcal{Q}_{1/4}\bar{\tilde{r}}^{\dot{1}} = \tilde{\Pi}\bar{\tilde{\rho}}_{2-}, \quad \mathcal{Q}_{1/4}\bar{\tilde{r}}^{\dot{2}} = \tilde{\Pi}\bar{\tilde{\rho}}_{1+} \,. \tag{6.30}$$

So far it is not hard to see the similarities between $\tilde{r}$ and r, $\tilde{\Pi}$ and Π. Moreover, using

$$\tilde{r}_{I+1,\dot{1}}\bar{\tilde{r}}^{\dot{1}}_{I+1} + \tilde{r}_{I+1,\dot{2}}\bar{\tilde{r}}^{\dot{2}}_{I+1} = \tilde{\Pi}\tilde{\nu}_{I+1}, \quad \bar{\tilde{r}}^{\dot{1}}_{I+1}\tilde{r}_{I+1,\dot{1}} + \bar{\tilde{r}}^{\dot{2}}_{I+1}\tilde{r}_{I+1,\dot{2}} = \tilde{\Pi}\tilde{\nu}_{I+2} \,, \tag{6.31}$$

we get the second variations

$$\begin{aligned}
\mathcal{Q}_{1/4}^2\tilde{r}_{\dot{1}} &= \tilde{\Pi}\left(\Pi\left(i\partial_\varphi\tilde{r}_{\dot{1}} + \tilde{\mathcal{A}}_{I+1}\tilde{r}_{\dot{1}} - \tilde{r}_{\dot{1}}\tilde{\mathcal{A}}_{I+2}\right) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\upsilon)_a(\eta\sigma^3\bar{\upsilon})_b\tilde{r}_{\dot{1}}\right) \\
\mathcal{Q}_{1/4}^2\tilde{r}_{\dot{2}} &= \tilde{\Pi}\left(\Pi\left(i\partial_\varphi\tilde{r}_{\dot{2}} + \tilde{\mathcal{A}}_{I+1}\tilde{r}_{\dot{2}} - \tilde{r}_{\dot{2}}\tilde{\mathcal{A}}_{I+2}\right) - \frac{2i}{k}\Pi(\tilde{\nu}_{I+1}\tilde{r}_{\dot{2}} - \tilde{r}_{\dot{2}}\tilde{\nu}_{I+2}) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\sigma^3\upsilon)_a(\eta\bar{\upsilon})_b\tilde{r}_{\dot{2}}\right) \\
\mathcal{Q}_{1/4}^2\bar{\tilde{r}}^{\dot{1}} &= \tilde{\Pi}\left(\Pi\left(i\partial_\varphi\bar{\tilde{r}}^{\dot{1}} + \tilde{\mathcal{A}}_{I+2}\bar{\tilde{r}}^{\dot{1}} - \bar{\tilde{r}}^{\dot{1}}\tilde{\mathcal{A}}_{I+1}\right) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\sigma^3\upsilon)_a(\eta\bar{\upsilon})_b\bar{\tilde{r}}^{\dot{1}}\right) \\
\mathcal{Q}_{1/4}^2\bar{\tilde{r}}^{\dot{2}} &= \tilde{\Pi}\left(\Pi\left(i\partial_\varphi\bar{\tilde{r}}^{\dot{2}} + \tilde{\mathcal{A}}_{I+2}\bar{\tilde{r}}^{\dot{2}} - \bar{\tilde{r}}^{\dot{2}}\tilde{\mathcal{A}}_{I+1}\right) - \frac{2i}{k}\Pi(\tilde{\nu}_{I+2}\bar{\tilde{r}}^{\dot{2}} - \bar{\tilde{r}}^{\dot{2}}\tilde{\nu}_{I+1}) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\upsilon)_a(\eta\sigma^3\bar{\upsilon})_b\bar{\tilde{r}}^{\dot{2}}\right) .
\end{aligned} \tag{6.32}$$

At this point all the supersymmetry variations of rotated twisted scalar fields have been figured out, we can then proceed to study the supersymmetric deformations of 1/4 BPS hyperloops in (6.21).

6.3.1 Deformations with $\tilde{\Pi} \neq 0$

We choose the starting point to be $\mathcal{L}_{1/4}$ in (6.21) preserving the supercharge $\mathcal{Q}_{1/4}$ defined in (6.27). Following [59, 2], we take a deformation of the form

$$\mathcal{L} = \mathcal{L}_{1/4} - i\mathcal{Q}_{1/4}G + B + C\,, \tag{6.33}$$

where G is off-diagonal and Grassmann-even, so linear in the (twisted) scalar fields, B is a diagonal bilinear and C is annihilated by $\mathcal{Q}_{1/4}$. BPS non-conformal loops with higher dimension insertions are also possible here, but again are not considered. On this condition, $\mathcal{Q}_{1/4}C = 0$ splits into two cases: when the supercharge annihilates some of the matter fields and when it does not. As explained in [2], we exclude the solutions in C that includes any BPS bosonic loop where the supersymmetry variation is simply zero. Thus C includes only matter fields from the twisted hypermultiplet that are annihilated by $\mathcal{Q}_{1/4}$.

However, when $\tilde{\Pi} \neq 0$, the only solutions to $\mathcal{Q}_{1/4}C = 0$ which is at most bilinear in the fields is a numerical matrix containing no fields. So here we fix the gauge of deformation by setting $C = 0$.

Moving to the off-diagonal $-i\mathcal{Q}_{1/4}G$ term, we take

$$G = \begin{pmatrix} 0 & \bar{b}^{\dot{a}}\tilde{r}_{\dot{a}} \\ b_{\dot{a}}\bar{\tilde{r}}^{\dot{a}} & 0 \end{pmatrix}\,, \tag{6.34}$$

where the parameters $\bar{b}^{\dot{a}}, b_{\dot{a}}$ may be functions of φ.

By splitting the connection $\mathcal{L}_{1/4}$ into the diagonal (bosonic) part $\mathcal{L}^B_{1/4}$ and off-diagonal (fermionic) part $\mathcal{L}^F_{1/4}$, the second supersymmetry variation of G can be written as

$$-i\mathcal{Q}^2_{1/4}G = \partial_\varphi(\Pi\tilde{\Pi}G) - i[\mathcal{L}^B_{1/4}, \Pi\tilde{\Pi}G] + i[H^2_{1/4}, G] - \hat{G}\,, \tag{6.35}$$

where we use $\mathcal{Q}_{1/4}H_{1/4} = i\Pi\tilde{\Pi}\mathcal{L}^F_{1/4}$ with $H_{1/4}$ given by $Q_{1/4}\mathcal{L} = \mathcal{D}^{\mathcal{L}}_\varphi H_{1/4}$ in (6.3)

$$H_{1/4} = \begin{pmatrix} 0 & \Pi\bar{\delta}^{\mathrm{i}}\tilde{r}_{I+1,\mathrm{i}} \\ \delta_{\mathrm{i}}\bar{\tilde{r}}^{\mathrm{i}}_{I+1} & 0 \end{pmatrix}\,, \tag{6.36}$$

and the remainder $\hat{G}$ is

$$\hat{G} = \begin{pmatrix} 0 & \Pi\tilde{\Pi}(\partial_\varphi\bar{b}^{\dot{a}})\tilde{r}_{\dot{a}} - i\lambda\tilde{\Pi}\bar{b}^{\dot{2}}\tilde{r}_{\dot{2}} \\ \partial_\varphi(\Pi\tilde{\Pi}b_{\dot{a}})\bar{\tilde{r}}^{\dot{a}} + i\lambda\tilde{\Pi}b_{\dot{2}}\bar{\tilde{r}}^{\dot{2}} & 0 \end{pmatrix}\,. \tag{6.37}$$

Since we are looking for the supersymmetric hyperloops with superconnections with the action of $\mathcal{Q}_{1/4}$ on it to be a total derivative

$$\mathcal{Q}_{1/4}\mathcal{L} = \mathcal{D}_\varphi^{\mathcal{L}_{1/4}} H_{1/4} - i\mathcal{Q}_{1/4}^2 G + \mathcal{Q}_{1/4} B = \mathcal{D}_\varphi^{\mathcal{L}}(H_{1/4} + \Delta H)\,, \tag{6.38}$$

by plugging in equation (6.35), we get that $\Delta H = \Pi\tilde{\Pi} G$ and $B = \{H_{1/4}, G\} + \Pi\tilde{\Pi} G^2$ on the condition $\hat{G} = 0$. Consequently, almost the same as [2], the deformation in form of (6.33) is

$$\mathcal{L} = \mathcal{L}_{1/4} - i\mathcal{Q}_{1/4} G + \{H_{1/4}, G\} + \Pi\tilde{\Pi} G^2 + C\,. \tag{6.39}$$

The vanishing $\hat{G}$ gives four differential equations for $\bar{b}^{\dot{a}}$ and $b_{\dot{a}}$

$$\begin{aligned} i\partial_\varphi(\Pi b_{\dot{1}}) &= 0, & i\partial_\varphi(\Pi \bar{b}^{\dot{1}}) &= (i\partial_\varphi \Pi)\bar{b}^{\dot{1}} \\ i\partial_\varphi(\Pi b_{\dot{2}}) &= \lambda b_{\dot{2}}, & i\partial_\varphi(\Pi \bar{b}^{\dot{2}}) &= (i\partial_\varphi \Pi - \lambda)\bar{b}^{\dot{2}}\,. \end{aligned} \tag{6.40}$$

As discussed in section 5.1 of [2], for

$$\hat{c}(\varphi) = \int_0^\varphi \lambda/\Pi d\varphi'\,, \tag{6.41}$$

when $e^{i\hat{c}(\varphi)}$ is single valued, i.e. $e^{i\hat{c}(2\pi)} = 1$, $\mathcal{L}$ in (6.39) may couple to all twisted scalars, otherwise it could only couple either to the pair $\tilde{r}_{\dot{1}}, \bar{\tilde{r}}^{\dot{1}}$ or to $\tilde{r}_{\dot{2}}, \bar{\tilde{r}}^{\dot{2}}$. Note that with the unfactorised $\eta_a^\imath, \bar{\eta}_a^\imath$, λ (6.23) is probably non-vanishing, so the periodical condition (6.41) is not guaranteed. However, we still focus on the case with two pairs of scalars, whose coefficients are given by

$$\bar{b}^{\dot{1}} = \frac{\bar{\beta}^{\dot{1}} - \bar{\delta}^{\dot{1}}}{\tilde{\Pi}}, \quad \bar{b}^{\dot{2}} = \frac{\bar{\beta}^{\dot{2}} e^{i\hat{c}(\varphi)}}{\tilde{\Pi}}, \quad b_{\dot{1}} = \frac{\beta_{\dot{1}} - \delta_{\dot{1}}}{\Pi\tilde{\Pi}}, \quad b_{\dot{2}} = \frac{\beta_{\dot{2}} e^{-i\hat{c}(\varphi)}}{\Pi\tilde{\Pi}}\,, \tag{6.42}$$

where $\bar{\beta}^{\dot{a}}, \beta_{\dot{a}}$ are constants, and to be consistent with the results in the later section 6.4, we put $\tilde{\Pi}$ in the denominator explicitly rather than absorb it into $\bar{\beta}^{\dot{a}}, \beta_{\dot{a}}$ since $\tilde{\Pi}$ is a constant parameter.

As a result, the deformed connection $\mathcal{L}$ in (6.39) is

$$\begin{pmatrix} \tilde{\mathcal{A}}_{1/4,I+1} - \frac{1}{2} & -i\bar{\beta}^{\dot{1}}\tilde{\rho}^2_{I+1,+} - i\bar{\beta}^{\dot{2}} e^{i\hat{c}(\varphi)}\tilde{\rho}^1_{I+1,-} \\ -i\frac{\beta_{\dot{1}}}{\Pi}\bar{\tilde{\rho}}_{I+1,2-} - i\frac{\beta_{\dot{2}}}{\Pi} e^{-i\hat{c}(\varphi)}\bar{\tilde{\rho}}_{I+1,1+} & \tilde{\mathcal{A}}_{1/4,I+2} + \bar{\Gamma} \end{pmatrix}\,, \tag{6.43}$$

where

$$\tilde{\mathcal{A}}_{1/4} = A_\varphi + \frac{i}{k\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2) + \frac{1}{\tilde{\Pi}}\tilde{M}^{\dot{a}}_{\dot{b}}\tilde{r}_{\dot{a}}\bar{\tilde{r}}^{\dot{b}}\,. \tag{6.44}$$

Note that the subscript 1/4 means this is the deformation away from 1/4 BPS loops rather than $\tilde{\mathcal{A}}_{1/4}$ itself preserves 1/4 BPS supersymmetry, and

$$\tilde{M}^{\dot{a}}_{\dot{b}} = \begin{pmatrix} -\frac{i}{k} + \bar{\beta}^{\dot{1}}\beta_{\dot{1}} & \bar{\beta}^{\dot{1}}\beta_{\dot{2}} e^{-i\hat{c}(\varphi)} \\ \bar{\beta}^{\dot{2}}\beta_{\dot{1}} e^{i\hat{c}(\varphi)} & \frac{i}{k} + \bar{\beta}^{\dot{2}}\beta_{\dot{2}} \end{pmatrix}\,. \tag{6.45}$$

After fixing a supercharge $\mathcal{Q}_{1/4}$, the possible space of hyperloops is given by four complex parameters $\bar{\beta}^{\dot{a}}, \beta_{\dot{a}}$ modded by $\mathbb{C}^\star$, which is a conifold. This is the same type of moduli space found in [58, 59, 2], but totally independent from them except for some joint points, for example (6.21), the starting point of deformation. Generally such loops preserve only one supercharge (6.27) thus being 1/16 BPS, but there are some special cases where they receive certain supersymmetry enhancements as the discussion in the following.

- Single node bosonic loops.

  We may decouple the nodes by simply setting $\bar{\beta}^{\dot{1}} = \bar{\beta}^{\dot{2}} = \beta_{\dot{1}} = \beta_{\dot{2}} = 0$. This eliminates all the fermions in the superconnection (6.43), thus it becomes block-diagonal with entries

$$\mathcal{A}^{\text{bos}}_{1/8} = A_\varphi + \frac{i}{k\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2) - \frac{i}{k\tilde{\Pi}}(\bar{\tilde{r}}^{\dot{1}}\tilde{r}_{\dot{1}} - \bar{\tilde{r}}^{\dot{2}}\tilde{r}_{\dot{2}})\,. \tag{6.46}$$

  Under the condition that $\eta^{\imath}_a, \bar{\eta}^{\imath}_a$ are unfactorised, these can only be 1/8 BPS bosonic loops, in a rotated version of (6.11). The two preserved loops are naturally the barred and unbarred parts of (6.27). Noticing that now we can write the double supersymmetric variations of the twisted scalar fields in (6.32) in terms of the bosonic loops

$$\begin{aligned}\mathcal{Q}^2_{1/4}\tilde{r}_{\dot{1}} &= \tilde{\Pi}\left(\Pi\left(i\partial_\varphi\tilde{r}_{\dot{1}} + \mathcal{A}^{\text{bos}}_{1/8,I+1}\tilde{r}_{\dot{1}} - \tilde{r}_{\dot{1}}\mathcal{A}^{\text{bos}}_{1/8,I+2}\right) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}v)_a(\eta\sigma^3\bar{v})_b\tilde{r}_{\dot{1}}\right)\\ \mathcal{Q}^2_{1/4}\bar{\tilde{r}}^{\dot{1}} &= \tilde{\Pi}\left(\Pi\left(i\partial_\varphi\bar{\tilde{r}}^{\dot{1}} + \mathcal{A}^{\text{bos}}_{1/8,I+2}\bar{\tilde{r}}^{\dot{1}} - \bar{\tilde{r}}^{\dot{1}}\mathcal{A}^{\text{bos}}_{1/8,I+1}\right) - \frac{1}{2}\epsilon^{ab}(\bar{\eta}\sigma^3 v)_a(\eta\bar{v})_b\bar{\tilde{r}}^{\dot{1}}\right)\,,\end{aligned} \tag{6.47}$$

  likewise for $\tilde{r}_{\dot{2}}, \bar{\tilde{r}}^{\dot{2}}$. This implies us another path to obtain all the loops in the form of (6.43), which is to deform from the bosonic loops (6.46) with proper supercharge (6.27). Actually, if we take

$$G = \begin{pmatrix} 0 & \frac{1}{\Pi}(\bar{\beta}^{\dot{1}}\tilde{r}_{\dot{1}} + \bar{\beta}^{\dot{2}}e^{i\hat{c}(\varphi)}\tilde{r}_{\dot{2}}) \\ \frac{1}{\Pi\tilde{\Pi}}(\beta_{\dot{1}}\bar{\tilde{r}}^{\dot{1}} + \beta_{\dot{2}}e^{-i\hat{c}(\varphi)}\bar{\tilde{r}}^{\dot{2}} & 0\end{pmatrix}, \quad c = \bar{\Gamma} + \frac{1}{2}\,, \tag{6.48}$$

  with $\bar{\Gamma}$ defined in (6.22), and the deformation

$$\text{diag}(\mathcal{A}^{\text{bos}}_I, \mathcal{A}^{\text{bos}}_{I+1}) \to \text{diag}(\mathcal{A}^{\text{bos}}_I, \mathcal{A}^{\text{bos}}_{I+1} + c) - i\mathcal{Q}G + \Pi\tilde{\Pi}G^2\,, \tag{6.49}$$

  we are able to recover arbitrary hyperloops in (6.43).

- Fermionic loops with non-trivial diagonal $\tilde{M}$.

  When turning on only one pair of $\bar{\beta}, \beta$'s, for example $\bar{\beta}^{\dot{1}}$ and $\beta_{\dot{1}}$, we get diagonal $\tilde{M}$ and the resulting loops receive a natural supersymmetry enhancement. In addition to the supercharge in (6.27), these loops preserve at least the other supercharge

$$\mathcal{Q}'_{1/4} = w_{\dot{b}}\eta^{\imath}_a Q^{\dot{b}a+}_{\imath} - \bar{w}_{\dot{b}}\bar{\eta}^{\imath}_a(\sigma^1)^{\bar{\imath}}_{\imath}Q^{\dot{b}a-}_{\bar{\imath}}\,. \tag{6.50}$$

To see this, in the case of $\mathcal{Q}'_{1/4}$ we change $\tilde{\Pi} \to -\tilde{\Pi}$ and $\tilde{M} \to -\tilde{M}$, with the same $\bar{\beta}, \beta$'s the connection remains invariant. In particular, when $\bar{\beta}^{\dot{1}}\beta_{\dot{1}} = \frac{2i}{k}$ (or $\bar{\beta}^{\dot{2}}\beta_{\dot{2}} = -\frac{2i}{k}$ in the case $\bar{\beta}^{\dot{2}}, \beta_{\dot{2}}$ are turned on), $\tilde{M}$ is proportional to the identity and restore $SU(2)_R$ R-symmetry, so the resulting loops become 1/4 BPS.

- "Fermionic latitude" loops and "Jordan normal form" fermionic loops.

  Another special case is the "fermionic latitude" loops constructed in [59], which here can be recovered with parameters in (6.17). Besides the latitude loops and their conjugates up to $SU(2)_L \times SU(2)_R \times SL_2(\mathbb{R})$ which are the symmetries preserved by a great circle, similar analysis to [3] tells that there is the other class of hyperloops, with $\eta_a^{\imath}$ or $\bar{\eta}_a^{\imath}$ in the non-trivial Jordan normal forms (6.19). With such choices of preserved supercharges, (6.43) can be viewed as fermionic deformations of (6.20).

6.3.2 Deformations with $\tilde{\Pi} = 0$

The other possible case is for

$$\tilde{\Pi} = \bar{w}_{\dot{1}} w_{\dot{2}} - \bar{w}_{\dot{2}} w_{\dot{1}} = 0\,, \tag{6.51}$$

where $\mathcal{Q}_{1/4}$ has a nontrivial kernel (6.28) (see (6.30) for the short proof), that brings out some novel cases. Although $\tilde{\Pi} = 0$, the four parameters $w_{\dot{a}}, \bar{w}_{\dot{a}}$ can not be all zero, otherwise the supercharge (6.27) just vanishes. We assume $\bar{w}_{\dot{1}} \neq 0$, and introduce the quotient parameter

$$\omega = \frac{w_{\dot{1}}}{\bar{w}_{\dot{1}}} = \frac{w_{\dot{2}}}{\bar{w}_{\dot{2}}}\,, \tag{6.52}$$

where the constant $\omega \in \mathbb{C} \cup \{\infty\}$[26].

The pairs of rotated fields defined in (6.28) are not linearly independent

$$\tilde{r}_{\dot{1}} = -\omega \tilde{r}_{\dot{2}}, \quad \bar{\tilde{r}}^{\dot{2}} = \omega \bar{\tilde{r}}^{\dot{1}}\,. \tag{6.53}$$

So in order to construct a new basis of the twisted scalar fields, besides

$$\tilde{r}_{\parallel} = -\tilde{r}_{\dot{2}}, \quad \bar{\tilde{r}}^{\parallel} = \bar{\tilde{r}}^{\dot{1}}\,, \tag{6.54}$$

we also need an orthogonal pair which are not annihilated by $\mathcal{Q}$. Define

$$\tilde{r}_{\perp} = \bar{w}_{\dot{1}} \tilde{q}_{\dot{1}} + \bar{w}_{\dot{2}} \tilde{q}_{\dot{2}}, \quad \bar{\tilde{r}}^{\perp} = \bar{w}_{\dot{2}} \bar{\tilde{q}}^{\dot{1}} - \bar{w}_{\dot{1}} \bar{\tilde{q}}^{\dot{2}}\,, \tag{6.55}$$

so that similar to (6.31), there are

$$\tilde{r}_{I+1,\parallel} \bar{\tilde{r}}^{\perp}_{I+1} + \tilde{r}_{I+1,\perp} \bar{\tilde{r}}^{\parallel}_{I+1} = \bar{\tilde{\Lambda}} \tilde{\nu}_{I+1}, \quad \bar{\tilde{r}}^{\perp}_{I+1} \tilde{r}_{I+1,\parallel} + \bar{\tilde{r}}^{\parallel}_{I+1} \tilde{r}_{I+1,\perp} = \bar{\tilde{\Lambda}} \tilde{\nu}_{I+2}\,, \tag{6.56}$$

[26] $\omega = \infty$ corresponds to $\bar{w}_{\dot{1}} = \bar{w}_{\dot{2}} = 0$, which is the case we avoid discussing where only the unbarred supercharges are preserved. Similarly for $\omega = 0$.

where

$$\bar{\bar{\Lambda}} = \bar{w}_{\dot{1}}^2 + \bar{w}_{\dot{2}}^2 \,. \tag{6.57}$$

Since $w_{\dot{a}}, \bar{w}_{\dot{a}}$ can not be all zero, $\bar{\bar{\Lambda}}$ is always nonvanishing. With it we get single and double supersymmetric transformations of $\tilde{\chi}_\perp, \bar{\tilde{\chi}}^\perp$

$$\begin{aligned}
&\mathcal{Q}_{1/4}\tilde{r}_\perp = \bar{\bar{\Lambda}}(\omega\tilde{\rho}_-^1 + \tilde{\rho}_+^2), \quad \mathcal{Q}_{1/4}\bar{\tilde{r}}^\perp = \bar{\bar{\Lambda}}(\omega\bar{\tilde{\rho}}_{2-} - \bar{\tilde{\rho}}_{1+}) \\
&\mathcal{Q}_{1/4}^2\tilde{r}_\perp = \bar{\bar{\Lambda}}\left(\omega\lambda\tilde{\chi}_\parallel + \frac{2i}{k}\omega\Pi(\nu_{I+1}\tilde{\chi}_\parallel - \tilde{\chi}_\parallel\nu_{I+2})\right) \\
&\mathcal{Q}_{1/4}^2\bar{\tilde{r}}^\perp = -\bar{\bar{\Lambda}}\left(\omega\lambda\bar{\tilde{\chi}}^\parallel - \frac{2i}{k}\omega\Pi(\nu_{I+2}\bar{\tilde{\chi}}^\parallel - \bar{\tilde{\chi}}^\parallel\nu_{I+1})\right) .
\end{aligned} \tag{6.58}$$

Now we proceed to reconsider the deformation (6.33), for which the same formalism (6.39) as in the $\tilde{\Pi} \neq 0$ case can be applied

$$\mathcal{L} = \mathcal{L}_{1/4} - i\mathcal{Q}_{1/4}G + \{H_{1/4}, G\} + C, \quad \mathcal{Q}_{1/4}C = 0 \,, \tag{6.59}$$

where H is the same as above (6.36), just in the new notations it becomes

$$H_{1/4} = \begin{pmatrix} 0 & \omega\Pi\bar{\delta}^{\dot{1}}_{I+1}\tilde{r}_{I+1,\parallel} \\ \delta_{I+1,\dot{1}}\bar{\tilde{r}}^\parallel_{I+1} & 0 \end{pmatrix} . \tag{6.60}$$

The action of $\mathcal{Q}_{1/4}$ on the superconnection $\mathcal{L}$ will again be a total derivative

$$\mathcal{Q}_{1/4}\mathcal{L} = \mathcal{D}_\varphi^{\mathcal{L}} H_{1/4} \,. \tag{6.61}$$

The most significant distinction between (6.59) and the previous case is C. Because of the fact that the supercharge $\mathcal{Q}_{1/4}$ in (6.27) annihilates the rotated twisted scalars $\tilde{\chi}_\parallel, \bar{\tilde{\chi}}^\parallel$, C may contain their bilinears as well as the numerical factor c, explicitly

$$C = \begin{pmatrix} \bar{\beta}^\parallel\tilde{r}_\parallel\bar{\tilde{r}}^\parallel & 0 \\ 0 & \beta_\parallel\bar{\tilde{r}}^\parallel r_\parallel + c \end{pmatrix} . \tag{6.62}$$

The new off-diagonal matrix G is comprised of the other pair of the rotated twisted scalar fields only

$$G = \begin{pmatrix} 0 & \bar{\beta}^\perp\tilde{r}_\perp \\ \beta_\perp\bar{\tilde{r}}^\perp & 0 \end{pmatrix} , \tag{6.63}$$

where the parameters $\bar{\beta}^\perp, \beta_\perp$ may be functions of φ. If the parmeters satisfy

$$\bar{\bar{\Lambda}}\omega\lambda\bar{\beta}^\perp = -c\omega\Pi\bar{\delta}^{\dot{1}}, \quad \bar{\bar{\Lambda}}\omega\lambda\beta_\perp = -c\delta_{\dot{1}}, \quad \bar{\beta}^\parallel = \beta_\parallel \,, \tag{6.64}$$

The resulting supersymmetric loops obtained by (6.59) are

$$\begin{pmatrix} \tilde{\mathcal{A}}^0_{1/4,I+1} - \frac{1}{2} & -i\bar{\bar{\Lambda}}\omega\bar{\beta}^\perp\tilde{\rho}^1_{I+1,-} - i(\bar{\delta}^{\dot{1}} + \bar{\bar{\Lambda}}\bar{\beta}^\perp)\tilde{\rho}^2_{I+1,+} \\ i\bar{\bar{\Lambda}}\beta_\perp\bar{\tilde{\rho}}_{I+1,1+} - i(\frac{\delta_{\dot{1}}}{\Pi} + \bar{\bar{\Lambda}}\omega\beta_\perp)\bar{\tilde{\rho}}_{I+1,2-} & \tilde{\mathcal{A}}^0_{1/4,I+2} + \bar{\Gamma} + c \end{pmatrix} , \tag{6.65}$$

where

$$\tilde{\mathcal{A}}^0_{1/4} = A_\varphi + \frac{i}{k\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2) + \tilde{M}^{\dot{a}}_{\dot{b}}\tilde{r}_{\dot{a}}\bar{\tilde{r}}^{\dot{b}}\,, \tag{6.66}$$

with $\dot{a}, \dot{b} = \perp, \parallel$ and

$$\tilde{M}^{\dot{a}}_{\dot{b}} = \begin{pmatrix} 0 & \frac{i}{k\bar{\tilde{\Lambda}}} + \bar{\beta}^\perp \delta_{\rm i} \\ \frac{i}{k\bar{\tilde{\Lambda}}} + \omega\Pi\bar{\delta}^{\rm i}\beta_\perp & \bar{\beta}^\parallel \end{pmatrix}\,, \tag{6.67}$$

with $\beta_\perp, \bar{\beta}^\perp, c$ solutions of (6.64). Because of the unfactorised $\eta^\imath_a, \bar{\eta}^\imath_a$, the resulting loops are independent from the $\Pi = 0$ ones in [2], in which case the factorisation of $\eta^\imath_a, \bar{\eta}^\imath_a$ is required automatically by the condition $\Pi = 0$. Generically, these loops only preserve one supercharge, while just like $\Pi \neq 0$ case at some special points again we find supersymmetry enhancements.

It is shown in (6.64) that the solutions of parameters $\beta, \bar{\beta}$ with $\perp$ and $\parallel$ indices are independent from each other. The simplest case for loops with enhanced supersymmetry is when the superconnection is invariant under $SU(2)_R$, i.e. when $\bar{\beta}^\parallel = 0$ and $\bar{\beta}^\perp\delta_{\rm i} = \omega\Pi\bar{\delta}^{\rm i}\beta_\perp$, which is consistent with the solutions of (6.64). Since $\eta^\imath_a$ and $\bar{\eta}^\imath_a$ are unfactorized, it turns out this is the only enhancement that the resulting loops receive, thus being 1/8 BPS. The preserved supercharges are

$$\omega\eta^\imath_a Q^{\dot{1}a}_\imath + \bar{\eta}^\imath_a(\sigma^1)^{\bar{\imath}}_\imath Q^{\dot{1}a}_{\bar{\imath}}, \quad \omega\eta^\imath_a Q^{\dot{2}a}_\imath + \bar{\eta}^\imath_a(\sigma^1)^{\bar{\imath}}_\imath Q^{\dot{2}a}_{\bar{\imath}}\,. \tag{6.68}$$

Two further degenerations are when ω vanishes or goes to infinite. In both cases the preserved supercharges are composed entirely of either barred or unbarred ones which as mentioned in the introduction, are beyond the discussion of this paper.

6.4 3/8 BPS hyperloops

When one of $\eta^\imath_a, \bar{\eta}^\imath_a$ factorises while the other does not, the hyperloops in (6.21) are enhanced to 3/8 BPS. As an example, we focus on the case where the factorised ones are $\bar{\eta}^\imath_a$, i.e. $\bar{\eta}^\imath_a = \bar{y}^\imath\bar{u}_a$. Following the definition of Π in (6.29), now it turns into $\Pi = (\bar{y}v)\epsilon^{ab}\bar{u}_a(\eta\bar{v})_b$, where the same factor $\bar{y}v$ appears also in r^2 and $\bar{r}_1$ (6.13). This allows us to define the corresponding "factorised" parameters and rotated fields without such a common factor

$$\Pi^f = \epsilon^{ab}\bar{u}_a(\eta\bar{v})_b, \quad r^{f2} = \bar{u}_a q^a, \quad \bar{r}^f_1 = \epsilon^{ab}\bar{u}_a\bar{q}_b\,, \tag{6.69}$$

so that the bilinears of untwisted fields in (6.22) remains unchanged with factorised Π^f

$$\frac{1}{\Pi^f}(r^1\bar{r}^f_1 - r^{f2}\bar{r}_2) = \frac{1}{\Pi}(r^1\bar{r}_1 - r^2\bar{r}_2)\,. \tag{6.70}$$

However, if we furthermore want to express the fermions in (6.21) with factorised parameters as well, because of partial derivatives in the supersymmtric condition (6.3), the shift part in the connection has to be 0 adaptively. Explicitly

$$\mathcal{L}_{3/8} = \begin{pmatrix} \tilde{\mathcal{A}}_{\varphi,I+1} - \frac{1}{2} & -i\bar{\delta}^{\dot{1}}\tilde{\rho}_{+}^{2f} \\ -i\frac{\delta_{\dot{1}}}{\Pi^f}\bar{\tilde{\rho}}_{2-} & \tilde{\mathcal{A}}_{\varphi,I+2} \end{pmatrix} , \tag{6.71}$$

where the factorised rotated fermions $\tilde{\rho}_{+}^{f2}$ is defined by $\tilde{\rho}_{+}^{f2} = \bar{u}_a\tilde{\psi}_{+}^{a}$. Note that since (6.71) is just a $U(1)$ symmetry transformed version of (6.21), we can also take the latter one directly with factorised $\bar{\eta}_a^{\imath}$ and unfactorised $\eta_a^{\imath}$, as well as the original rotated fields defined in (6.13), (6.25) and (6.28). Besides, the periodical condition (6.41) is always satisfied, though the shift is generally non-zero. The real reason that drives us to turn to (6.71) rather than sticking with (6.21) is that instead of the four preserved supercharges in (6.26), now we have six

$$\eta_a^{\imath}Q_{\imath}^{\dot{1}a}, \quad \eta_a^{\imath}Q_{\imath}^{\dot{2}a}, \quad \bar{u}_aQ_{\bar{l}}^{\dot{1}a}, \quad \bar{u}_aQ_{\bar{r}}^{\dot{1}a}, \quad \bar{u}_aQ_{\bar{l}}^{\dot{2}a}, \quad \bar{u}_aQ_{\bar{r}}^{\dot{2}a} , \tag{6.72}$$

which can be packaged into a superposition

$$\mathcal{Q}_{3/8} = w_{\dot{b}}\eta_a^{\imath}Q_{\imath}^{\dot{b}a} + \bar{w}_{\dot{b}}^{\imath}\bar{u}_a(\sigma^1)_{\imath}^{\bar{\imath}}Q_{\bar{\imath}}^{\dot{b}a} . \tag{6.73}$$

Instead, the factorised version of 1/4 BPS supercharge (6.27) is just $\mathcal{Q}_{1/4}^{f} = w_{\dot{b}}\eta_a^{\imath}Q_{\imath}^{\dot{b}a+} + \bar{w}_{\dot{b}}\bar{y}^{\imath}\bar{u}_a(\sigma^1)_{\imath}^{\bar{\imath}}Q_{\bar{\imath}}^{\dot{b}a-}$, which is identical to (6.73) if and only if $\bar{w}_{\dot{b}}^{\imath}$ factorises. So to be more general, we would like to consider both of the cases with factorised and unfactorised $\bar{w}_{\dot{b}}^{\imath}$, and the proper gauge we should choose is the one in which superconnection can be written as (6.71).

There are other fields and parameters that are "factorised" as well along with the supercharges

$$\tilde{r}_{\dot{2}}^{f} = \epsilon^{\dot{a}\dot{b}}(\bar{w}v)_{\dot{a}}\tilde{q}_{\dot{b}}, \quad \bar{\tilde{r}}^{f\dot{1}} = (\bar{w}v)_{\dot{a}}\bar{\tilde{q}}^{\dot{a}}, \quad \bar{\tilde{\rho}}_{1+}^{f} = \epsilon^{ab}\bar{u}_a\bar{\tilde{\psi}}_{b+}, \quad \tilde{\Pi}^f = \epsilon^{\dot{a}\dot{b}}(\bar{w}v)_{\dot{a}}w_{\dot{b}} . \tag{6.74}$$

Loosely summarizing, the phases $v, \bar{v}$ dropping out of $\bar{\eta}$ are picked up by $\bar{w}$ in this case. In this way, (6.31) now becomes

$$\tilde{r}_{I+1,\dot{1}}\bar{\tilde{r}}_{I+1}^{f\dot{1}} + \tilde{r}_{I+1,\dot{2}}^{f}\bar{\tilde{r}}_{I+1}^{\dot{2}} = \tilde{\Pi}^f\tilde{\nu}_{I+1}, \quad \bar{\tilde{r}}_{I+1}^{f\dot{1}}\tilde{r}_{I+1,\dot{1}} + \bar{\tilde{r}}_{I+1}^{\dot{2}}\tilde{r}_{I+1,\dot{2}}^{f} = \tilde{\Pi}^f\tilde{\nu}_{I+2} . \tag{6.75}$$

Analogous to (6.30), in the factorised notation, the supersymmetry variations of the scalar fields are

$$\mathcal{Q}_{3/8}\tilde{r}_{\dot{1}} = \tilde{\Pi}^f\tilde{\rho}_{+}^{f2}, \quad \mathcal{Q}_{3/8}\tilde{r}_{\dot{2}}^{f} = \tilde{\Pi}^f\tilde{\rho}_{-}^{1}, \quad \mathcal{Q}_{3/8}\bar{\tilde{r}}^{f\dot{1}} = \tilde{\Pi}^f\bar{\tilde{\rho}}_{2-}, \quad \mathcal{Q}_{3/8}\bar{\tilde{r}}^{\dot{2}} = \tilde{\Pi}^f\bar{\tilde{\rho}}_{1+}^{f} , \tag{6.76}$$

and the second variations

$$
\begin{aligned}
\mathcal{Q}^2_{3/8}\tilde{r}_{\dot{1}} &= \tilde{\Pi}^f\left(\Pi^f\left(i\partial_\varphi\tilde{r}_{\dot{1}} + \tilde{\mathcal{A}}_{I+1}\tilde{r}_{\dot{1}} - \tilde{r}_{\dot{1}}\tilde{\mathcal{A}}_{I+2}\right) - \frac{1}{2}\epsilon^{ab}\bar{u}_a(\eta\sigma^3\bar{v})_b\tilde{r}_{\dot{1}}\right)\\
\mathcal{Q}^2_{3/8}\tilde{r}^f_{\dot{2}} &= \tilde{\Pi}^f\left(\Pi^f\left(i\partial_\varphi\tilde{r}^f_{\dot{2}} + \tilde{\mathcal{A}}_{I+1}\tilde{r}^f_{\dot{2}} - \tilde{r}^f_{\dot{2}}\tilde{\mathcal{A}}_{I+2} + \frac{1}{2}\tilde{r}^f_{\dot{2}}\right) - \frac{2i}{k}\Pi^f(\tilde{\nu}_{I+1}\tilde{r}^f_{\dot{2}} - \tilde{r}^f_{\dot{2}}\tilde{\nu}_{I+2})\right)\\
\mathcal{Q}^2_{3/8}\bar{\tilde{r}}^{f\dot{1}} &= \tilde{\Pi}^f\left(\Pi^f\left(i\partial_\varphi\bar{\tilde{r}}^{f\dot{1}} + \tilde{\mathcal{A}}_{I+2}\bar{\tilde{r}}^{f\dot{1}} - \bar{\tilde{r}}^{f\dot{1}}\tilde{\mathcal{A}}_{I+1} + \frac{1}{2}\bar{\tilde{r}}^{f\dot{1}}\right)\right)\\
\mathcal{Q}^2_{3/8}\bar{\tilde{r}}^{\dot{2}} &= \tilde{\Pi}^f\left(\Pi^f\left(i\partial_\varphi\bar{\tilde{r}}^{\dot{2}} + \tilde{\mathcal{A}}_{I+2}\bar{\tilde{r}}^{\dot{2}} - \bar{\tilde{r}}^{\dot{2}}\tilde{\mathcal{A}}_{I+1}\right) - \frac{2i}{k}\Pi^f(\tilde{\nu}_{I+2}\bar{\tilde{r}}^{\dot{2}} - \bar{\tilde{r}}^{\dot{2}}\tilde{\nu}_{I+1}) - \frac{1}{2}\epsilon^{ab}\bar{u}_a(\eta\sigma^3\bar{v})_b\bar{\tilde{r}}^{\dot{2}}\right).
\end{aligned}
\tag{6.77}
$$

Then we can easily check that similar to the 1/2 and 1/4 cases, here we have $\mathcal{Q}_{3/8}H_{3/8} = i\Pi^f\tilde{\Pi}^f\mathcal{L}^F_{3/8}$ with

$$
H_{3/8} = \begin{pmatrix} 0 & \bar{\delta}^{\dot{1}}\Pi^f\tilde{r}_{\dot{1}} \\ \delta_{\dot{1}}\bar{\tilde{r}}^{f\dot{1}} & 0 \end{pmatrix}
\tag{6.78}
$$

which is equal to the factorised (6.36).

6.4.1 Deformations with $\tilde{\Pi}^f \neq 0$

A general supersymmetric deformation from the 3/8 BPS loops (6.71) is totally the same as (6.39), except that everything is in the factorised notation now

$$
\mathcal{L} = \mathcal{L}_{3/8} - i\mathcal{Q}_{3/8}G + \{H_{3/8}, G\} + \Pi^f\tilde{\Pi}^f G^2 + C\,,
\tag{6.79}
$$

where we take same coefficients $\bar{b}, b$ in G as in (6.34), but of the factorised rotated fields $\tilde{r}_{\dot{1}}, \tilde{r}^f_{\dot{2}}, \bar{\tilde{r}}^{f\dot{1}}$ and $\bar{\tilde{r}}^{\dot{2}}$. Again we have $\mathcal{Q}_{3/8}$ acting on $\mathcal{L}$ to be a covariant derivative

$$
\mathcal{Q}_{3/8}\mathcal{L} = \mathcal{D}^{\mathcal{L}}_\varphi(H_{3/8} + \Pi^f\tilde{\Pi}^f G)\,,
\tag{6.80}
$$

as long as

$$
\begin{aligned}
i\partial_\varphi(\Pi^f\tilde{\Pi}^f b_{\dot{1}}) &= 0, \qquad & i\partial_\varphi(\tilde{\Pi}^f\bar{b}^{\dot{1}}) &= 0\\
i\partial_\varphi(\tilde{\Pi}^f b_{\dot{2}}) &= -\tilde{\Pi}^f b_{\dot{2}}, \qquad & i\partial_\varphi(\Pi^f\tilde{\Pi}^f\bar{b}^{\dot{2}}) &= \Pi^f\tilde{\Pi}^f\bar{b}^{\dot{2}}\,.
\end{aligned}
\tag{6.81}
$$

The solutions are

$$
\bar{b}^{\dot{1}} = \frac{\bar{\beta}^{\dot{1}} - \bar{\delta}^{\dot{1}}}{\tilde{\Pi}^f}, \quad \bar{b}^{\dot{2}} = \frac{\bar{\beta}^{\dot{2}}e^{-i\varphi}}{\Pi^f\tilde{\Pi}^f}, \quad b_{\dot{1}} = \frac{\beta_{\dot{1}} - \delta_{\dot{1}}}{\Pi^f\tilde{\Pi}^f}, \quad b_{\dot{2}} = \frac{\beta_{\dot{2}}e^{i\varphi}}{\tilde{\Pi}^f}\,,
\tag{6.82}
$$

which leads to supersymmetric loops that preserve (6.73)

$$
\begin{pmatrix} \tilde{\mathcal{A}}_{3/8,I+1} - \frac{1}{2} & -i\bar{\beta}^{\dot{1}}\tilde{\rho}^{f2}_+ - i\frac{\bar{\beta}^{\dot{2}}e^{-i\varphi}}{\Pi^f}\tilde{\rho}^1_- \\ -i\frac{\beta_{\dot{1}}}{\Pi^f} - i\beta_{\dot{2}}e^{i\varphi}\bar{\tilde{\rho}}^f_{1+} & \tilde{\mathcal{A}}_{3/8,I+2} \end{pmatrix}\,,
\tag{6.83}
$$

where

$$\tilde{\mathcal{A}}_{3/8} = A_\varphi + \frac{i}{k\Pi^f}(r^1\bar{r}_1^f - r^{f2}\bar{r}_2) + \frac{1}{\tilde{\Pi}^f}\tilde{M}_{\dot{b}}^{f\dot{a}}\tilde{r}_{\dot{a}}^{(f)}\bar{\tilde{r}}^{(f)\dot{b}}\,, \tag{6.84}$$

with

$$\tilde{M}_{\dot{b}}^{f\dot{a}} = \begin{pmatrix} -\frac{i}{k} + \bar{\beta}^{\dot{1}}\beta_{\dot{1}} & \Pi^f\bar{\beta}^{\dot{1}}\beta_{\dot{2}}e^{i\varphi} \\ \frac{1}{\Pi^f}\beta_{\dot{1}}\bar{\beta}^{\dot{2}}e^{-i\varphi} & \frac{i}{k} + \bar{\beta}^{\dot{2}}\beta_{\dot{2}} \end{pmatrix}\,. \tag{6.85}$$

After fixing $\mathcal{Q}_{3/8}$, the moduli space of hyperloops in (6.83) is another conifold independent from (6.43) and those in the previous references [58, 59, 2]. They are somehow at an unexplored state between the deformed 1/4 BPS and 1/2 BPS loops. The special points with supersymmetric enhancement are very similar to those deforming from 1/4 BPS loops (6.21), while allowing more possibilities. An example is the novel "Factorised w and $\bar{z}$" bosonic loops found in [3] are included in the moduli space of (6.83). Explicitly, from the choice $\bar{\beta}^{\dot{1}} = \bar{\beta}^{\dot{2}} = \beta_{\dot{1}} = \beta_{\dot{2}} = 0$, we get single node bosonic loops

$$A^{\text{bos}}_{3/16?} = A_\varphi + \frac{i}{k\Pi^f}(r^1\bar{r}_1^f - r^{f2}\bar{r}_2) - \frac{i}{k\tilde{\Pi}^f}(\tilde{r}_{\dot{1}}\bar{\tilde{r}}^{f\dot{1}} - \tilde{r}_{\dot{2}}^f\bar{\tilde{r}}^{\dot{2}})\,. \tag{6.86}$$

Compared to the 1/8 BPS bosonic loops in (6.46), we might think that the resulting loops are 3/16 BPS naively. However, this is only true when $\bar{w}_b^{\imath}$ factorise. If not, since $\eta_a^{\imath}$ is unfactorised, these loops are also 1/8 BPS, preserving the barred and unbarred supercharges in (6.73) separately. Especially, taking $\bar{u}_a = \delta_a^1, \eta_1^l = \eta_2^r = 1, w_a = \delta_a^2, \bar{w}_1^l = 1, \bar{w}_2^r = -1$, we recover the "Factorised w and $\bar{z}$" 1/8 BPS bosonic loop.

Again by deforming the bosonic loops above, we are able to obtain any fermionic loops in (6.84). This also implies us that the fermionic partners of the "Factorised w and $\bar{z}$" loops are included in our discussion.

6.4.2 Deformations with $\tilde{\Pi}^f = 0$

The other case is for

$$\tilde{\Pi}^f = (\bar{w}v)_{\dot{1}}w_{\dot{2}} - (\bar{w}v)_{\dot{2}}w_{\dot{1}} = 0\,. \tag{6.87}$$

Similar to (6.52), we define the quotient parameter

$$\omega^f = \frac{(\bar{w}v)_{\dot{1}}}{w_{\dot{1}}} = \frac{(\bar{w}v)_{\dot{2}}}{w_{\dot{2}}}\,. \tag{6.88}$$

However, instead of a constant in (6.52), here it takes the form $a + be^{i\varphi}$ with a, b free constant parameters. Since $\bar{\tilde{r}}^{f\dot{1}}, \tilde{r}_{\dot{1}}$ are linearly dependant with $\bar{\tilde{r}}^{\dot{2}}, \tilde{r}_{\dot{2}}^f$ in this case, we construct new orthogonal bases

$$\tilde{r}_{\parallel}^f = \tilde{r}_{\dot{1}}, \quad \bar{\tilde{r}}^{f\parallel} = \bar{\tilde{r}}^{\dot{2}}, \quad \tilde{r}_{\perp}^f = w_{\dot{1}}\tilde{q}_{\dot{1}} + w_{\dot{2}}\tilde{q}_{\dot{2}}, \quad \bar{\tilde{r}}^{f\perp} = w_{\dot{2}}\bar{\tilde{q}}^{\dot{1}} - w_{\dot{1}}\bar{\tilde{q}}^{\dot{2}}\,, \tag{6.89}$$

where the scalars with parallel indices are annihilated by the supercharge (6.73). Define

$$\tilde{\Lambda} = w_{\dot{1}}^2 + w_{\dot{2}}^2\,, \tag{6.90}$$

in terms of which similar identities as (6.56) exist, and supersymmetric variations of the new scalar bases are

$$
\begin{aligned}
&\mathcal{Q}_{3/8}\tilde{r}^f_{\perp} = \tilde{\Lambda}(\tilde{\rho}^1_- + \omega^f \tilde{\rho}^{f2}_+), \quad \mathcal{Q}_{3/8}\bar{\tilde{r}}^{f\perp} = \tilde{\Lambda}(\bar{\tilde{\rho}}_{2-} - \omega^f \bar{\tilde{\rho}}^f_{1+}) \\
&\mathcal{Q}^2_{3/8}\tilde{r}^f_{\perp} = -\tilde{\Lambda}\Pi^f\omega^f\left((i\partial_\varphi \log\frac{\omega^f}{\Pi^f}+1)\tilde{r}^f_{\|} - \frac{2i}{k}(\tilde{\nu}_{I+1}\tilde{r}^f_{\|} - \tilde{r}^f_{\|}\tilde{\nu}_{I+2})\right) \\
&\mathcal{Q}^2_{3/8}\bar{\tilde{r}}^{f\perp} = \tilde{\Lambda}\omega^f\Pi^f\left((i\partial_\varphi \log\frac{\omega^f}{\Pi^f}+1)\bar{\tilde{r}}^{f\|} + \frac{2i}{k}(\tilde{\nu}_{I+2}\bar{\tilde{r}}^{f\|} - \bar{\tilde{r}}^{f\|}\tilde{\nu}_{I+1})\right).
\end{aligned}
\tag{6.91}
$$

The deformation is the same as (6.79) with $\tilde{\Pi}^f = 0$, where H is also (6.78) in the replacement of $\tilde{r}_{\dot{1}} \to \tilde{r}_{\|}$, $\bar{\tilde{r}}^{f\dot{1}} \to \omega^f \bar{\tilde{r}}^{\|}$. With $\bar{\beta}^{\perp}, \bar{\beta}_{\perp}$ satisfying

$$
c\bar{\delta}^{\dot{1}}\Pi^f = -i\bar{\beta}^{\perp}\tilde{\Lambda}\Pi^f\omega^f\partial_\varphi \log\left(\frac{\omega^f}{\Pi^f}e^{-i\varphi}\right), \quad c\delta_{\dot{1}}\omega^f = -i\beta_{\perp}\tilde{\Lambda}\Pi^f\omega^f\partial_\varphi \log\left(\frac{\omega^f}{\Pi^f}e^{-i\varphi}\right), \tag{6.92}
$$

the resulting loops are

$$
\begin{pmatrix}
\tilde{\mathcal{A}}^0_{3/8,I+1} - \frac{1}{2} & -i\tilde{\Lambda}\bar{\beta}^{\perp}\tilde{\rho}^1_- - i(\bar{\delta}^{\dot{1}} + \tilde{\Lambda}\omega^f\bar{\beta}^{\perp})\tilde{\rho}^{f2}_+ \\
i\tilde{\Lambda}\omega^f\beta_{\perp}\bar{\tilde{\rho}}^f_{1+} - i(\frac{\delta_{\dot{1}}}{\Pi^f} + \tilde{\Lambda}\beta_{\perp})\bar{\tilde{\rho}}_{2-} & \tilde{\mathcal{A}}^0_{3/8,I+2} + c
\end{pmatrix}, \tag{6.93}
$$

where

$$
\tilde{\mathcal{A}}^0_{3/8} = A_\varphi + \frac{i}{k\Pi^f}(r^1\bar{r}^f - r^{f2}\bar{r}_2) + \tilde{M}^{f\dot{a}}_{\dot{b}}\tilde{r}^f_{\dot{a}}\bar{\tilde{r}}^{f\dot{b}}, \tag{6.94}
$$

with

$$
\tilde{M}^{f\dot{a}}_{\dot{b}} = \begin{pmatrix} 0 & \frac{i}{k\tilde{\Lambda}} + \delta_{\dot{1}}\omega^f\bar{\beta}^{\perp} \\ \frac{i}{k\tilde{\Lambda}} + \bar{\delta}^{\dot{1}}\Pi^f\beta_{\perp} & \bar{\beta}^{\|} \end{pmatrix}. \tag{6.95}
$$

Again generally the hyperloops obtained above are 1/16 BPS that are independent from (6.65) and the $\Pi = 0$ ones in [2]. Analogous to section 6.3.2, there are points of enhanced supersymmetry on the moduli space as well. Those loops with $\bar{\beta}^{\|} = 0$ and $\delta_{\dot{1}}\omega^f\bar{\beta}^{\perp} = \bar{\delta}^{\dot{1}}\Pi^f\beta_{\perp}$ which is consistent with solutions of (6.92) are enhanced to be 1/8 BPS[27], preserving two supercharges

$$
\eta^{\imath}_a Q^{\dot{1}a}_{\imath} + \omega^{f\imath}\bar{u}_a(\sigma^1)^{\bar{\imath}}_{\imath} Q^{\dot{1}a}_{\bar{\imath}}, \quad \eta^{\imath}_a Q^{\dot{2}a}_{\imath} + \omega^{f\imath}\bar{u}_a(\sigma^1)^{\bar{\imath}}_{\imath} Q^{\dot{2}a}_{\bar{\imath}}, \tag{6.96}
$$

where $\omega^{f\imath}$ are components of ω^f with $\omega^f = \omega^{fl}v_l + \omega^{fr}v_r$, according to its definition in (6.88).

Acknowledgements

We would like to thank N. Drukker for suggesting us to write this paper, for his support and inspirational comments throughout the project, and for his great help in improving the draft. And we would like to acknowledge related collaboration with N. Drukker, M. Probst, M. Tenser and D. Trancanelli. The work of ZK is supported by CSC grant No. 201906340174. ZK would like to thank the University of Barcelona and the DESY theory group for their hospitality in the course of this work.

[27] In the $\Pi^f \neq 0$ analogy, one might expect to find 3/16 BPS loops in this case as well, but it is in fact impossible because of the preserved $SU(2)_R$ R-symmetry.

7 1/3 BPS loops and defect CFTs in ABJM theory

This section is based on [5] with minor edits.

7.1 Introduction

ABJM theory [87] has a rich spectrum of line operators including the 1/2 BPS loop [110] and 1/6 BPS loops. The latter may be bosonic [60–62] with only a single gauge field or include fermi fields like the 1/2 BPS ones [64,66,68,58]. There are also Wilson loops preserving fewer supercharges [172, 69, 59, 2, 3, 70], though they are not conformal. Finally there are vortex loops [111] that are 1/2 BPS or 1/3 BPS (though there should also be less supersymmetric versions).

A natural question that many experts have tried to address, is whether there are also 1/3 BPS Wilson loops in this theory. Given that a vortex loop exists, there is an appropriate superalgebra. Indeed, the $\mathfrak{osp}(6|4)$ superalgebra of ABJM is broken to $\mathfrak{su}(1,1|3) \oplus \mathfrak{u}(1)$ by the 1/2 BPS line and to $\mathfrak{su}(1,1|1) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)^2$ by the 1/6 BPS loops (for the bosonic loop, one $\mathfrak{u}(1)$ is enhanced to another $\mathfrak{su}(2)$). The 1/3 BPS algebra is $\mathfrak{su}(1,1|2) \oplus \mathfrak{u}(1)^2$. This latter algebra (up to a $\mathfrak{u}(1)$ factor) is also the symmetry of the 1/2 BPS loops of $\mathcal{N} = 4$ Chern-Simons theories, which is a hint for our construction.

The 1/2 BPS Wilson loops of $\mathcal{N} = 4$ theories [65, 63] have a degeneracy of pairs of loops preserving the same eight supercharges. Choosing then eight of the twelve supercharges of a 1/2 BPS loop W_1^+ of ABJM that generate an $\mathfrak{su}(1,1|2)$ subalgebra, there should be another Wilson loop, W_4^- preserving the same supercharges. This second Wilson loop is also 1/2 BPS, but the linear combination $W_{1/3} = n_1 W_1^+ + n_4 W_4^-$ is 1/3 BPS. Explicit expressions for W_1^+ and W_4^- are presented in Appendix 7.A: (7.66), (7.67).

Defining an operator as a linear combination of other ones may not seem fundamental, and one may raise the objection that they should each be studied independently. One way to see that this is not the case is that this linear combination arises naturally when considering Wilson loops based on superconnections larger than that of the 1/2 BPS loop. Such larger constructions with repeated entries from the same gauge field were proposed in [69, 59, 2] and give rise also to operators that cannot be expressed in the block-diagonal form of $n_1 W_1^+ + n_4 W_4^-$. While the 1/3 BPS loop itself can be written this way, it can be deformed into non-diagonal loops, so operator insertions into $W_{1/3}$ cannot be factorised as insertions into W_1^+ and those in W_4^-.

Having realised this 1/3 BPS line, we turn to studying its properties and in particular the defect CFT for operator insertions along it [118, 101, 121, 122, 95]. Part of this analysis relies on the explicit realisation of $W_{1/3}$ presented here and part is based on representation theory of the superconformal group, so is valid for any 1/3 BPS line operator including the vortex loop of [111] or any further line operators that may be found in the future.

The displacement and tilt operators are insertions that arise from broken translation and R-symmetry, respectively. As reviewed in Appendix 7.B, the two point functions of these operators are related to bremsstrahlung functions [119,133]. ABJM theory has a rich spectrum of such functions [192,198,199,141], and the case of the 1/3 BPS Wilson loop is even richer.

Any 1/3 BPS line breaks the conformal group $\mathfrak{so}(4,1) \to \mathfrak{so}(2,1) \oplus \mathfrak{so}(2)$, just as the 1/2 BPS or any other conformal line operator, so has two displacement operators from the broken translations. The $\mathfrak{su}(4)$ R-symmetry is broken to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)^2$ with five different pairs of tilt operators: a conjugate pair denoted $\mathbb{O}$ and $\bar{\mathbb{O}}$ for the broken generators between the two $\mathfrak{u}(1)$s and the others relating each of them to $\mathfrak{su}(2)$ (they are denoted as $\mathbb{O}^a$, $\bar{\mathbb{O}}_a$, $\mathbb{O}_a$, $\bar{\mathbb{O}}^a$ with $a \in \{2,3\}$). These are really different operators with different normalisations, i.e. there are three bremsstrahlung functions related to R-symmetry breaking in addition to the one for a real cusp.

Following [141,200], we use in Section 7.4 Ward identities and the explict form of the tilt operators to find relations between the bremsstrahlung functions. $\mathbb{O}$ is in the same multiplet with the displacement $\mathbb{D}$, so the associated bremsstrahlung functions are clearly related. For the other tilt operators, their sum is equal to that of $\mathbb{O}$.

In the case when the bremsstrahlung functions for $\mathbb{O}^a$ and $\mathbb{O}_a$ are equal (for $n_1 = n_4$), they are half of that of $\mathbb{O}$ or $\mathbb{D}$. A similar relation exists between the bremsstrahlung functions for the tilt and displacement of the bosonic loop, but here we find a very simple setting of the same phenomenon and a far easier proof of it.

Another natural object that arises in this context is a permutation operator, which we denote by σ, that replaces the connection of W_1^+ with that of W_4^-. This is the most clear manifestation of the nontrivial interplay between the two Wilson loops. This operator preserves two supercharges and has vanishing conformal dimension, so is topological. This enriches the spectrum of protected operators on $W_{1/3}$, but can also be considered as an operator on the 1/2 BPS W_1^+. We study some of its properties, but leave most of them for future work.

Going back to the five pairs of tilt operators, they are exactly marginal operators on $W_{1/3}$ and in Section 7.5 we study the deformation of the loop by them. We follow [1] to calculate the geometry of the resulting defect conformal manifold and precisely match it with the coset $SU(4)/S(U(2) \times U(1) \times U(1))$.

Some background information and many technical details are relegated to the appendices.

7.2 Realising 1/3 BPS Wilson loops

We construct here 1/3 BPS Wilson loops in ABJM theory and then recall some subtle features of general loops in this theory that play an important role for these new loops.

The simplest 1/2 BPS loop W_1^+ is formed out of $\mathcal{L}_1^+$ (7.66)

$$W_1^+ = \operatorname{Tr}\mathcal{P}\exp\int_{-\infty}^{\infty} i\mathcal{L}_1^+\,dx\,. \tag{7.1}$$

Here we take it to be a straight line in the x^3 direction (which we denote as x). For the circular loop there is subtlety of taking the trace or supertrace [110, 58], but here we are taking the infinite line, so really it is an open line. We write trace, and if adapting to a circle, one should include a twist operator (7.3) or following the convensions of [58], use a supertrace.

Another 1/2 BPS Wilson line is W_4^-, made out of the superconnection $\mathcal{L}_4^-$ (7.67). Each of W_1^+ and W_4^- preserves twelve supercharges, and when they are along the same line, they have eight supercharges in common.

To define the 1/3 BPS loop, we take a bigger structure combining both superconnections

$$W_{1/3} = \operatorname{Tr}\mathcal{P}\exp i\int_{-\infty}^{\infty}\operatorname{diag}(\underbrace{\mathcal{L}_1^+,\cdots,\mathcal{L}_1^+}_{n_1},\underbrace{\mathcal{L}_4^-,\cdots,\mathcal{L}_4^-}_{n_4})\,dx\,. \tag{7.2}$$

To avoid confusion, each $\mathcal{L}_i^\pm$ is an $(N_1|N_2)$ supermatrix, so this is not a $(2n_1N_1|2n_4N_2)$ supermatrix, but rather $((n_1+n_4)N_1|(n_1+n_4)N_2)$. With this diagonal structure, this loop can also be written as $n_1W_1^+ + n_4W_4^-$.

This loop on its own is 1/3 BPS, resolving this long-standing question. One may wonder whether there are other 1/3 BPS loops with non-diagonal structure, as there are many constructions of BPS Wilson loops that do not respect it. We made an extensive and systematic search, based on the techniques of [59, 2] and all the 1/3 BPS loops we found could be diagonalised to (7.2).

This construction of Wilson loops out of supermatrices has an $S(GL(n_1+n_4)\times GL(n_1+n_4))$ global symmetry, which was pointed out in [58, 69]. If we reorder the superconnection in a way that all $A_x^{(1)}$ are at the top left and $A_x^{(2)}$ at the bottom right, this group acts by independently rotating the n_1+n_4 copies of $A_x^{(1)}$ and of $A_x^{(2)}$, see [69] for details. This global symmetry is important in the analysis of the space of BPS Wilson loops, as the Wilson loop is a trace, so we should really identify operators related by conjugation. It is also this action that allows us to diagonalise all other 1/3 BPS loops we found to the same form as (7.2).

In general, this action is not a local symmetry. The simplest manifestation of that is in the case of a single $\mathcal{L}_1^+$, where the group is simply $GL(1)=\mathbb{C}^*$ and it acts on the 2×2 structure within $\mathcal{L}_1^+$ (7.66) as conjugation by elements like

$$T = \begin{pmatrix} I_{N_1} & 0 \\ 0 & -I_{N_2} \end{pmatrix}. \tag{7.3}$$

This has the effect of changing the signs $\alpha\to-\alpha$, $\bar\alpha\to-\bar\alpha$. The local action of this operator was studied in [201], where it was found to be a nontrivial operator in the defect CFT of the

1/2 BPS line. Though it has vanishing classical dimension, it is not BPS, so its dimension receives quantum corrections.

We focus instead on a diagonal $SL(n_1+n_4)$ subgroup which acts simultaneously on $A_x^{(1)}$ and $A_x^{(2)}$, not modifying $\mathcal{L}_1^+$ and $\mathcal{L}_4^-$. This is the obvious group acting by conjugation on the n_1+n_4 matrix of superconnections in (7.2).

7.2.1 Permutation operators

Of the diagonal $SL(n_1+n_4)$ action on the matrix in (7.2), an $S(GL(n_1)\times GL(n_4))$ subgroups is in fact a local symmetry, as the superconnection is proportional to the identity in those blocks. Other group elements change the form of the connection and are nontrivial operations on the Wilson line and can be viewed as operators in a 1d defect CFT.

To keep the gauge fields on the diagonal, the group elements we employ are permutations, changing the order of the entries. Explicitly for the case of $n_1=n_4=1$, there is a single non-trivial permutation

$$\sigma=\begin{pmatrix}0 & I_{N_1+N_2}\\ I_{N_1+N_2} & 0\end{pmatrix},\qquad \sigma\begin{pmatrix}\mathcal{L}_1^+ & 0\\ 0 & \mathcal{L}_4^-\end{pmatrix}\sigma=\begin{pmatrix}\mathcal{L}_4^- & 0\\ 0 & \mathcal{L}_1^+.\end{pmatrix} \tag{7.4}$$

As both $\mathcal{L}_1^+$ and $\mathcal{L}_4^-$ have the same gauge-group structure, there is no obstruction to doing this, and it is particularly nice since W_1^+ and W_4^- share eight supercharges. Furthermore, as we show in Appendix 7.E.1, this combination preserves half the supercharges shared by the two lines.

Another natural operator is $\tau=\mathrm{diag}(I,-I)$, satisfying $\tau\sigma\tau=-\sigma$. This is different from T of (7.3), which acts within a single block of these matrices, so on a single $\mathcal{L}_i^\pm$. In this setting there are then two basic T-like operators, $\mathrm{diag}(T,1)$, $\mathrm{diag}(1,T)=\sigma\,\mathrm{diag}(T,1)\sigma$, and one can also multiply them with σ and τ.

Unlike T, the permutation σ and $\tau\sigma$ are protected local operators. As their conformal dimension vanishes, they are topological and correlation functions of any other operators do not depend on the exact position where the permutation happens, as long as it does not cross any of the other operators.

σ, T and τ are "line changing operators", similar to boundary changing operators in 2d CFTs.[28] The most studied such operators in Wilson lines are cusps, where the direction of the line changes, or there is a change in some internal parameters [203, 204, 94, 132, 192]. Indeed in both $\mathcal{N}=4$ SYM in 4d and in ABJM theory those were studied extensively and some cusps were shown to be BPS [190, 132, 192]. T is similar to a non-BPS cusp and σ to the BPS cusp. But unlike the usual cusps, both T and σ are discrete operations, so one cannot study them in a small or large angle expansion.

[28]It is also natural to relate σ to permutation branes, see e.g. [202].

7.2.2 1/2 BPS loop with alternating superconnections

The operation of replacing part of a line with another connection arises naturally from the permutation symmetry above, but does not require large supermatrices. To see that consider

$$\sigma^+ = \frac{1}{2}(1+\tau)\sigma = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \frac{1}{2}(1-\tau)\sigma = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}. \tag{7.5}$$

Clearly $(\sigma^+)^2 = (\sigma^-)^2 = 0$, so we should avoid that. On the other hand, $\sigma^+\sigma^-$ and $\sigma^-\sigma^+$ are projectors on the top or bottom parts of $\mathrm{diag}(\mathcal{L}_1^+, \mathcal{L}_4^-)$. Inserting this into the 1/3 BPS line we find

$$W_{1/3}[\sigma^+\sigma^-(0)] = \mathrm{Tr}\,\mathcal{P}e^{i\int_{-\infty}^0 \begin{pmatrix} \mathcal{L}_1^+(x) & 0 \\ 0 & \mathcal{L}_4^-(x) \end{pmatrix} dx} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \mathcal{P}e^{i\int_0^\infty \begin{pmatrix} \mathcal{L}_1^+(x) & 0 \\ 0 & \mathcal{L}_4^-(x) \end{pmatrix} dx} = W_1^+ \,. \tag{7.6}$$

So this reduces the 1/3 BPS loop to the 1/2 BPS one. Inserting $\sigma^-\sigma^+$ reproduces W_4^-.

As stated, both σ and τ are protected topological operators, and hence also $\sigma^\pm$. We can therefore separate the two insertions and in particular move σ^- to $x \to \infty$, leaving us with the operator

$$W_{1/3}[\sigma^+(0)] = \mathrm{Tr}\,\mathcal{P}\left[\exp\int_{-\infty}^0 i\mathcal{L}_1^+(x)\,dx\, \exp\int_0^\infty i\mathcal{L}_4^-(x)\,dx\right] = W_1^+[\sigma]. \tag{7.7}$$

This loop starts with a single superconnection $\mathcal{L}_1^+$ and switches at $x=0$ to $\mathcal{L}_4^-$. In the last expression, we employed the notation $W_1^+[\sigma]$, where σ is now an insertion in the 1/2 BPS loop that changes the connection. We can then denote $\sigma^- \simeq \bar\sigma$, where it is assumed that there is first a σ insertion. On it's own $\bar\sigma$ is a good insertion in W_4^-.

Unlike $W_{1/3}$ (7.2), the line W_1^+ (7.6) is 1/2 BPS, but if we insert σ into it as in (7.7), it is natural to analyse it in the same context as the 1/3 BPS line. In the following we study both objects: the true 1/3 BPS line constructed from a larger superconection and the 1/2 BPS line with the topological operator σ in its spectrum and splitting the 1/2 BPS supermultiplets to 1/3 BPS ones.

A subtlety when writing expressions like (7.7) in terms of ABJM fields, is that one should treat the σ's as a book-keeping device indicating where the connection and spectrum of insertions changes. One is not meant to implement a substitution rule like $\sigma\bar\psi_+^1\bar\sigma = \bar\psi_-^4$. Such notations may also possible, but they are not what we use here.

Not all line operators are Wilson lines, for example the vortex loops of [111], and in those cases we may not be able to write the expression on the right hand side of (7.7) explicitly. For that reason, we try to rely as much as possible on algebra, rather than on the explicit realisation of $W_{1/3}$ and the operator insertions.

7.3 Displacement multiplets of 1/3 BPS line operators

Among all operator insertions into the Wilson loop, the displacement operator and its super-partners are special, as they arise from broken global symmetries. The conservation equation for translation, supersymmetry and R-transformations are violated by the Wilson lines.

We study here how the displacement multiplets of 1/2 BPS line defects constructed in [195, 141, 95] split into 1/3 BPS multiplets. Most of the analysis is based on the breaking of global symmetries, so valid for any 1/3 BPS line operator.

7.3.1 First 1/2 BPS line

The 1/2 BPS defect along the $x = x_3$ axis preserves the rigid 1d conformal group, rotation around the line, and an $SU(3) \times U(1)$ R-symmetry, rotating $I, J = 2, 3, 4$ indicated by i, j. In addition, it preserves the supercharges Q_+^{12}, Q_+^{13}, Q_+^{14}, Q_-^{23}, Q_-^{34}, Q_-^{24} and the corresponding S's. A realisation of such an operator is W_1^+ with the superconnection in (7.66).

For all lines along x, the broken translation generators are the components $T^{\mu 1}$ and $T^{\mu 2}$ of the stress tensor. The other symmetries broken by this particular line include[29] the six components of the supercurrents $S_-^{\mu 1i}$ and $S_+^{\mu ij}$ and the 6 components of the R current $J^\mu{}_1{}^i$ and $J^\mu{}_i{}^1$. The conservation equations for the currents are then

$$\begin{aligned}
\partial_\mu T^{\mu\nu}(x) W_1^+ &= \delta(x_1)\delta(x_2)\delta_n^\nu\, W_1^+[\mathbb{D}^n(x)]\,, \qquad n \in \{2,3\}\,,\\
\partial_\mu S_\alpha^{\mu IJ}(x) W_1^+ &= \delta(x_1)\delta(x_2)\, W_1^+[\delta_1^I\delta_\alpha^- \bar{\mathbb{\Lambda}}^J(x) + \epsilon^{1IJK}\delta_\alpha^+ \mathbb{\Lambda}_K(x)]\,,\\
\partial_\mu J^\mu{}_I{}^J(x) W_1^+ &= \delta(x_1)\delta(x_2)\, W_1^+[\delta_I^1 \mathbb{O}^J(x) + \delta_1^J \bar{\mathbb{O}}_I(x)]\,.
\end{aligned} \tag{7.8}$$

Together, the operators on the right hand side form most of the displacement multiplet [195, 95] including the displacement itself $\mathbb{D} = \mathbb{D}^1 - i\mathbb{D}^2$, a fermionic operator $\mathbb{\Lambda}_i$ and the tilt $\mathbb{O}^i$. In fact there is one element missing, $\mathbb{F}$, which is fermionic and the lowest weight state in the multiplet. There are eight further operators in the complex conjugate multiplet.

The action of the preserved supersymmetries on the multiplet are [141, 95]

$$\{Q_+^{1i}, \mathbb{F}\} = \mathbb{O}^i\,, \quad [Q_+^{1i}, \mathbb{O}^j] = \epsilon^{ijk}\mathbb{\Lambda}_k\,, \quad \{Q_+^{1i}, \mathbb{\Lambda}_j\} = -2\delta_j^i \mathbb{D}\,, \quad [Q_+^{1i}, \mathbb{D}] = 0\,. \tag{7.9}$$

From the Jacobi identities for the superalgebra one also finds $[Q_-^{ij}, \mathbb{F}] = 0$ and

$$[Q_-^{ij}, \mathbb{O}^k] = -2i\epsilon^{ijk}\mathcal{D}_x\mathbb{F}\,, \quad \{Q_-^{ij}, \mathbb{\Lambda}_k\} = 2i\delta_k^i \mathcal{D}_x \mathbb{O}^j - 2i\delta_k^j \mathcal{D}_x \mathbb{O}^i\,, \quad [Q_-^{ij}, \mathbb{D}] = i\epsilon^{ijk}\partial_x \mathbb{\Lambda}_k\,. \tag{7.10}$$

$\mathcal{D}_x$ is an appropriate covariant derivative along the line operator.

Explicit expressions for the operators in terms of the fields of ABJM theory are presented in [95] and also in Appendix 7.E. These operators can also be identified with fluctuations of the sigma-model describing an AdS_2 string in $AdS_4 \times \mathbb{CP}^3$ [193].

[29] Broken rotations, special conformal transformations and superconformal generators all vanish at the origin, so do not give further operaotrs.

7.3.2 Second 1/2 BPS line

The second line we consider also preserves the conformal group along x, rotation around the line, and an $SU(3) \times U(1)$ R-symmetry, rotating $I, J = 1, 2, 3$, now indicated as $\hat{\imath}, \hat{\jmath}$. It preserves the supercharges Q_+^{12}, Q_+^{13}, Q_+^{23}, Q_-^{34}, Q_-^{24}, Q_-^{14} and the corresponding S's. A realisation of such an operator is W_4^- with the superconnection in (7.67).

We can write the action of the symmetries broken by W_4^- on that loop as

$$\begin{aligned}
\partial_\mu T^{\mu\nu}(x) W_4^- &= \delta(x_1)\delta(x_2)\delta_n^\nu W_4^- [\mathbb{D}^n(x)]\,, \qquad n \in \{2,3\}\,,\\
\partial_\mu S_\alpha^{\mu IJ}(x) W_4^- &= \delta(x_1)\delta(x_2)\, W_4^- [\epsilon^{IJK4}\delta_\alpha^- \bar{\mathbb{\Lambda}}_I(x) + \delta_4^J \delta_\alpha^+ \mathbb{\Lambda}^K(x)]\,,\\
\partial_\mu J^\mu{}_I{}^J(x) W_4^- &= \delta(x_1)\delta(x_2)\, W_4^- [\delta_4^J \mathbb{O}_I(x) + \delta_I^4 \bar{\mathbb{O}}^J(x)]\,.
\end{aligned} \tag{7.11}$$

These operators fit into the displacement multiplet (and its conjugate) for the appropriate $\mathfrak{su}(1,1|3)$ superalgebra. Compared to the previous case we need to exchange $1 \leftrightarrow 4$, though one should also take into account that the spinors change chirality as do some signs in the matrix $M^I{}_J$.

The analogue of (7.9) is now

$$\{Q_+^{\hat{\imath}\hat{\jmath}}, \Finv\} = \epsilon^{\hat{\imath}\hat{\jmath}\hat{k}}\mathbb{O}_{\hat{k}}\,, \quad [Q_+^{\hat{\imath}\hat{\jmath}}, \mathbb{O}_{\hat{k}}] = \delta_{\hat{k}}^{\hat{\imath}}\mathbb{\Lambda}^{\hat{\jmath}} - \delta_{\hat{k}}^{\hat{\jmath}}\mathbb{\Lambda}^{\hat{\imath}}\,, \quad \{Q_+^{\hat{\imath}\hat{\jmath}}, \mathbb{\Lambda}^{\hat{k}}\} = -2\epsilon^{\hat{\imath}\hat{\jmath}\hat{k}}\mathbb{D}\,, \quad [Q_+^{\hat{\imath}\hat{\jmath}}, \mathbb{D}] = 0\,, \tag{7.12}$$

and the analogue of (7.10) is

$$[Q_-^{\hat{\imath}4}, \mathbb{O}_{\hat{\jmath}}] = -2i\delta_{\hat{\jmath}}^{\hat{\imath}}\mathcal{D}_x \Finv\,, \quad \{Q_-^{\hat{\imath}4}, \mathbb{\Lambda}^{\hat{\jmath}}\} = 2i\epsilon^{\hat{\imath}\hat{\jmath}\hat{k}}\mathcal{D}_x\mathbb{O}_{\hat{k}}\,, \quad [Q_-^{\hat{\imath}4}, \mathbb{D}] = i\mathcal{D}_x\mathbb{\Lambda}^{\hat{\imath}}\,. \tag{7.13}$$

7.3.3 Decompsition into 1/3 BPS multiplets

The 1/2 BPS displacement multiplets are in the representation $\mathcal{B}^{1/2}_{3/2,0,0}$ of their respective $\mathfrak{su}(1,1|3)$ in the notations of [195] and $\mathbf{L\bar{A}_1}$ with primary $[\frac{3}{2}]^{0,0}_{1/2}$ in the notation of [104]. This representation splits into two representations of $\mathfrak{su}(1,1|2)$ denoted as $\mathbf{L\bar{A}_1}$ with primaries $[\frac{1}{2}]^0_{1/2}$ and $[1]^0_1$ in the notations of [104].

A simple way to see this in practice is to match the symmetries broken by both W_1^+ and W_4^- or only one of them. The symmetries broken by W_1^+ and preserved by W_4^- are

$$Q_+^{23}\,, \quad Q_-^{14}\,, \quad J_1{}^a\,, \quad J_a{}^1\,, \qquad a = 2,3\,. \tag{7.14}$$

Those give rise to the operators

$$Q_+^{23} \rightsquigarrow \mathbb{\Lambda}_4\,, \quad J_1{}^2 \rightsquigarrow \mathbb{O}^2\,, \quad J_1{}^3 \rightsquigarrow \mathbb{O}^3\,, \quad \mathbb{F}\,, \tag{7.15}$$

and their complex conjugates. We include $\mathbb{F}$ to complete the multiplet and in the following often omit the subscript 4 from the singlet $\mathbb{\Lambda}$. We call this the tilt multiplet.

Likewise the symmetries broken by W_4^- and not by W_1^+ are

$$Q_+^{14}\,, \quad Q_-^{23}\,, \quad J_4{}^a\,, \quad J_a{}^4\,, \qquad a = 2,3\,. \tag{7.16}$$

Those give rise to the operators

$$Q_+^{14} \rightsquigarrow \reflectbox{$\mathbb{\Lambda}$}^4\,, \quad J_2{}^4 \rightsquigarrow \reflectbox{$\mathbb{O}$}_2\,, \quad J_3{}^4 \rightsquigarrow \reflectbox{$\mathbb{O}$}_3\,, \quad \reflectbox{$\mathbb{F}$}\,, \tag{7.17}$$

and the complex conjugate multiplet. We name this the tlit multiplet, to distinguish from the tilt.

The symmetries broken by both W_1^+ and W_4^- are

$$P_1\,, \quad P_2\,, \quad Q_-^{12}\,, \quad Q_-^{13}\,, \quad Q_+^{24}\,, \quad Q_+^{34}\,, \quad J_1{}^4\,, \quad J_4{}^1\,, \tag{7.18}$$

In the case of W_1^+ they correspond to

$$P_1 - iP_2 \rightsquigarrow \mathbb{D}\,, \quad Q_+^{24} \rightsquigarrow -\mathbb{\Lambda}_3\,, \quad Q_+^{34} \rightsquigarrow \mathbb{\Lambda}_2\,, \quad J_1{}^4 \rightsquigarrow \mathbb{O}^4\,, \tag{7.19}$$

and in the case of W_4^-

$$P_1 - iP_2 \rightsquigarrow \reflectbox{$\mathbb{D}$}\,, \quad Q_+^{24} \rightsquigarrow \reflectbox{$\mathbb{\Lambda}$}^2\,, \quad Q_+^{34} \rightsquigarrow \reflectbox{$\mathbb{\Lambda}$}^3\,, \quad J_1{}^4 \rightsquigarrow \reflectbox{$\mathbb{O}$}_1\,. \tag{7.20}$$

7.3.4 Multiplets of $W_{1/3}$

In the case of a 1/3 BPS line operator, based on symmetry breaking alone, we should have the combination of terms in (7.8) and (7.11)

$$\begin{aligned}
\partial_\mu T^{\mu\nu}(x)W_{1/3} &= \delta(x_1)\delta(x_2)\delta_n^\nu\, W_{1/3}\big[\mathbb{D}^n(x)\big]\,, \qquad n \in \{2,3\}\,, \\
\partial_\mu S_\alpha^{\mu IJ}(x)W_{1/3} &= \delta(x_1)\delta(x_2)\, W_{1/3}\big[\epsilon^{1aI4}\delta_4^J\delta_\alpha^+\mathbb{A}_a(x) + \delta_1^I\delta_a^J\delta_\alpha^-\bar{\mathbb{A}}^a(x) \\
&\quad + \delta_2^I\delta_3^J\big(\delta_\alpha^+\mathbb{\Lambda}(x) + \delta_\alpha^-\bar{\reflectbox{$\mathbb{\Lambda}$}}(x)\big) + \delta_1^I\delta_4^J\big(\delta_\alpha^+\reflectbox{$\mathbb{\Lambda}$}(x) + \delta_\alpha^-\bar{\mathbb{\Lambda}}(x)\big)\big]\,, \\
\partial_\mu J^\mu{}_I{}^J(x)W_{1/3} &= \delta(x_1)\delta(x_2)\, W_{1/3}\big[\delta_I^1\delta_4^J\mathbb{O}(x) + \delta_I^4\delta_1^J\bar{\mathbb{O}}(x) + \delta_I^1\delta_a^J\mathbb{O}^a(x) \\
&\quad + \delta_I^a\delta_1^J\bar{\reflectbox{$\mathbb{O}$}}_a(x) + \delta_I^a\delta_4^J\reflectbox{$\mathbb{O}$}_a(x) + \delta_I^4\delta_a^J\bar{\mathbb{O}}^a(x)\big]\,.
\end{aligned} \tag{7.21}$$

For $W_{1/3}$ in (7.2), where the connection is a larger supermatrix, the operators on the right hand side are now matrices. In the $n_1 = n_4 = 1$ case, the operators form the tilt multiplet are

$$\begin{pmatrix} \mathbb{F} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbb{O}^a & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbb{\Lambda}_4 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.22}$$

The tlit multiplet is

$$\begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{F}$} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{O}$}_a \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \reflectbox{$\mathbb{\Lambda}$}^1 \end{pmatrix}, \tag{7.23}$$

and the displacement multiplet is

$$\mathbb{O} = \begin{pmatrix} \mathbb{O}^4 & 0 \\ 0 & \reflectbox{$\mathbb{O}$}_1 \end{pmatrix}, \quad \mathbb{A}_a = \begin{pmatrix} \mathbb{\Lambda}_a & 0 \\ 0 & \epsilon_{ab}\reflectbox{$\mathbb{\Lambda}$}^b \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} \mathbb{D} & 0 \\ 0 & \reflectbox{$\mathbb{D}$} \end{pmatrix}. \tag{7.24}$$

We do not introduce different notation for the matrices in (7.22) and (7.23) and at times below refer to the entire larger matrices with the same letter as the operator inside. It should be clear from the context, which of those is meant.

The action of the preserved generators on the different operators are presented in Appendix 7.D. Explicit expressions for these operators in terms of the ABJM fields are presented in Appendix 7.E.2.

In addition to the multiplets inherited from the 1/2 BPS displacement multiplet, we have the permutation multiplet constructed in Appendix 7.E.1

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbb{\Sigma}^a = \begin{pmatrix} 0 & \bar{G}^a - G^a \\ G^a - \bar{G}^a & 0 \end{pmatrix}, \qquad o\,. \tag{7.25}$$

For the expression for o, see (7.94). We can also think of them more abstractly as a short representation of the 1/3 BPS algebra with a primary with labels $[0]_0^0$ in the $\mathbf{A_1\bar{A}_1}$ multiplet the notations of [104]. Unlike the fields in the other multiplets, this operator is real.

We can of course form composites of these operators, which include the combinations like $\mathbb{F}\mathbb{\Lambda}$ as well as off-diagonal entries arising from σ times another operator. As usual, if two operators do not share supercharges, the composite would not be protected. An example of that is σ and $\mathbb{O}$.

We can also endow the operators with Chan-Paton factors

$$\text{⅂}_1{}^1 \simeq \begin{pmatrix} \text{⅂} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{⅂}_1{}^2 \simeq \begin{pmatrix} 0 & \text{⅂} \\ 0 & 0 \end{pmatrix}, \quad \text{⅂}_2{}^1 \simeq \begin{pmatrix} 0 & 0 \\ \text{⅂} & 0 \end{pmatrix}, \quad \text{⅂}_2{}^2 \simeq \begin{pmatrix} 0 & 0 \\ 0 & \text{⅂} \end{pmatrix}. \tag{7.26}$$

In particular operators like $\text{⅂}_1{}^1 = \sigma \text{⅂}_2{}^2 \sigma$ are inserted into the $\mathcal{L}_1^+$ line and enable our construction in the next subsection.

7.3.5 1/3 BPS multiplets of the 1/2 BPS Wilson line

The 1/2 BPS line W_1^+ has a displacement multiplet, as presented in Section 7.3.1. It combines the 1/3 BPS displacement multiplet with $\mathbb{O}^4$, $\mathbb{\Lambda}_a$ and $\mathbb{D}$ as well as the 1/3 BPS tilt with $\mathbb{F}$, $\mathbb{O}^a$ and $\mathbb{\Lambda}$ (7.15).

As presented in Section 7.2.2, The operators σ^+ and σ^- (or σ and $\bar{\sigma}$) (7.5) are also natural insertions in W_1^+. They are in the 1/3 BPS multiplet (7.25) and as σ changes the superconnection from $\mathcal{L}_1^+$ to $\mathcal{L}_4^-$, we can then insert operators naturally living on W_4^-. In particular this applies to the tlit operators, (7.17) as

$$\sigma \text{⅂}\,, \qquad \sigma \mathbb{O}_a\,, \qquad \sigma \mathbb{\Lambda}\,. \tag{7.27}$$

We can adjoin to all these operators $\bar{\sigma}$ from the right, so they become insertions into the W_1^+ loop without a change in connection.

Some care is required in analysing σ and composites like $\sigma\daleth$ or $\sigma\daleth\bar{\sigma}$. Recall that a special feature of the ABJM Wilson lines is that the preserved supercharges do not annihilate the connections in Appendix 7.A but give total derivatives (7.86). When acting on the entire line, these total derivative integrate to zero, hence the loops are BPS.

When acting on the line with insertions, we find extra boundary terms. As discussed in Appendix 7.E.4, the action of a supercharge on an odd supermatrix insertion in W_1^- is covariantised to

$$\tilde{Q}_+^{1a}\bullet = Q_+^{1a}\bullet - \{\bar{G}^a, \bullet\}\,, \qquad \tilde{Q}_-^{a4}\bullet = Q_-^{a4}\bullet - \{G^a, \bullet\}\,, \tag{7.28}$$

with $\bar{G}^a$ and G^a in (7.87). In the case of W_4^- the roles of $\bar{G}^a$ and G^a are reversed (7.86).

In evaluating the variation of σ, the direct action by Q_α^{IJ} is trivial and we only have the covariant part, with that from W_1^+ on the left and from W_4^- on the right. This is the source of the terms in the expression for $\mathbb{\Sigma}$ (7.25).

An $\daleth$ insertion into W_4^- is annihilated by three supercharges of which two are shared by W_1^+. Yet, when inserting it as $\sigma\daleth\bar{\sigma}$ into W_1^+, there are different total derivative terms. In fact, no supercharges annihilate it and only the combination of $Q_+^{1a} + Q_-^{a4}$ acting on it gives $\epsilon^{ab}\sigma\mathbb{O}_b\bar{\sigma}$.

This construction seems to introduces several new marginal operators into the 1/2 BPS loop: $\sigma\mathbb{O}_a\bar{\sigma}$, $o\bar{\sigma}$ and $(\mathcal{D}_x\sigma)\bar{\sigma}$ (7.93). Unlike $\mathbb{O}^2$, $\mathbb{O}^3$ and $\mathbb{O}^4$, these operators do not arise from broken global symmetries, so it is not guaranteed that they are indeed exactly marginal. We leave this question for further study.

Of particular note is the operator $(\mathcal{D}_x\sigma)\bar{\sigma}$, the descendant of σ (7.93), which is an infinitesimal deformation in the direction of the ABJM version of the loops described in Section 6.3.2 of [2]. The loops described there are classically conformal, but conformality is not guaranteed by the preserved supercharges. The question raised in the last paragraph is another avatar of the question of whether these loops are truly conformal.

For explicit expressions in terms of the ABJM fields, see Appendix 7.E.2.

7.4 Two point functions

For the operators arising from broken symmetries, as in (7.3.4), their normalisations are fixed by the normalisation of the conserved currents. We study here the relations between the normalisations of the different operators and their relation to the bremsstrahlung functions of these loops.

7.4.1 Ward identities

From conformal symmetry we know that the correlators of the operators in the displacement multiplet take the form

$$\langle\!\langle \mathbb{D}(0)\bar{\mathbb{D}}(x)\rangle\!\rangle = \frac{C_{\mathbb{D}}}{x^4}\,, \tag{7.29}$$

$$\langle\!\langle \mathbb{A}_a(0)\bar{\mathbb{A}}^b(x)\rangle\!\rangle = \frac{C_{\mathbb{A}_a}\delta_a^b}{x^3}\,, \tag{7.30}$$

$$\langle\!\langle \mathbb{O}(0)\bar{\mathbb{O}}(x)\rangle\!\rangle = \frac{C_{\mathbb{O}}}{x^2}\,. \tag{7.31}$$

The notation $\langle\!\langle \bullet \cdots \bullet \rangle\!\rangle$ represents the expectation value of the $\bullet$ insertions into the line normalized by the expectation value of the line without insertions.

The coefficients $C_{\mathbb{D}}$ and $C_{\mathbb{O}}$ are fixed from the definition of the operators and the normalisation of the broken currents in (7.21). They are also related to the bremsstrahlung functions of the line operators, as in (7.74), (7.75). The relations between them can be found from the ward identity for supersymmetry, following [141, 200].

Starting with the vanishing correlator $\langle\!\langle \mathbb{A}_2(0)\,\bar{\mathbb{D}}(x)\rangle\!\rangle = 0$ and acting with the preserved supercharge Q_+^{12}, using (7.9), we find

$$-2\langle\!\langle \mathbb{D}(0)\,\bar{\mathbb{D}}(x)\rangle\!\rangle = \langle\!\langle \mathbb{A}_2(0)\,\mathcal{D}_x\bar{\mathbb{A}}^2(x)\rangle\!\rangle\,, \tag{7.32}$$

where $\mathcal{D}_x$ is an appropriate covariant derivative along the Wilson line. This gives $2C_{\mathbb{D}} = 3C_{\mathbb{A}_a}$.

Likewise starting with $\langle\!\langle \mathbb{O}(0)\,\bar{\mathbb{A}}^3(x)\rangle\!\rangle = 0$ and acting with the preserved supercharge Q_+^{12} as in (7.10), we find

$$-\langle\!\langle \mathbb{A}_3(0)\,\bar{\mathbb{A}}^3(x)\rangle\!\rangle = 2\langle\!\langle \mathbb{O}\,\mathcal{D}_x\bar{\mathbb{O}}(x)\rangle\!\rangle\,, \tag{7.33}$$

or $C_{\mathbb{A}_a} = 4C_{\mathbb{O}}$. Combining the two, we find $C_{\mathbb{D}} = 6C_{\mathbb{O}}$.

These expressions were already derived in [95] from a superspace representation of the displacement multiplet in the case of the 1/2 BPS loop and they are not modified in the 1/3 BPS case.

For the operators in the tilt and tlit multiplets

$$\langle\!\langle \mathbb{\Lambda}(0)\bar{\mathbb{\Lambda}}(x)\rangle\!\rangle = \frac{C_{\mathbb{\Lambda}}}{x^3}\,, \qquad \langle\!\langle \mathbb{V}(0)\,\bar{\mathbb{V}}(x)\rangle\!\rangle = \frac{C_{\mathbb{V}}}{x^3}\,, \tag{7.34}$$

$$\langle\!\langle \mathbb{O}^a(0)\bar{\mathbb{O}}_b(x)\rangle\!\rangle = \frac{C_{\mathbb{O}^a}\delta_b^a}{x^2}\,, \qquad \langle\!\langle \mathbb{O}_a(0)\,\bar{\mathbb{O}}^b(x)\rangle\!\rangle = \frac{C_{\mathbb{O}_a}\delta_a^b}{x^3}\,, \tag{7.35}$$

$$\langle\!\langle \mathbb{F}(0)\bar{\mathbb{F}}(x)\rangle\!\rangle = \frac{C_{\mathbb{F}}}{x}\,. \qquad \langle\!\langle \mathbb{T}(0)\,\bar{\mathbb{T}}(x)\rangle\!\rangle = \frac{C_{\mathbb{T}}}{x^3}\,. \tag{7.36}$$

To find relations among those, we start with the vanishing correlator $\langle\!\langle \mathbb{O}^3(0)\,\bar{\mathbb{\Lambda}}(x)\rangle\!\rangle$ and act with the preserved supercharge Q_+^{12}, which yields

$$-\langle\!\langle \mathbb{\Lambda}(0)\,\bar{\mathbb{\Lambda}}(x)\rangle\!\rangle = 2\langle\!\langle \mathbb{O}^3(0)\,\mathcal{D}_x\bar{\mathbb{O}}_3(x)\rangle\!\rangle \quad\Rightarrow\quad C_{\mathbb{\Lambda}} = 4C_{\mathbb{O}^a}\,. \tag{7.37}$$

Then taking $\langle\!\langle \mathbb{F}(0)\,\bar{\mathbb{O}}_2(x) \rangle\!\rangle$ and acting with the preserved supercharge Q_+^{12}, we find

$$-\langle\!\langle \mathbb{O}(0)\,\bar{\mathbb{O}}(x) \rangle\!\rangle = 2\langle\!\langle \mathbb{F}^2(0)\,\mathcal{D}_x\bar{\mathbb{F}}_2(x) \rangle\!\rangle\,, \tag{7.38}$$

This gives $C_{\mathbb{O}^a} = 2C_{\mathbb{F}}$ and finally $C_{\mathbb{A}} = 8C_{\mathbb{F}}$.

The expressions for the tlit multiplet are identical, but $C_{\mathbb{O}^a}$ does not have to be equal to $C_{\mathbb{O}^a}$. Likewise, for the 1/2 BPS loop we know that $C_{\mathbb{A}} = C_{\mathbb{A}_a}$ and $C_{\mathbb{O}^a} = C_{\mathbb{O}}$, but this does not necessarily hold for 1/3 BPS operators, as we discuss in the next section.

7.4.2 Relations accross multiplets

We can go further and relate the different multiplets to each-other, using the explicit representation of $W_{1/3}$ (7.2) and its expression in terms of 1/2 BPS loops. We consider the case of n_1 copies of $\mathcal{L}_1^+$ and n_4 copies of $\mathcal{L}_4^-$, but for simplicity write them as 2×2 matrices.

Using the representation in (7.22), the two point function of the operators from the tilt multiplets can be related to those of the 1/2 BPS loop W_1^+ as

$$C_{\mathbb{O}^a}^{1/3}\delta_b^a = \left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^a(0) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbb{O}}_b(\infty) & 0 \\ 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle = \frac{n_1\delta_b^a}{n_1+n_4} C_{\mathbb{O}}^{1/2}\,. \tag{7.39}$$

This is a simple consequence of having n_1 insertions. Likewise for $\mathbb{O}_a$ and $\mathbb{O}$, we have

$$C_{\mathbb{O}_a}^{1/3} = \frac{n_4}{n_1+n_4} C_{\mathbb{O}}^{1/2}\,, \qquad C_{\mathbb{O}}^{1/3} = C_{\mathbb{O}}^{1/2}\,. \tag{7.40}$$

In particular

$$C_{\mathbb{O}^a}^{1/3} + C_{\mathbb{O}_a}^{1/3} = C_{\mathbb{O}}^{1/3}\,. \tag{7.41}$$

Though we derived this from the expressions in (7.2), we expect this relation to hold for any 1/3 BPS loop.

The story is very different when considering the operators inserted into the 1/2 BPS Wilson loop with the aid of σ. In that case $C_{\mathbb{O}^a} = C_{\mathbb{O}^4} = C_{\mathbb{O}}^{1/2}$, as these are simply the usual tilt operators of the 1/2 BPS line. If we look at $\langle\!\langle \sigma\mathbb{O}_a\bar{\sigma}(0)\,\sigma\bar{\mathbb{O}}^b\bar{\sigma}(x) \rangle\!\rangle$, assume $C_\sigma = 1$ to cancel the middle $\sigma(0)\bar{\sigma}(x)$ and move the other σ's far away, then it is natural to expect that this too is $C_{\mathbb{O}}^{1/2}$. This indicates that in this case all three normalisation constants are equal to $C_{\mathbb{O}}^{1/2}$, though this deserves more careful study.

7.4.3 Bremsstrahlung functions of $W_{1/3}$

As reviewed in Appendix 7.B, the normalisation constants are related to the bremsstrahlung functions arising from nearly straight cusps.

We can characterise cusps of the 1/3 BPS loop (7.2) by an angle ϕ and an R symmetry $SU(4)$ matrix U. When this matrix is close to the identity we can write it in terms of the symmetry breaking generators as

$$U = I + i\theta {J_1}^4 + i\theta'_a {J_1}^a + i\theta''^b {J_b}^4\,. \tag{7.42}$$

Then the cusp anomalous dimension takes the form (where we omit the indices from the θ)

$$\Gamma(\phi, U) \simeq B_{\theta}^{1/3}\theta^2 + \theta'^2 B_{\theta'}^{1/3} + \theta''^2 B_{\theta''}^{1/3} - \phi^2 B_{\phi}^{1/3} \,. \tag{7.43}$$

$W_{1/3}$ has therefore four brensstrahlung functions and the usual relations (7.74) and (7.75) give

$$B_{\phi}^{1/3} = \frac{1}{24} C_{\mathbb{D}}^{1/3} \,, \quad B_{\theta}^{1/3} = \frac{1}{4} C_{\mathbb{O}}^{1/3} \,, \quad B_{\theta'}^{1/3} = \frac{1}{4} C_{\mathbb{O}^a}^{1/3} \,, \quad B_{\theta''}^{1/3} = \frac{1}{4} C_{\mathbb{O}_a}^{1/3} \,. \tag{7.44}$$

The relation after (7.33) and (7.41) then lead to the equalities

$$B_{\phi}^{1/3} = B_{\theta}^{1/3} = B_{\theta'}^{1/3} + B_{\theta''}^{1/3} \,. \tag{7.45}$$

This allows us to write (7.43) in terms of only two independent functions

$$\Gamma(\phi, U) \simeq (\theta^2 + \theta'^2 - \phi^2) B_{\theta'}^{1/3} + (\theta^2 + \theta''^2 - \phi^2) B_{\theta''}^{1/3} \,. \tag{7.46}$$

Furthermore, we can rely on the relation to the 1/2 BPS loop (7.40) to write this in terms of the 1/2 BPS bremsstrahlung function $B_{\phi}^{1/2}$ and n_1, n_4 as

$$\Gamma(\phi, U) \simeq \left(\theta^2 - \phi^2 + \frac{n_1}{n_1 + n_4} \theta'^2 + \frac{n_4}{n_1 + n_4} \theta''^2 \right) B_{\phi}^{1/2} \,. \tag{7.47}$$

This relation can be seen as a generalisation of that found for the 1/6 BPS bosonic loop, where $2B_{\theta}^{\text{bos}} = B_{\phi}^{\text{bos}}$ [140, 195]. To see the relation, take $\theta = \theta'' = 0$ and $n_1 = n_4$ in (7.47) and then identify θ' with the angle in the 1/6 BPS cusp.

7.5 Defect conformal manifolds

We identified multiple marginal operators living on the 1/3 BPS line as well as possible new marginal operators on the 1/2 BPS line. Such operators allow to deform the defect along a defect conformal manifold, the space of all connected conformal defects. For complex marginal operators Φ_i and $\bar{\Phi}_{\bar{\imath}}$ we define the coordinates ζ^i and $\bar{\zeta}^{\bar{\imath}}$ and express the deformed line as

$$W_{\zeta,\bar{\zeta}}[\bullet \cdots \bullet] = W \left[\bullet \cdots \bullet \, \exp \int dx \left(\zeta^i \Phi_i(x) + \bar{\zeta}^{\bar{\imath}} \bar{\Phi}_{\bar{\imath}}(x) \right) \right] . \tag{7.48}$$

This space of dCFTs is endowed with the Zamolodchikov metric

$$g_{i\bar{\jmath}}(\zeta, \bar{\zeta}) = W_{\zeta,\bar{\zeta}}[\Phi_i(0) \bar{\Phi}_{\bar{\jmath}}(\infty)] \,, \qquad \bar{\Phi}_{\bar{\jmath}}(\infty) = \lim_{x \to \infty} x^2 \bar{\Phi}_{\bar{\jmath}}(x) \,. \tag{7.49}$$

Clearly at $\zeta = \bar{\zeta} = 0$ the metric is given by expressions like $C_{\mathbb{O}^a}$, $C_{\mathbb{O}_a}$ and $C_{\mathbb{O}}$ (7.31), (7.35). According to [115, 96] the curvature is as in Riemann normal coordinates; the second derivative of the metric with respect to the coordinates, leading to the integrated 4-point function (3.55) and (3.58). Again we have to take care of the ordering of points, which should be by increasing argument which depends on the value of 0 and 1 or η (3.27).

7.5.1 The case of $W_{1/3}$

Of all the marginal operators, the simplest ones are those that arise from global symmetry breaking. In the case of 1/3 BPS line operators, they break the global symmetry group $OSp(6|4)$ to $SU(1,1|2)\times U(1)\times U(1)$. The $SU(4)$ R-symmetry group is broken to $SU(2)\times U(1)\times U(1)$. This indicates that the space of allowed 1/3 BPS loops is (at least) the coset

$$\mathcal{M} = SU(4)/S(U(2)\times U(1)\times U(1))\,. \tag{7.50}$$

This is a 10 dimensional manifold (or 5 complex-dimensional).

Symmetry breaking gives rise to the tilt, tlit and displacement multiplets and they contain five complex operators of dimension one, $\mathbb{O}^a$, $\mathbb{O}_a$ and $\mathbb{O}$. To conform with the notation in (7.48), we label the marginal operators collectively as

$$\Phi_i \simeq \{\mathbb{O}^2, \mathbb{O}^3, \mathbb{O}, \mathbb{O}_2, \mathbb{O}_3\}\,, \qquad \bar\Phi_{\bar\imath} \simeq \{\bar{\mathbb{O}}_2, \bar{\mathbb{O}}_3, \bar{\mathbb{O}}, \bar{\mathbb{O}}^2, \bar{\mathbb{O}}^3\}\,, \qquad i, \bar\imath = 1,\cdots,5\,. \tag{7.51}$$

For finite ζ^1, ζ^2, the $\mathcal{L}_1^+$ entries in the line (7.2) are rotated into another one with $\mathcal{L}_{1'}^+$. Finite ζ^4 and ζ^5 change the $\mathcal{L}_4^+$ block.

The nonvanishing components of the metric are (7.39), (7.40)

$$g_{i\bar\jmath} = \begin{cases} C^{1/3}_{\mathbb{O}^a}\delta_{i\bar\jmath} = \frac{n_1}{n_1+n_4} C^{1/2}_{\mathbb{O}}\delta_{i\bar\jmath}\,, & i,\bar\jmath = 1,2\,. \\ C^{1/3}_{\mathbb{O}} = C^{1/2}_{\mathbb{O}}\,, & i,\bar\jmath = 3\,, \\ C^{1/3}_{\mathbb{O}_a}\delta_{i\bar\jmath} = \frac{n_4}{n_1+n_4} C^{1/2}_{\mathbb{O}}\delta_{i\bar\jmath}\,, & i,\bar\jmath = 4,5\,. \end{cases} \tag{7.52}$$

To calculate the curvature we use (2.29), where we insert the operators (7.22), (7.23) and (7.24) into the superconnection. For example, for $i=k=1$ and $\bar\imath=\bar l=1$

$$\begin{aligned} R_{1\bar 1 1\bar 1} = \int_{-\infty}^{+\infty} dx_1\, dx_2 \Bigg[&\left\langle\!\!\left\langle \begin{pmatrix}\bar{\mathbb{O}}_2(x_1) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbb{O}^2(x_2) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbb{O}^2(0) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{\mathbb{O}}_2(\infty) & 0\\ 0 & 0\end{pmatrix}\right\rangle\!\!\right\rangle_c \\ &- \left\langle\!\!\left\langle \begin{pmatrix}\mathbb{O}^2(x_1) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbb{O}^2(x_2) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{\mathbb{O}}_2(0) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar{\mathbb{O}}_2(\infty) & 0\\ 0 & 0\end{pmatrix}\right\rangle\!\!\right\rangle_c \Bigg]. \end{aligned} \tag{7.53}$$

We write here 2×2 matrices, but they should be larger, as appropriate. This expression involves only the insertions of the tilt operators of W_1^+ into W_1^+, so is the same as in [1], except for the normalisation, which is $n_1/(n_1+n_4)$, because there are no insertions into the W_4^- block. The 4-point function in the case of the 1/2 BPS Wilson loop was calculated in [95] and the integral was evaluated in [1] with the final expression (accounting for the normalisation) being

$$R_{1\bar 1 1\bar 1} = 2g_{1\bar 1} = 2C^{1/3}_{\mathbb{O}^a} = \frac{2n_1}{n_1+n_4} C^{1/2}_{\mathbb{O}}\,. \tag{7.54}$$

In Appendix 7.F we calculate the Riemann tensor of the coset (7.50) in a matching coordinate system and write down all its nonzero components. Indeed $R_{1\bar 1 1\bar 1} = 2g_{1\bar 1}$, as in the CFT calculation above. In the same way we can match all the components of the curvature for $c = a + b$ (7.127) except for terms mixing $1, 2$ and $4, 5$ indices, such as $R_{1\bar{4}4\bar{1}}$ and $R_{1\bar{2}4\bar{5}}$.

In those cases, plugging the expressions from (7.22) and (7.23) into (2.29) would give something like (7.53), but with two non-zero entries at the top left and two on the bottom right, which seems to vanish.

To fix that, we need another ingredient ignored so far.[30] The expression for the tilt and tlit in (7.22), (7.23) are the terms arising from symmetry breaking, as in (7.21). If symmetries are not broken, then there should be a conserved current along the line. In the case of the preserved supercharges these are the total derivatives in (7.86), where $\bar{G}^a$ and G^a can be considered as supercurrents along the line. For the R-symmetry charges

$$[J_1{}^a, W_{1/3}] = \int dx\, W_{1/3} \left[\begin{pmatrix} \mathbb{O}^a & \partial_x \Gamma_1{}^a \\ \partial_x \Gamma_1{}^a & \partial_x \Gamma_1{}^a \end{pmatrix} (x) \right], \tag{7.55}$$

where $\Gamma_1{}^a$ are $SU(4)$ generators and serve as 1d conserved currents (they should be written as $\Gamma^x{}_1{}^a$, but we omit the repetitive superscript). Their derivative vanishes, since this symmetry is preserved for those three entries, but these expressions are important to reproduce the missing components of the curvature.

Looking at the first term in the first line of (2.29) in the case of $R_{1\bar{4}4\bar{1}}$, we get the integrand

$$\left\langle\!\!\left\langle \begin{pmatrix} \partial_x \Gamma_2{}^4 & \partial_x \Gamma_2{}^4 \\ \partial_x \Gamma_2{}^4 & \mathbb{O}_2 \end{pmatrix} (x_1) \begin{pmatrix} \partial_x \Gamma_4{}^2 & \partial_x \Gamma_4{}^2 \\ \partial_x \Gamma_4{}^2 & \bar{\mathbb{O}}^2 \end{pmatrix} (x_2) \begin{pmatrix} \mathbb{O}^2 & \partial_x \Gamma_1{}^2 \\ \partial_x \Gamma_1{}^2 & \partial_x \Gamma_1{}^2 \end{pmatrix} (0) \begin{pmatrix} \bar{\mathbb{O}}_2 & \partial_x \Gamma_2{}^1 \\ \partial_x \Gamma_2{}^1 & \partial_x \Gamma_2{}^1 \end{pmatrix} (\infty) \right\rangle\!\!\right\rangle_c . \tag{7.56}$$

The derivatives $\partial_x \Gamma$ vanish at the points 0 and ∞, so we keep there only $\mathbb{O}^2$ and $\bar{\mathbb{O}}_2$. We then ignore $\mathbb{O}_2$ and $\bar{\mathbb{O}}^2$ from the first two terms, since they give disconnected contributions. Integration over the remaining $\partial_x \Gamma$, for the ordering $0 < x_1 < x_2$ gives

$$\begin{aligned} &- \left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^2 & 0 \\ 0 & 0 \end{pmatrix} (0) \begin{pmatrix} \Gamma_2{}^4 & \Gamma_2{}^4 \\ \Gamma_2{}^4 & 0 \end{pmatrix} (0) \begin{pmatrix} \Gamma_4{}^2 & \Gamma_4{}^2 \\ \Gamma_4{}^2 & 0 \end{pmatrix} (\infty) \begin{pmatrix} \bar{\mathbb{O}}_2 & 0 \\ 0 & 0 \end{pmatrix} (\infty) \right\rangle\!\!\right\rangle_c \\ &= - \left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^2 \Gamma_2{}^4 & \mathbb{O}^2 \Gamma_2{}^4 \\ 0 & 0 \end{pmatrix} (0) \begin{pmatrix} \Gamma_4{}^2 \bar{\mathbb{O}}_2 & 0 \\ \Gamma_4{}^2 \bar{\mathbb{O}}_2 & 0 \end{pmatrix} (\infty) \right\rangle\!\!\right\rangle_c \sim - \left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^4(0) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbb{O}}_4(\infty) & 0 \\ 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle . \end{aligned} \tag{7.57}$$

It is natural to consider only the off-diagonal terms as contributing to the connected part of the correlator, as the other terms would arise also in the case of the 1/2 BPS loop in [1].

For a general $W_{1/3} = n_1 W_1^+ + n_4 W_4^-$, we would get contributions from $n_1 n_4$ off-diagonal entries, giving the answer

$$R_{1\bar{4}4\bar{1}} = -n_4 C^{1/3}_{\mathbb{O}^a} = -n_1 C^{1/3}_{\mathbb{O}_a} = -\frac{n_1 n_4}{n_1 + n_4} C^{1/2}_{\mathbb{O}} , \tag{7.58}$$

[30]We thank V. Schomerus for clarifying this point to us.

in agreement with (7.127). One would expect another contribution from the rotations of $\mathbb{O}_2$ and $\bar{\mathbb{O}}^2$, but one can see that there is no such term in (2.29). In that expression, symmetry was used to reduce four terms to two, so we could recover the other contribution and divide them both by 2. In any case, they are identical, so the expression in (7.58) is correct.

Another case is $R_{1\bar{5}4\bar{2}}$, where the same calculation yields

$$-\left\langle\!\!\left\langle \begin{pmatrix} \mathbb{O}^2{\Gamma_2}^4 & \mathbb{O}^2{\Gamma_2}^4 \\ 0 & 0 \end{pmatrix}(0) \begin{pmatrix} {\Gamma_4}^3\bar{\mathbb{O}}_3 & 0 \\ {\Gamma_4}^3\bar{\mathbb{O}}_3 & 0 \end{pmatrix}(\infty) \right\rangle\!\!\right\rangle_c . \tag{7.59}$$

with the same result as in (7.58), in agreement with (7.127). Terms like $R_{1\bar{5}5\bar{1}}$ vanish in (7.127) and this is true also from the field theory side, since ${\Gamma_4}^3$ does not act on $\bar{\mathbb{O}}_2$. It is easy to verify then that such arguments exactly reproduces all terms in (7.127).

The results presented above are for the marginal operators arising from broken global symmetries. Those are guaranteed to be marginal. We mentioned above possible other marginal operators, like insertions of $\sigma\mathbb{O}_a\bar{\sigma}$ in W_1^+ or o and $(\mathcal{D}_x\sigma)$ from the σ multiplet in $W_{1/3}$ or for $n_1 > 1$ an insertion of $\mathbb{O}_a$ into only one of the $\mathcal{L}_1^+$ blocks. We postpone the question of whether they are exactly marginal as well as the resulting conformal manifolds to future work.

7.6 Discussion

We found an explicit realisation of a 1/3 BPS Wilson line operator in ABJM theory in terms of a large superconnection, combining two 1/2 BPS Wilson lines, and discussed general properties of 1/3 BPS line operators. Many of these results are valid for any 1/3 BPS loop, including the vortex loop of [111]. We have not attempted to verify them by detailed microscopic calculations in that setting, as the explicit forms of defect operators on the vortex loop may be subtle, given that there is a singularity along the line.

The entire discussion was for the straight line operator, but it carries over to the case of the circle. The preserved and broken symmetries are related by conjugation and we do not think that there are any subtleties in our calculation due to the difference between compact and non-compact loops. Of course, when we consider only one σ insertion in W_1^+, we should remember the $\bar{\sigma}$ at infinity, when mapping to the circle.

The circular Wilson loop has a finite expectation value that can be calculated using localization [205, 187–189]. Given that $W_{1/3} = n_1 W_1^+ + n_4^- W_4^-$, the expression for the 1/3 BPS loop is exactly the same as the 1/2 BPS one.

The operators σ, $\bar{\sigma}$ are a side product of our construction and should be studied more fully on their own right. They are presented in Section 7.2.1, Section 7.2.2 and Appendix 7.E.1. There is also the τ operator and T (studied already in [201]) and the relations among them should be examined more closely. For example, whether anything changes if we replace σ by $\sigma\tau$. With those operators under control, one could then try to study operators like $\sigma\mathbb{O}_a\bar{\sigma}$ and whether they are truly marginal.

Our analysis of the relation between the normalisation constants in Sections 7.4.1 and 7.4.2 is modeled closely after [141]. There this was done for the 1/6 BPS bosonic Wilson loop, preserving the superalgebra $\mathfrak{su}(1,1|1)$ and in addition a bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The supermultiplets are much shorter, one has the complex displacement $\mathbb{D}$ and a superpartner $\mathbb{A}$ (and their descendents). There are also four complex twist operators in the $(\mathbf{2}, \bar{\mathbf{2}})$ representation and each has a superpartner.

In that case, the symmetry guaranties that the normalisation factors of all of the displacements $C_{\mathbb{O}}$ are equal, and a similar result to Section 7.4.2 shows that they are half what they would be if they were in the same multiplet as $\mathbb{D}$ (as below (7.33)), so $C_{\mathbb{D}} = 12C_{\mathbb{O}}$ and the two bremsstrahlung functions are related by this factor of 2.

The defect conformal manifold constructed in Section 7.5 is a generalisation of that in [1]. It is higher dimensional and not a symmetric space. Technically we also had to take care of the seemingly vanishing mixed curvature terms, which required the inclusion of the conserved R-symmetry currents on the line.

For the bosonic loops and their four tilts, the conformal manifold is four complex dimensional, and should be $SU(4)/S(U(2) \times U(2)) = \mathbb{Gr}_2(\mathbb{C}^4)$, the Grassmannian for complex 2-planes in $\mathbb{C}^4$. Since the preserved symmetry includes the $S(U(2) \times U(1) \times U(1))$ of the 1/3 BPS loops studied here, our conformal manifold is a $\mathbb{CP}^1$ bundle over this Grassmannian. Shrinking the fibers would give the base, in the same way we can reduce our conformal manifold in Section 7.5.1 to that of the 1/2 BPS loop, $\mathbb{CP}^3$, by simply taking $n_1 \to 0$ or $n_4 \to 0$.

In the case of our 5 complex dimensional conformal manifold the size of the $\mathbb{CP}^1$, which is fixed by $C_{\mathbb{O}}$ is related to the other two length scales via $C_{\mathbb{O}} = C_{\mathbb{O}^a} + C_{\mathbb{O}_a}$ (7.41), so it cannot be shrunk, without also shrinking the base.

Interestingly, this shrinking can be realised with the aid of the 1/6 BPS fermionic loops [64, 58]. They all preserve the same $\mathfrak{su}(1,1|1)$ superalgebra of the bosonic loop, but the bosonic symmetry is only $SU(2) \times U(1) \times U(1)$, enhancing to $SU(3) \times U(1)$ at the 1/2 BPS points and $S(U(2) \times U(2))$ at the bosonic point.

The general 1/6 BPS loop still has one complex displacement and a superpartner. There should then be five complex tilts, as in the case of the 1/3 BPS loop. The two doublets form multiplets $\{\mathbb{O}^a, \mathbb{\Lambda}^a\}$, $\{\mathbb{O}^a, \mathbb{\Lambda}^a\}$ and the singlet is now in a different multiplet $\{\mathbb{F}, \mathbb{O}\}$. This last multiplet is not in the spectrum of the bosonic loop and this tilt generates motion along the $\mathbb{CP}^1$, so we expect its normalisation $C_{\mathbb{O}}$, which starts as $C_{\mathbb{D}}/6$ at the 1/2 BPS point, to vanish as we approach the bosonic loop. Presumably there are still relations like $C_{\mathbb{O}^a} + C_{\mathbb{O}_a} = C_{\mathbb{D}}/6$, as in the case of the 1/3 BPS loop studied here.

The fact that the singlet tilt is in the same multiplet with $\mathbb{F}$ is consistent with the breaking the 1/2 BPS multiplet (7.9), but cannot arise from the 1/3 BPS loop, where there are a pair $\mathbb{F}$ and $\reflectbox{$\mathbb{F}$}$ in different multiplets without the singlet tilt. This is another indication that there is no 1/3 BPS loop in the same muduli space of 1/6 BPS loops based on 2×2

superconnections unrelated to 1/2 BPS loops.

Another family of 1/6 BPS loops that have previously not been studied are based on superconnections $\mathcal{L}_1^+$ and $\mathcal{L}_2^+$, where the latter, unlike $\mathcal{L}_4^-$, is the direct $SU(4)$ rotation of $\mathcal{L}_1^+$. Unlike the 1/3 BPS loops, one can continuously rotate W_1^+ into W_2^+ while preserving 4 supercharges. These have the same bosonic symmetries as the generic 1/6 BPS loops, so are a simpler setting to study this system and one can redo our calculation in Section 7.5, again relying on the 4-point functions that were calculated for the 1/2 BPS loop.

A better understanding of the space of line operators in the field theory could help in identifying the holographic duals, which is still an open question [60, 58, 178, 197]. To this we now add the puzzle of the holographic dual of the 1/3 BPS Wilson line. This is unlikely to be the 1/3 BPS solution of [111], but rather a superposition of two strings at different points on $\mathbb{CP}^3$.

Other natural questions are the values of the normalisation constants (bremsstrahlung functions) for arbitrary 1/6 BPS loops, which are not bosonic. Likewise, one could push the analysis here to theories with $\mathcal{N} = 4$ supersymmetry [83,84], and their equally rich spectrum of line operators [91, 63–67, 206, 59, 2, 3]. We hope to address some of these questions, and many more that arise from this work, in the near future.

Acknowledgements

We would like to acknowledge related collaboration with M. Probst, M. Tenser and D. Trancanelli. We are also grateful to G. Bliard, G. Papadopoulos, M. Preti, S. Salamon, V. Schomerus and P. Soresina for extremely useful discussions. ND is supported by STFC under the grants ST/T000759/1 and ST/P000258/1 and by the National Science Foundation under Grant No. NSF PHY-1748958. The work of ZK is supported by CSC grant No. 201906340174. ND would like to thank École Polytechnique Fédérale de Lausanne, the Kavli Institute for Theoretical Physics, UCSB, the DESY theory group and Humboldt University for their hospitality in the course of this work. ZK would like to thank T. Fiol, the University of Barcelona, the DESY theory group and Humboldt University for their hospitality.

7.A Some 1/6 and 1/2 BPS Wilson lines

We present here the BPS Wilson loops that are used in our analysis. All of them are straight lines along the x_3 axis denoted as x. The first is the bosonic Wilson loop [60–62] which is the ABJM analogue of the Gaiotto-Yin loop in $\mathcal{N} = 2$ theories [76]. It is 1/6 BPS, preserving the four supercharges Q_+^{12}, Q_-^{34}, S_+^{12}, S_-^{34}. It is given as

$$W_{\text{bos}} = \operatorname{Tr} \mathcal{P} \exp\left(\int i\mathcal{A}_{\text{bos}}\, dx \right), \tag{7.60}$$

where in the case of a loop in the first gauge group

$$\mathcal{A}_{\rm bos} = A_x^{(1)} - \frac{2\pi i}{k} M^I{}_J C_I \bar{C}^J \,, \qquad M = {\rm diag}(-1,-1,1,1)\,, \tag{7.61}$$

and similarly for the second group.

There is a large moduli space of Wilson loops preserving these supercharges [58,59]. One first needs to elevate the bosonic Wilson loop to couple to both gauge groups as

$$W_{\rm bos} = {\rm Tr}\,\mathcal{P}\exp\left(\int i\mathcal{L}_{\rm bos}\,dx\right), \qquad \mathcal{L}_{\rm bos} = \begin{pmatrix} \mathcal{A}^{(1)}_{\rm bos} & 0 \\ 0 & \mathcal{A}^{(2)}_{\rm bos} \end{pmatrix}, \tag{7.62}$$

Then we can deform it as (w_I and $\bar{w}^I$ are not necessarily complex conjugate)

$$\mathcal{L} = \mathcal{L}_{\rm bos} - i(Q_+^{12} + Q_-^{34})\mathcal{G} + 2\mathcal{G}^2\,, \qquad \mathcal{G} = \begin{pmatrix} 0 & \bar{w}^I C_I \\ w_I \bar{C}^I & 0 \end{pmatrix}. \tag{7.63}$$

The explicit action of the supercharges on the fields is given in Appendix 2.21. The resulting loop is 1/6 BPS for arbitrary constant w_1, w_2, $\bar{w}^1$, $\bar{w}^2$ and the other vanishing or vice versa. Modding out by a $\mathbb{C}^*$ action discussed in Section 7.2, the moduli space is two copies of the conifold [58,69].

In the case with $w_3 = w_4 = \bar{w}^3 = \bar{w}^4 = 0$ we write those loops as in (7.62) with

$$\mathcal{L} = \begin{pmatrix} A_x^{(1)} + \alpha\bar{\alpha} M^I{}_J C_I \bar{C}^J & -i\bar{w}^1\bar{\psi}^2_+ + i\bar{w}^2\bar{\psi}^1_+ \\ iw_1^+\psi_2^+ - iw_2\psi_1^+ & A_x^{(2)} + \alpha\bar{\alpha} M^I{}_J C_I \bar{C}^J \end{pmatrix}, \quad M^I{}_J = (M_{\rm bos})^I{}_J + \frac{2}{\alpha\bar{\alpha}}\bar{w}^I w_J\,, \tag{7.64}$$

with $\alpha\bar{\alpha} = -2\pi i/k$. In the other case we have

$$\mathcal{L} = \begin{pmatrix} A_x^{(1)} + \frac{\alpha\bar{\alpha}}{2} k M^I{}_J C_I \bar{C}^J & -i\bar{w}^3\bar{\psi}^4_- + i\bar{w}^4\bar{\psi}^3_- \\ -iw_3\psi_4^- + iw_4^-\psi_3^- & A_x^{(2)} + \frac{\alpha\bar{\alpha}}{2} M^I{}_J C_I \bar{C}^J \end{pmatrix}, \quad M^I{}_J = (M_{\rm bos})^I{}_J + \frac{2}{\alpha\bar{\alpha}}\bar{w}^I w_J\,. \tag{7.65}$$

Within this space, the loops with $w_1\bar{w}^1 + w_2\bar{w}^2 = \alpha\bar{\alpha}$ are 1/2 BPS as are those with $w_3\bar{w}^3 + w_4\bar{w}^4 = -\alpha\bar{\alpha}$. The particular cases that are used in the body of the paper are:

$\boldsymbol{W_1^+}$: Taking $w_2 = \alpha$ and $\bar{w}^2 = \bar{\alpha}$ satisfying $\bar{\alpha}\alpha = -2\pi i/k$, and all others vanishing we get a loop with $SU(3)$ symmetry among indices 2, 3, 4

$$\mathcal{L}_1^+ = \begin{pmatrix} A_x^{(1)} + \alpha\bar{\alpha}(M_1)^I{}_J C_I \bar{C}^J & i\bar{\alpha}\bar{\psi}^1_+ \\ -i\alpha\psi_1^+ & A_x^{(2)} + \alpha\bar{\alpha}(M_1)^I{}_J \bar{C}^J C_I \end{pmatrix}, \quad M_1 = {\rm diag}(-1,1,1,1)\,. \tag{7.66}$$

Explicitly, W_1^+ preserves Q_+^{12}, Q_+^{13}, Q_+^{14}, Q_-^{34}, Q_-^{24}, Q_-^{23} and the corresponding superconformal generators.

$\boldsymbol{W_4^-}$: The loop with $SU(3)$ symmetry among indices 1, 2, 3 has

$$\mathcal{L}_4^- = \begin{pmatrix} A_x^{(1)} + \alpha\bar{\alpha}(M_4)^I{}_J C_I \bar{C}^J & i\bar{\alpha}\bar{\psi}_-^4 \\ -i\alpha\psi_4^- & A_x^{(2)} + \alpha\bar{\alpha}(M_4)^I{}_J \bar{C}^J C_I \end{pmatrix}, \quad M_4 = \mathrm{diag}(-1,-1,-1,1)\,. \tag{7.67}$$

W_4^- preserves Q_+^{12}, Q_+^{13}, Q_+^{23}, Q_-^{34}, Q_-^{24}, Q_-^{14} and the corresponding S's. It shares 8 supercharges with W_1^+.

7.B Cusps, bremsstrahlungs and displacements

In this appendix we review the necessary background on cusped Wilson loops, the small angle limit giving the bremsstrahlung functions and their relation to displacement and tilt operators.

7.B.1 Cusped Wilson loops

A cusped Wilson loop is comprised of two semi-infinite rays meeting at an angle ϕ such that $\phi = 0$ is a straight line. We can parametrise the curve as

$$x^\mu(x) = \begin{cases} (0,0,x)\,, & x < 0\,, \\ (0, x\sin\phi, x\cos\phi)\,, & x > 0\,. \end{cases} \tag{7.68}$$

Generically such loops suffer from logarithmic divergences [203, 204], which means that the singular point obtains an anomalous dimension $\Gamma(\phi)$. For small angles this should be an even function, so to lowest order

$$\Gamma(\phi) = -B_\phi \phi^2 + \mathcal{O}(\phi^4)\,, \tag{7.69}$$

and B^ϕ is known as the bremsstrahlung function.

For loops coupling to scalar fields or fermions, we can also change those at the same point. With the structure of the 1/2 BPS loops, we can use the expressions in (7.64) and take

$$\big(w_1(x), w_2(x), \bar{w}^1(x), \bar{w}^2(x)\big) = \begin{cases} \alpha(0,-1,0,-1)\,, & x < 0\,, \\ \bar{\alpha}(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, -\cos\frac{\theta}{2})\,, & x > 0\,. \end{cases} \tag{7.70}$$

This would also lead to an anomalous dimension

$$\Gamma(\theta) = B_\theta \theta^2 + \mathcal{O}(\theta^4)\,. \tag{7.71}$$

More generally we have a function of both ϕ and θ.

One may wonder why the case of a straight line with a nonzero θ there is an anomaly, given that all the loops in (7.64) share four supercharges. The reason is that the supersymmetry variation of the superconnection $\mathcal{L}$ does not vanish, but is a total derivative c.f. (7.86). For different θ, these total derivatives are different, so leave a boundary term at the location the change occures. In the special case of $\theta = \phi$ the two rays also share supercharges and the boundary terms cancel, so the combined system is BPS [192]. In this case there should not be any anomaly and from the small angle expansions (7.69) and (7.71) we conclude that $B_\phi^{1/2} = B_\theta^{1/2}$.

For the bosonic loops (7.60), (7.61) the ϕ cusp is as above and for the θ cusp we can take M along the second ray to be

$$M_{\text{bos}}^{\theta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{7.72}$$

In this case, $\phi = \theta$ is not a BPS configuration so the relation between B_ϕ^{bos} and B_θ^{bos} remained unclear until it was proven [198, 199, 141] that

$$B_\phi^{\text{bos}} = 2B_\theta^{\text{bos}} . \tag{7.73}$$

We are left with two independent functions B_ϕ^{bos} and $B_\phi^{1/2}$. The first is expressed as the derivative of the n-wound Wilson loop, which can be evaluated using localisation [207, 139]. The second can be related to the so-called latitude Wilson loop [194, 140, 195, 176]. They are also related via the framing anomaly factor that arises in calculating Wilson loops in Chern-Simons theory [194, 174].

7.B.2 Displacement and Twist

The bremsstrahlung function of $\mathcal{N} = 4$ SYM in 4d was defined in [119], where it was also related to the exact expectation value of the circular Wilson loop [184, 185, 205] via the exact expectation value of other BPS Wilson loops [208, 170, 169]. The bremsstrahlung function is related to the two point functions of the displacement operator, and with enough supersymmetry, also of its superpartner, the tilt [200].

In the context of ABJM theory, the relation between the bremsstrahlung function and the two point functions of displacement operators was presented in [195]. We do not repeat the derivation here, but the result is that the normalisation $C_{\mathbb{D}}$ in (7.29) is related to B^ϕ as[31]

$$C_{\mathbb{D}} = 24B_\phi . \tag{7.74}$$

Such expression are valid for any BPS conformal loop, so the 1/2 BPS one, the bosonic loop and the 1/3 BPS loop as well.

[31] This is for the complex $\mathbb{D} = \mathbb{D}_1 + i\mathbb{D}_2$, so double the usual expression in [119].

A similar argument relates the two point function of the tilt $\mathbb{O}$ to B^θ. Specifically [95],

$$C_{\mathbb{O}} = 4B_\theta\,. \tag{7.75}$$

In the case of the 1/2 BPS loop, this is consistent with $B^\phi_{1/2} = B^\theta_{1/2}$ and $C_{\mathbb{D}} = 6C_{\mathbb{O}}$.

7.C Algebras and subalgebras

We present here the subalgebras preserving various Wilson loops. We follow closely the notations in [195] so do not impose reality conditions. In [195] some factors of i were introduced in describing the $\mathfrak{su}(1,1|3)$ algebra. We refrain from doing that to avoid confusion and also do not introduce separate notations for the subalgebras associated to W_1^+ and w_4^- or spell out their commutation relations, as they are all directly inherited from the original algebra. We could have imposed the reality condition on $\mathfrak{osp}(6|4)$ that would be appropriate for a theory in $\mathbb{R}^{2,1}$, but the benefit of that extra works seems marginal compared with consistency with [195, 95].

7.C.1 1/2 BPS $\mathfrak{su}(1,1|3)$ subalgebras

For the 1/2 BPS loop W_1^+, the preserved supercharges are Q_+^{12}, Q_+^{13}, Q_+^{14}, Q_-^{23}, Q_-^{24}, Q_-^{34}, and likewise S_+^{12}, etc.

Choosing $\gamma_3 = \sigma_3$ and $(\sigma_3)_+{}^+ = 1$, their anticommutators give the bosonic generators

$$P_3\,,\quad K_3\,,\quad M_{12} + 2D\,,\quad J_i{}^j - \frac{1}{3}\delta_i^j J_k{}^k\,,\quad i,j,k \in \{2,3,4\}\,. \tag{7.76}$$

Since $[P_3, K_3] = 2D$, we get separately this generator and M_{12}. The full algebra can be easily read off from the commutators in Appendix 2.1.2 and can also be found in [195].

Inside $\mathfrak{osp}(6|4)$ there is an extra $\mathfrak{u}(1)$ symmetry $J_1{}^1$ (or being pedantic about the tracelessness condition $J_1{}^1 - J_i{}^i/3$) that commutes with this $\mathfrak{su}(1,1|3)$. This generator acts nontrivially on the off-diagonal entries in $\mathcal{L}_1^+$, the fermionic fields $\bar\psi_+^1$ and ψ_1^+. It's action on $\mathcal{L}_1^+$ is the commutator with the supermatrix $T = \mathrm{diag}(I, -I)$ (7.3) (studied recently in [201]). M_{12} has a similar action on the fermions, so the combination $M_{12} + J_1{}^1/2$ acts trivially on the superconnection. Still each generator is a symmetry of the Wilson loop W_1^+, since their action either vanishes, or can be expressed as a total derivative of τ, which integrates to zero [110, 2].

For the second 1/2 BPS loop, W_4^-, the preserved supercharges are Q_+^{12}, Q_+^{13}, Q_+^{23}, Q_-^{14}, Q_-^{24}, Q_-^{34}, and likewise S_+^{12}, etc.

Their algebra closes onto the bosonic generators

$$P_3\,,\quad K_3\,,\quad D\,,\quad M_{12}\,,\quad J_{\hat\imath}{}^{\hat\jmath} - \frac{1}{3}\delta_{\hat\imath}^{\hat\jmath} J_{\hat k}{}^{\hat k}\,,\quad \hat\imath,\hat\jmath,\hat k \in \{1,2,3\}\,. \tag{7.77}$$

And again, $M_{12} - J_4{}^4/2$ generates an extra central $\mathfrak{u}(1)$ symmetry.

7.C.2 1/3 BPS $\mathfrak{su}(1,1|2)$ subalgebra

The supercharges preserved by both W_1^+ and W_4^- are Q_+^{12}, Q_+^{13}, Q_-^{24}, Q_-^{34}, and likewise S_+^{12}, etc.

Their algebra close onto $\mathfrak{su}(1,1|2)$ and in particular the bosonic generators

$$P_3\,, \quad K_3\,, \quad D\,, \quad M_{12}\,, \quad J_2{}^3\,, \quad J_3{}^2\,, \quad J_2{}^2 - J_3{}^3\,. \tag{7.78}$$

Though not generated separately by the supercharges, the intersection of the two algebras 7.76 and 7.77 includes also $M_{1/2} + J_1{}^1/2$ and $J_1{}^1 - J_4{}^4$.

7.C.3 Broken and unbroken generators

The table below lists all generators in $\mathfrak{osp}(6|4)$ and whether they are broken by W_1^+, W_4^- and/or $W_{1/3}$.

Generator	W_1^+	W_4^-	$W_{1/3}$
P_3, K_3, D	✓	✓	✓
P_1, P_2, K_1, K_2	✗	✗	✗
M_{12}	✓	✓	✓
M_{13}, M_{23}	✗	✗	✗
$J_2{}^3$, $J_3{}^2$, $J_2{}^2 - J_3{}^3$	✓	✓	✓
$J_1{}^1 - \frac{1}{2}J_2{}^2 - \frac{1}{2}J_3{}^3$, $J_4{}^4 - \frac{1}{2}J_2{}^2 - \frac{1}{2}J_3{}^3$	✓	✓	✓
$J_1{}^2$, $J_1{}^3$, $J_2{}^1$, $J_3{}^1$	✓	✗	✗
$J_4{}^2$, $J_4{}^3$, $J_2{}^4$, $J_3{}^4$	✗	✓	✗
$J_1{}^4$, $J_4{}^1$	✗	✗	✗
Q_+^{12}, Q_+^{13}, Q_-^{24}, Q_-^{34}	✓	✓	✓
Q_+^{14}, Q_-^{23}	✓	✗	✗
Q_-^{14}, Q_+^{23}	✗	✓	✗
Q_-^{12}, Q_-^{13}, Q_+^{24}, Q_+^{34}	✗	✗	✗
S_+^{12}, S_+^{13}, S_-^{24}, S_-^{34}	✓	✓	✓
S_+^{14}, S_-^{23}	✓	✗	✗
S_-^{14}, S_+^{23}	✗	✓	✗
S_-^{12}, S_-^{13}, S_+^{24}, S_+^{34}	✗	✗	✗

7.D Multiplet structure

We list here the explicit action of the preserved supercharges on the tilt and displacement multiplets

7.D.1 The tilt multiplets

with $a = 2, 3$ and $\epsilon^{23} = -\epsilon_{23} = 1$

$$\begin{aligned}
\{Q_+^{1a}, \mathbb{F}\} &= \mathbb{O}^a\,, & [Q_+^{1a}, \mathbb{O}^b] &= \epsilon^{ab}\mathbb{\Lambda}\,, & & \\
& & [S_-^{a4}, \mathbb{O}^b] &= 2\epsilon^{ab}\mathbb{F}\,, & \{S_-^{a4}, \mathbb{\Lambda}\} &= -2\mathbb{O}^a\,, \\
& & [Q_-^{a4}, \mathbb{O}^b] &= 2i\epsilon^{ab}\mathcal{D}_x\mathbb{F}\,, & \{Q_-^{a4}, \mathbb{\Lambda}\} &= -i2\mathcal{D}_x\mathbb{O}^a\,,
\end{aligned} \tag{7.79}$$

$$\begin{aligned}
\{Q_-^{a4}, \bar{\mathbb{F}}\} &= \epsilon^{ab}\bar{\mathbb{O}}_b\,, & [Q_-^{a4}, \bar{\mathbb{O}}_b] &= -\delta_b^a\bar{\mathbb{\Lambda}}\,, & & \\
& & [S_+^{1a}, \bar{\mathbb{O}}_b] &= -2\delta_b^a\bar{\mathbb{F}}\,, & \{S_+^{1a}, \bar{\mathbb{\Lambda}}\} &= -2\epsilon^{ab}\bar{\mathbb{O}}_b\,, \\
& & [Q_+^{1a}, \bar{\mathbb{O}}_b] &= 2i\delta_b^a\mathcal{D}_x\bar{\mathbb{F}}\,, & \{Q_+^{1a}, \bar{\mathbb{\Lambda}}\} &= 2i\epsilon^{ab}\mathcal{D}_x\bar{\mathbb{O}}_b\,,
\end{aligned} \tag{7.80}$$

7.D.2 The tlit multiplets

$$\begin{aligned}
\{Q_+^{1a}, \Finv\} &= \epsilon^{ab}\mathbb{O}_b\,, & [Q_+^{1a}, \mathbb{O}_b] &= -\delta_b^a\mathbb{\Lambda}\,, & & \\
& & [S_-^{a4}, \mathbb{O}_b] &= -2\delta_b^a\Finv\,, & \{S_-^{a4}, \mathbb{\Lambda}\} &= -2\epsilon^{ab}\mathbb{O}_b\,, \\
& & [Q_-^{a4}, \mathbb{O}_b] &= -2i\delta_b^a\mathcal{D}_x\Finv\,, & \{Q_-^{a4}, \mathbb{\Lambda}\} &= -2i\epsilon^{ab}\mathcal{D}_x\mathbb{O}_b\,,
\end{aligned} \tag{7.81}$$

$$\begin{aligned}
\{Q_-^{a4}, \bar{\Finv}\} &= \bar{\mathbb{O}}^a\,, & [Q_-^{a4}, \bar{\mathbb{O}}^b] &= \epsilon^{ab}\bar{\mathbb{\Lambda}}\,, & & \\
& & [S_+^{1a}, \bar{\mathbb{O}}^b] &= 2\epsilon^{ab}\bar{\Finv}\,, & \{S_+^{1a}, \bar{\mathbb{\Lambda}}\} &= -2\bar{\mathbb{O}}^a\,, \\
& & [Q_+^{1a}, \bar{\mathbb{O}}^b] &= 2i\epsilon^{ab}\mathcal{D}_x\bar{\Finv}\,, & \{Q_+^{1a}, \bar{\mathbb{\Lambda}}\} &= -i2\mathcal{D}_x\bar{\mathbb{O}}^a\,,
\end{aligned} \tag{7.82}$$

7.D.3 Displacement multiplets

$$\begin{aligned}
[Q_+^{1a}, \mathbb{O}] &= -\epsilon^{ab}\mathbb{A}_b\,, & \{Q_+^{1a}, \mathbb{A}_b\} &= -2\delta_b^a\mathbb{D}\,, & & \\
& & \{S_-^{a4}, \mathbb{A}_b\} &= 2\delta_b^a\mathbb{O}\,, & [S_-^{a4}, \mathbb{D}] &= -2\epsilon^{ab}\mathbb{A}_b\,, \\
& & \{Q_-^{a4}, \mathbb{A}_b\} &= 2i\delta_b^a\mathcal{D}_x\mathbb{O}\,, & [Q_-^{a4}, \mathbb{D}] &= -i\epsilon^{ab}\mathcal{D}_x\mathbb{A}_b\,,
\end{aligned} \tag{7.83}$$

$$\begin{aligned}
[Q_-^{a4}, \bar{\mathbb{O}}] &= \bar{\mathbb{A}}^a\,, & \{Q_-^{a4}, \bar{\mathbb{A}}^b\} &= -2\epsilon^{ab}\bar{\mathbb{D}}\,, & & \\
& & \{S_+^{1a}, \bar{\mathbb{A}}^b\} &= 2\epsilon^{ab}\bar{\mathbb{O}}\,, & [S_+^{1a}, \bar{\mathbb{D}}] &= -2\bar{\mathbb{A}}^a\,, \\
& & \{Q_+^{1a}, \bar{\mathbb{A}}^b\} &= -2i\epsilon^{ab}\mathcal{D}_x\bar{\mathbb{O}}\,, & [Q_+^{1a}, \bar{\mathbb{D}}] &= i\mathcal{D}_x\bar{\mathbb{A}}^a\,,
\end{aligned} \tag{7.84}$$

7.E Explicit expressions in terms of ABJM fields

7.E.1 The σ multiplet

The permutation operator σ in the 1/3 BPS loop is given in (7.4). Let us start with a more general $GL(2)$ matrix g (or more precisely $g \in I_{2\times 2} \otimes GL(2)_{\mathbb{C}}$) acting by conjugation as

$$\begin{pmatrix} \mathcal{L}_1^+ & 0 \\ 0 & \mathcal{L}_4^- \end{pmatrix} \to g \begin{pmatrix} \mathcal{L}_1^+ & 0 \\ 0 & \mathcal{L}_4^- \end{pmatrix} g^{-1} \tag{7.85}$$

To examine its variation under supersymmetry, we recall that [110]

$$\begin{aligned} [Q_+^{1a}, i\mathcal{L}_1^+] &= \mathcal{D}_x^{\mathcal{L}_1^+}\bar{G}^a\,, \qquad & [Q_+^{1a}, i\mathcal{L}_4^-] &= \mathcal{D}_x^{\mathcal{L}_4^-}G^a\,,\\ [Q_-^{a4}, i\mathcal{L}_1^+] &= \mathcal{D}_x^{\mathcal{L}_1^+}G^a\,, \qquad & [Q_-^{a4}, i\mathcal{L}_4^-] &= \mathcal{D}_x^{\mathcal{L}_4^-}\bar{G}^a\,, \end{aligned} \tag{7.86}$$

where

$$G^a = \begin{pmatrix} 0 & 2i\bar{\alpha}\epsilon^{ab}C_b \\ 0 & 0 \end{pmatrix}, \qquad \bar{G}^a = \begin{pmatrix} 0 & 0 \\ -2i\alpha\bar{C}^a & 0 \end{pmatrix}, \tag{7.87}$$

Integrating the total derivaties, we find the boundary terms

$$Q_+^{1a}W[g] = W\left[\begin{pmatrix} \bar{G}^a & 0 \\ 0 & G^a \end{pmatrix} g - g\begin{pmatrix} \bar{G}^a & 0 \\ 0 & G^a \end{pmatrix}\right], \tag{7.88}$$

as this is a local action, we can identify the action of the preserved supercharges on g as

$$[Q_+^{1a}, g] = \left[\begin{pmatrix} \bar{G}^a & 0 \\ 0 & G^a \end{pmatrix}, g\right], \qquad [Q_-^{a4}, g] \;\; = \left[\begin{pmatrix} G^a & 0 \\ 0 & \bar{G}^a \end{pmatrix}, g\right]. \tag{7.89}$$

A nicer action arises from the sum and difference of the supercharges

$$\begin{aligned} [Q_+^{1a} + Q_-^{a4}, g] &= \left[\begin{pmatrix} \bar{G}^a + G^a & 0 \\ 0 & \bar{G}^a + G^a \end{pmatrix}, g\right] = 0 \\ [Q_+^{1a} - Q_-^{a4}, g] &= \left[\begin{pmatrix} \bar{G}^a - G^a & 0 \\ 0 & -(\bar{G}^a - G^a) \end{pmatrix}, g\right] = (\bar{G}^a - G^a) \otimes [\tau, g]. \end{aligned} \tag{7.90}$$

In the last expression we view g as a 2×2 matrix and use the tensor symbol explicitly. We also use

$$\tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{7.91}$$

Clearly for a diagonal g all variations cancel, so we can focus on off-diagonal g, which are linear combinations of σ and $\tau\sigma$. We find then the descendents of σ and $\tau\sigma$ as

$$\begin{aligned} \mathbb{Z}^a &= \frac{1}{2}[Q_+^{1a} - Q_-^{a4}, \sigma] = (\bar{G}^a - G^a) \otimes \tau\sigma\,, \\ \mathbb{Z}^a &= \frac{1}{2}[Q_+^{1a} - Q_-^{a4}, \tau\sigma] = (\bar{G}^a - G^a) \otimes \sigma\,. \end{aligned} \tag{7.92}$$

Looking at the second variation, first acting with the sum, then with the proper covariant derivative (7.116) we find

$$\begin{aligned} \frac{1}{2}\{Q_+^{1a} + Q_-^{a4}, \mathbb{Z}^b\} &= \epsilon^{ab}\begin{pmatrix} -2\bar{\alpha}\alpha C_c\bar{C}^c & i\bar{\alpha}(\bar{\psi}_-^4 - \bar{\psi}_+^1) \\ -i\alpha(\psi_4^+ - \psi_1^+) & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \tau\sigma \\ &= \epsilon^{ab}\begin{pmatrix} 0 & \mathcal{L}_4^- - \mathcal{L}_1^+ \\ \mathcal{L}_1^+ - \mathcal{L}_4^- & 0 \end{pmatrix} = \epsilon^{ab}\mathcal{D}_x\sigma\,. \end{aligned} \tag{7.93}$$

Here $C_c\bar{C}^c = C_2\bar{C}^2 + C_3\bar{C}^3$ and the result is the covariant derivative of σ, in agreement with the algebra (2.19). We find a similar result for $\mathbb{X}^a$.

Acting with the other combinations of supercharges we find the descendant

$$\begin{aligned}\epsilon^{ab}o = \frac{1}{2}\{Q_+^{1a} - Q_-^{a4}, \mathbb{\Sigma}^b\} = \epsilon^{ab}\begin{pmatrix} 2\bar{\alpha}\alpha C_c\bar{C}^c & 0 \\ 0 & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \sigma \\ - \epsilon^{ab}\begin{pmatrix} 0 & i\bar{\alpha}(\bar{\psi}_+^1 + \bar{\psi}_-^4) \\ i\alpha(\bar{\psi}_+^1 + \bar{\psi}_-^4) & 0 \end{pmatrix} \otimes \tau\sigma\,.\end{aligned} \tag{7.94}$$

The expressions become a bit easier when starting with $\sigma^\pm = (\sigma \pm \tau\sigma)/2$

$$\begin{aligned} o^+ &= \frac{1}{8}\{Q_+^{12} - Q_-^{24}, \mathbb{\Sigma}^3 + \mathbb{X}^3\} = \begin{pmatrix} 2\bar{\alpha}\alpha C_c\bar{C}^c & -i\bar{\alpha}(\bar{\psi}_+^1 + \bar{\psi}_-^4) \\ -i\alpha(\bar{\psi}_+^1 + \bar{\psi}_-^4) & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \sigma^+\,, \\ o^- &= \frac{1}{8}\{Q_+^{12} - Q_-^{24}, \mathbb{\Sigma}^3 - \mathbb{X}^3\} = \begin{pmatrix} 2\bar{\alpha}\alpha C_c\bar{C}^c & i\bar{\alpha}(\bar{\psi}_+^1 + \bar{\psi}_-^4) \\ i\alpha(\bar{\psi}_+^1 + \bar{\psi}_-^4) & -2\bar{\alpha}\alpha\bar{C}^c C_c \end{pmatrix} \otimes \sigma^-\,. \end{aligned} \tag{7.95}$$

7.E.2 The tilt multiplets

We can act by the broken generators ${J_1}^a$, ${J_a}^1$, ${J_a}^4$ and ${J_4}^a$ on $\mathcal{L}_1^+$ and $\mathcal{L}_4^-$ to find the tilt operators

$$\begin{aligned} \mathbb{O}^a &= \begin{pmatrix} -2\bar{\alpha}\alpha C_1\bar{C}^a & i\bar{\alpha}\bar{\psi}_+^a \\ 0 & -2\bar{\alpha}\alpha\bar{C}^a C_1 \end{pmatrix}, \qquad & \bar{\mathbb{O}}_a &= \begin{pmatrix} 2\bar{\alpha}\alpha C_a\bar{C}^1 & 0 \\ i\alpha\psi_a^+ & 2\bar{\alpha}\alpha\bar{C}^1 C_a \end{pmatrix}, \\ \bar{\mathbb{O}}^a &= \begin{pmatrix} 2\bar{\alpha}\alpha C_4\bar{C}^a & i\bar{\alpha}\bar{\psi}_-^a \\ 0 & 2\bar{\alpha}\alpha\bar{C}^a C_4 \end{pmatrix}, & \mathbb{O}_a &= \begin{pmatrix} -2\bar{\alpha}\alpha C_a\bar{C}^4 & 0 \\ i\alpha\psi_a^- & -2\bar{\alpha}\alpha\bar{C}^4 C_a \end{pmatrix}. \end{aligned} \tag{7.96}$$

By matching the fermionic parts of (see Appendix 7.D)

$$\begin{aligned} \{\tilde{Q}_+^{1a}, \mathbb{F}\} &= \mathbb{O}^a\,, \qquad & \{\tilde{Q}_-^{a4}, \bar{\mathbb{F}}\} &= \epsilon^{ab}\bar{\mathbb{O}}_b\,, \\ \{\tilde{Q}_+^{1a}, \reflectbox{$\mathbb{F}$}\} &= \epsilon^{ab}\mathbb{O}_b\,, & \{\tilde{Q}_-^{a4}, \bar{\reflectbox{$\mathbb{F}$}}\} &= \bar{\mathbb{O}}^a\,. \end{aligned} \tag{7.97}$$

we get

$$\begin{aligned} \mathbb{F} &= \begin{pmatrix} 0 & i\bar{\alpha}C_1 \\ 0 & 0 \end{pmatrix}, \qquad & \bar{\mathbb{F}} &= \begin{pmatrix} 0 & 0 \\ i\alpha\bar{C}^1 & 0 \end{pmatrix}, \\ \bar{\reflectbox{$\mathbb{F}$}} &= \begin{pmatrix} 0 & -i\bar{\alpha}C_4 \\ 0 & 0 \end{pmatrix}, & \reflectbox{$\mathbb{F}$} &= \begin{pmatrix} 0 & 0 \\ -i\alpha\bar{C}^4 & 0 \end{pmatrix}. \end{aligned} \tag{7.98}$$

To make these expressions work, one needs to use the form of the variation on odd supermatrices in (7.116).

The covariant supercharges acting on Grassmann odd matrices like $\mathbb{F}$ and $\bar{\mathbb{F}}$ inserted into W_1^+ as (see the discussion in Appendix 7.E.4)

$$\tilde{Q}_+^{1a}\bullet = Q_+^{1a}\bullet - \{\bar{G}^a, \bullet\}\,, \quad \tilde{Q}_-^{a4}\bullet = Q_-^{a4}\bullet - \{G^a, \bullet\}\,, \tag{7.99}$$

with $\bar{G}^a$ and G^a in (7.87). For the tlit operators inserted into W_4^- we need to use the corresponding covariantisation with the roles of $\bar{G}^a$ and G^a reversed.

We can then check that conversely (with a mixed anti-commutator for the even and odd entries in $\mathcal{L}_1^+$)

$$[\tilde{Q}_-^{a4}, \mathbb{O}^b] = 2i\epsilon^{ab}(\partial_x \mathbb{F} + i[\mathcal{L}_1^+, \mathbb{F}\}) \,, \tag{7.100}$$

and likewise should be the case for the other operators, in accordance with (7.79) and (7.80).

We can carry over the tlit operators $\Finv$, $\mathbb{O}_a$ and $\mathbb{\Lambda}$ to be insertions in W_1^+. We denote those operators as $\sigma\Finv\bar{\sigma}$, etc., but in terms of the field expressions, they have the same form as above. The difference is that when acting on them with a preserved charge, we need to use instead the appropriate covariantisation for W_1^+.

Since $Q_+^{1a} + Q_-^{a4}$ annihilates σ (7.90), these operators have the same covariantisation with $G^a - \bar{G}^a$ inside both W_1^+ and W_4^-, so acting with them on $\sigma\bar{\Finv}\bar{\sigma}$ we find

$$\begin{aligned} [\tilde{Q}_+^{1a} + \tilde{Q}_-^{a4}, \sigma\Finv\bar{\sigma}] &= \epsilon^{ab}\sigma\mathbb{O}_b\bar{\sigma} \,, \\ [\tilde{Q}_+^{1a} + \tilde{Q}_-^{a4}, \sigma\bar{\Finv}\bar{\sigma}] &= \sigma\bar{\mathbb{O}}^a\bar{\sigma} \,, \end{aligned} \tag{7.101}$$

which is the appropriate covariantisation for operators in W_4^-.

Acting with Q_+^{1a} and Q_-^{a4} according to (7.79), (7.80) and the corresponding covariant derivatives (7.113), we get the remainging operators in the tilt multiplets

$$\begin{aligned} \mathbb{\Lambda} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_+^2\bar{C}^3 - \bar{\psi}_+^3\bar{C}^2 + C_1\psi_4^- & -\frac{1}{\alpha}(D_1 - iD_2)C_4 \\ 2i\alpha(\bar{C}^3C_1\bar{C}^2 - \bar{C}^2C_1\bar{C}^3) & \psi_4^-C_1 - \bar{C}^2\bar{\psi}_+^3 + \bar{C}^3\bar{\psi}_+^2 \end{pmatrix}, \\ \bar{\mathbb{\Lambda}} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_-^4\bar{C}^1 + C_2\psi_3^+ - C_3\psi_2^+ & 2i\bar{\alpha}(C_3\bar{C}^1C_2 - C_2\bar{C}^1C_3) \\ \frac{1}{\bar{\alpha}}(D_1 + iD_2)\bar{C}^4 & \bar{C}^1\bar{\psi}_-^4 + \psi_3^+C_2 - \psi_2^+C_3 \end{pmatrix}. \end{aligned} \tag{7.102}$$

D_1 and D_2 are covariant derivatives (with the usual connections $A_\mu^{(1)}$ and $A_\mu^{(2)}$ in the transverse $\mu = 1, 2$ directions.

Likewise from $[Q_+^{1a}, \mathbb{O}_b] = -\delta_b^a\mathbb{\Lambda}$ and $[Q_-^{a4}, \bar{\mathbb{O}}^b] = \epsilon^{ab}\bar{\mathbb{\Lambda}}$ we get

$$\begin{aligned} \mathbb{\Lambda} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_+^1\bar{C}^4 + C_2\psi_3^- - C_3\psi_2^- & 2i\bar{\alpha}(C_2\bar{C}^4C_3 - C_3\bar{C}^4C_2) \\ -\frac{1}{\bar{\alpha}}(D_1 - iD_2)\bar{C}^1 & \bar{C}^4\bar{\psi}_+^1 + \psi_3^-C_2 - \psi_2^-C_3 \end{pmatrix}, \\ \bar{\mathbb{\Lambda}} &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_-^2\bar{C}^3 + C_4\psi_1^+ - \bar{\psi}_-^3\bar{C}^2 & \frac{1}{\alpha}(D_1 + iD_2)C_1 \\ 2i\alpha(\bar{C}^2C_4\bar{C}^3 - \bar{C}^3C_4\bar{C}^2) & \bar{C}^3\bar{\psi}_-^2 + \psi_1^+C_4 - \bar{C}^2\bar{\psi}_-^3 \end{pmatrix}. \end{aligned} \tag{7.103}$$

7.E.3 The dispalcement multiplet

It is easy to get the tilt operator in the displacement multiplet by replacing a in (7.96) with 4 and 1 we find

$$\begin{aligned} \mathbb{O}^4 &= \begin{pmatrix} -2\bar{\alpha}\alpha C_1\bar{C}^4 & i\bar{\alpha}\bar{\psi}_+^4 \\ 0 & -2\bar{\alpha}\alpha\bar{C}^4C_1 \end{pmatrix}, & \bar{\mathbb{O}}_4 &= \begin{pmatrix} 2\bar{\alpha}\alpha C_4\bar{C}^1 & 0 \\ i\alpha\psi_4^+ & 2\bar{\alpha}\alpha\bar{C}^1C_4 \end{pmatrix}, \\ \bar{\mathbb{O}}^1 &= \begin{pmatrix} 2\bar{\alpha}\alpha C_4\bar{C}^1 & i\bar{\alpha}\bar{\psi}_-^1 \\ 0 & 2\bar{\alpha}\alpha\bar{C}^1C_4 \end{pmatrix}, & \mathbb{O}_1 &= \begin{pmatrix} -2\bar{\alpha}\alpha C_1\bar{C}^4 & 0 \\ i\alpha\psi_1^- & -2\bar{\alpha}\alpha\bar{C}^4C_1 \end{pmatrix}. \end{aligned} \tag{7.104}$$

$\mathbb{O}$ and $\bar{\mathbb{O}}$ are then expressed in terms of those as in (7.24).

Then we find the explicit expressions for $\mathbb{\Lambda}_a$, $\bar{\mathbb{\Lambda}}^a$, $\mathbb{\Lambda}^a$ and $\bar{\mathbb{\Lambda}}_a$ as

$$\begin{aligned} \mathbb{\Lambda}_a &= -2\bar{\alpha}\alpha \begin{pmatrix} C_1\psi_a^- - \epsilon_{ab}(\bar{\psi}_+^b\bar{C}^4 - \bar{\psi}_+^4\bar{C}^b) & -\frac{1}{\alpha}(D_1 - iD_2)C_a \\ -2i\alpha\epsilon_{ab}(\bar{C}^4C_1\bar{C}^b - \bar{C}^bC_1\bar{C}^4) & \psi_a^-C_1 - \epsilon_{ab}(\bar{C}^4\psi_+^b - \bar{C}^b\bar{\psi}_+^4) \end{pmatrix} \\ \bar{\mathbb{\Lambda}}^a &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_-^a\bar{C}^1 - \epsilon^{ab}(C_4\psi_b^+ + C_b\psi_4^+) & 2i\bar{\alpha}\epsilon^{ab}(C_4\bar{C}^1C_b - C_b\bar{C}^1C_4) \\ \frac{1}{\bar{\alpha}}(D_1 + iD_2)\bar{C}^a & \bar{C}^1\bar{\psi}_-^a - \epsilon^{ab}(\psi_b^+C_4 + \psi_4^+C_b) \end{pmatrix} \end{aligned} \tag{7.105}$$

and

$$\begin{aligned} \mathbb{\Lambda}^a &= -2\bar{\alpha}\alpha \begin{pmatrix} \bar{\psi}_+^a\bar{C}^4 - \epsilon^{ab}(C_1\psi_b^- + C_b\psi_1^-) & 2i\bar{\alpha}\epsilon^{ab}(C_b\bar{C}^4C_1 - C_1\bar{C}^4C_b) \\ -\frac{1}{\bar{\alpha}}(D_1 - iD_2)\bar{C}^a & \bar{C}^4\bar{\psi}_+^a - \epsilon^{ab}(\psi_b^-C_1 + \psi_1^-C_b) \end{pmatrix} \\ \bar{\mathbb{\Lambda}}_a &= -2\bar{\alpha}\alpha \begin{pmatrix} C_4\psi_a^+ - \epsilon_{ab}(\bar{\psi}_-^b\bar{C}^1 - \bar{\psi}_-^1\bar{C}^b) & \frac{1}{\alpha}(D_1 + iD_2)C_a \\ -2i\alpha\epsilon_{ab}(\bar{C}^bC_4\bar{C}^1 - \bar{C}^1C_4\bar{C}^b) & \psi_a^+C_4 - \epsilon_{ab}(\bar{C}^1\bar{\psi}_-^b - \bar{C}^b\bar{\psi}_-^1) \end{pmatrix} \end{aligned} \tag{7.106}$$

The expressions for $\mathbb{D}$ and $\mathbb{C}$ can then be found by further action with the supercharges.

7.E.4 Subtlety in covariant derivatives of supermatrices

For a Grassmann-even matrix $\mathcal{O}$ inserted into a Wilson line

$$W[\mathcal{O}(0)] = \operatorname{Tr}\mathcal{P}\left[\left(\exp\int_{-\infty}^{0} i\mathcal{L}(x)dx\right)\mathcal{O}(0)\left(\exp\int_{0}^{\infty} i\mathcal{L}(x)dx\right)\right], \tag{7.107}$$

as well as a Grassmann-even symmetry generator δ with $\delta(i\mathcal{L}) = \mathcal{D}_x^{\mathcal{L}}\mathcal{G}$, the variation of the Wilson line is

$$\begin{aligned} \delta W[\mathcal{O}(0)] &= W\left[\left(\int_{-\infty}^{0}\mathcal{D}_x^{\mathcal{L}}\mathcal{G}(x')\,dx'\,\mathcal{O}(0) + \delta\mathcal{O}(0) + \mathcal{O}(0)\int_{0}^{\infty}\mathcal{D}_x^{\mathcal{L}}\mathcal{G}(x')\,dx'\right)\right] \\ &= W\left[\delta\mathcal{O} + \mathcal{G}\mathcal{O} - \mathcal{O}\mathcal{G})(0)\right]. \end{aligned} \tag{7.108}$$

So that we can define a covariant symmetry $\tilde{\delta}$ acting by

$$\tilde{\delta}\mathcal{O} = \delta\mathcal{O} + \mathcal{G}\mathcal{O} - \mathcal{O}\mathcal{G}\,. \tag{7.109}$$

Turning to the case of Grassmann-odd operators, for example, the Grassmann-odd Q and G

$$G = \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix}, \tag{7.110}$$

with even g_{ij}. We can use a unit Grassmannian θ to repackage them into even objects $\delta = \theta Q$ and $\mathcal{G} = \theta G$.

In the case where $\mathcal{O}$ is an even supermatrix (like $\mathbb{O}^a$ and $\mathbb{D}$)

$$\mathcal{O} = \begin{pmatrix} B_{11} & F_{12} \\ F_{21} & B_{22} \end{pmatrix}, \tag{7.111}$$

so the covariant action of Q can be found from (7.109) to be

$$\tilde{Q}\mathcal{O} = \begin{pmatrix} QB_{11} + g_{12}F_{21} + F_{12}g_{21} & QF_{12} + g_{12}B_{22} - B_{11}g_{12} \\ QF_{21} + g_{21}B_{11} - B_{22}g_{21} & QB_{22} + g_{21}F_{12} + F_{21}g_{12} \end{pmatrix} . \tag{7.112}$$

In other words,

$$\tilde{Q}\mathcal{O} = Q\mathcal{O} + \{G, \mathcal{O}_F\} + [G, \mathcal{O}_B] \,, \tag{7.113}$$

where $\mathcal{O}_B$ and $\mathcal{O}_F$ are the bosonic and fermionic parts of $\mathcal{O}$.

The other case is for an odd supermatrix

$$\mathcal{O}' = \begin{pmatrix} F_{11} & B_{12} \\ B_{21} & F_{22} \end{pmatrix} . \tag{7.114}$$

We take an odd ϵ such that $\mathcal{O} = \epsilon\mathcal{O}'$ is an even supermatrix. Then plugging this into (7.109), we get

$$\tilde{Q}\mathcal{O}' = \begin{pmatrix} QF_{11} - (g_{12}B_{21} + B_{12}g_{21}) & QB_{12} - (g_{12}F_{22} - F_{11}g_{12}) \\ QB_{21} - (g_{21}F_{11} - F_{22}g_{21}) & QF_{22} - (g_{21}B_{12} + B_{21}g_{12}). \end{pmatrix} \tag{7.115}$$

In short

$$\tilde{Q}\mathcal{O}' = Q\mathcal{O}' - \{G, \mathcal{O}'_B\} - [G, \mathcal{O}'_F]. \tag{7.116}$$

7.F The geometry of $SU(4)/S(U(2) \times U(1) \times U(1))$

As explained in Section 7.5, the defect conformal manifold is the coset $SU(4)/S(U(2) \times U(1) \times U(1))$ and the integrated 4-point functions of the tilt operators are related to the curvature of this manifold. We follow [209] (see also [210]) to describe this coset and evaluate the Riemann tensor.

We start by choosing explict generators of $SU(4)$ in terms of the 4×4 matrices, α_{ab} with entry 1 at location ab. The generators of $S(U(2) \times U(1) \times U(1))$ are the three diagonal ones and $\sqrt{2}\alpha_{12}$ and $\sqrt{2}\alpha_{21}$. Note that this is not a Hermitian basis, but we normalise them such multiplying by the hermitian conjugate and tracing gives 2. We denote them collectively as h_A with $A = 1, \ldots, 5$.

The remaining generators are

$$\begin{aligned} &m_1 = \sqrt{2}\alpha_{13} \,, \quad m_2 = \sqrt{2}\alpha_{23} \,, \quad m_3 = \sqrt{2}\alpha_{41} \,, \quad m_4 = \sqrt{2}\alpha_{42} \,, \quad m_5 = \sqrt{2}\alpha_{43} \,, \\ &m_{\bar{1}} = \sqrt{2}\alpha_{31} \,, \quad m_{\bar{2}} = \sqrt{2}\alpha_{32} \,, \quad m_{\bar{3}} = \sqrt{2}\alpha_{14} \,, \quad m_{\bar{4}} = \sqrt{2}\alpha_{24} \,, \quad m_{\bar{5}} = \sqrt{2}\alpha_{34} \,. \end{aligned} \tag{7.117}$$

We denote them collectively as m_i. One can then define structure constants such that

$$[h_A, h_B] = f_{AB}{}^C h_C \,, \qquad [h_A, m_i] = f_{Ai}{}^j m_j \,, \qquad [m_i, m_j] = f_{ij}{}^A h_A + f_{ij}{}^k m_k \,. \tag{7.118}$$

We do not wish to write explicit coordinates on the coset, but in any (local) representation in terms of group elemets g, the Maurer-Cartan form on the coset can then be decomposed as $g^{-1}dg = \ell^i m_i + \Omega^A h_A$. The metric on the coset can then be written as

$$ds^2 = g_{ij}\ell^i \ell^j \,, \tag{7.119}$$

and this metric is $SU(4)$ invariant if g_{AB} are constants and satisfy

$$f_{Ai}{}^k g_{jk} + f_{Aj}{}^k g_{ik} = 0 \,. \tag{7.120}$$

In our case the possible solutions are

$$g_{1\bar{1}} = g_{\bar{1}1} = g_{2\bar{2}} = g_{\bar{2}2} = a \,, \qquad g_{3\bar{3}} = g_{\bar{3}3} = c \,, \qquad g_{4\bar{4}} = g_{\bar{4}4} = g_{5\bar{5}} = g_{\bar{5}5} = b \,. \tag{7.121}$$

In terms of the dCFT data (7.31), (7.35), those are

$$a = C_{\mathbb{O}_a} \,, \qquad b = C_{\mathbb{O}^a} \,, \qquad c = C_{\mathbb{O}} \,. \tag{7.122}$$

The Levi-Civita connection is then given as

$$C_k{}^i{}_j = \frac{1}{2}\left(g^{il} f_{lj}{}^m g_{km} + g^{il} f_{lk}{}^m g_{jm} + f_{kj}{}^i\right) , \tag{7.123}$$

And the Riemann tensor is

$$R^i{}_{jkl} = \left(C_k{}^i{}_m C_l{}^m{}_j - C_l{}^i{}_m C_k{}^m{}_j - C_m{}^i{}_j f_{kl}{}^m - f_{Aj}{}^i f_{kl}{}^A\right) . \tag{7.124}$$

The full explanation of these expressions and their implementation for other cosets can be found in [209].

Lowering the first index and plugging in the metric and structure constants, we find that up to the usual symmetries of the Riemann tensor, the nonzero components of the form $R_{i\bar{j}k\bar{l}}$ are

$$\begin{aligned}
&R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = 2a \,, \qquad R_{1\bar{1}2\bar{2}} = R_{1\bar{2}2\bar{1}} = a \,, \\
&R_{1\bar{1}3\bar{3}} = R_{2\bar{2}3\bar{3}} = -\frac{(a+b-c)^2 - 4ab}{4b} \,, \\
&R_{1\bar{1}4\bar{4}} = R_{1\bar{2}4\bar{5}} = R_{2\bar{1}5\bar{4}} = R_{2\bar{2}5\bar{5}} = \frac{(a+b-c)^2 - 4ab}{4c} \,, \\
&R_{1\bar{4}4\bar{1}} = R_{1\bar{5}4\bar{2}} = R_{2\bar{4}5\bar{1}} = R_{2\bar{5}5\bar{2}} = \frac{(a-b+c)(a-b-c)}{4c} \,, \\
&R_{3\bar{3}4\bar{4}} = R_{3\bar{3}5\bar{5}} = -\frac{(a+b-c)^2 - 4ab}{4a} \,, \\
&R_{1\bar{3}3\bar{1}} = R_{2\bar{3}3\bar{2}} = \frac{a+c-b}{2} \,, \qquad R_{3\bar{4}4\bar{3}} = R_{3\bar{5}5\bar{3}} = \frac{b+c-a}{2} \,, \\
&R_{3\bar{3}3\bar{3}} = 2c \,, \qquad R_{4\bar{4}4\bar{4}} = R_{5\bar{5}5\bar{5}} = 2b \,, \qquad R_{4\bar{4}5\bar{5}} = R_{4\bar{5}5\bar{4}} = b \,.
\end{aligned} \tag{7.125}$$

There are also nonvanishing $R_{ij\bar{k}\bar{l}}$ components

$$
\begin{aligned}
&R_{1\bar{3}1\bar{3}} = R_{2\bar{3}2\bar{3}} = \frac{(a+b-c)(b+c-a)}{4b}\,, \qquad R_{3\bar{4}3\bar{4}} = R_{3\bar{5}3\bar{5}} = \frac{(a+b-c)(a+c-b)}{4a}\,. \\
&R_{1\bar{4}1\bar{4}} = R_{1\bar{4}\bar{2}\bar{5}} = R_{2\bar{5}1\bar{4}} = R_{2\bar{5}\bar{2}\bar{5}} = -\frac{a+b-c}{2}\,,
\end{aligned}
\tag{7.126}
$$

Such terms are incompatible with a Kähler structure and they all vanish for $\gamma = \alpha+\beta$, which is in fact the case for the 1/3 BPS loop (7.41). With that condition also (7.125) simplifies to

$$
\begin{aligned}
&\frac{1}{2}R_{1\bar{1}1\bar{1}} = \frac{1}{2}R_{2\bar{2}2\bar{2}} = R_{1\bar{1}2\bar{2}} = R_{1\bar{2}2\bar{1}} = R_{1\bar{1}3\bar{3}} = R_{2\bar{2}3\bar{3}} = R_{1\bar{3}3\bar{1}} = R_{2\bar{3}3\bar{2}} = a\,, \\
&R_{1\bar{1}4\bar{4}} = R_{1\bar{2}4\bar{5}} = R_{2\bar{1}5\bar{4}} = R_{2\bar{2}5\bar{5}} = R_{1\bar{4}4\bar{1}} = R_{1\bar{5}4\bar{2}} = R_{2\bar{4}5\bar{1}} = R_{2\bar{5}5\bar{2}} = -\frac{ab}{a+b}\,, \\
&R_{3\bar{3}3\bar{3}} = 2(a+b)\,, \\
&R_{3\bar{3}4\bar{4}} = R_{3\bar{3}5\bar{5}} = R_{3\bar{4}4\bar{3}} = R_{3\bar{5}5\bar{3}} = R_{4\bar{4}5\bar{5}} = R_{4\bar{5}5\bar{4}} = \frac{1}{2}R_{4\bar{4}4\bar{4}} = \frac{1}{2}R_{5\bar{5}5\bar{5}} = b\,.
\end{aligned}
\tag{7.127}
$$

8 Conclusions and outlook

As we already discussed in the introduction, while conformal defects and defect conformal manifolds are ubiquitous and important, the latter remains relatively unexplored. This thesis is intended to invigorate the realm of conformal defects of dimension one or two and their exactly marginal deformations.

The main results of this thesis can be divided into two parts. In section 3 and 7.5, we construct the "trivial" defect conformal manifolds generated by the breaking of the global R-symmetry. We check the Riemann curvature (2.29), (2.30) given by an integrated four-point function against known 4-point functions in four different examples and find a match with the curvature of the metric as in (3.26). In the first three cases, the defect conformal manifolds are symmetric spaces, S^5, $\mathbb{CP}^3$ and S^4, that have just a single scale, which is accessible by a single integral of the 4-point function. While for the last one, the 1/3 BPS defects break the symmetry in more interesting ways and have multiple sectors of defect exactly marginal operators with different 2-pt functions, giving rise to defect conformal manifold with more interesting metrics.

Though these examples are in supersymmetric theories, symmetry breaking defects exist in many CFTs including the non-supersymmetric ones. Recently the line and surface defects in $O(N)$ model are attracting considerable attention, see [50–56, 117, 211–213] and references therein. In an ongoing work [214], we check that our analysis indeed applies there, against the dCFT data and four-point functions obtained in [51]. We also expect that similar calculations can be carried out in 2d CFTs, for example [112], which we leave for future work.

Another promising direction to develop the formalism is to calculate the curvature tensor of more general vector bundles over the defect conformal manifold. Instead of 4-point functions of just defect exactly marginal operators living in the tangent bundle, this requires insertions of two marginal and two other operators. Such cases have been studied in CFTs without any defect in [97]. Besides, analogous constraints should also be found for higher point functions [138, 215, 216].

Then in the other sections, we focus on 3d theories where line operators are known to have multiple marginal couplings. We adopt the algorithm 1 that allows us to exhaust all the connected components of the space of BPS Wilson loops in any three-dimensional $\mathcal{N} = 4$ Chern-Simons-matter theory on S^3. Among them, a subset of the conformal loops underlie the "non-trivial" conformal manifolds. Matching their geometry with that of the moduli space could provide some other integral constraints, as a complement to the symmetry breaking case. However, this has not been done so far.

The most direct reason is that the associated correlation functions of exactly marginal operators are not known, including both the 2-point functions that give the Zamolodchikov metric and the 4-point functions producing the curvature tensor. Since correlation functions can be calculated by the means of standard Witten diagrams [122], the real problem is to find holographic duals of these BPS Wilson loops as well as their marginal deformations, which have not been found yet.

A comparable example for reference is the conformal Wilson loops in ABJM theory, whose moduli space is two copies of conifolds [58]. Among the moduli space there is a special family of Wilson loops interpolating between the 1/6 BPS bosonic loop and the 1/2 BPS loop. Their holographic duals are studied in [178, 197], described as strings in $AdS_4 \times \mathbb{CP}^3$ with different boundary conditions. More precisely, the dual description of 1/2 BPS loops is given in terms of Dirichlet boundary conditions for all the $\mathbb{CP}^3$ directions [95], and that of bosonic loops is of Neumann boundary conditions for a $\mathbb{CP}^1 \subset \mathbb{CP}^3$ [60, 139]. Concomitantly, the holographic interpolating BPS loops are interpreted in accordance with mixed boundary conditions that interpolate between Dirichlet and Neumann. The 2- and 4-point correlation functions of excitations in the strong coupling limit are indeed calculated there as well, however, they are not consistent with the conformal symmetries. All of these outstanding problems imply us that there is still a long way to go.

Another more basic question is about the general properties of defect conformal manifold, though we know very little about it so far. Before we move to the discussion of defect conformal manifolds, let us collect some known facts of the case without defect firstly.

The conformal manifolds of 2d CFTs are especially interesting, since when the CFTs are the worldsheet of a string theory, the conformal manifolds play the role of the moduli space of vacua. Some well-known examples include the free theories where the conformal manifolds are homogeneous of constant negative curvature [97], and the $\mathcal{N} = (2, 2)$ SCFTs, whose geometry is that of a smooth compact complex Kähler manifold [217] with trivial Kähler class [218].

In four dimensions, most known conformal manifolds are either non-compact or comprised of a single point. A famous example of the non-compact case is the the space of marginal couplings of $\mathcal{N} = 4$ super Yang-Mills theory with an exact Zamolodchikov metric $d\tau d\bar{\tau}/(\mathrm{Im}\tau)^2$, where $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. And for the latter case, an example is QCD in the Banks-Zaks phase where there are no exactly marginal deformation, and the resulting conformal manifold is just a single point. Compactness provides a better control on the conformal manifolds that are systematically studied in [219]. Additionally, in the theories with supersymmetry, it is proved in [220] that the supersymmetric conformal manifolds are necessarily Kähler. In particular, for $\mathcal{N} = 1$ theories, the conformal manifold is the quotient of the space of marginal couplings by the complexified continuous global symmetry group [36].

The 3d $\mathcal{N} = 2$ supersymmetric theories is closely related to $\mathcal{N} = 1$ theories in 4d. Among them, the conformal manifold of $\mathcal{N} = 2$ super Chern-Simons-matter theory is given by a symplectic quotient [221, 33]. More generally, it is proved that in $d \geq 3$ the supersymmetric conformal manifolds with at least four supercharges are Kähler-Hodge.[32]

[32] A Kähler-Hodge manifold is a Kähler manifold for which the flux of the Kähler 2-form through any

As we mentioned in the introduction, the construction of defect exactly marginal deformations, especially the trivial ones, requires no supersymmetry. Consequently, the space of defect conformal manifolds should be even wider than the one without defect. So a series of questions arise: in which cases the defect conformal manifolds are compact, or connected, or Kähler, or something we can put definite limits on in any way? All the defect conformal manifolds we study in this thesis: S^5, S^4, $\mathbb{CP}^3$ and $SU(4)/S(U(2)\times U(1)\times U(1))$ together with the two in [57]: $O(p+q)/(O(p)\times O(q))$ and $U(p+q)/(U(p)\times U(q))$ are coset spaces thus homogeneous. Though these examples are all compact, it is not always true for an arbitrary quotient space G/G'. Defects that break the global symmetry in more interesting ways will give more diverse geometric structures, for example the noncompactness. Moreover, when it comes to the cases including non-trivial defect exactly marginal deformations, such as the conifold comprised of BPS Wilson loops in ABJM theory [58], we further lose the restrictions for the space to be homogeneous, and the situation shall become even more complicated.

A consequent question is, instead of the explicit theories and defects we focused on in this thesis, is it possible to bypass the concrete cases and use abstract tools such as the algebraic approach to classify the defect conformal manifolds? For example, the classification of unitary superconformal line defects in $3 \leq d \leq 6$ unitary superconformal field theories has already been studied in [104], just relying on the superconformal symmetry and its associated unitary representations. The conclusion is that in SCFTs of $d > 3$, superconformal lines preserving transverse rotations (or sufficient supersymmetry) admit no marginal deformations, while in 3d, there is a much richer structure and the superconformal lines do permit marginal deformations. It will be very interesting to see its generalization to higher dimensional defects, or to defects in non-supersymmetric theories. It will be particularly exciting to compare the results with those of conformal manifolds in the absence of defects, when we have answers to our questions one day.

As a final remark, the field of defect conformal field theory and defect conformal manifold is rich, inviting, and mostly unexplored. We hope this thesis may share our interests and provide a bit of motivations and tools for the study of defects in the future.

2-circle is an integer.

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