This electronic thesis or dissertation has been downloaded from the King's Research Portal at https://kclpure.kcl.ac.uk/portal/

Ordinary parts and local-global compatibility at ℓ = p

Hevesi, Bence

Awarding institution:
King's College London

The copyright of this thesis rests with the author and no quotation from it or information derived from it may be published without proper acknowledgement.

END USER LICENCE AGREEMENT

Unless another licence is stated on the immediately following page this work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International licence. https://creativecommons.org/licenses/by-nc-nd/4.0/

You are free to copy, distribute and transmit the work

Under the following conditions:

- Attribution: You must attribute the work in the manner specified by the author (but not in any way that suggests that they endorse you or your use of the work).
- Non Commercial: You may not use this work for commercial purposes.
- No Derivative Works - You may not alter, transform, or build upon this work.

Any of these conditions can be waived if you receive permission from the author. Your fair dealings and other rights are in no way affected by the above.

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 25. Dec. 2024

Ordinary parts and local-global compatibility at $\ell = p$

Bence Hevesi

A thesis presented for the degree of
Doctor of Philosophy
at
King's College London

Supervised by
Ana Caraiani and Fred Diamond

Department of Mathematics
Faculty of Natural, Mathematical and Engineering Sciences
King's College London
United Kingdom

June 2024

Papa emlékére

Abstract

In this thesis, we prove local-global compatibility results at $\ell = p$ for the torsion automorphic Galois representations constructed by Scholze, generalising the work of Caraiani–Newton. In particular, we verify, up to a nilpotent ideal, the local-global compatibility conjecture at $\ell = p$ of Gee–Newton in the case of imaginary CM fields under some technical assumptions.

The key new ingredient is a local-global compatibility result for Q-ordinary self-dual automorphic representations for arbitrary parabolic subgroups.

Acknowledgements

First and foremost, I would like to thank my advisor Ana Caraiani for introducing me to beautiful pieces of mathematics, sharing her knowledge with generosity and listening to my ideas with great interest. Besides the plenty of mathematics I learned from her throughout these years, she spent just as much of her time showing me the ways of being a mathematician. I certainly would not be standing here without her support and guidance.

I am deeply grateful to my advisor Fred Diamond for the enlightening mathematical discussions, all the useful feedback on my work and for always asking excellent questions. I am also indebted to James Newton for the many illuminating conversations and for constantly showing interest in my work.

I am grateful to Richard Taylor and Shaun Stevens for agreeing to be my examiners, carefully reading my thesis and providing detailed and useful feedback.

Over the last four years, I had the privilege to be a part of the vibrant number theory community of London. I am grateful for the incredibly welcoming, friendly and stimulating environment they provided. I am especially thankful to Lambert A'Campo, Martin Ortiz and Matteo Tamiozzo for our mathematical discussions and their feedback on earlier versions of this text.

I am grateful to my close friends for constantly reminding me that there is much to life besides mathematics.

Last but not least, words cannot express my gratitude towards my family for their unconditional love and support.

This work was supported by the Engineering and Physical Sciences Research Council [EP/S021590/1]. The EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London, King's College London and Imperial College London.

Contents

Chapter 1

Background and introduction

1.1 Langlands reciprocity

In this thesis, we make progress towards Langlands reciprocity by verifying new cases of local-global compatibility at p. Given a number field F, a rational prime p and an integer $n \geq 1$, Langlands reciprocity, explicated by Clozel [Clo90] and combined with the conjecture of Fontaine–Mazur, predicts a precise correspondence between certain automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ and n-dimensional p-adic Galois representations of F. Let us start this thesis with carefully explaining the mentioned correspondence by breaking it into several separate conjectures, the first of which is concerned with associating Galois representations to automorphic forms.

Conjecture 1.1.1 (Construction of Galois representations)**.** Consider a field isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, and an *algebraic* cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$. There is a (necessarily unique) irreducible Galois representation

$$r_t(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

that is unramified at every finite place $v \nmid p$ of F where π_v is unramified, and the Satake parameters of $t^{-1}\pi_v|\det|_v^{\frac{1-n}{2}}$ match with the eigenvalues of the geometric Frobenius Frob_v acting on $r_t(\pi)$. In other words, $r_t(\pi)$ satisfies local-global compatibility at unramified places of π.

We now explain the statement of the conjecture in more detail.

Algebraic automorphic representations

First, Conjecture 1.1.1 is concerned with so-called *algebraic* automorphic representations. This is a rationality (or indeed, algebraicity) condition on the infinity component of our automorphic representations introduced[1] by

---

[1] Clozel's motivation for introducing this notion was to single out the class of cuspidal automorphic rerpesentations that should correspond to absolutely irreducible pure motives of rank n under Langlands's envisioned correspondence [Lan79].

Clozel in [Clo90]. Given an automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, it is, by definition, an irreducible $(\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C}, K_\infty) \times \mathrm{GL}_n(\mathbf{A}_F^\infty)$-subquotient $\pi_\infty \otimes \pi_f$ of the space of automorphic forms $\mathcal{A}(\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)$. Here $\mathfrak{g}_\infty = \oplus_{v|\infty}\mathfrak{g}_v$ is the Lie algebra of $(\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)(\mathbf{R}) = \prod_{v|\infty}\mathrm{GL}_n(F_v)$, and $K_\infty := \prod_{v|\infty} K_v$ is the product of maximal compact subgroups K_v of $\mathrm{GL}_n(F_v)$. In particular, π_∞ is an irreducible admissible $(\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C}, K_\infty)$-module. Therefore, by a version of Schur's lemma, the centre $Z(\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C})$ acts via a character

$$\omega_{\pi_\infty} : Z(\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C}) \to \mathbf{C},$$

called the *infinitesimal character* of π_∞. We can express this in simpler terms, using the Harish-Chandra isomorphism. Let $T_n \subset B_n \subset \mathrm{GL}_n$ denote the usual choice of Borel subgroup of upper-triangular matrices and torus of diagonal matrices. Set $\mathfrak{t}_\infty$ to be the Lie algebra of $(\mathrm{Res}_{F/\mathbf{Q}}T_n)(\mathbf{R})$ and W to be the Weyl group $W((\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)_{\mathbf{C}}, (\mathrm{Res}_{F/\mathbf{Q}}T_n)_{\mathbf{C}})$. Then Harish-Chandra sets up an isomorphism[2]

$$Z(\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C}) \cong \mathcal{U}(\mathfrak{t}_\infty \otimes_{\mathbf{R}} \mathbf{C})^W.$$

In particular, one identifies ω_{π_∞} with a W-orbit in $\mathfrak{t}_\infty^* := \mathrm{Hom}_{\mathbf{R}}(\mathfrak{t}_\infty, \mathbf{C})$. Inside $\mathfrak{t}_\infty^*$, we have as a W-invariant lattice the group of characters $X^*((\mathrm{Res}_{F/\mathbf{Q}}T_n)_{\mathbf{C}})$ by sending a character to its derivative. In particular, its shift by the half-sum of positive roots $X^*((\mathrm{Res}_{F/\mathbf{Q}}T_n)_{\mathbf{C}}) - \rho$ is again W-invariant.

Definition 1.1.2. Let $\pi = \pi_\infty \otimes \pi_f$ be an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. We say that π is *algebraic* if, under the normalised Harish-Chandra isomorphism, the W-orbit ω_{π_∞} lies within $X^*((\mathrm{Res}_{F/\mathbf{Q}}T_n)_{\mathbf{C}}) - \rho$ where ρ is the usual half-sum of positive roots of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$.

Remark 1.1.3. Another way of introducing the notion of algebraicity is using Langlands's archimedean local correspondence [Lan89]. Namely, for an archimedean place $v : F \hookrightarrow \mathbf{C}$, irreducible admissible $(\mathfrak{g}_v \otimes_{\mathbf{R}} \mathbf{C}, K_v)$-modules π_v are parametrised by semisimple representations $\mathrm{rec}(\pi_v) : W_{F_v} \to \mathrm{GL}_n(\mathbf{C})$ where W_{F_v} denotes the Weil group of F_v. Depending on v being a complex or a real place, W_{F_v} is given by $\mathbf{C}^\times$ or the nonsplit extension of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ by $\mathbf{C}^\times$. In either case, $\mathbf{C}^\times$ is canonically a subgroup of W_{F_v}. Then an automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is algebraic if and only if $\mathrm{rec}(\pi_v|\cdot|_{\mathbf{C}}^{\frac{1-n}{2}})|_{\mathbf{C}^\times} : \mathbf{C}^\times \to \mathrm{GL}_n(\mathbf{C})$ is an algebraic representation for every archimedean place v of F. In other words, if it is of the form $\mathrm{rec}(\pi_v|\cdot|_{\mathbf{C}}^{\frac{1-n}{2}})|_{\mathbf{C}^\times} \cong \oplus_{i=1}^n \chi_{r_i,s_i}$ for some pairs of integers $(r_i, s_i) \in \mathbf{Z}^2$ where $\chi_{r_i,s_i} : \mathbf{C}^\times \to \mathbf{C}^\times$, $z \mapsto z^{r_i}\bar{z}^{s_i}$. For details on the dictionary between the two definitions, see the remark on p.90 of [Clo90].

---

[2] We consider here the normalisation of the isomorphism that is independent of the choice of the Borel subgroup. In particular, we take W-invariants with respect to the natural action, not the dot action.

Example 1.1.4. Any cuspidal Hecke eigenform f of weight k and character $\psi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ gives rise to a cuspidal automorphic representation $\pi(f)$ of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ (cf. [Gel75], §5) with central character ψ and $\pi(f)_\infty$ given by the unique infinite-dimensional subrepresentation of the normalised parabolic induction[3] $\text{n-Ind}_{B_2(\mathbf{R})}^{\mathrm{GL}_2(\mathbf{R})}(|\cdot|^{\frac{k-1}{2}} \otimes |\cdot|^{\frac{1-k}{2}})$. One can then compute that, under the normalised Harish-Chandra isomorphism, $\omega_{\pi(f)_\infty}$ is given by the S_2-orbit of the $\mathbf{C}$-algebra map $\mathcal{U}(\mathfrak{t}_\mathbf{C}) \cong \mathbf{C}[Z,H]^{S_2} \to \mathbf{C}$, sending

$$Z \mapsto 0,\ H \mapsto k-1.$$

Here $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and S_2 acts by interchanging H and $-H$. On the other hand, one sees that the lattice $X^*(T_\mathbf{C}) - \rho \subset \mathfrak{t}^*_\mathbf{C} \cong \mathrm{Hom}_\mathbf{C}(\langle Z, H\rangle_\mathbf{C}, \mathbf{C})$ is given by the $\mathbf{C}$-linear maps sending

$$Z \mapsto a,\ H \mapsto b$$

with $a, b \in \mathbf{Z}$ such that $a-b$ is odd. In particular, after observing that the infinitesimal character of $|\det|^{\frac{1}{2}}$ sends Z to 1 and H to 0, we see that $\pi(f)^{\mathrm{coh}} := \pi(f) \otimes |\det|^{\frac{2-k}{2}}$ is an example of an algebraic automorphic representation. For $k \geq 2$, the system of Hecke eigenvalues of $\pi(f)^{\mathrm{coh}}$ is the one we see appearing in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\mathbf{C}^2)$ under the Eichler–Shimura isomorphism.

We note that automorphic representations associated with cuspidal Hecke newforms exhaust a huge portion of all algebraic automorphic representations. Namely, one sees that the rest of the cuspdidal algebraic automorphic representations for $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ are associated with algebraic Maass forms.

Satake parameters

To make sense of Conjecture 1.1.1, we first recall the definition of *Satake parameters* of unramified smooth irreducible representations of $\mathrm{GL}_n(F_v)$ for v a finite place of F. To keep the discussion in context, fix an automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $\mathrm{GL}_n(\mathbf{A}_F)$. A theorem of Flath [Fla79] shows that the smooth irreducible $\mathbf{C}$-representation π_f of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$ decomposes as a restricted tensor product $\bigotimes'_v \pi_v$ for some unique collection of irreducible smooth $\mathbf{C}$-representations π_v of $\mathrm{GL}_n(F_v)$ where v runs over finite places of F. Moreover, for all but finitely many finite places v of F, π_v is unramified, meaning that it admits a fixed vector under the compact open subgroup $\mathrm{GL}_n(\mathcal{O}_{F_v}) \subset \mathrm{GL}_n(F_v)$. Fix such a place v of F, and consider the usual induced action of the spherical Hecke algebra[4]

$$\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))_\mathbf{C} :=$$

---

[3] In other words, it is the (limit of) discrete series of type $(k-1, 0)$.

[4] The algebra structure is given by the usual convolution product.

$$\{f : \mathrm{GL}_n(\mathcal{O}_{F_v})\backslash \mathrm{GL}_n(F_v)/\mathrm{GL}_n(\mathcal{O}_{F_v}) \to \mathbf{C} \text{ compactly supported}\}$$

on $\pi_v^{\mathrm{GL}_n(\mathcal{O}_{F_v})}$ given by convolution. This action makes the space of fixed vectors a simple module over the Hecke algebra, and the representation π_v can be recovered from this Hecke module.

On the other hand, the (normalised) Satake isomorphism sets up an isomorphism (see [Car79], Theorem 4.1 for instance)

$$\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))_{\mathbf{C}} \cong \mathcal{H}(T_n(F_v), T_n(\mathcal{O}_{F_v}))_{\mathbf{C}}^{W(\mathrm{GL}_n,T_n)} \cong \mathbf{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]^{S_n}.$$

In particular, $\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))_{\mathbf{C}}$, turning out to be commutative, acts through $\pi_v^{\mathrm{GL}_n(\mathcal{O}_{F_v})}$ via a $\mathbf{C}$-algebra homomorphism

$$\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))_{\mathbf{C}} \to \mathbf{C}$$

that, under the Satake isomorphism, corresponds to a $\mathrm{GL}_n(\mathbf{C})$-conjugacy class $c(\pi_v)$ of a diagonal matrix $\mathrm{diag}(\alpha_1, ..., \alpha_n) \in \mathrm{GL}_n(\mathbf{C})$ that we call the *Satake parameter* of π_v. Finally, note that this construction sets up a correspondence between unramified irreducible smooth $\mathbf{C}$-representations of $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_n(\mathbf{C})$-conjugacy classes of semisimple elements in $\mathrm{GL}_n(\mathbf{C})$.

Then the determining property of $r_t(\pi)$ is that for every finite place $v \nmid p$ of F for which π_v is unramified, the characteristic polynomial of the Satake parameter $t^{-1}c(\pi_v \otimes |\det|_v^{\frac{1-n}{2}})$ matches with the characteristic polynomial of the geometric Frobenius Frob_v acting on $r_t(\pi)$.

More classically, this condition is also often expressed in terms of the usual generators $T_{v,i} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))_{\mathbf{C}}$, $i = 1, ..., n$ given by the characteristic function of the double coset

$$\mathrm{GL}_n(\mathcal{O}_{F_v})\mathrm{diag}(\varpi_v, ..., \varpi_v, 1, ..., 1)\mathrm{GL}_n(\mathcal{O}_{F_v})$$

where the first i elements in the diagonal are given by a choice of uniformiser $\varpi_v \in \mathcal{O}_{F_v}$ and the rest by 1. Namely, after unravelling the normalised Satake isomorphism, one finds that, if we denote by $a_{v,i} \in \overline{\mathbf{Q}}_p$ the eigenvalue of $T_{v,i}$ acting on $t^{-1}\pi_v$, then the characteristic polynomial of Frob_v on $r_t(\pi)$ is asked to coincide with

$$X^n - a_{v,1}X^{n-1} + ... + (-1)^j q_v^{\frac{j(j-1)}{2}} a_{v,j}X^{n-j} + ... + (-1)^n q_v^{\frac{n(n-1)}{2}} a_{v,n} \in \overline{\mathbf{Q}}_p[X].$$

Example 1.1.5. Going back to the case of modular forms, consider a normalised cuspidal newform $f = \sum_{n\geq 1} a_n q^n$ of weight k, level $\Gamma_1(N)$, and character ψ. Then $\pi(f)^{\mathrm{coh}}$ is of level $K_1(N)$, and therefore is unramified at any $\ell \nmid N$. After unravelling the construction of $\pi(f)$, one finds that, for $\ell \nmid N$, the Satake parameter $c(\pi(f)_\ell^{\mathrm{coh}})$ is given by the conjugacy class of $\mathrm{diag}(\ell^{-\frac{1}{2}}\alpha_\ell, \ell^{-\frac{1}{2}}\beta_\ell)$, where α_ℓ and β_ℓ are the roots of the polynomial $X^2 - a_\ell X + \ell \cdot \ell^{k-2}\psi(\ell)$. In particular, Conjecture 1.1.1 predicts, for any isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, the existence of an irreducible Galois representation

$$\rho_{f,t} := r_t(\pi(f)^{\mathrm{coh}}) : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

unramified at every $\ell \nmid Np$ with characteristic polynomial of the geometric Frobenius Frob_ℓ given by

$$X^2 - t^{-1}(a_\ell)X + \ell^{k-1}t^{-1}(\psi(\ell)).$$

This is the more classical formulation of (part of) Langlands reciprocity for cuspidal newforms, by now a theorem of Eichler–Shimura ($k = 2$), Deligne ($k \geq 2$) and Deligne–Serre ($k = 1$).

We note that the computation of the Satake parameters $c(\pi(f)_\ell^{\mathrm{coh}})$ sheds some light on the presence of the twist $|\cdot|^{\frac{1-n}{2}}$ in the conjecture. Namely, it is the collection of the Satake parameters of $\pi(f)^{\mathrm{coh}} \otimes |\cdot|^{-\frac{1}{2}}$ that reflects the rationality properties of f.

Conjecture 1.1.1 allows us to associate a unique Galois representation $r_t(\pi)$ to any algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ characterised by matching the local behaviour of the Galois representation with that of the automorphic representation at the unramified (non-p-adic) finite places of the latter. In fact, as we saw, for such finite places $v \nmid p$ of F, π_v amounts to the same piece of data as $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ as explained by the Satake isomorphism. Therefore, thanks to the strong multiplicity one theorem for GL_n/F, the automorphic analogue of Chebotarev's density theorem, we obtain an injection of sets of isomorphism classes

$$\left\{ \begin{array}{l} \text{algebraic cuspidal automorphic} \\ \text{representations of } \mathrm{GL}_n(\mathbf{A}_F) \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{continuous } p\text{-adic } n\text{-dimensional} \\ \text{almost everywhere unramified} \\ \text{irreducible Galois representations of } F \end{array} \right\}.$$

There are two obvious questions that one can raise.

i. Does $r_t(-)$ set up a more precise correspondence? Namely, can we write down π_v for all finite places v of F in terms of $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$? Conversely, how much of $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ can be reconstructed from π_v?

ii. Can we describe the image?

Let us first focus on the first of these questions that is a vague formulation of the problem of *local-global compatibility*. In order to turn it into a precise conjecture, we first need to discuss how to relate p-adic Galois representations of the ℓ-adic field F_v to smooth irreducible $\mathbf{C}$-representations of $\mathrm{GL}_n(F_v)$. Even though the answer will differ depending on whether $\ell \neq p$ or $\ell = p$, in both cases the main ingredient bridging the two worlds will be the local Langlands correspondence, a vast generalisation of (consequences of) local class field theory (the $n = 1$ case) and the discussed unramified correspondence realised by the Satake isomorphism.

The local Langlands correspondence

To explain the local correspondence, let ℓ be a rational prime and $K/\mathbf{Q}_\ell$ be a finite extension, and denote by q the cardinality of its residue field. For Ω an algebraically closed field, set $\mathrm{Irr}_\Omega(\mathrm{GL}_n(K))$ to be the set of isomorphism classes of irreducible smooth Ω-representations of $\mathrm{GL}_n(K)$. The correspondence matches such representations with so-called n-dimensional Frobenius-semisimple Weil–Deligne representations of K over Ω. To define these gadgets, we have to work with the Weil group W_K of K, as opposed to the whole Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$. Recall that W_K is the subgroup of G_K sitting in a short exact sequence

$$0 \to I_K \to W_K \to \mathrm{Frob}_K^{\mathbf{Z}} \to 0$$

where I_K is the inertia subgroup of G_K and Frob_K denotes the geometric Frobenius. Moreover, it is endowed with the topology making $I_K \subset W_K$ an open subgroup.

Definition 1.1.6. A Weil–Deligne representation of K over Ω is a triple (r, N, V) where

- V is a finite-dimensional Ω-vector space,
- r is a representation $W_K \to \mathrm{GL}(V)$ with open kernel, and
- N is a nilpotent endomorphism

subject to the compatibility that, for every $\sigma \in W_K$, we have

$$r(\sigma)Nr(\sigma)^{-1} = q^{-d(\sigma)}N$$

where $\sigma \equiv \mathrm{Frob}_K^{d(\sigma)} \mod I_K$. We say that (r, N, V) is n-dimensional if $\dim_\Omega V = n$. Moreover, we call (r, N, V) Frobenius-semisimple if r is semisimple.

Remark 1.1.7. Given a Weil–Deligne representation (r, N, V) of K, we can introduce its so-called *Frobenius-semisimplification* $(r, N, V)^{F-ss} := (r^{ss}, N, V)$ where $r^{ss} : W_K \to \mathrm{GL}(V)$ is the semisimple representation obtained from r by setting $r^{ss}(\varphi) \in \mathrm{GL}(V)$ to be the semisimple part of $r(\varphi)$ for a fixed lift φ of Frob_K, and keeping $r^{ss}|_{I_K} = r|_{I_K}$ unchanged. One easily checks that the definition is independent of our choice of lift of Frobenius.

We further define the semisimplification of (r, N, V) to be $(r, N, V)^{ss} := (r^{ss}, 0, V)$.

Denote by $\mathrm{WD}_\Omega^n(K)$ the set of isomorphism classes of n-dimensional Frobenius-semisimple representations of K over Ω. One then has the celebrated local Langlands correspondence for $\mathrm{GL}_n(K)$.

Theorem 1.1.8 (Harris–Taylor, Henniart)**.** *There exists a unique collection of bijections*

$$\mathrm{rec}_K : \mathrm{Irr}_{\mathbf{C}}(\mathrm{GL}_n(K)) \to \mathrm{WD}^n_{\mathbf{C}}(K)$$

for every $n \geq 1$ such that, for $n = 1$ it is induced by composition with the Artin map[5] *of local class field theory, compatible with character twists, central characters, taking contragradient to the dual Weil–Deligne representation, and matches $L-$ and $\epsilon-$factors of pairs.*

In fact, we will work with the arithmetic (or Tate) normalisation $\mathrm{rec}^T_K(-) := \mathrm{rec}_K((-) \otimes |\det|_K^{\frac{1-n}{2}})$ instead. The advantage of this normalisation is that it is compatible with automorphisms of $\mathbf{C}$. In particular, it provides a unique correspondence

$$\mathrm{rec}^T_K : \mathrm{Irr}_{\Omega}(\mathrm{GL}_n(K)) \longleftrightarrow \mathrm{WD}^n_{\Omega}(K)$$

over *arbitrary* Ω isomorphic to $\mathbf{C}$ (independent of the chosen isomorphism), and the rationality properties of $\pi \in \mathrm{Irr}_{\Omega}(\mathrm{GL}_n(K))$ will match with that of $\mathrm{rec}^T_K(\pi)$.

With the local Langlands correspondence in hand, to make our first question more precise, we have to relate $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ to Weil–Deligne representations. This is where the case of $\ell = p$ becomes significantly more subtle than the case of $\ell \neq p$. Let us first consider the latter.

Local-global compatibility at $\ell \neq p$

Let ℓ be a rational prime different from p, and $v|\ell$ a place of F. As it turns out, when ℓ differs from p, p-adic Galois representations of an ℓ-adic field are just special cases of Weil–Deligne representations. More precisely, there is a fully faithful functor $\mathrm{WD}(-)$ from the category of continuous p-adic Galois representations of F_v to the category of Weil–Deligne representations of F_v over $\overline{\mathbf{Q}}_p$. The point is that Grothendieck's ℓ-adic monodromy theorem shows that on any such Galois representation some open subgroup $I \subset I_{F_v}$ will act unipotently and, therefore, by taking its logarithm, we can rewrite this piece of information as a nilpotent endomorphism N.

We can then finally state local-global compatibility away from p.

Conjecture 1.1.9 (Local-global compatibility at $\ell \neq p$)**.** Assume Conjecture 1.1.1, and fix an isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. Let π be an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and $v \nmid p$ be a finite place of F. We then have an isomorphism

$$\mathrm{WD}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F-ss} \cong \mathrm{rec}^T(t^{-1}\pi_v).$$

---

[5] We normalise the Artin map so that it sends uniformisers to lifts of the *geometric* Frobenius.

Notice that Conjecture 1.1.9 indeed generalises the unramified case as the characteristic polyonomial of Frob_v acting on $\mathrm{rec}^T(t^{-1}\pi_v)$ is exactly given by the characteristic polynomial of the Satake parameters of $t^{-1}\pi_v \otimes |\det|_v^{\frac{1-n}{2}}$.

Example 1.1.10. For f a cuspidal newform of level $\Gamma_0(N)$, consider $\pi(f)^{\mathrm{coh}}$. Then, for $\ell \neq p$, a rational prime with $\ell | N$ but $\ell^2 \nmid N$, we see that $\pi(f)^{\mathrm{coh}}_\ell$ admits an Iwahori fixed vector. In particular, it is either an unramified principal series, an unramified twist of the trivial representation or a special series representation (i.e. an unramified twist of the Steinberg representation). Since f is a newform, $\pi(f)^{\mathrm{coh}}_\ell$ cannot be unramified and, therefore, it must be a special series $\mathrm{St} \otimes (\chi \circ \det)$ for some unramified character $\chi : \mathbf{Q}_\ell^\times \to \mathbf{C}^\times$. Then Conjecture 1.1.9 predicts that $\mathrm{WD}(\rho_{f,t}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)})^{F-ss}$ is isomorphic to the Weil–Deligne representation (r, N, V) where (r, V) is the representation[6] $\chi \oplus \chi|\cdot|_\ell^{-1}$, and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In particular, arbitrarily small open subgroups of the wild inertia at ℓ are expected to have a non-trivial action on $\rho_{f,t}$.

Local-global compatibility at $\ell = p$

Now let $v|p$ be a finite place of F, and set $K = F_v$. When trying to mimic the prime-to-p case, the first problem we run into is that, in general, p-adic Galois representations of K cannot be realised as Weil–Deligne representations as they amount to a significantly more complicated collection of data thanks to the possibly highly non-trivial action of the wild inertia subgroup.[7] There is, however, a subcategory of p-adic Galois representations of K singled out by Fontaine called *de Rham* Galois representations. To such representations a p-adic version of the monodromy theorem (first proved by Berger) applies and, by a construction of Fontaine, de Rham Galois representations of K do have an associated Weil–Deligne representation. However, we must warn the reader that in this p-adic setup a (de Rham) Galois representation cannot be reconstructed from its associated Weil–Deligne representation. Without going into the details, a de Rham Galois representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is determined by a so-called filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module[8] $D_{\mathrm{pst}}(\rho)$[9] (cf. [Fon94], §5.6.3). The associated Weil–Deligne representation $\mathrm{WD}(\rho)$ is constructed from this semilinear algebra gadget by forgetting the filtration.

---

[6] We consider smooth characters of $\mathbf{Q}_\ell^\times$ as characters of $W_{\mathbf{Q}_\ell}$ by composing them with the Artin map of local class field theory (sending ℓ to the geometric Frobenius).

[7] We can already think of the example of the p-adic cyclotomic character.

[8] Here L is some suitably large finite extension of K.

[9] This is a finite free $L_0 \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$-module where $L/L_0/\mathbf{Q}_p$ is the maximal unramified intermediate extension. It is equipped with an (arithmetic Frobenius)-semilinear automorphism φ, an L-semilinear Galois action, a nilpotent endomorphism N and an $L \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$-linear filtration on $L \otimes_{L_0} D_{\mathrm{pst}}(\rho)$.

The upshot of the previous paragraph is that one arrives at an obvious guess for local-global compatibility at $\ell = p$ if we are to believe that $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is de Rham. However, when reciprocity is formulated in terms of Galois representations, it stays slightly mysterious why this de Rham property should be expected. The author therefore feels the urge to motivate this expectation by going back to the more classical motivic formulation of Langlands reciprocity [Lan79] as was explicated in [Clo90]. Namely, the a priori stronger conjecture of Langlands–Clozel predicts that cuspidal algebraic automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ should admit associated irreducible pure motives of rank n over F with coefficients in some suitable number field $E \subset \mathbf{C}$ (cf. [Clo90], Conjecture 4.5). Whatever the category of motives over F might be, every smooth projective variety over F provides an example and each motive comes with a collection of realisation functors (Betti, ℓ-adic, de Rham,...) that, for varieties, specialise to taking their respective cohomology groups. Moreover, these realisation functors come with all the extra structures and comparison theorems that the corresponding cohomology theories admit. In particular, $r_t(\pi)$ is expected to be realised as the p-adic realisation of the conjectural motive corresponding to π (and t) and therefore should be de Rham by the p-adic de Rham comparison theorem.

Before finally stating a local-global compatibility conjecture at $\ell = p$, there is one more subtle point that is to be explained. The reader might have already noticed that we made no mention of "local-global compatibility at ∞". Furthermore, soon after trying to come up with a guess for what such a compatibility should look like, one easily gets stuck when realising that for an archimedean place w of F, $\mathrm{Gal}(\overline{F}_w/F_w)$ is either the trivial group (when w is complex) or the group of two elements $\{1, c\}$ (when w is real). However, as Langlands's parametrisation of irreducible admissible $\mathrm{GL}_n(F_w)$-representations shows, π_w is described by an n-dimensional complex representation of the Weil group W_{F_w} that fits into a short exact sequence

$$0 \to \mathbf{C}^\times \to W_{F_w} \to \mathrm{Gal}(\overline{F}_w/F_w) \to 0.$$

Therefore, we have no chance of reconstructing π_w from $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_w/F_w)}$! More precisely, at real places, the action of the complex conjugation c on $\mathrm{rec}(\pi_w)$ should be possible to be matched with its action on $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ and indeed there is a precise conjecture of this kind (see [BG14], Conjecture 3.2.1). However, it is not clear how to tell $\mathrm{rec}(\pi_w)|_{\mathbf{C}^\times}$ from some local data of $r_t(\pi)$. The reason for raising this issue just now is that in fact it is $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ that is expected to mirror the shape of $\mathrm{rec}(\pi_w)|_{\mathbf{C}^\times}$ for $v|p$ the place induced by w under t. In particular, when reciprocity is formulated in terms of p-adic Galois representations, (part of) local-global compatibility at ∞ is incorporated into local-global compatibility at p. Let us first explain briefly the prediction and then we will provide some motivation as well.

Consider an archimedean place $w : F \hookrightarrow \mathbf{C}$, set $v|p$ to be the corresponding p-adic place under the identification t, and denote by $\iota_w : F_v \hookrightarrow \overline{\mathbf{Q}}_p$

the induced embedding. Recall that, since π_w is algebraic, $\mathrm{rec}(\pi_w|\cdot|_{\mathbf{C}}^{\frac{1-n}{2}}) \cong \oplus_{i=1}^n \chi_{r_{w,i},s_{w,i}}$ for some tuple of integers $(r_{w,1}, ..., r_{w,n}, s_{w,1}, ..., s_{w,n}) \in \mathbf{Z}^{2n}$ (see Remark 1.1.3 for the notation) with $r_{w,1} \leq ... \leq r_{w,n}$. On the other hand, $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is de Rham, and, in particular, Hodge–Tate, and we obtain its ι_w-Hodge–Tate weights $\lambda_{\iota_w,1} \leq ... \leq \lambda_{\iota_w,n}$ by recording the corresponding jumps of the filtration on $D_{\mathrm{pst}}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})_L$. Then local-global compatibility predicts that $(s_{w,n}, ..., s_{w,1}) = (\lambda_{\iota_w,1}, ..., \lambda_{\iota_w,n})$.

Remark 1.1.11. Note that thanks to purity of π_w (see Clozel's "purity lemma" [Clo90], Lemma 4.9), $(s_{w,1}, ..., s_{w,n})$ already determines $\mathrm{rec}(\pi_w|\cdot|_w^{\frac{1-n}{2}})|_{\mathbf{C}^\times}$.

Again, to motivate this expectation, we revisit the motivic formulation of reciprocity. Given π and $w : F_w \hookrightarrow \mathbf{C}$ as before, [Clo90], Conjecture 4.5 tells us that there should be an associated motive M. Its Betti realisation with respect to the embedding $w : F \hookrightarrow \mathbf{C}$ admits a Hodge decomposition $H_{B,w}(M) = \oplus_{r+s=d} H^{r,s}$ where d is the purity weight of π and, therefore, of M. Moreover, if w is a real place, there is an action of $\mathrm{Gal}(\overline{F}_w/F_w)$ on $H_{B,w}(M)$ induced by the action on the motive (i.e. by complex conjugation on $X \times_F \mathbf{C}$ for a motive induced by a smooth proper F-variety X) and not on the coefficients. This Galois action interchanges the components $H^{r,s}$ and $H^{s,r}$, and all of its eigenvalues lie in $\{+1, -1\}$. The Hodge numbers then give rise to an algebraic representation $\oplus_{r+s=d} \chi_{-r,-s}^{\dim_{\mathbf{C}} H^{r,s}} : \mathbf{C}^\times \to \mathrm{GL}_n(\mathbf{C})$. In the complex case we treat this as a representation of W_{F_w} and denote it by $r(M, w)$. In the real case, by a recipe of Serre ([Ser69], [Clo06] p.17), the action of complex conjugation on $H_{B,w}(M)$ allows us to extend the given representation of $\mathbf{C}^\times$ to a representation $r(M, w) : W_{F_w} \to \mathrm{GL}_n(\mathbf{C})$. Then local-global compatibility predicts that there is an isomorphism $\mathrm{rec}(\pi_w|\det|_{\mathbf{C}}^{\frac{1-n}{2}}) \cong r(M, w)$ (cf. [Clo90], 4.3.3).

Now the point is that the p-adic realisation sees the Hodge numbers, and in the real case, even the action of $\mathrm{Gal}(\overline{F}_w/F_w)$ through comparison theorems. In the case of the former, this is a consequence of the Hodge–Tate comparison theorem that shows that the Hodge numbers of $H_{B,w}(M)$ can be read off from the ι_w-Hodge–Tate weights $\lambda_{\iota_w,1} \leq ... \leq \lambda_{\iota_w,n}$ of the $\mathrm{Gal}(\overline{F}_v/F_v)$-action on the p-adic realisation of the motive. More precisely, one obtains that $r(M, w)|_{\mathbf{C}^\times} \cong \oplus_{i=1}^n \chi_{-d-\lambda_{\iota_w,i},\lambda_{\iota_w,i}}$. In the case of the latter, it follows already from the (properties of the) comparison theorem between the Betti and p-adic realisations.

We can now finally state a hopefully well-motivated conjecture.

Conjecture 1.1.12 (Local-global compatibility at $\ell = p$)**.** Assume Conjecture 1.1.1, and fix an isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. Let π be an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and $v|p$ be a finite place of F.

Then $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is de Rham. Moreover, we have an isomorphism

$$\mathrm{WD}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F-ss} \cong t^{-1}\mathrm{rec}^T(\pi_v), \tag{1.1.1}$$

and, for every embedding $\iota : F \hookrightarrow \overline{\mathbf{Q}}_p$ inducing v, the ι-Hodge–Tate weights $\lambda_{\iota,1} \leq ... \leq \lambda_{\iota,n}$ of $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ satisfy

$$\mathrm{rec}(\pi_{t\iota}|\det|_{\mathbf{C}}^{\frac{1-n}{2}})|_{\mathbf{C}^\times} \cong \oplus_{i=1}^{n}\chi_{-d-\lambda_{\iota,i},\lambda_{\iota,i}}$$

where $\chi_{r,s} : \mathbf{C}^\times \to \mathbf{C}^\times$, $z \mapsto z^r\bar{z}^s$, and d is the purity weight of $\pi_{t\iota}$.

Example 1.1.13. Let $f = \sum_{n\geq 1} a_n q^n$ be a normalised cuspidal newform of weight k and level $\Gamma_0(N)$. Assume that $p \nmid N$. In particular, $\pi(f)_p^{\mathrm{coh}}$ is unramified, so $\mathrm{rec}^T(\pi(f)_p^{\mathrm{coh}})$ is unramified as well, meaning that the action of the inertia subgroup $I_{\mathbf{Q}_p}$ is trivial and the monodromy N is zero. Consequently, according to Conjecture 1.1.12, the same should hold for $\mathrm{WD}(\rho_{f,t}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)})$. One concludes from this that $\rho_{f,t}$ must be crystalline meaning in particular that we have an isomorphism $D_{\mathrm{cris}}(\rho_{f,t}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}) \cong \mathrm{WD}(\rho_{f,t}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)})$ where we treat the former as an unramified Weil–Deligne representation by inflating the action of the crystalline Frobenius (playing the role of the arithmetic Frobenius). After unravelling 1.1.1, we then obtain that the trace of the inverse of the crystalline Frobenius of $\rho_{f,t}$ is expected to coincide with $t^{-1}a_p$.

As for the Hodge–Tate weights of $\rho_{f,t}$, we indicated already that, under the normalised Harish–Chandra isomorphism, the infinitesimal character of $\pi(f)_\infty^{\mathrm{coh}}$ is given by $(\frac{1}{2}, \frac{3}{2}-k)$. In particular, one sees that $\mathrm{rec}(\pi(f)_\infty^{\mathrm{coh}}|\det|^{-\frac{1}{2}})|_{\mathbf{C}^\times} \cong \chi_{0,1-k} \oplus \chi_{1-k,0}$ (cf. [Clo90], p.90). Then Conjecture 1.1.12 predicts that the Hodge–Tate weights of $\rho_{f,t}$ should be $(k-1, 0)$.

We are now ready to answer our first question. As we can see from Conjecture 1.1.9 and Conjecture 1.1.12 (and the discussion beforehand), for every place v of F, π_v is expected to be determined by the corresponding local behaviour of $r_t(\pi)$. In fact, in the case of $v \nmid p, \infty$, π_v and $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ determine each other. When $v|p$, however, $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ not only determines π_v, but the infinitesimal characters of the corresponding archimedean factors of π as well. On the other hand, π_v and the corresponding archimedean factors of π are still not sufficient to reconstruct $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$, only up to passing to the associated graded on $D_{\mathrm{pst}}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})$. Wondering what the filtration should correspond to on the automorphic side is the starting point of the p-adic Langlands program, and is beyond the scope of this thesis.

Automorphy

We finally discuss our second question, the question of automorphy of Galois representations. On the way of investigating our first question, we observed that in fact every irreducible continuous p-adic Galois representation coming from an algebraic cuspidal automorphic representation has a special local behaviour. Following Fontaine–Mazur, we isolate such Galois representations by calling them *geometric*.

Definition 1.1.14. Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a continuous representation. We call ρ *geometric* if

i. it is unramified at all but finitely many places, and

ii. is de Rham at p.

The striking conjecture of Fontaine–Mazur says that every irreducible geometric p-adic Galois representation is motivic.

Conjecture 1.1.15 (Fontaine–Mazur)**.** Any irreducible geometric Galois representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

can be realised as the $\mathrm{Gal}(\overline{F}/F)$-equivariant subquotient of

$$H^i(X \times_F \overline{F}, \overline{\mathbf{Q}}_p)(j)^{10}$$

for some smooth proper variety X/F, $i \in \mathbf{Z}_{\geq 0}$, and $j \in \mathbf{Z}$.

The Fontaine–Mazur conjecture combined with Langlands's prediction and the observations of Clozel (cf. [Clo90], Question 4.16) tells us what the answer to our second question should be.

Conjecture 1.1.16 (Automorphy)**.** For every irreducible geometric p-adic Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, and isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, there is an algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\rho \cong r_t(\pi)$.

Example 1.1.17. Let E/F be an elliptic curve without CM (i.e. $\mathrm{End}(E) \cong \mathbf{Z}$). As is well-known, the dual of its Tate module $T_pE = \varprojlim_{n\geq 1} E[p^n](\overline{F})$ is isomorphic to $H^1_{\text{ét}}(E \times_F \overline{F}, \mathbf{Z}_p)$ as a continuous $\mathrm{Gal}(\overline{F}/F)$-module. By inverting p and fixing a basis, we obtain a 2-dimensional geometric Galois representation

$$\rho_{E,p} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p).$$

A theorem of Serre says that E having no CM means that $\rho_{E,p}$ must be irreducible. By looking at the Hodge numbers of E, we see that all of its labelled Hodge–Tate weights are given by $(0, 1)$. In particular, for every choice of isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, Conjecture 1.1.16 predicts the existence of an algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ with all of its archimedean components having infinitesimal character $(\frac{1}{2}, -\frac{1}{2})$.[11]

---

[10] Here $(-)(j)$ denotes the Tate twist of the Galois action by the j^{th} power of the cyclotomic character.

[11] Note that this is the infinitesimal character of the trivial representation of $\mathrm{GL}_2(F_w)$, i.e. the representation with highest weight $(0, 0)$.

On the other hand, at any $v \nmid p$ finite place of F where E has good reduction, $\rho_{E,p}$ is unramified and by the Lefschetz fixed point formula for E,

$$\mathrm{tr}\rho_{E,p}(\mathrm{Frob}_v) = 1 + |\mathcal{O}_{F_v}/\varpi_v| - |E(\mathcal{O}_{F_v}/\varpi_v)| =: a_v(E).$$

In particular, the L-function of E is exactly the L-function of $\rho_{E,p}$ that is the L-function of π. In other words, E is expected to be modular, and therefore, its L-function should enjoy the properties of an automorphic L-function.

In the case of $F = \mathbf{Q}$, algebraic cuspidal automorphic representations with infinitesimal character $(\frac{1}{2}, -\frac{1}{2})$ are exactly the ones of the form $\pi(f)^{\mathrm{coh}}$ for weight 2 normalised cuspidal newforms. In this case, the conjecture translates exactly to modularity of elliptic curves $E/\mathbf{Q}$ (cf. [Wil95], [TW95], [Bre+01]) in the classical sense: the existence of a newform $f = \sum_{n\geq 1} a_n(f)q^n$ of weight 2 matching the dconductor of E with the level of f and satisfying $a_\ell(f) = a_\ell(E)$ at all primes ℓ where E has good reduction.

Combining Conjecture 1.1.1, Conjecture 1.1.9, Conjecture 1.1.12 and Conjecture 1.1.16 we obtain Langlands reciprocity.

Conjecture 1.1.18 (Langlands reciprocity)**.** For any choice of integer $n \geq 1$, and isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, there is a (necessarily unique) bijection of sets of isomorphism classes

$$\left\{ \begin{array}{l} \text{algebraic cuspidal automorphic} \\ \text{representations of } \mathrm{GL}_n(\mathbf{A}_F) \end{array} \right\} \xrightarrow{r_t(-)} \left\{ \begin{array}{l} \text{irreducible geometric} \\ n\text{-dimensional } p\text{-adic} \\ \text{Galois representations of } F \end{array} \right\}$$

satisfying local-global compatibility at every place of F.

1.2 Known cases of Langlands reciprocity

After a general introduction to the statement of reciprocity, we now turn to discussing known cases of Conjecture 1.1.18 with a focus on progress towards Conjecture 1.1.12. However, we do not attempt to do enough justice to the history of Langlands reciprocity here, instead provide an incomplete list of the major achievements of the subject most relevant to this thesis.

1.2.1 The case of GL_1/F

The only integer $n \geq 1$ for which Langlands reciprocity is known in its entirety is $n = 1$. For GL_1/F for a general number field F, Conjecture 1.1.18 is about matching algebraic Hecke characters of F with 1-dimensional Galois representations of F and it can be deduced from Class Field Theory as for instance is explained in [Far11].

The correspondence is set up by first noting that the global Artin reciprocity map $\mathrm{Art}_F : \mathbf{A}_F^\times/F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ induces an isomorphism of profinite groups $\mathbf{A}_F^\times/\overline{(F_\infty^\times)^\circ F^\times} = \pi_0(\mathbf{A}_F^\times/F^\times) \cong \mathrm{Gal}(F^{\mathrm{ab}}/F)$. In particular, every finite order (i.e. weight 0 algebraic) Hecke character gives rise to a finite order character $\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathbf{C}^\times \cong^{t^{-1}} \overline{\mathbf{Q}}_p^\times$. For general algebraic Hecke characters, one notes that the finite part of the Hecke character must always land in some number field E and the induced algebraic character of $F_\infty^\times$ descends to an algebraic character $(\mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m)_E \to \mathbf{G}_{m,E}$. This allows for exchanging the algebraic character at ∞ to an algebraic character at p on the cost of landing in a p-adic coefficient field.

We note that local-global compatibility away from p then follows from compatibility between local and global Class Field Theory. At $\ell = p$ one further needs to check that the induced local character is potentially semistable (in fact potentially crystalline) with the right Hodge–Tate weights to verify the predicted compatibility.

We mention that the construction is even proved to factor through the category of motives, matching algebraic Hecke characters with rank 1 motives generated by potentially CM abelian varieties and Artin motives (cf. [Sch88], Chapter I, §4, [Far11], Proposition 7.4).

1.2.2 The case of $\mathrm{GL}_2/\mathbf{Q}$

The next group for which significant progress has been made is $\mathrm{GL}_2/\mathbf{Q}$. However, already in this case, Conjecture 1.1.18 is far from known in its entirety. The most developed case is that of *regular* algebraic cuspidal automorphic representations.

Definition 1.2.1. Let F be a number field, and $n \geq 1$ be an integer. An algebraic automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is called *regular* if, for every place $w \mid \infty$ of F, if we write $\mathrm{rec}(\pi_w \otimes |\det|_{\mathbf{C}}^{\frac{1-n}{2}})|_{\mathbf{C}^\times} = \oplus_{i=1}^n \chi_{p_{w,i},q_{w,i}}$[12], the integers $p_{w,i}$ are all distinct for $i = 1, ..., n$. Equivalently, π is regular if π_∞ has the same infinitesimal character as a highest weight representation of $(\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)_{\mathbf{R}}$.

We say that a regular algebraic automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is of weight $\lambda \in (\mathbf{Z}^n)^{\mathrm{Hom}(F,\mathbf{C})}$ if the infinitesimal character of π_∞ coincides with that of the *dual* of the highest weight representation of $(\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)_{\mathbf{R}}$ of highest weight λ.

Example 1.2.2. Given a cuspidal newform of weight $k \geq 2$, the infinitesimal character of $\pi(f)_\infty^{\mathrm{coh}}$ coincides with that of $(\mathrm{Sym}^{k-2}\mathbf{C}^2)^\vee$. Conversely, every regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ is of the form $\pi(f)^{\mathrm{coh}} \otimes |\det|_{\mathbf{C}}^m$ for a unique normalised cuspidal newform of some weight $k \geq 2$ and an integer $m \in \mathbf{Z}$.

---

[12] See Remark 1.1.3 for the notation.

By work of plenty of mathematicians, we have the following result towards Conjecture 1.1.18.

Theorem 1.2.3. *For any choice of field isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, there is an injective map of sets of isomorphism classes*

$$\left\{ \begin{array}{l} \textit{regular algebraic cuspidal automorphic} \\ \textit{representations of } \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}) \end{array} \right\} \xrightarrow{r_t(-)} \left\{ \begin{array}{l} \textit{irreducible odd geometric} \\ \textit{2-dimensional } p\textit{-adic} \\ \textit{Galois representations of } \mathbf{Q} \\ \textit{with distinct Hodge–Tate weights} \end{array} \right\}$$

satisfying local-global compatibility at every rational prime. Moreover, it is known to be a bijection as long as $p \geq 5$.

Let us indicate the construction of $r_t(-)$ in the weight 0 case (i.e. in the case of weight 2 cuspidal newforms), already going back to Eichler–Shimura. The first important point is that the Eichler–Shimura isomorphism tells us that, for every weight 0 cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$, π_f sits in

$$H := \varinjlim_{K} H^1(X_K(\mathbf{C}), \mathbf{C})$$

as a $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty})$-equivariant direct summand with multiplicity 2 where $X_K/\mathbf{Q}$ is the compactified modular curve. On the other hand, X_K being a smooth algebraic variety over $\mathbf{Q}$ already, $H \otimes_{\mathbf{C},t^{-1}} \overline{\mathbf{Q}}_p$ is naturally a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty})$-module. Therefore, we get a 2-dimensional Galois representation of $\mathbf{Q}$

$$r_t(\pi) := \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty})}(\pi_f, H \otimes_{\mathbf{C},t^{-1}} \overline{\mathbf{Q}}_p).$$

Now enters the Eichler–Shimura relation to show that $r_t(\pi)$ is the right candidate for verifying Conjecture 1.1.1. More precisely, the Eichler–Shimura relation tells us that, for $N \geq 3$ and $\ell \nmid Np$, the geometric Frobenius acting on $H^1_{\text{ét}}(X_{K_1(N),\overline{\mathbf{Q}}}, \overline{\mathbf{Q}}_p)$ satisfies the polynomial

$$X^2 - T_{\ell,1}X + \ell T_{\ell,2}.$$

In particular, $r_t(\pi)(\mathrm{Frob}_\ell)$ is killed by $X^2 - a_{\ell,1}(\pi)X + \ell a_{\ell,2}(\pi)$ and with some extra work (see for instance [Con10]) one sees that it is in fact its characteristic polynomial.

The construction for general (dominant) weight $(k-2, 0)$ is due to Deligne and goes along similar lines but considering cohomology with coefficients in the local system attached to the algebraic representation $\mathrm{Sym}^{k-2}\mathbf{C}^2$ of $\mathrm{GL}_2/\mathbf{Q}$ and the étale local system attached to its p-adic variant (see [Del71]).

Local-global compatibility of $r_t(\pi)$ away from p is a theorem of Carayol [Car86]. The fact that the modular Galois representation is de Rham with the right Hodge–Tate weights follows from the de Rham comparison theorem and Falting's p-adic Eichler–Shimura decomposition [Fal87]. The rest of the compatibility at p is a result of Saito [Sai97] reducing the question to Carayol's work.

Finally, the fact that the associated Galois representations are irreducible is a theorem of Ribet (cf. [Rib77], Theorem (2.3)).

Remark 1.2.4. The modularity part of Theorem 1.2.3 has a lot more complicated history starting with the work of Wiles [Wil95] and Taylor–Wiles [TW95]. They proved that in the residually irreducible case suitable local-global compatibility and residual modularity together imply modularity in the Barsotti–Tate case by introducing the so-called Taylor–Wiles method. Their argument was generalised to certain potentially Barsotti–Tate and crystalline cases with small weight as well[13] (see for instance [Dia96], [Dia97], [CDT99], [DFG04]). Kisin later introduced a modification [Kis09a] making it flexible enough to succeed even in cases when the local deformation ring at p is not formally smooth.

More general modularity results (in the residually irreducible case) were proved by Kisin [Kis09b] and Emerton [Eme11].

In the residually reducible case there is the work of Skinner–Wiles [SW99] for ordinary representations. In general, it was established by Pan in his thesis work [Pan22].

However, the proof of all of the more uniform results ([Kis09b], [Eme11], [Pan22]) rely on the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. In particular, these arguments as of now have no chance of being generalised beyond the case of GL_2/F with p being completely split in F.

Nevertheless, we mention that recently a major breakthrough has been made by Pan [Pan23] that reproves the modularity results of [Eme11] without the use of the p-adic local Langlands correspondence giving some hope for generalisations to other groups.

Remark 1.2.5. We also note that, in light of Example 1.1.4, the irregular algebraic automorphic representations all come either from weight 1 modular forms or algebraic Maass forms. In the case of the latter very little is known. In the case of the former, even though these forms can only be found in coherent cohomology, Deligne–Serre [DS74] showed that one can associate Galois representations to them by using the Hasse invariant to find mod p congruences with higher weight modular forms. All of these modular Galois representations will be odd and have finite image and in particular have Hodge–Tate weights $(0, 0)$. For modularity results in this direction see [Pan20], Theorem 1.0.5 and the references in *loc. cit.* Remark 1.0.6.

---

[13] All of these cases are so that the local deformation rings at p are formally smooth integrally.

1.2.3 GL_n/F, the self-dual case

As we saw, in the case of $\mathrm{GL}_2/\mathbf{Q}$ the main point that allowed us to have a good grip on the problem of reciprocity was the existence of a tower of Shimura varieties (that of the modular curves) with Betti cohomology groups realising our automorphic representation. In particular, in light of the Eichler–Shimura isomorphism, we had to assume that our algebraic automorphic representation satisfies a certain regularity condition. This is not special to $\mathrm{GL}_2/\mathbf{Q}$.

To make this more precise, let $G/\mathbf{Q}$ be a connected reductive group. Then it is a theorem of Matsushima when G is compact mod centre at infinity and of Franke ([Fra98], Theorem 18) for general G that a cuspidal automorphic representation π of $G(\mathbf{A}_{\mathbf{Q}})$ contributes to the Betti cohomology of the associated locally symmetric spaces if and only if π is cohomological in the sense of [BG19], Definition 7.2.1. On the other hand, for $G = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$, π is cohomological if and only if it is regular algebraic.

In particular, in the hope of generalising the construction of Galois representations to GL_n/F, we restrict ourselves to regular algebraic automorphic representations. However, even after making this assumption, another issue is waiting to be dealt with: The locally symmetric spaces associated with GL_n/F typically do not give rise to Shimura varieties, only possess the structure of a real manifold. One can already see this happen for instance in the case of $\mathrm{GL}_2/\mathbf{Q}(\sqrt{-2})$ where the induced locally symmetric spaces are unions of Bianchi manifolds, having real dimension 3 leaving no chance for them to carry a complex algebraic structure.

The key idea that still allows for some progress is to involve another major player from the Langlands's programme, functoriality. Given an imaginary CM number field F with totally real subfield F^+ and complex conjugation c, we can consider certain n-dimensonal quasi-split unitary group U/F^+. These groups do admit Shimura data and so bring us one step closer to carry out the strategy of $\mathrm{GL}_2/\mathbf{Q}$ for their automorphic representations.

On the other hand, they are related to general linear groups by being forms thereof. Namely, we have an identification $\mathrm{Res}_{F/F^+}(U \times_{F^+} F)(\mathbf{A}_{F^+}) = U(\mathbf{A}_F) \cong \mathrm{GL}_n(\mathbf{A}_F)$. Now enters functoriality in the form of automorphic quadratic base change for U. A theorem of Clozel–Labesse[14] asserts that every cuspidal cohomological automorphic representation Π of $U(\mathbf{A}_{F^+})$ admits a base change $\mathrm{BC}_{F/F^+}(\Pi)$, an automorphic representation of $\mathrm{Res}_{F/F^+}(U \times_{F^+} F)(\mathbf{A}_{F^+})$, and if θ is the automorphism $1 \otimes c$ of $\mathrm{Res}_{F/F^+}(U \times_{F^+} F)$, then cohomological cuspidal automorphic representations π of $\mathrm{Res}_{F/F^+}(U \times_{F^+} F)(\mathbf{A}_{F^+})$ satisfying $\pi \cong \pi \circ \theta$ are all of the form $\mathrm{BC}_{F/F^+}(\Pi)$.[15] The automorphism θ under the isomorphism $\mathrm{Res}_{F/F^+}(U \times_{F^+} F)(\mathbf{A}_{F^+}) \cong \mathrm{GL}_n(\mathbf{A}_F)$ is given by

---

[14] To be more precise, their unconditional results only hold under some mild constraints that in practice can be be assumed to be satisfied.

[15] Note that Π is typically not unique here.

$g \mapsto^t (g^{-c})$ motivating the following definition.

Definition 1.2.6. Let F be an imaginary CM number field with complex conjugation c, and π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. We call π conjugate self-dual if there is an isomorphism

$$\pi^\vee \cong \pi \circ c$$

as automorphic representations.

In particular, for conjugate self-dual regular algebraic cuspidal automorphic representations π one can at least find their system of Hecke eigenvalues in the cohomology of unitary Shimura varieties (that does admit a Galois action) by considering their descent to the corresponding unitary group.

One then would like to pick a suitable unitary group U/F^+ of rank n, set $G := \mathrm{Res}_{F^+/\mathbf{Q}} U$, consider the ($\xi^\vee$-cohomological) descent Π of π to U, and define $r_t(\pi)$ to be the Galois module

$$\mathrm{Hom}_{G(\mathbf{A}_{\mathbf{Q}}^\infty)}(\Pi^\infty, \varinjlim_{K \subset G(\mathbf{A}_{\mathbf{Q}}^\infty)} H^d(\mathrm{Sh}_K(\mathbf{C}), W_{t,\xi}))$$

where $\{\mathrm{Sh}_K\}_K$ is a tower of Shimura varieties attached to G, d is the middle degree for the Shimura variety,[16] and $W_{t,\xi}$ is the p-adic local system induced by the algebraic representation ξ of $G_{\mathbf{C}}$. Guided by the Kottwitz conjecture, the quasi-split unitary group with signature $(1, n-1)$ at one, and $(0, n)$ at the rest of the archimedean places is the right group to pick(provided that it exists).[17] The existence of such a U can be ensured as long as n and $[F^+ : \mathbf{Q}]$ are not simultaneously even. In the latter case one goes around this issue by working with the isobaric sum $\pi \boxplus \chi$ for some suitable self-dual character χ.

A subtlety to be pointed out is that to check that the characteristic polynomials of the Frobenii are the right ones one has to employ something significantly more complicated than the Eichler–Shimura relation called the Langlands–Kottwitz–Rapoport method. This is a systematic programme that expresses the L-function of the Shimura variety in terms of automorphic representations for the unitary group and its endoscopic groups. For how this works in the case of the modular curve (and to compare its nature to that of the Eichler–Shimura relation) the reader can consult [Sch13] and for a survey on the method see [Zhu20].

---

[16] The reason for considering only the middle degree originates in the combination of the expectation that the tempered spectrum is concentrated in the middle degree for Shimura varieties and the Ramanujan–Petersson conjecture predicting that cuspidal (globally generic) automorphic representations should all be tempered at every place.

[17] The corresponding Shimura varieties are often referred to as Harris–Taylor Shimura varieties as they were famously used in [HT01] to construct the first examples of $r_t(\pi)$ in the self-dual case in great generality and use them to prove the local Langlands correspondence for GL_n.

Without attempting to explain the many complications one has to face to make this strategy work, we point the reader to the excellent surveys [Shi11], [CS23], §2.2 on the subject and the references therein.

As the culmination of the effort of several mathematicians starting with the work of Harris–Taylor [HT01] we have the following theorem.

Theorem 1.2.7. *Let F be an imaginary CM field with complex conjugation c. Let π be a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ. Then for any isomorphism $t : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ there is a continuous semisimple Galois representation*

$$r_t(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

satisfying the following conditions:

i. We have an isomorphism $r_t(\pi)^c \cong r_t(\pi)^\vee(1-n)$.

ii. For each p-adic place v of F, $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is de Rham and, for each embedding $\iota : F_v \hookrightarrow \overline{\mathbf{Q}}_p$, the labelled ι-Hodge–Tate weights are given by

$$\mathrm{HT}_\iota(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}) = \{\lambda_{t\circ\iota,n} < \lambda_{t\circ\iota,n-1} + 1 < \ldots < \lambda_{t\circ\iota,1} + n - 1\}.$$

iii. For each finite place v of F, we have

$$\mathrm{WD}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F-ss} \cong \mathrm{rec}_{F_v}(t^{-1}\pi_v|\det|_v^{\frac{1-n}{2}}).$$

Remark 1.2.8. We note that Theorem 1.2.7 is not concerned with irreducibility of the associated Galois representations and indeed, it is not known in general. Nevertheless, $r_t(\pi)$ is proved to be irreducible as long as π_v is square integrable for some finite place $v \nmid p$ of F (cf. [TY07], Corollary B).

On the other hand, besides the question of irreducibility, Theorem 1.2.7 completely settles Conjecture 1.1.1, Conjecture 1.1.9 and Conjecture 1.1.12 under the extra regularity and self-duality assumption.

Even though in this self-dual setup the construction of Galois representations satisfying full local-global compatibility is established, the problem of automorphy is still far from solved. However, the Taylor–Wiles–Kisin method generalises well ([CHT08]) and allows for a great source of automorphy lifting theorems. For some of the sharper results of this kind (in the residually irreducible case) the reader can look at [Tho12], Theorem 7.1, and [Bar+14], Theorem B. Moreover, in [Bar+14] it is shown that using their automorphy lifting results, one can deduce potential automorphy (i.e. automorphy after possible base change along some finite Galois extension L/F) of certain irreducible geometric Galois representations.

1.2.4 GL_n/F, beyond the self-dual case

Relaxing the self-duality condition in Theorem 1.2.7 proved to require a genuinely new and more indirect construction. As we saw, the self-duality condition was key to find the corresponding systems of Hecke eigenvalues in the étale cohomology of Shimura varieties. However, as was made precise in [JT], certain non-self-dual geometric irreducible Galois representations (under some standard conjectures) cannot be found in the cohomology of Shimura varieties.

Another new phenomenon that comes with going beyond the self-dual case is the presence of genuine "torsion automorphic forms".[18] We elaborate on the significance of extending Langlands reciprocity to torsion classes at the end of the subsection.

Construction of Galois representations

The first major step in establishing reciprocity in this generality was made in [Har+16] and shortly after independently in [Sch15]. The authors constructed $r_t(\pi)$ for arbitrary regular algebraic cuspidal automorphic representation π of GL_n over imaginary CM fields. Let us now briefly sketch their argument. We will follow the construction of [Sch15] as this is the perspective on which this thesis builds on.

In light of [JT] perhaps not surprisingly, the construction goes via p-adic congruences and reduces the question to the self-dual case. The two main ideas are the introduction of a robust Eisenstein series construction for the Betti cohomology of locally symmetric spaces and a novel automorphic result concerning congruences between automorphic forms appearing in completed cohomology and classical cusp forms for groups giving rise to (Hodge type) Shimura varieties. We start by explaining the first of these on a regular algebraic cuspidal automorphic representation π of GL_n/F for some imaginary CM field F/F^+.

Consider the quasi-split unitary group $\widetilde{G} = U(n,n)/F^+$ associated with the imaginary quadratic extension F/F^+, a form of GL_{2n}. The group $\widetilde{G}$ has a maximal parabolic subgroup P (called the Siegel parabolic) with Levi quotient $\mathrm{Res}_{F/F^+}\mathrm{GL}_n$. From this point on we impose as a blanket assumption the existence of an imaginary quadratic subfield $F_0 \subset F$.[19] The end product of Scholze's construction is, for every integer $m \geq 1$, an "Eisenstein series" for $\widetilde{G}$ associated with π mod p^m appearing in the Betti cohomology of the

---

[18] To support this sentence mathematically, we invite the reader to look at for instance [BV13] where, in the defect 1 case (such as the case of GL_2 over an imaginary quadratic field), they prove exponential growth of torsion in the cohomology of locally symmetric spaces as we deepen the level while the space of characteristic 0 automorphic forms contributing to the cohomology stays small.

[19] This is only a technical point ensuring that we have access to unconditional base change along F/F^+ for $\widetilde{G}$ cf. [Shi14].

corresponding unitary Shimura variety. To make this more precise, set T to be the set of finite places v of F where either π ramifies or v is p-adic. Then π is determined by the associated system of Hecke eigenvalues

$$\psi : \mathbf{T}^T := \mathbf{Z}[\mathrm{GL}_n(\mathbf{A}_F^{\infty,T})//\mathrm{GL}_n(\widehat{\mathcal{O}}_F^{\infty,T})] \to \overline{\mathbf{Z}}_p.$$

Moreover, $\psi_m := \psi \mod p^m$ factors through the Hecke action on the Betti cohomology group $H^i(X_K, \mathbf{Z}/p^m)$ of the GL_n/F-locally symmetric space for some integer $i \geq 0$ and level subgroup $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$. Then, abstractly, the "Eisenstein series" for $\widetilde{G}$ induced by ψ_m is the system of Hecke eigenvalues

$$\widetilde{\psi_m} : \widetilde{\mathbf{T}}^T := \mathbf{Z}[\widetilde{G}(\mathbf{A}_{F^+}^{\infty,T})//\widetilde{G}(\widehat{\mathcal{O}}_{F^+}^{\infty,T})] \to \overline{\mathbf{Z}}_p/p^m$$

obtained as the mod p^m reduction of the system of Hecke eigenvalues attached to $\mathrm{Ind}_{P(\mathbf{A}_{F^+}^{\infty,T})}^{\widetilde{G}(\mathbf{A}_F^{\infty,S})}\pi_f^T$. Scholze's observation is that $\widetilde{\psi_m}$ always appears in the compactly supported (degree $i+1$) or ordinary (degree i) Betti cohomology of the unitary Shimura variety with p-torsion coefficients.

The proof of this fact is a topological argument making use of the Borel–Serre boundary of the unitary Shimura variety. Namely, the Borel–Serre boundary admits a stratification labelled by $\widetilde{G}(F^+)$-conjugacy classes of (proper) parabolic subgroups with the open strata corresponding to conjugacy classes of *maximal* parabolic subgroups (cf. [NT16], §3.1.2). The stratum corresponding to P contains as an open subspace a torus bundle over X_K (cf. [Sch15], §V.2, [Clo22]). The point is then that the natural map induced between the cohomology of the Borel–Serre boundary and the (interior) cohomology of X_K descends the (unnormalised) Satake transform $\mathcal{S} : \widetilde{\mathbf{T}}^T \to \mathbf{T}^T$. Moreover, the Satake transform has the property that $\widetilde{\psi_m}$ is given by precomposing ψ_m with $\mathcal{S}$. To conclude, one relates the cohomology of the Borel-Serre boundary to the cohomology of the Shimura variety via the usual excision long exact sequence.

The other ingredient is the main automorphic result of [Sch15] whose proof occupies most of *loc.cit.* It states that every system of Hecke eigenvalues appearing in the compactly supported completed cohomology of a Hodge type Shimura variety (like the unitary Shimura varieties in our consideration) is lifted by the Hecke eigensystem of a finite linear combination of classical cuspidal eigenforms of the corresponding group.[20] As for its proof, as mentioned this is the main result of [Sch15] using the full strength of the most significant innovation of the paper: the introduction and analysis of perfectoid Shimura varieties and their Hodge–Tate period map.

One can now construct $r_t(\pi)$ with relative ease. The Eisenstein series construction tells us that, for every integer $m \geq 1$, we can find $\widetilde{\psi_m} = \psi_m \circ \mathcal{S}$ in the compactly supported cohomology of the Shimura variety, and so also

---

[20] For a precise formulation, see *loc. cit.* Theorem IV.3.1.

in its compactly supported completed cohomology. In particular, $\widetilde{\psi}_m$ is glued from a finite collection of classical cusp forms for $\widetilde{G}(\mathbf{A}_{F^+})$. By making use of the trace formula and Theorem 1.2.7, we can attach to these cusp forms $2n$-dimensional Galois representations of F.[21] Since $\widetilde{\psi_m}$ is glued from these forms, we obtain a $2n$-dimensional determinant $D_{\widetilde{\psi_m}} : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Z}}_p/p^m$ (in the sense of Chenevier [Che09]) matching Frobenii away from T with the Satake parameters of $\widetilde{\psi}_m$. We can pass to the limit over m to obtain $D_{\widetilde{\psi}} : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Z}}_p$, the $2n$-dimensional determinant associated with $\widetilde{\psi}$. We can then look at the unique semisimple Galois representation $r_{\widetilde{\psi}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$ associated with it. By unravelling the effect of the Satake transform on the Satake parameters, we further obtain that, for $v \notin T$,

$$(r_{\widetilde{\psi}}|_{W_{F_v}})^{\mathrm{ss}} \cong t^{-1}\mathrm{rec}_{F_v}(\pi_v|\det|^{\frac{1-n}{2}}) \oplus t^{-1}\mathrm{rec}_{F_v}(\pi_v|\det|^{\frac{1-n}{2}})^{\vee,c}(1-2n) \quad (1.2.1)$$

where $(-)$ denotes the usual twist by the p-adic cyclotomic character. Then the expectation would be to obtain a global factorisation of $r_{\widetilde{\psi}}$ of this kind as then the first factor was forced to be $r_t(\pi)$. This is achieved by a twisting argument. Namely, one notices that twisting π by $|\det|^M$ for some integer M and looking at the induced system $\psi(M)$ allows us to tell apart the two factors of 1.2.1 for $r_{\widetilde{\psi(M)}}$ for any fixed place $v \notin T$. Then an elementary group theory argument shows (cf. [Har+16], §7) that, for any sufficiently large M, $r_{\widetilde{\psi(M)}} \cong r_t(\pi|\det|^M) \oplus r_t(\pi|\det|^M)^{\vee,c}(1-2n)$. The construction of $r_t(\pi)$ is then finished by setting $r_t(\pi) := r_t(\pi|\det|^M)(-M)$.

Torsion automorphic Galois representations

As was already hinted at, the cohomology groups $H^*(X_K, \mathbf{Z}/p^m)$ can contain Hecke eigenclasses that do not lift to characteristic 0 (cf. [BV13]). One is tempted to study these Hecke eigenclasses as well.

Definition 1.2.9. Given a system of Hecke eigenvalues $\psi : \mathbf{T}^T \to A$ valued in some local Artinian $\mathbf{Z}_p$-algebra, we call ψ automorphic if it factors through the natural Hecke action on $H^*(X_K, \mathbf{Z}/p^m)$ for some level subgroup $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ and integer $m \geq 1$.

The novelty of Scholze's construction is that it deals with arbitrary automorphic Hecke eigensystems.

Theorem 1.2.10 (Scholze). *Given an automorphic system of Hecke eigenvalues $\psi : \mathbf{T}^T \to A$, there is an associated n-dimensional determinant D_ψ :*

---

[21] What we mean by this is that the corresponding cuspidal automorphic representations base change along F/F^+ to (isobaric sums) of regular algebraic conjugate self-dual cuspidal automorphic representations of GL_m/F for some integers $m \leq 2n$ (cf. [Shi14]). These then admit associated Galois representations with full local-global compatibility above split places of F^+ thanks to Theorem 1.2.7.

$\mathrm{Gal}(\overline{F}/F) \to A'$ *unramified outside* T*, and matching characteristic polynomial of* Frob_v *with Satake parameters at* $v \notin T$*. Here* A' *is a local Artinian quotient* $A \to A'$ *with kernel* $I \subset A$ *having nilpotence degree bounded by an integer only depending on* n *and* $[F : \mathbf{Q}]$*.*

In particular, every automorphic mod p *Hecke eigensystem* $\overline{\psi} : \mathbf{T}^T \to \overline{\mathbf{F}}_p$ *admits an associated continuous semisimple Galois representation* $r_{\overline{\psi}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$*.*

Remark 1.2.11. From now on we will largely ignore the subtle point that the associated determinant is only defined up to a nilpotent ideal and will simply write A' for the quotient. However, whenever we appeal to this notation, the nilpotence degree of the kernel of $A \to A'$ will always satisfy the property of stated in the theorem.

Note that, as we have seen, the nilpotent ideal causes no issues for applications to classical reciprocity.

We also note that Newton–Thorne [NT16] has proved that the bound on the nilpotence degree can always be chosen to be 4 in Scholze's theorem. However, this bound is only known for r_ψ without knowing local-global compatibility and in the soon to be discussed local-global compatibility results the nilpotence degree might be larger than this.

Since the deformation theory of absolutely irreducible determinants coincides with the deformation theory of the corresponding Galois representations, the following class of automorphic Hecke eigensystems is of special interest.

Definition 1.2.12. Given an automorphic Hecke eigensystem $\psi : \mathbf{T}^T \to A$, we call it non-Eisenstein if the Galois representation associated with the corresponding mod p system of Hecke eigenvalues $\overline{\psi} : \mathbf{T}^T \to A/\mathfrak{m}_A$ is absolutely irreducible.

For such automorphic Hecke eigensystems, we will denote by r_ψ the Galois representation associated with D_ψ.

Local-global compatibility at $\ell \neq p$

With the non-self-dual automorphic Galois representations of [Har+16] and [Sch15] at our disposal, the question of local-global compatibility in this generality was waiting to be answered. Away from p in characteristic 0 the problem was almost completely solved in the thesis work of Varma [Var14]. More precisely, she proved that at any finite place $v \nmid p$ the isomorphism of Conjecture 1.1.9 holds up to semisimplification and was able to bound the nilpotence of the monodromy of $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ by that of π_v. She achieves the semisimple local-global compatibility by keeping track of the local Hecke action at ramified non-p-adic places in the construction of [Har+16] proving that $r_{\widetilde{\psi}}$ satisfies 1.2.1 at these places as well.[22] Then a refinement of the "separation argument" of the previous paragraph allows her to conclude local-global compatibility.

For torsion automorphic Hecke eigensystems the best known results away from p are limited to cases relevant to the Taylor–Wiles method (see [All+23], §3, [MT22]).

Remark 1.2.13. In light of Varma's result, to completely settle Conjecture 1.1.9 in the regular algebraic case over imaginary CM fields, the only thing left to prove is that the monodromy operators of $r_t(\pi)$ and π_v match for $v \nmid p$. Such results have been obtained in [AN21], [Yan21] and [Mat24] with the general strategy of first establishing a fine enough potential automorphy result to reduce the problem to Varma's result (see the introduction of [AN21]).

Local-global compatibility at $\ell = p$: New congruences between Eisenstein series and cusp forms

The question of local-global compatibility at $\ell = p$ is more subtle. In [Sch15] the Eisenstein series $\widetilde{\psi_m}$ are realised in completed cohomology allowing for no control over the level and weight of the cusp forms we find congruences with. Similarly, in [Har+16] $\widetilde{\psi}$ is found in some space of p-adic cusp forms leading to the same issue. Therefore, to tackle our problem, we need a source of congruences with better control at p.

Such a source has been provided by the novel work of Caraiani–Scholze [CS19]. The existence of these finer congruences forces us to impose additional conditions on our mod p system of Hecke eigenvalues.

Definition 1.2.14. Given a continuous Galois representation

$$\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_h(k)$$

for some finite field extension $k/\mathbf{F}_p$, we call $\overline{r}$ *decomposed generic* if there is a prime $\ell \neq p$ splitting completely in F such that for every place $v \mid \ell$ in F, $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is unramified and the eigenvalues $\alpha_1, ..., \alpha_n$ of $\overline{r}(\mathrm{Frob}_v)$ satisfy $\alpha_i/\alpha_j \neq \ell$ for $i \neq j$. Given a mod p system of Hecke eigenvalues $\overline{\psi}$, we further call $\overline{\psi}$ decomposed generic if $r_{\overline{\psi}}$ satisfies that condition.

We further need the notion of an irreducible algebraic representation ξ of the group $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\mathbf{C}}$ being CTG ("cohomologically trivial for $G = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$") (cf. [All+23], Definition 4.3.5). Instead of giving the precise

---

[22] More precisely, [Har+16] in fact realises the finer Eisenstein series given by $\mathrm{Ind}_{P(\mathbf{A}_{F^+}^{\infty,p})}^{\widetilde{G}(\mathbf{A}_{F^+}^{\infty,p})}\pi_f^p$ as a p-adic cusp form and Varma tracks the Hecke action on this Eisenstein series at ramified places as well and matches it with that of the congruent cusp forms.

definition, we only mention that it ensures that there are no (classical) $\xi^\vee$-cohomological Eisenstein-series for $\widetilde{G}$ coming from regular algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$.

Now set $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$ to be a (sufficiently small) compact open subgroup. From now on, $E/\mathbf{Q}_p$ will denote a suitably large finite field extension with ring of integers $\mathcal{O}$ and a choice of uniformiser ϖ. In particular, E contains the image of all embeddings $F \hookrightarrow \overline{\mathbf{Q}}_p$. Let ξ be an irreducible algebraic representation of $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\mathbf{C}} \cong \prod_{w|\infty} \mathrm{GL}_{2n}/\mathbf{C}$. To ξ one can attach an $\mathcal{O}$-local system $\mathcal{V}_\xi$ on the Borel–Serre boundary $\partial\widetilde{X}_{\widetilde{K}}$ of the Shimura variety with level $\widetilde{K}$. Then a vague form of the corollary of [CS19] to congruences between cusp forms and Eisenstein series is as follows.

Corollary 1.2.15 ([CS19],[Kos21],[All+23])**.** *Given an automorphic system of Hecke eigenvalues* $\psi : \mathbf{T}^T \to A$*. Assume that the Eisenstein series* $\widetilde{\psi} : \widetilde{\mathbf{T}}^T \to A$ *factors through the middle degree cohomology*

$$H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_\xi)$$

of the Borel–Serre boundary $\partial\widetilde{X}_{\widetilde{K}}$ *of the* $U(n,n)/F^+$*-Shimura variety of level* $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$*. Then* $\widetilde{\psi}$ *lifts to a finite linear combination of classical* $\xi^\vee$*-cohomological level* $\widetilde{K}$ *cuspidal eigenforms of* $\widetilde{G}$ *if the following are satisfied.*

i. The associated mod p Hecke eigensystem $\overline{\psi}$ *is non-Eisenstein,*

ii. the associated mod p Eisenstein series $\widetilde{\overline{\psi_m}}$ *is decomposed generic, and*

iii. the algebraic representation ξ *is CTG.*

Local-global compatibility at $\ell = p$: The Fontaine–Laffaille degree shifting argument

The use of the mentioned new congruences led to the breakthrough of the ten authors [All+23] on local-global compatibility at p. We now spell out the main innovation of [All+23], a new construction of Eisenstein series they call the *degree shifting argument.*

From now on, we further assume that p splits in our imaginary quadratic subfield $F_0 \subset F$. In particular, every p-adic place $\bar{v}$ of F^+ splits in F, and for a place $v \mid p$ of F we will write $\bar{v}$ for the place in F^+ lying below it.

We remind the reader that, for a highest weight vector λ for $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$, we are interested in proving instances of Conjecture 1.1.12 for cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ. To λ, one can associate an E-local system V_λ on X_K, and according to Franke's formula, the cuspidal part of the cohomology $H^*(X_K, V_\lambda)$, as a Hecke module, computes all such π's of level K. One can define an $\mathcal{O}$-lattice $\mathcal{V}_\lambda \subset V_\lambda$ and so more generally we are interested in local-global compatibility for non-Eisenstein Hecke eigensystems appearing in $H^*(X_K, \mathcal{V}_\lambda) = \varprojlim_{m\geq 1} H^*(X_K, \mathcal{V}_\lambda/\varpi^m)$. In light of Corollary 1.2.15, a first attempt in the hope of progress on such local-global compatibility is investigating the following problem.

Problem 1.2.16. Fix the following collection of data.

- An integer $0 \leq q \leq d-1$;[23]
- A place $\bar{v}$ of F^+ dividing p;
- A compact open subgroup $\widetilde{K}'_{\bar{v}} \subset \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$;
- An irreducible algebraic representation $\xi'_{\bar{v}} = (\xi'_\iota)_{\iota : F^+_{\bar{v}} \hookrightarrow \overline{\mathbf{Q}}_p}$ of $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\overline{\mathbf{Q}}_p}$.

Consider an automorphic Hecke eigensystem $\psi : \mathbf{T}^T \to A$ that satisfies assumption i) and ii) of Corollary 1.2.15. Assume that ψ appears in

$$H^q(X_K, \mathcal{V}_\lambda/\varpi^m)$$

for some integer m, with

- level K satisfying $\widetilde{K}'_{\bar{v}} \cap (\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c})) = K_v \times K_{v^c}$ and
- highest weight vector $\lambda = (\lambda_\iota)_{\iota : F \hookrightarrow \overline{\mathbf{Q}}_p}$ such that $\xi'_{\bar{v}}$ is associated with $(-w_0\lambda_{v^c}, \lambda_v)$.[24]

Can we find the associated Eisenstein series $\widetilde{\psi} : \widetilde{\mathbf{T}}^T \to A'$ in $H^d(\partial \widetilde{X}_{\widetilde{K}}, \mathcal{V}_\xi)$ for some CTG weight $\xi = (\xi_\iota)_{\iota : F^+ \hookrightarrow \overline{\mathbf{Q}}_p}$ such that, $\widetilde{K}_{\bar{v}} = \widetilde{K}'_{\bar{v}}$ and $\xi_{\bar{v}} = \xi'_{\bar{v}}$?

The authors of [All+23] then prove the following.

Proposition 1.2.17 (Fontaine–Laffaille degree shifting, [All+23], Propostion 4.4.1)**.** *Fix a place $v \cdot v^c = \bar{v}$ of F^+ dividing p, and an integer $\frac{d}{2} - 1 \leq q \leq d-1$. Assume that the following are satisfied.*

i. There is a place $\bar{v}' \neq \bar{v}$ in F^+ dividing p such that

$$\sum_{\bar{v}'' \neq \bar{v}, \bar{v}'} [F^+_{\bar{v}''} : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}]$$

where the sum runs over p-adic places of F^+;

ii. $p > n^2$;

---

[23] We note that the top degree for X_K is $d-1$.

[24] In particular, $(-w_0\lambda_{v^c}, \lambda_v)$ must be dominant. In applications to local-global compatibility this condition can always be achieved after twisting our Hecke eigensystem by a large enough power of the determinant character and from now on we will typically ignore this condition in our discussions.

iii. p is unramified in F.

Then Problem 1.2.16 has an affirmative answer as long as $U(\mathcal{O}_{F^+_{\bar{v}}}) \subset \widetilde{K}'_{\bar{v}}$.

Solving Problem 1.2.16 already has strong applications to local-global compatibility in the *"crystalline case"* i.e. when $\widetilde{K}_{\bar{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$. Namely, in characteristic 0, one obtains the following result towards Conjecture 1.1.12.

Corollary 1.2.18. *Let F be an imaginary CM field and assume that*

i. $p > n^2$;

ii. p is unramified in F.

Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ *such that*

- $\overline{r_t(\pi)}$ *is non-Eisenstein and decomposed generic;*
- $\pi^{\mathrm{GL}_n(\mathcal{O}_{F_v})}$ *and* $\pi^{\mathrm{GL}_n(\mathcal{O}_{F_{v^c}})}$ *are both non-zero.*

Then $r_t(\pi)$ is crystalline both at v and v^c with the expected Hodge–Tate weights.

Remark 1.2.19. If we further assumed that the coordinates of the highest weight vector $\lambda_{\bar{v}}$ are not too far apart in the sense of Fontaine–Laffaille theory so that integrally we had a workable notion of weight, one could obtain (with significantly more work) results on local global compatibility for genuine torsion classes as well (cf. [All+23], Theorem 4.5.1).

We note that this is our reason to name the method after Fontaine–Laffaille even though in characteristic 0 it is sufficient to handle arbitrary weights.

We now elaborate on the proof of Proposition 1.2.17. Note that the realisation of Eisenstein series appearing in [Sch15] is of no use for us when investigating Problem 1.2.16. Namely, if ψ appears in degree $i(\leq d-1)$, Scholze's construction realises $\widetilde{\psi}$ in the degree i boundary cohomology. This is due to the purely topological nature of his argument.

Therefore, in [All+23] it was key to change the perspective on the cohomology of locally symmetric spaces. Instead of working with cohomology, they work with the corresponding complexes in the relevant derived category, allowing for tools from representation theory. Favouring this point of view allows for a richer pool of congruences and more exotic realisations of Eisenstein series.

Sketch of proof of Proposition 1.2.17. Set ξ to be so that $\xi_{\bar{v}'}$ is arbitrary, $\xi_{\bar{v}} = \xi'_{\bar{v}}$ and ξ is trivial elsewhere. Denote by $\lambda'_{\bar{w}}$ the highest weight vector for

$\mathrm{Res}_{F_w/\mathbf{Q}_p}\mathrm{GL}_n \times \mathrm{Res}_{F_{w^c}/\mathbf{Q}_p}\mathrm{GL}_n$ corresponding to $\xi_{\bar{w}}$.[25] It suffices to prove that $\widetilde{\psi}$ can be found in $H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_\xi)$ for some $\widetilde{K}$ as in Problem 1.2.16 (cf. [All+23], Lemma 4.3.6).

Recall that $\partial\widetilde{X}_{\widetilde{K}}$ admits a (Hecke equivariant) stratification $\coprod_{\{Q\}}\partial\widetilde{X}^Q_{\widetilde{K}}$ with open strata corresponding to maximal parabolic subgroups. It is enough to find $\widetilde{\psi}$ in $H^d(\partial\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_\xi)$. This follows from the fact that ψ is non-Eisenstein and so $r_{\widetilde{\psi}}$ is of length 2 (see [All+23], Theorem 2.4.2)

To make some further reductions, we further recall that the strata of the Borel–Serre boundary admit *natural*[26] decompositions

$$\partial\widetilde{X}^Q_{\widetilde{K}} \cong \coprod_{g\in Q(\mathbf{A}^\infty_{F^+})\backslash\widetilde{G}(\mathbf{A}^\infty_{F^+})/\widetilde{K}} X^Q_{\widetilde{K}_{Q,g}} \tag{1.2.2}$$

into opens given by Q-locally symmetric spaces with level $\widetilde{K}_{Q,g} := Q(\mathbf{A}^\infty_{F^+}) \cap g\widetilde{K}g^{-1}$. In particular, $H^d(X^P_{\widetilde{K}_P}, \mathcal{V}_\xi)$ is a Hecke-equivariant direct summand of $H^d(\partial\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_\xi)$ where $\widetilde{K}_P := \widetilde{K}_{P,1}$. Therefore, it suffices to find $\widetilde{\psi}$ in the former cohomology group.

Now the key observation that allows us to shift the degree is that there is a Hecke-equivariant isomorphism

$$R\Gamma(X^P_{\widetilde{K}_P}, \mathcal{V}_\xi) \cong R\Gamma(X_{K'}, \mathcal{V}^U_\xi) \tag{1.2.3}$$

where $K' = \widetilde{K} \cap \mathrm{GL}_n(\mathbf{A}_F)$, and $\mathcal{V}^U_\xi$ is a certain complex of sheaves on $X_{K'}$ we define now. Set ξ° to be the $\mathcal{O}[\widetilde{K}_p]$-module associated with ξ (see [Ger18], Definition 2.3 for details).[27] Then $\mathcal{V}^U_\xi = \varprojlim_{m\geq 1}\mathcal{V}^U_\xi(m)$ where $\mathcal{V}^U_\xi(m)$ is the complex of sheaves on $X_{K'}$ associated with the complex

$$R\Gamma_{\mathrm{cont}}(U(\widehat{\mathcal{O}}_{F^+}), \xi^\circ/\varpi^m) \tag{1.2.4}$$

of $\mathcal{O}/\varpi^m[K']$-modules (where we inflate ξ° from $\widetilde{K}_p$ to $\widetilde{K}$).

The proof of 1.2.3 follows from combining two observations:

- $R\Gamma(X^P_{\widetilde{K}_P}, \mathcal{V}_\xi/\varpi^m)$ is the derived $\widetilde{K}_P$-invariants of (derived) completed[28] cohomology of P with ξ°-coefficients;

- The completed cohomology of U is trivial so the completed cohomology of P becomes isomorphic with that of its Levi quotient $\mathrm{Res}_{F/F^+}\mathrm{GL}_n$.

---

[25] Again, this means that $\xi_{\bar{w}}$ has highest weight vector $(-w_0\lambda_{w^c}, \lambda_w)$.

[26] The decomposition is natural in the sense that it is compatible with the natural maps between the corresponding strata for different level subgroups $\widetilde{K}$, so the $\widetilde{G}(\mathbf{A}^\infty_{F^+})$-action on the full tower permutes the members of the decomposition accordingly.

[27] Our local system $\mathcal{V}_\xi$ is associated with this particular representation.

[28] We complete with respect to every finite place.

To further simplify 1.2.4, we note that continuous $U^0_{\bar{w}} := U(\mathcal{O}_{F^+_{\bar{w}}})$-cohomology with $\mathcal{O}/\varpi^m$-coefficients is trivial for $\bar{w} \nmid p$. Moreover, $R\Gamma_{\mathrm{cont}}(U^0_{\bar{w}}, \xi^\circ_{\bar{w}}/\varpi^m)$ always admits $\mathcal{V}_{\lambda'_{\bar{w}}}[0]$ as a K'-equivarant direct summand. We therefore see that 1.2.4 further admits $R\Gamma_{\mathrm{cont}}(\prod_{\bar{v}'' \neq \bar{v}, \bar{v}'} U^0_{\bar{v}''}, \mathcal{O}/\varpi^m) \otimes \xi^\circ$ as K'-equivariant direct summand. In particular, $\mathcal{V}^U_\xi$ admits the corresponding complex of sheaves $\mathcal{V}^{\{\bar{v},\bar{v}'\}}_U \otimes \mathcal{V}_\xi \cong \varprojlim_{m \geq 1} \mathcal{V}^{\{\bar{v},\bar{v}'\}}_U(m) \otimes \mathcal{V}_\xi$ as a direct summand reducing our problem to finding ψ in the *hypercohomology* group

$$\mathbf{H}^d(X_{K'}, \mathcal{V}^{\{\bar{v},\bar{v}'\}}_U \otimes \mathcal{V}_{\lambda'})$$

for the highest weight vector λ' corresponding to ξ.

Lemma 1.2.20 (Key Lemma, [All+23], Lemma 4.2.2, Lemma 4.2.3)**.** *Under the assumptions ii) and iii) of Proposition 1.2.17, we have an isomorphism*

$$\mathcal{V}^{\{\bar{v},\bar{v}'\}}_U \cong \oplus_{j \geq 0} H^j(\mathcal{V}^{\{\bar{v},\bar{v}'\}}_U)[-j]$$

of complexes of sheaves on $X_{K'}$.

Moreover, for each j, $\mathcal{V}^j_U := H^j(\mathcal{V}^{\{\bar{v},\bar{v}'\}}_U)$ is a finite direct sum of explicit sheaves on $X_{K'}$ associated with highest weight representations for $\mathrm{Res}_{F^+/\mathbf{Q}}\mathrm{GL}_n$ *trivial at $\bar{v}$. Finally, assuming i) of Proposition 1.2.17, $\mathcal{V}^j_U$ is non-zero for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor$.*

Thanks to Lemma 1.2.20, we have a Hecke equivariant identification

$$\mathbf{H}^d(X_{K'}, \mathcal{V}^{\{\bar{v},\bar{v}'\}}_U \otimes \mathcal{V}_{\lambda'}) \cong \oplus_{j \geq 0} H^{d-j}(X_{K'}, \mathcal{V}^j_U \otimes \mathcal{V}_{\lambda'}) \tag{1.2.5}$$

and by picking j so that $d - j = q$, the problem is reduced to finding ψ in the corresponding direct summand. This is still not an obvious task as ψ is an eigensystem in cohomology with p-torsion coefficients, with level K and weight λ. However, the levels K, K' and weights $\mathcal{V}_\lambda$, $\mathcal{V}^j_U \otimes \mathcal{V}_{\lambda'}$ agree at $\bar{v}$ and so by an elaborate but elementary argument (cf. [All+23], Proposition 4.4.1) relying on congruences, repeatedly changing the weight and level at prime-to-$\bar{v}$ places allows one to conclude. $\square$

Local-global compatibility at $\ell = p$: The ordinary degree shifting argument

Even though Proposition 1.2.17 is sufficient to make significant progress on Conjecture 1.1.12 in the crystalline case, it is too coarse to obtain similarly strong results for more general level subgroups. Moreover, it is certainly not enough to track down the p-adic Hodge theoretic properties of torsion automorphic Galois representations once we leave the safety of the Fontaine–Laffaille setup.

However, there is another popular case where the integral theory is robust enough for satisfactory progress that was considered in [All+23]. This is the case of *ordinary* automorphic representations.

Fix a finite place $v|p$ of F and $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. For integers $0 \leq a \leq b$, $1 \leq b$, set $\mathrm{Iw}_v(a,b) \subset \mathrm{GL}_n(F_v)$ to be the Iwahori subgroup of matrices that are congruent to strict upper triangular matrices modulo ϖ_v^a and to upper triangular matrices modulo ϖ_v^b. Let $B_n = T_n N_n \subset \mathrm{GL}_n$ be the Borel subgroup of upper triangular matrices with its usual Levi decomposition. Set $u_v := \mathrm{diag}(\varpi_v^{n-1}, \varpi_v^{n-2}, ..., 1) \in T_n(F_v)$ for some fixed choice of uniformiser $\varpi_v \in \mathcal{O}_{F_v}$.

Definition 1.2.21. A regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ is called t-ordinary at v if $t^{-1}\pi_v^{\mathrm{Iw}_v(b,b)} \neq 0$ for some $b \geq 1$ and the (λ_v-normalised) U_v-operator

$$U_v := (-w_0\lambda_v(u_v)) \cdot [\mathrm{Iw}_v(b,b)u_v\mathrm{Iw}_v(b,b)]^{29}$$

acting on $t^{-1}\pi_v^{\mathrm{Iw}_v(b,b)}$ has an eigenvalue lying in $\overline{\mathbf{Z}}_p^\times$.

Denote by $\pi_v^{\text{t-ord}} \subset \varinjlim_{b\geq 1} t^{-1}\pi_v^{\mathrm{Iw}_v(b,b)}$ the generalised eigenspace of vectors with such U_v-eigenvalues.

Remark 1.2.22. As it turns out, $\pi_v^{\text{t-ord}}$ has a rather rich structure. Namely, we can rewrite $\varinjlim_{b\geq 1} t^{-1}\pi_v^{\mathrm{Iw}_v(b,b)}$ as $t^{-1}\pi_v^{N_v^0}$ for $N_v^0 = N(\mathcal{O}_{F_v})$ to see that it admits a smooth action of the open submonoid $T_v^+ := \{t \in T(F_v) \mid tN_v^0t^{-1} \subset N_v^0\} \subset T_n(F_v)$ via Hecke action of the double coset operators $[N_v^0 t N_v^0]$. Then $\pi_v^{\text{t-ord}}$ is in fact a T_v^+-equivariant direct summand. Moreover, it is the admissible smooth $\langle T_v^+, u_v^{-1}\rangle = T_n(F_v)$-representation we obtain by passing to the generalised eigenspace with eigenvalues with p-adic valuation that of $(w_0\lambda_v)(u_v)$. Finally, a simple argument (cf. [Ger18], Lemma 5.4) shows that in fact it must be a 1-dimensional $T_n(F_v)$-representation.

Then the expectation is that ordinary automorphic representations satisfy the following compatibility with the local Langlands correspondence.

Conjecture 1.2.23 (Ordinary local-global compatibility)**.** Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ that is t-ordinary at v. Write the 1-dimensional $T_n(F_v)$-representation $\pi_v^{\text{t-ord}}$ as $\chi_1 \otimes ... \otimes \chi_n$.[30] Then there is an isomorphism

$$r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)} \sim \begin{pmatrix} \Psi_1 & * & ... & * \\ 0 & \Psi_2 & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \Psi_n \end{pmatrix}$$

such that, for $1 \leq i \leq n$, $\Psi_i : \mathrm{Gal}(\overline{F}_v/F_v) \to \overline{\mathbf{Q}}_p^\times$ is the potentially crystalline character with Hodge–Tate weights $\lambda_{v,n-i} + i - 1$ and associated Weil–Deligne representation $\chi_i \circ \mathrm{Art}_{F_v}^{-1}$.

---

[29] By this expression we mean a scalar (namely $-w_0\lambda_v(u_v) \in \overline{\mathbf{Q}}_p^\times$) multiple of a double coset operator.

[30] Note that to formulate the conjecture it is essential that $\pi_v^{\text{t-ord}}$ is 1-dimensional.

The validity of Conjecture 1.2.23 lies in the fact that it can be deduced from Conjecture 1.1.12 (cf. [Tho15], Theorem 2.4). The above conjecture admits a very simple extension to the torsion setup (see [All+23], Theorem 5.5.1). Namely, in characteristic 0, the characters Ψ_i can be read off from the weight vector λ_v, the the U_v-eigenvalues and the Hecke action of the Iwasawa algebra that admit obvious integral analogues. Moreover, the notion of upper triangular Galois representations admits well-behaved and easy to control extension to the integral setup.

To set up such local-global compatibility, we certainly have to investigate Problem 1.2.16 in the case when $\widetilde{K}_{\bar{v}}$ is a deep enough Iwahori subgroup $\widetilde{\mathrm{Iw}}_{\bar{v}}(a,b) \subset \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}}) \cong \mathrm{GL}_{2n}(\mathcal{O}_{F_v})$. However, we need more than that. Namely, it is crucial for us to realise $\widetilde{\psi}$ in the ordinary part of the relevant middle degree boundary cohomology. Moreover, to be able to track the action of the Iwasawa algebra and the U_v-operator at v, we need to also include the corresponding eigenvalues in our Hecke eigensystem ψ and introduce the corresponding finer abstract Eisenstein series.

To make this more precise, we introduce the enlarged Iwasawa algebra

$$\mathcal{O}[[T_p^+]] := \mathcal{O}[[\mathrm{Res}_{F/\mathbf{Q}} T_n(\mathbf{Z}_p)]] \otimes_{\mathcal{O}[\mathrm{Res}_{F/\mathbf{Q}} T_n(\mathbf{Z}_p)]} \mathcal{O}[\prod_{v|p} T_v^+]$$

that contains both the usual Iwasawa algebra $\mathcal{O}[[\mathrm{Res}_{F/\mathbf{Q}} T_n(\mathbf{Z}_p)]]$ and the U_p-operators. We then introduce the ordinary Hecke algebra $\mathbf{T}^{T,\mathrm{ord}} := \mathbf{T}^T \otimes_{\mathcal{O}} \mathcal{O}[[T_p^+]]$. The enlarged Hecke algebra acts on cohomology $H^*(X_K, \mathcal{V}_\lambda/\varpi^m)$ with arbitrary Iwahori level at p with $u_v \in \mathcal{O}[[T_p^+]]$ playing the role of the (λ_v-normalised) U_v-operator. By passing to the maximal direct summand where the action of u_v is invertible for each $v|p$, we obtain the ordinary part of cohomology

$$H^*(X_K, \mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}}. \tag{1.2.6}$$

Definition 1.2.24. An ordinary (abstract) system of Hecke eigenvalues (for GL_n/F) is an algebra map

$$\psi_m^{\mathrm{ord}} : \mathbf{T}^{T,\mathrm{ord}} \to A$$

valued in some finite local Artinian $\mathbf{Z}_p$-algebras. It is called automorphic when it appears in 1.2.6 for some level and weight.

Define $\mathcal{O}[[\widetilde{T}_p^+]]$ similarly and set $\widetilde{\mathbf{T}}^{T,\mathrm{ord}} := \widetilde{\mathbf{T}}^T \otimes_{\mathcal{O}} \mathcal{O}[[\widetilde{T}_p^+]]$. To prove cases of ordinary local-global compatibility, one would like to find some suitably defined finer Eisenstein series $\widetilde{\psi^{\mathrm{ord}}}$ for $\widetilde{\mathbf{T}}^{T,\mathrm{ord}}$ in the ordinary middle degree boundary cohomology for $\widetilde{G}$ with the right weight and level as before.

In [All+23], to achieve this, the authors rely heavily on Emerton's representation theoretic point of view on ordinary parts (cf. [Eme10a], [Eme10b]) and the full strength of Hida theory in the Betti setting. After assuming that ψ^{ord} is non-Eisenstein, one reduces the question to finding $\widetilde{\psi^{\mathrm{ord}}}$ in the middle degree cohomology of $\partial\widetilde{X}^P_{\widetilde{K}}$ just as in the crystalline case. However, to take advantage of the representation theoretic features of the cohomology of the Borel–Serre boundary, one does not restrict further to $X^P_{\widetilde{K}_P}$. Namely, thanks to the internal structure 1.2.2 of $\partial\widetilde{X}^P_{\widetilde{K}}$, we have a $\widetilde{G}(\mathbf{A}^\infty_{F^+})$-equivariant isomorphism

$$\varinjlim_{\widetilde{K}} H^*(\partial\widetilde{X}^P_{\widetilde{K}}, \mathcal{O}/\varpi^m) \cong \mathrm{Ind}_{P(\mathbf{A}^\infty_{F^+})}^{\widetilde{G}(\mathbf{A}^\infty_{F^+})}(\varinjlim_{\widetilde{K}_P} H^*(X^P_{\widetilde{K}_P}, \mathcal{O}/\varpi^m)).$$

Moreover, the isomorphism already holds on the level of complexes computing these cohomology groups. In particular, by taking $\widetilde{K}^p$-invariants, passing to the right direct summand in the Mackey formula, and remembering that completed cohomology of P and $G = \mathrm{Res}_{F/F^+}\mathrm{GL}_n$ are naturally isomorphic, we are left with finding our Eisenstein series in

$$R^d\Gamma(\widetilde{\mathrm{Iw}}_p(b,c), (\mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)}\pi(K^p, m)) \otimes \mathcal{V}_\xi)^{\mathrm{ord}}. \tag{1.2.7}$$

Here $\pi(K^p, m)$ is the suitable complex of smooth $\mathcal{O}/\varpi^m[G(F_p^+)]$-modules that computes mod ϖ^m completed cohomology of tame level K^p for G and $F_p^+ := \prod_{\bar{v}|p} F_{\bar{v}}^+$.

Then the ordinary degree shifting is done by computing the parts of 1.2.7 corresponding to each "Bruhat stratum" of $\mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)}(-)$. To make this slightly more precise, recall that for each place $\bar{v}$ of F^+ dividing p we have the Bruhat stratification

$$\widetilde{G}(F_{\bar{v}}^+) = \coprod_{w \in W_{\bar{v}}^P} P(F_{\bar{v}}^+)w\widetilde{B}(F_{\bar{v}}^+)$$

labelled by the shortest representatives $W_{\bar{v}}^P \subset W(\widetilde{G}_{F_{\bar{v}}^+}, \widetilde{T}_{F_{\bar{v}}^+})$ of the quotient of Weyl groups $W(G_{F_{\bar{v}}^+}, T_{n,F_{\bar{v}}^+}) \setminus W(\widetilde{G}_{F_{\bar{v}}^+}, \widetilde{T}_{F_{\bar{v}}^+})$. The functor $\mathrm{Ind}_{P(F_{\bar{v}}^+)}^{\widetilde{G}(F_{\bar{v}}^+)}(-)$ admits a filtration by the subfunctors given by functions supported on the opens $\widetilde{G}_{\geq i} = \coprod_{\ell_{\bar{v}}(w) \geq i} P(F_{\bar{v}}^+)w\widetilde{B}(F_{\bar{v}}^+)$. Employing the properties of ordinary parts one shows that it yields a filtration of the functor $H^j(\widetilde{\mathrm{Iw}}_{\bar{v}}(a,b), \mathrm{Ind}_{P(F_{\bar{v}}^+)}^{\widetilde{G}(F_{\bar{v}}^+)}(-))^{\mathrm{ord}}$ with subquotients labelled by w and (up to an explicit twist depending on w) computed as the degree $j - \ell(w)$[31] ordinary cohomology of $\mathrm{Iw}_{\bar{v}}(a,b)$. A vague form of the end product of these observations for 1.2.7 is as follows.

Lemma 1.2.25 (Key Lemma, cf. [All+23] Proposition 5.3.8, Theorem 5.4.3)**.** *Fix* $w = (w_{\bar{v}}) \in \prod_{\bar{v}|p} W_{\bar{v}}^P$. *Then 1.2.7 admits*

$$H^{d-\ell(w)}(X_K, \mathcal{V}_{\lambda_w}/p^{m'}) \tag{1.2.8}$$

---

[31] Here $\ell(-)$ is the length function of the *absolute* Weyl group $W((\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\overline{\mathbf{Q}}_p})$ and w is viewed as an element of the absolute Weyl group via the natural map $W(\widetilde{G}_{F_{\bar{v}}^+}, \widetilde{T}_{F_{\bar{v}}^+}) \to W((\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\overline{\mathbf{Q}}_p})$.

as a $\widetilde{\mathbf{T}}^{T,\mathrm{ord}}$*-equivariant subquotient for* λ_w *an explicit weight vector depending on* w *and* ξ *equipped with the* $\widetilde{\mathbf{T}}^{T,\mathrm{ord}}$*-action via an explicit map*

$$\mathcal{S}^{\mathrm{ord},w,\xi} : \widetilde{\mathbf{T}}^{T,\mathrm{ord}} \to \mathbf{T}^{T,\mathrm{ord}}$$

depending only on w *and* ξ *and satisfying* $\mathcal{S}^{\mathrm{ord},w,\xi}|_{\widetilde{\mathbf{T}}^S} = \mathcal{S}$.

Remark 1.2.26. Without giving the definition of λ_w and the (w,ξ)-twisted transfer maps $\mathcal{S}^{\mathrm{ord},w,\xi}$ we only mention that their interaction is exactly the one expected by local-global compatibility.

Thanks to the above discussion (and Remark 1.2.26), one can settle ordinary local-global compatibility for any non-Eisenstein and decomposed generic ordinary ψ^{ord} that is automorphic of weight λ_w for some w and CTG ξ and appears in degree $\ell(w)$. With considerably more work relying on congruences, making use of the independence of weight property of Hida theory, and several tricks and elementary combinatorial arguments, the authors of [All+23] manage to reduce the case of arbitrary weight and cohomological degree to these cases.

The work of Caraiani–Newton

Most recently, Caraiani–Newton [CN23] went way further and relaxed the assumptions on p and the weights in the torsion crystalline local-global compatibility results of [All+23].

Theorem 1.2.27 (Caraiani–Newton)**.** *Let* $\psi : \mathbf{T}^T \to A$ *be a system of Hecke eigenvalues factoring through some* $H^*(X_K, \mathcal{V}_\lambda/\varpi^m)$ *with* $K_{\bar{v}} = \mathrm{GL}_n(\mathcal{O}_{F_v}) \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})$ *for a place* $\bar{v} = v \cdot v^c$ *of* F^+ *dividing* p. *Assume that*

i. the mod p *Hecke eigensystem* $\overline{\psi}$ *is non-Eisenstein and decomposed generic,*

ii. and assumption (i) of Proposition 1.2.17 holds.

Then, if we write $r_\psi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(A')$ *for the associated Galois representation constructed in [Sch15],* $r_\psi|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ *admits a crystalline lift with labelled Hodge–Tate weights* $(\lambda_{t\circ\iota,1} + n - 1, ..., \lambda_{t\circ\iota,n})_{\iota:F_v \hookrightarrow \overline{\mathbf{Q}}_p}$.[32]

To prove such a local-global compatibility result, they significantly generalise and combine the Fontaine–Laffaille and ordinary degree shifting arguments to make use of the features of both methods.

Let us first explain their vast improvement of the Fontaine–Laffaile degree shifting argument. The main weakness of the corresponding argument in [All+23] originates in their Key Lemma 1.2.20. One of the main new innovations of [CN23] was the replacement of the key lemma by the following more general statement.

---

[32] To be more precise, one might need to change A' to be smaller quotient for the statement to hold. However, the point is that the kernel of $A \to A'$ still satisfies the property from Theorem 1.2.10.

Lemma 1.2.28 (Key Lemma 2.0, [CN23], Lemma 2.3.17)**.** *Given an integer $m \geq 1$, there is $M(m) \geq m$ such that there is an isomorphism*

$$\mathcal{V}_U^{\{\bar{v},\bar{v}'\}}(m) \cong \oplus_{j\geq 0} H^j(\mathcal{V}_U^{\{\bar{v},\bar{v}'\}}(m))$$

as complexes of sheaves on $X_{K'}$ as long as $K'_{\bar{v}''} \subset K_{\bar{v}''}(M(m)) := \ker(G(\mathcal{O}_{F^+_{\bar{v}''}}) \to G(\mathcal{O}_{F^+_{\bar{v}''}}/\varpi_{\bar{v}''}^{M(m)}))$ for every p-adic place $\bar{v}'' \neq \bar{v}', \bar{v}$ of F^+.

Moreover, under the assumption i) of Proposition 1.2.17, $\mathcal{V}_U^j(m)$ is nonzero for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor$.

Finally, $\mathcal{V}_U^j(m) := H^j(\mathcal{V}_U^{\{\bar{v},\bar{v}'\}}(m))$ is a direct sum of finitely many copies of $\mathcal{V}_{\underline{0}}/p^m$ for $\underline{0}$ the trivial highest weight vector.

Using Lemma 1.2.28 however comes with the difficulty that $\mathbf{H}^d(X_{K'}, \mathcal{V}_U^{\{\bar{v},\bar{v}'\}} \otimes \mathcal{V}_{\lambda'})$ does not decompose as in 1.2.5 anymore. Therefore, we only have $\mathbf{T}^S$-equivariant spectral sequence

$$H^{d-j}(X_{K'}, \mathcal{V}_U^j \otimes \mathcal{V}_{\lambda'}) \Rightarrow \mathbf{H}^d(X_{K'}, \mathcal{V}_U^{\{\bar{v},\bar{v}'\}} \otimes \mathcal{V}_{\lambda'}) \tag{1.2.9}$$

that is not known to degenerate. However, 1.2.9 admits a map of spectral sequence towards the analogous one for $\mathcal{V}_U^{\{\bar{v},\bar{v}'\}}(m)$ that will split thanks to Lemma 1.2.28. Using this, on the cost of a very impressive homological algebra argument, Caraiani–Newton relax assumptions ii) and iii) in Proposition 1.2.17 and so also in Corollary 1.2.18.

Remark 1.2.29. We mention that the improved Fontaine–Laffaille degree shifting appears already in the thesis work of A'Campo [ACa23], attributing Lemma 1.2.28 to Caraiani–Newton. In particular, the improved version of Corollary 1.2.18 (relaxing ii) and iii)) is already obtained in [ACa23].

However, for genuine torsion Hecke eigenclasses the congruences of the kind appearing in Proposition 1.2.17 are only sufficient to prove (part of) crystalline local-global compatibility when integral p-adic Hodge theory is robust enough (e.g. the Fontaine–Laffaille case). For instance, in order for the authors of [All+23] to conclude, it was crucial to have access to a notion of "torsion crystalline" Galois representations with a trackable set of Hodge–Tate weights that is closed under taking subquotients, a feature of Fontaine–Laffaille theory.

Moreover, already formulating a conjecture in general is not obvious. Nevertheless, one possible formulation is to ask that the *local* representation at $\bar{v}$ admits a crystalline lift with the expected Hodge–Tate weights. Another great insight of Caraiani and Newton was that this can be achieved by finding congruences to cusp forms that are P-ordinary at $\bar{v}$ with maximal parahoric level.

Namely, one can introduce the notion of $t-Q_{\bar{v}}$-ordinary $\underset{\sim}{\xi}^\vee$-cohomological automorphic cuspidal automorphic representations $\widetilde{\pi}$ of $\widetilde{G}(\mathbf{A}_{F^+})$ for every standard parabolic subgroup $Q_{\bar{v}} \subset \widetilde{G}_{F_{\bar{v}}^+}$. For $Q_{\bar{v}} = \widetilde{B}_{\bar{v}}$ we recover the notion of being t-ordinary at $\bar{v}$. For $Q_{\bar{v}} = P_{\bar{v}} = G_{F_{\bar{v}}^+} U_{F_{\bar{v}}^+}$ the unnormalised $U_{\bar{v}}$-operators are given by the double coset operators $[\mathcal{P}_{\bar{v}}(a,b)\widetilde{u}_{\bar{v}}\mathcal{P}_{\bar{v}}(a,b)]$ where $\mathcal{P}_{\bar{v}}(a,b) \subset \widetilde{G}(\mathcal{O}_{F_{\bar{v}}^+})$ is one of the $P_{\bar{v}}$-parahoric level subgroups, and $\widetilde{u}_{\bar{v}} := \mathrm{diag}(\varpi_v, ..., \varpi_v, 1, ..., 1) \in \mathrm{GL}_{2n}(F_v) \cong \widetilde{G}(F_{\bar{v}}^+)$ is the element with the first n entries being ϖ_v and the rest being 1. The role of $\widetilde{T}_{\bar{v}}^+$ is now played by $G_{\bar{v}}^+ := \{g \in G(F_{\bar{v}}^+) \mid gU_{\bar{v}}^0 g^{-1} \subset U_{\bar{v}}^0\}$. Then $\widetilde{\pi}$ is called $t - P_{\bar{v}}$-ordinary if the analogous $\xi_{\bar{v}}$-normalised $U_{\bar{v}}$-operator acting on $t^{-1}\widetilde{\pi}_{\bar{v}}^{\mathcal{P}_{\bar{v}}(b,b)}$ for some deep enough parahoric level subgroup has an eigenvalue in $\overline{\mathbf{Z}}_p^{\times}$. One can then similarly introduce $\widetilde{\pi}_{\bar{v}}^{t-P_{\bar{v}}\text{-ord}}$, a $G_{\bar{v}}^+$-equivariant direct summand of $t^{-1}\widetilde{\pi}_{\bar{v}}^{U_{\bar{v}}^0}$ and a smooth admissible representation of $\langle G_{\bar{v}}^+, \widetilde{u}_{\bar{v}}^{-1}\rangle = G(F_{\bar{v}}^+)$. Then the motivation for finding a congruence with a $t - P_{\bar{v}}$-ordinary cusp form is the following result.

Theorem 1.2.30 (Caraiani–Newton, [CN23], Theorem 3.1.2)**.** *Let* $\widetilde{\pi}$ *be a* $\xi^{\vee}$*-cohomological cuspidal automorphic representation of* $\widetilde{G}(\mathbf{A}_{F^+})$ *that is* $t - P_{\bar{v}}$*-ordinary of maximal parahoric level* $\mathcal{P}_{\bar{v}}(0,1)$*. Let* $-w_0\widetilde{\lambda}_{\bar{v}} = (-w_0\widetilde{\lambda}_{\iota})_{\iota : F_{\bar{v}}^+ \hookrightarrow \overline{\mathbf{Q}}_p}$ *be the lowest weight vector of* $t^{-1}\xi_{\bar{v}}$*. Then we have an isomorphism*

$$r_t(\widetilde{\pi})|_{\mathrm{Gal}(\overline{F}_v/F_v)} \sim \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}.$$

with $\rho_1, \rho_2 : \mathrm{Gal}(\overline{F}_v/F_v) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ *are crystalline with the expected Hodge–Tate weights. Moreover,* $(t^{-1}\widetilde{\pi}_{\bar{v}})^{\mathcal{P}_{\bar{v}}(0,1)}$ *is* 1*-dimensional and*

$$(\det \rho_1) \circ \mathrm{Art}_{F_v} = t^{-1}\widetilde{\pi}_{\bar{v}}^{\mathcal{P}_{\bar{v}}(0,1)} \otimes (-w_0\widetilde{\lambda}_{\bar{v}})$$

where $F_v^{\times}$ *acts on the latter via the first factor of the centre of* G *under the isomorphism* $Z_G(F_{\bar{v}}^+) \cong \mathbf{G}_m(F_v) \times \mathbf{G}_m(F_v) \subset \mathrm{GL}_{2n}(F_v)$*.*

Sketch of proof of Theorem 1.2.27. Consider the abstract Hecke algebras

$$\mathbf{T}^{T,\bar{v}\text{-ord}} := \mathbf{T}^T \otimes_{\mathbf{Z}} \mathcal{O}[Z_G(F_{\bar{v}}^+)/Z_G(\mathcal{O}_{F_{\bar{v}}^+})],\ \widetilde{\mathbf{T}}^{T,\bar{v}\text{-ord}} := \widetilde{\mathbf{T}}^T \otimes_{\mathbf{Z}} \mathcal{O}[Z_{G,\bar{v}}^+/Z_G(\mathcal{O}_{F_{\bar{v}}^+})]$$

for $Z_{G,\bar{v}}^+ = Z_G(F_{\bar{v}}^+) \cap G_{\bar{v}}^+$. Note that we simply added the $G_{\bar{v}}$-ordinary (i.e. central elements), respectively $P_{\bar{v}}$-ordinary $U_{\bar{v}}$-operators at maximal parahoric level. These then act on $H^*(X_K, \mathcal{V}_\lambda/\varpi^m)$ with $K_{\bar{v}}$ hyperspecial, and on $H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_\xi)$ with $\widetilde{K}_{\bar{v}} = \mathcal{P}_{\bar{v}}(0,1)$ via the $\lambda_{\bar{v}}$- respectively, $\xi_{\bar{v}}$-normalised $U_{\bar{v}}$-operators.

By developing a $P_{\bar{v}}$-ordinary analogue of Lemma 1.2.25, and simultaneously performing a $P_{\bar{v}}$-ordinary degree shifting at $\bar{v}$ (by degree 0) and the improved Fontaine–Laffaille degree shifting away from $\bar{v}$ and $\bar{v}'$, Caraiani–Newton prove their main degree shifting result (cf. [CN23], Proposition 4.2.6).

To briefly formulate the end product of this, let $\psi^{\bar{v}\text{-ord}}$ be a Hecke eigensystem for $\mathbf{T}^{T,\bar{v}\text{-ord}}$ extending ψ and appearing in $H^*(X_K, \mathcal{V}_\lambda/\varpi^m)$. After some reduction steps as before, their degree shifting result shows that the (suitably twisted) Eisenstein series $\widetilde{\psi}^{\bar{v}\text{-ord}}$ for $\widetilde{\mathbf{T}}^{T,\bar{v}\text{-ord}}$ lifts to the Hecke eigensystem of some finite linear combination of $t - P_{\bar{v}}$-ordinary cuspidal automorphic representations $\widetilde{\pi}_1, ..., \widetilde{\pi}_k$ of $\widetilde{G}(\mathbf{A}_{F^+})$ of weight ξ and level $\widetilde{K}$ with $\xi_{\bar{v}}$ corresponding to $(-w_0\lambda_{v^c}, \lambda_v)$ and $\widetilde{K}_{\bar{v}} = \mathcal{P}_{\bar{v}}(0,1)$. By Theorem 1.2.30, $r_t(\widetilde{\pi}_i)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ admits an n-dimensional crystalline subrepresentation $\rho_{1,i}$ with the right Hodge–Tate weights for the expected lift of $r_\psi|_{\mathrm{Gal}(\overline{F}_v/F_v)}$.

To conclude, one uses that, by the last part of Theorem 1.2.30, $\det \rho_{1,i}$ is compatible with the $U_{\bar{v}}$-operator for $\widetilde{G}$ that, under the twisted Satake transform, is the central action of ϖ_v for G. This central action can easily be shown to be compatible with $\det \overline{r_\psi}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$, showing that $\det \rho_{1,i}$ and $\det \overline{r_\psi}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ match for every i. After a twisting argument putting the Jordan–Hölder constituents of $\overline{r_\psi}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ and $\overline{r_\psi}^{\vee,c}(1-2n)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ in sufficiently generic position, this forces $\prod_i \rho_{1,i}$ to be a lift of r_ψ (cf. [CN23], §3.2). □

Significance of torsion: Automorphy beyond the self-dual case

In the articles [Sch15], [All+23] and [CN23] a significant amount of work went into developing reciprocity for torsion automorphic forms as well. One of the main motivations for this comes from making progress on automorphy beyond the self-dual case.

Recall that in the case of $\mathrm{GL}_2/\mathbf{Q}$ the main source of establishing automorphy was the Taylor–Wiles–Kisin method. When trying to generalise this method to GL_n/F, one notices that it only succeeds in the cases when F is totally real and $n = 2$. The point is that its success relies on a so-called "numerical coincidence" for the group $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ that only holds in these special cases.

Still, one can make work their method in self-dual situations essentially because the mentioned numerical coincidence does hold for unitary groups to which we can transfer our self-dual automorphic representations using the trace formula (see [CHT08]).

However, without the self-duality condition it was a long standing problem to find the right modification that works even when the classical numerical coincidence fails to hold. The solution to this was laid out in the beautiful work of Calegari–Geraghty [CG18]. They established a vast generalisation of the Taylor–Wiles–Kisin method to GL_n/F relying on a list of conjectures on torsion reciprocity (see *loc. cit.*, Conjecture B). These can be divided into three main groups:

i. Existence of Galois representation associated with torsion Hecke eigenclasses appearing in the integral cohomology of GL_n/F-locally symmetric spaces.

ii. Local–global compatibility for these automorphic Galois representations.

iii. Vanishing of the non-Eisenstein part of mod p cohomology of the GL_n/F-locally symmetric spaces outside the "Borel–Wallach range" $[q_0, q_0 + l_0]$.

Assuming that F is imaginary CM, the first of these conjectures was (up to a nilpotent ideal) settled in [Sch15].

Keeping the assumption on the field, the torsion local-global compatibility conjectures formulated in [CG18] were treated in [All+23] allowing them to prove automorphy lifting results beyond the self-dual case by bypassing the vanishing conjecture. These automorphy lifting results, among other things, were sufficient for them to prove potential automorphy of elliptic curves over CM fields.

Most recently, in [CN23] with the significantly stronger torsion crystalline local-global compatibility theorems at their disposal Caraiani and Newton even proved modularity for a large portion of these elliptic curves.

Perhaps more relevant to this thesis is the work of Gee–Newton [GN20]. Combining ideas of [Eme11] and [CG18], they show that, assuming standard conjectures on completed cohomology and some rather sharp local-global compatibility at p, one can make progress on the Fontaine–Mazur–Langlands–Clozel conjecture when $n = 2$ and p splits completely in F. Settling their conjectures on local-global compatibility and their generalisations is the content of this thesis.

1.3 Statement of results and method of proof

In this thesis, we push the ideas of [All+23] and [CN23] on local-global compatibility at p close to their limits. Our main result in characteristic 0 is as follows.

Theorem 1.3.1. *Let F be an imaginary CM field, $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$ be any isomorphism, and π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\mathbf{C})}$. Assume that*

- *$\overline{r_t(\pi)}$ is irreducible and decomposed generic in the sense of [CS19] (see Definition 2.9.7).*

Then, for any p-adic place $v|p$ of F, and $\iota : F_v \hookrightarrow \overline{\mathbf{Q}}_p$, $r_t(\pi)$ is de Rham at v, with labelled ι-Hodge–Tate weights $\lambda_{t\circ\iota,n} < ... < \lambda_{t\circ\iota,1} + n - 1$ and we have

$$\mathrm{WD}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F-ss} \preceq t^{-1}\mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}). \tag{1.3.1}$$

Remark 1.3.2. For the notion of "$\preceq$" in 1.3.1, see §2.6. It in particular means that semisimple local-global compatibility holds, i.e. we have an isomorphism

$$\mathrm{WD}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{ss} \cong (t^{-1}\mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}))^{ss}.$$

On top of this it says that, in some precise sense, the monodromy of the LHS is at least as nilpotent as the monodromy of the RHS. As a consequence, assuming that monodromy of $\mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}})$ is in fact 0, we obtain full local-global compatibility

$$\mathrm{WD}(r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F-ss} \cong t^{-1}\mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}}).$$

We deduce Theorem 1.3.1 from a more general result that also deals with the torsion automorphic Galois representations constructed in [Sch15]. In this generality, formulating a precise conjecture already requires some work that we summarise now assuming that F is an imaginary CM field and for details we point the reader to §5.

Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good compact open subgroup (in the sense of [All+23], §2.1.1) with $K_p = \prod_v \mathrm{GL}_n(\mathcal{O}_{F_v})$ and let X_K be the corresponding locally symmetric space for GL_n/F. Let $E \subset \overline{\mathbf{Q}}_p$ be a large enough finite field extension so that the images of all embeddings $F \hookrightarrow \overline{\mathbf{Q}}_p$ lie in E, and set $\mathcal{O}$ to be its ring of integers, and $\varpi \in \mathcal{O}$ be a choice of uniformiser. Given a highest weight vector $\lambda = (\lambda_v)_{v|p}$ for $(\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)_E$, and a so-called Weil–Deligne inertial type[33] $\tau = (\tau_v)_{v|p}$ at p (see §2.6 for a definition), we obtain an $\mathcal{O}$-local system $\mathcal{V}_{(\lambda,\tau)}$ on X_K. The corresponding Betti cohomology groups $H^*(X_K, \mathcal{V}_{(\lambda,\tau)})$ admit an action of an abstract Hecke algebra $\mathbf{T}^{\lambda,\tau} = \mathbf{T}^T \otimes_{\mathcal{O}} \mathfrak{z}^\circ_{\lambda,\tau}$. Here $\mathbf{T}^T$ is the usual abstract spherical Hecke algebra (over $\mathcal{O}$) acting at the prime-to-T unramified places and $\mathfrak{z}^\circ_{\lambda,\tau} = \otimes_{v|p}\mathfrak{z}^\circ_{\lambda_v,\tau_v}$ is an $\mathcal{O}$-flat algebra consisting of Hecke operators at p admitting a natural identification $\mathfrak{z}^\circ_{\lambda_v,\tau_v}[1/p] \cong \mathfrak{z}_{\tau_v}$ where the latter denotes the Bernstein centre corresponding to τ_v. Denote by $\mathbf{T}^{\lambda,\tau}(K)$ the quotient of $\mathbf{T}^{\lambda,\tau}$ acting faithfully on $H^*(X_K, \mathcal{V}_{(\lambda,\tau)})$. Given a maximal ideal $\mathfrak{m} \subset \mathbf{T}^{\lambda,\tau}(K)$, Theorem 1.2.10 attaches a continuous semisimple Galois representation

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbf{T}^{\lambda,\tau}(K)/\mathfrak{m})$$

to the system of Hecke eigenvalues $\mathbf{T}^{\lambda,\tau} \to \mathbf{T}^{\lambda,\tau}(K)/\mathfrak{m}$ induced by quotienting out by $\mathfrak{m}$. Assuming that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible (in other words, $\mathfrak{m}$ is non-Eisenstein), Theorem 1.2.10 further provides a lift to a continuous representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbf{T}^{\lambda,\tau}(K)_{\mathfrak{m}}/I),$$

associated with the Hecke eigensystem $\mathbf{T}^{\lambda,\tau} \to \mathbf{T}^{\lambda,\tau}(K)_{\mathfrak{m}}$. Here $I \subset \mathbf{T}^{\lambda,\tau}(K)_{\mathfrak{m}}$ is a nilpotent ideal with $I^4 = 0$ (for the bound on the nilpotence degree, see [NT16]). Both of these representations are determined by the property that they satisfy local-global compatibility away from T.

One then seeks an integral version of local-global compatibility for $\rho_{\mathfrak{m}}$. As discussed already, in cases when the integral theory on the Galois side is

---

[33] This is simply, for each p-adic place v of F, a choice of an isomorphism class of a pair of an inertial type and a monodromy operator.

robust enough, Calegari–Geraghty in [CG18] formulated and the authors of [All+23] proved such local-global compatibility conjectures. These however only treated a handful of cases. The first signs (known to the author) of a more general (and sharper) conjecture (in the potentially crystalline case) appeared in [Car+16b] and a precise formulation (in the crystalline case) can be found in [GN20], Conjecture 5.1.12. Caraiani and Newton recently proved a large part of the conjecture of Gee–Newton (cf. [CN23], Theorem 1.3).

The two key ideas going into [GN20], Conjecture 5.1.12 are the use of Kisin's potentially semistable (p-torsion free) local deformation rings and the interpolation of the semisimple local Langlands over the generic fibre of these deformation rings. The first of these ingredients provides a workable notion of being "torsion potentially semistable of a given weight and inertial type". Moreover, by p-adically rescaling, we can test compatibility of the action of Hecke operators at p using the interpolation map.

Namely, for $v|p$ setting $\rho_v := \rho_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$, an $\mathcal{O}$-flat quotient $R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}$ of the unrestricted local framed deformation ring $R^{\square}_{\overline{\rho}_v}$ is constructed in [Kis07]. It is characterised by the property that its $\overline{E}$-points $\rho : \mathrm{Gal}(\overline{F}_v/F_v) \to \mathrm{GL}_n(\overline{E})$ are exactly those lifts of $\overline{\rho}_v$ that are potentially semistable with Hodge–Tate weights determined by λ_v (by the usual ρ-shift) and such that the Weil–Deligne inertial type of $\mathrm{WD}(\rho)$ is bounded by τ_v. Moreover, on the generic fiber one can construct a map

$$\eta : \mathfrak{z}_{\tau_v} \to R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}[1/p]$$

interpolating the (Tate-normalised) semisimple local Langlands correspondence. Although η does not necessarily send $\mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$ into $R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}$, we can introduce the subring

$$\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v} := \eta^{-1}(R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}) \cap \mathfrak{z}^{\circ}_{\lambda_v,\tau_v} \subset \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}.$$

This ring still has the property $\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}[1/p] = \mathfrak{z}_{\tau_v}$. Local-global compatibility can then be phrased by asking that there exists a (necessarily unique) dotted arrow making the diagram

$$\begin{tikzcd}
R^{\square}_{\overline{\rho}_v} \arrow[r, "\rho_v"] \arrow[d, two heads] & \mathbf{T}^{\lambda,\tau}(K)_{\mathfrak{m}}/I \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v} \arrow[ur, dotted] \arrow[d, hook] & \mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v} \arrow[l, "\eta|_{\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}}"'] \arrow[u, "\mathrm{nat}"'] \arrow[d, hook] \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}[1/p] & \mathfrak{z}_{\tau_v} \arrow[l, "\eta"']
\end{tikzcd} \tag{1.3.2}$$

commutative. Here the map nat denotes the natural map towards the faithful Hecke algebra (along the inclusion $\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v} \subset \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$) and the inclusions in the bottom are the ones induced by inverting p. Our main result is then (roughly) as follows (cf. Theorem 5.3.4).

Theorem 1.3.3. *Let F be an imaginary CM field that contains an imaginary quadratic field F_0 in which p splits with totally real subfield F^+. Assume that the finite set of places T is as in Theorem 5.3.4. Let $\bar{v}$ be a p-adic place in F^+. Assume the following:*

i. There is a p-adic place $\bar{v}'$ of F^+ such that $\bar{v} \neq \bar{v}'$ and

$$\sum_{\bar{v}'' \neq \bar{v}, \bar{v}'} [F^+_{\bar{v}''} : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}]$$

where the sum runs over p-adic places of F^+;

ii. $\overline{\rho}_{\mathfrak{m}}$ is decomposed generic.

Then, up to possibly enlarging I, there is a (necessarily unique) dotted arrow making (1.3.2) commutative.

Remark 1.3.4. The further constraint on T in the statement of Theorem 5.3.4 is a mild one that can be always achieved by enlarging T. This condition is so that we can appeal to the unconditional base change results of [Shi14] and it is already present in [Sch15].

Assumption i) is essential to our methods and already appears in [All+23], and [CN23]. In particular, this rules out the case of F being an imaginary quadratic field. Using Theorem 1.3.1, it seems plausible to weaken this assumption, for instance allowing one to prove the theorem for $n = 2$, and $[F : \mathbf{Q}] = 4$.

Finally, assumption ii) on $\overline{\rho}_{\mathfrak{m}}$ is crucial to be able to use the vanishing results of [CS19] and so to appeal to Corollary 1.2.15.

Remark 1.3.5. Let us elaborate on Theorem 1.3.3 in the crystalline case. Note that this is the case when τ_v is trivial (i.e. when the inertial type is the trivial representation of the inertia subgroup and the monodromy is the 0 matrix). Indeed, one has then $\mathcal{V}_{(\lambda_v, \tau_v)} = \mathcal{V}_{\lambda_v}$ and so we are in the setup of [CN23]. Their main local-global compatibility theorem (Theorem 1.2.27) then proves the existence of a dotted arrow making the top triangle of (1.3.2) commute. In particular, Theorem 1.3.3 is new even in the crystalline case.

To spell out the meaning of the second half of the diagram (1.3.2), note that in the crystalline case $\mathfrak{z}_{\tau_v} \cong E[T_1, ..., T_n^{\pm 1}]$ is the usual spherical Hecke algebra, and $\mathfrak{z}^{\circ}_{\lambda_v, \tau_v} \subset \mathfrak{z}_{\tau_v}$ is some suitable subalgebra acting on $H^*(X_K, \mathcal{V}_{(\lambda,\tau)})$. The interpolation map η then simply sends T_j to the universal function $a^{\mathrm{univ}}_{v,j} \in R^{\lambda_v, 0}_{\overline{\rho}_v}[1/p]$ such that, for every $x : R^{\lambda_v, 0}_{\overline{\rho}_v}[1/p] \to \overline{\mathbf{Q}}_p$ with corresponding Galois representation ρ_x,

$$X^n - a^{\mathrm{univ}}_{v,1} X^{n-1} + ... + (-1)^j q_v^{\frac{j(j-1)}{2}} a^{\mathrm{univ}}_{v,j} X^{n-j} + ... + (-1)^n q_v^{\frac{n(n-1)}{2}} a^{\mathrm{univ}}_{v,n}$$

specialises along x to the characteristic polynomial of Frob_v acting on $\mathrm{WD}(\rho_x)$. Therefore, the commutativity of the lower triangle of (1.3.2) roughly says that, beyond $\rho_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ being crystalline with the right Hodge–Tate weights, its crystalline Frobenius too is compatible with the local Langlands correspondence.

In particular, in characteristic 0, for a given automorphic representation π, this proves that the inverse of the crystalline Frobenius on the filtered φ-module associated with $r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ coincides with the Frobenius on $t^{-1}\mathrm{rec}(\pi_v \otimes |\det|_v^{\frac{1-n}{2}})$.

Although in general for proving small $R = \mathbf{T}$ results it often suffices to prove factorisation through the right potentially semistable deformation rings at p, we remark that understanding the crystalline Frobenius is for instance essential to make the big $R = \mathbf{T}$ results of [GN20], §5 unconditional.

In the rest of the introduction, we briefly explain our approach to Theorem 1.3.3 focusing on what it adds to the proof of [CN23].

We saw in §1.2.4 that in *loc. cit.* the main ingredients[34] going into the proof of Theorem 1.2.27 were

i. a new robust degree shifting argument combining their improved Fontaine–Laffaille style and P-ordinary degree shifting arguments,

ii. and a P-ordinary local-global compatibility result in characteristic 0 (Theorem 1.2.30).

Because of the limitations of ii), from the Hecke action at p they could only read off the inertial type and the central character (under local Langlands) of the constructed crystalline lift. In particular, even if they could keep track of the action of a larger Hecke algebra at p in their degree shifting argument, they had no chance to conclude anything stronger.

The main technical innovation of this thesis is the following vastly improved Q-ordinary local-global compatibility result (cf. Theorem 4.3.1 and Theorem 4.3.4).[35]

Theorem 1.3.6. *Let π be a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\mathbf{C})}$, $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$ be a fixed identification and $v|p$ be a p-adic place of F. Let $Q \subset \mathrm{GL}_n$ be the standard parabolic subgroup corresponding to a partition $n = n_1 + \ldots + n_t$, and denote by M its Levi quotient. Assume that $t^{-1}\pi_v$ is Q-ordinary of weight λ_v.*

Then the Q-ordinary part $(t^{-1}\pi_v)^{Q\text{-ord}}$ is irreducible as a smooth representation of $M(F_v) = \mathrm{GL}_{n_1}(F_v) \times \ldots \times \mathrm{GL}_{n_t}(F_v)$. Write $(t^{-1}\pi_v)^{Q\text{-ord}} = \pi_1 \otimes \ldots \otimes \pi_t$.

---

[34] We must point out that their most novel results are all part of the first of these ingredients and there is an obvious imbalance between i) and ii) in terms of difficulty. However, for the sake of explaining the new ingredients of this thesis, it was most natural to the author to group them in this manner.

[35] The reader might find it useful to compare it with Conjecture 1.2.23 and Theorem 1.2.30.

Then there is moreover an isomorphism

$$r_t(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)} \sim \begin{pmatrix} \rho_1 & * & ... & * \\ 0 & \rho_2 & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \rho_t \end{pmatrix}$$

where, for $1 \leq j \leq t$,

$$\rho_j : \mathrm{Gal}(\overline{F}_v/F_v) \to \mathrm{GL}_{n_j}(\overline{\mathbf{Q}}_p)$$

is potentially semistable such that, for every embedding $\iota : F_v \hookrightarrow \overline{\mathbf{Q}}_p$, *the labelled* ι*-Hodge–Tate weights of* ρ_j *are given by*

$$\lambda_{t\circ\iota,n+1-(n_1+...+n_j)}+n_1+...+n_j-1 > ... > \lambda_{t\circ\iota,n+1-(n_1+...+n_{j-1}+1)}+n_1+...+n_{j-1}$$

and we have an isomorphism

$$\mathrm{WD}(\rho_j)^{F-ss} \cong \mathrm{rec}(\pi_j \otimes |\det|_v^{n_1+...+n_{j-1}} \otimes |\det|_v^{\frac{1-n_j}{2}}).$$

By specialising the above theorem to the case of the Siegel parabolic subgroup $P_{(n,n)}$ inside GL_{2n}, and comparing it with Theorem 1.2.30, we see that in the notation of *loc. cit.*, Theorem 1.3.6 allows us to read off the whole of ρ_1 from the Hecke action of $\widetilde{\pi}_{\bar{v}}^{t-P_{\bar{v}}\text{-ord}}$, not just its inertial type and determinant. Therefore, by carrying out the degree shifting argument with the larger Hecke algebras $\widetilde{\mathbf{T}}^T \otimes \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$, and $\mathbf{T}^T \otimes \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$ one obtains Theorem 1.3.3.

To achieve such a degree shifting, our main technical results are a generalisation of the Hida theory developed in [All+23], [CN23] and an improved Hecke equivariant Q-ordinary degree shifting argument. Having these results in hand, with some care (most notably a tracking of Hecke operators at p under Poincaré duality for X_K with p-torsion coefficients), one checks that the improved P-ordinary degree shifting argument can be combined with the Fontaine–Laffaille style degree shifting argument of Caraiani–Newton.

Remark 1.3.7. Theorem 1.3.6 generalises the ordinary local-global compatibility result [Ger18], Corollary 2.7.8 (see also [Tho15], Theorem 2.4) to the case of a general standard parabolic subgroup. The main extra difficulty is having a generalisation of [Ger18], Lemma 5.4 2) for Q-ordinary parts for a general standard parabolic subgroup. This is the first part of Theorem 1.3.6 and is a question purely in the realm of smooth representations of p-adic reductive groups. This part of the theorem we prove in a rather general setup in §4.2 that might be of independent interest.

Remark 1.3.8. As it does not require any significant extra work, we in fact also produce (locally at p) Q-ordinary lifts with expected shape of Galois representations associated with Q-ordinary torsion Hecke eigenclasses for any standard parabolic subgroup $Q \subset \mathrm{GL}_n$ (see Proposition 6.3.1). Although we do not investigate it further, after formulating a suitable Q-ordinary local-global compatibility conjecture, these Q-ordinary lifts seem to have the potential to make progress on such conjectures.

The organisation of the thesis is as follows. In Chapter 2, we collect the preliminaries on the cohomology of locally symmetric spaces and introduce the necessary local systems and the corresponding Hecke actions for the group GL_n/F and the quasi-split unitary group $\widetilde{G}$. In Chapter 3, we recollect and generalise Hida theory in the Betti setting for general standard parabolic subgroups and carry out the computation of ordinary parts of the Bruhat strata of parabolic induction, further keeping track of the Hecke actions. In Chapter 4, we revisit and further develop a theory of ordinary parts for locally algebraic representations. Moreover, we finish the chapter with proving Theorem 1.3.6. In Chapter 5, we formulate a torsion local-global compatibility conjecture following [GN20]. In Chapter 6, we execute the strategy outlined in the introduction to prove Theorem 1.3.3 and conclude the chapter with deducing Theorem 1.3.1. In the Appendix, we make a brief recollection on the smooth representation theory of GL_n over a p-adic field and prove a couple of relevant technical lemmas that are only used at the end of §4.2.

Notation and Conventions

Given a number field F, we will denote by $S(F)$ its set of finite places and by $S_p(F)$ its set of p-adic places. We set G_F to be the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ and for a finite set $S \subset S(F)$ we denote by $G_{F,S}$ the quotient of G_F corresponding to the maximal Galois extension of F, unramified outside S. For $v \in S(F)$, set F_v to be the v-adic completion of F, fix a choice of uniformiser ϖ_v and set $k_v := \mathcal{O}_{F_v}/\varpi_v$ to be its residue field. Set $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$. Moreover, $I_{F_v} \subset G_{F_v}$ will denote its inertia subgroup and set $\mathrm{Frob}_v \in G_{F_v}/I_{F_v}$ to be the geometric Frobenius. We denote by $\mathbf{A}_F$ the ring of adeles of F and for S a finite set of places of F we denote by $\mathbf{A}_F^S$ its prime-to-S part and set $\widehat{\mathcal{O}}_{F,S} := \prod_{v\in S}\mathcal{O}_{F_v}$.

For G a reductive group over a number field F and a finite set $S \subset S(F)$, we will denote by $G^S := G(\mathbf{A}_F^{S\cup\infty})$ and by $G_S := G(\prod_{v\in S} F_v)$. Moreover, if $S = S_p(F)$, we only write G^p, respectively G_p.

Let G be a linear algebraic group over $\mathcal{O}_L$ where $\mathcal{O}_L$ is the ring of integers of a finite extension $L/\mathbf{Q}_p$. Let $\varpi_L \in \mathcal{O}_L$ be a choice of uniformiser. For $n \in \mathbf{Z}_{\geq 0}$, denote $\ker(G(\mathcal{O}_L) \to G(\mathcal{O}_L/\varpi_L^n))$ by G^n.

For G a reductive group over a finite extension $L/\mathbf{Q}_p$ with parabolic subgroup $Q = M \ltimes N \subset G$, we denote by $\delta_Q : M(L) \to \mathbf{Q}^\times$ the corresponding modulus character $x \mapsto |\det(\mathrm{ad}(x)|\mathrm{Lie}N)|_L$. For a smooth representation σ of $M(L)$, we denote by $\text{n-Ind}_{Q(L)}^{G(L)}\sigma$ the normalised parabolic induction

$\mathrm{Ind}_{Q(L)}^{G(L)}\delta_Q^{1/2}\sigma$. For a smooth $\overline{\mathbf{Q}}_p$-representation π of $G(L)$, we will denote by $J_Q(\pi)$ its *unnormalised* Jacquet module associated with Q. In general we denote by c-Ind compact induction for smooth representations.

For a smooth irreducible representation π of $\mathrm{GL}_n(L)$ with supercuspidal support $(\mathrm{GL}_{n_1}(L) \times ... \times \mathrm{GL}_{n_k}(L), \pi_1 \otimes ... \otimes \pi_k)$, we set $\mathrm{SC}(\pi) := \{\pi_1, ..., \pi_k\}$.

Set W_L to be the Weil group of L and write $\mathrm{Art}_L : L^\times \xrightarrow{\sim} W_L^{\mathrm{ab}}$ for the Artin map of local class field theory normalised by sending uniformisers to lifts of the geometric Frobenius.

We denote by rec_L the local Langlands correspondence for L. If it's clear from the context, we will just write rec instead. Moreover, for π an irreducible admissible $\mathrm{GL}_n(L)$-representation, we set $\mathrm{rec}^T(\pi) = \mathrm{rec}(\pi \otimes |\cdot|_L^{\frac{1-n}{2}})$. Then rec^T commutes with $\mathrm{Aut}(\mathbf{C})$ and therefore rec^T makes sense over $\overline{\mathbf{Q}}_p$ by choosing an abstract isomorphism $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. In the literature, this is often called the *Tate normalisation* of the local Langlands correspondence.

We fix an algebraic closure $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$ and denote by val_p the p-adic valuation normalised by setting $\mathrm{val}_p(p) = 1$. By convention, the ι-Hodge–Tate weights of the p-adic cyclotomic character $\epsilon : G_L \to \mathbf{Z}_p^\times$ are -1.

For an ℓ-adic Galois representation $\rho : G_L \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ with $\ell \neq p$, we denote by $\mathrm{WD}(\rho)$ the associated Weil–Deligne representation. For a de Rham p-adic Galois representation $\rho : G_L \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, we denote by $\mathrm{WD}(\rho)$ the associated Weil–Deligne representation provided by the recipe of Fontaine.

For a Weil–Deligne representation (r, N), we denote by $(r, N)^{F-ss}$ its Frobenius semisimplification and by $(r, N)^{ss}$ its semisimplification.

For a ring R, we denote by $D(R)$ the derived category of left R-modules and by $D^+(R)$ resp. $D^b(R)$ the bounded below resp. bounded derived category. Given moreover a locally profinite group G, we will denote by $\mathrm{Mod}_{\mathrm{sm}}(R[G])$ the category of smooth $R[G]$-modules and denote by $D_{\mathrm{sm}}(R[G])$ its derived category and by $D_{\mathrm{sm}}^+(R[G])$ resp. $D_{\mathrm{sm}}^b(R[G])$ its bounded below resp. bounded derived category.

Given a topological group G, and a topological space X with a continuous right action of G, we denote by $\mathrm{Sh}_G(X)$ the category of G-equivariant sheaves on X in the sense of [NT16], Definition 2.22, (2). Moreover, for a ring R, we denote by $\mathrm{Sh}_G(X, R)$ the category of G-equivariant sheaves of R-modules on X.

For G/L a split reductive group with a choice of a Borel subgroup B and a maximal torus T, denote by w_0^G the longest element in the Weyl group $W_G := W(G, T)$. For a standard parabolic subgroup $Q \subset G$ with Levi decomposition $M \ltimes N$, set W^Q to be the set of minimal length representatives of W_G/W_Q. We denote by w_0^Q the longest element in W^Q that is, in fact, given by $w_0^G w_0^M$. Similar notations apply to ${}^QW$. Moreover, for another standard parabolic subgroup $Q' \subset G$ with Levi decomposition $M' \ltimes N'$, denote by ${}^{Q'}W^Q$ the set of minimal length representatives of $W_{Q'} \backslash W_G / W_Q$.

For an integer $n \geq 1$, we denote by $\mathbf{Z}_+^n \subset \mathbf{Z}^n$ the subset of n-tuples of integers $(k_1, ..., k_n)$ satisfying $k_1 \geq ... \geq k_n$.

Chapter 2

Preliminaries

We collect the necessary preliminaries regarding locally symmetric spaces and their cohomology following [CN23], §2. The main differences compared to the setup of [All+23], §2 are the use of different infinite level locally symmetric spaces, which take into account the profinite topology on the adelic part.

2.1 Locally symmetric spaces

Consider a number field F and a connected linear algebraic group G over F, with a model over $\mathcal{O}_F$, still denoted by G. One can then associate with $\mathrm{Res}_{F/\mathbf{Q}}G$ its symmetric space X^G, a homogeneous $G(F \otimes_{\mathbf{Q}} \mathbf{R})$-space, as is defined in [BS73], §2. By [BS73], Lemma 2.1, X^G is unique up to isomorphism of homogeneous $G(F \otimes_{\mathbf{Q}} \mathbf{R})$-spaces.

Given a good compact open subgroup $K_G \subset G(\mathbf{A}_F^\infty)$ in the sense of [All+23], §2.1.1, we can form the corresponding locally symmetric space

$$X_{K_G} := G(F)\backslash(X^G \times G(\mathbf{A}_F^\infty)/K_G),$$

a smooth orientable Riemannian manifold. Borel–Serre then construct a partial compactification $\overline{X}^G$ of X^G (cf. [BS73], §7.1) and form the compactified locally symmetric space

$$\overline{X}_{K_G} := G(F)\backslash(\overline{X}^G \times G(\mathbf{A}_F^\infty)/K_G),$$

an orientable compact smooth manifold with corners and with interior X_{K_G}. We will denote the corresponding boundary by $\partial X^G := \overline{X}^G \setminus X^G$ resp. $\partial X_{K_G} := \overline{X}_{K_G} \setminus X_{K_G}$.

Following [CN23], we define the sets

$$\mathfrak{X}_G := \varprojlim_{K_G} X_{K_G},\ \overline{\mathfrak{X}}_G := \varprojlim_{K_G} \overline{X}_{K_G},\ \partial\mathfrak{X}_G = \varprojlim_{K_G} \partial X_{K_G}$$

where the limits run over good subgroups of $G(\mathbf{A}_F^\infty)$, and make them into topological spaces by endowing them with the projective limit topology. The latter two then become compact Hausdorff spaces, being projective limits of such. They all are equipped with the natural continuous right action of $G(\mathbf{A}_F^\infty)$. Moreover, the induced action of any good subgroup $K_G \subset G(\mathbf{A}_F^\infty)$ on $\overline{\mathfrak{X}}_G$, and $\partial\mathfrak{X}_G$ is free in the sense of [NT16], Definition 2.23. As explained in [CN23], §2.1.1, we can also introduce these spaces as the quotient spaces

$$\mathfrak{X}_G = G(F) \setminus X^G \times G(\mathbf{A}_F^\infty),\ \overline{\mathfrak{X}}_G = G(F) \setminus \overline{X}^G \times G(\mathbf{A}_F^\infty),\ \text{and}$$
$$\partial\mathfrak{X}_G = G(F) \setminus \partial X^G \times G(\mathbf{A}_F^\infty)$$

where the group $G(\mathbf{A}_F^\infty)$ is equipped with its locally profinite topology. We remind the reader that, before [CN23], the more common choice was to use the discrete topology of $G(\mathbf{A}_F^\infty)$ instead (see [NT16]). They yield different spaces of course, but the produced finite level cohomology groups (with the induced Hecke actions) compare well (cf. [CN23], Lemma 2.1.5). We denote by $j : \mathfrak{X}_G \hookrightarrow \overline{\mathfrak{X}}_G$ the natural open immersion.

For later use we introduce some further notation. Set S to be a finite set of finite places of F and $K_G^S \subset G(\mathbf{A}_F^{S\cup\{\infty\}})$ be a compact open subgroup that extends to some good subgroup of $G(\mathbf{A}_F^\infty)$. We then set

$$\overline{X}_{K_G^S} := \overline{\mathfrak{X}}_G/K_G^S = \varprojlim_{K_{G,S}} \overline{X}_{K_G^S K_{G,S}},\ \text{and}\ \partial X_{K_G^S} := \partial\mathfrak{X}_G/K_G^S = \varprojlim_{K_{G,S}} \partial X_{K_G^S K_{G,S}}$$

where the limits run over compact open subgroups $K_{G,S} \subset G(\mathbf{A}_F^\infty)$ such that $K_G^S K_{G,S}$ is good.

2.2 Coefficient systems and Hecke operators

For S a finite set of finite places of F, we set $G^S := G(\mathbf{A}_F^{S\cup\{\infty\}})$ and $G_S := G(\mathbf{A}_{F,S})$, and similarly, for a good subgroup $K_G \subset G(\mathbf{A}_F^\infty)$, we set $K_G^S := \prod_{v\notin S} K_{G,v}$ and $K_{G,S} := \prod_{v\in S} K_{G,v}$.

Let R be a commutative ring. Note that both $\mathfrak{X}_G$, and $\overline{\mathfrak{X}}_G$ admit a continuous right action of $G^S \times K_{G,S}$. In particular, we can consider the corresponding categories of $G^S \times K_{G,S}$-equivariant sheaves of R-modules $\mathrm{Sh}_{G^S\times K_{G,S}}(\mathfrak{X}_G, R)$, $\mathrm{Sh}_{G^S\times K_{G,S}}(\overline{\mathfrak{X}}_G, R)$ in the sense of [NT16], Definition 2.22, (2). Given a smooth $R[K_{G,S}]$-module $\mathcal{V}$, the formalism of [All+23] attaches to it a $G^S \times K_{G,S}$-equivariant sheaf $\mathcal{V} \in \mathrm{Sh}_{G^S\times K_{G,S}}(\mathfrak{X}_G, R)$ resp. $\mathcal{V} \in \mathrm{Sh}_{G^S\times K_{G,S}}(\overline{\mathfrak{X}}_G, R)$. Namely, one inflates $\mathcal{V}$ to get an element of $\mathrm{Mod}_{\mathrm{sm}}(R[G^S \times K_{G,S}])$ which, by [NT16], Lemma 2.25, is equivalent to the category $\mathrm{Sh}_{G^S\times K_{G,S}}(*, R)$ and we then pull it back along $f : \mathfrak{X}_G \to *$ resp. $\overline{f} : \overline{\mathfrak{X}}_G \to *$. As explained in [Sch98], the categories $\mathrm{Sh}_{G^S\times K_{G,S}}(\mathfrak{X}_G, R)$ and $\mathrm{Sh}_{G^S\times K_{G,S}}(\overline{\mathfrak{X}}_G, R)$ have enough injectives and both $f_!$ and $\overline{f}_! = \overline{f}_*$ land in $\mathrm{Sh}_{G^S\times K_{G,S}}(*, R) \cong \mathrm{Mod}_{\mathrm{sm}}(R[G^S \times K_{G,S}])$. Therefore, we see that both

$$R\Gamma(\overline{\mathfrak{X}}_G, j_!\mathcal{V}) = {}^1R(\overline{f}_* \circ j_!)\mathcal{V} = Rf_!\mathcal{V}$$

---

[1] Here we use that $j_!$ preserves injective objects.

and

$$R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}) = R\overline{f}_*\mathcal{V} = R\overline{f}_!\mathcal{V}$$

lie in $D^+_{\mathrm{sm}}(R[G^S \times K_{G,S}])$. By taking derived invariants, we end up with

$$R\Gamma(K_G, R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V})) \text{ and } R\Gamma(K_G, R\Gamma(\overline{\mathfrak{X}}_G, j_!\mathcal{V}))$$

in $D^+(\mathcal{H}(G^S, K^S_G) \otimes_{\mathbf{Z}} R)$ where $\mathcal{H}(G^S, K^S_G) := \mathbf{Z}[K^S_G \backslash G^S / K^S_G]$ is the prime-to-S Hecke algebra i.e., the additive group of integer valued K^S_G-biinvariant functions on G^S equipped with the convolution product.

On the other hand, by descent (cf. [NT16], Lemma 2.24), $\mathcal{V}$ and $j_!\mathcal{V}$ give rise to sheaves on $\overline{X}_{K_G}$, which, by abuse of notation, we denote by the same letter. Then, by combining the fact that j is a homotopy equivalence and [CN23], Proposition 2.1.3, we obtain natural isomorphisms

$$R\Gamma(X_{K_G}, \mathcal{V}) \cong R\Gamma(\overline{X}_{K_G}, \mathcal{V}) \cong R\Gamma(K_G, R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}))^{\sim}$$

and

$$R\Gamma_c(X_{K_G}, \mathcal{V}) := R\Gamma(\overline{X}_{K_G}, j_!\mathcal{V}) \cong R\Gamma(K_G, R\Gamma(\overline{\mathfrak{X}}_G, j_!\mathcal{V}))^{\sim}$$

in $D^+(R)$, where $(-)^{\sim}$ denotes the forgetful functor. In particular, we have a ring homomorphism

$$\mathcal{H}(G^S, K^S_G) \otimes_{\mathbf{Z}} R \to \mathrm{End}_{D^+(R)}(R\Gamma_{(c)}(X_{K_G}, \mathcal{V})).$$

The same observations also apply to $R\Gamma(\partial X_{K_G}, \mathcal{V})$.

Finally, assuming that R is Noetherian and $K'_G \subset K_G$ is a good normal subgroup, one sees, by writing down the complex explicitly in terms of a well-chosen simplicial complex, that $R\Gamma_{(c)}(X_{K'_G}, \mathcal{V})$ is in fact a perfect object in $D^+(R[K_G/K'_G])$ (cf. [CN23], Lemma 2.1.6).

We now turn our attention to completed cohomology following the viewpoint of [CN23]. For this we fix $m \in \mathbf{Z}_{\geq 1}$ and set $R = \mathcal{O}/\varpi^m$ where $\mathcal{O}$ is the ring of integers of a finite field extension $E/\mathbf{Q}_p$. Moreover, let $S \subset S_p := S_p(F)$ be a subset of the set of p-adic places of F.

Definition 2.2.1. Given $\mathcal{V} \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S_p}])$, we define its completed cohomology (with compact support) at S of level K^S_G to be

$$\pi(K^S_G, \mathcal{V}) := R\Gamma(K^S_G, R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}))^{\sim} \cong {}^2 R\Gamma(\overline{X}_{K^S_G}, \mathcal{V}) \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S}])$$

resp.

$$\pi_c(K^S_G, \mathcal{V}) := R\Gamma(K^S_G, R\Gamma(\overline{\mathfrak{X}}_G, j_!\mathcal{V}))^{\sim} \cong R\Gamma(\overline{X}_{K^S_G}, j_!\mathcal{V}) \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S}]).$$

Note that if $\mathcal{V}$ is inflated from an element $\mathcal{V}^S \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S_p \backslash S}])$, then the S-completed cohomology complexes in fact lie in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G_S])$.

---

[2] This identification is proved exactly the same way as in the proof of [CN23], Proposition 2.1.3.

For any set of finite places $S_p \subset T$, $\mathcal{H}(G^T, K_G^T) \otimes_{\mathbf{Z}} \mathcal{O}/\varpi^m$ acts on the completed cohomology complexes (with compact support) as it does so on $R\Gamma(K_G^S, R\Gamma(\overline{\mathfrak{X}}_G, (j_!)\mathcal{V}))$. Moreover, one similarly defines boundary completed cohomology

$$\pi_\partial(K_G^S, \mathcal{V}) := R\Gamma(\partial X_{K_G^S}, \mathcal{V}) \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S}]).$$

Note that [CN23], Lemma 2.1.7 justifies the use of the term completed cohomology i.e., it shows that, after taking cohomology groups, we get back Emerton's completed cohomology as defined in [Eme06b].

Finally, we have the usual phenomenon of completed cohomology at S being independent of the weight at S (cf. [CN23], Lemma 2.1.8). To state this more precisely, let $\mathcal{V} \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S_p\setminus S} \times \Delta_S])$ where $\Delta_S \subset G_S$ is a submonoid containing a compact open subgroup $U_S \subset G_S$ and assume that $\mathcal{V}$ is $\mathcal{O}/\varpi^m$-flat. We can view $\mathcal{V}$ as a $G^{S_p} \times K_{G,S_p\setminus S} \times U_S$-equivariant sheaf on $\overline{\mathfrak{X}}_G$.

Lemma 2.2.2. *We have canonical isomorphisms*

$$R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}) \cong R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \mathcal{V},$$

and

$$R\Gamma(\partial\mathfrak{X}_G, \mathcal{V}) \cong R\Gamma(\partial\mathfrak{X}_G, \mathcal{O}/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \mathcal{V}$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G^{S_p} \times K_{G,S_p\setminus S} \times U_S])$.

Proof. This is [CN23], Lemma 2.1.8. □

Following [CN23], we use the lemma above to define the object

$$R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}) := R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \mathcal{V} \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G^{S_p} \times K_{G,S_p\setminus S} \times \Delta_S])$$

in a way that is independent of the choice of U_S. In particular, we can endow $R\Gamma(X_{K_G}, \mathcal{V})$ with a natural $\mathcal{H}(G^{S_p}, K_G^{S_p}) \otimes \mathcal{H}(\Delta_S, U_S)$-action for any choice of compact open subgroup $U_S \subset G_S$

We also have a version of Lemma 2.2.2 with $\mathcal{V}$ being a complex of sheaves with bounded cohomology. Namely, consider $\mathcal{V} \in D^b_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S_p\setminus S}])$ and denote also by $\mathcal{V}$ the associated object in $D^b(\mathrm{Sh}_{G^{S_p\setminus S} \times K_{G,S_p\setminus S}}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m))$. One can then make sense of the derived tensor product

$$R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbf{L}}_{\mathcal{O}/\varpi^m} - : D^b_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S_p\setminus S}]) \to D^b_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G^{S_p\setminus S} \times K_{G,S_p\setminus S}])$$

as explained on page 13 of [CN23].[3] Then *loc. cit.* Lemma 2.1.9 shows that there is a canonical isomorphism

$$R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbf{L}}_{\mathcal{O}/\varpi^m} \mathcal{V} \xrightarrow{\sim} R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V})$$

---

[3] Note that it is the consideration of the derived tensor product that forces us to switch here to the bounded derived category.

in $D^b_{\mathrm{sm}}(G^{S_p\backslash S} \times K_{G,S_p\backslash S})$.

We finally define cohomology complexes with $\mathcal{O}$-coefficients by taking homotopy limit. For this we start with an $\mathcal{O}[K_{G,S}]$-module $\mathcal{V}$, finite free as an $\mathcal{O}$-module. We set

$$R\Gamma_{(c)}(X_{K_G}, \mathcal{V}) := \varprojlim_m R\Gamma_{(c)}(X_{K_G}, \mathcal{V}/\varpi^m) \in D^+(\mathcal{O}),$$

where the projective limit is understood as a homotopy limit. One can endow these complexes with an action of the Hecke algebra $\mathcal{H}(G^S, K_G^S) \otimes_{\mathbf{Z}} \mathcal{O}$ (see the footnote on Page 14 of [CN23] for a discussion on how it relates to the Hecke action defined in [NT16]).

The next lemma explains that after taking cohomology we get back classical Betti cohomology with $\mathcal{O}$-coefficients. Such an argument will be used at several places whenever the Mittag–Leffler condition holds.

Lemma 2.2.3. *Let $K_G \subset G(\mathbf{A}_F^\infty)$ be a good subgroup, S be a finite set of finite places of F and $\mathcal{V}$ an $\mathcal{O}[K_{G,S}]$-module, finite free as an $\mathcal{O}$-module. Then, for every $j \in \mathbf{Z}_{\geq 0}$, we have a natural identification*

$$H^j(R\Gamma_{(c)}(X_{K_G}, \mathcal{V})) \cong H^j_{(c)}(X_{K_G}, \mathcal{V}).$$

Proof. By [Sta24, Lemma 0CQE], it suffices to prove that the higher inverse limit

$$R^1 \varprojlim_m H^j_{(c)}(X_{K_G}, \mathcal{V}/\varpi^m)$$

vanishes. This vanishing is ensured once we prove that the inverse system

$$\{H^j_{(c)}(X_{K_G}, \mathcal{V}/\varpi^m)\}_{m\geq 0}$$

satisfies Mittag-Leffler. To see this, we note that, by [CN23], Lemma 2.1.6, all the appearing cohomology groups are finite. Therefore, the images of all the transition maps in the inverse system must stabilise after finitely many steps. ☐

We now recall the definition of the unnormalised Satake transform. Assume that G is reductive and $P = M \rtimes N \subset G$ is a parabolic subgroup with its Levi decomposition. Given a good subgroup $K_G \subset G(\mathbf{A}_F^\infty)$, set $K_P = K_G \cap P(\mathbf{A}_F^\infty)$, $K_N = K_G \cap N(\mathbf{A}_F^\infty)$ and $K_M = \mathrm{im}(K_P \to M(\mathbf{A}_F^\infty))$. We call K_G *decomposed* with respect to $P = M \rtimes N$, if $K_P = K_M \rtimes K_N$; equivalently, if $K_M = K_G \cap M(\mathbf{A}_F^\infty)$.

Assume K_G is decomposed with respect to $P = M \rtimes N$ and let S be a finite set of finite places such that, for $v \notin S$, $K_{G,v}$ is a hyperspecial maximal compact in $G(F_v)$. Then we have the usual maps on Hecke algebras

$$r_P : \mathcal{H}(G^S, K_G^S) \to \mathcal{H}(P^S, K_P^S) \text{ and } r_M : \mathcal{H}(P^S, K_P^S) \to \mathcal{H}(M^S, K_M^S)$$

given by "restriction to P" and "integration along N", respectively (cf. [NT16], 2.2.3, 2.2.4). We then can define $\mathcal{S} := r_M \circ r_P$, the unnormalised Satake transform.

2.3 Hecke algebras of types

We will make use of the Hecke algebra and Bernstein centre action at p with respect to locally algebraic types. For this we briefly recall the content of Appendix A.1 and A.4 of [Vig04]. First we consider the following general setup. Let R be a commutative ring, G be a locally profinite group, $K \subset G$ an open subgroup and σ be a smooth $R[K]$-module, finitely generated over R.

Given σ, we can define the corresponding Hecke algebra

$$\mathcal{H}(\sigma) := \mathrm{End}_{R[G]}(\text{c-Ind}_K^G \sigma).$$

For $\pi \in \mathrm{Mod}_{\mathrm{sm}}(R[G])$, $\mathcal{H}(\sigma)$ acts on the space of invariants

$$\mathrm{Hom}_{R[G]}(\text{c-Ind}_K^G \sigma, \pi) = \mathrm{Hom}_{R[K]}(\sigma, \pi) = \sigma^\vee \otimes_{R[K]} \pi$$

on the right. We also note that $\mathcal{H}(\sigma)$ can be identified with the convolution algebra of compactly supported functions $f : G \to \mathrm{End}_R(\sigma)$ satisfying $f(k_1 g k_2) = \sigma(k_1) f(g) \sigma(k_2)$ for every $k_1, k_2 \in K$ and $g \in G$. The isomorphism is realised by acting with the convolution algebra on c-Ind$_K^G \sigma$ via convolution. From this description it is clear that $\mathcal{H}(\sigma)$ is spanned over R by elements represented by pairs $[g, \psi]$ where $g \in G$ and $\psi \in \mathrm{End}_R(\sigma)$ such that

$$\sigma(k) \circ \psi = \psi \circ \sigma(g^{-1} k g) \tag{2.3.1}$$

for every $k \in K \cap gKg^{-1}$. More precisely, such a pair gives rise to the function $G \to \mathrm{End}_R(\sigma)$ supported on KgK that sends g to ψ. Moreover, under the mentioned identification, $[g, \psi]$ acts on $\phi \in \mathrm{Hom}_{R[K]}(\sigma, \pi)$ by the formula

$$\phi \cdot [g, \psi] : v \mapsto \sum_i \pi(g_i)^{-1} \phi([g, \psi](g_i) v)$$

for $v \in \sigma$ and $KgK = \coprod_i K g_i$.

Note that we have an anti-involution

$$\mathcal{H}(\sigma) \xrightarrow{\sim} \mathcal{H}(\sigma^\vee),$$

$$[g, \psi] \mapsto [g^{-1}, \psi^t].$$

Moreover, $[h, \chi] = [g^{-1}, \psi^t] \in \mathcal{H}(\sigma^\vee)$ acts on $\sum_j f_j \otimes p_j \in \sigma^\vee \otimes_{R[K]} \pi = (\sigma^\vee \otimes_R \pi)^K$ via the formula

$$[h, \chi] \cdot (\sum_j f_j \otimes p_j) = \sum_i \left(\sum_j [h, \chi](h_i) f_j \otimes \pi(h_i)(p_j) \right) =$$

$$= \sum_i \left(\sum_j [g^{-1}, \psi^t](g_i^{-1}) f_j \otimes \pi(g_i^{-1})(p_j) \right)$$

for $\coprod_i h_iK = KhK = Kg^{-1}K = \coprod_i g_i^{-1}K = (\coprod_i Kg_i)^{-1}$. This gives rise to a left action of $\mathcal{H}(\sigma^\vee)$ on $\sigma^\vee \otimes_{R[K]} \pi$. Moreover, as the computation shows, the anti-isomorphism $\mathcal{H}(\sigma) \xrightarrow{\sim} \mathcal{H}(\sigma^\vee)$ intertwines the two actions under the identification $\mathrm{Hom}_{R[K]}(\sigma, \pi) \cong \sigma^\vee \otimes_{R[K]} \pi$.

We are now ready to equip the cohomology of locally symmetric spaces with a Hecke action at p. For this, we revisit the setup of §2.1, and G will again denote a connected linear algebraic group G over F admitting a model over $\mathcal{O}_F$. Before stating the lemma, we remind the reader that, for a compact open subgroup $K_G \subset G(\mathbf{A}_F^\infty)$ and a smooth $\mathcal{O}/\varpi^m[K_{G,S}]$-module σ, we introduced a $G^S \times K_{G,S}$-equivariant sheaf on $\overline{\mathfrak{X}}_G$ and, by descent, a sheaf on $\overline{X}_{K_G}$. Moreover, by abuse of notation, we denoted all of these objects by σ.

Lemma 2.3.1. *Let $S \subset S(F)$ be a finite set of finite places, $K_G \subset G(\mathbf{A}_F^\infty)$ a good subgroup, and $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_{G,S}])$, finite free as an $\mathcal{O}/\varpi^m$-module. Then the diagram of derived functors*

$$\begin{tikzcd}
D^+\mathrm{Sh}_{G(\mathbf{A}_F^\infty)}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \arrow[r, "R\Gamma(\overline{\mathfrak{X}}_G, -)"] \arrow[d, "\text{forget}"'] & D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(\mathbf{A}_F^\infty)]) \arrow[r, "R\mathrm{Hom}_{\mathcal{O}/\varpi^m[K_G]}(\sigma, -)"] & D^+(\mathcal{H}(G^S, K_G^S) \times \mathcal{H}(\sigma^\vee)) \arrow[d, "\text{forget}"] \\
D^+\mathrm{Sh}_{K_G}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \arrow[r, "\text{descent}"'] & D^+\mathrm{Sh}(\overline{X}_{K_G}, \mathcal{O}/\varpi^m) \arrow[r, "R\Gamma(\overline{X}_{K_G}, - \otimes_{\mathcal{O}/\varpi^m} \sigma^\vee)"'] & D^+(\mathcal{O}/\varpi^m)
\end{tikzcd}$$

is commutative.[4]

Proof. Throughout the proof, we repeatedly use [Wei94], Corollary 10.8.3 without further mention. According to [NT16], Lemma 2.28, $\Gamma(\overline{\mathfrak{X}}_G, -)$ preserves injectives. Moreover, the forgetful functor

$$(-)^\sim : D^+\mathrm{Sh}_{G(\mathbf{A}_F^\infty)}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \to D^+\mathrm{Sh}_{K_G}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m)$$

is exact and preserves injectives by [Sch98], §3, Corollary 3.[5] In particular, we get that the composition of the functors on the top of the square is naturally isomorphic to the functor

$$R\Gamma(K_G, R\Gamma(\overline{\mathfrak{X}}_G, (-)^\sim) \otimes_{\mathcal{O}/\varpi^m} \sigma^\vee) : D^+\mathrm{Sh}_{G(\mathbf{A}_F^\infty)}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \to D^+(\mathcal{O}/\varpi^m).$$

On the other hand, the proof of Lemma 2.2.2 (cf. [CN23], Lemma 2.1.8) with arbitrary $\mathcal{G} \in D^+\mathrm{Sh}_{K_G}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m)$ in place of $\mathcal{O}/\varpi^m$ shows that we have a natural isomorphism

$$R\Gamma(\overline{\mathfrak{X}}_G, -) \otimes_{\mathcal{O}/\varpi^m} \sigma^\vee \cong R\Gamma(\overline{\mathfrak{X}}_G, - \otimes_{\mathcal{O}/\varpi^m} \sigma^\vee)$$

---

[4] Here by the lower right horizontal arrow we mean the composition of first tensoring over $\mathcal{O}/\varpi^m$ with $\sigma^\vee \in \mathrm{Sh}(\overline{X}_{K_G}, \mathcal{O}/\varpi^m)$ and then applying derived invariants to the obtained object.

[5] Note that a running assumption in [Sch98] is that the ring of coefficients is $\mathbf{C}$. However, one sees that the proof of *loc. cit.* goes through without a change also with $\mathcal{O}/\varpi^m$-coefficients.

of derived functors $D^+\mathrm{Sh}_{K_G}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \to D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_G])$.

Finally, we note that the descent functor respects tensor products since the pullback functor (its inverse) does. We then obtain a natural isomorphism of derived functors

$$R\Gamma(K_G, R\Gamma(\overline{\mathfrak{X}}_G, - \otimes_{\mathcal{O}/\varpi^m} \sigma^\vee)) \cong R\Gamma(\overline{X}_{K_G}, - \otimes_{\mathcal{O}/\varpi^m} \sigma^\vee).$$

By putting these observations together, we can conclude. □

We spell out a generalisation of [All+23], Proposition 2.2.22 that will be used to make twisting arguments in the proof of local-global compatibility analogous to the ones in *loc. cit.*, Corollary 4.4.8. Let $G = \mathrm{GL}_{n,F}$, and $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and $\chi : G_F \to \mathcal{O}^\times$ be a continuous character such that $\chi \circ \mathrm{Art}_{F_v}$ is trivial on $\det(K_v)$ for each finite place $v \notin T$ of F. Let $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_p])$, finite free as an $\mathcal{O}/\varpi^m$-module. Set $\sigma_\chi : K_p \to \mathcal{O}^\times$ to be the continuous character defined by

$$(k_v)_{v\in S_p} \mapsto \prod_{v\in S_p} \chi(\mathrm{Art}_{F_v}(\det(k_v))).$$

Define the isomorphism of $\mathcal{O}$-algebras

$$f_\chi : \mathcal{H}(G^T, K^T) \otimes_{\mathbf{Z}} \mathcal{H}(\sigma^\vee) \to \mathcal{H}(G^T, K^T) \otimes_{\mathbf{Z}} \mathcal{H}(\sigma^\vee \otimes \sigma_{\chi^{-1}})$$

sending a function $f : \mathrm{GL}_n(\mathbf{A}_F) \to \mathrm{End}(\sigma^\vee)$ lying in the source of f_χ to the function $f_\chi(f) : g \mapsto \chi(\mathrm{Art}_F(\det(g)))^{-1} f(g)$.

Lemma 2.3.2. *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and $\chi : G_F \to \mathcal{O}^\times$ be a continuous character such that $\chi \circ \mathrm{Art}_{F_v}$ is trivial on $\det(K_v)$ for each finite place $v \notin T$ of F. Let $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_p])$, finite free as an $\mathcal{O}/\varpi^m$-module. Then there is an isomorphism*

$$R\Gamma(X_K, \sigma^\vee) \cong R\Gamma(X_K, \sigma^\vee \otimes \sigma_{\chi^{-1}})$$

in $D^+(\mathcal{O}/\varpi^m)$ that is $\mathcal{H}(G^T, K^T) \otimes_{\mathbf{Z}} \mathcal{H}(\sigma^\vee)$-equivariant when we consider its usual action on the left and the one induced by pre-composition with f_χ on the right.

Proof. To see this, we introduce some constructions. For any $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-equivariant sheaf $\mathcal{F} \in \mathrm{Sh}_{\mathrm{GL}_n(\mathbf{A}_F^\infty)}(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{O}/\varpi^m)$, consider the map

$$\Gamma(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{F}) \to \Gamma(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{F}), \tag{2.3.2}$$

$$s \mapsto s^\chi$$

defined by the formula $s^\chi((x, g)) := \chi(\mathrm{Art}_F(\det(g)))s((x, g))$. An easy computation shows that 2.3.2 becomes $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-equivariant when we twist the target by the character $g \mapsto \chi(\mathrm{Art}_F(\det(g)))^{-1}$. In particular, 2.3.2 descends to a map

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[K]}(\sigma, \Gamma(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{F})) \to \mathrm{Hom}_{\mathcal{O}/\varpi^m[K]}(\sigma \otimes \sigma_\chi, \Gamma(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{F})). \quad (2.3.3)$$

After unravelling the definition of the Hecke action, one sees that this amounts to saying that the induced map 2.3.3 satisfies the desired Hecke-equivariance of the lemma.

We can conclude by choosing an injective resolution $\mathcal{O}/\varpi^m \to \mathcal{I}^\bullet$ in the category $\mathrm{Sh}_{\mathrm{GL}_n(\mathbf{A}_F^\infty)}(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{O}/\varpi^m)$ and applying the previous observation with the choice of $\mathcal{F} = \mathcal{I}^i$ for every $i \in \mathbf{Z}_{\geq 0}$. Indeed, this is because, by Lemma 2.3.1, applying the forgetful functor to

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[K]}(\sigma, \Gamma(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{I}^\bullet)), \text{ and } \mathrm{Hom}_{\mathcal{O}/\varpi^m[K]}(\sigma \otimes \sigma_\chi, \Gamma(\overline{\mathfrak{X}}_{\mathrm{GL}_n}, \mathcal{I}^\bullet))$$

computes $R\Gamma(X_K, \sigma^\vee)$, and $R\Gamma(X_K, \sigma^\vee \otimes \sigma_{\chi^{-1}})$, respectively. Moreover, the Hecke actions come from these identifications. □

2.4 Hecke equivariance of Poincaré duality

In the proof of our local-global compatibility results, we appeal to the Poincaré duality isomorphism for the cohomology of the locally symmetric spaces attached to GL_n/F. To keep track of the Hecke algebra actions during this process, it is crucial to verify that Poincaré duality is equivariant with respect to suitable Hecke actions on the two sides. This is already checked for instance in [NT16], Proposition 3.7 for unramified prime-to-p places. We will need a version of this Hecke equivariancy for the Hecke algebra actions at p. This will need slightly more care and will be handled in this section.

We return to our setup in §2.2. We let $S \subset S_p(F)$ to be a set of p-adic places, $K \subset G(\mathbf{A}_F^\infty)$ a good subgroup. Let $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_S])$, which we also assume to be finite free over $\mathcal{O}/\varpi^m$. Then, by the previous section, for any G_S-equivariant sheaf $\mathcal{G}$ on $\overline{X}_{K^S}$ in the sense of [NT16], Section 2.4, the complex

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[K_S]}(\sigma, R\Gamma(\overline{X}_{K^S}, \mathcal{G}))$$

lies in $D^+(\mathcal{H}(\sigma)^{\mathrm{op}}) = D^+(\mathcal{H}(\sigma^\vee))$. Then, if we let $\pi : \overline{X}_{K^S} \to \overline{X}_K$ to be the natural projection and $\overline{f} : \overline{X}_{K^S} \to *$, Lemma 2.3.1 implies that there is an induced morphism of algebras

$$\mathcal{H}(\sigma^\vee) \to \mathrm{End}_{D^+(\mathcal{O}/\varpi^m)}(R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)).$$

To prove Hecke equivariancy of Poincaré duality for $R\Gamma(X_K, \sigma^\vee)$, we give a different description of this action as in [NT16], Lemma 2.19. As we are working with places above p, which are not assumed to be unramified, we need to refine the constructions of [NT16].

We pick an element $[g, \psi] \in \mathcal{H}(\sigma)^{\mathrm{op}}$ as in the previous section. We define an action of this element on cohomology. Let $K' := K^S K'_S$ where $K'_S = K_S \cap g^{-1} K_S g$. By abuse of notation, whenever $K''_S \subset K_S$ is a compact open subgroup, we denote by π the projection $\overline{X}_{K^S} \to \overline{X}_{K^S K''_S}$. We consider the correspondence

$$\begin{tikzcd}
& \overline{X}_{K'} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\overline{X}_K & & \overline{X}_K
\end{tikzcd}$$

where p_1 is the natural projection and p_2 is given by the map $\overline{X}_{K'} \xrightarrow{\cdot g^{-1}} \overline{X}_{K^S \cdot g K'_S g^{-1}}$ followed by the natural projection. By the intertwining property 2.3.1 of ψ, it descends to a map

$$p_1^* \pi_*^{K_S} \overline{f}^* \sigma = \pi_*^{K'_S} \overline{f}^* \sigma \xrightarrow{\psi} (g^{-1})^* \pi_*^{g K'_S g^{-1}} \overline{f}^* \sigma = p_2^* \pi_*^{K_S} \overline{f}^* \sigma,$$

giving a cohomological correspondence. For any G_S-equivariant sheaf of $\mathcal{O}/\varpi^m$-modules $\mathcal{G}$ on $\overline{X}_{K^S}$, we get an induced map

$$\Psi : R\Gamma(\overline{X}_{K'}, p_2^* \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)) \cong R\mathrm{Hom}_{\mathrm{Sh}(\overline{X}_{K'}, \mathcal{O}/\varpi^m)}(p_2^* \pi_*^{K_S} \overline{f}^* \sigma, p_2^* \pi_*^{K_S} \mathcal{G}) \to$$

$$R\mathrm{Hom}_{\mathrm{Sh}(\overline{X}_{K'}, \mathcal{O}/\varpi^m)}(p_1^* \pi_*^{K_S} \overline{f}^* \sigma, p_1^* \pi_*^{K_S} \mathcal{G}) \cong R\Gamma(\overline{X}_{K'}, p_1^* \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)),$$

where the map in the middle is induced by the map ψ in the first component and by the multiplication $g^{-1} : p_2^* \pi_*^{K_S} \mathcal{G} \cong p_1^* \pi_*^{K_S} \mathcal{G}$ in the second component.

We then obtain an endomorphism

$$\theta([g, \psi]) \in \mathrm{End}_{D^+(\mathcal{O}/\varpi^m)}(R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)))$$

defined by

$$R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)) \xrightarrow{p_2^*} R\Gamma(\overline{X}_{K'}, p_2^* \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)) \xrightarrow{\Psi}$$

$$\to R\Gamma(\overline{X}_{K'}, p_1^* \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)) = R\Gamma(\overline{X}_K, p_{1,*} p_1^* \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)) \xrightarrow{\mathrm{tr}}$$

$$\to R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee))$$

where the last map is the trace map coming from the canonical map induced by the adjoint pair $(p_{1,*} = p_{1,!}, p_1^* = p_1^!)$.

Lemma 2.4.1. *The endomorphism* $\theta([g, \psi])$ *of* $R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee))$ *coincides with the one induced by* $[g, \psi]$ *via the recipe of Lemma 2.3.1.*

Proof. Note that we have a natural isomorphism

$$R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^* \sigma^\vee)) \cong R\mathrm{Hom}_{\mathrm{Sh}(\overline{X}_K, \mathcal{O}/\varpi^m)}(\pi_*^{K_S} \overline{f}^* \sigma, \pi_*^{K_S} \mathcal{G})$$

in $D^+(\mathcal{O}/\varpi^m)$. Indeed, this follows from the fact that the functor

$$\pi_*^{K_S} : \mathrm{Sh}_{K_S}(\overline{X}_{K^S}, \mathcal{O}/\varpi^m) \to \mathrm{Sh}(\overline{X}_K, \mathcal{O}/\varpi^m)$$

is an equivalence of categories with inverse π^* and pullbacks commute with tensor products. We obtain analogous descriptions of the other two complexes appearing in the definition of $\theta([g,\psi])$.

We now pick an injective resolution $\mathcal{G} \to \mathcal{I}^\bullet$ in $D^+(\mathrm{Sh}_{G_S}(\overline{X}_{K^S}, \mathcal{O}/\varpi^m))$. Notice that, by [Sch98], §3, Corollary 3, it gives rise to an injective resolution in $D^+(\mathrm{Sh}_{K_S}(\overline{X}_{K^S}, \mathcal{O}/\varpi^m))$, and $D^+(\mathrm{Sh}_{K'_S}(\overline{X}_{K^S}, \mathcal{O}/\varpi^m))$. Moreover, since the functors $\pi_*^{K_S}$, $p_1^*\pi_*^{K_S}$, and $p_2^*\pi_*^{K_S}$ are all equivalences, $\pi_*^{K_S}\mathcal{I}^\bullet$, $p_1^*\pi_*^{K_S}\mathcal{I}^\bullet$, and $p_2^*\pi_*^{K_S}\mathcal{I}^\bullet$ are injective resolutions. If we combine this with the discussion of the previous paragraph, we see that

$$\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{I}^\bullet \otimes_{\mathcal{O}/\varpi^m} \overline{f}^*\sigma^\vee)) = \mathrm{Hom}_{K_S}(\sigma, \mathcal{I}^\bullet(\overline{X}_{K^S}))$$

computes $R\Gamma(\overline{X}_K, \pi_*^{K_S}(\mathcal{G} \otimes_{\mathcal{O}/\varpi^m} \overline{f}^*\sigma^\vee))$. Analogous observations apply to the other two complexes appearing in the definition of $\theta([g,\psi])$. We reduced the lemma to comparing the two actions in

$$\mathrm{End}_{\mathcal{O}/\varpi^m}(\mathrm{Hom}_{K_S}(\sigma, \mathcal{I}^i(\overline{X}_{K^S})))$$

for every $i \in \mathbf{Z}$. The lemma then follows from the concrete description of the effect of the trace map on global sections and an easy unravelling of the definitions. □

Remark 2.4.2. Note that our discussion applies also to the case of $R\Gamma_c(X_K, \sigma^\vee)$. To see this, denote by $j : \mathfrak{X}_G \hookrightarrow \overline{\mathfrak{X}}_G$ the natural open immersion and by $\overline{f} : \overline{\mathfrak{X}}_G \to *$ the projection to the point. Then the claim follows from the fact that we have an isomorphism

$$j_!j^*\overline{f}^*\sigma^\vee \cong j_!\mathcal{O}/\varpi^m \otimes_{\mathcal{O}/\varpi^m} \overline{f}^*\sigma^\vee$$

of $G(\mathbf{A}_F^\infty)$-equivariant sheaves. This identification follows from an equivariant version of [KS94], Proposition 2.5.13.

Corollary 2.4.3. *Given $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[K_S])$, finite free as an $\mathcal{O}/\varpi^m$-module. The Verdier duality isomorphism*

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m}(R\Gamma_c(X_K, \sigma^\vee), \mathcal{O}/\varpi^m) \cong R\Gamma(X_K, \sigma)[\dim_{\mathbf{R}} X_K]$$

is equivariant with respect to the natural left action of $\mathcal{H}(\sigma)$ on the right and the one induced by the anti-isomorphism

$$\mathcal{H}(\sigma) \xrightarrow{\sim} \mathcal{H}(\sigma^\vee),$$

$$[g,\psi] \mapsto [g^{-1}, \psi^t]$$

on the left.

Proof. Just as in the proof of [NT16], Proposition 3.7, this follows from the functoriality of Verdier duality and Lemma 2.4.1 taking into account that passing to duals interchanges pullbacks with traces in the definition of $\theta([g,\psi])$. □

2.5 The quasi-split unitary group

From now on we specialise to our setup of interest. The two groups we will be interested in are the quasi-split unitary group $U(n,n)$ and the general linear group appearing as its Levi subgroup. In particular, we fix an integer $n \geq 2$ and an imaginary CM field F with maximal totally real subfield $F^+ \subset F$. Denote by $c \in \mathrm{Gal}(F/F^+)$ its complex conjugation and set $\overline{S}_p := S_p(F^+)$ resp. $S_p := S_p(F)$. Consider the $2n \times 2n$ matrix

$$J_n := \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}$$

where Ψ_n denotes the $n \times n$ matrix with 1's on the anti-diagonal and 0's elsewhere. We then set $\widetilde{G}/\mathcal{O}_{F^+}$ to be the group scheme that, for an $\mathcal{O}_{F^+}$-algebra R, has R-points given by

$$\widetilde{G}(R) = \{g \in \mathrm{GL}_{2n}(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F) \mid {}^t g J_n g^c = J_n\}$$

where ${}^t(-)$ denotes the transpose matrix. This is an integral model of the quasi-split unitary group $U(n,n)/F^+$, a form of GL_{2n}, splitting after base change to F. In particular, it becomes reductive after base change to $\mathcal{O}_{F^+_{\bar{v}}}$ for $\bar{v}$ a finite place of F^+ which is unramified in F.

We let $P \subset \widetilde{G}$ to be the Siegel parabolic consisting of block upper triangular matrices with blocks of size $n \times n$. Let $P = G \ltimes U$ be a Levi decomposition such that G is given by the closed subgroup of block diagonal matrices. Then G can be identified with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$ as in [NT16], Lemma 5.1. Namely, if we denote by $(-)^*$ the anti-involution of $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$ given by $A^* = \psi_n^t A^c \psi_n^{-1}$ then, by its very definition, $P \subset \widetilde{G}$ can be identified with the subgroup of $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_{2n}$ of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^{-*} & B \\ 0 & D \end{pmatrix}$$

where $D \in \mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$ without any condition and B is so that $B^* = B$. Under this identification, the subgroup defined by $B = 0$ is the Levi subgroup G. Then $\begin{pmatrix} D^{-*} & B \\ 0 & D \end{pmatrix} \mapsto D$ gives the identification $G \cong \mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$.

We will write $\widetilde{X}$ for the symmetric space $X^{\widetilde{G}}$ and $\widetilde{X}_{\widetilde{K}}$ for the associated locally symmetric space for a good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$. Similarly, we denote by X the symmetric space X^G and write X_K for the associated locally symmetric space for a good subgroup $K \subset G(\mathbf{A}^\infty_{F^+}) = \mathrm{GL}_n(\mathbf{A}^\infty_F)$.

Write $T \subset \widetilde{B} \subset \widetilde{G}$ for the subgroup consisting, respectively, of the diagonal and upper triangular matrices of $\widetilde{G}$. These form a maximal torus and a Borel subgroup of $\widetilde{G}$. Moreover, $B = \widetilde{B} \cap G \subset G$ is the Borel subgroup of upper triangular matrices.

Recall that, for a place $\bar{v}$ of F^+ splitting in F, a choice of place $v \mid \bar{v}$ in F gives a canonical isomorphism $\iota_v : \widetilde{G}(F^+_{\bar{v}}) \cong \mathrm{GL}_{2n}(F_v)$. Indeed, there is an isomorphism $F^+_{\bar{v}} \otimes_{F^+} F \cong F_v \times F_{v^c}$ and ι_v is the projection to the first factor of the natural inclusion $\widetilde{G}(F^+_{\bar{v}}) \subset \mathrm{GL}_{2n}(F_v) \times \mathrm{GL}_{2n}(F_{v^c})$. Under ι_v, $P(F^+_{\bar{v}})$ is identified with the standard parabolic subgroup $P_{(n,n)}(F_v) \subset \mathrm{GL}_{2n}(F_v)$ of block upper triangular matrices of type (n, n) and $G(F^+_{\bar{v}})$ with its standard Levi subgroup of block diagonal matrices. Similarly, $\widetilde{B}(F^+_{\bar{v}})$ is identified with the subgroup of upper triangular matrices and $T(F^+_{\bar{v}})$ with the diagonal matrices. Moreover, for any parabolic subgroup $\widetilde{B}_{F^+_{\bar{v}}} \subset \widetilde{Q} \subset P_{F^+_{\bar{v}}}$, $\widetilde{Q}(F^+_{\bar{v}})$ is identified with a standard parabolic subgroup $P_{(n_1,...,n_t)}(F_v) \subset \mathrm{GL}_{2n}(F_v)$ where $(n_1, ..., n_t)$ refines (n, n). Let $\widetilde{Q} = \widetilde{M} \ltimes \widetilde{N}$ its standard Levi decomposition and set $M = G \cap \widetilde{M}$.

Note that, since the inclusion

$$G(F^+_{\bar{v}}) = \mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}) \hookrightarrow \mathrm{GL}_{2n}(F_v)$$

under ι_v is given by

$$(A, B) \mapsto \begin{pmatrix} (\Psi_n {}^t B^{-1} \Psi_n)^c & 0 \\ 0 & A \end{pmatrix}, \tag{2.5.1}$$

we have $M(F^+_{\bar{v}}) = M_{(n_{k+1},...,n_t)}(F_v) \times M_{(n_k,...,n_1)}(F_{v^c}) \hookrightarrow G(F^+_{\bar{v}})$ where $1 \leq k \leq t$ is so that $n_1 + ... + n_k = n$. We set $\theta_n : \mathrm{GL}_n(F_{v^c}) \cong \mathrm{GL}_n(F_v)$ to be the map $B \mapsto (\Psi_n {}^t B^{-1} \Psi_n)^c$ above.

2.6 Inertial local Langlands for GL_n

Set $L = F_v$ for some $v \in S_p(F)$. According to [BD84], the category $\mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])$ admits a direct sum decomposition

$$\oplus_\Omega \mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])[\Omega]$$

into so-called Bernstein blocks. In terms of the local Langlands correspondence, two *irreducible* representation $\pi_1, \pi_2 \in \mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])$ correspond to the same Bernstein block if and only if

$$\mathrm{rec}(\pi_1)^{ss}|_{I_L} \cong \mathrm{rec}(\pi_2)^{ss}|_{I_L},$$

and a general representation $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])$ lies in Ω if each of its Jordan–Hölder constituents does. Moreover, the centre $\mathfrak{z}_\Omega$ of the category $\mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])[\Omega]$ is called the *Bernstein centre* possessing the following property. Being the centre of the Bernstein block, it acts on each object lying in Ω. In particular, for any *irreducible* π lying in Ω, the natural action induces a character $\chi_\pi : \mathfrak{z}_\Omega \to \overline{E}$. Then, given a pair of irreducible objects

$\pi_1, \pi_2 \in \mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])[\Omega]$, $\chi_{\pi_1} = \chi_{\pi_2}$ if and only if π_1 and π_2 have the same supercuspidal support (cf. [BD84]).[6]

Work of Bushnell–Kutzko [BK99] shows that, given any Bernstein block Ω, there is always a pair (J, σ) of a compact open $J \subset G(\mathcal{O}_L)$ and an irreducible $\overline{E}$-representation of J such that $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])$ lies in Ω if and only if it is generated by its σ-isotypic vectors. Such a pair is then called a *semisimple Bushnell–Kutzko type* for the block Ω. Given a semisimple Bushnell–Kutzko type (J, σ), [BK98], Theorem 4.3 shows that taking σ-invariants sets up an equivalence of categories

$$\mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])[\Omega] \xrightarrow{\sim} \mathrm{Mod}(\mathcal{H}(\sigma)),$$

$$\pi \mapsto \mathrm{Hom}_J(\sigma, \pi) \cong \mathrm{Hom}_{G(L)}(\text{c-Ind}_J^{G(L)}\sigma, \pi)$$

where $\mathcal{H}(\sigma) := \mathrm{End}_{G(L)}(\text{c-Ind}_J^{G(L)}\sigma)$. In particular, we see that the action of the Bernstein centre on $\text{c-Ind}_J^{G(L)}\sigma$ identifies $\mathfrak{z}_\Omega$ with the centre $Z(\mathcal{H}(\sigma))$ of $\mathcal{H}(\sigma)$. Finally, given a Bernstein block Ω, as was observed in [Car+16b], §3.13,[7] after possibly enlarging E, the Bernstein centre $\mathfrak{z}_\Omega$ admits a model $\mathfrak{z}_{\Omega,E}$ over E acting on any $\pi \in \mathrm{Mod}_{\mathrm{sm}}(E[G(L)])$ such that $\pi \otimes_E \overline{\mathbf{Q}}_p$ lies in Ω. Moreover, for any type (J, σ) for Ω with a model σ_E over E, the natural action $\mathfrak{z}_{\Omega,E} \to \mathrm{End}_{G(L)}(\text{c-Ind}_J^{G(L)}\sigma_E) = \mathcal{H}(\sigma_E)$ induces an isomorphism $\mathfrak{z}_{\Omega,E} \cong Z(\mathcal{H}(\sigma_E))$ as one notes by combining [Car+16b], (3.15), Lemma 3.18 and Proposition 3.23. In particular, from now on for a given Bernstein block Ω we always assume that E is sufficiently large so that the centre is already defined over E.

In [SZ99], the authors refine the Bernstein decomposition of $\mathrm{Mod}_{\mathrm{sm}}(\overline{E}[G(L)])$ to a "stratification" of the category and construct a type theory with respect to this stratification as we will discuss now. We state their results in terms of the local Langlands correspondence following [BC09], §6.5.

Definition 2.6.1. We define a *Weil–Deligne inertial type* (of L over E) to be an isomorphism class of pairs $\tau = (\rho_\tau, N_\tau)$ such that $\rho_\tau : I_L \to \mathrm{GL}_n(E)$ is a representation of the inertia subgroup $I_L \subset W_L$ with open kernel, $N_\tau \in M_n(E)$ is a nilpotent matrix such that there exists a Weil–Deligne representation (r, N) of L and an isomorphism $(r, N)|_{I_L} \cong (\rho_\tau, N_\tau)$.

Given a nilpotent matrix $N \in M_m(E)$, its Jordan normal form gives rise to a partition P_N of m. A partition P can be viewed uniquely as a decreasing function $P : \mathbf{Z}_{>0} \to \mathbf{Z}_{\geq 0}$ with finite support (where P is a partition of $\sum_{i \in \mathbf{Z}_{>0}} P(i)$). Given two nilpotent matrices $N_1, N_2 \in M_m(E)$, we write $N_1 \preceq N_2$ if and only if $\sum_{1 \leq i \leq k} P_{N_1}(i) \leq \sum_{1 \leq i \leq k} P_{N_2}(i)$ for every $k \in \mathbf{Z}_{\geq 1}$. We record the following observation (cf. [BC09], Proposition 7.8.1).

---

[6] For a brief overview of these results stated with more care, see [Hel16], §3.

[7] See in particular *loc. cit.* Proposition 3.23.

Proposition 2.6.2. *Let $N_1, N_2 \in M_m(E)$ be two nilpotent matrices. Then $N_1 \preceq N_2$ if and only if, for all $i \in \mathbf{Z}_{\leq 1}$, we have*

$$\mathrm{rank}(N_1^i) \leq \mathrm{rank}(N_2^i).$$

Given a Weil–Deligne inertial type $\tau = (\rho_\tau, N_\tau)$ of L over E, and a finite dimensional irreducible $\overline{\mathbf{Q}}_p$-representation $\theta : I_L \to \mathrm{GL}(V_\theta)$ with open kernel, we can consider the θ-isotypic component $\rho_\tau[\theta] : I_L \to \mathrm{GL}(V_\tau[\theta])$ of $\rho_\tau \otimes_E \overline{\mathbf{Q}}_p$. As N_τ commutes with the action of I_L, it restricts to a nilpotent endomorphism $N_\tau[\theta] \in \mathrm{End}(V_\tau[\theta])$.

Definition 2.6.3. Given two Weil–Deligne inertial types τ_1, τ_2 of L, we write $\tau_1 \preceq \tau_2$ if $\rho_{\tau_1} \cong \rho_{\tau_2}$ and $N_{\tau_1}[\theta] \preceq N_{\tau_2}[\theta]$ for every irreducible $\overline{\mathbf{Q}}_p$-representation $\theta : I_L \to \mathrm{GL}(V_\theta)$ with open kernel. Moreover, given two Weil–Deligne representations r_1, r_2 of L, we write $r_1 \preceq r_2$ if $r_1|_{I_L} \preceq r_2|_{I_L}$.

We note that the partial order (on Weil–Deligne representations) appearing in Definition 2.6.3 is the one defined in [BC09], Definition 6.5.1,[8] [Var14], Defintion 8.3 and [Hun+18], Definition 2.5.3, respectively.

Theorem 2.6.4 ([SZ99])**.** *Let τ be a Weil–Deligne inertial type. Then there is a smooth irreducible E-representation $\sigma(\tau)$ of $G(\mathcal{O}_L)$ such that, for any irreducible smooth representation π of $G(L)$, the following hold.*

i. If $\pi|_{G(\mathcal{O}_L)}$ contains $\sigma(\tau)$, then $\mathrm{rec}(\pi)|_{I_L} \preceq \tau$;

ii. if $\mathrm{rec}(\pi)|_{I_L} \cong \tau$, then $\pi|_{G(\mathcal{O}_L)}$ contains $\sigma(\tau)$ with multiplicity one;

iii. if $\mathrm{rec}(\pi)|_{I_L} \preceq \tau$ and π is generic, then $\pi|_{G(\mathcal{O}_L)}$ contains $\sigma(\tau)$.

Proof. See [Hun+18], Theorem 2.5.4 and the references therein. □

We point out that the Theorem makes no mention about the uniqueness of $\sigma(\tau)$. Throughout this thesis, given a Weil–Deligne inertial type τ, we work with the $\sigma(\tau)$ constructed in [SZ99].

Remark 2.6.5. Notice that when $\tau = (\rho_\tau, N_\tau)$ is so that $N_\tau = 0$, we obtain the potentially crystalline inertial local Langlands of [Car+16b], Theorem 3.7.

Note that each Weil–Deligne inertial type τ gives rise to an inertial type ρ_τ in the classical sense and henceforth to a Bernstein block Ω. Moreover, if (J, σ) is a type for Ω, then, by construction, $\sigma(\tau)$ is a direct summand of $\text{c-Ind}_J^{G(\mathcal{O}_L)}\sigma$. In particular, $\mathfrak{z}_\Omega$ acts on $\text{c-Ind}_{G(\mathcal{O}_L)}^{G(L)}\sigma(\tau)$. One sees that this, in fact, is a faithful action,[9] yielding an injection $\mathfrak{z}_\Omega \hookrightarrow \mathfrak{z}_\tau := Z(\mathcal{H}(\sigma(\tau)))$.

---

[8] They introduce the partial order for irreducible smooth representations of $\mathrm{GL}_n(L)$ but, under the local Langlands correspondence, it translates to our definition (cf [BC09], Proposition 7.8.1).

[9] In fact, [Pyv20a], Theorem 7.1 shows that $\mathcal{H}(\sigma(\tau))$ is a free module over $\mathfrak{z}_\Omega$.

2.7 Local systems on $\widetilde{X}_{\widetilde{K}}$ and X_K

We now introduce the local systems we will be working with. These will be constructed from locally algebraic representations $\sigma_{\mathrm{alg}} \otimes \sigma_{\mathrm{sm}}$ where the algebraic part σ_{alg} will encode the weight of the automorphic representations considered (i.e. their shape at ∞) and the smooth part σ_{sm} will pin down their inertial type at p.

Our integral coefficient systems will depend on the choice of a prime p. We further set $E/\mathbf{Q}_p$ to be our coefficient field, a sufficiently large subfield of $\overline{\mathbf{Q}}_p$ finite over $\mathbf{Q}_p$ such that $\mathrm{Hom}(F, E) = \mathrm{Hom}(F, \overline{\mathbf{Q}}_p)$, and denote by $\mathcal{O}$ its ring of integers. Fix a choice of uniformiser $\varpi \in \mathcal{O}$ as well.

We start by introducing our locally algebraic representations for G. For each p-adic place v of F, we consider a standard (possibly non-proper) parabolic subgroup $Q_v \subset \mathrm{GL}_n$ with standard Levi decomposition $Q_v = M_v \ltimes N_v$ and consider the corresponding parahoric subgroup scheme ${}^v\mathcal{Q}$ of GL_n. For $S \subset S_p$, we set $Q_S := \prod_{v \in S} Q_v$ and $Q_p := Q_{S_p}$. Set $\mathcal{Q}_v := {}^v\mathcal{Q}(\mathcal{O}_{F_v}) \subset \mathrm{GL}_n(\mathcal{O}_{F_v})$ to be the corresponding parahoric subgroup. For a finite set of p-adic places $S \subset S_p$, we set $\mathcal{Q}_S := \prod_{v \in S} \mathcal{Q}_v \subset \mathrm{GL}_n(\widehat{\mathcal{O}}_{F,S})$. Moreover, for any $v \in S_p$, and integers $c \geq b \geq 0$ with $c \geq 1$, we denote by $\mathcal{Q}_v(b, c) \subset \mathcal{Q}_v$ the subgroup of matrices which are block upper triangular modulo ϖ_v^c and block unipotent modulo ϖ_v^b, where $\varpi_v \in \mathcal{O}_{F_v}$ is some choice of uniformiser. Extend the definition in the obvious way to define $\mathcal{Q}_S(b, c)$. In particular, we have $\mathcal{Q}_v(0, 1) = \mathcal{Q}_v$. Note that $\mathcal{Q}_v(b, c)$ admits an Iwahori decomposition $\overline{N}_v^c M_v^b N_v^0$[10] and therefore the formalism of [All+23], 2.1.9 applies.

Such parahoric subgroups will be our level subgroups for which we introduce locally algebraic representations. These representations will then yield local systems convenient for the development of Q_p-ordinary Hida theory. After taking Q_p-ordinary parts, the cohomology of these local systems will encode integral Q_p-ordinary automorphic representations with prescribed weights at ∞ and inertial types at p. When Q_p is taken to be $\prod_v \mathrm{GL}_n$, this will simply mean pinning down the weight of the automorphic and the inertial type at p of the whole automorphic representation.

We first take care of the algebraic part. As usual, the character group of $(\mathrm{Res}_{F^+/\mathbf{Q}} T)_E = (\mathrm{Res}_{F/\mathbf{Q}} T_n)_E$, for $T_n \subset \mathrm{GL}_n$ the subgroup of diagonal matrices, can be identified with $(\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$. Denote by $\mathbf{Z}^n_+ \subset \mathbf{Z}^n$ the subset of tuples $(k_1, ..., k_n)$ satisfying

$$k_1 \geq ... \geq k_n.$$

A character $\lambda = (\lambda_{\iota,i}) \in (\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$ will then be $(\mathrm{Res}_{F^+/\mathbf{Q}} B)_E = (\mathrm{Res}_{F/\mathbf{Q}} B_n)_E$-dominant if and only if, it lies in $(\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$. In other words, for every $\iota \in \mathrm{Hom}(F, E)$, we have

$$\lambda_{\iota,1} \geq ... \geq \lambda_{\iota,n}.$$

---

[10] Recall from 1.3 that for H being any of the groups $\overline{N}_v, M_v$, or N_v, and $n \in \mathbf{Z}_{\leq 0}$, H^n denotes the subgroup of matrices of $H(\mathcal{O}_{F_v})$ that reduce to the identity modulo ϖ_v^n.

Given $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, highest weight theory provides an integral representation of $\prod_{\iota:F\hookrightarrow E}\mathrm{GL}_n(\mathcal{O})$. This will simply be the representation constructed for instance in [Ger18], §2.2. More precisely, if $B_n \subset \mathrm{GL}_n$ is the standard Borel of upper triangular matrices, $w_{0,n}$ denotes the longest element of the Weyl group of GL_n and $\iota \in \mathrm{Hom}(F,E)$, we consider the algebraic induction

$$\xi_{\lambda_\iota} := (\mathrm{Ind}_{B_n}^{\mathrm{GL}_n} w_{0,n}\lambda_\iota)_{/\mathcal{O}} := \{f \in \mathcal{O}[\mathrm{GL}_n] \mid f(bg) = (w_{0,n}\lambda_\iota)(b)f(g)$$

$$\text{for every } \mathcal{O} \to R, b \in B_n(R), g \in \mathrm{GL}_n(R)\},$$

and set $\mathcal{V}_{\lambda_\iota} := \xi_{\lambda_\iota}(\mathcal{O})$, $V_{\lambda_\iota} := \mathcal{V}_{\lambda_\iota} \otimes_{\mathcal{O}} E$. We then set $\mathcal{V}_\lambda := \otimes_{\iota,\mathcal{O}}\mathcal{V}_{\lambda_\iota}$ and $V_\lambda := \mathcal{V}_\lambda \otimes_{\mathcal{O}} E$. Note that V_λ is the highest weight representation of $(\mathrm{Res}_{F^+/\mathbf{Q}}G)_E = (\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n)_E \cong \prod_{\iota:F\hookrightarrow E}\mathrm{GL}_{n,E}$ of highest weight λ, a finite dimensional E-representation. Moreover, $\mathcal{V}_\lambda \subset V_\lambda$ is a $G(\mathcal{O})$-stable $\mathcal{O}$-lattice. In particular, for every $m \in \mathbf{Z}_{\geq 1}$, $\mathcal{V}_\lambda/\varpi^m$ is a smooth $\mathcal{O}/\varpi^m[\prod_{v\in S_p}\mathrm{GL}_n(\mathcal{O}_{F_v})]$-module[11] under the product of diagonal embeddings $\mathrm{GL}_n(\mathcal{O}_{F_v}) \hookrightarrow \prod_{\iota:F_v\hookrightarrow E}\mathrm{GL}_n(\mathcal{O})$, finite free over $\mathcal{O}/\varpi^m$, and the formalism of §2.2 applies.

For a dominant weight $\lambda_v \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F_v,E)}$, set $\mathcal{V}_{w_0^{Q_v}\lambda_v} = \otimes_{\iota:F_v\hookrightarrow E}\mathcal{V}_{w_0^{Q_v}\lambda_\iota}$ to be the representation of $\prod_{\iota:F_v\hookrightarrow E}M_v(\mathcal{O})$ associated with $w_0^{Q_v}\lambda_v$ by the previous procedure where $w_0^{Q_v} = w_0^{M_v}w_0^{\mathrm{GL}_n}$ denotes the product of the longest Weyl group element of M_v, and of GL_n. Concretely, if we assume that $Q_v = P_{(n_1,..,n_k)}$, then $w_0^{Q_v}\lambda_v = (\lambda_v^{n_1},...,\lambda_v^{n_k})$, where

$$\lambda_v^{n_i} = (\lambda_\iota^{n_i})_{\iota:F_v\hookrightarrow E} = (\lambda_{\iota,n+1-(n_1+...+n_i)},...,\lambda_{\iota,n+1-(n_1+...+n_{i-1}+1)})_\iota \in (\mathbf{Z}_+^{n_i})^{\mathrm{Hom}(F_v,E)},$$ [12]

and

$$\mathcal{V}_{w_0^{Q_v}\lambda_v} = \mathcal{V}_{\lambda_v^{n_1}} \otimes ... \otimes \mathcal{V}_{\lambda_v^{n_k}}.$$

We then have the analogue of [CN23], Lemma 2.1.12 with identical proof.

Lemma 2.7.1. *The natural* $\prod_{\iota:F_v\hookrightarrow E}Q_v(\mathcal{O})$*-equivariant map*

$$\mathcal{V}_{\lambda_v} \to \mathcal{V}_{w_0^{Q_v}\lambda_v}$$

given by evaluation of functions at the identity is a surjection.

We now turn to the smooth part which will be given by inflating to parahoric level the types of Schneider–Zink. Therefore, the smooth part will in fact depend on the choice of Q_v. For $v \in S_p$ consider a standard parabolic Q_v and assume that it corresponds to a partition $(n_1,...,n_k)$ of n.

Definition 2.7.2. We call a tuple $\underline{\tau_v} = (\tau_{v,i})_{i=1,...,k}$ consisting of (E-valued) n_i-dimensional Weil–Deligne inertial types $\tau_{v,i}$ for F_v to be an inertial type of type Q_v.

---

[11] In particular, it becomes a smooth $\mathcal{O}/\varpi^m[\mathcal{Q}_{S_p}]$-module.

[12] Here we use the convention $n_0 = 0$.

Given an inertial type $\underline{\tau_v}$ of type Q_v, Schneider–Zink provides a smooth irreducible E-representation $\sigma(\tau_{v,i})$ of $\mathrm{GL}_{n_i}(\mathcal{O}_{F_v})$ for every $i = 1, ..., k$ (cf. Theorem 2.6.4). In particular, we obtain an irreducible smooth E-representation

$$\sigma(\underline{\tau_v}) := \otimes_{i=1,...,k}\sigma(\tau_{v,i})$$

of $M_v^0 = M_v(\mathcal{O}_{F_v})$. We fix a choice of M_v^0-stable $\mathcal{O}$-lattice $\sigma(\underline{\tau_v})^\circ \subset \sigma(\underline{\tau_v})$. We set $c_v \geq 0$ to be the smallest integer such that $M_v^{c_v} = \ker(M_v^0 \to \prod_{v\in S_p} M_v(\mathcal{O}_{F_v}/\varpi^{c_v}))$ acts trivially on $\sigma(\underline{\tau_v})^\circ$. Then $\mathcal{Q}_v(0, c_v)$ acts on $\sigma(\underline{\tau_v})^\circ$ by sending

$$\begin{pmatrix} A_1 & . & . & . & * \\ . & A_2 & & & . \\ . & & . & & . \\ . & & & . & . \\ * & . & . & . & A_k \end{pmatrix} \in \mathcal{Q}_v(0, c_v)$$

to $\sigma(\underline{\tau_v})^\circ(A_1, ..., A_k)$. Since the corresponding map

$$\mathcal{Q}_v(0, c_v) \to M_v^0/M_v^{c_v} = M_v(\mathcal{O}_{F_v}/\varpi_v^{c_v})$$

is easily checked to be a group homomorphism, this indeed defines a group action. We then denote by $\widetilde{\sigma(\underline{\tau_v})}^{\circ}$ the representation $\sigma(\underline{\tau_v})^\circ$ viewed as an $\mathcal{O}[\mathcal{Q}_v(0, c_v)]$-module.

We then set our locally algebraic representation associated with the data $(Q_{S_p}, \lambda, \underline{\tau}) := (Q_v, \lambda_v, \underline{\tau_v})_{v\in S_p}$ to be

$$\mathcal{V}_{(\lambda,\underline{\tau})}^{Q_p} := \mathcal{V}_\lambda \otimes_{\mathcal{O}} (\bigotimes_{v\in S_p,\mathcal{O}} \widetilde{\sigma(\underline{\tau_v})}^{\circ})^\vee.$$

This gives rise to a $\mathcal{O}/\varpi^m[\mathcal{Q}_p(0, c_p)]$-module for $c_p := \max\{c_v\}_{v\in S_p}$. By abuse of notation, we will use the same notation for the induced local systems on $X_K, \overline{X}_K$ and ∂X_K, respectively, for $K \subset G(\mathbf{A}_{F^+}^\infty)$ any good subgroup with $K_p \subset \mathcal{Q}_p(0, c_p)$.

Note that when, at each place $v \in S_p$, $Q_v = \mathrm{GL}_n$, these local systems simplify. In this case we will abbreviate the notation to $\mathcal{V}_{(\lambda,\tau)} = \mathcal{V}_\lambda \otimes_{\mathcal{O}} \sigma(\tau)^{\circ,\vee}$. We denote the corresponding locally algebraic type by $\sigma(\lambda, \tau)^\circ := (\mathcal{V}_{(\lambda,\tau)})^\vee$. By Lemma 2.3.1, the corresponding mod ϖ^m cohomology groups are obtained by taking invariants of completed cohomology

$$R\Gamma_{(c)}(X_K, \mathcal{V}_{(\lambda,\tau)}/\varpi^m) \cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[K_p]}(\sigma(\lambda, \tau)^\circ/\varpi^m, \pi_{(c)}(K^p, \mathcal{O}/\varpi^m)) \cong$$

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[G_p]}(\text{c-Ind}_{K_p}^{G_p}\sigma(\lambda, \tau)^\circ/\varpi^m, \pi_{(c)}(K^p, \mathcal{O}/\varpi^m)).$$

As a consequence, we get a natural right action of the Hecke algebra $\mathcal{H}(\sigma(\lambda, \tau)^\circ) = \mathrm{End}_{G_p}(\text{c-Ind}_{K_p}^{G_p}\sigma(\lambda, \tau)^\circ)$ on $R\Gamma_{(c)}(X_K, \mathcal{V}_{(\lambda,\tau)}/\varpi^m)$. In particular, as we will explain in 5.3, we obtain a natural action of an "integral Bernstein centre" $\mathfrak{z}_{\lambda,\tau}^\circ := \mathcal{H}(\sigma(\lambda, \tau)^\circ) \cap \mathfrak{z}_\Omega$. In fact, we will see that we have a similar description for general Q after taking Q-ordinary parts due to independence of level and weight, yielding natural Hecke actions at $\underline{p}$.

We now turn to treating the case of $\widetilde{G}$. From now on we introduce the following running assumption on F.

Assumption 2.7.3. *Assume that our imaginary CM field F/F^+ is so that every $\overline{v} \in \overline{S}_p = S_p(F^+)$ splits in F.*

Therefore we can write $\bar{v} = v \cdot v^c$ in F for each $\bar{v} \in \overline{S}_p$. In particular, we fix a choice of a preferred place $v \mid \overline{v}$ in F. This fixes a lift $\iota : F \hookrightarrow E$ for every embedding $\overline{\iota} : F^+ \hookrightarrow E$. Therefore, it induces an identification

$$(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_E = \prod_{\mathrm{Hom}(F^+,E)} \mathrm{GL}_{2n,E}$$

and an identification of the character group of $(\mathrm{Res}_{F^+/\mathbf{Q}}T)_E$ with $(\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$. This identifies a weight $\lambda = (\lambda_{\iota,i}) \in (\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$ with $\tilde{\lambda} = (\tilde{\lambda}_{\overline{\iota},i})$ where

$$\tilde{\lambda}_{\overline{\iota}} = (-w_0^{\mathrm{GL}_n}\lambda_{\iota c}, \lambda_\iota) = (-\lambda_{\iota c,n}, ..., -\lambda_{\iota c,1}, \lambda_{\iota,1}, ..., \lambda_{\iota,n}).$$

Note that the $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{B})_E$-dominant weights are precisely given by $(\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$. For such weights we can therefore define $\mathcal{V}_{\tilde{\lambda}} \subset V_{\tilde{\lambda}}$. For every $m \in \mathbf{Z}_{\geq 1}$, we then obtain a smooth $\mathcal{O}/\varpi^m[\prod_{\bar{v}\in\overline{S}_p}\widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})]$-module $\mathcal{V}_{\tilde{\lambda}}/\varpi^m$, finite free as an $\mathcal{O}/\varpi^m$-module. These cover the algebraic parts of our locally algebraic representations.

For $\widetilde{G}$ we will only take ordinary parts at a certain subset $\overline{S} \subset \overline{S}_p$ where we wish to prove local-global compatibility using the degree shifting argument and will vary the level at the rest of the places. Therefore, we fix such a set $\overline{S} \subset \overline{S}_p$ of p-adic places and will introduce locally algebraic representations that are only non-algebraic at $\overline{S}$. Namely, if for $\bar{v} \in \overline{S}$ given a tuple $(Q_{v'}, \lambda_{v'}, \underline{\tau_{v'}})_{v'|\bar{v}}$ as before such that the weight $\tilde{\lambda}_{\bar{v}}$ associated with $(\lambda_v, \lambda_{v^c})$ is dominant (i.e. lies in $(\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$), we introduce a tuple $(\widetilde{Q}_{\bar{v}}, \tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}} = (\underline{\tau_v}, \underline{\tau_{v^c}}))$ where $\widetilde{Q}_{\bar{v}} = \widetilde{M}_{\bar{v}} \ltimes \widetilde{N}_{\bar{v}} \subset \widetilde{G}_{F^+_{\bar{v}}}$ is the standard parabolic subgroup with $\widetilde{Q}_{\bar{v}} \cap G_{F^+_{\bar{v}}} = Q_v \times Q_{v^c} \subset \mathrm{GL}_{n,F_v} \times \mathrm{GL}_{n,F_{v^c}}$. As before, we denote by ${}^{\bar{v}}\widetilde{\mathcal{Q}}$ the corresponding parahoric subgroup scheme of $\widetilde{G}_{F^+_{\bar{v}}} \cong^{\iota_v} \mathrm{GL}_{2n,F_v}$. We set $\widetilde{\mathcal{Q}}_{\bar{v}} = {}^{\bar{v}}\widetilde{\mathcal{Q}}(\mathcal{O}_{F^+_{\bar{v}}})$ with Iwahori decomposition $\overline{\widetilde{N}}^1_{\bar{v}}\widetilde{M}^0_{\bar{v}}\widetilde{N}^0_{\bar{v}}$. We define the identification

$$\iota_v^{w_0} : \mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}) \xrightarrow{\sim} \mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_v),$$

$$(A, B) \mapsto (A, \theta_n B).$$

We then set $\widetilde{Q}^{w_0}_{\bar{v}} = \widetilde{M}^{w_0}_{\bar{v}} \ltimes \widetilde{N}^{w_0}_{\bar{v}} \subset \widetilde{G}_{F^+_{\bar{v}}}$ to be the standard parabolic subgroup associated with the Levi $\iota_v^{w_0}\widetilde{M}_{\bar{v}} \subset \mathrm{GL}_{2n,F_v}$ under the identification ι_v. Set

$$\tilde{\sigma}(\underline{\tau_{\bar{v}}})^\circ := \sigma(\underline{\tau_v})^\circ \otimes_{\mathcal{O}} (\theta_n^{-1})^* \sigma(\underline{\tau_{v^c}})^\circ \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}[\widetilde{M}_{\bar{v}}^{w_0,0}]).^{13}$$

Note that here $\theta_n : \mathrm{GL}_n(F_{v^c}) \cong \mathrm{GL}_n(F_v)$, as previously defined, is given by the map $B \mapsto (\Psi_n {}^t B^{-1} \Psi_n)^c$ and we are applying the pullback along θ_n^{-1}. For $\bar{v} \in \overline{S}$, define

$$\mathcal{V}_{(\tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}}^{w_0}} := \mathcal{V}_{\tilde{\lambda}_{\bar{v}}} \otimes_{\mathcal{O}} \widetilde{\tilde{\sigma}(\underline{\tau_{\bar{v}}})^\circ}^{\vee}$$

where $\tilde{\lambda}_{\bar{v}}$ have the obvious meaning of considering the weights corresponding to all the embeddings inducing the given place.

Finally, given a tuple $(\widetilde{Q}_{\overline{S}}, \tilde{\lambda}_{\overline{S}}, \underline{\tau_{\overline{S}}}) = (\widetilde{Q}_{\bar{v}}, \tilde{\lambda}_{\bar{v}}, \underline{\tau}_{\bar{v}})_{\bar{v} \in \overline{S}}$ coming from a tuple $(Q_p, \lambda, \underline{\tau})$ as above, and a dominant weight $\tilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ for $\widetilde{G}$ extending $\tilde{\lambda}_{\overline{S}}$, we set

$$\mathcal{V}_{(\tilde{\lambda}, \underline{\tau})}^{\widetilde{Q}_{\overline{S}}^{w_0}} := \Big(\bigotimes_{\bar{v} \notin \overline{S}, \mathcal{O}} \mathcal{V}_{\tilde{\lambda}_{\bar{v}}} \Big) \otimes_{\mathcal{O}} \Big(\bigotimes_{\bar{v} \in \overline{S}, \mathcal{O}} \mathcal{V}_{(\tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}}^{w_0}} \Big),$$

a locally algebraic $\mathcal{O}$-representation of $\left(\prod_{\bar{v} \in \overline{S}_p \setminus \overline{S}} \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})\right) \times \widetilde{\mathcal{Q}}_{\overline{S}}^{w_0}(0, c_p)$ for an appropriate integer $c_p \geq 1$ as before. In particular, $\mathcal{V}_{(\tilde{\lambda}, \underline{\tau})}^{\widetilde{Q}_{\overline{S}}^{w_0}} / \varpi^m$ becomes a smooth $\left(\prod_{\bar{v} \in \overline{S}_p \setminus \overline{S}} \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})\right) \times \widetilde{\mathcal{Q}}_{\overline{S}}^{w_0}(0, c_p)$-module. When, for every $\bar{v} \in \overline{S}$, the parabolic at $\bar{v}$ is the Siegel one, we will abbreviate the notation to $\mathcal{V}_{(\tilde{\lambda}, \tau)}^{\overline{S}}$. Again, by abuse of notation we will denote identically the local systems they induce on locally symmetric spaces.

Example 2.7.4. For the convenience of the reader, we spell out an example of $\widetilde{Q}_{\bar{v}}$, $\widetilde{Q}_{\bar{v}}^{w_0}$, and $\widetilde{\sigma}(\underline{\tau_{\bar{v}}})^\circ$.

Let $n = 3$, fix $\bar{v} \in \overline{S}$ and write $\bar{v} = v \cdot v^c$. We set $Q_v = \mathrm{GL}_3$ and $Q_{v^c} = P_{(1,2)} \subset \mathrm{GL}_{3,F_{v^c}}$, the standard parabolic subgroup with standard Levi subgroup $M_{(1,2)} = \mathrm{GL}_1 \times \mathrm{GL}_2 \subset \mathrm{GL}_{3,F_{v^c}}$. Then $\widetilde{Q}_{\bar{v}} \subset \widetilde{G}_{F^+_{\bar{v}}}$ is the standard parabolic subgroup with standard Levi subgroup

$$\widetilde{M}_{\bar{v}} = \mathrm{GL}_3 \times M_{(1,2)} \subset \mathrm{GL}_{3,F_v} \times \mathrm{GL}_{3,F_{v^c}}.$$

In particular, $\iota_v \widetilde{Q}_{\bar{v}} = P_{(2,1,3)} \subset \mathrm{GL}_{6,F_v}$. Moreover, $\iota_v^{w_0}\widetilde{M}_{\bar{v}} = M_{(3,2,1)} \subset \mathrm{GL}_{6,F_v}$. Comsequently, $\widetilde{Q}_{\bar{v}}^{w_0} \subset \widetilde{G}_{F^+_{\bar{v}}}$ is the standard parabolic subgroup satisfying $\iota_v \widetilde{Q}_{\bar{v}}^{w_0} = P_{(3,2,1)} \subset \mathrm{GL}_{6,F_v}$. Therefore, its standard Levi subgroup is

$$\widetilde{M}_{\bar{v}}^{w_0} = M_{(2,1)} \times \mathrm{GL}_3 \subset \mathrm{GL}_{3,F_v} \times \mathrm{GL}_{3,F_{v^c}}.$$

Further consider Weil–Deligne inertial types $\underline{\tau_v} = (\tau_v)$, and $\underline{\tau_{v^c}} = (\tau_{v^c,1}, \tau_{v^c,2})$ and (a choice of) associated smooth $\mathcal{O}$-representations $\sigma(\underline{\tau_v})^\circ = \sigma(\tau_v)^\circ$, and

---

[13] We note that $\tilde{\sigma}(\underline{\tau_{\bar{v}}})^\circ$ is a representation of $\iota_v \widetilde{M}_{\bar{v}}^{w_0,0} \cong \iota_v^{w_0}\widetilde{M}_{\bar{v}}^0 \subset \mathrm{GL}_{2n}(\mathcal{O}_{F_v})$ and we view it as a representation of $\widetilde{M}_{\bar{v}}^{w_0,0}$ via ι_v.

$\sigma(\underline{\tau_{v^c}})^\circ = \sigma(\tau_{v^c,1})^\circ \otimes \sigma(\tau_{v^c,2})^\circ$ of $\mathrm{GL}_3(\mathcal{O}_{F_v})$, and $M_{(1,2)}(\mathcal{O}_{F_{v^c}}) = \mathrm{GL}_1(\mathcal{O}_{F_{v^c}}) \times \mathrm{GL}_2(\mathcal{O}_{F_{v^c}})$, respectively. Then $\widetilde{\sigma}(\underline{\tau_{\bar{v}}})^\circ$, as a representation of $\iota_v \widetilde{M}_{\bar{v}}^{w_0,0} = \mathrm{GL}_3(\mathcal{O}_{F_v}) \times \mathrm{GL}_2(\mathcal{O}_{F_v}) \times \mathrm{GL}_1(\mathcal{O}_{F_v})$, is given by

$$\sigma(\tau_v)^\circ \otimes \left((\theta_2^{-1})^* \sigma(\tau_{v^c,2})^\circ \otimes (\theta_1^{-1})^* \sigma(\tau_{v^c,1})^\circ\right).$$

In particular, as a representation of $\widetilde{M}_{\bar{v}}^{w_0,0} = \mathrm{GL}_2(\mathcal{O}_{F_v}) \times \mathrm{GL}_1(\mathcal{O}_{F_v}) \times \mathrm{GL}_3(\mathcal{O}_{F_{v^c}})$ via the identification ι_v, it is given by

$$\left((\theta_2^{-1})^* \sigma(\tau_{v^c,2})^\circ \otimes (\theta_1^{-1})^* \sigma(\tau_{v^c,1})^\circ\right) \otimes (\theta_3)^* \sigma(\tau_{v,3})^\circ.$$

We also introduce a local system on $\widetilde{X}_{\widetilde{K}}$ suitable for a *dual* version of the degree shifting argument in §6.1. For this we assume now that, for each $\bar{v} \in \overline{S}$, $\tilde{\lambda}_{\bar{v}} := (\lambda_v, -w_0^{\mathrm{GL}_n} \lambda_{v^c})$ is dominant for $\widetilde{G}$.[14] Then, for $\bar{v} \in \overline{S}$, we set

$$\tilde{\sigma}(\underline{\tau_{\bar{v}}})^{\circ,w_0} := (\theta_n^{-1})^* \sigma(\underline{\tau_{v^c}})^\circ \otimes_{\mathcal{O}} \sigma(\underline{\tau_v})^\circ \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}[\widetilde{M}_{\bar{v}}^0]),^{15}$$

and

$$\mathcal{V}_{(\tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}}, w_0^P} := \mathcal{V}_{w_0^P \tilde{\lambda}_{\bar{v}}} \otimes_{\mathcal{O}} (\tilde{\sigma}(\underline{\tau_{\bar{v}}})^{\circ,w_0})^{\sim,\vee} \in \mathrm{Mod}(\mathcal{O}[\widetilde{\mathcal{Q}}(0, c_p)]).$$

For any dominant weight $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ extending $\widetilde{\lambda}_{\overline{S}}$, we can then set

$$\mathcal{V}_{(\tilde{\lambda}, \underline{\tau})}^{\widetilde{Q}_{\overline{S}}, w_0^P} := \Big(\bigotimes_{\bar{v} \notin \overline{S}, \mathcal{O}} \mathcal{V}_{\tilde{\lambda}_{\bar{v}}}\Big) \otimes_{\mathcal{O}} \Big(\bigotimes_{\bar{v} \in \overline{S}, \mathcal{O}} \mathcal{V}_{(\tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}},})}^{\widetilde{Q}_{\bar{v}}, w_0^P}\Big)$$

an $\mathcal{O}$-representation of $\left(\prod_{\bar{v} \in \overline{S}_p \setminus \overline{S}} \widetilde{G}(\mathcal{O}_{F_{\bar{v}}^+})\right) \times \widetilde{\mathcal{Q}}_{\overline{S}}(0, c_p)$. We denote identically the corresponding local systems induced on locally symmetric spaces for $\widetilde{G}$.

2.8 Explicit Hecke operators

Next we spell out the explicit formula for the usual unramified Hecke operators and U_p-operators as for instance in [NT16], [All+23] and [CN23]. Fix for any v finite place of F a uniformiser $\varpi_v \in \mathcal{O}_{F_v}$. We start by introducing the usual explicit Hecke operators at unramified places. Let v be a finite place of F, and $1 \leq i \leq n$ an integer. Write $T_{v,i} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))$ for the double coset operator

$$T_{v,i} = [\mathrm{GL}_n(\mathcal{O}_{F_v}) \mathrm{diag}(\varpi_v, ..., \varpi_v, 1, ..., 1) \mathrm{GL}_n(\mathcal{O}_{F_v})]$$

---

[14] Note that this identification $(\mathbf{Z}^n)^{\mathrm{Hom}(F,E)} \cong (\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$ is *not* the one used in the previous paragraph.

[15] Again, the representation present is a representation of $\iota_v \widetilde{M}_{\bar{v}}^0$ and we view it as a representation of $\widetilde{M}_{\bar{v}}^0$ via ι_v. Note that in particular, it is simply the representation $\sigma(\underline{\tau_v})^\circ \otimes_{\mathcal{O}} \sigma(\underline{\tau_{v^c}})^\circ$ of $\widetilde{M}_{\bar{v}}^0$.

where ϖ_v appears exactly i times in the diagonal. We define the polynomial

$$P_v(X) = X^n - T_{v,1}X^{n-1} + ... + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + ...$$
$$+ q_v^{n(n-1)/2} T_{v,n} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))[X]$$

where recall that $q_v = |\mathcal{O}_{F_v}/\varpi_v|$. Note that $P_v(X)$ corresponds to the characteristic polynomial of the Frobenius element acting on $\mathrm{rec}^T_{F_v}(\pi_v)$ for π_v any unramified representation of $\mathrm{GL}_n(F_v)$.

If $\bar{v}$ is a finite place of F^+, unramified in F, v is a choice of place of F above it, and $1 \leq j \leq 2n$ is an integer, then we denote by $\widetilde{T}_{v,j} \in \mathcal{H}(\widetilde{G}(F^+_{\bar{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})) \otimes_{\mathbf{Z}} \mathbf{Z}[q_{\bar{v}}^{-1}]$ the Hecke operator denoted by $T_{G,v,j}$ in [NT16], Proposition-Definition 5.2. In particular, $q_v^{-j(2n-j)/2}\widetilde{T}_{v,j}$ is the operator corresponding to the ith symmetric polynomial in $2n$ variables under the dual map on Hecke algebras corresponding to the unramified endoscopic transfer from $\widetilde{G}(F^+_{\bar{v}})$ to $\mathrm{GL}_{2n}(F_v)$ (where we also apply the (normalised) Satake isomorphism). We then define the polynomial

$$\widetilde{P}_v(X) = X^{2n} - \widetilde{T}_{v,1}X^{2n-1} + ... + (-1)^j q_v^{j(j-1)/2} \widetilde{T}_{v,j} X^{2n-j} + ...$$
$$+ q_v^{2n(2n-1)/2} \widetilde{T}_{v,2n} \in \mathcal{H}(\widetilde{G}(F^+_{\bar{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})) \otimes_{\mathbf{Z}} \mathbf{Z}[q_{\bar{v}}^{-1}][X].$$

This then corresponds to the characteristic polynomial of the Frobenius element acting on $\mathrm{rec}^T_{F_v}(\pi_v)$, where π_v is the base change with respect to $F_v/F^+_{\bar{v}}$ of any unramified representation $\sigma_{\bar{v}}$ of $\widetilde{G}(F^+_{\bar{v}})$.

We finally describe the effect of the unnormalised Satake transform $\mathcal{S} = r_G \circ r_P$ (for the notation, see the end of section 2.2) at unramified places. We use the following notation: for $f(X)$ a polynomial of degree d, with constant term a unit a_0, set $f^\vee(X) := a_0^{-1} X^d f(X^{-1})$. Therefore, $f^\vee(X)$ is the monic polynomial with zeroes given by the inverse of the zeroes of $f(X)$. We then have the following.

Proposition 2.8.1. *Let v be a finite place of F, unramified above the place $\bar{v}$ of F^+. Then the unnormalised Satake transform*

$$\mathcal{S} : \mathcal{H}(\widetilde{G}(F^+_{\bar{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})) \to \mathcal{H}(G(F^+_{\bar{v}}), G(\mathcal{O}_{F^+_{\bar{v}}}))$$

sends $\widetilde{P}_v(X)$ to $P_v(X) q_v^{n(2n-1)} P^\vee_{v^c}(q_v^{1-2n}X)$.

Proof. This follows from the explicit formula for $\mathcal{S}$ given in [NT16], Proposition-Definition 5.3. □

We now turn to discussing the U_p-operators we will consider. As before, we assume that every p-adic place $\bar{v} \in \overline{S}_p$ splits in F and we fix a choice of $v|\bar{v}$ in S_p. We start with the case of G. Consider a tuple $(Q_{S_p}, \lambda, \underline{\tau}) = (Q_v, \lambda_v, \underline{\tau_v})_{v \in S_p}$ with Levi decomposition $Q_v = M_v \ltimes N_v$. Set X_{Q_v} to be the set of B_n-dominant cocharacters in $X_*(Z(M_v))$. Concretely, if $Q_v = P_{(n_1,\dots,n_k)} \subset \mathrm{GL}_n$, then it identifies in $X_*(T_n)^+ = \mathbf{Z}_+^n$ with the elements with jumps only at the indices $n_1 + ... + n_j$ for $1 \leq j \leq k$. For $c \geq 1$, we then define the subset $\Delta_{\mathcal{Q}_v}(c) \subset \mathrm{GL}_n(F_v)$ given by

$$\Delta_{\mathcal{Q}_v}(c) := \coprod_{\nu \in X_{Q_v}} \mathcal{Q}_v(0,c)\nu(\varpi_v)\mathcal{Q}_v(0,c).$$

By [CN23], Lemma 2.1.15, the elements $\nu(\varpi_v)$ are $\mathcal{Q}_v(0,c)$-positive in the sense of *loc. cit.*. In particular, $\Delta_{\mathcal{Q}_v}(c)$ forms a monoid and $\Delta_{M_v}^+ := \Delta_{\mathcal{Q}_v}(c) \cap M_v(F_v) \subset M_v(F_v)^+ = \{m \in M_v(F_v) \mid mN_v^0 m^{-1} \subset N_v^0 \text{ and } m^{-1}\overline{N}_v^1 m \subset \overline{N}^1\}$ so the formalism of [All+23], 2.1.9 applies. Set Δ_{M_v} to be the group generated by $\Delta_{M_v}^+$. One proves that the map

$$\mathcal{H}(\Delta_{M_v}^+, M_v^0) \to \mathcal{H}(\Delta_{\mathcal{Q}_v}(c), \mathcal{Q}_v(0,c)),$$

$$[M_v^0 \nu(\varpi_v) M_v^0] \mapsto [\mathcal{Q}_v(0,c)\nu(\varpi_v)\mathcal{Q}_v(0,c)]$$

is an isomorphism of Hecke algebras (cf. [BK98], Corollary 6.12). In particular, the latter is commutative.

We introduce our distinguished element, the "U_p-operator at v" in our Hecke algebras. Namely, if $Q_v = P_{(n_1,...,n_k)}$, set

$$u_v^{Q_v} := \mathrm{diag}(\varpi^{k-1}, ..., \varpi^{k-1}, \varpi^{k-2}, ..., \varpi, 1, ..., 1) \in \Delta_{\mathcal{Q}_v}$$

where, for $1 \leq j \leq k$, ϖ^{j-1} appears exactly n_j times. Then, for $c \geq 1$, we denote by $U_v^{Q_v} \in \mathcal{H}(\Delta_{\mathcal{Q}_v}(c), \mathcal{Q}_v(0,c))$ the double coset operator $[\mathcal{Q}_v(0,c)u_v^Q \mathcal{Q}_v(0,c)]$. Note that $\Delta_{M_v} = \Delta_{M_v}^+[u_v^{Q_v,\pm 1}]$.

We introduce the usual λ-twisted action of $\Delta_{\mathcal{Q}_v}(c_p)$[16] on $\mathcal{V}_{(\lambda_v,\underline{\tau_v})}^{Q_v}$ which will then yield our action of U_p-operators on the corresponding cohomology groups by the formalism of [All+23], 2.1.9. Define the character $\alpha_\lambda^{Q_v} : \Delta_{\mathcal{Q}_v} \to E^\times$ by setting it to be trivial on $\mathcal{Q}_v$ and, for $\nu \in X_{Q_v}$, sending $\nu(\varpi_v)$ to

$$\prod_{\iota : F_v \hookrightarrow E} \iota(\varpi_v)^{\langle \nu, w_0^G \lambda_\iota \rangle}.$$

We then view $\mathcal{V}_{(\lambda_v,\underline{\tau_v})}^{Q_v}$ as an $\mathcal{O}[\Delta_{\mathcal{Q}_v}(c_p)]$-module by inflating the $\mathcal{Q}_v(0,c_p)$-action on the factor $(\widetilde{\sigma(\underline{\tau_v})}^{\circ})^\vee$ and acting on the factor $\mathcal{V}_{\lambda_v}$ by the recipe

$$g \cdot^{\lambda,Q_v} x := \alpha_\lambda^{Q_v}(g)^{-1} g \cdot x$$

where $- \cdot -$ is the usual action of $g \in \Delta_{\mathcal{Q}_v}(c_p)$ on $x \in \mathcal{V}_{\lambda_v}$.

Lemma 2.8.2. *The $\mathcal{O}[\Delta_{\mathcal{Q}_v}(c_p)]$-module structure on $\mathcal{V}_{(\lambda_v,\underline{\tau_v})}^{Q_v}$ makes sense i.e., the twisted action of $\Delta_{\mathcal{Q}_v}(c_p)$ on V_{λ_v} preserves the lattice $\mathcal{V}_{\lambda_v}$.*

---

[16] For the integer $c_p \geq 1$ introduced in the previous section.

Proof. This follows from the fact that V_{λ_v} has lowest weight $w_0^G\lambda_v$. More precisely, one uses [Ger18], Lemma 2.2 to conclude. $\square$

As mentioned, the formalism of [All+23], 2.1.9 applies here. In particular, for $S_p \subset T$ any finite set of finite places, and a choice of good subgroup $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ with $K_v = \mathcal{Q}_v(b, c)$ for some $0 \leq b \leq c$ with $c_p \leq c$, for every $v \in S_p$, we have a canonical homomorphism

$$\mathcal{H}(G^T, K^T) \otimes_{\mathbf{Z}} \mathcal{H}(\Delta_{\mathcal{Q}_p(c_p)}, K_p) \to \mathrm{End}_{D^+(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}^{Q_p}_{(\lambda, \underline{\tau})})).$$

In particular, the above constructed U_p-operators act on the cohomology complex $R\Gamma(X_K, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})})$.

For later use, set $\mathbf{T}^T(K, \lambda, \underline{\tau})$ to be the image of $\mathcal{H}(G^T, K^T) \otimes_{\mathbf{Z}} \mathcal{O}$ inside the ring $\mathrm{End}_{D^+(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}))$, a finite $\mathcal{O}$-algebra.

Next we consider the case of $\widetilde{G}$. Fix a subset of p-adic places $\overline{S} \subset \overline{S}_p$ and, for each $\bar{v} \in \overline{S}_p$, fix a choice $v \mid \bar{v}$ in S_p. Consider a tuple $(\widetilde{Q}_{\overline{S}}, \tilde{\lambda}_{\overline{S}}, \underline{\tau_{\overline{S}}}) := (\widetilde{Q}_{\bar{v}}, \tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})_{\bar{v} \in \overline{S}}$ as before. We can then similarly define the set of $\widetilde{B}$-dominant cocharacters $X_{\widetilde{Q}^{w_0}_{\bar{v}}} \subset X_*(Z(\widetilde{M}^{w_0}_{\bar{v}}))$ and the corresponding open submonoids $\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}}(c) \subset \widetilde{G}(F^+_{\bar{v}})$. The rest of the conclusions of [CN23], Lemma 2.1.15 will again apply. We further set $\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}}(c_p) := \prod_{\bar{v} \in \overline{S}} \widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}}(c_p)$. Again, we introduce a notation for our U_p-operator in $\mathcal{H}(\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}}(c_p), \widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(0, c_p))$. For $v|\bar{v}$, we set $u_{\bar{v}}^{\widetilde{Q}^{w_0}_v} := \iota_v^{-1} u_v^{\iota_v \widetilde{Q}^{w_0}_{\bar{v}}}$ where $\iota_{\bar{v}} \widetilde{Q}^{w_0}_v \subset \mathrm{GL}_{2n}/F_v$ is the standard parabolic subgroup corresponding to $\widetilde{Q}^{w_0}_{\bar{v}}$ under ι_v. Then set $U_{\bar{v}}^{\widetilde{Q}^{w_0}_{\bar{v}}} \in \mathcal{H}(\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}}(c_p), \widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(0, c_p))$ to be the double coset operator $[\widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(0, c) u_v^{\widetilde{Q}^{w_0}_{\bar{v}}} \widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(0, c_p)]$. We note that as the notation suggests, $U_{\bar{v}}^{\widetilde{Q}^{w_0}_{\bar{v}}}$ is independent of the choice of $v|\bar{v}$.

Moreover, by the exact same recipe as before, we equip $\mathcal{V}^{\widetilde{Q}_{\overline{S}}}_{(\tilde{\lambda}, \underline{\tau})}$ with an $\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}}(c_p)$-module structure extending the natural action of $\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}(0, c_p)$. This yields a canonical homomorphism of algebras

$$\mathcal{H}(\widetilde{G}^T, \widetilde{K}^T) \otimes_{\mathbf{Z}} \mathcal{H}(\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}}(c_p), \widetilde{K}_{\overline{S}}) \to \mathrm{End}_{D^+(\mathcal{O})}(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda}, \underline{\tau})}))$$

for any finite set of finite places $S_p \subset T$ with $T = T^c$ and good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$ with $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(b, c')$ for some $0 \leq b \leq c'$ with $c_p \leq c'$ for every $\bar{v} \in \overline{S}$. Therefore, the constructed U_p-operators act on the complex $R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda}, \underline{\tau})})$.

We finally set $\mathbf{T}^T(\widetilde{K}, \tilde{\lambda}, \underline{\tau})$ to be the corresponding faithful quotient of $\mathcal{H}(\widetilde{G}^T, \widetilde{K}^T) \otimes_{\mathbf{Z}} \mathcal{O}$.

2.9 Automorphic Galois representations

In this section, we recall the well-known results which we will need later about Galois representations attached to (mod p) automorphic forms. We also state the consequence of the vanishing results of Caraiani–Scholze that serves as the key ingredient for proving local-global compatibility in the style of [All+23].

Recall that π is called a regular algebraic conjugate self-dual cuspidal automorphic representation (RACSDCAR) of $\mathrm{GL}_n(\mathbf{A}_F)$ if it is a regular algebraic cuspidal automorphic representation (RACAR) of $\mathrm{GL}_n(\mathbf{A}_F)$ satisfying $\pi^c \cong \pi^\vee$ where $(.)^\vee$ denotes the contragradient of the representation.

By work of many people, such automorphic representations admit associated Galois representations satisfying local-global compatibility at every place.

Theorem 2.9.1. [HT01; TY07; Shi11; CH13; Clo13; Bar+12; Bar+11; Car12; Car14] *Let π be a RACSDCAR of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\mathbf{C})}$. Then for any isomorphism $t : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ there is a continuous semisimple Galois representation*

$$r_t(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

satisfying the following conditions:

i. We have an isomorphism $r_t(\pi)^c \cong r_t(\pi)^\vee(1-n)$.

ii. For each p-adic place v of F, $r_t(\pi)|_{G_{F_v}}$ is potentially semistable and for each embedding $\iota : F_v \hookrightarrow \overline{\mathbf{Q}}_p$ we have

$$\mathrm{HT}_\iota(r_t(\pi)|_{G_{F_v}}) = \{\lambda_{t\circ\iota,n}, \lambda_{t\circ\iota,n-1}+1, ..., \lambda_{t\circ\iota,1}+n-1\}.$$

iii. For each finite place v of F, we have

$$\mathrm{WD}(r_t(\pi)|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}^T(t^{-1}\pi_v).$$

Once combined with Shin's base change result [Shi14] for automorphic representations of $\widetilde{G}$, we obtain the following.

Theorem 2.9.2. *Suppose that F contains an imaginary quadratic field. Let $\tilde{\pi}$ be a ξ-cohomological cuspidal automorphic representation of $\widetilde{G}(\mathbf{A}_{F^+})$ for some irreducible algebraic representation ξ of $\widetilde{G}_{\mathbf{C}} \cong \prod_{\mathrm{Hom}(F^+,\mathbf{C})} \mathrm{GL}_{2n,\mathbf{C}}$. For any field isomorphism $t : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$, there exists a continuous, semisimple Galois representation*

$$r_t(\tilde{\pi}) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$$

satisfying the following conditions:

i. *For each prime* $\ell \neq p$*, unramified in* F*, above which* $\tilde{\pi}$ *is unramified, and for each place* v *of* F *dividing* ℓ*,* $r_t(\tilde{\pi})|_{G_{F_v}}$ *is unramified and the characteristic polynomial of* $r_t(\mathrm{Frob}_v)$ *coincides with image of* $\widetilde{P}_v(X)$ *in* $\overline{\mathbf{Q}}_p[X]$ *corresponding to the base change of* $t^{-1}(\tilde{\pi}_{\bar{v}})$*.*

ii. *For each place* v *of* F *dividing* p*,* $r_t(\tilde{\pi})$ *is potentially semistable, and for each embedding* $\iota : F \hookrightarrow \overline{\mathbf{Q}}_p$*, the* ι*-labelled Hodge–Tate weights are given by*

$$\tilde{\lambda}_{\iota,1} + 2n - 1 > \tilde{\lambda}_{\iota,2} + 2n - 2 > ... > \tilde{\lambda}_{\iota,2n},$$

where $\tilde{\lambda} \in (\mathbf{Z}^{2n}_{+})^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ *is the highest weight of the representation* $t^{-1}(\xi \otimes \xi)^{\vee}$ *of* GL_{2n} *over* $\overline{\mathbf{Q}}_p$*.*

iii. *If* $F_0 \subset F$ *is an imaginary quadratic field and* ℓ *is a prime (possibly* $\ell = p$*) that splits in* F_0*, then for each place* $v \mid \ell$ *of* F *lying above a place* $\bar{v}$ *of* F^+*, there is an isomorphism*

$$\mathrm{WD}(r_t(\tilde{\pi})|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}^T(\tilde{\pi}_{\bar{v}} \circ \iota_v).$$

Proof. For the proof see [All+23], Theorem 2.3.3 and the references therein. □

Theorem 2.9.3. *Assume that* F *contains an imaginary quadratic field. Let* $\mathfrak{m} \subset \mathbf{T}^T(K, \lambda, \underline{\tau})$ *be a maximal ideal. Suppose that the finite set of places* T *is so that* $T = T^c$ *and further satisfies the following condition.*

- *Given a finite place* v *not lying in* T*, denote its residual characteristic by* ℓ*. Then either* ℓ *is unramified in* F *and* T *contains no* ℓ*-adic places, or* ℓ *splits in some imaginary quadratic subfield* $F_0 \subset F$*.*

Then there exists a continuous semisimple Galois representation

$$\bar{\rho}_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^T(K, \lambda, \underline{\tau})/\mathfrak{m})$$

such that for each finite place v *of* F *not lying in* T*, the characteristic polynomial of* $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ *coincides with the image of* $P_v(X)$ *in* $(\mathbf{T}^T(K, \lambda, \underline{\tau})/\mathfrak{m})[X]$*.*

Proof. This is proved the same way as [All+23], Theorem 2.3.5. □

Definition 2.9.4. We say that a maximal ideal $\mathfrak{m} \subset \mathbf{T}^T(K, \lambda, \underline{\tau})$ is non-Eisenstein if $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Theorem 2.9.5. *Assume that* F *and* T *satisfies the conditions of Theorem 2.9.3. Let* $\mathfrak{m} \subset \mathbf{T}^T(K, \lambda, \underline{\tau})$ *be a non-Eisenstein maximal ideal. There exists an integer* $N \geq 1$*, depending only on* n *and* $[F : \mathbf{Q}]$*, an ideal* $I \subset \mathbf{T}^T(K, \lambda, \underline{\tau})$ *satisfying* $I^N = 0$*, and a continuous group homomorphism*

$$\rho_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^T(K, \lambda, \underline{\tau})/I)$$

such that, for each finite place v *of* F *not lying in* T*, the characteristic polynomial of* $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ *coincides with the image of* $P_v(X)$ *in* $(\mathbf{T}^T(K, \lambda, \underline{\tau})/I)[X]$*.*

Proof. This is proved the same way as [Sch15], Corollary 5.4.4. □

Theorem 2.9.6. *Assume that F and T satisfies the conditions of Theorem 2.9.3. Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^T(\widetilde{K}, \tilde{\lambda}, \underline{\tau})$ be a maximal ideal. Then there is a continuous, semisimple Galois representation*

$$\bar{\rho}_{\widetilde{\mathfrak{m}}} : G_{F,T} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}^T(\widetilde{K}, \tilde{\lambda}, \underline{\tau})/\widetilde{\mathfrak{m}})$$

such that for each finite place $v \notin T$ of F, the characteristic polynomial of $\bar{\rho}_{\widetilde{\mathfrak{m}}}(\mathrm{Frob}_v)$ is given by the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}^T(\widetilde{K}, \tilde{\lambda}, \underline{\tau})/\widetilde{\mathfrak{m}})[X]$.

Proof. The same proof applies as in [CN23], Theorem 2.1.26 noting that the first step is to pass to deep enough level where $\mathcal{V}_{(\tilde{\lambda},\underline{\tau})}^{\widetilde{Q}_{\overline{S}}^{w_0}}/\varpi$ is trivialised. □

Finally, we discuss the key technical condition we need to have access to the vanishing result of Caraiani–Scholze.

Definition 2.9.7. A continuous representation $\bar{\rho} : G_F \to \mathrm{GL}_m(\mathcal{O}/\varpi)$ is called *decomposed generic* if there exists a prime ℓ different from p such that:

i. ℓ splits completely in F;

ii. for every place v of F dividing ℓ, $\bar{\rho}|_{G_{F_v}}$ is unramified and the eigenvalues $\alpha_1, ..., \alpha_m$ of $\bar{\rho}(\mathrm{Frob}_v)$ satisfy $\alpha_i/\alpha_j \neq \ell$ for $i \neq j$.

Remark 2.9.8. As explained in [All+23], Lemma 4.3.2, once we know that $\bar{\rho}$ is decomposed generic, an argument using Chebotarev's density theorem shows that there are infinitely many choices of ℓ as in Definition 2.9.7.

Theorem 2.9.9. [CS19; Kos21] *Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^T(\widetilde{K}, \tilde{\lambda}, \underline{\tau})$ be a maximal ideal such that the associated Galois representation $\bar{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic. If we set $d = \dim_{\mathbf{C}} \widetilde{X}_{\widetilde{K}}$, we have a $\widetilde{\mathbf{T}}^T$-equivariant diagram*

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{(\tilde{\lambda},\underline{\tau})}^{\widetilde{Q}_{\overline{S}}^{w_0}}[1/p])_{\widetilde{\mathfrak{m}}} \hookleftarrow H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{(\tilde{\lambda},\underline{\tau})}^{\widetilde{Q}_{\overline{S}}^{w_0}})_{\widetilde{\mathfrak{m}}} \twoheadrightarrow H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{(\tilde{\lambda},\underline{\tau})}^{\widetilde{Q}_{\overline{S}}^{w_0}})_{\widetilde{\mathfrak{m}}}.$$

Proof. This follows from the main result of [CS19] as explained in [All+23], Theorem 4.3.3 except the extra conditions appearing in [CS19] that $[F^+ : \mathbf{Q}] \geq 2$ and that the length of $\bar{\rho}_{\widetilde{\mathfrak{m}}}$ is at most 2. These conditions were removed in [Kos21]. □

Chapter 3

Q-ordinary Hida theory

In the first part of the chapter, based on [All+23], §5.2 and [CN23], §2.2, we spell out Q-ordinary Hida theory for the Betti cohomology of the locally symmetric space $X_{\widetilde{K}}$ and X_K, where Q will be an arbitrary standard parabolic subgroup. Finally, we close the chapter with a computation of Q-ordinary parts of certain Bruhat strata of parabolic induction. This is completely analogous to [All+23], §5.3 and [CN23], §2.3. More precisely, the contents of §3.1, §3.2, and §3.4 are carried out for the Borel subgroup in [All+23], §5.2, and for the Siegel parabolic subgroup in [CN23], §2.2. Moreover, the content of §3.5 has been worked out for the Borel subgroup in [All+23], §5.3, and for the Siegel parabolic subgroup in [CN23], §2.3. Finally, Hida theory with dual coefficients for the opposite parabolic is considered in [CN23] that we generalise in §3.3. A seemingly new result in §3.3 is a Q-ordinary and Hecke equivariant Verdier duality with $\mathcal{O}/\varpi^m$-coefficients (see Proposition 3.3.5).

3.1 Ordinary parts of smooth representations

In this text we use several incarnations of ordinary parts. To track the representation theory throughout our arguments, it is often crucial to take the point of view of Emerton [Eme10a], [Eme10b] on taking ordinary parts. We recollect here his approach. In fact, following [All+23], we consider a modified version that is more convenient to do homological algebra with and coincides with the original definition on admissible representations which exhaust all the objects which we will consider here.

The setup for the section is as follows. We let $L/\mathbf{Q}_p$ be a finite field extension, G/L a connected reductive group. Set $Q \subset G$ to be a parabolic subgroup with a Levi decomposition $Q = M \ltimes N$ and denote by $\overline{Q} = M \ltimes \overline{N}$ the opposite parabolic subgroup. Denote by $Z_M \subset M$ the centre of the Levi factor. Assume that $\mathcal{Q} \subset G(L)$ is a compact open subgroup which admits an Iwahori decomposition

$$\overline{N}^1 \times M^0 \times N^0 \xrightarrow{\sim} \mathcal{Q} \xleftarrow{\sim} N^0 \times M^0 \times \overline{N}^1$$

with respect to Q in the sense of [All+23], §2.1.9. Let

$$... \subset \mathcal{Q}(b,b) \subset ... \subset \mathcal{Q}(1,1) \subset \mathcal{Q}(0,1) = \mathcal{Q} \subset G(L)$$

be a cofinal family of compact open subgroups of $G(L)$ such that each $\mathcal{Q}(b,b)$ is normal in $\mathcal{Q}$ and $\mathcal{Q}(b,b)$ admits an Iwahori decomposition

$$\overline{N}^b M^b N^0 \xrightarrow{\sim} \mathcal{Q}(b,b)$$

with respect to Q. Then, by setting $\mathcal{Q}(b,c) = \overline{N}^c M^b N^0$ for $1 \leq b \leq c$, one checks that we get a compact open subgroup of $\mathcal{Q}$ (admitting an Iwahori decomposition).

Example 3.1.1. For $G = \mathrm{GL}_{n,L}$, and $Q = M \ltimes N \subset G$ a parabolic subgroup standard with respect to the Borel subgroup of upper triangular matrices, we can consider the corresponding parahoric group scheme $\mathcal{Q}^{\mathrm{sch}} \subset \mathrm{GL}_n$. Then $\mathcal{Q} := \mathcal{Q}^{\mathrm{sch}}(\mathcal{O}_L)$ admits an Iwahori decomposition $\overline{N}^1 M^0 N^0$, where $M^0 = M(\mathcal{O}_L)$, $N^0 = N(\mathcal{O}_L)$, and $\overline{N}^1$ is $\ker(\overline{N}(\mathcal{O}_L) \to \overline{N}(\mathcal{O}_L/\varpi_L)$. If Q is the parabolic subgroup corresponding to the partition $(n_1, ..., n_t)$ of n, $\mathcal{Q}$ will be the subgroup of matrices in $G(\mathcal{O}_L)$ that are upper block triangular of type $(n_1, ..., n_t)$ modulo ϖ_L. For integers $0 \leq b \leq c$ with $c \geq 1$, we can then set $\mathcal{Q}(b,c)$ to be the subgroup of matrices in $G(\mathcal{O}_L)$ that are block upper triangular of type $(n_1, ..., n_t)$ modulo ϖ_L^c, and block unipotent of type $(n_1, ..., n_t)$ modulo ϖ_L^b.

However, we consider this more general setup since it allows us to for instance set M^1 to be $\ker(M(\mathcal{O}_L) \to M(\mathcal{O}_L/\varpi_L^d)$ for some arbitrary integer $d \geq 0$. Moreover, we have the freedom to choose N^0 to be other compact open subgroups in $N(\mathcal{O}_L)$ that are preserved under conjugation by M^0. All of this allows the formalism to be more flexible and saves some space in introducing notations and applying it in later chapters.

Remark 3.1.2. Note that an easy argument using the Iwahori decomposition shows that M^0 normalises $N^0 = N(L) \cap \mathcal{Q}$ and each $\overline{N}^c = \overline{N}(L) \cap \mathcal{Q}(0,c)$ for $c \geq 1$.

We set

$$M^+ := \{m \in M(L) \mid mN^0m^{-1} \subset N^0 \text{ and } m^{-1}\overline{N}^1 m \subset \overline{N}^1\}^1$$

and define $Z_M^+ := Z_M(L) \cap M^+$. By Remark 3.1.2, we see that in fact both $M^+ \subset M(L)$ and $Z_M^+ \subset Z_M(L)$ are open submonoids. We make the following assumption that will be satisfied in our context.

---

[1] We note that this is not the definition that appears in [Eme10a], [Eme10b] as there the second condition on the elements $m \in M^+$ is not present. However, this is needed for us in order to compare the two natural Hecke actions of our monoids on finite level (see Propostion 3.3.5). We also note that this extra condition is already present for instance in §2.1.9 of [All+23].

Hypothesis 3.1.3. *For any* $m \in M^+$*, and* $c \geq 1$*, we have* $m^{-1}\overline{N}^c m \subset \overline{N}^c$*.*

Denote by $M^+ \times_{Z_M^+} Z_M(L)$ the quotient of the monoid $M^+ \times Z_M(L)$ by the equivalence relation generated by $(mz^+, z) \sim (m, z^+z)$ for $m \in M^+$, $z^+ \in Z_M^+$ and $z \in Z_M(L)$.

Lemma 3.1.4. *The morphism of monoids* $M^+ \times Z_M(L) \to M(L)$ *given by multiplication factors through* $M^+ \times_{Z_M^+} Z_M(L)$ *and descends to an isomorphism.*

Proof. This is essentially Proposition 3.3.6 of [Eme06a]. Since our definition of M^+ is slightly different, we provide a sketch of proof here.

We start by picking $z_p \in Z_M^+$ such that $\{z_p^k N^0 z_p^{-k}\}_{k\geq 0}$ forms a basis of neighborhoods of 0. Such a $z_p \in Z_M^+$ exists by (the discussion above [Eme10b], Lemma 3.1.3, and) [Eme10b] Lemma 3.1.3 (2),(3) and Lemma 3.1.4, (3). Note that z_p^{-1} and Z_M^+ generate $Z_M(L)$ as a monoid. Indeed, given $z \in Z_M(L)$, then for $k \geq 0$ large enough, $z_p^k z$ will satisfy the assumption of *loc. cit.* Lemma 3.1.3, (3) and Lemma 3.1.4, (3), and, in particular, will lie in Z_M^+.

We further see that M^+ and z_p^{-1} generate $M(L)$ as a monoid. To see this, pick $m \in M(L)$ and assume that we have

$$mN^0m^{-1} \subset z_p^{-k}N^0z_p^k \text{ and } m^{-1}\overline{N}^1 m \subset z_p^k\overline{N}^1 z_p^{-k}$$

for a large enough integer $k \geq 0$. Such k always exists by *loc. cit.* Lemma 3.1.3, (2) and Lemma 3.1.4, (2). Then $z_p^k m = m z_p^k$ clearly lies in M^+.

One can now conclude just as in the proof of [Eme06a], Proposition 3.3.6. □

From now on, we fix a $z_p \in Z_M^+$ as in the proof of Lemma 3.1.4. We set $\Delta_M^+ \subset M^+$ to be any open submonoid containing M^0 and z_p. Moreover, for $c \geq 1$, we set $\mathcal{Q}^+(0,c) = \overline{N}^c M^+ N^0$ and $\Delta_{\mathcal{Q}}(c) = \overline{N}^c \Delta_M^+ N^0$. Then [All+23], Lemma 2.1.10 says that $\mathcal{Q}^+(0,c)$ is also a monoid, open in $G(L)$. Moreover, $\mathcal{Q}\mathcal{Q}^+(0,c)\mathcal{Q} = \mathcal{Q}^+(0,c)$, and $\mathcal{Q}^+ \cap M(L) = M^+$. The same conclusion holds for $\Delta_{\mathcal{Q}}(c)$ too, since the assumption $M^0 \subset \Delta_M^+$ ensures that the proof of *loc. cit.* Lemma 2.1.10 applies. Moreover, set $Q^+ = \mathcal{Q}^+(0,c) \cap Q(L) = M^+ \ltimes N^0$ and $\Delta_Q = \Delta_{\mathcal{Q}}(c) \cap Q(L) = \Delta_M^+ \ltimes N^0$. We finally set Δ_M to be the monoid generated by Δ_M^+ and z_p^{-1}.

Given $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$, we can consider the Hecke action of Δ_M^+ on the space of N^0-invariants $\Gamma(N^0, \pi)$. Namely, for $m \in \Delta_M^+$ and $v \in \Gamma(N^0, \pi)$, the action is given by

$$m \cdot v := \sum_{n \in N^0/mN^0m^{-1}} nmv.$$

We then obtain a left exact functor

$$\Gamma(N^0, -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M^+])$$

and, in particular, a derived functor[2]

$$R\Gamma(N^0, -) : D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q]) \to D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta^+_M]).$$

We also have the exact localisation functor

$$(-)^{Q\text{-ord}} : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta^+_M]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M])$$

induced by the inclusion $\Delta^+_M \subset \Delta_M$. We then obtain the functor of "taking Q-ordinary parts at infinite level"

$$D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q]) \to D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M]),$$
$$\pi \mapsto R\Gamma(N^0, \pi)^{Q\text{-ord}}.$$

We now consider several versions of taking Q-ordinary parts at finite level and will compare them. To work in the generality we need, we will take invariants with respect to general "types" and not only the trivial representation. In particular, set σ to be a smooth $\mathcal{O}/\varpi^m[M^0]$-modules, finite free over $\mathcal{O}/\varpi^m$. We will often abuse the notation and confuse σ with $\mathrm{Inf}_{M^0}^{M^0 \ltimes N^0}\sigma$. We define the Hecke algebras $\mathcal{H}(\sigma)^{\Delta^+_M} \subset \mathcal{H}(\sigma)^+ \subset \mathcal{H}(\sigma)$ as the subalgebra generated by functions supported on Δ^+_M, respectively on M^+. Finally, note that $[z_p, \mathrm{id}]$ lies $\mathcal{H}(\sigma)^{\Delta^+_M}$ and is a central element. Combined with Lemma 3.1.4, it easily implies that $\mathcal{H}(\sigma)^+[[z_p, \mathrm{id}]^{-1}] = \mathcal{H}(\sigma)$. We set $\mathcal{H}(\sigma)^{\Delta_M} = \mathcal{H}(\sigma)^{\Delta^+_M}[[z_p, \mathrm{id}]^{-1}]$.

The first candidate for ordinary parts at level σ is as follows. For $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$ we apply the functor

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, -) : D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M]) \to D^+_{\mathrm{sm}}(\mathcal{H}(\sigma)^{\Delta_M})$$

to $R\Gamma(N^0, \pi)^{Q\text{-ord}}$. Here the functor $R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, -)$ is constructed by taking the usual left Hecke action of $\mathcal{H}(\sigma)^{\Delta_M}$ on the space of $\sigma^\vee$-invariants.

In order to define the other candidate, we introduce the functor

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q]) \to \mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta^+_M})$$

where the target is the category of left $\mathcal{H}(\sigma)^{\Delta^+_M}$-modules. We spell out the definition of $\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, -)$. Let $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\Delta_Q)$, pick $[m, \psi] \in \mathcal{H}(\sigma)^{\Delta^+_M}$ and $\phi \in \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, \pi)$. The action is defined by setting

$$[m, \psi] \cdot \phi : v \mapsto \sum_{n\tilde{m} \in N^0 \rtimes M^0 / m(N^0 \rtimes M^0)m^{-1} \cap (N^0 \rtimes M^0)} \pi(n\tilde{m}m)\phi(\psi^t \circ \sigma^\vee((n\tilde{m})^{-1})v).$$

Note that we have identifications of sets

$$N^0 \rtimes M^0 / \left(m(N^0 \rtimes M^0)m^{-1} \cap (N^0 \rtimes M^0)\right) \cong \tag{3.1.1}$$
$$\left\{(n, \tilde{m}) \mid n \in N^0/\tilde{m}mN^0(\tilde{m}m)^{-1}, \tilde{m} \in M^0/mM^0m^{-1} \cap M^0\right\} \cong$$
$$(M^0/mM^0m^{-1} \cap M^0) \times N^0/mN^0m^{-1}.$$

In particular, we obtain the following lemma.

---

[2] Here we use [All+23], Lemma 5.2.4 to see that both categories considered are abelian with enough injectives.

Lemma 3.1.5. *We have a natural equivalence of functors*

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,-)\circ\Gamma(N^0,-)\cong\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0\ltimes N^0]}(\sigma^\vee,-):\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])\to$$

$$\to\mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M^+}).$$

In particular, for $\pi\in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$*, we have a natural isomorphism*

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,R\Gamma(N^0,\pi))\cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0\ltimes N^0]}(\sigma^\vee,\pi)$$

in $D^+(\mathcal{H}(\sigma)^{\Delta_M^+})$*.*

Proof. As the underlying $\mathcal{O}/\varpi^m$-modules of the two functors evaluated on some $\pi\in\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$ clearly coincide, to see the first part, we only have to check that the Hecke actions match up. This follows from the definitions and the identifications 3.1.1.

The second part follows from [Wei94], Corollary 10.8.3 as soon as we verify the fact that $\Gamma(N^0,-)$ carries injectives to $\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,-)$-acyclics. To see this we argue just as in the proof of [All+23], Lemma 5.2.7, (1). Namely, we have the exact forgetful functors

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])\xrightarrow{\alpha}\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M^0\ltimes N^0]),$$

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M^+])\xrightarrow{\beta}\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M^0])\text{ and}$$

$$\mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M^+})\xrightarrow{\gamma}\mathrm{Mod}(\mathcal{O}/\varpi^m).$$

Moreover, [All+23], Lemma 5.2.4 says that α and β preserve injectives. Now pick an injective $\mathcal{I}\in\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$. We would like to show that, for $i\geq 1$,

$$R^i\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,\Gamma(N^0,\mathcal{I}))=0.$$

As the previous discussion shows, we can equivalently verify that

$$\gamma R^i\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,\Gamma(N^0,\mathcal{I}))=R^i\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,\Gamma(N^0,\alpha\mathcal{I}))=0.$$

However, $\Gamma(N^0,-)$, as a functor $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M^0\ltimes N^0])\to\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M^0])$, preserves injectives as it possesses an exact left adjoint given by inflation. Therefore, $\alpha\mathcal{I}$ being injective, $R^i\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,\Gamma(N^0,\alpha\mathcal{I}))$ vanishes. □

We further consider the functor

$$(-)^{Q\text{-ord}}:\mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M^+})\to\mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M})$$

defined by localising along $\mathcal{H}(\sigma)^{\Delta_M^+}\subset\mathcal{H}(\sigma)^{\Delta_M}$.

Lemma 3.1.6. *The functor*

$$(-)^{Q\text{-ord}}:\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M^+])\to\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M])$$

sends injectives to $\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee,-)$*-acyclics.*

Proof. Set Δ'^{+}_{M} to be the monoid generated by M^0 and z_p. Then, for any integer $b \geq 0$, $\mathcal{H}(\Delta'^{+}_{M}, M^b) = \mathcal{O}/\varpi^m[\Delta'^{+}_{M}/M^b] \cong \mathcal{O}/\varpi^m[M^0/M^b][z_p]$ is a polynomial ring over the Noetherian ring $\mathcal{O}/\varpi^m[M^0/M^b]$. In particular, localisation along $\mathcal{H}(\Delta'^{+}_{M}, M^b) \hookrightarrow \mathcal{H}(\Delta'_M, M^b) \cong \mathcal{O}/\varpi^m[M^0/M^b][z_p^{\pm 1}]$ preserves injectives for every integer $b \geq 0$.[3] Therefore, the proof of [All+23], Lemma 5.2.7, (2) applies and we get that $(-)^{Q\text{-ord}}$, as a functor

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta'^{+}_{M}]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta'_M]),$$

preserves injectives.

We claim that the forgetful functors

$$\alpha : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta^{+}_{M}]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta'^{+}_{M}]),$$

$$\beta : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_M]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta'_M])$$

preserve injectives. Note that once this is verified, running the argument of the second part of the proof of Lemma 3.1.5 with γ being the forgetful functor $\mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M}) \to \mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta'_M})$ would allow us to conclude.

To see that α, respectively β preserves injectives, we note that it admits the functor $\mathcal{O}/\varpi^m[\Delta^{+}_{M}] \otimes_{\mathcal{O}/\varpi^m[\Delta'^{+}_{M}]} -$, respectively $\mathcal{O}/\varpi^m[\Delta_M] \otimes_{\mathcal{O}/\varpi^m[\Delta'_M]} -$ as a left adjoint. Therefore, it's enough to see that the latter functors are exact. This follows from the fact that $\mathcal{O}/\varpi^m[\Delta^{+}_{M}]$, respectively $\mathcal{O}/\varpi^m[\Delta_M]$ is free as a right $\mathcal{O}/\varpi^m[\Delta'^{+}_{M}]$- , respectively $\mathcal{O}/\varpi^m[\Delta'_M]$-module with set of generators given by a set of representatives of Δ_M/Δ'_M. ☐

Our other candidate for ordinary parts of some $\pi \in D^{+}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$ is the composition $R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, \pi)^{Q\text{-ord}}$. Using Lemma 3.1.5 and Lemma 3.1.6 we see that the two candidates in fact coincide.

Corollary 3.1.7. *Given $\pi \in D^{+}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$, we have a natural isomorphism*

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, R\Gamma(N^0, \pi)^{Q\text{-ord}}) \cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, \pi)^{Q\text{-ord}}$$

in $D^{+}(\mathcal{H}(\sigma)^{\Delta_M})$.

Proof. Exactness of $(-)^{Q\text{-ord}}$, Lemma 3.1.6, and an argument just as in the proof of [All+23], Lemma 5.2.6 implies that we have

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, -) \circ (-)^{Q\text{-ord}} \cong$$

$$R(\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, -) \circ (-)^{Q\text{-ord}}) \cong$$

$$R((-)^{Q\text{-ord}} \circ \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, -)) \cong$$

---

[3] Indeed, this is true for arbitrary localisation of the form $R[x] \hookrightarrow R[x^{\pm 1}]$ for R a (not necessarily commutative) left Noetherian ring.

$$(-)^{Q\text{-ord}} \circ R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, -).$$

In particular, for $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_Q])$, we get a natural isomorphism

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, R\Gamma(N^0,\pi)^{Q\text{-ord}}) \cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, R\Gamma(N^0,\pi))^{Q\text{-ord}}.$$

As a consequence, Lemma 3.1.5 implies that there is a natural isomorphism

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, R\Gamma(N^0,\pi)^{Q\text{-ord}}) \cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0\ltimes N^0]}(\sigma^\vee, \pi)^{Q\text{-ord}}.$$

□

We now further assume that the action of M^1 on σ is trivial. Note that it is not a serious restriction since σ is assumed to be finite and free over $\mathcal{O}/\varpi^m$ and, by the second paragraph of Example 3.1.1, we are free to change M^1 to be a smaller compact open subgroup so that it acts trivially on σ. For $c \geq 1$, set $\widetilde{\sigma}$ to be the smooth $\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]$-module defined by the map $\mathcal{Q}(0,c) \to \mathcal{Q}(0,c)/\mathcal{Q}(c,c) \cong M^0/M^c$ i.e., for $\bar{n}mn \in \mathcal{Q}(0,c)$, we set $\widetilde{\sigma}(\bar{n}mn) = \sigma(m)$. Consider the subalgebras $\mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}}(c)} \subset \mathcal{H}(\widetilde{\sigma})^+ \subset \mathcal{H}(\widetilde{\sigma})$ generated by functions supported on $\Delta_{\mathcal{Q}}(c)$ and $\mathcal{Q}^+(0,c)$, respectively. We then have the following observation.

Lemma 3.1.8. *For any $c \geq 1$, there is an isomorphism of algebras*

$$t_c : \mathcal{H}(\sigma)^{\Delta_M^+} \to \mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}}(c)}$$

such that, for any $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}}(c)])$, the inclusion

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, \pi) \hookrightarrow \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, \Gamma(N^0,\pi))$$

intertwines the action of $\mathcal{H}(\sigma)^{\Delta_M^+}$ on the source via t_c with the action of $\mathcal{H}(\sigma)^{\Delta_M^+}$ on the target.

Proof. Note that $\mathcal{H}(\sigma)^{\Delta_M^+}$ is freely generated as an $\mathcal{O}/\varpi^m$-module by elements of the form $[m,\psi]$ with $m \in \Delta_M^+$ running through any choice of set of representatives for $M^0 \setminus \Delta_M^+/M^0$ and ψ is an element of $\mathrm{End}_{\mathcal{O}/\varpi^m}(\sigma)$ intertwining m. Similarly, $\mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}}(c)}$ is freely generated as an $\mathcal{O}/\varpi^m$-module by elements of the form $[q,\widetilde{\psi}]$ with $q \in \Delta_{\mathcal{Q}}(c)$ running through any choice of set of representatives for $\mathcal{Q}(0,c) \setminus \Delta_{\mathcal{Q}}(c)/\mathcal{Q}(0,c)$ and $\widetilde{\psi}$ is an element of $\mathrm{End}_{\mathcal{O}/\varpi^m}(\widetilde{\sigma})$ intertwining q. We definte t_c by sending a generator $[m,\psi] \in \mathcal{H}(\sigma)^{\Delta_M^+}$ to the function that is supported on $\mathcal{Q}(0,c)m\mathcal{Q}(0,c)$ and sends m to ψ regarded as an element of $\mathrm{End}_{\mathcal{O}/\varpi^m}(\widetilde{\sigma})$. In other words, $t_c([m,\psi]) = [m,\psi] \in \mathcal{H}(\widetilde{\sigma})$.

We check that $t_c([m,\psi])$ indeed lies in $\mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}}(c)}$. To do so, we pick $k \in \mathcal{Q}(0,c) \cap m\mathcal{Q}(0,c)m^{-1}$ and verify that $\widetilde{\sigma}(k) \circ \psi = \psi \circ \widetilde{\sigma}(m^{-1}km)$. Write $k = nh\bar{n} = mn'h'\bar{n}'m^{-1} = (mn'm^{-1})(mh'm^{-1})(m\bar{n}'m^{-1})$ for $n, n' \in N^0$, $h, h' \in M^0$, and $\bar{n}, \bar{n}' \in \overline{N}^c$. Using the Iwahori decomposition for $\mathcal{Q}(0,c)$, it is easy to see that we must have $n = mn'm^{-1}$, $h = mh'm^{-1}$, and $\bar{n} = m\bar{n}'m^{-1}$. In particular, we have

$$\psi \circ \widetilde{\sigma}(m^{-1}km) = \psi \circ \widetilde{\sigma}(m^{-1}nmm^{-1}hmm^{-1}\bar{n}m) =$$

$$\psi \circ \widetilde{\sigma}(n'h'\bar{n}') = \psi \circ \sigma(h') =$$

$$\sigma(h) \circ \psi = \widetilde{\sigma}(nh\bar{n}) \circ \psi =$$

$$\widetilde{\sigma}(k) \circ \psi.$$

We further claim that t_c yields an isomorphism $t_c : \mathcal{H}(\sigma)^{\Delta_M^+} \xrightarrow{\sim} \mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}}(c)}$ of $\mathcal{O}/\varpi^m$-modules. To see this, we have to prove that

i. the map $M^0 \setminus \Delta_M^+/M^0 \to \mathcal{Q}(0,c) \setminus \Delta_{\mathcal{Q}}(c)/\mathcal{Q}(0,c)$ is bijective, and

ii. for $m \in \Delta_M^+$, $[m, \psi] \in \mathcal{H}(\sigma)$ if and only if $[m, \psi] \in \mathcal{H}(\widetilde{\sigma})$ (where in the latter case we treat ψ as an element of $\mathrm{End}_{\mathcal{O}/\varpi^m}(\widetilde{\sigma})$).

The first claim follows from the observation that every $q_1mq_2 \in \Delta_{\mathcal{Q}}(c)$ can be written uniquely as $nm'\overline{n}'$ for $n \in N^0$, $m' \in M^0mM^0$, and $\overline{n} \in \overline{N}^c$. This further follows from the existence of an Iwahori decomposition for $\mathcal{Q}(0,c)$ and the fact that $\Delta_M^+ \subset M^+$. The "only if" direction of the second claim is exactly the well-definedness of t_c that we already have checked. The other direction follows from the inclusion $M^0 \cap mM^0m^{-1} \subset \mathcal{Q}(0,c) \cap m\mathcal{Q}(0,c)m^{-1}$.

Before proving that t_c also respects the algebra structure, we check the last claim. To do this, we pick an element $\phi \in \mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, \pi) \subset \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, \Gamma(N^0, \pi))$, and compute, first the action of $[m, \psi]$,

$$[m,\psi] \cdot \phi : v \mapsto \sum_{\tilde{m} \in M^0/mM^0m^{-1} \cap M^0} \pi^{N^0}(\tilde{m}m)\phi(\psi^t \circ \sigma^\vee(\tilde{m}^{-1})v) =$$

$$\sum_{\tilde{m} \in M^0/mM^0m^{-1} \cap M^0} \pi(\tilde{m}) \sum_{n \in N^0/mN^0m^{-1}} \pi(nm)\phi(\psi^t \circ \widetilde{\sigma}^\vee(n^{-1}\tilde{m}^{-1})v) =$$

$$\sum_{(\tilde{m},n) \in (M^0/mM^0m^{-1} \cap M^0) \times (N^0/mN^0m^{-1})} \pi(\tilde{m}nm)\phi(\psi^t \circ \widetilde{\sigma}^\vee(n^{-1}\tilde{m}^{-1})v).$$

On the other hand, we have

$$t_c([m,\psi]) \cdot \phi : v \mapsto \sum_{q \in \mathcal{Q}(0,c)/\mathcal{Q}(0,c) \cap m\mathcal{Q}(0,c)m^{-1}} \pi(qm)\phi(\psi^t \circ \widetilde{\sigma}^\vee(q^{-1})v).$$

Therefore, it suffices to prove that the inclusion $M^0 \ltimes N^0 \hookrightarrow \mathcal{Q}(0,c)$ descends to a bijection

$$(M^0/mM^0m^{-1} \cap M^0) \times (N^0/mN^0m^{-1}) \cong \mathcal{Q}(0,c)/\mathcal{Q}(0,c) \cap m\mathcal{Q}(0,c)m^{-1}.$$

This follows from the Iwahori decomposition, the fact that $m \in \Delta_M^+ \subset M^+$ and 3.1.1.

We now easily deduce that t_c respects the algebra structure. Indeed, note that if we set $\pi := \text{c-Ind}_{\mathcal{Q}(0,c)}^{G(L)} \widetilde{\sigma}^\vee$, the action of $\mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}(0,c)}}$ on π gives an embedding

$$\mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}(0,c)}} \hookrightarrow \mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, \pi).$$

In particular, t_c must be an algebra homomorphism as, by the previous paragraph, t_c yields an algebra action of $\mathcal{H}(\sigma)^{\Delta_M^+}$ on $\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, \pi)$, that factors through the faithful algebra action of the target of t_c. □

As a consequence of Lemma 3.1.8, for every $c \geq 1$, we get a left exact functor

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}}]) \to \mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M^+}).$$

Lemma 3.1.9. *For any $c \geq 1$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, -)^{Q\text{-ord}} \cong \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, -)^{Q\text{-ord}}$$

of functors

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}}]) \to \mathrm{Mod}(\mathcal{H}(\sigma)^{\Delta_M}).$$

Proof. By the exactness of $(-)^{Q\text{-ord}}$, for $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}}])$, we have an inclusion

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, \pi)^{Q\text{-ord}} \hookrightarrow \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, \pi)^{Q\text{-ord}} \tag{3.1.2}$$

and, by Lemma 3.1.8, it is $\mathcal{H}(\sigma)^{\Delta_M}$-equivariant. In particular, we are left to check that 3.1.2 is a bijection. Given $\phi \in \mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, \pi)$, by smoothness of π and the fact that $\sigma^\vee$ is finite free as an $\mathcal{O}/\varpi^m$-module, ϕ lies in $\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c')]}(\widetilde{\sigma}^\vee, \pi)$ for some $c' > c$. By induction, it suffices to prove that, for some integer $k \geq 1$,

$$[z_p, \mathrm{id}]^k \cdot \phi \in \mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c'-1)]}(\widetilde{\sigma}^\vee, \pi).$$

This, for instance, is proved in [Eme10a], Lemma 3.3.1. □

The following Corollary then summarises the observations of the subsection.

Corollary 3.1.10. *Given $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}}])$, we have natural isomorphisms*

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0]}(\sigma^\vee, R\Gamma(N^0, \pi)^{Q\text{-ord}}) \cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0 \ltimes N^0]}(\sigma^\vee, \pi)^{Q\text{-ord}} \cong$$

$$\cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[\mathcal{Q}(0,c)]}(\widetilde{\sigma}^\vee, \pi)^{Q\text{-ord}}$$

in $D^+(\mathcal{H}(\sigma)^{\Delta_M})$.

3.2 Q-ordinary Hida theory for G

We revisit the setup from §2.7. In particular, F will be a CM field and $G = \mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}}\mathrm{GL}_n$. Moreover, we fix a tuple $(Q_p, \lambda, \underline{\mathcal{T}}) = (Q_v, \lambda_v, \underline{\mathcal{T}_v})_{v\in S_p}$ as in §2.7. We will work with good subgroups $K \subset G(\mathbf{A}_{F^+}^\infty)$ with K^p being fixed and K_p of the form

$$\mathcal{Q}_p(b,c) = \prod_{v\in S_p} \mathcal{Q}_v(b,c) \subset \prod_{v\in S_p} \mathrm{GL}_n(\mathcal{O}_{F_v})$$

for $c \geq b \geq 0$ with $c \geq c_p$ where $\mathcal{Q}_v(b,c)$ is the parahoric level subgroup corresponding to Q_v, just as before. We denote such a good subgroup by $K(b,c) \subset G(\mathbf{A}_{F^+}^\infty)$. Recall that, given a local system $\mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}$ as in §2.7, $\mathcal{H}(\Delta_{\mathcal{Q}_p}(c), \mathcal{Q}_p(b,c))$ acts on $R\Gamma(X_{K(b,c)}, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p})$ via endomorphisms in $D^+(\mathcal{O})$ (cf. §2.8).[4] In particular, for $v \in S_p$, the corresponding U_p-operator $U_v^{Q_v}$ acts on the mentioned complex. Following [KT17], §2.4, we set

$$R\Gamma(X_{K(b,c)}, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p})^{Q_p\text{-ord}}$$

to be the maximal direct summand of $R\Gamma(X_{K(b,c)}, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p})$ on which $U_v^{Q_v}$ acts invertibly for each $v \in S_p$. This will then be an object of $D^+(\mathcal{O}[\mathcal{Q}_p(0,c)/\mathcal{Q}_p(b,c)])$ with an action of the spherical Hecke algebra $\mathbf{T}^T$.

On the other hand, we can apply the formalism of §3.1 with the parabolic subgroup $Q_p \subset G_p$, compact opens given by $\mathcal{Q}_p(b,c)$, open submonoid of G_p given by $\Delta_{M_p}^+$ (from §2.8) and σ being the trivial $\Delta_{M_p}^+$-module. Note that in this case $\mathcal{H}(\sigma)^{\Delta_{M_p}^+} = \mathcal{H}(\Delta_{M_p^+}, M_p^0) \otimes_{\mathbf{Z}} \mathcal{O}/\varpi^m = \mathcal{O}/\varpi^m[\Delta_{M_p}^+/M_p^0]$, $\mathcal{H}(\sigma)^{\Delta_{M_p}} = \mathcal{O}/\varpi^m[\Delta_{M_p}/M_p^0]$ and $\mathcal{H}(\widetilde{\sigma})^{\Delta_{\mathcal{Q}(c)}} \cong \mathcal{H}(\Delta_{\mathcal{Q}_p}, \mathcal{Q}_p(0,c)) \otimes_{\mathbf{Z}} \mathcal{O}/\varpi^m$. One notes that, since the cohomology groups of $R\Gamma(X_{K(b,c)}, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}/\varpi^m)$ are finite $\mathcal{O}/\varpi^m$-modules, the notions of taking Q_p-ordinary parts in the sense of [KT17] and in the sense of [Eme10a] coincide. For an argument see the proof of [All+23], Proposition 5.2.15. In other words, we have the following.

Lemma 3.2.1. *For any $m \geq 1$, $c \geq b \geq 0$ with $c \geq c_p$, there is a natural $\mathbf{T}^T$-equivariant isomorphism*

$$R\Gamma(X_{K(b,c)}, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}/\varpi^m)^{Q_p\text{-ord}} \xrightarrow{\sim} R\Gamma(\mathcal{Q}_p(b,c), \pi(K^p, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}/\varpi^m))^{Q_p\text{-ord}}$$

in $D^+(\mathcal{O}/\varpi^m[M_p^0/M_p^b])$, induced by the natural isomorphism

$$R\Gamma(X_{K(b,c)}, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}/\varpi^m) \cong R\Gamma(\mathcal{Q}_p(b,c), \pi(K^p, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}/\varpi^m)).$$

We now highlight the two important features of Hida theory for Betti cohomology. The first is usually referred to as the *independence of level* property. Consider completed cohomology

$$\pi(K^p, \mathcal{V}_{(\lambda,\underline{\mathcal{T}})}^{Q_p}/\varpi^m) \in D_{\mathrm{sm}}^+(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}_p}(c_p)]).$$

Recall that it is equipped with an action of $\mathbf{T}^T$.

---

[4] We emphasise that here $\Delta_{\mathcal{Q}_p}(c)$ denotes the monoid introduced in §2.8.

Definition 3.2.2. We set Q_p-ordinary completed cohomology to be

$$\pi^{Q_p\text{-ord}}(K^p, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m) := R\Gamma(N_p^0, \pi(K^p, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau,N})}/\varpi^m))^{Q_p\text{-ord}} \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}]).$$

By Corollary 3.1.10 and Lemma 3.2.1, for integers $m \geq 1$, $c \geq b \geq 0$ with $c \geq c_p$, we have

$$R\Gamma(M_p^b, \pi^{Q_p\text{-ord}}(K^p, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)) \cong R\Gamma(X_{K(b,c)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}$$

in $D^+(\mathcal{O}/\varpi^m[M_p^0/M_p^b])$. As an immediate consequence, we can deduce the independence of level property.

Corollary 3.2.3. (Independence of level) *For integers $m \geq 1$, and $c \geq b \geq 0$ with $c \geq c_p$, the natural $\mathbf{T}^T$-equivariant morphism*

$$R\Gamma(X_{K(b,\max\{c_p,b\})}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}} \to$$

$$R\Gamma(X_{K(b,c)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}$$

is an isomorphism in $D^+(\mathcal{O}/\varpi^m[M_p^0/M_p^b])$.

Note that the analogous statement also holds for compactly supported and boundary cohomology.

As a consequence of independence of level (or rather the statements behind its proof), we can deduce that the Q_p-ordinary part of the cohomology complexes $R\Gamma(X_{K(0,c)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})})$ is given by taking invariants of Q_p-ordinary completed cohomology with respect to the corresponding smooth types of the Levi subgroup.

Corollary 3.2.4. *For every integer $c \geq c_p$, we have a natural $\mathbf{T}^T$-equivariant isomorphism*

$$R\Gamma(X_{K(0,c)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}} \cong$$

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_p^0]}(\sigma(\underline{\tau})^\circ/\varpi^m, \pi^{Q_p\text{-ord}}(K^p, \mathcal{V}_\lambda/\varpi^m))$$

in $D^+(\mathcal{O}/\varpi^m)$ induced by Corollary 3.1.10, Lemma 3.2.1 and Lemma 2.2.2.

Proof. By Corollary 3.1.10 and Lemma 3.2.1, we have a natural isomorphism

$$R\Gamma(X_{K(0,c)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau,N})}/\varpi^m)^{Q_p\text{-ord}} \cong R\Gamma(M_p^0, \pi^{Q_p\text{-ord}}(K^p, \mathcal{V}_{(\lambda,\underline{\tau})}/\varpi^m))$$

in $D^+(\mathcal{O}/\varpi^m)$. Recall that by definition we have

$$\pi^{Q_p\text{-ord}}(K^p, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m) = R\Gamma(N_p^0, \pi(K^p, \mathcal{V}_\lambda/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \widetilde{\sigma(\underline{\tau})}^{\circ,\vee}/\varpi^m)^{Q_p\text{-ord}}. \tag{3.2.1}$$

Since $- \otimes_{\mathcal{O}/\varpi^m} \widetilde{\sigma(\underline{\tau})}^{\circ,\vee}/\varpi^m$ is exact and has $- \otimes_{\mathcal{O}/\varpi^m} \widetilde{\sigma(\underline{\tau})}^{\circ}/\varpi^m$ as an exact left adjoint, [Wei94], Corollary 10.8.3 applies proving that 3.2.1 is naturally isomorphic to

$$\pi^{Q_p\text{-ord}}(K^p, \mathcal{V}_\lambda/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \sigma(\underline{\tau})^{\circ,\vee}/\varpi^m =$$

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m}(\sigma(\underline{\tau})^\circ/\varpi^m, \pi^{Q_p\text{-ord}}(K^p, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m))$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}])$ when $\sigma(\underline{\tau})^\circ$ is viewed as an $\mathcal{O}/\varpi^m[\Delta_{M_p}]$-module via inflation from M_p^0. Another application of [Wei94], Corollary 10.8.3 to

$$\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_p^0]}(\sigma(\underline{\tau})^\circ/\varpi^m, -) = \Gamma(M_p^0, -) \circ \mathrm{Hom}_{\mathcal{O}/\varpi^m}(\sigma(\underline{\tau})^\circ/\varpi^m, -)$$

finishes the proof. □

We now turn to discussing the second feature of Hida theory called *independence of weight.* Recall the representation

$$\mathcal{V}_{w_0^{Q_p}\lambda} = \otimes_{v\in S_p}\mathcal{V}_{w_0^{Q_v}\lambda_v} \in \mathrm{Mod}(\mathcal{O}[\prod_{v\in S_p} M_v^0])$$

from Lemma 2.7.1. View it as a Δ_{M_p}-module via inflation.

Proposition 3.2.5. (Independence of weight) *For any integers $m \geq 1$, $c \geq c_p$, subset $\overline{S} \subset \overline{S}_p$, and complex $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{\mathcal{Q}_{\overline{S}}}(c)])$, the map introduced in Lemma 2.7.1 induces an isomorphism*

$$R\Gamma(N^0_{\overline{S}}, \pi \otimes_{\mathcal{O}/\varpi^m} \mathcal{V}_{\lambda_{\overline{S}}}/\varpi^m)^{Q_{\overline{S}}\text{-ord}} \xrightarrow{\sim} R\Gamma(N^0_{\overline{S}}, \pi)^{Q_{\overline{S}}\text{-ord}} \otimes_{\mathcal{O}/\varpi^m} \mathcal{V}_{w_0^{Q_{\overline{S}}}\lambda_{\overline{S}}}/\varpi^m$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_{\overline{S}}}])$. In particular, we have a natural isomorphism

$$\pi^{Q_p\text{-ord}}(K^p, \mathcal{V}_\lambda/\varpi^m) \xrightarrow{\sim} \pi^{Q_p\text{-ord}}(K^p, \mathcal{O}/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \mathcal{V}_{w_0^{Q_p}\lambda}/\varpi^m$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}])$.

Proof. The same argument as in the proof of [CN23], Proposition 2.2.15 applies here. □

We now can deduce that $R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}$ admits a natural Hecke action at p corresponding to the data $(\lambda, \underline{\tau})$. Namely, we set

$$\sigma(\lambda, \underline{\tau})^\circ := \mathcal{V}^\vee_{w_0^{Q_p}\lambda} \otimes_{\mathcal{O}} \sigma(\underline{\tau})^\circ,$$

a locally algebraic $\mathcal{O}$-representation of M_p^0.

Corollary 3.2.6. *We have a natural isomorphism*

$$R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}} \cong$$

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_p^0]}(\sigma(\lambda, \underline{\tau})^\circ/\varpi^m, \pi^{Q_p\text{-ord}}(K^p, \mathcal{O}/\varpi^m))$$

in $D^+(\mathcal{O}/\varpi^m)$ induced by Corollary 3.2.4, and Proposition 3.2.5. In particular, we have an induced algebra homomorphism

$$\mathcal{H}(\sigma(\lambda, \underline{\tau})^{\circ,\vee}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m \to \mathrm{End}_{D^+(\mathcal{O}/\varpi^m)}(R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}).$$

Proof. The first part of the statement follows immediately from Corollary 3.2.4 and Proposition 3.2.5.

For the second part, note that the formalism of §3.1 with the choice M_p^+ for the role of $\Delta^+_{M_p}$ implies that $\pi^{Q_p\text{-ord}}(K^p, \mathcal{O}/\varpi^m)$ can be viewed as an object in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_p])$. We then have

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_p^0]}(\sigma(\lambda,\underline{\tau})^\circ/\varpi^m, \pi^{Q_p\text{-ord}}(K^p,\mathcal{O}/\varpi^m)) \in D^+(\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m)).$$

Moreover, the forgetful functor

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G_p]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta^+_{M_p} \ltimes N_p^0])$$

sends injectives to $\Gamma(N_p^0,-)$-acyclics by [Eme10b], Proposition 2.1.11. Consequently, an application of [Wei94], Corollary 10.8.3 shows that the ordinary completed cohomology complex $\pi^{Q_p\text{-ord}}(K^p,\mathcal{O}/\varpi) \in D^+(\mathrm{Mod}_{\mathrm{sm}}[\Delta_{M_p}])$ can be computed by applying $R\Gamma(N_p^0,-)^{Q_p\text{-ord}}$ to the completed cohomology complex $\pi(K^p,\mathcal{O}/\varpi^m) \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G_p])$ followed by an application of the forgetful functor $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_p]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}])$. On the other hand, [Eme10b], Proposition 2.1.2 shows that the forgetful functor

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_p]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_p^0])$$

preserves injectives. In particular, another application of [Wei94], Corollary 10.8.3 yields an algebra homomorphism

$$\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m) \to \mathrm{End}_{D^+(\mathcal{O}/\varpi^m)}(R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}).$$

Moreover, we have a natural morphism

$$\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}) \to \mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m)$$

induced by the short exact sequence

$$0 \to \text{c-Ind}_{M_p^0}^{M_p}\sigma(\lambda,\underline{\tau})^{\circ,\vee} \xrightarrow{\varpi^m\cdot} \text{c-Ind}_{M_p^0}^{M_p}\sigma(\lambda,\underline{\tau})^{\circ,\vee} \to \text{c-Ind}_{M_p^0}^{M_p}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m) \to 0$$

and the proof is finished. □

We note that the action of $\mathcal{H}(\sigma,\underline{\tau})^{\circ,\vee}$ on $R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}$ admits another description. Namely, by independence of level, we can pass to level $K(0,m)$ and then, by independence of weight, we get an identification

$$R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}} \cong R\Gamma(X_{K(0,m)}, \widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m})^{Q_p\text{-ord}}$$

in $D^+(\mathcal{O}/\varpi^m)$. On the other hand, we have

$$R\Gamma(X_{K(0,m)}, \widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m}) \cong R\mathrm{Hom}_{Q_p(0,m)}(\widetilde{\sigma(\lambda,\underline{\tau})^{\circ}/\varpi^m}, \pi(K^p,\mathcal{O}/\varpi^m))$$

in $D^+(\mathcal{O}/\varpi^m)$ and the latter naturally lives in $D^+(\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m)^+)$ according to §3.1. Therefore, we get a natural action of $\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m)^+$ on the complex $R\Gamma(X_{K(0,m)}, \widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m})$ and, using Corollary 3.1.10, one sees that the induced action of $\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m)$ on $R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}$ is the one constructed in Corollary 3.2.6. An upshot of this observation is that this way the Hecke action can be described using the formalism of §2.4.

3.3 Hida theory with dual coefficients

Given a tuple $(Q_p, \lambda, \underline{\tau})$ as in §3.2, and an integer $m \geq 1$, we also discuss $\overline{Q}_p$-ordinary Hida theory for $\mathcal{V}^{Q_p,\vee}_{\lambda,\underline{\tau}}/\varpi^m$ on X_K where $K \subset G(\mathbf{A}^{\infty}_{F^+})$ is a good subgroup such that $K_p = \mathcal{Q}_p(0, \widetilde{c})$ with $\widetilde{c} \geq c_p$. This is developed by applying the formalism of §3.1 with parabolic subgroup $\overline{Q}_p = M_p \ltimes \overline{N}_p \subset G_p$, $N^0 := \overline{N}^{\widetilde{c}}_p$, $M^b := M^b_p$, $\overline{N}^c := N^c_p$, $\Delta^+_M := (\Delta^+_{M_p})^{-1}$, $z_p := u^{Q_p,-1}_p$, and $\sigma := \sigma(\lambda, \underline{\tau})^{\circ,\vee}$. We can then introduce $\overline{Q}_p$-ordinary parts of the level $K(0, \widetilde{c})$ cohomology of $\mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m$ by inverting the Hecke operator attached to $u^{Q_p,-1}_p$. We can also introduce $\overline{Q}_p$-ordinary completed cohomology

$$\pi^{\overline{Q}_p\text{-ord}}(K^p, \mathcal{V}^{Q_p,\vee}_{(\lambda,\tau)}/\varpi^m) := R\Gamma(\overline{N}^{\widetilde{c}}_p, \pi(K^p, \mathcal{V}^{Q_p,\vee}_{(\lambda,\tau)}/\varpi^m))^{\overline{Q}_p\text{-ord}} \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}]).$$

Given this setup, the formalism of §3.1 combined with the short exact sequence

$$0 \to \mathcal{V}^{\vee}_{w_0^{Q_p}\lambda} \to \mathcal{V}^{\vee}_{\lambda} \to \mathcal{K}^{\vee}_{\lambda} \to 0$$

induced by taking duals of the surjection of Lemma 2.7.1 yields the independence of weight property for dual coefficients.

Proposition 3.3.1. (Independence of weight) *For any integer $m \geq 1$, there is a natural isomorphism*

$$\pi^{\overline{Q}_p\text{-ord}}(K^p, \mathcal{V}^{\vee}_{\lambda}/\varpi^m) \xrightarrow{\sim} \pi^{\overline{Q}_p\text{-ord}}(K^p, \mathcal{O}/\varpi^m) \otimes_{\mathcal{O}/\varpi^m} \mathcal{V}^{\vee}_{w_0^{Q_p}\lambda}/\varpi^m$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}])$.

Remark 3.3.2. Note that even though the definition of $\overline{Q}_p$-ordinary completed cohomology seems to depend on the choice of $\tilde{c} \geq c_p$, it in fact is independent of this choice up to natural isomorphism. To see this, consider two integers $\tilde{c}_1, \tilde{c}_2 \geq c_p$. Then, by independence of weight for $i = 1, 2$, we have natural isomorphisms

$$R\Gamma(\overline{N}^{\tilde{c}_i}_p, \pi(K^p, \mathcal{V}^{Q_p,\vee}_{(\lambda,\tau)}/\varpi^m))^{\overline{Q}_p\text{-ord}} \cong R\Gamma(\overline{N}^{\tilde{c}_i}_p, \pi(K^p, \mathcal{O}/\varpi^m))^{\overline{Q}_p\text{-ord}} \otimes_{\mathcal{O}/\varpi^m} \sigma(\lambda, \underline{\tau})^{\circ}/\varpi^m$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_{M_p}])$. In particular, we can reduce the question to one with trivial coefficients. In that case, $\pi(K^p, \mathcal{O}/\varpi^m)$ in fact lies in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\prod_{v \in S_p} \overline{Q}_v(F_v)])$ and the proof of [Eme10a], Proposition 3.1.12 shows that there is a natural isomorphism

$$R\Gamma(\overline{N}^{\tilde{c}_1}_p, \pi(K^p, \mathcal{O}/\varpi^m))^{\overline{Q}_p\text{-ord}} \cong R\Gamma(\overline{N}^{\tilde{c}_2}_p, \pi(K^p, \mathcal{O}/\varpi^m))^{\overline{Q}_p\text{-ord}} \qquad (3.3.1)$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_p])$, showing the claim. We note that the same argument shows that instead of $\overline{N}^{\tilde{c}}_p$, we could have taken any compact open $\overline{N}^{\circ}_p \subset \overline{N}^{c_p}_p$ that is preserved under conjugation by M^0_p.

As a consequence of Remark 3.3.2, we obtain an independence of level property for dual coefficients.

Proposition 3.3.3 (Independence of level)**.** *For integers $m \geq 1$, and $\tilde{c} \geq c_p$, the natural $\mathbf{T}^T$-equivariant morphism*

$$R\Gamma(X_{K(0,c_p)}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)^{\overline{Q}_p\text{-ord}} \to R\Gamma(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)^{\overline{Q}_p\text{-ord}}$$

is an isomorphism in $D^+(\mathcal{O}/\varpi^m)$.

Due to the independence of weight property, and Corollary 3.1.10, we can introduce the relevant Hecke action at p.

Corollary 3.3.4. *We have a natural isomorphism*

$$R\Gamma(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)^{\overline{Q}_p\text{-ord}} \cong$$

$$\cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_p^0]}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m, \pi^{\overline{Q}_p\text{-ord}}(K^p, \mathcal{O}/\varpi^m))$$

in $D^+(\mathcal{O}/\varpi^m)$. In particular, we have a natural algebra homomorphism

$$\mathcal{H}(\sigma(\lambda,\underline{\tau})^\circ) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m \to \mathrm{End}_{D^+(\mathcal{O}/\varpi^m)}(R\Gamma(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)^{\overline{Q}_p\text{-ord}}).$$

Proof. The proof is identical to that of Corollary 3.2.6. Namely, by applying Corollary 3.1.10, and the obvious analogue of Lemma 3.2.1, we get an isomorphism

$$R\Gamma(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)^{\overline{Q}_p\text{-ord}} \cong R\Gamma(M_p^0, \pi^{\overline{Q}_p\text{-ord}}(K^p, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)).$$

Then the isomorphism we seek is induced by the dual of the surjection

$$\mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m \to \widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m}. \tag{3.3.2}$$

$\square$

We mention that the introduced Hecke action once again has a slightly different description just as explained at the end of §3.2.

Finally, we deduce Hecke-equivariance of Poincaré duality for Q_p-ordinary cohomology with $\mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m$-coefficients.

Proposition 3.3.5. *The Verdier duality isomorphism*

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m}(R\Gamma_c(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m), \mathcal{O}/\varpi^m) \cong R\Gamma(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)[\dim_{\mathbf{R}} X_K]$$

induces an isomorphism

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m}(R\Gamma_c(X_{K(0,\tilde{c})}, \mathcal{V}^{Q_p,\vee}_{(\lambda,\underline{\tau})}/\varpi^m)^{\overline{Q}_p\text{-ord}}, \mathcal{O}/\varpi^m) \cong$$

$$\cong R\Gamma(X_{K(0,\widetilde{c})}, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_p\text{-ord}}[\dim_{\mathbf{R}} X_K]$$

in $D^+(\mathcal{O}/\varpi^m)$*. Moreover, the latter isomorphism is equivariant with respect to the natural left action of* $\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$ *on the RHS and the one induced by the anti-isomorphism*

$$\mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m) \xrightarrow{\sim} \mathcal{H}(\sigma(\lambda,\underline{\tau})^{\circ}/\varpi^m),$$

$$[g,\psi] \mapsto [g^{-1},\psi^t]$$

on the LHS.

Proof. The first part follows from applying Corollary 2.4.3 with $\sigma = \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m$ and noting that $U_p^{Q_p} = [u_p^{Q_p}, u_p^{Q_p} \cdot (-)] \in \mathcal{H}(\mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)$.

To see the second part, we reduce the question to the case when $\tilde{c} \geq m$ using independence of level (cf. Corollary 3.2.3, Proposition 3.3.3). In particular, $\widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m}$ makes sense as a representation of $\mathcal{Q}(0,\widetilde{c})$! Then, by independence of weight and naturality of Verdier duality applied to the $\mathcal{Q}(0,\widetilde{c})$-equivariant surjection

$$\mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m \to \widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m}, \tag{3.3.3}$$

the mentioned Verdier duality isomorphism for ordinary parts is also induced by the Verdier duality isomorphism

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m}(R\Gamma_c(X_{K(0,\widetilde{c})}, \widetilde{\sigma(\lambda,\underline{\tau})^{\circ}/\varpi^m}), \mathcal{O}/\varpi^m) \cong R\Gamma(X_{K(0,\widetilde{c})}, \widetilde{\sigma(\lambda,\underline{\tau})^{\circ,\vee}/\varpi^m}). \tag{3.3.4}$$

However, the Verdier duality isomorphism 3.3.4 satisfies the desired Hecke-equivariance by Corollary 2.4.3 and Corollary 3.1.10.[5] □

3.4 $\widetilde{Q}$-ordinary Hida theory for $\widetilde{G}$

As everything mentioned in the previous two sections applies verbatim for the group $\widetilde{G}$ at split p-adic places of F^+, we will only set up the notations and explain the relevant results. We revisit the setup of the corresponding part of §2.7. In particular, we remind the reader of Assumption 2.7.3. Fix a subset of p-adic places $\overline{S} \subset \overline{S}_p$ and a lift $v \mid \bar{v}$ for each $\bar{v} \in \overline{S}_p$ in S_p. Consider a tuple $(Q_p, \lambda, \underline{\tau})$ as previously and consider a corresponding tuple $(\widetilde{Q}_{\overline{S}}, \tilde{\lambda}_{\overline{S}}, \underline{\tau}) := (\widetilde{Q}_{\bar{v}}, \tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})_{\bar{v}\in\overline{S}}$ as in §2.7.[6] Further set $\tilde{\lambda}$ to be some extension of $\tilde{\lambda}$ to a $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{B})_E$-dominant weight of $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_E$. We then similarly form the parahoric level subgroups

$$\widetilde{\mathcal{Q}}_{\overline{S}}(b,c) = \prod_{\bar{v}\in\overline{S}} \widetilde{\mathcal{Q}}_{\bar{v}}(b,c) \subset \prod_{\bar{v}\in\overline{S}} \widetilde{G}(\mathcal{O}_{F^+_{\bar{v}}})$$

---

[5] Note that at this step we used that our Hecke action at p matches up with the Hecke action constructed at the end of §3.2.

[6] Note that in particular we are implicitly assuming that $\widetilde{\lambda}_{\overline{S}}$ is dominant.

for integers $0 \leq b \leq c$ with $c \geq c_p$. For the rest of the section, we fix a prime-to-$\overline{S}$ good level subgroup $\widetilde{K}^{\overline{S}} \subset \widetilde{G}(\mathbf{A}_{F^+}^{\overline{S}\cup\{\infty\}})$ and set $\widetilde{K}(b,c)$ to be the good subgroup $\widetilde{K}^{\overline{S}}\widetilde{\mathcal{Q}}_{\overline{S}}(b,c) \subset \widetilde{G}(\mathbf{A}_{F^+}^{\infty})$. We then freely borrow the notation of §2.8. In particular, we have an open submonoid $\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}}(c_p) \subset \widetilde{G}_{\overline{S}}$, and can set $\widetilde{\Delta}^{+}_{\widetilde{M}^{w_0}_{\overline{S}}} := \widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}}(c_p) \cap \widetilde{M}^{w_0}_{\overline{S}} \subset \widetilde{M}^{w_0,+}_{\overline{S}}$, and $\widetilde{\Delta}_{\widetilde{M}^{w_0}_{\overline{S}}} := \widetilde{\Delta}^{+}_{\widetilde{M}^{w_0}_{\overline{S}}}[u_{\bar{v}}^{\widetilde{Q}^{w_0}_{v},\pm 1} \mid \bar{v} \in \overline{S}]$.

For $c \geq b \geq 0$ with $c \geq c_p$, the Hecke algebra $\mathcal{H}(\widetilde{\Delta}_{\widetilde{\mathcal{Q}}^{w_0}_{\overline{S}}}(c), \widetilde{\mathcal{Q}}_{\overline{S}^{w_0}}(b,c))$ acts on $R\Gamma(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda},\underline{\tau})})$ via endomorphisms in $D^{+}(\mathcal{O})$. In particular, we introduce the $\widetilde{Q}^{w_0}_{\overline{S}}$-ordinary parts of the complex $R\Gamma(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda},\underline{\tau})})^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}}$ as the maximal direct summand on which each $U_{\bar{v}}^{\widetilde{Q}^{w_0}_{\bar{v}}}$ acts invertibly.

On the other hand, the formalism of §3.1 can be applied with the choices $Q = \widetilde{Q}^{w_0}_{\overline{S}}$, $N^0 = \widetilde{N}^{w_0,0}_{\overline{S}}$, $M^b = \widetilde{M}^{w_0,b}_{\overline{S}}$, $\overline{N}^c = \overline{\widetilde{N}}^{w_0,c}_{\overline{S}}$, and σ being the trivial $\widetilde{\Delta}^{+}_{\widetilde{M}^{w_0}_{\overline{S}}}$-module. One can then compare the two constructions and see that the analogue of Lemma 3.2.1 holds.

We can further introduce $\widetilde{Q}^{w_0}_{\overline{S}}$-ordinary completed cohomology

$$\pi^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}}(\widetilde{K}^{\overline{S}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda},\underline{\tau})}/\varpi^m) :=$$

$$R\Gamma(\widetilde{N}^{w_0,0}_{\overline{S}}, \pi(\widetilde{K}^{\overline{S}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda},\underline{\tau,N})}/\varpi^m))^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}} \in D^{+}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{\Delta}_{\widetilde{M}^{w_0}_{\overline{S}}}]).$$

The analogues of the independence of level and weight property hold once again. To emphasise the normalisations and introduce the necessary notations for later, we spell out the statement of the latter.

Proposition 3.4.1 (Independence of weight)**.** *For any integer $m \geq 1$, there is a natural isomorphism*

$$\pi^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}}(\widetilde{K}^{\overline{S}}, \mathcal{V}_{\tilde{\lambda}}/\varpi^m) \xrightarrow{\sim} \pi^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}}(\widetilde{K}^{\overline{S}}, \mathcal{V}_{\tilde{\lambda}^{\overline{S}}}) \otimes_{\mathcal{O}/\varpi^m} \mathcal{V}_{w_0^{\widetilde{Q}^{w_0}_{\overline{S}}}\widetilde{\lambda}_{\overline{S}}}/\varpi^m$$

in $D^{+}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{\Delta}_{\widetilde{M}^{w_0}_{\overline{S}}}])$.

Here we denoted by $\widetilde{\lambda}^{\overline{S}}$ the dominant weight for $\widetilde{G}$ that is trivial at $\overline{S}$ and coincides with $\widetilde{\lambda}$ outside $\overline{S}$. Similarly, $\widetilde{\lambda}_{\overline{S}}$ is the dominant weight for $\widetilde{G}$ that is trivial outside $\overline{S}$ and coincides with $\widetilde{\lambda}$ at $\overline{S}$. Also, analogously to the previous section, we view $\mathcal{V}_{w_0^{\widetilde{Q}^{w_0}_{\overline{S}}}\widetilde{\lambda}_{\overline{S}}}/\varpi^m$ as an $\mathcal{O}/\varpi^m[\widetilde{\Delta}_{\widetilde{M}^{w_0}_{\overline{S}}}]$-module via inflation from $\widetilde{M}^{w_0,0}_{\overline{S}}$.

We can then also deduce the analogues of Corollary 3.2.4 and Corollary 3.2.6. In particular, $R\Gamma(\widetilde{X}_{\widetilde{K}(0,c)}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda},\underline{\tau})}/\varpi^m)^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}}$ admits a natural Hecke action at p. Namely, set

$$\tilde{\sigma}(\tilde{\lambda}_{\overline{S}}, \underline{\tau}_{\overline{S}})^{\circ} := \mathcal{V}^{\vee}_{w_0^{\widetilde{Q}^{w_0}_{\overline{S}}} \tilde{\lambda}_{\overline{S}}} \otimes_{\mathcal{O}} \tilde{\sigma}(\underline{\tau}_{\overline{S}})^{\circ} \in \mathrm{Mod}(\mathcal{O}[\widetilde{M}^{w_0,0}_{\overline{S}}]).$$

Then there is a natural algebra homomorphism

$$\mathcal{H}(\tilde{\sigma}(\tilde{\lambda}_{\overline{S}}, \underline{\tau}_{\overline{S}})^{\circ,\vee}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m \to \mathrm{End}_{D^+(\mathcal{O}/\varpi^m)}(R\Gamma(\widetilde{X}_{\widetilde{K}(0,c)}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\overline{S}}}_{(\tilde{\lambda},\underline{\tau})}/\varpi^m)^{\widetilde{Q}^{w_0}_{\overline{S}}\text{-ord}}).$$

Remark 3.4.2. We note that if we set $\widetilde{S} := \{v \mid \bar{v} \in \overline{S}\}$, we have $\mathcal{V}_{w_0^{\widetilde{Q}^{w_0}_{\overline{S}}} \tilde{\lambda}_{\overline{S}}} = \mathcal{V}_{w_0^{Q_{\widetilde{S}}} \lambda_{\widetilde{S}}} \otimes_{\mathcal{O}} \mathcal{V}_{-w_0^{Q_{\widetilde{S}^c}} \lambda_{\widetilde{S}^c}}$, and, by the proof of [CN23], Lemma 2.2.14, we get

$$\tilde{\sigma}(\tilde{\lambda}_{\overline{S}}, \underline{\tau}_{\overline{S}})^{\circ} = \sigma(\lambda_{\widetilde{S}}, \underline{\tau}_{\widetilde{S}})^{\circ} \otimes_{\mathcal{O}} (\theta_n^{-1})^{*} \sigma(\lambda_{\widetilde{S}^c}, \underline{\tau}_{\widetilde{S}^c})^{\circ} \in$$
$$\mathrm{Mod}(\mathcal{O}[\left(\prod_{v \in \widetilde{S}} M_v(\mathcal{O}_{F_v})\right) \times \left(\prod_{v^c \in \widetilde{S}^c} (\theta_n M_{v^c})(\mathcal{O}_{F_v})\right)].$$

Finally, one can develop $\overline{\widetilde{Q}}_{\overline{S}}$-ordinary Hida theory for $\mathcal{V}^{\widetilde{Q}_{\overline{S}}, w_0^P, \vee}_{(\tilde{\lambda},\underline{\tau})}/\varpi^m$ (see the end of §2.7 for the notation) just as in §3.3. In particular, one arrives to Hecke-equivariance of Poincaré duality at p for ordinary cohomology of $\widetilde{X}_{\widetilde{K}}$.

3.5 Ordinary parts of the Bruhat stratification

Here we again closely follow [CN23] on computing the ordinary parts of certain Bruhat strata of parabolic inductions in the derived category. As we often will not need serious changes in the proofs, we sometimes only state the results we need and indicate how to deduce them from the arguments of [All+23] and [CN23].

As in [CN23], 2.3.1, we restrict ourselves to a completely local setup. Let $L/\mathbf{Q}_p$ be a finite field extension with ring of integers $\mathcal{O}_L$ and a choice of uniformiser ϖ_L. Let G/L be a split connected reductive group with a split maximal torus $T \subset G$ and Weyl group $W = W(G,T)$. Fix a Borel subgroup $T \subset B$ and two standard parabolic subgroups $B \subset Q_1, Q_2 \subset G$ with Levi decomposition $Q_1 = M_1 \ltimes N_1$ and $Q_2 = M_2 \ltimes N_2$, respectively. We denote by W_{Q_i} the Weyl group of M_i and by ${}^{Q_1}W^{Q_2} \subset W$ the set of minimal length representatives of $W_{Q_1} \backslash W / W_{Q_2}$. For $w \in W$ we denote its length by $\ell(w) \in \mathbf{Z}_{\geq 0}$. Recall that $G(L)$ admits a stratification (with respect to its p-adic topology) called the Bruhat stratification[7]

$$G(L) = \coprod_{w \in {}^{Q_1}W^{Q_2}} Q_1(L) w Q_2(L) = \coprod_{w \in {}^{Q_1}W^{Q_2}} S_w$$

---

[7] For a reference in this generality, see [Hau18], Lemma 2.1.2. However, note that *loc. cit.* considers the opposite parabolic $\overline{Q}_1$ on the left instead, so the closure relations are reversed there.

with closure relations given by the Bruhat order

$$\overline{S_w} = \coprod_{w \geq w' \in {}^{Q_1}W^{Q_2}} S_{w'}.$$

In particular, for $i \in \mathbf{Z}_{\geq 0}$, the subset

$$G_{\geq i} := \coprod_{\ell(w) \geq i} S_w \subset G(L)$$

is open.

We first recall how the Bruhat stratification provides a "stratification" of the exact functor

$$\mathrm{Ind}_{Q_1(L)}^{G(L)} : D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)]) \to D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(L)])$$

in a suitable sense. We define the functor

$$I_{\geq i} : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)])) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_2(L)])$$

which sends π to the $Q_2(L)$-stable subspace of functions in $\mathrm{Ind}_{Q_1(L)}^{G(L)}\pi$ which are supported at $G_{\geq i}$. For $w \in {}^{Q_1}W^{Q_2}$, we can further define the functor

$$I_w : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)])) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_2(L)]))$$

which sends π to the set of locally constant functions $f : S_w \to \pi$ which are compactly supported modulo $Q_1(L)$ and left invariant with respect to elements of $Q_1(L)$ equipped with the left $Q_2(L)$-action given by right multiplication on the source. Finally, for $w \in {}^{Q_1}W^{Q_2}$, consider also the open subspace $S_w^\circ := Q_1(L)wM_2(L)N_2(\mathcal{O}_L) \subset S_w$. We then define the functor $I_w^\circ : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+ \ltimes N_2(\mathcal{O}_L)])$ by setting $I_w^\circ(\pi) \subset I_w(\pi)$ to be the subset of functions supported on S_w°. Each of these functors are exact as the argument of [All+23], Proposition 5.3.1 shows.

For every $i \in \mathbf{Z}_{\geq 0}$ and $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)])$, the natural inclusion of functors $I_{\geq i+1} \subset I_{\geq i}$ induces a distinguished triangle

$$I_{\geq i+1}(\pi) \to I_{\geq i}(\pi) \to \oplus_{\ell(w)=i} I_w(\pi) \to I_{\geq i+1}(\pi)[1] \tag{3.5.1}$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_2(L)])$. For the proof of this, see [Hau18], Lemma 2.2.1.

Given a $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L])$, finite free over $\mathcal{O}/\varpi^m$, we consider the functor

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), -)^{Q_2\text{-ord}}) : D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_2(L)]) \to D^+(\mathcal{H}(\sigma^\vee)).$$

We can apply this functor to 3.5.1 to get a distinguished triangle

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_{\geq i+1}(\pi))^{Q_2\text{-ord}}) \to$$

$$\to R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_{\geq i}(\pi))^{Q_2\text{-ord}}) \to$$
$$\to \oplus_{\ell(w)=i} R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_w(\pi))^{Q_2\text{-ord}}) \to$$
$$\to R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_{\geq i+1}(\pi))^{Q_2\text{-ord}})[1].$$

The argument of [CN23], Proposition 2.3.4 shows that taking long exact sequence of cohomology of the distinguished triangle above gives a $\mathcal{H}(\sigma^\vee)$-equivariant short exact sequence

$$0 \to R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_{\geq i+1}(\pi))^{Q_2\text{-ord}}) \to$$
$$\to R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_{\geq i}(\pi))^{Q_2\text{-ord}}) \to$$
$$\to \oplus_{\ell(w)=i} R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), I_w(\pi))^{Q_2\text{-ord}}) \to 0$$

for every $j \in \mathbf{Z}_{\geq 0}$. To be more precise, the proof of Proposition 2.3.4 in *loc. cit.* relies on their Lemma 2.3.5. We state the obvious generalisation we need. However, we omit the proof as it can be proved just as the version in *loc. cit.*

Lemma 3.5.1. *For any $i \in \mathbf{Z}_{\geq 0}$, there are decompositions*

$$G_{\geq i} = U_1^m \coprod U_2^m$$

into open and closed subsets, indexed by $m \in \mathbf{Z}_{\geq 1}$, that are $Q_1(L)$-invariant on the left and $Q_2(\mathcal{O}_L)$-invariant on the right such that

$$G_{\geq i+1} = \bigcup_{m \geq 1} U_1^m.$$

To apply this line of argument, we compute

$$R\Gamma(N_2(\mathcal{O}_L), I_w(\pi))^{Q_2\text{-ord}}$$

for a class of $w \in {}^{Q_1}W^{Q_2}$. First we note the following lemma that is essentially [All+23], Lemma 5.3.4 (see also [CN23], Lemma 2.3.6).

Lemma 3.5.2. *Let $w \in {}^{Q_1}W^{Q_2}$ such that $wM_2(L)w^{-1} \subset M_1(L)$. Then:*

i. I_w° takes injectives to $\Gamma(N_2(\mathcal{O}_L), -)$-acyclics.

ii. Let $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m)[Q_1(L)])$. Then there is a natural isomorphism

$$R\Gamma(N_2(\mathcal{O}_L), I_w^\circ(\pi))^{Q_2\text{-ord}} \xrightarrow{\sim} R\Gamma(N_2(\mathcal{O}_L), I_w(\pi))^{Q_2\text{-ord}}.$$

Proof. Note that the assumption on w implies that

$$S_w^\circ = Q_1(L)wN_2(\mathcal{O}_L).$$

Knowing this, the proof of [All+23], Lemma 5.3.4 applies verbatim. □

If $w \in {}^{Q_1}W^{Q_2}$ such that $wM_2(L)w^{-1} \subset M_1(L)$, we define

$$N_{2,w}^{\circ} := Q_1(L) \cap wN_2(\mathcal{O}_L)w^{-1},$$

a compact subgroup of $Q_1(L)$. We define the functor

$$\Gamma(N_{2,w}^{\circ}, -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+])$$

where an element $m \in M_2(L)^+$ acts on a $v \in \pi^{N_{2,w}^{\circ}}$ by the formula

$$m \cdot v := \sum_{n \in N_{2,w}^{\circ}/wmw^{-1}N_{2,w}^{\circ}wm^{-1}w^{-1}} nwmw^{-1} \cdot v$$

where on the right we consider the natural action of $wmw^{-1} \in Q_1(L)$. One checks easily that our assumption on w ensures that this formula makes sense meaning that we have $wmw^{-1}N_{2,w}^{\circ}wm^{-1}w^{-1} \subset N_{2,w}^{\circ}$.

Lemma 3.5.3. *Let $w \in {}^{Q_1}W^{Q_2}$ such that $wM_2(L)w^{-1} \subset M_1(L)$ and consider $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)])$. Then we have a natural isomorphism*

$$R\Gamma(N_2(\mathcal{O}_L), I_w^{\circ}(\pi)) \cong R\Gamma(N_{2,w}^{\circ}, \pi)$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+])$.

Proof. Again, it can be proved by running the proof of the analogous statement [All+23], Lemma 5.3.5, keeping in mind our assumption on w. For the reader's convenience, we recall the argument here.

By the first part of Lemma 3.5.2, it suffices to give a natural isomorphism of underived functors

$$\Gamma(N_2(\mathcal{O}_L), I_w^{\circ}(-)) \cong \Gamma(N_{2,w}^{\circ}, -).$$

For $\pi \in \mathrm{Mod}_{\mathrm{sm}}(Q_1(L))$, the map will send an $N_2(\mathcal{O}_L)$-invariant function

$$f : Q_1(L)wN_2(\mathcal{O}_L) \to \pi$$

to $f(w) \in \pi^{N_{2,w}^{\circ}}$. This visibly gives an isomorphism of underlying $\mathcal{O}/\varpi^m$-modules with inverse sending $v \in \pi^{N_{2,w}^{\circ}}$ to $f_v : pwn \mapsto p \cdot v$.

We are left with checking that this defines an $M_2(L)^+$-equivariant map. In other words, for $f \in \Gamma(N_2(\mathcal{O}_L), I_w^{\circ}(\pi))$ and $m \in M_2(L)^+$, we need to see that

$$\sum_{n \in N_2(\mathcal{O}_L)/mN_2(\mathcal{O}_L)m^{-1}} f(wnm) = \sum_{\tilde{n} \in N_{2,w}^{\circ}/wmw^{-1}N_{2,w}^{\circ}(wmw^{-1})^{-1}} \tilde{n}wmw^{-1}f(w). \tag{3.5.2}$$

Note that the association

$$N_{2,w}^{\circ}/wmw^{-1}N_{2,w}^{\circ}(wmw^{-1})^{-1} \to N_2(\mathcal{O}_L)/mN_2(\mathcal{O}_L)m^{-1}, \tag{3.5.3}$$

$$\tilde{n} \mapsto w^{-1}\tilde{n}w$$

is injective. Moreover, for $n = w^{-1}\tilde{n}w$ lying in the image, we have $f(wnm) = \tilde{n}wmw^{-1}f(w)$. In particular, to see that 3.5.2 holds, it suffices to prove that $f(wnm) \neq 0$ only if n lies in the image of 3.5.3. So assume that $wnm \in S_w^\circ = Q_1(L)wN_2(\mathcal{O}_L)$. Accordingly, we write it in the form $wnm = qwn'$ with $q \in Q_1(L)$ and $n' \in N_2(\mathcal{O}_L)$. On the other hand,

$$n = w^{-1}qwn'm^{-1} = (w^{-1}qwm^{-1})(mn'm^{-1}) \in N_2(\mathcal{O}_L)$$

and our assumptions imply that $mn'm^{-1} \in N_2(\mathcal{O}_L)$. Therefore, we get

$$w^{-1}qwm^{-1} = w^{-1}(qwm^{-1}w^{-1})w \in w^{-1}Q_1(L)w \cap N_2(\mathcal{O}_L) = w^{-1}N_{2,w}^\circ w.$$

□

Corollary 3.5.4. *Let* $w \in {}^{Q_1}W^{Q_2}$ *such that* $wM_2(L)w^{-1} \subset M_1(L)$ *and* $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[Q_1(L)])$. *There is a natural isomorphism*

$$R\Gamma(N_2(\mathcal{O}_L), I_w(\pi))^{Q_2\text{-ord}} \cong R\Gamma(N_{2,w}^\circ, \pi)^{Q_2\text{-ord}}$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)])$.

Finally, we compute $R\Gamma(N_{2,w}^\circ, \pi)^{Q_2\text{-ord}}$ as in [All+23], Lemma 5.3.7. For the rest of the section, we assume that $G = \mathrm{GL}_n$ with $T = T_n$, the torus consisting of diagonal matrices and $B = B_n$ the Borel consisting of upper triangular matrices. We also introduce some further notation. Given $w \in {}^{Q_1}W^{Q_2}$ such that $wM_2(L)w^{-1} \subset M_1(L)$, set $Q_w := wQ_2w^{-1} \cap M_1 \subset M_1$ with Levi quotient $M_w = wM_2w^{-1}$ and unipotent radical $N_w = wN_2w^{-1} \cap M_1$. Consider the character

$$\chi_w : M_2(L) \to \mathcal{O}^\times,$$
$$m \mapsto \frac{\mathrm{Norm}_{L/\mathbf{Q}_p} \det_L(\mathrm{Ad}(m^w) \mid_{\mathrm{Lie} N_w(L)})^{-1}}{|\mathrm{Norm}_{L/\mathbf{Q}_p} \det_L(\mathrm{Ad}(m^w) \mid_{\mathrm{Lie} N_w(L)})|_p}$$

where we set $m^w := wmw^{-1}$. Introduce the equivalence of categories

$$\tau_w : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[wM_2(L)w^{-1}])$$

sending π to $\tau_w(\pi)$ with underlying $\mathcal{O}/\varpi^m$-module π but with the twisted action $\tau_w(\pi)(m) = \pi(w^{-1}mw)$. Finally, set $N_w^\circ = N_w(\mathcal{O}_L)$ and $N_{2,w,N_1}^\circ := N_{2,w}^\circ \cap N_1(L)$.

Lemma 3.5.5. *Assume that* $G = \mathrm{GL}_n$ *with* $T = T_n$, $B = B_n$ *and let* $w \in {}^{Q_1}W^{Q_2}$ *such that* $wM_2(L)w^{-1} \subset M_1(L)$. *Then, for any* $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_1(L)])$, *there is a natural isomorphism between*

$$R\Gamma(N_{2,w}^\circ, \mathrm{Inf}_{M_1(L)}^{Q_1(L)}\pi)^{Q_2\text{-ord}}$$

and

$$\mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}/\varpi^m} \tau_w^{-1} R\Gamma(N_w^\circ, \pi)^{Q_w\text{-ord}}[-\mathrm{rk}_{\mathbf{Z}_p} N_{2,w,N_1}^\circ]$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)])$.

Proof. Before starting the proof, we note that the argument is just an obvious generalisation of the proof of [All+23], Lemma 5.3.7 and so we kept the structure of their argument and sometimes will refer to it as *loc. cit.*

We set the monoid $M_2(L)^+ \ltimes_w N^\circ_{2,w}$ to be $M_2(L)^+ \times N^\circ_{2,w}$ with the action $(m,1)(1,n) = (1, m^w n (m^w)^{-1})(m,1)$. Consider the short exact sequence

$$0 \to N^\circ_{2,w,N_1} \to N^\circ_{2,w} \to N^\circ_w \to 0$$

and note that it is obviously equivariant for the $M_2(L)^+$-action on each groups via $m \mapsto wmw^{-1}$. Then just as in *loc. cit.*, we basically write $R\Gamma(N^\circ_{2,w}, \mathrm{Inf}^{Q_1(L)}_{M_1(L)} -)^{Q_2\text{-ord}}$ as "$R\Gamma(N^\circ_w, -)^{Q_w\text{-ord}} \circ R\Gamma(N^\circ_{2,w,N_1}, \mathrm{Inf}^{Q_1(L)}_{M_1(L)} -)^{Q_1\text{-ord}}$". To make this precise, we need to introduce some functors. Denote by

$$\mathrm{Res}^w : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_1(L)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+ \ltimes_w N^\circ_{2,w}])$$

the composite of $\mathrm{Inf}^{Q_1(L)}_{M_1(L)}$ with the functor that sends $\pi \in \mathrm{Mod}_{\mathrm{sm}}(Q_1(L))$ to itself as an $\mathcal{O}/\varpi^m$-module with the action $\mathrm{Res}^w(\pi)(mn) = \pi(m^w n)$. Further set

$$\alpha : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+ \ltimes_w N^\circ_w]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_w(L)^+ \ltimes N^\circ_w])$$

to be the functor defined by composing the equivalence

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+ \ltimes_w N^\circ_w]) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{sm}}[wM_2(L)^+w^{-1} \ltimes N^\circ_w])$$

with the localisation

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[wM_2(L)^+w^{-1} \ltimes N^\circ_w]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_w(L)^+ \ltimes N^\circ_w])$$

induced by the inclusion $wM_2(L)^+w^{-1} \subset M_w(L)^+$. Analogous construction defines a functor β such that they fit into a commutative diagram (up to natural equivalence)

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+ \ltimes_w N^\circ_w]) \arrow[r, "\alpha"] \arrow[d, "\Gamma(N^\circ_w{,}-)"] & \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_w(L)^+ \ltimes N^\circ_w]) \arrow[d, "\Gamma(N^\circ_w{,}-)"] \\
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+]) \arrow[r, "\beta"] & \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_w(L)^+])
\end{tikzcd}$$

where the vertical arrows are defined the usual way by considering the corresponding Hecke actions. A reasoning similar to the proof of Lemma 3.1.6 shows that α takes injectives to $\Gamma(N^\circ_w, -)$-acyclics. Therefore, by checking things on underived functors, we have

$$R\Gamma(N^\circ_{2,w}, \mathrm{Inf}^{Q_1(L)}_{M_1(L)}\pi)^{Q_2\text{-ord}} \cong$$

$$\tau_w^{-1}(\beta R\Gamma(N^\circ_w, R\Gamma(N^\circ_{2,w,N_1}, \mathrm{Res}^w(\pi)))^{Q_w\text{-ord}} \cong$$

$$\tau_w^{-1}(R\Gamma(N_w^\circ, \alpha R\Gamma(N_{2,w,N_1}^\circ, \mathrm{Res}^w \pi)))^{Q_w\text{-ord}}.$$

Since N_{2,w,N_1}° acts trivially on $\mathrm{Res}^w\pi$, we get

$$R\Gamma(N_{2,w,N_1}^\circ, \mathrm{Res}^w\pi) \cong \mathrm{Res}^w\pi \otimes_{\mathcal{O}/\varpi^m} R\Gamma(N_{2,w,N_1}^\circ, \mathcal{O}/\varpi^m)$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(L)^+ \ltimes_w N_w^\circ])$. In particular, to compute $\alpha R\Gamma(N_{2,w,N_1}^\circ, \mathrm{Res}^w\pi)$, it suffices to compute $\alpha R\Gamma(N_{2,w,N_1}^\circ, \mathcal{O}/\varpi^m)$.
Claim: We have natural isomorphisms

$$\alpha R\Gamma(N_{2,w,N_1}^\circ, \mathcal{O}/\varpi^m) \cong \tau_w \mathcal{O}/\varpi^m(\chi_w)[-\mathrm{rk}_{\mathbf{Z}_p} N_{2,w,N_1}^\circ] \cong$$

$$\alpha \mathcal{O}/\varpi^m(\chi_w)[-\mathrm{rk}_{\mathbf{Z}_p} N_{2,w,N_1}^\circ]$$

in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_w(L)^+ \ltimes N_w^\circ])$ where, by abuse of notation we consider χ_w as an $M_2(L)^+ \ltimes_w N_w^\circ$-module by inflation.

Proof of claim. Assume that $Q_1 = P_{(n_1,\dots,n_t)}$. Then we can write

$$N_1(\mathcal{O}_L) \cong \prod_{1\le i<j\le t} N_{i,j}^\circ$$

where $N_{i,j}^\circ \subset N_1(\mathcal{O}_L)$ corresponds to the entries lying in $[n_{i-1}+1, n_i] \times [n_{j-1}+1, n_j]$ with the convention that $n_0 = 0$. This induces an isomorphism

$$N_{2,w,N_1}^\circ \cong \prod_{1\le i<j\le t} N_{2,w,ij}^\circ.$$

Set $r_{i,j} := \mathrm{rk}_{\mathbf{Z}_p} N_{2,w,ij}^\circ$ and, for $1 \le k \le t$,

$$z_{p,k} := \mathrm{diag}(p, \dots, p, 1, \dots, 1)$$

where the first $n_1 + \dots + n_k$ entries in the diagonal are given by p and the rest by 1. Note that $z_{p,k}$ is in the centre of $M_1(L)$, so it lies in $M_w(L)^+$ and it is invertible there, in particular, its Hecke action on $\alpha R\Gamma(N_{2,w,N_1}^\circ, \mathcal{O}/\varpi^m)$ is invertible.

Moreover, Künneth formula gives

$$R\Gamma(N_{2,w,N_1}^\circ, \mathcal{O}/\varpi^m) \cong \bigotimes_{1\le i<j\le t} R\Gamma(N_{2,w,ij}^\circ, \mathcal{O}/\varpi^m)$$

and the Hecke action of $z_{p,k}$ is given by multiplying by the scalar

$$[N_{2,w,N_1}^\circ : z_{p,k} N_{2,w}^\circ z_{p,k}^{-1}] = p^{\sum_{i<k<j} r_{i,j}}$$

the tensor product of maps

$$m_{i,j}^k : R\Gamma(N_{2,w,ij}^\circ, \mathcal{O}/\varpi^m) \to R\Gamma(N_{2,w,ij}^\circ, \mathcal{O}/\varpi^m)$$

induced by multiplication by p on $N_{2,w,ij}^\circ$ if $i < k < j$ and by $\mathrm{id} : N_{2,w,ij}^\circ \to N_{2,w,ij}^\circ$ otherwise. In the first case this means that, for $0 \le d_{i,j} \le r_{i,j}$,

$H^{d_{i,j}}(m^k_{i,j})$ is multiplication by $p^{-d_{i,j}}$ and it is multiplication by 1 otherwise. In particular, $\alpha(\bigotimes_{i<j} H^{d_{i,j}}(N^\circ_{2,w,ij}, \mathcal{O}/\varpi^m)) \neq 0$ only when $d_{i,j} = r_{i,j}$ for each $1 \leq i < j \leq t$. Note that $\sum_{i<j} r_{i,j} = \mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}$. Therefore, we have

$$\alpha R\Gamma(N^\circ_{2,w,N_1}, \mathcal{O}/\varpi^m) \cong \alpha H^{\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}}(N^\circ_{2,w,N_1}, \mathcal{O}/\varpi^m)[-\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}].$$

Moreover, just as in the proof of [Hau16], Proposition 3.1.8, using [Eme10b], Proposition 3.5.6 and the description of the corestriction map on top degree cohomology (cf. [Eme10b], Lemma 3.5.10), we see that the latter is, as an $\mathcal{O}/\varpi^m[M_w(L)^+]$-module, given by $\alpha\mathcal{O}/\varpi^m(\chi_w)[-\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}]$. □

Putting everything together, we get

$$R\Gamma(N^\circ_{2,w}, \mathrm{Inf}^{Q_1(L)}_{M_1(L)}\pi)^{Q_2\text{-ord}} \cong$$

$$\tau_w^{-1} R\Gamma(N^\circ_w, \alpha(\mathcal{O}/\varpi^m(\chi_w) \otimes \mathrm{Res}^w \pi))^{Q_w\text{-ord}}[-\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}] \cong$$
$$\tau_w^{-1}(\tau_w(\mathcal{O}/\varpi^m(\chi_w))) \otimes_{\mathcal{O}/\varpi^m} \tau_w^{-1} R\Gamma(N^\circ_w, \pi)^{Q_w\text{-ord}}[-\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}] \cong$$
$$\mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}/\varpi^m} \tau_w^{-1} R\Gamma(N^\circ_w, \pi)^{Q_w\text{-ord}}[-\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}].$$

□

Combining the results of the section, we obtain the following.

Corollary 3.5.6. *Assume that $G = \mathrm{GL}_n$, $T = T_n$ and $B = B_n$. Let $w \in {}^{Q_1}W^{Q_2}$ such that $wM_2(L)w^{-1} \subset M_1(L)$. Then, for every $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_1(L)])$, $\sigma \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)])$ finite free as an $\mathcal{O}/\varpi^m$-module, and $j \in \mathbf{Z}_{\geq 0}$, the group*

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, R\Gamma(N_2(\mathcal{O}_L), \mathrm{Ind}^{G(L)}_{Q_1(L)}\pi)^{Q_2\text{-ord}})$$

admits

$$R^{j-\mathrm{rk}_{\mathbf{Z}_p} N^\circ_{2,w,N_1}}\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_2(\mathcal{O}_L)]}(\sigma, \mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}/\varpi^m} \tau_w^{-1} R\Gamma(N^\circ_w, \pi)^{Q_w\text{-ord}})$$

as a $\mathcal{H}(\sigma)$-equivariant subquotient.

For the reader's convenience, we spell out the case of interest for our application in proving local-global compatibility. For this we introduce some notation that hopefully makes it easier to motivate how Corollary 3.5.6 will be applied. In particular, consider $\widetilde{G} = \mathrm{GL}_{2n}$, and set $P = P_{(n,n)} \subset \widetilde{G}$ to be the Siegel parabolic with Levi decomposition $P = G \ltimes U$. Moreover, set $\widetilde{Q} \subset P$ to be any standard[8] parabolic subgroup with Levi decomposition $\widetilde{M} \ltimes \widetilde{N}$. Set $\widetilde{Q} \cap G = Q_c \times Q \subset G = \mathrm{GL}_n \times \mathrm{GL}_n$ and denote their Levi decompositions by $M_c \ltimes N_c$, and $M \ltimes N$, respectively. Note that $\widetilde{M} = M_c \times M$. Denote by $\widetilde{Q}^{w_0} \subset P \subset \widetilde{G}$ the standard parabolic subgroup with Levi decomposition

---

[8] Standard with respect to the Borel of upper triangular matrices.

$\widetilde{M}^{w_0} \ltimes \widetilde{N}^{w_0}$ where $\widetilde{M}^{w_0} = M \times M_c \subset \mathrm{GL}_n \times \mathrm{GL}_n$. Pick $\widetilde{\sigma} = \sigma \otimes \sigma_c \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{M}^{w_0}(\mathcal{O}_L]) = \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[M(\mathcal{O}_L) \times M_c(\mathcal{O}_L)])$, finite free as an $\mathcal{O}/\varpi^m$-module. Finally, note that $w_0^P \in {}^P W^{\widetilde{Q}^{w_0}}$. Then a direct application of Corollary 3.5.6 with $Q_1 = P$, $Q_2 = \widetilde{Q}^{w_0}$, and $w = w_0^P$ gives.

Corollary 3.5.7. *For every* $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(L)])$*, and* $j \in \mathbf{Z}_{\geq 0}$*, the group*

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}^{w_0}(\mathcal{O}_L)]}(\widetilde{\sigma}, R\Gamma(\widetilde{N}^{w_0}(\mathcal{O}_L), \mathrm{Ind}_{P(L)}^{\widetilde{G}(L)}\pi)^{\widetilde{Q}^{w_0}\text{-ord}})$$

admits

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M(\mathcal{O}_L)\times M_c(\mathcal{O}_L)]}(\sigma \otimes \sigma_c, R\Gamma(N(\mathcal{O}_L) \times N_c(\mathcal{O}_L), \tau^{-1}_{w_0^P}\pi)^{Q\times Q_c\text{-ord}})$$

as a $\mathcal{H}(\sigma) \times \mathcal{H}(\sigma_c)$*-equivariant subquotient.*

We also deduce a dual statement computing $\overline{\widetilde{Q}}$-ordinary parts of a Bruhat stratum of $\mathrm{Ind}_{P(L)}^{\widetilde{G}(L)}\pi$.

Corollary 3.5.8. *For every* $\pi \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(L)])$*, and integer* $j \in \mathbf{Z}_{\geq 0}$*, the group*

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}(\mathcal{O}_L)]}(\tau_{w_0^P}\widetilde{\sigma}, R\Gamma(\overline{\widetilde{N}}^1, \mathrm{Ind}_{P(L)}^{\widetilde{G}(L)}\pi)^{\overline{\widetilde{Q}}\text{-ord}})$$

admits

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_c(\mathcal{O}_L)\times M(\mathcal{O}_L)]}(\sigma_c \otimes \sigma, R\Gamma(\overline{N}_c^1 \times \overline{N}^1, \pi)^{\overline{Q_c}\times\overline{Q}\text{-ord}})$$

as a $\mathcal{H}(\sigma_c) \times \mathcal{H}(\sigma)$*-equivariant subquotient.*

Proof. We reduce it to Corollary 3.5.6. Set $\tilde{z}_p := u_p^{\widetilde{Q}} \in Z^+_{\widetilde{M}}$ to be the usual element defining the U_p-operator with respect to $\widetilde{Q}$ and $\widetilde{w}_0 := w_0^{\widetilde{Q}}$. Set $\widetilde{Q}^{\tilde{w}_0} = \widetilde{M}^{\tilde{w}_0} \ltimes \widetilde{N}^{\tilde{w}_0}$ to be the standard parabolic subgroup with Levi subgroup $\tilde{w}_0^{-1}\widetilde{M}\tilde{w}_0$. Note that $\overline{N}^{\circ} := \tilde{z}_p\widetilde{w}_0\widetilde{N}^{\tilde{w}_0}(\mathcal{O}_L)(\tilde{z}_p\widetilde{w}_0)^{-1} \subset \overline{\widetilde{N}}^1$ and, by Remark 3.3.2, we have a natural ($\mathcal{H}(\tau_{w_0^P}\widetilde{\sigma})$-equivariant) isomorphism

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}(\mathcal{O}_L)]}\left(\tau_{w_0^P}\widetilde{\sigma}, R\Gamma(\overline{\widetilde{N}}^1, \mathrm{Ind}_{P(L)}^{\widetilde{G}(L)}\pi)^{\overline{\widetilde{Q}}\text{-ord}}\right) \cong$$

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}(\mathcal{O}_L)]}\left(\tau_{w_0^P}\widetilde{\sigma}, R\Gamma(\overline{N}^{\circ}, \mathrm{Ind}_{P(L)}^{\widetilde{G}(L)}\pi)^{\overline{\widetilde{Q}}\text{-ord}}\right).$$

Moreover, multiplication by $\tilde{z}_p\widetilde{w}_0$ sets up a $\mathcal{H}(\tau^P_{w_0}\widetilde{\sigma})$-equivariant isomorphism between the latter and

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}^{\tilde{w}_0}(\mathcal{O}_L)]}\left(\tau_{w_0^{Q\times Q_c}}\widetilde{\sigma}, R\Gamma(\widetilde{N}^{\tilde{w}_0}(\mathcal{O}_L), \mathrm{Ind}_{P(L)}^{\widetilde{G}(L)}\pi)^{\widetilde{Q}^{\tilde{w}_0}\text{-ord}}\right) \tag{3.5.4}$$

where the Hecke action on 3.5.4 is through the isomorphism $\mathcal{H}(\tau_{w_0^P}\widetilde{\sigma}) \cong \mathcal{H}(\tau_{w_0^{Q\times Q_c}}\widetilde{\sigma})$ induced by $\tilde{w}_0$. Set $N^{\tilde{w}_0} \times N_c^{\tilde{w}_0} := \widetilde{N}^{\tilde{w}_0} \cap G$ and note that $N^{\tilde{w}_0}$, respectively $N_c^{\tilde{w}_0}$ is the unipotent radical corresponding to $M^{\tilde{w}_0} := w_0^Q M w_0^{Q,-1}$, respectively $M_c^{\tilde{w}_0} := w_0^{Q_c} M_c w_0^{Q_c,-1}$. Then Corollary 3.5.6 shows that 3.5.4 admits

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_c^{\tilde{w}_0,0}\times M^{\tilde{w}_0,0}]}\left(\tau_{w_0^{Q_c}}\sigma_c \times \tau_{w_0^Q}\sigma, R\Gamma(N_c^{\tilde{w}_0}(\mathcal{O}_L) \times N^{\tilde{w}_0}(\mathcal{O}_L), \pi)^{Q_c^{\tilde{w}_0}\times Q^{\tilde{w}_0}\text{-ord}}\right) \tag{3.5.5}$$

as a $\mathcal{H}(\tau_{w_0^{Q\times Q_c}}\widetilde{\sigma})$-equivariant subquotient. Note that $\widetilde{w}_0 = (w_0^{Q_c}, w_0^Q)w_0^P = w_0^P(w_0^Q, w_0^{Q_c})$. Therefore, multiplication by $(w_0^{Q_c}u_p^{Q_c}, w_0^Q u_p^Q)$ and another application of Remark 3.3.2 sets up a $\mathcal{H}(\widetilde{\sigma}) \cong \mathcal{H}(\tau_{\tilde{w}_0}\widetilde{\sigma})$-equivariant isomorphism between 3.5.5 and

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_c(\mathcal{O}_L)\times M(\mathcal{O}_L)]}\left(\tau_{w_0^P}\widetilde{\sigma}, R\Gamma(\overline{N}_c^1 \times \overline{N}^1, \pi)^{\overline{Q}_c\times\overline{Q}\text{-ord}}\right) \cong$$

$$R^j\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_c(\mathcal{O}_L)\times M(\mathcal{O}_L)]}(\sigma_c \otimes \sigma, R\Gamma(\overline{N}_c^1 \times \overline{N}^1, \pi)^{\overline{Q_c}\times\overline{Q}\text{-ord}}).$$

□

Chapter 4

Q-ordinary parts in characteristic 0

In this chapter, we discuss the notion of taking ordinary parts of smooth admissible representations of a p-adic reductive group that arise as local components of cohomological automorphic representations (see [Ger18], §5.1 for instance). In fact, we take a slightly more involved approach and define ordinary parts of locally algebraic representations in terms of Emerton's slope 0 part. It will allow us to compare it to the corresponding Jacquet module and to our previous notion of ordinary parts. The latter will be useful in the endgame of proving our local-global compatibility results. We will then prove our main characteristic 0 result regarding Q-ordinary parts of Q-ordinary locally algebraic representations of GL_n. Finally, we close the chapter with deducing Q-ordinary local-global compatibility for regular algebraic cuspidal automorphic representations of GL_n in the conjugate self-dual case.

4.1 Ordinary parts of locally algebraic representations

We briefly summarise the notion of ordinary parts for locally algebraic representations as introduced in [BD20], §4.3 (see also [Eme11], §5.6) and how it compares to Emerton's ordinary part functor from [Eme10a] when the representation admits an invariant lattice. We revisit the setup of §3.1 and without further notice will use the introduced notation. However, we further assume that G is split over L and fix a choice of maximal torus T and a Borel subgroup B containing it. Finally, Q is now assumed to be standard with respect to B. Recall that our coefficient field is $E/\mathbf{Q}_p$, a finite extension, large enough, so that $[L : \mathbf{Q}_p] = \mathrm{Hom}(L, E)$. Then, given a $(\mathrm{Res}_{L/\mathbf{Q}_p} B)_E$-dominant weight $\lambda \in \mathrm{Hom}((\mathrm{Res}_{L/\mathbf{Q}_p} T)_E, \mathbf{G}_{m,E}) \cong \oplus_{\iota : L \hookrightarrow E} \mathrm{Hom}(T_E, \mathbf{G}_{m,E})$, set $V_\lambda = \otimes_\iota V_{\lambda_\iota}$ to be the corresponding absolutely irreducible algebraic E-representation of $G(L)$. Note that the dual $V_\lambda^\vee$ is isomorphic to $V_{\lambda^\vee}$ where

$\lambda^\vee := -w_0^G \lambda$.

Definition 4.1.1. An E-representation Π of $G(L)$ is called locally algebraic of weight λ if it is locally $V_{\lambda^\vee}$-algebraic in the sense of [Eme17], Definition 4.2.1 such that the smooth vectors $\mathrm{Hom}(V_{\lambda^\vee}, \Pi)_{\mathrm{sm}} = \varinjlim_{K\to 1} \mathrm{Hom}_K(V_{\lambda^\vee}, \Pi)$ form an admissible smooth representation of $G(L)$.

We note that our definition admits a more intrinsic formulation (cf. [Eme17] Definition 6.3.9, Proposition 6.3.10).

Remark 4.1.2. Note that any locally algebraic E-representation Π of weight λ is of the form $\pi \otimes V_{\lambda^\vee}$ with π a smooth admissible E-representation (cf. [Eme17], Proposition 4.2.4). The functor $\mathrm{Hom}(V_{\lambda^\vee}, -)_{\mathrm{sm}}$ sets up a natural equivalence between the category of locally algebraic E-representations of $G(L)$ of weight λ and the category of smooth admissible E-representations of $G(L)$. However, the reason one might want to appeal to the notion of locally algebraic representations is that it yields a different notion of p-adic integrality. Indeed, this is our motivation to work with locally algebraic representations.

Remark 4.1.3. Note that, V_λ being an algebraic E-representation of $\mathrm{Res}_{L/\mathbf{Q}_p} G$,

$$V_\lambda^{N^0} \cong V_\lambda^{N(L)}$$

as subspaces of V_λ. Moreover, the latter is the absolutely irreducible representation of $M(L)$ associated with λ viewed as a $(\mathrm{Res}_{L/\mathbf{Q}_p} B \cap M)_E$-dominant weight for $(\mathrm{Res}_{L/\mathbf{Q}_p} M)_E$ (see [Cab84]). In particular, the former is naturally an $M(L)$-representation. As a corollary, one sees that for a locally algebraic E-representation $\Pi = \pi \otimes V_{\lambda^\vee}$ of weight λ, we have an induced identification $\Pi^{N^0} \cong \pi^{N^0} \otimes V_{\lambda^\vee}^{N(L)}$. Moreover, under this isomorphism, the Hecke action of M^+ on Π^{N^0} coincides with the M^+-action on $\pi^{N^0} \otimes V_{\lambda^\vee}^{N(L)}$ given by the usual Hecke action on the first factor and the natural action on the algebraic part (see the proof of [Eme17], Proposition 4.3.6).

We now introduce the notion of finite slope and slope 0 parts of Π^{N^0}. For $b \geq 1$, consider the *finite dimensional* E-vector space

$$\Pi_b := \pi^{\mathcal{Q}(b,b)} \otimes V_{\lambda^\vee}^{N(L)} \subset \Pi^{N^0}.$$

By Hypothesis 3.1.3, it is a Z_M^+-invariant subspace. Denote by B_b the E-subalgebra of $\mathrm{End}_E(\Pi_b)$ generated by Z_M^+. This is an Artinian E-algebra and as such, it decomposes into a product of local Artinian E-algebras indexed by its maximal ideals. We then say that a maximal ideal $\mathfrak{m} \subset B_b$ is of *finite slope* if the image of Z_M^+ in B_b is disjoint from $\mathfrak{m}$. For such an $\mathfrak{m}$, the composition

$$Z_M^+ \to B_b \twoheadrightarrow B_b/\mathfrak{m} \hookrightarrow \overline{\mathbf{Q}}_p$$

lands in $\overline{\mathbf{Q}}_p^\times$. We further say that a finite slope maximal ideal $\mathfrak{m} \subset B_b$ is of *slope zero* if the composition lands in $\overline{\mathbf{Z}}_p^\times$. Considering the corresponding factors of B_b induces Z_M^+-equivariant decompositions

$$\Pi_b \cong (\Pi_b)_{\mathrm{fs}} \oplus (\Pi_b)_{\mathrm{null}} \cong (\Pi_b)_0 \oplus (\Pi_b)_{>0} \tag{4.1.1}$$

into finite slope and slope 0 parts and their complements. Note that $(\Pi_b)_0 \subset (\Pi_b)_{\mathrm{fs}}$ and that the Z_M^+-action uniquely extends to a $Z_M(L)$-action on both by definition. The decompositions in 4.1.1 are easily checked to be compatible when we vary b. Therefore, by passing to the colimit over $b \geq 1$, we get Z_M^+-equivariant decompositions

$$\Pi^{N^0} \cong (\Pi^{N^0})_{\mathrm{fs}} \oplus (\Pi^{N^0})_{\mathrm{null}} \cong (\Pi^{N^0})_0 \oplus (\Pi^{N^0})_{>0}.$$

Moreover, in the colimit, the decompositions are preserved by the M^+-action. In particular, Lemma 3.1.4 and Remark 4.1.3 shows that $(\Pi^{N^0})_{\mathrm{fs}}$ and $(\Pi^{N^0})_0$ become locally algebraic E-representations of $M(L)$ of weight $w_0^M w_0^G \lambda = w_0^Q \lambda$.

First, we compare the finite slope part with the classical (unnormalised) Jacquet functor. This is essentially the theory of canonical liftings that goes back to Casselman (cf. [Cas95]).

Proposition 4.1.4. *We have a natural $M(L)$-equivariant isomorphism*

$$(\Pi^{N^0})_{\mathrm{fs}} \otimes \delta_Q \cong J_Q(\pi) \otimes V_{\lambda^\vee}^{N(L)}$$

induced by the natural M^+-equivariant (surjective) map

$$\Pi^{N^0} \otimes \delta_Q \to J_Q(\pi) \otimes V_{\lambda^\vee}^{N(L)}.$$

Proof. This can be deduced from [Eme06a], §4.3. We sketch an argument here. It is an easy consequence of the definitions that $(\Pi^{N^0})_{\mathrm{fs}} = (\pi^{N^0})_{\mathrm{fs}} \otimes V_{\lambda^\vee}^{N(L)}$ and $(\Pi^{N^0})_{\mathrm{null}} = (\pi^{N^0})_{\mathrm{null}} \otimes V_{\lambda^\vee}^{N(L)}$. Moreover, one sees that $(\pi^{\mathcal{Q}(b,b)})_{\mathrm{null}} \subset \pi^{\mathcal{Q}(b,b)}$ is the subspace of vectors on which, for some choice of $z_p \in Z_M^+$[1] as in Lemma 3.1.4, the action of z_p is nilpotent. Using this description of $(\pi^{N^0})_{\mathrm{null}}$, one sees that $(\pi^{N^0})_{\mathrm{null}}$ is exactly the kernel of the natural $\pi^{N^0} \to J_Q(\pi)$. Moreover, [Eme06a], Proposition 4.3.4 i) shows that $\pi^{N^0} \otimes \delta_Q \to J_Q(\pi)$ is an M^+-equivariant surjection (see also [Cas95], Theorem 3.3.3 and Lemma 4.1.1). Note that in [Eme06a] we don't see the appearance of δ_Q. This is due to the fact that they work with a different normalisation of the Hecke action on π^{N^0} (see *loc. cit.* Definition 3.4.1). Combining these observations, we get the proposition. □

---

[1] Hence for any choice of such z_p.

Remark 4.1.5. To avoid confusion, we point out that what is denoted by J_Q in [Eme06a] is Emerton's locally analytic Jacquet functor. It can be applied to a certain class of locally analytic representations Π of $G(L)$ and defined by the analogous formula $(\Pi^{N^0})_{\mathrm{fs}}$ for a suitable notion of finite slope parts in this setup. Without elaborating on it any further, we just note that it is easily extracted from [Eme06a], §4.3 that the constructions of [BD20], §4.3 are all compatible with the ones of Emerton. In other words, for Π a locally algebraic E-representation of $G(L)$ we have $J_Q(\Pi) \cong (\Pi^{N^0})_{\mathrm{fs}} \otimes \delta_Q$ where the former is in the sense of Emerton and the latter is in the sense of Breuil–Ding. Again, the character δ_Q appears because of the different normalisations of the Hecke action. From now on, we freely use the notation $J_Q(\Pi)$ for Π a locally algebraic representation to denote its Jacquet module in the sense of Emerton. In light of Proposition 4.1.4, it recovers the classical Jacquet functor.

We now turn to discussing $(\Pi^{N^0})_0$. We would like to compare it with Emerton's ordinary parts for smooth admissible $\mathcal{O}/\varpi^m[G(L)]$-modules (cf. [Eme10a]). In order to have a chance to compare the two notions, we assume that our Π is also a unitary representation of $G(L)$. By this we mean that there is an $\mathcal{O}$-lattice $\Pi^\circ \subset \Pi$[2] that is invariant under the action of $G(L)$. The given lattice induces $\mathcal{O}$-lattices $\Pi_b^\circ := \Pi^\circ \cap \Pi_b \subset \Pi_b$ which are necessarily finite and free as $\mathcal{O}$-modules. Then, for an integer $b \geq 1$, set A_b to be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(\Pi_b^\circ)$ generated by Z_M^+. This is a finite $\mathcal{O}$-algebra and, in particular, we have $A_b \cong \prod_{\mathfrak{n}} (A_b)_{\mathfrak{n}}$, where the propduct runs over maximal ideals in A_b. Call a maximal ideal $\mathfrak{n} \subset A_b$ *ordinary* if the image of Z_M^+ in A_b is disjoint from $\mathfrak{n}$. This gives a Z_M^+-equivariant decomposition

$$\Pi_b^\circ \cong (\Pi_b^\circ)_{\mathrm{ord}} \oplus (\Pi_b^\circ)_{\mathrm{nonord}}$$

and, just as before, the Z_M^+-action on $(\Pi_b^\circ)_{\mathrm{ord}}$ extends uniquely to a $Z_M(L)$-action. By passing to the colimit over $b \geq 1$, we obtain an M^+-equivariant decomposition

$$\Pi^{\circ, N^0} \cong \mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^\circ) \oplus \mathrm{NOrd}_Q^{\mathrm{lalg}}(\Pi^\circ).$$

Therefore, $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^\circ)$ is naturally a representation of $M(L)$. Then [BD20], Lemma 4.10 shows that we have a natural isomorphism

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^\circ) \otimes_{\mathcal{O}} E \cong (\Pi^{N^0})_0$$

of locally algebraic E-representations of $M(L)$. In particular, the former is independent of the choice of lattice $\Pi^\circ \subset \Pi$ and N^0 and the latter is unitary. Motivated by this, introduce the following notation.

Definition 4.1.6. Let Π be a locally algebraic E-representation of $G(L)$ of weight λ. We then define its Q-ordinary part

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) := (\Pi^{N^0})_0,$$

a locally algebraic E representation of $M(L)$ of weight $w_0^Q \lambda$.

---

[2] In other words, an $\mathcal{O}$-submodule that spans Π over E and contains no E-line.

In the rest of the section, we will justify that when Π is assumed to be unitary our locally algebraic Q-ordinary parts does behave like an ordinary part. In the next section we will see that in fact under some a priori milder assumption it still behaves like an ordinary part functor.

From now on, for the rest of the section, we assume that Π is unitary and choose an $\mathcal{O}$-lattice Π°. For an integer $m \geq 1$, we can also introduce analogous constructions for Π_b°/ϖ^m to get a Z_M^+-equivariant decomposition $\Pi_b^\circ/\varpi^m \cong (\Pi_b^\circ/\varpi^m)_{\mathrm{ord}} \oplus (\Pi_b^\circ/\varpi^m)_{\mathrm{nonord}}$. The natural map then induces an isomorphism (cf. [BD20], 4.16)

$$(\Pi_b^\circ)_{\mathrm{ord}}/\varpi^m \xrightarrow{\sim} (\Pi_b^\circ/\varpi^m)_{\mathrm{ord}}. \tag{4.1.2}$$

On the other hand, Π°/ϖ^m is a smooth admissible $\mathcal{O}/\varpi^m[G(L)]$-module so one can take its Q-ordinary part in the sense of Emerton. Then [BD20], Lemma 4.14 shows that we have a natural injection

$$\varinjlim_{b\geq 1}(\Pi_b^\circ/\varpi^m)_{\mathrm{ord}} \hookrightarrow \mathrm{Ord}_Q(\Pi^\circ/\varpi^m) \tag{4.1.3}$$

where the latter is Emerton's Q-ordinary part (cf. [Eme10a], Definition 3.1.7).[3] However, due to the presence of group cohomology, 4.1.3 is not necessarily surjective. Combining 4.1.2 and 4.1.3, one gets (cf. [BD20], Lemma 4.16) a natural $M(L)$-equivariant injection

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^\circ) \hookrightarrow \mathrm{Ord}_Q(\Pi^\circ) := \varprojlim_m \mathrm{Ord}_Q(\Pi^\circ/\varpi^m). \tag{4.1.4}$$

Remark 4.1.7. The map 4.1.4 is clearly not surjective in general: The source is a locally algebraic representation, the target is a lattice in an E-Banach space representation. A more sensible question to ask is whether it has dense image or not. One notes that it does have dense image as long as 4.1.3 is surjective, see [BD20], Remark 4.15 and Remark 4.17.

Given a unitary admissible E-Banach space representation Π of $G(L)$, with a unit ball Π°, then $\varprojlim_m \mathrm{Ord}_Q(\Pi^\circ/\varpi^m)$ is what Emerton calls $\mathrm{Ord}_Q(\Pi^\circ)$ in [Eme10a]. Moreover, one can then set $\mathrm{Ord}_Q(\Pi) := \mathrm{Ord}_Q(\Pi^\circ) \otimes_{\mathcal{O}} E$.[4] We borrow his notation. We conclude the section with discussing a corollary of the adjunction property of [BD20], §4.4. Before stating the corollary, we introduce the following notation. Let Π be any E-representation of $G(L)$ and W be an algebraic E-representation of $G(L)$. We then denote by $\Pi^{W\text{-lalg}} \subset \Pi$ the $G(L)$-invariant subspace of locally W-algebraic vectors in Π in the sense of [Eme17], Proposition-Definition 4.2.6.

---

[3] We note here that since we are in an admissible situation, Emerton's ordinary part simplifies to localising $(\Pi^\circ/\varpi^m)^{N^0}$ along $\mathcal{O}[z_p] \hookrightarrow \mathcal{O}[z_p^{\pm 1}]$ for any z_p as in Lemma 3.1.4 (see [Eme10b], Lemma 3.2.1).

[4] One checks easily that it is independent of the choice of unit ball.

Proposition 4.1.8. *Let* Π *be a unitary admissible*[5] E*-Banach space representation of* $G(L)$*. Then, for any choice of* $(\mathrm{Res}_{L/\mathbf{Q}_p}B)_E$*-dominant weight* λ*, we have a natural* $M(L)$*-equivariant identification*

$$\mathrm{Ord}_Q(\Pi)^{V_{\lambda^\vee}^{N(L)}\text{-lalg}} \cong \mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^{V_{\lambda^\vee}\text{-lalg}}).$$

Proof. Before proving the proposition, we first remark that the statement indeed makes sense. In other words, we need to check that $\Pi^{V_{\lambda^\vee}\text{-lalg}}$ is indeed a locally algebraic representation (of weight λ) in the sense of this text. Note that the functor of taking locally $V_{\lambda^\vee}$-algebraic vectors factors through the functor $\Pi \mapsto \Pi^{\mathrm{la}}$ of taking locally analytic vectors. By [Eme17], Proposition 6.2.2 and Proposition 6.2.4, we see that Π^{la} is an admissible locally analytic representation of $G(L)$. By *loc. cit.* Proposition 6.3.6, we see that $\Pi^{\mathrm{la},V_{\lambda^\vee}\text{-lalg}} \cong \Pi^{V_{\lambda^\vee}\text{-lalg}}$ is again an admissible locally analytic representation. Finally, the claim follows from *loc. cit.* Proposition 6.3.10 by noting that $\mathrm{End}_{E[G(L)]}(\mathrm{V}_{\lambda^\vee}) = E$.

Now the proposition easily follows from [BD20] §4 as we explain now. We first apply [BD20], Lemma 4.18 with the choice $V = \Pi^{V_{\lambda^\vee}\text{-lalg}} \subset W = \Pi$, $W^\circ = \Pi^\circ$ and $V^\circ = V \cap W^\circ$. It gives an injection

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^{V_{\lambda^\vee}\text{-lalg}}) \hookrightarrow \mathrm{Ord}_Q(\Pi). \tag{4.1.5}$$

As $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi^{V_{\lambda^\vee}\text{-lalg}})$ is a locally algebraic representation of $M(L)$ of weight $w_0^Q\lambda$, it necessarily lands in $\mathrm{Ord}_Q(\Pi)^{V_{\lambda^\vee}^{N(L)}\text{-lalg}}$.

For the other inclusion, note that $\mathrm{Ord}_Q(\Pi)$ is again a unitary admissible E-Banach space representation by [Eme10a], Theorem 3.4.8 (and [Eme17], Proposition 6.5.7). Therefore, we see that $\mathrm{Ord}_Q(\Pi)^{V_{\lambda^\vee}^{N(L)}\text{-lalg}} \cong \pi \otimes V_{\lambda^\vee}^{N(L)}$ is a locally algebraic representation of weight $w_0^Q\lambda$. Consider the natural inclusion

$$\iota_v : \pi \otimes V_{\lambda^\vee}^{N(L)} \cong \mathrm{Ord}_Q(\Pi)^{V_{\lambda^\vee}^{N(L)}\text{-lalg}} \hookrightarrow \mathrm{Ord}_Q(\Pi).$$

Then [BD20], Proposition 4.21 applied to $f := \iota_v$ yields a map

$$\mathrm{Ind}_{Q(L)}^{G(L)}\pi \otimes V_{\lambda^\vee} \to \Pi$$

and ι_v can be reconstructed by precomposing the map

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\mathrm{Ind}_{Q(L)}^{G(L)}\pi \otimes V_{\lambda^\vee}) \to \mathrm{Ord}_Q(\Pi) \tag{4.1.6}$$

induced by *loc. cit.* Lemma 4.18 with the inclusion

$$\pi \otimes V_{\lambda^\vee}^{N(L)} \hookrightarrow \mathrm{Ord}_Q^{\mathrm{lalg}}(\mathrm{Ind}_{Q(L)}^{G(L)}\pi \otimes V_{\lambda^\vee})$$

of *loc. cit.* Lemma 4.19. But 4.1.6 factors through 4.1.5 and the proof is finished. □

---

[5] For the notion of admissible E-Banach space representations, see [ST06], and [Eme17], §6.

4.2 Ordinary parts of weakly admissible representations

We introduce the notion of weakly admissible locally algebraic representations, an a priori weaker integrality notion than unitariness. We then prove that the notion of being Q-ordinary behaves well for such representations. For the rest of the section, we assume that $G = \mathrm{GL}_n$, T is the torus of diagonal matrices and B is the Borel subgroup of upper-triangular matrices. In particular, $Q = P_{(n_1,...,n_t)}$ for a partition $n_1 + ... + n_t = n$. Fix a choice of uniformiser $\varpi_L \in \mathcal{O}_L$ and recall that e denotes the ramification index of L. Moreover, for integers $0 \le b \le c$ with $c \ge 1$, let $\mathcal{Q}(b,c) \subset G(\mathcal{O}_L)$ be the corresponding parahoric subgroup associated with Q. Then, by the Iwahori decomposition, $\mathcal{Q}(b,c) = \overline{N}^c M^b N^0$. For $1 \le i \le n$, set

$$u_p^{(i)} := \mathrm{diag}(\varpi_L, ..., \varpi_L, 1, ..., 1)$$

where the first i elements are given by ϖ_L and the rest by 1. Then $u_p^{(n_1)} \cdot .... \cdot u_p^{(n_1+...+n_{t-1})} \in Z_M^+$ is our choice for the role of z_p from the proof of Lemma 3.1.4. For $x \in L^\times$, set $\langle x \rangle := \mathrm{diag}(x, ..., x) \in G(L)$.

Definition 4.2.1. Let Π be a locally algebraic E-representation of $G(L)$ of weight λ. We say that Π is weakly admissible (of weight λ) if, for any standard parabolic subgroup $Q' = M' \ltimes N' \subset G$ and locally algebraic character $\chi : Z_{M'}(L) \to \overline{\mathbf{Q}}_p^\times$, such that $\mathrm{Hom}_{Z_{M'}(L)}(\chi, J_{Q'}(\Pi)) \neq 0$, we have

$$\mid \chi(z)\delta_{Q'}^{-1}(z) \mid_p \le 1,$$

for every $z \in Z_{M'}^+$. We denote the category of such representations by $\mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L))$.

Remark 4.2.2. This is what Hu refers to as Emerton's condition in [Hu09], where he proves the "easy" direction of the Breuil–Schneider conjecture.[6] The main result (Théorème 1.2) of [Hu09] shows that, at least when Π is of the form $\pi \otimes V_{\lambda^\vee}$ with π irreducible and generic, Π is weakly admissible if and only if $\mathrm{rec}^T(\pi)$ comes from a de Rham Galois representation with Hodge–Tate weights given by the usual ρ-shift of λ. The latter means that the (φ, N, G_L)-module associated with $\mathrm{rec}^T(\pi)$ admits a filtration with jumps given by the ρ-shift of λ that makes it a weakly admissible filtered (φ, N, G_L)-module.[7] The way Hu applies it to the Breuil–Schneider conjecture is that if Π is unitary

---

[6] In fact, he asks for this condition to be satisfied for *any* parabolic subgroup $Q' \subset G$, but as it is explained in *loc. cit.* Remarque 2.11, it suffices to check it for standard parabolic subgroups.

[7] To avoid confusion, note that the original formulation of the Breuil–Schneider conjecture uses different normalisations. Namely, one instead asks whether the representation $\pi(n-1) \otimes V_{\lambda^\vee + n-1}$ admits a unitary completion. As the character $(|\det|\det)^{n-1}$ is unitary, regardless of the mismatch, the two different normalisations yield equivalent conjectures.

then one deduces easily that it must be weakly admissible (see [Eme06a], Lemma 4.4.2).

Using Emerton's condition to define weak admissibility has the advantage of being easy to keep track of while proving results regarding weakly admissible representations. However, we record here an equivalent condition that has the advantage that in the case of our examples appearing in this text it will clearly be satisfied.

Lemma 4.2.3. *Let $\Pi = \pi \otimes V_{\lambda^\vee}$ be a locally algebraic E-representation of $G(L)$ of weight λ and $P_{(n'_1,\ldots,n'_h)} = Q' = M' \ltimes N' \subset G$ a standard parabolic subgroup. The following two conditions are equivalent:*

i. For each $1 \leq i \leq h$, the generalised eigenvalues of the double coset operator $[N'^0 u_p^{(n_1+\ldots+n_i)} N'^0]$ acting on $\Pi^{N'^0}$ lie in $\overline{\mathbf{Z}}_p$.

ii. For any locally algebraic character $\chi : Z_M(L) \to \overline{\mathbf{Q}}_p^\times$ with

$$\mathrm{Hom}_{Z_M(L)}(\chi, J_{Q'}(\Pi)) \neq 0,$$

we have

$$|\chi(z)\delta_{Q'}^{-1}(z)|_p \leq 1$$

for every $z \in Z_M^+$.

In particular, Π is weakly admissible if and only if it satisfies i) for every choice of standard parabolic subgroup $Q' \subset G$.

Proof. We first make a few observations about condition *ii*). Note that χ is of the form $\chi_{\mathrm{sm}} \otimes \chi_{\mathrm{alg}}$. Moreover, $J_{Q'}(\Pi) = J_{Q'}(\pi) \otimes V_{(w_0^Q \lambda)^\vee}$ and $V_{(w_0^Q \lambda)^\vee}$ has constant central character sending $u_p^{(n'_1+\ldots+n'_i)}$ to

$$-w_0^G \lambda(u_p^{(n'_1+\ldots+n'_i)}) = \prod_{\iota : L \hookrightarrow E} \iota(\varpi_L)^{-(\lambda_{\iota,n}+\ldots+\lambda_{\iota,n+1-(n_1+\ldots+n_i)})}.$$

Therefore, χ is forced to be a smooth twist of $-w_0^G \lambda$.

Moreover, after extending π to a $\overline{\mathbf{Q}}_p$-representation, we can apply [Cas95], Proposition 2.1.9 to get a direct sum decomposition

$$J_{Q'}(\pi) = \bigoplus_{\chi_{\mathrm{sm}} : Z_{M'}(L) \to \overline{\mathbf{Q}}_p^\times} J_{Q'}(\pi)_{\chi_{\mathrm{sm}}} \tag{4.2.1}$$

of the $M(L)$-representation into generalised eigenspaces with respect to all the possible central characters. Here we used that both base change to $\overline{\mathbf{Q}}_p$ and $J_{Q'}$ preserves admissibility. In particular, we see that $J_{Q'}(\pi)_{\chi_{\mathrm{sm}}} \neq 0$ if and only if $\mathrm{Hom}_{Z_{M'}(L)}(\chi_{\mathrm{sm}}(-w_0^G \lambda), J_{Q'}(\Pi)) \neq 0$.

We also note that the decomposition 4.2.1 is a refinement of the decomposition of $J_{Q'}(\pi)$ into generalised eigenspaces with respect to the actions of the $u_p^{(n_1'+\ldots+n_i')}$'s. On the other hand, it is easy to check that $\chi|_{Z^0_{M'}}$ and $\delta_{Q'}|_{Z^0_{M'}}$ necessarily land in $\overline{\mathbf{Z}}_p^\times$. So the inequality $|\chi(z)\delta_{Q'}^{-1}(z)|_p \leq 1$ is satisfied for all $z \in Z^+_{M'}$ if and only if it is satisfied for $z = u_p^{(n_1'+\ldots+n_i')}$ for all $1 \leq i \leq h$.

Finally, we finish the proof by recalling that the natural map $\Pi^{N'^0} \otimes \delta_{Q'} \to J_{Q'}(\Pi)$ is $Z^+_{M'}$-equivariant, and its kernel is the subspace on which the U_p-operators act nilpotently. □

Remark 4.2.4. As was promised, using Lemma 4.2.3, we can find a large source of examples of weakly admissible representations. Namely, given a number field F, we can consider any regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ. Given a p-adic place $v \in S_p(F)$, set $L = F_v$, and look at the component π_v, a smooth admissible $\mathbf{C}$-representation of $G(L)$. After fixing an identification $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, we can find a model of $t^{-1}\pi_v$ over some large enough field extension $E/\mathbf{Q}_p$. Since $t^{-1}\pi_v$ can then be found as a $\mathrm{GL}_n(F_v)$-equivariant direct summand in $(\varinjlim_{K_v \subset \mathrm{GL}_n(\mathcal{O}_{F_v})} H^*(X_{K^vK_v}, \mathcal{V}_\lambda)) \otimes_{\mathcal{O}} E$ and the *normalised* U_p-operators already act integrally on the latter by [CN23], Lemma 2.1.17, we see that Lemma 4.2.3, i) is satisfied for any standard parabolic subgroup, making $t^{-1}\pi_v \otimes_E V_{\lambda^\vee}$ into a weakly admissible E-representation of $G(L)$.

Definition 4.2.5. Let Π be a weakly admissible E-representation of $G(L)$ of weight λ. We say that Π is Q-ordinary if $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) \neq 0$. We denote the subcategory of such by $\mathrm{Mod}_E^{Q\text{-ord},\lambda}(G(L)) \subset \mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L))$.

Remark 4.2.6. We also note that both the notion of weakly admissible and the notion of ordinary generalises to any split reductive group over L and we will be speaking of weakly admissible and ordinary representations of Levi subgroups $M(L)$ of $G(L)$ without further explanation.

We would also like to talk about parabolic induction for locally algebraic representations.

Definition 4.2.7. Let $\Pi_M = \pi_M \otimes V_{(w_0^G\lambda)^\vee}$ be a weakly admissible E-representation of $M(L)$ of weight $w_0^Q\lambda$. We then set its parabolic induction to be

$$\mathrm{Ind}_Q^G(\Pi_M) := (\mathrm{Ind}_{Q(L)}^{G(L)}\pi_M) \otimes V_{\lambda^\vee},$$

a locally algebraic E-representation of $G(L)$ of weight λ.

Note that Ind_Q^G can only be applied to locally algebraic representations with weight of the form $w_0^Q\lambda$.

The key result of the section is the following theorem.

Theorem 4.2.8. *Let Π_M be a weakly admissible E-representation of $M(L)$ of weight $w_0^Q\lambda$. Then $\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)$ is a weakly admissible E-representation of $G(L)$ of weight λ and we have a natural isomorphism*

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)) \cong \mathrm{Ord}_M^{\mathrm{lalg}}(\Pi_M). \tag{4.2.2}$$

In particular, if Π_M *is* M*-ordinary, then* $\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)$ *is* Q*-ordinary and if* Π_M *is also absolutely irreducible, we have* $\mathrm{Ord}_Q^{\mathrm{lalg}}(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)) \cong \Pi_M$.

Before starting the proof, we prove some preliminary technical lemmas. The proof of these lemmas and the proof of Theorem 4.2.8 was intentionally written in a way so that it clearly generalises to split reductive groups other than GL_n. However, we kept the assumption $G = \mathrm{GL}_n$ for the whole section as later we will appeal to the specific features of the representation theory of $\mathrm{GL}_n(L)$, such as the Bernstein–Zelevinsky classification.

Introduce the following notation. Denote by Σ_G the set of roots of (G, T) and by $\Sigma_G^+ \subset \Sigma_G$ the subset of B-dominant roots. Given a standard parabolic subgroup $M' \ltimes N' = Q' \subset G$, analogous notations apply to M'. Further denote by $\Sigma_{N'} \subset \Sigma_G^+$ the roots corresponding to N'. Note that $\Sigma_G^+ = \Sigma_{M'}^+ \coprod \Sigma_{N'}$ and $\Sigma_G = -\Sigma_N \coprod \Sigma_{M'} \coprod \Sigma_{N'}$.

Lemma 4.2.9. *Let* $M \ltimes N = Q, M' \ltimes N' = Q' \subset G$ *be two standard parabolic subgroups,* $\nu \in X_*(Z_{M'})$ *be a cocharacter that is moreover* B*-dominant. Consider an element* $w \in {}^{Q'}W^Q$ *and define the standard parabolic subgroups* $M_{w^{-1}} \ltimes N_{w^{-1}} = Q_{w^{-1}}^M := M \cap w^{-1}Q'w \subset M$ *and* $M'_w \ltimes N'_w = Q_w^{M'} := M' \cap wQw^{-1} \subset M'$. *Then* $w^{-1}\nu$ *is in* $X_*(Z_{M_{w^{-1}}})$ *and is a* $B \cap M$*-dominant cocharacter. In particular, the isomorphism* $M'_w(L) \xrightarrow{\sim} M_{w^{-1}}(L)$, $m \mapsto w^{-1}mw$ *restricts to a map* $Z_{M'}^+ \to Z_{M_{w^{-1}}}^+$.[8]

Proof. To see that $w^{-1}\nu$ lies in $X_*(Z_{M_{w^{-1}}})$, note that, by [Hau18], Lemma 2.1.1, (ii), we have $M_{w^{-1}} = M \cap w^{-1}M'w$ and $M'_w = M' \cap wMw^{-1}$. In particular, conjugation by w^{-1} restricts to an isomorphism $M'_w \xrightarrow{\sim} M_{w^{-1}}$ from which the claim follows.

To verify that $w^{-1}\nu$ is $B \cap M$-dominant, we have to check that for any $\alpha \in \Sigma_M^+$, we have $\langle w^{-1}\nu, \alpha\rangle \geq 0$. Equivalently, we prove that $\langle \nu, w\alpha\rangle \geq 0$. Since $w \in {}^{Q'}W^Q$, we know that $w\alpha \in \Sigma_G^+$ so the inequality follows from the fact that ν is B-dominant.

For the last claim of the lemma, note that any $z \in Z_{M'}^+$ must be of the form $\nu(\varpi_L) \cdot z'$ for some B-dominant cocharacter $\nu \in X_*(Z_{M'})$ and $z' \in Z_{M'}^0$. $\square$

Lemma 4.2.10. *Let* $M \ltimes N = Q$, $M' \ltimes N' = Q'$, $w \in {}^{Q'}W^Q$, $Q_{w^{-1}}^M = M_{w^{-1}} \ltimes N_{w^{-1}}$ *and* $Q_w^{M'} = M'_w \ltimes N'_w$ *as in Lemma 4.2.9. Further consider the standard parabolic subgroups* $Q_{w^{-1}} = M_{w^{-1}} \ltimes (N_{w^{-1}}N)$, $Q'_w = M'_w \ltimes (N'_wN') \subset G$. *Then, for any* $z \in Z_{M'}^+$, *we have*

$$\mathrm{val}_p(\delta_{Q'_w}(z)) \leq \mathrm{val}_p(\delta_{Q_{w^{-1}}}(w^{-1}zw)).$$

Moreover, if $w \neq 1$, *we can find* $z \in Z_{M'}^+$ *such that* $\mathrm{val}_p(\delta_{Q'_w}(z)) < \mathrm{val}_p(\delta_{Q_{w^{-1}}}(w^{-1}zw))$.

---

[8] Note that $M_{w^{-1}}$ is considered as a Levi subgroup in M and $Z_{M_{w^{-1}}}^+$ has its meaning accordingly.

Proof. Recall that we have

$$\delta_{Q'_w}(z) = \left| \prod_{\alpha \in \Sigma_{N'_w} \coprod \Sigma_{N'}} \alpha(z) \right|_L$$

and

$$\delta_{Q_{w^{-1}}}(w^{-1}zw) = \left| \prod_{\alpha \in \Sigma_{N_{w^{-1}}} \coprod \Sigma_N} \alpha(w^{-1}zw) \right|_L = \left| \prod_{\alpha \in \Sigma_{N_{w^{-1}}} \coprod \Sigma_N} w\alpha(z) \right|_L .$$

We also note that $w \in {}^{Q'}W^Q$ means that we have

i. $w^{-1}(-\Sigma_G^+) \cap \Sigma_G^+ \subset \Sigma_N$, and

ii. $(-\Sigma_G^+) \cap w(\Sigma_G^+) \subset -\Sigma_{N'}$.

Moreover, by [Hau18], Lemma 2.1.1, (ii), w sets up a bijection between $\Sigma_{M_{w^{-1}}}$ and $\Sigma_{M'_w}$. In particular, we have

$$w(\Sigma_{N_{w^{-1}}} \coprod \Sigma_N) \subset \Sigma_{N'_w} \coprod \Sigma_{N'} \coprod -\Sigma_{N'}.$$

From this the first part of the lemma follows.

For the last part, we assume that $1 \neq w$ and would like to find $z \in Z_{M'}^+$ such that $\delta_{Q'_w}(z) < \delta_{Q_{w^{-1}}}(w^{-1}zw)$. Note that if we found an $\alpha \in \Sigma_N$ such that $w\alpha \in -\Sigma_{N'}$, then for any choice of a B-dominant cocharacter $\nu \in X_*(Z_{M'})$ with $\langle \nu, w\alpha \rangle < 0$, $\nu(\varpi)$ would be a suitable choice for z. The existence of such an α is the content of [AHV19], Lemma 5.13. □

Proof of Theorem 4.2.8. We first prove that $\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)$ is weakly admissible. To do so, we fix a standard parabolic subgroup $Q' = M' \ltimes N' \subset G$ and write $\Pi_M = \pi_M \otimes V_{-w_0^G\lambda}$. We would like to understand the $Z_{M'}^+$-action on $J_{Q'}(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M))$. By applying the geometric lemma (cf. [BZ77], Lemma 2.12) to $J_{Q'}(\mathrm{Ind}_{Q(L)}^{G(L)}(\delta_Q \otimes \pi_M))$, we obtain an $M'(L)$-equivariant filtration of $J_{Q'}(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M))$ with subquotients given by

$$I_w^{Q'} := \left(\delta_{Q'}^{1/2} \mathrm{Ind}_{Q_w^{M'}}^{M'} \left(\delta_{Q_w^{M'}}^{1/2} w^* \big(\delta_{Q_{w^{-1}}^M}^{-1/2} J_{Q_{w^{-1}}^M} (\delta_Q^{1/2} \pi_M) \big) \right) \right) \otimes V_{(w_0^Q \lambda)^\vee}{}^{9}$$

for $w \in {}^{Q'}W^Q$ where $M'_w \ltimes N'_w = Q_w^{M'} := M' \cap wQw^{-1} \subset M'$, $M_{w^{-1}} \ltimes N_{w^{-1}} = Q_{w^{-1}}^M := M \cap w^{-1}Q'w$, and $w^*(-)$ denotes the pullback along the isomorphism $M_{w^{-1}}(L) \xrightarrow{\sim} M'_w(L)$, $m \mapsto wmw^{-1}$.[10] To check weak admissibility,

---

[9] We warn the reader that we consider unnormalised versions of parabolic induction and the Jacquet functor as opposed to [BZ77], where each of the functors are normalised, hence the appearance of several modulus characters in the definition of $I_w^{Q'}$.

[10] For the fact that it is indeed an isomorphism, see [Hau18], Lemma 2.1.1, (ii).

it suffices to see that for any $w \in {}^{Q'}W^{Q}$, any locally algebraic character $\chi = \chi_{\mathrm{sm}}(-w_0^G\lambda) : Z_{M'}(L) \to \overline{\mathbf{Q}}_p^\times$ with a $Z_{M'}(L)$-equivariant embedding $\chi \hookrightarrow I_w^{Q'}$ and any $z \in Z_{M'}^+$, we have

$$\mathrm{val}_p(\chi_{\mathrm{sm}}(z)) \geq \mathrm{val}_p(w_0^G\lambda(z)) + \mathrm{val}_p(\delta_{Q'}(z)). \tag{4.2.3}$$

Fix such a Weyl group element w, character χ, and embedding $\chi \hookrightarrow I_w^{Q'}$. Set $I_w^{Q',\mathrm{sm}}$ to be the smooth part of $I_w^{Q'}$. Fix a $z \in Z_{M'}^+$ and compute

$$I_w^{Q',\mathrm{sm}}(z) = \delta_{Q'}^{1/2}(z)\delta_{Q_w^{M'}}^{1/2}(z)\delta_{Q_{w^{-1}}^M}^{-1/2}(w^{-1}zw)J_{Q_{w^{-1}}^M}(\delta_Q^{1/2}\pi_M)(w^{-1}zw) =$$

$$= \delta_{Q'}^{1/2}(z)\delta_{Q_{w^{-1}}^M}^{-1/2}(w^{-1}zw)\delta_Q^{1/2}(w^{-1}zw)J_{Q_{w^{-1}}^M}(\pi_M)(w^{-1}zw).$$

Here we used that z is a central element in $M'(L)$. Since $w \in {}^{Q'}W^{Q}$, Lemma 4.2.9 shows that $w^{-1}zw \in Z_{M_{w^{-1}}}^+$. In particular, by weak admissibility of Π_M, we see that for any $Z_{M_{w^{-1}}}(L)$-equivariant embedding $\widetilde{\chi} = \widetilde{\chi}_{\mathrm{sm}}(-w_0^G\lambda) \hookrightarrow J_{Q_{w^{-1}}^M}(\Pi_M) = J_{Q_{w^{-1}}^M}(\pi_M) \otimes V_{-w_0^G\lambda}$, we have

$$\mathrm{val}_p(\widetilde{\chi}_{\mathrm{sm}}(w^{-1}zw)) \geq \mathrm{val}_p(\delta_{Q_{w^{-1}}^M}(w^{-1}zw)) + \mathrm{val}_p(w_0^G\lambda(w^{-1}zw)).$$

Our chosen embedding $\chi \hookrightarrow I_w^{Q'}$ then induces a $w^{-1}Z_{M'}(L)w$-equivariant embedding $\widetilde{\chi}'_{\mathrm{sm}} := (w^{-1})^*(\chi_{\mathrm{sm}}\delta_{Q'}^{-1/2})\delta_{Q_{w^{-1}}^M}^{1/2}\delta_Q^{-1/2} \hookrightarrow J_{Q_{w^{-1}}^M}(\pi_M)$. As a consequence, there exists a character $\widetilde{\chi} := \widetilde{\chi}_{\mathrm{sm}}(-w_0^G\lambda) : Z_{M_{w^{-1}}}(L) \to \overline{\mathbf{Q}}_p^\times$ with an embedding $\widetilde{\chi} \hookrightarrow J_{Q_{w^{-1}}^M}(\Pi_M)$ such that $\widetilde{\chi}_{\mathrm{sm}}|_{w^{-1}Z_{M'}(L)w} = \widetilde{\chi}'_{\mathrm{sm}}$. Pick any such $\widetilde{\chi}$ and compute

$$\mathrm{val}_p\big(\chi_{\mathrm{sm}}(z)\delta_{Q'}^{-1/2}(z)\delta_{Q_{w^{-1}}^M}^{1/2}(w^{-1}zw)\delta_Q^{-1/2}(w^{-1}zw)\big) = \mathrm{val}_p(\widetilde{\chi}_{\mathrm{sm}}(w^{-1}zw)) \tag{4.2.4}$$

$$\geq \mathrm{val}_p(\delta_{Q_{w^{-1}}^M}(w^{-1}zw)) + \mathrm{val}_p(w_0^G\lambda(w^{-1}zw)) \geq \mathrm{val}_p(\delta_{Q_{w^{-1}}^M}(wzw^{-1})) + \mathrm{val}_p(w_0^G\lambda(z)).$$

Here the last inequality follows from the fact that λ is $(\mathrm{Res}_{L/\mathbf{Q}_p}B)_E$-dominant. After rearranging 4.2.4, we get

$$\mathrm{val}_p(\chi_{\mathrm{sm}}(z)) \geq \frac{1}{2}\Big(\mathrm{val}_p(\delta_{Q_{w^{-1}}^M}(w^{-1}zw)) + \mathrm{val}_p(\delta_Q(w^{-1}zw)) - \mathrm{val}_p(\delta_{Q'}(z))\Big) +$$

$$+ \mathrm{val}_p(\delta_{Q'}(z)) + \mathrm{val}_p(w_0^G\lambda(z)).$$

In particular, it suffices to prove that

$$\mathrm{val}_p\Big(\delta_{Q_{w^{-1}}^M}(w^{-1}zw)\delta_Q(w^{-1}zw)\delta_{Q'}^{-1}(z)\Big) \geq 0.$$

To see this, denote by $Q_{w^{-1}} \subset G$ the standard parabolic subgroup $M_{w^{-1}} \ltimes (N_{w^{-1}}N)$ and by $Q'_w \subset G$ the standard parabolic subgroup $M'_w \ltimes (N'_w N')$. Then one has $\delta_{Q'}(z) = \delta_{Q'_w}(z)$ and $\delta_{Q_{w^{-1}}^M}(w^{-1}zw)\delta_Q(w^{-1}zw) = \delta_{Q_{w^{-1}}}(w^{-1}zw)$.

In particular, we only need to see that $\mathrm{val}_p(\delta_{Q'_w}(z)) \leq \mathrm{val}_p(\delta_{Q_{w^{-1}}}(w^{-1}zw))$. This is the content of the first part of Lemma 4.2.10. We conclude that $\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)$ is indeed weakly admissible.

We note that we essentially already proved the rest of the theorem, too. Namely, recall that we have an $M(L)$-equivariant split injection

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)) \overset{\oplus}{\hookrightarrow} J_Q(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)) \otimes \delta_Q^{-1} \tag{4.2.5}$$

with image given by the slope 0 part (cf. §4.1). By the geometric lemma applied to the RHS of 4.2.5, we get a surjection

$$J_Q(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)) \otimes \delta_Q^{-1} \twoheadrightarrow I_1^Q \otimes \delta_Q^{-1} = \Pi_M \tag{4.2.6}$$

with its kernel K admitting a filtration with subquotients given by $I_w^Q \otimes \delta_Q^{-1}$ for $1 \neq w \in {}^QW^Q$. In particular, if we prove that, for any $1 \neq w \in {}^QW^Q$, the slope zero part $(\delta_Q^{-1} \otimes I_w^Q)_0$ is trivial, then we immediately obtain that 4.2.2 holds. Indeed, we then can conclude by applying slope 0 part to the short exact sequence

$$0 \to K \to J_Q(\mathrm{Ind}_Q^G(\delta_Q \otimes \Pi_M)) \otimes \delta_Q^{-1} \to \Pi_M \to 0.$$

By the proof of Lemma 4.2.3, to prove the claimed description of K, it is enough to check that for any $1 \neq w \in {}^QW^Q$ and any locally algebraic character $\chi = \chi_{\mathrm{sm}}(-w_0^G\lambda) : Z_M(L) \to \overline{\mathbf{Q}}_p^\times$ with an embedding $\chi \hookrightarrow I_w \otimes \delta_Q^{-1}$, there is a $z \in Z_M^+$ such that

$$\mathrm{val}_p(\chi_{\mathrm{sm}}(z)) > \mathrm{val}_p(w_0^G\lambda(z)).$$

Based on the previous paragraph of the proof, to see this, we only need to prove that we can find $z \in Z_M^+$ such that $\mathrm{val}_p(\delta_{Q_w}(z)) < \mathrm{val}_p(\delta_{Q_{w^{-1}}}(w^{-1}zw))$. This is the content of the second part of Lemma 4.2.10. □

Theorem 4.2.8 shows that we obtain a functor

$$\mathrm{Ind}_Q^G(- \otimes \delta_Q) : \mathrm{Mod}_E^{\mathrm{wa},w_0^Q\lambda}(M(L)) \to \mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L))$$

that restricts to a functor

$$\mathrm{Mod}_E^{M\text{-}\mathrm{ord},w_0^Q\lambda}(M(L)) \to \mathrm{Mod}_E^{Q\text{-}\mathrm{ord},\lambda}(G(L)).$$

Furthermore, we already constructed a functor

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(-) : \mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L)) \to \mathrm{Mod}_E^{\mathrm{wa},w_0^Q\lambda}(M(L)).$$

We introduce the notion of *Z-integral* weakly admissible representations which, by Theorem 4.2.8, form the essential image of $\mathrm{Ord}_Q^{\mathrm{lalg}}$.

Definition 4.2.11. Call a weakly admissible E-representation Π of $G(L)$ (of weight λ) Z-integral if $\mathrm{Ord}_G^{\mathrm{lalg}}(\Pi) = \Pi$. We denote the corresponding subcategory by $\mathrm{Mod}_E^{Z\text{-int},\lambda}(G(L)) \subset \mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L))$.

Lemma 4.2.12. *The pair of functors* $(\mathrm{Ord}_Q^{\mathrm{lalg}}, \mathrm{Ind}_Q^G(- \otimes \delta_Q))$*, between the categories* $\mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L))$ *and* $\mathrm{Mod}_E^{Z\text{-int},w_0^Q\lambda}(M(L))$*, forms an adjoint pair. The association in one direction sends, for* $\Pi \in \mathrm{Mod}_E^{\mathrm{wa},\lambda}(G(L))$*,* $\Pi_M \in \mathrm{Mod}_E^{\mathrm{wa},w_0^Q\lambda}(M(L))$*, a map* $f : \Pi \to \mathrm{Ind}_Q^G(\Pi_M \otimes \delta_Q)$ *to the composition* $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) \xrightarrow{g_1} \mathrm{Ord}_Q^{\mathrm{lalg}}(\mathrm{Ind}_Q^G(\Pi_M \otimes \delta_Q)) \xrightarrow{g_2} \Pi_M$ *where* g_1 *is induced by applying* $\mathrm{Ord}_Q^{\mathrm{lalg}}$ *to* f *and* g_2 *is the isomorphism of Theorem 4.2.8.*

Proof. By Frobenius reciprocity, maps of the form $f : \Pi \to \mathrm{Ind}_Q^G(\Pi_M \otimes \delta_Q)$ are in natural bijection with maps of the form $g : J_Q(\Pi) \to \Pi_M \otimes \delta_Q$. Since Π_M is Z-integral, any such g must factor through the split surjection

$$J_Q(\Pi) \twoheadrightarrow \mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) \otimes \delta_Q. \tag{4.2.7}$$

Moreover, from the obtained map $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) \otimes \delta_Q \to \Pi_M \otimes \delta_Q$, g can be recovered by precomposing it with 4.2.7. Finally, twisting by δ_Q^{-1} certainly sets up a bijection of Hom sets.

That this association coincides with the one in the statement follows from an easy diagram chase, the definition of Frobenius reciprocity and the definition of the isomorphism 4.2.2. □

We record an immediate corollary that was the main motivation to prove Theorem 4.2.8.

Corollary 4.2.13. *Let* Π *be an absolutely irreducible* Q*-ordinary* E*-representation of* $G(L)$ *of weight* λ*. Then* $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi)$ *is an absolutely irreducible locally algebraic* E*-representation of* $M(L)$ *of weight* $w_0^Q\lambda$*.*

Proof. Since $J_Q(\Pi)$ is of finite finite length, at the very least, so is $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi)$. After possibly enlarging E, we can pick an absolutely irreducible quotient

$$\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) \twoheadrightarrow \Pi_M. \tag{4.2.8}$$

In particular, Π_M is Z-integral. Adjunction then gives a non-trivial map $g : \Pi \to \mathrm{Ind}_Q^G(\Pi_M \otimes \delta_Q)$ that must then be an embedding. By Lemma 4.2.12, we recover 4.2.8 by applying $\mathrm{Ord}_Q^{\mathrm{lalg}}(-)$ to g and postcomposing it with 4.2.2. In particular, due to left exactness of $\mathrm{Ord}_Q^{\mathrm{lalg}}$, 4.2.8 must also be an injection. □

As a consequence, for any absolutely irreducible object $\Pi \in \mathrm{Mod}_E^{Q\text{-ord},\lambda}(G(L))$, there is a unique absolutely irreducible $\Pi_M \in \mathrm{Mod}_E^{\mathrm{wa},w_0^Q\lambda}(M(L))$ such that Π embeds into $\mathrm{Ind}_Q^G(\Pi_M \otimes \delta_Q)$. We can then make the following definition.

Definition 4.2.14. Given an absolutely irreducible $\Pi \in \mathrm{Mod}_E^{Q\text{-ord},\lambda}(G(L))$, we say that $\Pi_M \in \mathrm{Mod}_E^{\mathrm{wa},w_0^Q\lambda}(M(L))$ is its *Q-ordinary support* if it is the (necessarily unique) absolutely irreducible representation in $\mathrm{Mod}_E^{\mathrm{wa},w_0^Q\lambda}(M(L))$ with an embedding $\Pi \hookrightarrow \mathrm{Ind}_Q^G(\Pi_M \otimes \delta_Q)$.

A very useful feature of the notion of the Q-ordinary support is that it can be read off from Π by applying $\mathrm{Ord}_Q^{\mathrm{lalg}}(-)$.

For the rest of the section, we will be occupied with understanding the relation between Π and its Q-ordinary support in terms of the Bernstein–Zelevinsky and Langlands classifications. In particular, it is this point from which we make use of the assumption that $G = \mathrm{GL}_n$. For a quick review on the Bernstein–Zelevinsky classification and the relevant notations used in the rest of the section, we refer to the appendix.

Lemma 4.2.15. *Let $\Pi = \pi \otimes V_{\lambda^\vee}$ be an absolutely irreducible weakly admissible E-representation of $G(L)$ of weight λ that is Z-integral. Then, for any $\pi_{\mathrm{sc}} \in \mathrm{SC}(\pi)$[11], we have*[12]

$$\frac{1}{e}\sum_{\iota:L\hookrightarrow E}(\lambda_{\iota,n} + \frac{1-n}{2}) \leq \frac{\mathrm{val}_p(\pi_{\mathrm{sc}}(\langle\varpi\rangle))}{\deg(\pi_{\mathrm{sc}})} \leq \frac{1}{e}\sum_{\iota:L\hookrightarrow E}(\lambda_{\iota,1} + \frac{n-1}{2}). \tag{4.2.9}$$

Proof. Assume that $\pi = Z(\Delta_1, ...\Delta_k)$ for an ordered multiset of segments $\underline{\Delta} := (\Delta_1, ..., \Delta_k)$ with $\Delta_i := \Delta(\pi_i, r_i)$ for $i = 1, ..., k$. Denote by $Q_{\underline{\Delta}} \subset G$ the standard parabolic subgroup attached to the corresponding ordering of the *supercuspidal support* of π. For $\Delta = \Delta(\sigma, r)$, set

$$v_\Delta := \frac{\sum_{i=0}^{r-1}\mathrm{val}_p(\sigma(i)(\langle\varpi\rangle))}{r\deg(\sigma)} = \frac{\mathrm{val}_p(\sigma(\langle\varpi\rangle))}{\deg(\sigma)} + \frac{[L:\mathbf{Q}_p]}{e}\frac{1-r}{2},$$

the arithmetic mean value of the numbers $\frac{\mathrm{val}_p(\pi_{\mathrm{sc}}(\langle\varpi\rangle))}{\deg(\pi_{\mathrm{sc}})}$ for $\pi_{\mathrm{sc}} \in \mathrm{SC}(Z(\Delta))$.

Note that we can assume that $\underline{\Delta}$ is ordered so that

$$(v_{\Delta_1}, -r_1) \leq ... \leq (v_{\Delta_k}, -r_k) \tag{4.2.10}$$

with respect to the lexicographic ordering. Similarly, we can assume that $\underline{\Delta}$ is ordered so that

$$(v_{\Delta_1}, r_1) \leq ... \leq (v_{\Delta_k}, r_k). \tag{4.2.11}$$

Indeed, if Δ_i and Δ_j are linked for some $1 \leq i \neq j \leq k$, say Δ_i precedes Δ_j, then $v_{\Delta_j} < v_{\Delta_i}$ and, by well-orderedness of the multiset of segments in the sense of Bernstein–Zelevinsky, we must also have $j < i$. Otherwise, if Δ_i and Δ_j are not linked, for instance if $v_{\Delta_i} = v_{\Delta_j}$, we have the freedom of choosing the order.

---

[11] Recall that we denote by $\mathrm{SC}(\pi)$ the set $\{\pi_1, ..., \pi_k\}$ where $(\mathrm{GL}_{n_1}(L)\times ...\times \mathrm{GL}_{n_k}(L), \pi_1 \otimes ... \otimes \pi_k)$ is the supercuspidal support of π.

[12] We remind the reader to the notation $\langle\varpi\rangle = \mathrm{diag}(\varpi, ..., \varpi) \in G(L)$.

If we choose $\underline{\Delta}$ so that 4.2.10 is satisfied, we see that

$$v_{\Delta_1} + \frac{[L:\mathbf{Q}_p]}{e}\frac{1-r_1}{2} = \frac{\mathrm{val}_p(\pi_1(r_1-1)(\langle\varpi\rangle))}{\deg(\pi_1)} \le \frac{\mathrm{val}_p(\pi_{\mathrm{sc}}(\langle\varpi\rangle))}{\deg(\pi)}$$

for any $\pi_{\mathrm{sc}} \in \mathrm{SC}(\pi)$. Similarly, if we choose $\underline{\Delta}$ so that 4.2.11 is satisfied, we see that

$$\frac{\mathrm{val}_p(\pi_{\mathrm{sc}}(\langle\varpi\rangle))}{\deg(\pi_{\mathrm{sc}})} \le v_{\Delta_k} + \frac{r_k-1}{2} = \frac{\mathrm{val}_p(\pi_k)(\langle\varpi\rangle))}{\deg(\pi_k)}$$

for any $\pi_{\mathrm{sc}} \in \mathrm{SC}(\pi)$. In particular, it suffices to prove that, for any choice of ordering $\underline{\Delta}$, we have

$$\frac{\mathrm{val}_p((\delta_{Q_{\underline{\Delta}}}^{1/2} w_0^G\lambda)(u_p^{(\deg(\Delta_1))}))}{\deg(\Delta_1)} \le v_{\Delta_1} + \frac{[L:\mathbf{Q}_p]}{e}\frac{1-r_1}{2} \tag{4.2.12}$$

and

$$v_{\Delta_k} + \frac{[L:\mathbf{Q}_p]}{e}\frac{r_k-1}{2} \le \frac{\mathrm{val}_p((\delta_{Q_{\underline{\Delta}}}^{1/2} w_0^G\lambda)(u_p^{(n)}/u_p^{(n-\deg(\Delta_k))}))}{\deg(\Delta_k)}. \tag{4.2.13}$$

Indeed, an easy computation, combined with the regularity of λ and the equality

$$\mathrm{val}_p(\delta_{Q_{\underline{\Delta}}}^{1/2}(u_p^{(m)})) = \frac{[L:\mathbf{Q}_p]}{e}\sum_{i=1}^{m}(\frac{1-n}{2}+i-1) \text{ for every } 1 \le m \le n,$$

shows that the LHS of 4.2.9 is bounded by the LHS of 4.2.12 and the RHS of 4.2.13 is bounded by the RHS of 4.2.9.

To prove these inequalities, we note that, by Lemma A.0.4, $J_{Q_{\underline{\Delta}}}(\pi)$ admits $\delta_{Q_{\underline{\Delta}}}^{1/2}\Delta_1 \otimes ... \otimes \Delta_k$ as a quotient. In particular, weak admissibility of Π gives that

$$\mathrm{val}_p((\delta_{Q_{\underline{\Delta}}} w_0^G\lambda))(u_p^{(\deg(\Delta_1))}) \le \mathrm{val}_p(\Delta_1(\langle\varpi\rangle)\delta_{Q_{\underline{\Delta}}}^{1/2}(u_p^{(\deg(\Delta_1))}))$$

and 4.2.12 is proved.

To get 4.2.13, one similarly applies weak admissibility with the choice $z = u_p^{(n-\deg(\Delta_k))}$ and combines it with the equality

$$\mathrm{val}_p((\delta_{Q_{\underline{\Delta}}} w_0^G\lambda)(\langle\varpi\rangle)) = \mathrm{val}_p(J_{Q_{\underline{\Delta}}}(\pi)(\langle\varpi\rangle)) = \mathrm{val}_p((\delta_{Q_{\underline{\Delta}}}^{1/2}\Delta_1 \otimes ... \otimes \Delta_k)(\langle\varpi\rangle))$$

coming from the assumption that Π is absolutely irreducible and Z-integral, so it has a central character that must be integral, and the fact that $\delta_{Q_{\underline{\Delta}}}(\langle\varpi\rangle) = 1$. □

Lemma 4.2.16. *Let* $\Pi_M = \pi_M \otimes V_{(w_0^Q\lambda)^\vee}$ *be an absolutely irreducible weakly admissible E-representation of* $M(L)$ *of weight* $w_0^Q\lambda$ *that is* Z*-integral. If* $Q = M \ltimes N = P_{(n_1,...,n_t)}$ *and* $\delta_Q^{1/2} \otimes \pi_M = \pi_1 \otimes ... \otimes \pi_t$*, then, for* $1 \le i < j \le t$ *and* $\pi_{\mathrm{sc},i} \in \mathrm{SC}(\pi_i)$, $\pi_{\mathrm{sc},j} \in \mathrm{SC}(\pi_j)$*, we have*

$$\frac{\mathrm{val}_p(\pi_{\mathrm{sc},i}(\langle\varpi\rangle))}{\deg(\pi_{\mathrm{sc},i})} \le \frac{\mathrm{val}_p(\pi_{\mathrm{sc},j}(\langle\varpi\rangle))}{\deg(\pi_{\mathrm{sc},j})} - \frac{[L:\mathbf{Q}_p]}{e} = \frac{\mathrm{val}_p(\pi_{\mathrm{sc},j}(\langle\varpi\rangle))}{\deg(\pi_{\mathrm{sc},j})} + \mathrm{val}_p(|\varpi|_L).$$

Proof. Write $\Pi_M = \Pi_1 \otimes ... \otimes \Pi_t$. Then the lemma follows easily from applying Lemma 4.2.15 to each of the Π_i's for $1 \leq i \leq t$ and noting that, for $1 \leq i \leq t$,

$$\delta_Q^{1/2}|_{\mathrm{GL}_{n_i}(L)} = |\det|^{\frac{n-n_i}{2}-(n_1+...+n_{i-1})}.$$

$\square$

Combining Lemma A.0.3 with the results of the section, we are now ready to understand the relation between the Langlands classification of Π and its Q-ordinary support. This is best stated using the local Langlands correspondence.

Corollary 4.2.17. *Let Π be an absolutely irreducible $P_{(n_1,...,n_t)} = Q$-ordinary E-representation of $G(L)$ of weight λ with Q-ordinary support Π_M. Write $\Pi \otimes_E \overline{\mathbf{Q}}_p = \pi \otimes_{\overline{\mathbf{Q}}_p} V_{\lambda^\vee}$ and $\Pi_M \otimes_E \overline{\mathbf{Q}}_p = (\pi_1 \otimes ... \otimes \pi_t) \otimes_{\overline{\mathbf{Q}}_p} V_{-w_0^G\lambda}$. Fix an identification $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$ and assume further that π is t-preunitary (see the appendix). Then $\mathrm{rec}^T(\pi)$ admits a flag $0 = F_0 \subset F_1 \subset ... \subset F_t = \mathrm{rec}^T(\pi)$ of sub-Weil–Deligne representations such that, for $1 \leq j \leq t$, we have isomorphisms*

$$F_j/F_{j-1} \cong \mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})$$

where $\sum_j := n_1 + ... + n_j$. In other words, there is an isomorphism of Weil–Deligne representations

$$\mathrm{rec}^T(\pi) \sim \begin{pmatrix} \mathrm{rec}^T(\pi_1) & * & ... & * \\ 0 & \mathrm{rec}^T(\pi_2 \otimes |\cdot|^{-n_1}) & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \mathrm{rec}^T(\pi_t \otimes |\cdot|^{n_t-n}) \end{pmatrix}.$$

Proof. Note that adjunction applied to $\mathrm{Ord}_Q^{\mathrm{lalg}}(\Pi) \otimes |\cdot|^{\frac{1-n}{2}} = \Pi_M \otimes |\cdot|^{\frac{1-n}{2}} \xrightarrow{\mathrm{id}} \Pi_M \otimes |\cdot|^{\frac{1-n}{2}}$ gives an embedding

$$\pi \otimes |\cdot|^{\frac{1-n}{2}} \hookrightarrow \text{n-Ind}_{Q(L)}^{G(L)}(\delta_Q^{1/2}|_{\mathrm{GL}_{n_1}(L)} \otimes |\cdot|^{\frac{1-n}{2}} \otimes \pi_1) \otimes ... \otimes (\delta_Q^{1/2}|_{\mathrm{GL}_{n_t}(L)} \otimes |\cdot|^{\frac{1-n}{2}} \otimes \pi_t). \tag{4.2.14}$$

One computes that, for $1 \leq j \leq t$, we have

$$\delta_Q^{1/2}|_{\mathrm{GL}_{n_j}(L)} = |\cdot|^{\frac{n-n_j}{2}-\sum_{j-1}}.$$

Therefore, we compute

$$\delta_Q^{1/2}|_{\mathrm{GL}_{n_j}(L)} \otimes |\cdot|^{\frac{1-n}{2}} \otimes \pi_j = (\pi_j \otimes |\cdot|^{\frac{1-n_j}{2}}) \otimes |\cdot|^{\frac{-(n-n_j)}{2}} \otimes \delta_Q^{1/2} = (\pi_j \otimes |\cdot|^{\frac{1-n_j}{2}}) \otimes |\cdot|^{-\sum_{j-1}}.$$

This implies the existence of a filtration

$$0 = F_0^{\mathrm{ss}} \subset ... \subset F_t^{\mathrm{ss}} = \mathrm{rec}^T(\pi)^{\mathrm{ss}}$$

of representations of the Weil group with subquotients given by

$$\mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})^{\mathrm{ss}}.$$

In fact, this really is a direct sum decomposition as the underlying Weil group representation of $\mathrm{rec}^T(\pi)$ is semisimple.

By combining Lemma 4.2.16 and our unitariness assumption, we can apply Lemma A.0.3 to 4.2.14. After unravelling the construction of the reduction of the local Langlands correspondence to supercuspidal representations, this exactly says that the monodromy on the subquotients does not change. In particular, we can upgrade $F^{\mathrm{ss}}_\bullet$ to the desired filtration of Weil–Deligne representations of $\mathrm{rec}^T(\pi)$. □

To ease the notation in the upcoming sections, we introduce the following notation.

Definition 4.2.18. Denote by Π an absolutely irreducible Q-ordinary E-representation of $G(L)$ of weight λ and write $\Pi \otimes_E \overline{\mathbf{Q}}_p = \pi \otimes_E V_{\lambda^\vee}$. Then we set $\pi^{Q\text{-ord}}$ to be the smooth $\overline{\mathbf{Q}}_p$-representation of $M(L)$ such that $\mathrm{Ord}^{\mathrm{lalg}}(\Pi) \otimes_E \overline{\mathbf{Q}}_p = \pi^{Q\text{-ord}} \otimes_{\overline{\mathbf{Q}}_p} V_{-w_0^G\lambda}$.

4.3 A Q-ordinary local-global compatibility result

Finally, we use our observations about Q-ordinary representations to deduce Q-ordinary local-global compatibility for regular algebraic conjugate self-dual cuspidal automorphic representations (RACSDCAR). We fix an integer $n \geq 1$ with a partition $n_1 + ... + n_t$ and denote by $Q = M \ltimes N$ the corresponding parabolic subgroup of GL_n. We further consider the appropriate global setup i.e., F will denote a CM number field and $v|p$ is a fixed p-adic place. If π is a RACAR of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\mathbf{C})}$ and $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$ is a fixed isomorphism then $t^{-1}\pi_v$ can be realized over a finite extension $E/\mathbf{Q}_p$. Moreover, $t^{-1}\pi_v \otimes_E V_{t^{-1}\lambda^\vee}$ becomes a weakly admissible E-representation of $\mathrm{GL}_n(F_v)$ of weight $t^{-1}\lambda_v$ (see Remark 4.2.4). Therefore, we are in the situation of §4.2. We are then interested in proving the following local-global compatibility result.

Theorem 4.3.1. *Let π be a RACSDCAR of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\mathbf{C})}$, $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$ be a fixed isomorphism and $v|p$ be a p-adic place of F. Assume that $t^{-1}\pi_v \otimes_E V_{\lambda^\vee}$ is Q-ordinary. Write $\pi^{Q\text{-ord}} = \pi_1 \otimes ... \otimes \pi_t$ (see Definition 4.2.18). Then there is an isomorphism*

$$r_t(\pi)|_{G_{F_v}} \sim \begin{pmatrix} \rho_1 & * & ... & * \\ 0 & \rho_2 & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \rho_t \end{pmatrix}$$

where, for $1 \leq j \leq t$,

$$\rho_j : G_{F_v} \to \mathrm{GL}_{n_j}(\overline{\mathbf{Q}}_p)$$

is potentially semistable such that, for every embedding $\iota : F_v \hookrightarrow \overline{\mathbf{Q}}_p$, *the labelled* ι*-Hodge–Tate weights of* ρ_j *are given by*

$$\lambda_{t\circ\iota, n+1-(n_1+...+n_j)}+n_1+...+n_{j-1}+n_j-1 > ... > \lambda_{t\circ\iota,n+1-(n_1+...+n_{j-1}+1)}+n_1+...+n_{j-1}$$

and we have

$$\mathrm{WD}(\rho_j)^{F-ss} \cong \mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})$$

where $\sum_j := n_1 + ... + n_j$.

We highlight the following simple observation.

Lemma 4.3.2. *Let* π_v *and* $(\pi_1, ..., \pi_t)$ *as in Theorem 4.3.1. Then, for any lift of (geometric) Frobenius* $\varphi_v \in W_{F_v}$ *and* $1 \leq j \leq t$, *we have*

$$\mathrm{val}_p(\det(\mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})(\varphi_v))) = \frac{1}{e_v} \sum_{\iota:F_v \hookrightarrow \overline{\mathbf{Q}}_p} \sum_{i=n_1+...+n_{j-1}+1}^{n_1+...+n_j} (\lambda_{\iota,n+1-i}+i-1).$$

Proof. Recall that for σ a smooth admissible $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GL}_n(F_v)$, $\det \circ \mathrm{rec}(\sigma) = \mathrm{rec}(\omega_\sigma)$, where ω_σ denotes the central character of σ.

In particular, we have

$$\mathrm{val}_p(\det(\mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})(\varphi_v))) = \mathrm{val}_p((\pi_j \otimes |\cdot|^{\frac{1-n_j}{2}-\sum_{j-1}})(\mathrm{diag}(\varpi_v, ..., \varpi_v)))$$

for ϖ_v the image of φ_v under the local Artin map. To see that the RHS of the equation above is the desired number, we use that $(\pi_1 \otimes ... \otimes \pi_t) \otimes_E V_{-w_0^G\lambda}$ has (p-adically) integral central character. Namely, it implies that we have

$$\mathrm{val}_p(\pi_j(\mathrm{diag}(\varpi_v, ..., \varpi_v))) = \sum_{\iota:F_v \hookrightarrow E} \sum_{i=n_1+...+n_{j-1}+1}^{n_1+...+n_j} \lambda_{\iota,n+1-i} \cdot \mathrm{val}_p(\iota(\varpi_v)) =$$

$$\frac{1}{e_v} \sum_{\iota:F_v \hookrightarrow E} \sum_{i=n_1+...+n_{j-1}+1}^{n_1+...+n_j} \lambda_{\iota,n+1-i}. \tag{4.3.1}$$

Moreover, we have

$$\mathrm{val}_p(|\det(\mathrm{diag}(\varpi_v, ..., \varpi_v))|^{\frac{1-n_j}{2}-\sum_{j-1}}) = (\frac{1-n_j}{2} - \sum_{j-1}) \cdot n_j \cdot \mathrm{val}_p(|\varpi_v|_v) =$$

$$f_v \cdot \left(\sum_{i=n_1+...+n_{j-1}+1}^{n_1+...+n_j} (i-1) \right) = \frac{1}{e_v} |\mathrm{Hom}(F_v, \overline{\mathbf{Q}}_p)| \cdot \left(\sum_{i=n_1+...+n_{j-1}+1}^{n_1+...+n_j} (i-1) \right) \tag{4.3.2}$$

where in the last equality we used that $f_v = \frac{[F_v:\mathbf{Q}_p]}{e_v}$.

We conclude by combining 4.3.1 and 4.3.2.

□

Before starting the proof of Theorem 4.3.1, we recall some constructions from p-adic Hodge theory which we will make use of in the proof. Consider a potentially semistable p-adic Galois representation $\rho : G_{F_v} \to \mathrm{GL}_d(E)$ for some finite extension $E/\mathbf{Q}_p$ and for simplicity assume that it has distinct Hodge–Tate weights. Let K/F_v be a finite Galois extension such that $\rho|_{G_K}$ is semistable and enlarge E so that it contains the images of all embeddings $K \hookrightarrow \overline{\mathbf{Q}}_p$. Set $K_0 := W(\mathcal{O}_K/\mathfrak{m}_K)[1/p]$ and denote by σ its arithmetic Frobenius.

We can then apply Fontaine's construction which associates with ρ a filtered $(\varphi, N, K/F_v, E)$-module $D := D_{\mathrm{st},K}(\rho)$. This, by definition, is a free $K_0 \otimes_{\mathbf{Q}_p} E$-module with

i. a $\sigma \otimes 1$-semilinear automorphism φ of D;

ii. a $K_0 \otimes_{\mathbf{Q}_p} E$-linear endomorphism N of D such that $N\varphi = p\varphi N$;

iii. a K-semilinear, E-linear action of $\mathrm{Gal}(K/F_v)$ commuting with φ and N;

iv. and a filtration $\mathrm{Fil}_\bullet D_K$ of $D_K := D \otimes_{K_0} K$ by $K \otimes_{\mathbf{Q}_p} E$-submodules.

Moreover, we know that D is weakly admissible. In other words, "its Newton polygon lies over its Hodge polygon", i.e. in the notations of [Fon94], $t_N(D) = t_H(D)$ and for any sub-$(\varphi, N, K/F_v, E)$-module $D' \subset D$ equipped with the induced filtration from D we have $t_N(D') \geq t_H(D')$. See [Fon94], 4.4.1 and Definition 4.4.3.

We have identifications

$$D_K \cong \prod_{\iota : K \hookrightarrow E} D_\iota, \ \mathrm{Fil}_\bullet D_K = \prod_{\iota : K \hookrightarrow E} \mathrm{Fil}_\bullet D_\iota.$$

Then for every embedding $\iota : K \hookrightarrow E$, we have

$$\dim_E \mathrm{gr}^i \mathrm{Fil}_\bullet D_\iota = \begin{cases} 1, & \text{if } i = \lambda_{\iota|_{F_v},j} \text{for } j \in \{1, ..., d\} \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda_{\iota|_{F_v},1} > ... > \lambda_{\iota|_{F_v},d}$ are the labelled $\iota|_{F_v}$-Hodge–Tate-weights of ρ. In other words, the Hodge–Tate weights are encoded in the Hodge-filtration of D.

We can further associate to D a Weil–Deligne representation as follows. Given $g \in W_{F_v}$, let g act on D by $(g \mod W_K) \circ \varphi^{-\alpha(g)}$ where $\alpha(g)$ is given by the power of the arithmetic Frobenius given by the action of g on the residue field of $\overline{F}_v$. Note that if f_K resp. f_v denotes the inertia degree of $K/\mathbf{Q}_p$ resp. $F_v/\mathbf{Q}_p$, then we have that, for any lift of geometric Frobenius $\varphi_v \in W_{F_v}$, $\varphi_v^{f_K/f_v}$ acts on D by φ^{f_K}. This action is then $K_0 \otimes_{\mathbf{Q}_p} E$-linear by definition and we can consider the W_{F_v}- and N-invariant decomposition

$$D = \prod_{\iota_0 : K_0 \hookrightarrow E} D_{\iota_0}.$$

We then set $\mathrm{WD}(\rho) := D_{\iota_0}$ to be the associated Weil–Deligne representation over E for a choice of ι_0. As the notation suggests, it is independent of the choice of ι_0 and K up to isomorphism.

Proof of Theorem 4.3.1. We argue as in [Tho15], Theorem 2.4 which was based on [Ger18], Corollary 2.7.8. In particular, the key ingredients are local-global compatibility at v for $r_t(\pi)$, Fontaine's theory of weakly admissible modules and Corollary 4.2.17.

Let $E \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ such that $r_t(\pi)|_{G_{F_v}}$ lands in $\mathrm{GL}_n(E)$. First note that by Theorem 2.9.1 we have that $r_t(\pi)$ is potentially semistable at v. Let K/F_v be a finite Galois extension such that $r_t(\pi)|_{G_K}$ is semistable and enlarge E if necessary to assume that it contains all images of embeddings $K \hookrightarrow \overline{\mathbf{Q}}_p$. Set D to be the weakly admissible filtered $(\varphi, N, K/F_v, E)$-module associated with $r_t(\pi)|_{G_{F_v}}$. By Theorem 2.9.1 again, we further have an identification

$$\mathrm{WD}(r_t(\pi)|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}^T(t^{-1}\pi_v).$$

Therefore, by Corollary 4.2.17, we have an isomorphism

$$\mathrm{WD}(r_t(\pi)|_{G_{F_v}})^{F-ss} \sim \begin{pmatrix} \mathrm{rec}^T(\pi_1) & * & ... & * \\ 0 & \mathrm{rec}^T(\pi_2 \otimes |\cdot|^{-n_1}) & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \mathrm{rec}^T(\pi_t \otimes |\cdot|^{n_t - n}) \end{pmatrix}.$$

In particular, we obtain an isomorphism

$$\mathrm{WD}(r_t(\pi)|_{G_{F_v}}) \sim \begin{pmatrix} \mathrm{WD}_1 & * & ... & * \\ 0 & \mathrm{WD}_2 & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \mathrm{WD}_t, \end{pmatrix}$$

such that, for $1 \leq j \leq t$, $\mathrm{WD}_j^{F-ss} \cong \mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})$. Consequently, $D = D_{\mathrm{st},K}(r_t(\pi)|_{G_{F_v}})$ admits a flag

$$F_0 = 0 \subset F_1 \subset ... \subset F_t = D$$

of sub-$(\varphi, N, K/F_v, E)$-modules with F_j/F_{j-1} corresponding to $\mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})$.

We now can apply Fontaine's theorem about classifying weakly admissible filtered (φ, N)-modules. Namely, if we combine Lemma 4.3.2 with [Fon94], Theorem 5.6.7 and use that $\varphi_v^{f_K/f_v}$ acts on D by φ^{f_K}, we get that, for each $1 \leq j \leq t$, $F_j \subset D$ comes from a sub-G_{F_v}-representation $\widetilde{\rho}_j \subset r_t(\pi)|_{G_{F_v}}$. Moreover, their subquotients $\rho_j := \widetilde{\rho}_j/\widetilde{\rho}_{j-1}$ are clearly potentially semistable and have the right associated Weil–Deligne representations as in the statement of the theorem.

Finally, to find the Hodge–Tate weights, we argue by induction on $1 \leq j \leq t$. Assume that the claim holds for indices smaller than j. We can combine the fact that $t_N(F_j) = t_H(F_j)$, $\mathrm{WD}(\rho_j)^{F-ss} \cong \mathrm{rec}^T(\pi_j \otimes |\cdot|^{-\sum_{j-1}})$, Lemma 4.3.2 and the induction hypothesis to see that the sum of the Hodge–Tate weights of ρ_j is given by

$$\sum_{\iota: F_v \hookrightarrow \overline{\mathbf{Q}}_p} \sum_{i=n_1+...+n_{j-1}+1}^{n_1+...+n_j} (\lambda_{t\circ\iota, n+1-i} + i - 1).$$

Now the regularity of λ combined with Theorem 2.9.1 forces ρ_j to have the right Hodge–Tate weights. □

Remark 4.3.3. Note that the proof of Theorem 4.3.1 already works for automorphic representation π with π_v being pre-unitary (in the sense of the appendix) and admitting an associated Galois representation satisfying local-global compatibility at v. Therefore, our results will already hold for automorphic representations π which are pre-unitary at v and are isobaric sums of regular algebraic discrete conjugate self-dual automorphic representations. Indeed, this follows from Moeglin–Waldspurger's classification of discrete automorphic representations. This observation combined with Shin's base change result [Shi14] leads to the application below, which is one of the crucial ingredients in proving our main local-global compatibility results.

To be able to appeal to [Shi14], we assume for the rest of the section that F contains an imaginary quadratic field in which p splits. Fix a p-adic place $\bar{v}$ of F^+ and fix a place v of F above $\bar{v}$. Let $\widetilde{Q}_{\bar{v}} \subset P_{F^+_{\bar{v}}} \subset \widetilde{G}_{F^+_{\bar{v}}}$ be a parabolic subgroup with Levi decomposition $\widetilde{Q}_{\bar{v}} = \widetilde{M}_{\bar{v}} \ltimes \widetilde{N}_{\bar{v}}$. Then $\widetilde{Q}_{\bar{v}}(F^+_{\bar{v}})$ is identified under ι_v with $P_{(n_1,...,n_t)}(F_v) \subset \mathrm{GL}_{2n}(F_v)$ for some tuple of integers $(n_1, ..., n_k, n_{k+1}, ..., n_t)$, refining (n, n).

Theorem 4.3.4. *Let $\widetilde{\pi}$ be a ξ-cohomological cuspidal automorphic representation of $\widetilde{G}(\mathbf{A}_{F^+})$ as in Theorem 2.9.2. Denote by $\tilde{\lambda}$ the highest weight of the representation $(\xi \otimes \xi)^\vee$. Assume further that, for a fixed $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, $t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}) \otimes V_{t^{-1}\tilde{\lambda}_{\bar{v}}}$ is $\widetilde{Q}_{\bar{v}}$-ordinary of weight $t^{-1}\tilde{\lambda}_{\bar{v}}$. Write $(t^{-1}(\widetilde{\pi} \circ \iota_v^{-1}))^{\widetilde{Q}_{\bar{v}}\text{-ord}} = \widetilde{\pi}_1 \otimes ... \otimes \widetilde{\pi}_t$. Then there is an isomorphism*

$$r_t(\widetilde{\pi})|_{G_{F_v}} \sim \begin{pmatrix} \rho_1 & * & . & . & . & . & * & * \\ 0 & . & & & & & & * \\ . & & . & & & & & . \\ . & & & \rho_k & & & & . \\ . & & & & \rho_{k+1} & & & . \\ . & & & & & . & & . \\ 0 & & & & & & . & * \\ 0 & 0 & . & . & . & . & 0 & \rho_t \end{pmatrix}$$

with ρ_j *being potentially semistable and, for every* $\iota : F_v \hookrightarrow \overline{\mathbf{Q}}_p$, *it has labelled* ι*-Hodge–Tate weights*

$$\tilde{\lambda}_{\iota,2n+1-n_1+...+n_j} + n_1 + ... + n_j - 1 > ... > \tilde{\lambda}_{\iota,2n+1-n_1+...+n_{j-1}+1} + n_1 + ... + n_{j-1}.$$

Moreover, we have

$$\mathrm{WD}(\rho_j)^{F-ss} \cong \mathrm{rec}^T(\widetilde{\pi}_j \otimes |\cdot|^{-\sum_{j-1}})$$

where $\sum_j := n_1 + ... + n_j$.

Proof. This follows from Remark 4.3.3, and Theorem 2.9.2. □

Chapter 5

A torsion local-global compatibility conjecture

In this chapter, we formulate integral local-global compatibility conjectures in great generality and state our progress on proving them. Most of this, although only in more restrictive setups, was already formulated in [Car+16b], and [GN20].

For the rest of the chapter, set $G := \mathrm{GL}_{n/F}$ for a fixed number field F and an integer $n \geq 2$. Pick a subfield $E \subset \overline{\mathbf{Q}}_p$, finite over $\mathbf{Q}_p$, with ring of integers $\mathcal{O} \subset E$ and a choice of uniformiser $\varpi \in \mathcal{O}$. We assume that E is large enough so that $[F : \mathbf{Q}] = |\mathrm{Hom}(F, E)|$.

5.1 Local deformation rings

To formulate local-global compatibility that also keeps track of torsion, we make use of Kisin's potentially semistable local deformation rings.

Let $\mathrm{CNL}_{\mathcal{O}}$ denote the category of complete local Noetherian $\mathcal{O}$-algebras with residue field $k := \mathcal{O}/\varpi$. Set $L := F_v$ for some $v \in S_p(F)$. Denote by L_0 its maximal subfield that is unramified over $\mathbf{Q}_p$, and by $\overline{L}_0$ the maximal unramified extension of L_0. Similar notations apply to any finite field extension L'/L. Given a continuous Galois representation $\overline{\rho} : G_L \to \mathrm{GL}_n(k)$, we can form the framed deformation functor $D_{\overline{\rho}}^{\square} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Sets}$, sending a test object $(A, \mathfrak{m}_A)$ to the set of continuous homomorphisms $\rho : G_L \to \mathrm{GL}_n(A)$ with $\rho \otimes_A A/\mathfrak{m}_A = \overline{\rho}$. It is known to be represented by an object $R_{\overline{\rho}}^{\square} \in \mathrm{CNL}_{\mathcal{O}}$, that is called its *framed deformation ring*. We denote by $\rho^{\mathrm{univ}} : G_L \to \mathrm{GL}_n(R_{\overline{\rho}}^{\square})$ the universal lift, and, for $x : R_{\overline{\rho}}^{\square} \to A$ in $\mathrm{CNL}_{\mathcal{O}}$, by ρ_x its specialisation $\rho^{\mathrm{univ}} \otimes_{R_{\overline{\rho}}^{\square}, x} A$.

Consider an n-dimensional Weil–Deligne inertial type $\tau = (\rho_\tau, N_\tau)$ for L over E and a highest weight vector $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(L,E)}$ for $\mathrm{Res}_{L/\mathbf{Q}_p}\mathrm{GL}_n$. From now on we will make the following technical assumption.

Assumption 5.1.1. Assume that E is large enough so that there is a finite

Galois extension L_τ/L such that $\rho_\tau|_{I_{L_\tau}}$ is trivial with $|\mathrm{Hom}(L_\tau, E)| = |L_\tau : \mathbf{Q}_p|$. Fix such an L_τ/L.

In [Kis07], Kisin constructs a reduced $\mathcal{O}$-flat quotient $R_{\overline{\rho}}^{\lambda,\rho_\tau}$ of $R_{\overline{\rho}}^{\square}$ whose E'-points, for any finite extension E'/E, correspond to lifts $\rho : G_L \to \mathrm{GL}_n(E')$ of $\overline{\rho}$ such that ρ is potentially semistable with labelled Hodge–Tate weights $(\lambda_{\iota,1} + n - 1 > ... > \lambda_{\iota,n})_{\iota \in \mathrm{Hom}(L,E)}$ and $\mathrm{WD}(\rho)^{ss}|_{I_L} \cong \rho_\tau \otimes_E E'$. After introducing some terminology, we will further describe its B-valued points for a general finite E-algebra B.

To a given λ, we associate a p-adic Hodge type $\mathbf{v}_\lambda$ in the sense of [Kis07]. Consider an n-dimensional E-vector space D_E and write

$$D_{E,L} := D_E \otimes_{\mathbf{Q}_p} L \cong \oplus_{\iota : L \hookrightarrow E} D_{E,\iota}.$$

For each $\iota : L \hookrightarrow E$, we pick an (arbitrary) decreasing filtration $\mathrm{Fil}^\bullet D_{E,\iota}$ by sub-E-vector spaces so that $\dim_E \mathrm{gr}^i D_{E,\iota} \neq 0$ if and only if $i = \lambda_{\iota,n+j-1} + j - 1$ for $1 \leq j \leq n$ in which case the E-dimension is exactly 1. By taking the direct sum of the filtrations, we get a decreasing a decreasing filtration $\mathrm{Fil}^\bullet D_{E,L}$ on $D_{E,L}$ by $E \otimes_{\mathbf{Q}_p} L$-submodules. We set $\mathbf{v}_\lambda = \{D_E, \mathrm{Fil}^\bullet D_{E,L}\}$. For a finite E-algebra B, and a continuous de Rham representation ρ_B of G_L on a finite free rank n B-module V_B, we say that ρ_B has p-adic Hodge type $\mathbf{v}_\lambda$ if, for each $i \in \mathbf{Z}$, there is an isomorphism

$$\mathrm{gr}^i (V_B \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_L} \cong B \otimes_E \mathrm{gr}^i D_{E,L}$$

of $B \otimes_{\mathbf{Q}_p} L$-modules. In particular, any de Rham Galois representation $\rho : G_L \to \mathrm{GL}_n(E)$ with labelled Hodge–Tate weights $(\lambda_{\iota,1} + n - 1 > ... > \lambda_{\iota,n})_{\iota \in \mathrm{Hom}(L,E)}$ has p-adic Hodge type $\mathbf{v}_\lambda$.

Consider a finite E-algebra B, and a continuous potentially semistable representation ρ_B of G_L on a finite free B-module V_B of rank n. We explain the notion of ρ_B having inertial type ρ_τ. Assume for a second that B is local with residue field E'. Further assume that V_B becomes semistable as a representation of G_{L_τ}. Set

$$D_{\mathrm{st}}^{L_\tau}(V_B) = (V_B \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{G_{L_\tau}},$$

a finite free $B \otimes_{\mathbf{Q}_p} L_{\tau,0}$-module, forming a filtered $(\varphi, N, L_\tau/L, E)$-module. In particular, it admits a $B \otimes_{\mathbf{Q}_p} L_{\tau,0}$-linear action of $I_{L_\tau/L}$ that commutes with φ and N. Since higher cohomology of finite groups is trivial in characteristic 0, deformation theory tells us that the $I_{L_\tau/L}$-action on $D^{L_\tau}(V_B)$ comes as extension of scalars along $E' \otimes_{\mathbf{Q}_p} L_{\tau,0} \hookrightarrow B \otimes_{\mathbf{Q}_p} L_{\tau,0}$ of a representation over a rank n free $E' \otimes_{\mathbf{Q}_p} L_{\tau,0}$-module. Moreover, since the $I_{L_\tau/L}$-action commutes with φ, it further descends to a representation of $I_{L_\tau/L}$ on some n-dimensional E'-vector space P_{ρ_B}. We say that ρ_B is of inertial type ρ_τ if its restriction to G_{L_τ} is semistable and P_{ρ_B} with its $I_{L_\tau/L}$-action is equivalent to ρ_τ. One then easily extends the definition to general finite E-algebras using that every such algebra is a product of finite *local* E-algebras.

Finally, we prepare the upcoming result by making the following definition. Let A be an E-algebra. We then define a $(\varphi, N, L_\tau/L, A)$-module of rank n to be a projective $A \otimes_{\mathbf{Q}_p} L_{\tau,0}$-module of rank n equipped with

i. a Frobenius semilinear automorphism φ;

ii. an $A \otimes_{\mathbf{Q}_p} L_{\tau,0}$-linear endomorphism N satisfying $N\varphi = p\varphi N$;

iii. and an $L_{\tau,0}$-semilinear, A-linear $\mathrm{Gal}(L_\tau/L)$-action commuting with φ and N.

Theorem 5.1.2. *There is a unique $\mathcal{O}$-flat quotient $R^{\square}_{\overline{\rho}} \to R^{\lambda,\rho_\tau}_{\overline{\rho}}$ characterised by the property that an arbitrary map $x : R^{\square}_{\overline{\rho}} \to B$ to a finite E-algebra factors through $R^{\lambda,\rho_\tau}_{\overline{\rho}}$ if and only if ρ_x is potentially semistable of p-adic Hodge type $\mathbf{v}_\lambda$ and inertial type ρ_τ. Moreover, $R^{\lambda,\rho_\tau}_{\overline{\rho}}$ is reduced.*

Finally, there exists a rank n $(\varphi, N, L_\tau/L, R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p])$-module $D^{\lambda,\rho_\tau}_{\mathrm{st},L_\tau,\overline{\rho}}$ such that, for any finite E-algebra B, and map $x : R^{\lambda,\rho_\tau}_{\overline{\rho}} \to B$, we have a canonical isomorphism of $(\varphi, N, L_\tau/L, B)$-modules

$$D^{\lambda,\rho_\tau}_{\mathrm{st},L_\tau,\overline{\rho}} \otimes_{R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p],x} B \cong (\rho_x \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{G_{L_\tau}} = D^{L_\tau}_{\mathrm{st}}(\rho_x).$$

Proof. The first part, besides the reducedness, is [Kis07], Theorem 2.7.6. The reducedness follows from [BG19], Theorem 3.3.3.

The second part is implicit in [Kis07] and follows from *loc. cit.*, Theorem 2.5.5 and Proposition 2.7.2. Namely, by *loc. cit.* Theorem 2.5.5, there is a projective $R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p] \otimes_{\mathbf{Q}_p} L_{\tau,0}$-module $D^{\lambda,\rho_\tau}_{\mathrm{st},L_\tau,\overline{\rho}}$ of rank n with a Frobenius semilinear automorphism φ, and an $R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p] \otimes_{\mathbf{Q}_p} L_{\tau,0}$-linear endomorphism N such that $N\varphi = p\varphi N$. Moreover, for any $x : R^{\lambda,\rho_\tau}_{\overline{\rho}} \to B$ as in the statement, there is a canonical isomorphism

$$D^{\lambda,\rho_\tau}_{\mathrm{st},L_\tau,\overline{\rho}} \otimes_{R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p],x} B \cong (\rho_x \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{G_{L_\tau}} = D^{L_\tau}_{\mathrm{st}}(\rho_x). \tag{5.1.1}$$

respecting φ and N. Furthermore, the isomorphism above is induced by the $R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p] \otimes_{\mathbf{Q}_p} L_{\tau,0}$-linear isomorphism

$$D^{\lambda,\rho_\tau}_{\mathrm{st},L_\tau,\overline{\rho}} \xrightarrow{\sim} (\rho^{\mathrm{univ}} \otimes_{R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p]} B_{\mathrm{st},R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p]})^{G_{L_\tau}} \tag{5.1.2}$$

of [Kis07], Proposition 2.7.2. The RHS of 5.1.2 admits a natural $L_{\tau,0}$-semilinear and $R^{\lambda,\rho_\tau}_{\overline{\rho}}$-linear action of $\mathrm{Gal}(L_\tau/L)$, and this action commutes with φ and N, equipping $D^{\lambda,\rho_\tau}_{\mathrm{st},L_\tau,\overline{\rho}}$ with the structure of a $(\varphi, N, L_\tau/L, R^{\lambda,\rho_\tau}_{\overline{\rho}}[1/p])$-module. Therefore, by definition, 5.1.1 will also respect the $\mathrm{Gal}(L_\tau/L)$-action. □

Finally, we consider a slight refinement of Kisin's deformation rings. Set $R_{\overline{\rho}}^{\lambda,\preceq\tau}$ be the the $\mathcal{O}$-flat reduced quotient of $R_{\overline{\rho}}^{\lambda,\rho_\tau}$ corresponding to the Zariski closure of

$$S_{\preceq\tau} := \{x \in \text{m-Spec}(R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p]) \mid \mathrm{WD}(\rho_x)|_{I_L} \preceq \tau\} =$$

$$\{x : R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p] \to E_x \mid E_x/E \text{ finite extension}, \mathrm{WD}(\rho_x)|_{I_L} \preceq \tau\}$$

in $\mathrm{Spec}(R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p])$. We check the only property of $R_{\overline{\rho}}^{\lambda,\preceq\tau}$ that we will need.

Proposition 5.1.3. *An E-algebra map $x : R_{\overline{\rho}}^{\lambda,\rho_\tau} \to E_x$ for a finite field extension E_x/E factors through $R_{\overline{\rho}}^{\lambda,\preceq\tau}$ if and only if* $\mathrm{WD}(\rho_x)|_{I_L} \preceq \tau$.

Proof. Note that it suffices to check that $S_{\preceq\tau} \subset \text{m-Spec}(R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p])$ is Zariski closed. Indeed, as then, $R_{\overline{\rho}}^{\lambda,\rho_\tau}$ being a Jacobson ring, the set of closed points of the Zariski closure of $S_{\preceq\tau}$ in $\mathrm{Spec}(R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p])$ must be $S_{\preceq\tau}$ itself (cf. [Sta24, Lemma 005Z]).

We further note that we are free to enlarge E as one checks that $R_{\overline{\rho},\mathcal{O}}^{\lambda,\rho_\tau} \otimes_{\mathcal{O}} \mathcal{O}_{E'} \cong R_{\overline{\rho},\mathcal{O}_{E'}}^{\lambda,\rho_\tau}$ for any finite extension E'/E. In particular, we may assume that the isotypic decomposition

$$V_{\rho_\tau} \cong \oplus_\theta V_{\rho_\tau}[\theta]$$

is defined over E where V_{ρ_τ} is the representation space of ρ_τ and the sum runs over absolutely irreducible E-representations of $I_{L_\tau/L}$.

According to Theorem 5.1.2, we have a rank n $(\varphi, N, L_\tau/L, R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p])$-module $D := D_{\mathrm{st},L_\tau,\overline{\rho}}^{\lambda,\rho_\tau}$. In particular, for a fixed $\iota_0 : L_{\tau,0} \hookrightarrow E$, we can introduce

$$W := D \otimes_{R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p]\otimes_{\mathbf{Q}_p} L_{\tau,0},\iota_0} R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p],$$

a projective rank n $R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p]$-module with a linear endomorphism N, and a commuting linear action of $I_{L_\tau/L}$. For any ring map $R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p] \to R$, we use the abbreviation $W_R := W \otimes_{R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p]} R$. For a closed point $x : R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p] \to E_x$, we further denote by (W_x, N_x) the corresponding specialisation and note that it is isomorphic to $\mathrm{WD}(\rho_x)|_{I_L}$.

We now pick an irreducible component $Z = \mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p])$ and prove the following lemma from which the proposition follows.

Lemma 5.1.4. *There is a Weil–Deligne inertial type of the form $\tau' = (\rho_\tau, N_{\tau'})$ such that the locus on* m-Spec(A) *where* $\mathrm{WD}(\rho_x)|_{I_L} \sim \tau'$ *is open and dense. Moreover, the locus on* m-Spec(A) *where* $\mathrm{WD}(\rho_x)|_{I_L} \preceq \tau$ *is Zariski closed and only non-empty if $\tau \preceq \tau'$.*

Proof. To prove the first statement, set $F = \mathrm{Frac}A$, and consider the base change W_F, a finite free F-module of rank n. Consider the decomposition into isotypic components

$$W_F = \oplus_\theta W_F[\theta]$$

with respect to the action of the inertia subgroup. Set $d_\theta = \dim_F W_F[\theta] = \dim_E V_\rho[\theta]$.

Consider the intersection $W_A[\theta] := W_A \cap W_F[\theta]$ and note that it is preserved by N and the action of $I_{L_\tau/L}$ as it holds for both members of the intersection. Moreover, we have the equality $W_A[\theta] \otimes_A F = W_F[\theta]$. In particular, $W_A[\theta]$ is generically both a free module of rank d_θ and a direct summand of W_A. In other words, we can find f'_θ such that the base change $W_{A[1/f'_\theta]}[\theta] = W_A[\theta] \otimes_A A[1/f'_\theta]$ is finite free over $A[1/f'_\theta]$ and is a direct summand of $W_{A[1/f'_\theta]}$. Again, it is preserved by the monodromy and the action of the inertia subgroup. Denote the restriction of N to the obtained subspace by $N[\theta]$.

Now we can apply Lemma 7.8.10 of [BC09] to get $f''_\theta \in A$ such that, for $f_\theta = f'_\theta f''_\theta$, the equivalence class of $N[\theta]_x$ is independent of $x \in \text{m-Spec}(A[1/f_\theta])$. The first part of the lemma follows by noting that $N[\theta]_x = N_x[\theta]$, the latter denoting the restriction of the specialisation of N_x to $W_x[\theta]$ where $W_x \sim \mathrm{WD}(\rho_x)|_{I_L}$. Denote the resulting Weil–Deligne inertial type by $\tau' = (\rho_\tau, N_{\tau'})$.

Now consider an arbitrary point $x \in \text{m-Spec}(A)$ with corresponding maximal ideal $\mathfrak{m}_x \subset A$, Galois representation ρ_x and residue field E_x, a finite extension of E. We prove that $\mathrm{WD}(\rho_x)|_{I_L} \preceq \tau'$. Consider the localisation $W_{\mathfrak{m}_x} := W_{A_{\mathfrak{m}_x}}$, and its base change $W^\wedge_{\mathfrak{m}_x}$ along $A_{\mathfrak{m}_x} \to A^\wedge_{\mathfrak{m}_x}$. Write $F_x := \mathrm{Frac} A^\wedge_{\mathfrak{m}_x}$ and $W^\wedge_{\mathfrak{m}_x}[\theta] := W^\wedge_{\mathfrak{m}_x} \cap W_{F_x}[\theta]$.

Since $I_{L_\tau/L}$ is a finite group, it has trivial higher cohomology in characteristic 0. In particular, the deformation $\rho_{A^\wedge_{\mathfrak{m}_x}} : I_{L_\tau/L} \to \mathrm{End}_{A^\wedge_{\mathfrak{m}_x}}(W^\wedge_{\mathfrak{m}_x})$ of $\mathrm{WD}(\rho_x)^{ss}\,|_{I_L}$ must be isomorphic to the extension of scalars of the latter along $E_x \to A^\wedge_{\mathfrak{m}_x}$. In other words, we have an isomorphism $W^\wedge_{\mathfrak{m}_x} \cong W_x \otimes_{E_x} A^\wedge_{\mathfrak{m}_x}$ of $I_{L_\tau/L}$-representations. In particular, $W^\wedge_{\mathfrak{m}_x}[\theta] \subset W^\wedge_{\mathfrak{m}_x}$ is a free direct summand for all isotypic component.

Set $N_{A^\wedge_{\mathfrak{m}_x}}[\theta] = N_{A^\wedge_{\mathfrak{m}_x}}|_{W^\wedge_{\mathfrak{m}_x}[\theta]}$. Note that $N_{A^\wedge_{\mathfrak{m}_x}}[\theta] \otimes_{A^\wedge_{\mathfrak{m}_x}} F_x = N_{F_x}[\theta] \sim N_F[\theta] \sim N_{\tau'}[\theta]$. Moreover, $N_{A^\wedge_{\mathfrak{m}_x}}[\theta] \otimes_{A^\wedge_{\mathfrak{m}_x}} E_x = N_x[\theta]$. Therefore, using Proposition 2.6.2 one sees that $N_x[\theta] \preceq N_{\tau'}[\theta]$ (cf. [Dot21], Lemma 4.2 and Remark 4.4). Running over all isotypic components, we obtain $\mathrm{WD}(\rho_x)|_{I_L}(\sim (W_x, N_x)) \preceq \tau'$.

We have proved so far that $\mathrm{Spec}(A)$ contains a dense open $D(\prod_\theta f_\theta)$ with closed points satisfying $\mathrm{WD}(\rho_x)|_{I_L} \sim \tau'$ and that, for every $x \in \text{m-Spec}(A)$ we have $\mathrm{WD}(\rho_x)|_{I_L} \preceq \tau'$. Set $\mathrm{Spec}(A^{\preceq\tau}) \subset \mathrm{Spec}(A)$ to be an irreducible component of the Zariski closure of

$$S_{\preceq\tau} = \{x \in \text{m-Spec}(A) | \mathrm{WD}(\rho_x)\,|_{I_L} \preceq \tau\}$$

in $\mathrm{Spec}(A)$ with its induced reduced subscheme structure. To conclude, it suffices to see that every closed point of $\mathrm{Spec}(A^{\preceq\tau})$ is of type $\preceq \tau$.

By considering the base change $W_{A_{\preceq\tau}}$ with the induced action of the inertia subgroup and nilpotent operator, and running the previous argument, we see that there is a Weil–Deligne inertial type $\widetilde{\tau}$ and an open dense subspace $\widetilde{U} \subset \mathrm{Spec}(A^{\preceq\tau})$ such that

i. every closed point in $\widetilde{U}$ is of type $\sim \widetilde{\tau}$, and

ii. every closed point in $\mathrm{Spec}(A^{\preceq\tau})$ is of type $\preceq \widetilde{\tau}$.

In particular, we must have $\widetilde{\tau} \preceq \tau$ and the proof is finished. ☐

☐

5.2 Interpolation of local Langlands

An essential ingredient to formulate torsion local-global compatibility is the existence of a (semisimple) local Langlands correspondence over the generic fiber of our potentially semistable deformation rings. Such a correspondence was already defined in [Car+16b] for the potentially crystalline deformation rings of Kisin. However, as it is shown in [Pyv20b], one can define it for the whole potentially semistable deformation ring following the same strategy. Let τ be a Weil–Deligne inertial type, Ω be the corresponding Bernstein block, and $\overline{\rho} : G_L \to \mathrm{GL}_n(k)$ be a continuous Galois representation as in the previous section.

Theorem 5.2.1. *There is an E-algebra homomorphism*

$$\eta : \mathfrak{z}_\Omega \to R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p]$$

such that, for any $x \in \text{m-Spec}(R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p])$ with residue field E_x (necessarily a finite extension of E), the character

$$x \circ \eta : \mathfrak{z}_\Omega \to E_x$$

coincides with the one induced by the natural action of $\mathfrak{z}_\Omega$ on $\mathrm{rec}^{-1}(\mathrm{WD}(\rho_x)^{F-ss}) \otimes |\det|_L^{\frac{n-1}{2}}$. In particular, we also have a map $\eta : \mathfrak{z}_\Omega \to R_{\overline{\rho}}^{\lambda,\preceq\tau}[1/p]$ with the same property.

Proof. The result follows from (the proof of) [Pyv20b], Theorem 3.3. We note that he states the existence of a map with source being a Hecke algebra $\mathcal{H}(\sigma_{\min})$ for a certain type $\sigma_{\min}$ of Schneider–Zink. This Hecke algebra in fact is isomorphic to $\mathfrak{z}_\Omega$ and, in the proof of Theorem 3.3 of *loc. cit.*, the map is first constructed from $\mathfrak{z}_\Omega$ and it is only at the very end that it is precomposed with the identification $\mathcal{H}(\sigma_{\min}) \cong \mathfrak{z}_\Omega$.

The rough sketch is as follows. The two key ingredients in the proof are:

i. The existence of a pseudo-representation $T : W_L \to \mathfrak{z}_\Omega$ constructed by Chenevier interpolating local Langlands (cf. [Che09], Proposition 3.11);

ii. and the existence of a universal $(\varphi, N, L_\tau/L, E)$-module over $R_{\overline{\rho}}^{\lambda,\rho_\tau}[1/p] \otimes_{\mathbf{Q}_p} L_{\tau,0}$ (cf. Theorem 5.1.2).

Then, for $w \in W_L$ it is clear that $\eta(T(w))$ should be (some normalisation of) the trace of w acting on the universal $(\varphi, N, L_\tau/L, E)$-module. In particular, we have defined η for every element in the image of T. Moreover, [Car+16b], Lemma 4.5 shows that $\mathfrak{z}_\Omega$ is generated (as an E-vector space) by the image of T. Therefore, we can linearly extend the domain of η from the image of T to the whole Bernstein centre. □

Remark 5.2.2. It is interesting to investigate on a possible integral avatar of η. For instance, one can ask under what circumstance does η send $\mathfrak{z}^\circ_{\lambda,\tau} := Z(\mathcal{H}(\sigma(\lambda,\tau)^\circ)) \cap \mathfrak{z}_\Omega$ into $R^{\lambda,\preceq\tau}_{\overline{\rho}}$? Moreover, is there a subring in $R^{\lambda,\preceq\tau}_{\overline{\rho}}[1/p]$, say finite over $R^{\lambda,\preceq\tau}_{\overline{\rho}}$, such that $\mathfrak{z}^\circ_{\lambda,\tau}$ always lands in this ring? This turns out to be a rather subtle question, and the answer in most cases depends on some folklore conjectures. More precisely, in [Car+16b] they remark that, at least in the potentially crystalline case, $\eta(\mathfrak{z}^\circ_{\lambda,\tau})$ should land in the normalisation of $R^{\lambda,\preceq\tau}_{\overline{\rho}}$ inside its generic fiber. This prediction is supported by what they see after patching (cf. *loc. cit.* Lemma 4.18, 3), and Remark 4.21). However, their observation can only be made into a rigorous proof for the components of the local deformation rings for which we know automorphy.

In some very special cases, it seems plausible that one can prove that η sends $\mathfrak{z}^\circ_{\lambda,\tau}$ into the normalisation by other means. Namely, when $n = 2$, ρ_τ is a tame type, $\tau = (\rho_\tau, 0)$, and we look at the potentialy Barsotti–Tate deformation ring $R^{0,\preceq\tau}_{\overline{\rho}}$, one can appeal to the work of Caraiani–Emerton–Gee–Savitt to realise the normalisation of the local deformation ring as the direct image of the Kisin variety[1] (parametrising Breuil–Kisin models of the deformations of $\overline{\rho}$) along its map towards $\mathrm{Spec}(R^{0,\preceq\tau}_{\overline{\rho}})$. Moreover, using the construction of the map, the fact that the Kisin variety parametrises Breuil–Kisin models of the deformations appearing in $\mathrm{Spec}(R^{0,\preceq\tau}_{\overline{\rho}})$, and comparisons between Breuil–Kisin modules and Weil–Deligne representations associated to the corresponding Galois representations, it seems possible to show that the image of $\mathfrak{z}^\circ_{\lambda,\tau}$ already lies in the direct image of the Kisin variety.

There is some further evidence already present in the literature. Namely, [Car+16a] Lemma 2.15 (under some further assumptions on $\overline{\rho}$) shows that when $n = 2$, $\tau = (\mathbf{1}, 0)$, $L = \mathbf{Q}_p$ and λ is in the Fontaine–Laffaille range, η sends $\mathfrak{z}^\circ_{\lambda,\tau}$ into $R^{\lambda,\preceq\tau}$. Moreover, up to a suitable completion of $\mathfrak{z}^\circ_{\lambda,\tau}$, the integral avatar of η becomes an isomorphism. Note that in this case $R^{\lambda,\preceq\tau}_{\overline{\rho}}$ is a regular local ring hence normal. In particular, their result is compatible with the prediction of [Car+16b], Remark 4.21.

---

[1] These spaces were first constructed in [Kis09a], in the Barsotti–Tate case, and have been constructed over any potentially Barsotti–Tate deformation ring associated with a tame type in [Car+21], §5 (where it is denoted by $X_{\overline{\tau}}$).

5.3 The local-global compatibility conjecture

Fix a dominant weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$ for G, and a collection of n-dimensional Weil–Deligne inertial types $\tau = \{\tau_v\}_{v \in S_p(F)}$. Set $\lambda_v := (\lambda_\iota)_{\iota : F_v \hookrightarrow E} \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F_v,E)}$. Recall that we introduced the abstract spherical Hecke algebra $\mathbf{T}^T$ and the corresponding faithful Hecke algebra $\mathbf{T}^T(K,\lambda,\tau)$ acting on $R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)})$. We further introduce a refinement of the abstract Hecke algebra that also consists of Hecke operators at p. To do this, note that since $\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_{F_v})}^{\mathrm{GL}_n(F_v)} \sigma(\lambda_v,\tau_v)^\circ$ is finitely generated as an $\mathcal{O}[\mathrm{GL}_n(F_v)]$-module, we have $\mathcal{H}(\sigma(\lambda_v,\tau_v)^\circ)[1/p] = \mathcal{H}(\sigma(\lambda_v,\tau_v))$. We have an identification

$$\left(\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_{F_v})}^{\mathrm{GL}_n(F_v)} \sigma(\tau_v)\right) \otimes_E V_{\lambda_v^\vee} \cong \text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_{F_v})}^{\mathrm{GL}_n(F_v)} \sigma(\lambda_v,\tau_v),$$

$$f \otimes w \mapsto [g \mapsto f(g) V_{\lambda_v^\vee}(g) w].$$

We can then define a natural map

$$\mathcal{H}(\sigma(\tau_v)) \to \mathcal{H}(\sigma(\lambda_v,\tau_v)),$$

$$\phi \mapsto \phi \otimes \mathrm{id}_{V_{\lambda^\vee}}$$

that, thanks to [ST06], Lemma 1.4, is an isomorphism of E-algebras. In particular, we have $\mathfrak{z}_{\tau_v} = Z(\mathcal{H}(\sigma(\lambda_v,\tau_v)^\circ))[1/p]$. Let Ω_v denote the Bernstein block corresponding to τ_v and set $\mathfrak{z}_{\Omega_v}$ to be the corresponding Bernstein centre. We then set

$$\mathfrak{z}^\circ_{\lambda_v,\tau_v} := Z(\mathcal{H}(\sigma(\lambda_v,\tau_v)^\circ)) \cap \mathfrak{z}_{\Omega_v} \subset \mathfrak{z}_{\tau_v},$$

a commutative $\mathcal{O}$-subalgebra and note that $\mathfrak{z}^\circ_{\lambda_v,\tau_v}[1/p] = \mathfrak{z}_{\Omega_v}$. Finally, set

$$\mathbf{T}^{T,\lambda,\tau} := \mathbf{T}^T \otimes_{\mathcal{O}} \big(\bigotimes_{v \in S_p(F)} \mathfrak{z}^\circ_{\lambda_v,\tau_v} \big),$$

a commutative $\mathcal{O}$-algebra. As a consequence of Lemma 2.3.1, Frobenius reciprocity, and [NT16], Lemma 3.11, we get a map of $\mathcal{O}$-algebras

$$\mathbf{T}^{T,\lambda,\tau} \to \mathrm{End}_{D^+(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)}))$$

for every good subgroup $K \subset G(\mathbf{A}^\infty_{F^+})$ with $K_p = \prod_{\bar{v} \in \overline{S}_p} G(\mathcal{O}_{F^+_{\bar{v}}})$. Denote by $\mathbf{T}^{T,\lambda,\tau}(K^p)$ the corresponding faithful Hecke algebra $\mathbf{T}^{T,\lambda,\tau}(R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)}))$. Since $R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)})$ is a perfect complex in $D^+(\mathcal{O})$, $\mathbf{T}^{T,\lambda,\tau}(K^p)$ is a commutative finite $\mathcal{O}$-algebra. In particular, we obtain a decomposition

$$\mathbf{T}^{T,\lambda,\tau}(K^p) = \prod_{\mathfrak{m}} \mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}}$$

where we run over all maximal ideals of $\mathbf{T}^{T,\lambda,\tau}(K^p)$. Therefore, for each $\mathfrak{m}$, we have a corresponding $\mathbf{T}^{T,\lambda,\tau}$-equivariant direct summand $R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)})_{\mathfrak{m}}$ of $R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)})$ and the natural map $\mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}} \to \mathbf{T}^{T,\lambda,\tau}(R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)})_{\mathfrak{m}})$ becomes an isomorphism (cf. [NT16], §3.2).

We recall the following conjecture (cf. [CG18], Conjecture B).

Conjecture 5.3.1 (Construction of torsion Galois representations)**.** Let $K \subset G(\mathbf{A}_{F^+}^{\infty})$ be any good subgroup and $\mathfrak{m} \subset \mathbf{T}^T(K,\lambda,\tau) = \mathbf{T}^T(R\Gamma(X_K, \mathcal{V}_{(\lambda,\tau)})_{\mathfrak{m}})$ be a maximal ideal. Then there exists a continuous semisimple Galois representation

$$\overline{\rho}_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^T(K,\lambda,\tau)/\mathfrak{m})$$

such that, for each finite place $v \notin T$ of F, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(\mathbf{T}^T(K,\lambda,\tau)/\mathfrak{m})[X]$.

Moreover, if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, then it admits a lift to a continuous homomorphism

$$\rho_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^T(K,\lambda,\tau)_{\mathfrak{m}})$$

such that, for each finite place $v \notin T$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $\mathbf{T}^T(K,\lambda,\tau)_{\mathfrak{m}}[X]$.

Since we have a natural map $\mathbf{T}^T(K,\lambda,\tau) \to \mathbf{T}^{T,\lambda,\tau}(K^p)$, assuming Conjecture 5.3.1, a maximal ideal $\mathfrak{m} \subset \mathbf{T}^{T,\lambda,\tau}(K^p)$ gives rise to a continuous semisimple Galois representation

$$\overline{\rho}_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^{T,\lambda,\tau}(K^p)/\mathfrak{m}).$$

Moreover, if we denote by $\mathfrak{m}^T$ the induced maximal ideal of $\mathbf{T}^T$, we have a natural inclusion

$$\mathbf{T}^T(K,\lambda,\tau)_{\mathfrak{m}^T} \hookrightarrow \mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}}.$$

In particular, assuming that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, Conjecture 5.3.1 provides a lift

$$\rho_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}})$$

of $\overline{\rho}_{\mathfrak{m}}$ satisfying the assertion of Conjecture 5.3.1.

To state the local-global compatibility conjecture, we need to further introduce the subring

$$\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v} := \eta^{-1}(R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}) \cap \mathfrak{z}^{\circ}_{\lambda_v,\tau_v} \subset \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}.$$

Note that we still have the property $\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}[1/p] = \mathfrak{z}_{\Omega_v}$.

Conjecture 5.3.2 (Torsion local-global compatibility at $\ell = p$)**.** Assume Conjecture 5.3.1. Let $\mathfrak{m} \subset \mathbf{T}^{T,\lambda,\tau}(K^p)$ be a non-Eisenstein maximal ideal. For $v \in S_p(F)$, set $\overline{\rho}_v := \overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ and $\rho_v := \rho_{\mathfrak{m}}|_{G_{F_v}}$. Then, for any $v \in S_p(F)$, there is a (necessarily unique) dotted arrow making the following diagram commutative

$$\begin{tikzcd}
R^{\square}_{\overline{\rho}_v} \arrow[r, "\rho_v"] \arrow[d, two heads] & \mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}} \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v} \arrow[ur, dotted] \arrow[d, hook] & \mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v} \arrow[u, "\mathrm{nat}"'] \arrow[l, "\eta|_{\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}}"] \arrow[d, hook] \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}[1/p] & \mathfrak{z}_{\Omega_v}. \arrow[l, "\eta"]
\end{tikzcd}$$

Remark 5.3.3. In the diagram above, we denote by nat the canonical map coming from the fact that the target of the map is a faithful Hecke algebra. We also note that the complication of introducing the ring $\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}$ originates in the fact the η does not necessarily send $\mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$ into $R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}$ (see Remark 5.2.2). We could also, instead of using the lower part of the diagram, simply just ask that whenever $z \in \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$ such that $\eta(z)$ lies in $R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}$, then $\mathrm{nat}(z)$ coincides with $\rho_v \circ \eta(z)$.

Note that since we have $\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}[1/p] = \mathfrak{z}_{\Omega_v}$, whenever we specialise to an E-algebra $\mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}} \to B$, we get that the induced natural $\mathfrak{z}_{\Omega_v}$-action coincides with the one induced by η.

We finally state our theorem that settles a large part of Conjecture 5.3.2 in the case when F is assumed to be an imaginary CM field.

Theorem 5.3.4. *Let F be an imaginary CM field that contains an imaginary quadratic field F_0 in which p splits. Assume that T is stable under complex conjugation and satisfies:*

- *Let $v \notin T$ be a finite place of F, with residue characteristic ℓ. Then either T contains no ℓ-adic places and ℓ is unramified in F, or there exists an imaginary quadratic subfield of F in which ℓ splits.*

Let $K \subset G(\mathbf{A}_F^\infty)$ be a good compact open subgroup with K^T hyperspecial. Fix a place $\bar{v} \in S_p(F^+)$. Let $\mathfrak{m} \subset \mathbf{T}^{T,\lambda,\tau}(K^p)$ be a non-Eisenstein maximal ideal. Assume that:

i. There is a place $\bar{v}' \in S_p(F^+)$ such that $\bar{v} \neq \bar{v}'$ and

$$\sum_{\bar{v}'' \neq \bar{v},\bar{v}'} [F^+_{\bar{v}''} : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}]$$

where the sum runs over $\bar{v}'' \in S_p(F^+)$;

ii. and $\overline{\rho}_{\mathfrak{m}}$ is decomposed generic.

Then there exists an integer $N \geq 1$, depending only on n and $[F^+ : \mathbf{Q}]$, a nilpotent ideal $I \subset \mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}}$ with $I^N = 0$ and a continuous homomorphism

$$\rho_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}}/I)$$

lifting $\overline{\rho}_{\mathfrak{m}}$ such that

- *for each finite place $v \notin T$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $\mathbf{T}^{T,\lambda,\tau}(K^p)[X]$.*

Moreover, if, for $v|\bar{v}$*, we set* $\overline{\rho}_v := \overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ *and* $\rho_v := \rho_{\mathfrak{m}}|_{G_{F_v}}$*, there is a (necessarily unique) dotted arrow making the following diagram commutative*

$$\begin{tikzcd}
R^{\square}_{\overline{\rho}_v} \arrow[r, "\rho_v"] \arrow[d, two heads] & \mathbf{T}^{T,\lambda,\tau}(K^p)_{\mathfrak{m}}/I \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v} \arrow[ur, dotted] \arrow[d, hook] & \mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v} \arrow[l, "\eta|_{\mathfrak{z}^{\circ,\mathrm{int}}_{\lambda_v,\tau_v}}"'] \arrow[u, "\mathbf{nat}"'] \arrow[d, hook] \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}[1/p] & \mathfrak{z}_{\Omega_v}. \arrow[l, "\eta"']
\end{tikzcd}$$

Remark 5.3.5. We say a few words about the assumptions appearing in Theorem 5.3.4.

The assumption on T is already present in [Sch15] and it makes sure that, up to nilpotent ideal, Conjecture 5.3.1 is known to be true (see Theorem 2.9.3 and Theorem 2.9.5). It is a rather mild condition that can always be fulfilled after enlarging T.

Assumption (i) is a lot more serious and excludes the case of imaginary quadratic fields. Its appearance originates in the use of the Fontaine–Laffaille style degree shifting argument (cf. [CN23], Lemma 4.2.5).

Finally, (ii) is again essential for our methods to work. It ensures that we can appeal to the vanishing results of Caraiani–Scholze. To our knowledge this is the only known way of producing the necessary congruences between cusp forms for $U(n,n)_{F^+}$ and Eisenstein series coming from cusp forms for $\mathrm{GL}_{n,F}$.

Remark 5.3.6. Although we don't discuss it here, one could possibly formulate a more general conjecture treating the case when our choice of parabolic subgroup $Q_p = \prod_v Q_v \subset \prod_v \mathrm{GL}_{n,F_v}$ is not the trivial one, and the complexes appearing are Q_p-ordinary. However, to do this, one would need to introduce the appropriate potentially semistable Q_v-ordinary deformation rings. This should be possible using ideas of [Ger18], §3.3.

Nevertheless, in the next chapter we will prove a general result (Proposition 6.3.1) that has the potential to settle the more general conjecture too, once it's formulated.

Chapter 6

Proof of Theorem 5.3.4

In this chapter, we provide a proof of Theorem 5.3.4 refining the argument of [CN23]. In fact, we prove a local-global compatibility result for the more general Q_p-ordinary complexes considered in §3.2. For the rest of the chapter, we fix the following setup. Let $n \geq 2$ be an integer, F be an imaginary CM field, and $T \subset S(F)$ as in Theorem 5.3.4. In particular, we assume Assumption 2.7.3. Fix a choice of lift $v|\bar{v}$ in F for each $\bar{v} \in \overline{S}_p$. We consider the corresponding groups G, P, and $\widetilde{G}$ as before. Moreover, we fix a tuple $(Q_p, \lambda, \underline{\tau}) = (Q_v, \lambda_v, \underline{\tau_v})_{v \in S_p(F)}$ as in §2.7.

6.1 Degree shifting

One essential step in executing the strategy laid out in [All+23] is what they call the "degree shifting argument". The most robust version of which appears in [CN23]. In fact, besides some enrichment of the method, their argument is already sufficient for our proof. We first state the precise statement we need and then indicate the changes to the argument of [CN23] one needs to prove our generalisation of *loc. cit.* Proposition 4.2.6.

Fix a place $\bar{v} \in \overline{S}_p$, and assume that, under the identification of §2.7 $(\lambda_v, \lambda_{v^c})$ corresponds to a dominant weight $\tilde{\lambda}_{\bar{v}} = (-w_0^{\mathrm{GL}_n}\lambda_{v^c}, \lambda_v) \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F_{\bar{v}}^+, E)}$. Extend it to an arbitrary dominant weight $\widetilde{\lambda}$ for $\widetilde{G}$. Set $Q_{\bar{v}} := Q_v \times Q_{v^c} \subset G_{F_{\bar{v}}^+} = \mathrm{GL}_{n,F_v} \times \mathrm{GL}_{n,F_{v^c}}$, and consider the standard parabolic subgroup $\widetilde{Q}_{\bar{v}}^{w_0} = \widetilde{M}_{\bar{v}}^{w_0} \ltimes \widetilde{N}_{\bar{v}}^{w_0} \subset \widetilde{G}_{F_{\bar{v}}^+}$ associated to it (cf. §2.7). We then obtain a tuple $(\widetilde{Q}_{\bar{v}}^{w_0}, \tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})$. Write $Q_v = P_{(n_1,...,n_k)}$, and $Q_{v^c} = P_{(m_1,....,m_{k^c})}$. Accordingly, write

$$\underline{\tau_v} = (\tau_{v,1}, ..., \tau_{v,k}),\ \underline{\tau_{v^c}} = (\tau_{v^c,1}, ..., \tau_{v^c,k^c}),$$
$$w_0^{Q_v}\lambda_v = (\lambda_{v,1}, ..., \lambda_{v,k}) \in (\mathbf{Z}_+^{n_k})^{\mathrm{Hom}(F_v,E)} \times ... \times (\mathbf{Z}_+^{n_1})^{\mathrm{Hom}(F_v,E)}, \text{ and}$$
$$w_0^{Q_{v^c}}\lambda_{v^c} = (\lambda_{v^c,1}, ..., \lambda_{v^c,k^c}) \in (\mathbf{Z}_+^{m_{k^c}})^{\mathrm{Hom}(F_{v^c},E)} \times ... \times (\mathbf{Z}_+^{m_1})^{\mathrm{Hom}(F_{v^c},E)}.$$

Set

$$\mathfrak{z}^{\circ}_{\lambda_v,\underline{\tau_v}} := \otimes_{i=1}^{k}\mathfrak{z}^{\circ}_{\lambda_{v,i},\tau_{v,i}},\ \mathfrak{z}^{\circ}_{\lambda_{v^c},\underline{\tau_{v^c}}} := \otimes_{i=1}^{k^c}\mathfrak{z}^{\circ}_{\lambda_{v^c,i},\tau_{v^c,i}},$$

and introduce the abstract Hecke algebras

$$\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}} := \widetilde{\mathbf{T}}^{T} \otimes_{\mathcal{O}} \left(\mathfrak{z}^{\circ}_{\lambda_v,\underline{\tau_v}} \otimes \mathfrak{z}^{\circ}_{\lambda_{v^c},\underline{\tau_{v^c}}}\right),$$

$$\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}} := \mathbf{T}^{T} \otimes_{\mathcal{O}} \left(\mathfrak{z}^{\circ}_{\lambda_v,\underline{\tau_v}} \otimes \mathfrak{z}^{\circ}_{\lambda_{v^c},\underline{\tau_{v^c}}}\right).$$

By Corollary 3.2.6 and its analogue for $\widetilde{G}$, $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$ respectively, $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$ naturally acts on $R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}$ respectively, $R\Gamma(X_K, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})})^{Q_{\bar{v}}\text{-ord}}$ as long as $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(0,c)$, and $K_{\bar{v}} = \mathcal{Q}_{\bar{v}}(0,c)$ with $c \geq c_p$. More precisely, in the case of the former, the action of $\mathfrak{z}^{\circ}_{\lambda_{v^c},\underline{\tau_{v^c}}}$ is via the identification $\mathcal{H}(\sigma(\lambda_{v^c},\underline{\tau_{v^c}})^{\circ}) \cong \mathcal{H}((\theta_n^{-1})^{*}\sigma(\lambda_{v^c},\underline{\tau_{v^c}})^{\circ})$ induced by θ_n (see Remark 3.4.2). We introduce an extension of the unnormalised Satake transform

$$\mathcal{S}^{\bar{v}} := \mathcal{S} \otimes \mathrm{id} : \widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}} \to \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}.$$

Let $\mathfrak{m} \subset \mathbf{T}^{T}$ be a non-Eisenstein maximal ideal, and set $\widetilde{\mathfrak{m}} := \mathcal{S}^{*}(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^{T}$. By Theorem 2.9.6, we have an associated Galois representation

$$\overline{\rho}_{\widetilde{\mathfrak{m}}} : G_{F,T} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}^{T}/\widetilde{\mathfrak{m}})$$

and, by Proposition 2.8.1, we have $\overline{\rho}_{\widetilde{\mathfrak{m}}} = \overline{\rho}_{\mathfrak{m}} \oplus \overline{\rho}_{\mathfrak{m}}^{\vee,c}(1-2n)$. Introduce the faithful Hecke algebras

$$A(K,\lambda,\underline{\tau},q,\bar{v}) := \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}(H^{q}(X_K, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})})^{Q_{\bar{v}}\text{-ord}}_{\mathfrak{m}}),$$

$$A(K,\lambda,\underline{\tau},q,\bar{v},m) := \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}(H^{q}(X_K, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^{m})^{Q_{\bar{v}}\text{-ord}}_{\mathfrak{m}}), \text{ and}$$

$$\widetilde{A}(\widetilde{K},\tilde{\lambda},\underline{\tau_{\bar{v}}},\bar{v}) := \widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\tau_{\bar{v}}}(H^{d}(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}})$$

for $d = \dim_{\mathbf{C}} \widetilde{X}_{\widetilde{K}}$, the middle degree for the (Betti) cohomology of $\widetilde{X}_{\widetilde{K}}$ and integers $0 \leq q \leq d-1$.[1]

Given a good subgroup $K \subset G(\mathbf{A}^{\infty}_{F^+})$, a subset $\overline{S} \subset \overline{S}_p$, and an integer $m \in \mathbf{Z}_{\geq 1}$, define the subgroup $K(m,\overline{S}) \subset K$ by setting

$$K(m,\overline{S})_{\bar{v}} := K_{\bar{v}} \cap K^{m}_{\bar{v}}$$[2]

if $\bar{v} \in \overline{S}$ and $K(m,\overline{S})_{\bar{v}} = K_{\bar{v}}$ otherwise. Also, given a good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^{\infty}_{F^+})$, define the good subgroup $\widetilde{K}(m,\overline{S}) \subset \widetilde{K}$ by setting

$$\widetilde{K}(m,\overline{S})_{\bar{v}} := \widetilde{K}_{\bar{v}} \cap \mathcal{P}_{\bar{v}}(m,m)$$

if $\bar{v} \in \overline{S}$, and $\widetilde{K}(m,\overline{S})_{\bar{v}} := \widetilde{K}_{\bar{v}}$ otherwise.

---

[1] Note that the real dimension of X_K is $d-1$ so its Betti cohomology has top degree $d-1$.

[2] Recall that $K^{m}_{\bar{v}} := \ker(G(\mathcal{O}_{F^+_{\bar{v}}}) \to G(\mathcal{O}_{F^+_{\bar{v}}}/\varpi^{m}_{\bar{v}}))$.

Proposition 6.1.1 (Degree shifting). *Let $\bar{v}, \bar{v}'$ be two distinct places of $\overline{S}_p$. Let $\overline{S}_1 := \{\bar{v}'\}$, $\overline{S}_3 := \{\bar{v}\}$, and $\overline{S}_2 := \overline{S}_p \setminus \{\bar{v}, \bar{v}'\}$ their complement. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^{\infty})$ be a good subgroup and $m \in \mathbf{Z}_{\geq 1}$ be an integer. Assume that the following conditions are satisfied.*

i. We have
$$\sum_{\bar{v}'' \in \overline{S}_2} [F_{\bar{v}''}^+ : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}].$$

ii. For $\bar{v}'' \in \overline{S}_1 \cup \overline{S}_2$, we have $U(\mathcal{O}_{F_{\bar{v}''}^+}) \subset \widetilde{K}_{\bar{v}''}$, and $\widetilde{K}_{\bar{v}''} = \widetilde{K}(m, \overline{S}_1 \cup \overline{S}_2)_{\bar{v}''}$. Finally, we have $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}_{\bar{v}}^{w_0}(0, c_p)$.

iii. For each $\iota : F \hookrightarrow E$ inducing $\bar{v}$ or $\bar{v}'$, we have $-\lambda_{\iota c,1} - \lambda_{\iota,1} \geq 0$.

iv. $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.

Define a weight $\tilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ as follows: if $\iota : F^+ \hookrightarrow E$ does not induce either $\bar{v}$ or $\bar{v}'$, set $\tilde{\lambda}_\iota = 0$. Otherwise, set $\tilde{\lambda}_\iota = (-w_{0,n}\lambda_{\iota c}, \lambda_\iota)$. Set $K := (\widetilde{K}^{\bar{v}} \cap G(\mathbf{A}_{F^+}^{\infty,\bar{v}})) \cdot \big(\widetilde{\mathcal{Q}}_{\bar{v}}(0, c_p) \cap G(F_{\bar{v}}^+)\big) = (\widetilde{K}^{\bar{v}} \cap G(\mathbf{A}_{F^+}^{\infty,\bar{v}})) \cdot \big(\mathcal{Q}_v(0, c_p) \times \mathcal{Q}_{v^c}(0, c_p)\big)$.

Let $q \in [\lfloor \frac{d}{2} \rfloor, d-1]$. Then there exist an integer $m' \geq m$, an integer $N \geq 1$, a nilpotent ideal $I \subset A(K, \lambda, \underline{\tau}, q, \bar{v}, m)$ satisfying $I^N = 0$, and a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}} \arrow[r] \arrow[d, "\mathcal{S}^{\bar{v}}"] & \widetilde{A}(\widetilde{K}(m', \overline{S}_2), \tilde{\lambda}, \underline{\tau_{\bar{v}}}, \bar{v}) \arrow[d] \\
\mathbf{T}^{T,\lambda_{\bar{v}},\tau_{\bar{v}}} \arrow[r] & A(K, \lambda, \underline{\tau}, q, \bar{v}, m)/I.
\end{tikzcd}$$

Moreover, N can be chosen to only depend on n and $[F^+ : \mathbf{Q}]$.

As a preliminary step, one proves the following lemma (compare with [CN23], Proposition 4.2.2). This already consists of (one of) the ideas coming from the ordinary degree shifting argument. Moreover, this lemma is one of the points where the main result of [CS19], in particular the decomposed generic assumption gets used. Finally, for this lemma to hold it is crucial that $\mathfrak{m}$ is non-Eisenstein as otherwise there is possible contribution to the boundary cohomology from strata corresponding to parabolic subgroups other than P and consequently, [CN23], Corollary 4.1.9 could fail to hold.

Before stating the lemma, we need to introduce some notation. Given a subset $\overline{S} \subset \overline{S}_p$ and an integer $m \geq 1$, set $\mathcal{V}_U(\overline{S}, m) := R\Gamma(U(\mathcal{O}_{F^+,\overline{S}}), \mathcal{O}/\varpi^m) \in D_{\mathrm{sm}}^b(\mathcal{O}/\varpi^m[K_{\overline{S}}])$. We can view it as an object of $D_{\mathrm{sm}}^b(\mathcal{O}/\varpi^m[G^{\overline{S}} \times K_{\overline{S}}])$ via inflation. In particular, it gives rise to a bounded complex of $G^{\overline{S}} \times K_{\overline{S}}$-equivariant sheaves on $\overline{\mathfrak{X}}_G$ and descends to an object in $D^b(\mathrm{Sh}(X_K, \mathcal{O}/\varpi^m))$ for any good subgroup $K \subset G(\mathbf{A}_{F^+}^{\infty})$. It has locally constant cohomology sheaves $\mathcal{V}_U^j(\overline{S}, m)$ that are non-zero if and only if $j \in [0, n^2 \sum_{\bar{v} \in \overline{S}} [F_{\bar{v}}^+ : \mathbf{Q}_p]]$

according to Lemma 2.3.17 of [CN23] (combined with the Künneth formula). By *loc. cit.* Lemma 2.1.9, we have

$$R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m) \otimes^{\mathbf{L}} \mathcal{V}_U(\overline{S}, m)^3 \cong R\Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}_U(\overline{S}, m))$$

in $D^b_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G^{\overline{S}} \times K_{\overline{S}}])$. In particular, by the proof of Corollary 3.2.6, the complex $R\Gamma(X_K, \mathcal{V}_U(\overline{S}, m) \otimes \mathcal{V}^{Q_{\overline{S}_p \setminus \overline{S}}}_{(\lambda_{\overline{S}_p \setminus \overline{S}}, \underline{\tau_{\overline{S}_p \setminus \overline{S}}})}/\varpi^m)^{Q_{\bar{v}}\text{-ord}}$ carries a natural action of $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$ for any $\bar{v} \in \overline{S}_p \setminus \overline{S}$ such that $K_{\bar{v}} = \mathcal{Q}_{\bar{v}}(0, c_p)$. We can pass to the homotopy limit over m to get $\mathcal{V}_U(\overline{S}) \in D^b(\mathrm{Sh}(X_K, \mathcal{O}))$ which once again has locally constant cohomology sheaves $\mathcal{V}^j_U(\overline{S})$. Assuming that for $\bar{v} \in \overline{S}_p \setminus \overline{S}$ we have $K_{\bar{v}} = \mathcal{Q}_{\bar{v}}(0, c_p)$, we get a natural action of $\mathbf{T}^{T,\lambda_{\bar{v}},\tau_{\bar{v}}}$ on $R\Gamma(X_K, \mathcal{V}_U(\overline{S}) \otimes \mathcal{V}^{Q_{\overline{S}_p \setminus \overline{S}}}_{(\lambda_{\overline{S}_p \setminus \overline{S}}, \underline{\tau_{\overline{S}_p \setminus \overline{S}}})})^{Q_{\bar{v}}\text{-ord}}$ by passing to the limit over m.

Lemma 6.1.2. *Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^{\infty}_{F^+})$ be a good subgroup that is decomposed with respect to P such that, for each $\bar{v} \in \overline{S}_p$, $\widetilde{K}_{\bar{v}} \cap U(F^+_{\bar{v}}) = U(\mathcal{O}_{F^+_{\bar{v}}})$. Let $\mathfrak{m} \subset \mathbf{T}^T$ be a non-Eisenstein maximal ideal, let $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^T$ and assume that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.*

Fix places $\bar{v}, \bar{v}' \in \overline{S}_p$ and introduce $\overline{S}_1, \overline{S}_2$ and $\overline{S}_3$ as before. Let $(Q_p, \lambda, \underline{\tau})$ be a tuple as in §2.7, and let $(\widetilde{Q}^{w_0}_{\bar{v}}, \tilde{\lambda}, \underline{\tau_{\bar{v}}})$ be the tuple associated to it as in Proposition 6.1.1. Assume that $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}^{w_0}_{\bar{v}}(0, c_p)$ and define K as in Proposition 6.1.1. Then the Satake transform $\mathcal{S}^{\bar{v}} : \widetilde{\mathbf{T}}^{T,\tilde{\lambda}_{\bar{v}},\underline{\tau_{\bar{v}}}} \to \mathbf{T}^{T,\tilde{\lambda}_{\bar{v}},\underline{\tau_{\bar{v}}}}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}}) \to$$

$$\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}\Big(\mathbf{H}^d\big(X_K, \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}_U(\overline{S}_2) \otimes \mathcal{V}^{Q_{\bar{v}}}_{(\lambda_{\bar{v}}, \tau_{\bar{v}})}\big)^{Q_{\bar{v}}\text{-ord}}_{\mathfrak{m}}\Big)$$

where $\mathbf{H}^d$ denotes the degree d hypercohomology.

Proof. By Theorem 2.9.9, the genericity of $\widetilde{\mathfrak{m}}$ implies that we have a $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant surjection

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}} \twoheadrightarrow H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}}.$$

In particular, it suffices to prove that $\mathcal{S}^{\bar{v}}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}(H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}}) \to \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}\Big(\mathbf{H}^d\big(X_K, \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}_U(\overline{S}_2) \otimes \mathcal{V}^{Q_{\bar{v}}}_{(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})}\big)^{Q_{\bar{v}}\text{-ord}}_{\mathfrak{m}}\Big).$$

Consider $\pi_{\partial}(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m) := R\Gamma(\partial\widetilde{X}_{\widetilde{K}^{\bar{v}}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m) \in D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F^+_{\bar{v}})])$, the $\bar{v}$-completed boundary cohomology. We then have a $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant isomorphism

$$H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})}/\varpi^m)^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}} \cong$$

---

[3] See Page 13 of [CN23] for how to define this derived tensor product.

$$\mathbf{R}^d\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}_{\bar{v}}^{w_0,0}]}\Big(\widetilde{\sigma}(\tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ/\varpi^m, R\Gamma\big(\widetilde{N}_{\bar{v}}^{w_0}(\mathcal{O}_{F_{\bar{v}}^+}), \pi_\partial(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\widetilde{\lambda}^{\bar{v}}}/\varpi^m)\big)^{\widetilde{Q}_{\bar{v}}^{w_0}\text{-ord}}\Big).$$

On the other hand, [CN23], Corollary 4.1.9 shows that $\pi_\partial(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\widetilde{\lambda}^{\bar{v}}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$ admits

$$\mathcal{S}^*\mathrm{Ind}_{P(F_{\bar{v}}^+)}^{\widetilde{G}(F_{\bar{v}}^+)} R\Gamma(\overline{X}_{K^{\bar{v}}}, \mathcal{V}_{\lambda_{\bar{v}'}}/\varpi^m \otimes \mathcal{V}_U(\overline{S}_2, m))_{\mathfrak{m}}$$

as a $\widetilde{\mathbf{T}}^T$-equivariant direct summand in $D^+_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_{\bar{v}}^+)])$. To simplify notation, set $\pi := \pi(K^{\bar{v}}, \mathcal{V}_{\lambda_{\bar{v}'}}/\varpi^m \otimes \mathcal{V}_U(\overline{S}_2, m))_{\mathfrak{m}}$. Then, by the previous discussion, $H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}_{\bar{v}}^{w_0}}_{(\widetilde{\lambda}, \underline{\tau_{\bar{v}}})}/\varpi^m)_{\widetilde{\mathfrak{m}}}^{\widetilde{Q}_{\bar{v}}^{w_0}\text{-ord}}$ admits

$$\mathbf{R}^d\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}_{\bar{v}}^{w_0,0}]}\Big(\widetilde{\sigma}(\tilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ/\varpi^m, R\Gamma\big(\widetilde{N}_{\bar{v}}^{w_0}(\mathcal{O}_{F_{\bar{v}}^+}), \mathcal{S}^*\mathrm{Ind}_{P(F_{\bar{v}}^+)}^{\widetilde{G}(F_{\bar{v}}^+)}\pi\big)^{\widetilde{Q}_{\bar{v}}^{w_0}\text{-ord}}\Big)$$

as a $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant direct summand.

By Corollary 3.5.7, the latter admits

$$\mathbf{R}^d\mathrm{Hom}_{\mathcal{O}/\varpi^m[M_{\bar{v}}^0]}\Big(\sigma(\lambda_v, \underline{\tau_v})^\circ \otimes \sigma(\lambda_{v^c}, \underline{\tau_{v^c}})^\circ/\varpi^m, R\Gamma\big(N_v(\mathcal{O}_{F_v}) \times N_{v^c}(\mathcal{O}_{F_{v^c}}), \pi\big)^{Q_v \times Q_{v^c}\text{-ord}}\Big)$$

$$\cong \mathbf{H}^d(X_K, \mathcal{V}_{\lambda_{\bar{v}'}} \otimes \mathcal{V}_U(\overline{S}_2) \otimes \mathcal{V}^{Q_{\bar{v}}}_{(\lambda_{\bar{v}}, \tau_{\bar{v}})}/\varpi^m)_{\mathfrak{m}}^{Q_{\bar{v}}\text{-ord}}$$

as a $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant subquotient, giving the desired map with mod ϖ^m coefficients. Here we implicitly used the identification $\sigma(\lambda_v, \underline{\tau_v})^\circ \otimes \sigma(\lambda_{v^c}, \underline{\tau_{v^c}})^\circ \cong \tau_{w_0^P}^{-1}\widetilde{\sigma}(\widetilde{\lambda}_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ$ induced by $\iota_v : \widetilde{G}(F_{\bar{v}}^+) \cong \mathrm{GL}_{2n}(F_v)$.

We finally note that these identifications are compatible when we vary m, and, since all the cohomology groups appearing are finitely generated $\mathcal{O}$-modules, we can conclude by passing to the limit over $m \geq 1$. □

Proof of Proposition 6.1.1. This is a generalisation of [CN23], Proposition 4.2.6. More precisely, the role of the faithful Hecke algebras $A(K, \lambda, q)$, $A(K, \lambda, q, m)$, and $\widetilde{A}(\widetilde{K}, \tilde{\lambda}, \bar{v})$ of *loc. cit.* are now played by $A(K, \lambda, \underline{\tau}, q, \bar{v})$, $A(K, \lambda, \underline{\tau}, q, \bar{v}, m)$, and $\widetilde{A}(\widetilde{K}, \tilde{\lambda}, \underline{\tau_{\bar{v}}}, \bar{v})$. Consequently, in our case $\mathbf{T}$, respectively $\widetilde{\mathbf{T}}$ will denote $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$, respectively $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$ and our goal is to show the existence of non-negative integers $m' \geq m$, and N such that

$$\mathcal{S}^{\bar{v}}(\mathrm{Ann}_{\widetilde{\mathbf{T}}} H^d(\widetilde{X}_{\widetilde{K}(m',\overline{S}_2)}, \mathcal{V}^{\widetilde{Q}_{\bar{v}}^{w_0}}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})})_{\widetilde{\mathfrak{m}}}^{\widetilde{Q}_{\bar{v}}^{w_0}\text{-ord}})^N \subset \mathrm{Ann}_{\mathbf{T}} H^q(X_K, \mathcal{V}^{Q_p}_{(\lambda, \underline{\tau})}/\varpi^m)_{\mathfrak{m}}^{Q_{\bar{v}}\text{-ord}}.$$

Regardless the change of setup, the proof is identical to that of *loc. cit.* In particular, we only indicate the necessary new inputs for the argument to work in our case and direct the reader to [CN23] for the proof.

- The proof uses Poincaré duality at certain points. To be able to appeal to Poincaré duality in our case, we need that it is Hecke-equivariant also at $\bar{v}$. This is the content of Proposition 3.3.5.

- To deepen the level at certain steps, their proof uses the Hochschild–Serre spectral sequence, and we need to verify that all of these spectral sequences are $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant. However, this is clear as we only have to go deeper level at places in $\overline{S}_2$ and our Hecke operators are at places in $T \cup \{\bar{v}\}$.

- We also need to argue that the hypercohomology spectral sequences with respect to $\mathcal{V}_U(\overline{S}_2)$ and $\mathcal{V}_U(\overline{S}_2, m)$ (that are denoted in [CN23] by $E_n^{i,j}(\mathcal{O})$ resp. $E_n^{i,j}(\mathcal{O}/\varpi^m)$) are $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant. To see this, we note that the $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-action on the target of the map in Lemma 6.1.2 is induced by the identification

  $$R\Gamma(X_K, \mathcal{V}_{\lambda_{\bar{v}'}}/\varpi^m \otimes \mathcal{V}_U(\overline{S}_2, m) \otimes \mathcal{V}^{Q_{\bar{v}}}_{(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})})^{Q_{\bar{v}}\text{-ord}} \cong$$

  $$\cong R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0_{\overline{v}}]}(\sigma(\lambda_{\overline{v}}, \underline{\tau_{\overline{v}}})^\circ/\varpi^m, \pi^{Q_{\overline{v}}\text{-ord}}(K^{\overline{v}}, \mathcal{V}_{\lambda_{\overline{v}'}} \otimes \mathcal{V}_U(\overline{S}_2, m)))$$

  in $D^+(\mathcal{O}/\varpi^m)$. We can then construct $E_n^{i,j}(\mathcal{O}/\varpi^m)$ of *loc. cit.* by taking the hypercohomology spectral sequence of

  $$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0_{\overline{v}}]}(\sigma(\lambda_{\overline{v}}, \underline{\tau_{\overline{v}}})^\circ/\varpi^m, \pi^{Q_{\overline{v}}\text{-ord}}(K^{\overline{v}}, \mathcal{V}_{\lambda_{\overline{v}'}} \otimes -)) \cong$$

  $$R\left(\mathrm{Hom}_{\mathcal{O}/\varpi^m[M^0_{\overline{v}}]}\left(\sigma(\lambda_{\overline{v}}, \underline{\tau_{\overline{v}}})^\circ/\varpi^m, \Gamma\left(N^0_{\overline{v}}, \Gamma\left(K^{\overline{v}}, \Gamma(\overline{\mathfrak{X}}_G, \mathcal{V}_{\lambda_{\overline{v}'}} \otimes -)\right)\right)^{Q_{\bar{v}}\text{-ord}}\right)\right)$$

  applied to $\mathcal{V}_U(\overline{S}_2, m) \in D^b\mathrm{Sh}_{G^{\overline{S}_2} \times K_{\overline{S}_2}}(\overline{\mathfrak{X}}_G, \mathcal{O}/\varpi^m)$. In particular, it will be $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariant by construction. Since all the members of $E_n^{i,j}(\mathcal{O}/\varpi^m)$ are finite $\mathcal{O}/\varpi^m$-modules, the Mittag–Leffler condition is satisfied and, in particular, the limit of the spectral sequences $E_n^{i,j}(\mathcal{O}/\varpi^m)$ over $m \geq 1$ produces $E_n^{i,j}(\mathcal{O})$.

- As input, we use Lemma 6.1.2 instead of *loc. cit.* Proposition 4.2.2.

□

Just as in [CN23], we will need a dual version of the degree shifting argument. We only explain the setup and the statements here as the proofs are identical to that of Proposition 6.1.1, and Lemma 6.1.2. Set $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\tilde{\iota}}$, respectively $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\iota}$ to be the image of $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$, respectively $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$ under the anti-isomorphism

$$\tilde{\iota} : \widetilde{\mathbf{T}}^T \otimes_{\mathcal{O}} \mathcal{H}(\sigma(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ) \cong \widetilde{\mathbf{T}}^T \otimes_{\mathcal{O}} \mathcal{H}(\sigma(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})^{\circ,\vee}),$$

respectively

$$\iota : \mathbf{T}^T \otimes_{\mathcal{O}} \mathcal{H}(\sigma(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ) \cong \mathbf{T}^T \otimes_{\mathcal{O}} \mathcal{H}(\sigma(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})^{\circ,\vee})$$

given by $[g,\psi] \mapsto [g^{-1},\psi^t]$ on double coset operators.[4] We denote by $\mathcal{S}^{\bar{v},\vee} : \widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\tilde{\iota}} \to \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\iota}$ the extension of the unnormalised Satake transform given by $\mathcal{S}\otimes\mathrm{id}$. We consider a tuple $(Q_p,\lambda,\underline{\tau})$ as before, but now, just as in the last paragraph of §2.7, we assume that $\widetilde{\lambda} := (-w_0^{\mathrm{GL}_n}\lambda_{v^c},\lambda_v)$ is dominant for $\widetilde{G}$ (instead of $w_0^P\widetilde{\lambda}_{\bar{v}}$). We also pick some dominant weight $\tilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ for $\widetilde{G}$ extending $\tilde{\lambda}_{\bar{v}}$. We consider the dual local systems $\mathcal{V}_{(\lambda,\underline{\tau})}^{Q_p,\vee}$, and $\mathcal{V}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})}^{\widetilde{Q},w_0,\vee}$. Then $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\tilde{\iota}}$, respectively $\mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\iota}$ acts on $R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{V}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}},w_0,\vee}/\varpi^m)^{\overline{\widetilde{Q}_{\bar{v}}}\text{-ord}}$, respectively $R\Gamma(X_K,\mathcal{V}_{(\lambda,\underline{\tau})}^{Q_p,\vee}/\varpi^m)^{\overline{Q_{\bar{v}}}\text{-ord}}$ when $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup with $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}_{\bar{v}}(0,c_p)$, and $K = \widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty)$. In the case of the latter, it follows from Corollary 3.3.4. The action in the case of the former comes from the identification

$$R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{V}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}},w_0,\vee}/\varpi^m)^{\overline{\widetilde{Q}_{\bar{v}}}\text{-ord}} \cong$$

$$R\mathrm{Hom}_{\mathcal{O}/\varpi^m[\widetilde{M}_{\bar{v}}^0]}(\theta_n^{-1}\sigma(\lambda_{v^c},\tau_{v^c})^{\circ,\vee} \otimes \sigma(\lambda_v,\tau_v)^{\circ,\vee},\pi^{\overline{\widetilde{Q}_{\bar{v}}}\text{-ord}}(\widetilde{K}^{\bar{v}},\mathcal{V}_{\tilde{\lambda}^{\bar{v}}}^\vee/\varpi^m))$$

in $D^+(\mathcal{O}/\varpi^m)$ and the isomorphism $\mathcal{H}(\sigma(\lambda_v,\underline{\tau_v})^{\circ,\vee}) \otimes \mathcal{H}(\sigma(\lambda_{v^c},\underline{\tau_{v^c}})^{\circ,\vee}) \cong \mathcal{H}(\theta_n^{-1}\sigma(\lambda_{v^c},\underline{\tau_{v^c}})^{\circ,\vee}) \otimes \mathcal{H}(\sigma(\lambda_v,\underline{\tau_v})^{\circ,\vee})$ induced by ι_v.

Given a maximal ideal $\mathfrak{m} \subset \mathbf{T}^T$, set $\mathfrak{m}^\vee := \iota|_{\mathbf{T}^T}(\mathfrak{m}) \subset \mathbf{T}^T$. Define the faithful Hecke algebras

$$A^\vee(K,\lambda,\underline{\tau},q,\bar{v}) := \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\iota}(H^q(X_K,\mathcal{V}_{(\lambda,\underline{\tau})}^{Q_p,\vee})_{\mathfrak{m}^\vee}^{\overline{Q_{\bar{v}}}\text{-ord}}),$$

$$A^\vee(K,\lambda,\underline{\tau},q,\bar{v},m) := \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\iota}(H^q(X_K,\mathcal{V}_{(\lambda,\underline{\tau})}^{Q_p,\vee}/\varpi^m)_{\mathfrak{m}^\vee}^{\overline{Q_{\bar{v}}}\text{-ord}}), \text{ and}$$

$$\widetilde{A}^\vee(\widetilde{K},\tilde{\lambda},\underline{\tau_{\bar{v}}},\bar{v}) := \widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\tilde{\iota}}(H^d(\widetilde{X}_{\widetilde{K}},\mathcal{V}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}},w_0^P,\vee})_{\mathcal{S}^*\mathfrak{m}^\vee}^{\overline{\widetilde{Q}_{\bar{v}}}\text{-ord}}).$$

Then the dual version of degree shifting reads as follows.

Proposition 6.1.3 (Dual degree shifting). *Let $\bar{v},\bar{v}'$ be two distinct places of $\overline{S}_p$. Let $\overline{S}_1 := \{\bar{v}'\}$, $\overline{S}_3 := \{\bar{v}\}$, and $\overline{S}_2 := \overline{S}_p \setminus \{\bar{v},\bar{v}'\}$ their complement. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup and $m \in \mathbf{Z}_{\geq 1}$ be an integer. Assume that the following conditions are satisfied.*

i. We have

$$\sum_{\bar{v}'' \in \overline{S}_2} [F_{\bar{v}''}^+ : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}].$$

ii. For $\bar{v}'' \in \overline{S}_1 \cup \overline{S}_2$, we have $U(\mathcal{O}_{F_{\bar{v}''}^+}) \subset \widetilde{K}_{\bar{v}''}$, and $\widetilde{K}_{\bar{v}''} = \widetilde{K}(m,\overline{S}_1 \cup \overline{S}_2)_{\bar{v}''}$. Finally, we have $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}_{\bar{v}}(0,c_p)$.

iii. For each $\iota : F \hookrightarrow E$ inducing $\bar{v}$ or $\bar{v}'$, we have $\lambda_{\iota,n} + \lambda_{\iota c,n} \geq 0$.

---

[4] Recall that $\tilde{\iota}$ and ι are the maps intertwining the Hecke actions between the two sides of Poincaré duality.

iv. $\overline{\rho}_{\mathcal{S}^*\mathfrak{m}^\vee}$ *is decomposed generic.*

Define a weight $\tilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ *as follows: if* $\iota : F^+ \hookrightarrow E$ *does not induce either* $\bar{v}$ *or* $\bar{v}'$*, set* $\tilde{\lambda}_\iota = 0$*. Otherwise, set* $\tilde{\lambda}_\iota = (\lambda_\iota, -w_{0,n}\lambda_{\iota c})$*. Set* $K := \widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty)$*.*

Let $q \in [\lfloor \frac{d}{2} \rfloor, d-1]$*. Then there exists an integer* $m' \geq m$*, an integer* $N \geq 1$*, a nilpotent ideal* $I \subset A^\vee(K, \lambda, \underline{\tau}, q, \bar{v}, m)$ *satisfying* $I^N = 0$*, and a commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}},\widetilde{\iota}} \arrow[r] \arrow[d, "\mathcal{S}^{\bar{v},\vee}"] & \widetilde{A}^\vee(\widetilde{K}(m', \overline{S}_2), \tilde{\lambda}, \underline{\tau_{\bar{v}}}, \bar{v}) \arrow[d] \\
\mathbf{T}^{T,\lambda_{\bar{v}},\tau_{\bar{v}},\iota} \arrow[r] & A^\vee(K, \lambda, \underline{\tau}, q, \bar{v}, m)/I.
\end{tikzcd}$$

Moreover, N *can be chosen to only depend on* n *and* $[F^+ : \mathbf{Q}]$*.*

Proof. One can argue the same way as in the proof of Proposition 6.1.1. In particular, one first proves the dual analogue of Lemma 6.1.2 (see [CN23], Proposition 4.2.4 for how the dual statement might look like). This can be done analogously using *loc. cit.*, Lemma 4.2.3 (taking into account the discussion above the lemma), and replacing Corollary 3.5.7 by Corollary 3.5.8. □

6.2 Middle degree cohomology

As before, consider a collection of data

$$\bar{v} \in \overline{S}_p,\ (\lambda_v, \lambda_{v^c}) \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F_v,E)} \times (\mathbf{Z}_+^n)^{\mathrm{Hom}(F_{v^c},E)},$$

$$Q_v \times Q_{v^c} = P_{(n_1,\ldots,n_k)} \times P_{(m_1,\ldots,m_{k^c})} \subset G_{F_{\bar{v}}^+},$$

$$(\underline{\tau_v}, \underline{\tau_{v^c}}) = ((\tau_{v,1}, \ldots, \tau_{v,k}), (\tau_{v^c,1}, \ldots, \tau_{v^c,k^c})).$$

Let $((\Omega_{v,1}, \ldots, \Omega_{v,k}), (\Omega_{v^c,1}, \ldots, \Omega_{v^c,k^c}))$ to be the corresponding collection of Bernstein centres. Consider the tuple $(\widetilde{Q}_{\bar{v}}^{w_0}, \tilde{\lambda}_{\bar{v}} = (-w_{0,n}\lambda_{v^c}, \lambda_v), \underline{\tau_{\bar{v}}} = (\underline{\tau_v}, \underline{\tau_{v^c}}))$. Set $\tilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ to be an extension of $\tilde{\lambda}_{\bar{v}}$. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup such that $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}_{\bar{v}}(0, c_p)$. Let $\mathfrak{m} \subset \mathbf{T}^T$ be a non-Eisenstein maximal ideal and set $\widetilde{\mathfrak{m}} := \mathcal{S}^*(\mathfrak{m})$. Recall that $d = \dim_{\mathbf{C}}(\widetilde{X}_{\widetilde{K}})$. Then the goal of this short section is to decompose the $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}[1/p]$-module

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})}^{\widetilde{Q}_{\bar{v}}^{w_0}})_{\widetilde{\mathfrak{m}}}^{\widetilde{Q}_{\bar{v}}^{w_0}\text{-ord}}[1/p] \tag{6.2.1}$$

in terms of cuspidal automorphic representations for $\widetilde{G}$. As without further assumptions the possibility of some Eisenstein series for G contributing to 6.2.1 cannot be ruled out, following [All+23], we put an extra assumption on the weight $\tilde{\lambda}$ to ensure that only cuspidal automorphic representations for $\widetilde{G}$ contribute. We recall the definition here.

Definition 6.2.1. A weight $\widetilde{\lambda} \in (\mathbf{Z}^{2n}_{+})^{\mathrm{Hom}(F^{+},E)}$ is CTG ("cohomologically trivial for G") if it satisfies the following condition

- Given $w \in W^P$, define $\lambda_w = w(\tilde{\lambda} + \rho) - \rho$, viewed as an element of $(\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$ as usual where ρ denotes the half-sum of positive roots. For each $w \in W^P$ and $i_0 \in \mathbf{Z}$, there exists $\iota \in \mathrm{Hom}(F,E)$ such that $\lambda_{w,\iota} - \lambda^{\vee}_{w,\iota c} \neq (i_0, ..., i_0)$.

We recall that $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}[1/p]$ is naturally identified with

$$\widetilde{\mathbf{T}}^{T}[1/p] \otimes_E \left((\bigotimes_{i=1}^{k} \mathfrak{z}_{\Omega_{v,i}}) \otimes (\bigotimes_{i=1}^{k^c} \mathfrak{z}_{\Omega_{v^c,i}}) \right).$$

Proposition 6.2.2. *Assume that $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m})$ is decomposed generic, and $\tilde{\lambda}$ is CTG. Then, after possibly enlarging E,*

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\widetilde{\lambda},\underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}}[1/p]$$

is a semisimple $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}[1/p]$-module. Moreover, for any homomorphism

$$x : \widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}}) \to \overline{\mathbf{Q}}_p$$

and isomorphism $t : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$, there is a cuspidal automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbf{A}_{F^+})$ such that $t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}) \otimes V^{\vee}_{\tilde{\lambda}_{\bar{v}}}$ is $\widetilde{Q}^{w_0}_{\bar{v}}$-ordinary (in the sense of Definition 4.2.5) and x is induced by the natural Hecke action of

$$\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}[1/p] = \widetilde{\mathbf{T}}^{T}[1/p] \otimes_E \left((\bigotimes_{i=1}^{k} \mathfrak{z}_{\Omega_{v,i}}) \otimes (\bigotimes_{i=1}^{k^c} \mathfrak{z}_{\Omega_{v^c,i}}) \right)$$

on

$$(t^{-1}\widetilde{\pi}^{\{\infty\}\cup T})^{\widetilde{G}(\widehat{\mathcal{O}}^{T}_{F^+})} \otimes_{\overline{\mathbf{Q}}_p} \mathrm{Hom}_{\widetilde{M}^{w_0}_{\bar{v}}(F_v)} \left(\sigma(\underline{\tau_v}) \otimes (\theta_n^{-1})^* \sigma(\underline{\tau_{v^c}}), (t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}))^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}} \right).^{5} \tag{6.2.2}$$

Before starting the proof, recall that $\mathfrak{z}_{\Omega_{v^c,1}} \otimes ... \otimes \mathfrak{z}_{\Omega_{v^c,k^c}}$ acts on the second factor of the Hom in 6.2.2 via θ_n. Moreover, we emphasise that in the statement we are implicitly using that, by Corollary 4.2.13, $(t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}))^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}$ is irreducible, so $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}[1/p]$ indeed acts on 6.2.2 through scalars.

Proof. To prove the statement, we show that there is a $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}[1/p]$-equivariant direct sum decomposition

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})})^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}_{\widetilde{\mathfrak{m}}} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \cong$$

---

[5] For the definition of $(-)^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}$, see Definition 4.2.18.

$$\bigoplus_{\widetilde{\pi}} d(\widetilde{\pi})(t^{-1}\widetilde{\pi}^{\{\infty\}\cup T})^{\widetilde{G}(\widehat{\mathcal{O}}^T_{F^+})} \otimes_{\overline{\mathbf{Q}}_p} \mathrm{Hom}_{\widetilde{M}^{w_0}_{\bar{v}}(F_v)} \left(\sigma(\underline{\tau_v}) \otimes (\theta_n^{-1})^* \sigma(\underline{\tau_{v^c}}), (t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}))^{\widetilde{Q}^{w_0}_{\bar{v}}\text{-ord}}\right)$$

where the sum runs over cohomological cuspidal automorphic representations of $\widetilde{G}(\mathbf{A}_{F^+})$ of weight $\tilde{\lambda}$ and $d(\widetilde{\pi}) \geq 0$ denotes some integer. A $\widetilde{\mathbf{T}}^T$-equivariant decomposition of this kind is given in the proof of [All+23], Theorem 2.4.11. To see that this decomposition is also Hecke equivariant at $\bar{v}$, we unravel the definition of this action.

To ease the notation, set $Q := \widetilde{Q}^{w_0}_{\bar{v}} \subset \mathrm{GL}_{2n,F_v}$, $M := \widetilde{M}^{w_0}_{\bar{v}}$, and $\mathcal{V} := \mathcal{V}^{\widetilde{Q}^{w_0}_{\bar{v}}}_{(\tilde{\lambda},\underline{\tau_{\bar{v}}})}$. Recall that we have

$$R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V})^{Q\text{-ord}}_{\widetilde{\mathfrak{m}}} \cong \varprojlim_m R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}/\varpi^m)^{Q\text{-ord}}_{\widetilde{\mathfrak{m}}}.$$

Moreover, as explained in §3.4, the $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-action is induced by the identification

$$R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}/\varpi^m)^{Q\text{-ord}}_{\widetilde{\mathfrak{m}}} \cong R\mathrm{Hom}_{M^0}\left(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ/\varpi^m, \pi^{Q\text{-ord}}(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m)_{\widetilde{\mathfrak{m}}}\right).$$

Since $\widetilde{\mathfrak{m}}$ is decomposed generic, the cohomology of $\pi^{Q\text{-ord}}(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$ vanishes for degrees below d thanks to [CS19], Theorem 1.1. Therefore, a standard argument with a hypercohomology spectral sequence (combined with the previous observations) gives an identification

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}/\varpi^m)^{Q\text{-ord}}_{\widetilde{\mathfrak{m}}} \cong \mathrm{Hom}_{M^0}(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}})^\circ/\varpi^m, H^d(\pi^{Q\text{-ord}}(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m)_{\widetilde{\mathfrak{m}}})).$$

Another application of the vanishing result of [CS19] combined with a standard argument with a Hocschild–Serre spectral sequence gives a $\widetilde{\mathbf{T}}^T$-equivariant isomorphism of smooth $\mathcal{O}/\varpi^m[M(F_v)]$-modules

$$H^d(\pi^{Q\text{-ord}}(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m)_{\widetilde{\mathfrak{m}}}) \cong \mathrm{Ord}_Q(\widetilde{H}^d(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m)_{\widetilde{\mathfrak{m}}}).$$

Here $\widetilde{H}^d(\widetilde{K}^{\bar{v}}, \mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m)$ denotes the degree d $\bar{v}$-completed cohomology of level $\widetilde{K}^{\bar{v}}$ and weight $\mathcal{V}_{\tilde{\lambda}^{\bar{v}}}/\varpi^m$ and $\mathrm{Ord}_Q(-)$ is Emerton's ordinary part functor from §4.1. Using that thanks to finiteness of finite level cohomology Mittag–Leffler applies in our situation, we deduce an identification

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V})^{Q\text{-ord}}_{\widetilde{\mathfrak{m}}}[1/p] \cong \mathrm{Hom}_{M^0}\left(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}), \mathrm{Ord}_Q(\widetilde{H}^d(\widetilde{K}^{\bar{v}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}})\right).^6 \quad (6.2.3)$$

The $\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-action on the former then is induced from this identification. As a consequence of Proposition 4.1.8, 6.2.3 is further identified ($\widetilde{\mathbf{T}}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$-equivariantly) with

$$\mathrm{Hom}_{M^0}\left(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}), \mathrm{Ord}_Q(\widetilde{H}^d(\widetilde{K}^{\bar{v}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}})^{V_{-w_0^Q\tilde{\lambda}_{\bar{v}}}\text{-lalg}}\right) \cong$$

---

[6] For the definition of Ord_Q applied to E-Banach space representations, see the discussion below Remark 4.1.7.

$$\mathrm{Hom}_{M^0}\left(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\mathcal{T}_{\bar{v}}}), \mathrm{Ord}_Q^{\mathrm{lalg}}(\widetilde{H}^d(\widetilde{K}^{\bar{v}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}}^{V_{\tilde{\lambda}_{\bar{v}}^{\vee}}\text{-lalg}})\right).$$

Using Emerton's spectral sequence (cf. [Eme06b], Corollary 2.2.18), and the fact that $\widetilde{H}^*(\widetilde{K}^{\bar{v}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}}$ vanishes below the middle degree, we see that there is a $\widetilde{\mathbf{T}}^T$-equivariant isomorphism

$$\widetilde{H}^d(\widetilde{K}^{\bar{v}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}}^{V_{\tilde{\lambda}_{\bar{v}}^{\vee}}\text{-lalg}} \cong \left(\varinjlim_{\widetilde{K}'_{\bar{v}}} H^d(\widetilde{X}_{\widetilde{K}^{\bar{v}}\widetilde{K}'_{\bar{v}}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}}\right) \otimes_E V_{\tilde{\lambda}_{\bar{v}}^{\vee}}$$

of locally algebraic E-representations of $\mathrm{GL}_{2n}(F_v)$. Moreover, by the proof of [All+23], Theorem 2.4.11, we have a $\widetilde{\mathbf{T}}^T$-equivariant direct sum decomposition

$$\left(\varinjlim_{\widetilde{K}'_{\bar{v}}} H^d(\widetilde{X}_{\widetilde{K}^{\bar{v}}\widetilde{K}'_{\bar{v}}}, V_{\tilde{\lambda}^{\bar{v}}})_{\widetilde{\mathfrak{m}}}\right) \otimes_E \overline{\mathbf{Q}}_p \cong$$

$$\bigoplus_{\widetilde{\pi}} d(\widetilde{\pi})(t^{-1}\widetilde{\pi}^{\{\infty\}\cup T})^{\widetilde{G}(\widehat{\mathcal{O}}^T_{F^+})} \otimes t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1})$$

of smooth admissible $\overline{\mathbf{Q}}_p[\mathrm{GL}_{2n}(F_v)]$-modules.

In particular, to conclude, it suffices to prove that for any $\widetilde{\pi}$ appearing in the direct sum decomposition above, there is an identification

$$\mathrm{Hom}_{M^0}\left(\tilde{\sigma}(\underline{\mathcal{T}_{\bar{v}}}), t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1})^{Q\text{-ord}}\right) \cong \tag{6.2.4}$$

$$\mathrm{Hom}_{M^0}\left(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\mathcal{T}_{\bar{v}}}), \mathrm{Ord}_Q^{\mathrm{lalg}}\left(t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}) \otimes_E V_{\tilde{\lambda}_{\bar{v}}^{\vee}}\right)\right)$$

such that the isomorphism

$$\mathcal{H}(\tilde{\sigma}(\underline{\mathcal{T}_{\bar{v}}})) \xrightarrow{\sim} \mathcal{H}(\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\mathcal{T}_{\bar{v}}})), \tag{6.2.5}$$

$$\phi \mapsto \phi \otimes \mathrm{id}_{-w_0^Q\tilde{\lambda}_{\bar{v}}}$$

(provided by [ST06], Lemma 1.4) intertwines the Hecke actions on the two sides. After noting that $\mathrm{Ord}_Q^{\mathrm{lalg}}(t^{-1}(\tilde{\pi}_{\bar{v}} \circ \iota_v^{-1}) \otimes_E V_{\tilde{\lambda}_{\bar{v}}^{\vee}}) \cong (t^{-1}(\widetilde{\pi}_{\bar{v}} \circ \iota_v^{-1}))^{Q\text{-ord}} \otimes_E V_{-w_0^Q\tilde{\lambda}_{\bar{v}}}$, and that $\tilde{\sigma}(\lambda_{\bar{v}}, \underline{\mathcal{T}_{\bar{v}}}) = \tilde{\sigma}(\underline{\mathcal{T}_{\bar{v}}}) \otimes_E V_{-w_0^Q\tilde{\lambda}_{\bar{v}}}$, the isomorphism 6.2.4 is given by the natural isomorphism

$$\mathrm{Hom}_{M^0}(\tilde{\sigma}(\underline{\mathcal{T}_{\bar{v}}}), -) \cong \mathrm{Hom}_{M^0}(\tilde{\sigma}(\underline{\mathcal{T}_{\bar{v}}}) \otimes_E V_{-w_0^Q\tilde{\lambda}_{\bar{v}}}, - \otimes_E V_{-w_0^Q\tilde{\lambda}_{\bar{v}}}).$$

The induced identification is clearly $\mathcal{H}(\tilde{\sigma}(\underline{\mathcal{T}_{\bar{v}}}))$-equivariant when on the RHS of 6.2.4 we act through 6.2.5. $\square$

6.3 The end of the proof

We are now ready to prove our main local-global compatibility result in this level of generality. The reader is invited to compare it with [CN23], Proposition 4.2.13. The content of this result is, for every p-adic place $v \in S_p(F)$, constructing a characteristic zero lift of $\rho_{\mathfrak{m}}|_{G_{F_v}}$ with the right shape according to the tuple $(Q_v, \lambda_v, \underline{\tau}_v)$. We then show that in the case of $Q_v = \mathrm{GL}_n$ the existence of such a lift easily implies Theorem 5.3.4.

Proposition 6.3.1. *Assume that p splits in an imaginary quadratic subfield of F. Let $K \subset G(\mathbf{A}_{F^+}^{\infty})$ be a good subgroup and fix distinct places $\bar{v}, \bar{v}' \in \overline{S}_p$. Fix a preferred lift $v \mid \bar{v}$ in $S_p(F)$. Let $(Q_p, \lambda, \underline{\tau}) = (Q_v, \lambda_v, \underline{\tau}_v)_{v \in S_p(F)}$ be a tuple as in §2.7 and assume that $K_p \subset \mathcal{Q}_p(0, c_p)$ and $K_{\bar{v}} = \mathcal{Q}_v(0, c_p) \times \mathcal{Q}_{v^c}(0, c_p)$.*

Let $m \in \mathbf{Z}_{\geq 1}$ be an integer, and $\mathfrak{m} \subset \mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}}$ be a maximal ideal in the support of $H^*(X_K, \mathcal{V}_{(\lambda, \underline{\tau})}^{Q_p}/\varpi^m)^{Q_v \times Q_{v^c}\text{-ord}}$. Write

$$Q_v = P_{(n_1, \ldots, n_k), F_v} \subset \mathrm{GL}_{n, F_v}, \text{ and } Q_{v^c} = P_{(m_1, \ldots, m_{k^c}), F_{v^c}} \subset \mathrm{GL}_{n, F_{v^c}}.$$

Assume that:

i. We have
$$\sum_{\bar{v}'' \neq \bar{v}, \bar{v}'} [F_{\bar{v}''}^+ : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}]$$
where the sum runs over $\bar{v}'' \in S_p(F^+)$.

ii. The maximal ideal $\mathfrak{m}$ is non-Eisenstein such that $\overline{\rho}_{\mathfrak{m}}$ is decomposed generic.

iii. Let $v \notin T$ be a finite place of F, with residue characteristic ℓ. Then either T contains no ℓ-adic places and ℓ is unramified in F, or there exists an imaginary quadratic subfield of F in which ℓ splits.

Then, for each $q \in [0, d-1]$, there exists an integer $N \geq 1$, depending only on n and $[F^+ : \mathbf{Q}]$, a nilpotent ideal $I \subset \mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}}\left(H^q(X_K, \mathcal{V}_{(\lambda, \underline{\tau})}^{Q_p}/\varpi^m)_{\mathfrak{m}}^{Q_{\bar{v}}\text{-ord}}\right)$ with $I^N = 0$, and a continuous n-dimensional representation

$$\rho_{\mathfrak{m}} : G_{F,T} \to \mathrm{GL}_n\left(\mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}}\left(H^q(X_K, \mathcal{V}_{(\lambda, \underline{\tau})}^{Q_p}/\varpi^m)_{\mathfrak{m}}^{Q_{\bar{v}}\text{-ord}}\right)/I\right)$$

such that the following conditions hold:

i. For each finite place $v \notin T$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$.

ii. For $\tilde{v} = v, v^c$, the representation $\rho_{\mathfrak{m}}|_{G_{F_v}}$ has a lift to $\widetilde{\rho}_{\tilde{v}} : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\widetilde{A})$, where $\widetilde{A}$ is a finite flat local $\mathcal{O}$-algebra equipped with a $\mathfrak{z}_{\lambda_{\tilde{v}}, \underline{\tau_{\tilde{v}}}}^{\circ}$-algebra *structure such that* $\widetilde{A}[1/p] \cong \prod_x E$ *is a semisimple* E*-algebra with a morphism*

$$\overline{\mathcal{S}}^{\tilde{v}} : \widetilde{A} \to \mathbf{T}^{T,\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}} \left(H^q(X_K, \mathcal{V}^{Q_p}_{(\lambda,\underline{\tau})}/\varpi^m)^{Q_{\bar{v}}\text{-ord}}_{\mathfrak{m}}\right)/I$$

of $\mathfrak{z}^{\circ}_{\lambda_{\bar{v}},\underline{\tau_{\bar{v}}}}$*-algebras.*

iii. *For* $\tilde{v} = v, v^c$, $\tilde{\rho}_{\tilde{v}}[1/p]$ *is potentially semistable with labelled Hodge–Tate weights* $(\lambda_{\iota,1} + n - 1 > ... > \lambda_{\iota,n})_{\iota:F_{\tilde{v}} \hookrightarrow E}$.

iv. *Moreover, these lifts admit isomorphisms*

$$\tilde{\rho}_v[1/p] \sim \begin{pmatrix} \tilde{\rho}_{v,1} & * & ... & * \\ 0 & \tilde{\rho}_{v,2} & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \tilde{\rho}_{v,k} \end{pmatrix}, \tilde{\rho}_{v^c}[1/p] \sim \begin{pmatrix} \tilde{\rho}_{v^c,1} & * & ... & * \\ 0 & \tilde{\rho}_{v^c,2} & ... & * \\ . & . & . & . \\ . & . & . & . \\ 0 & ... & 0 & \tilde{\rho}_{v^c,k^c} \end{pmatrix}$$

such that, for $1 \leq i \leq k$, *and* $1 \leq j \leq k^c$, $\tilde{\rho}_{v,i}$, *respectively* $\tilde{\rho}_{v^c,j}$ *has Weil–Deligne inertial type bounded by* $\tau_{v,i}$, *respectively* $\tau_{v^c,j}$. *Moreover, the labelled Hodge–Tate weights of* $\tilde{\rho}_{v,i}$, *respectively* $\tilde{\rho}_{v^c,j}$ *are determined by the property that they are increasing as* i, *respectively* j *grows.*

v. *For* $\tilde{v} = v$ *or* v^c, *an integer* $1 \leq i \leq \tilde{k}$ *for* $\tilde{k} = k$ *or* $\tilde{k} = k^c$ *(depending on* $\tilde{v}$*), and a morphism* $x : \widetilde{A} \to E$ *the following property is satisfied. The morphism* $\mathfrak{z}_{\Omega_{\tilde{v},i}} \to E$ *induced by postcomposing the structure map* $\mathfrak{z}_{\Omega_{\tilde{v},i}} \to \widetilde{A}$ *with* x *coincides with the one induced by the natural action of* $\mathfrak{z}_{\Omega_{\tilde{v},i}}$ *on*

$$\mathrm{rec}^{T,-1}(\mathrm{WD}(x \circ \tilde{\rho}_{\tilde{v},i})) \otimes |\cdot|^{n_1+...+n_{i-1}}.^{7}$$

Proof. As the argument follows very closely the proof of [CN23], Proposition 4.2.13, we only provide a sketch. Since the existence of $\rho_{\mathfrak{m}}$ satisfying the first condition is known, we are free to enlarge T.[8] By various twisting arguments, and an application of the Hocschild–Serre spectral sequence, we can assume the following:

- $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.
- $K = K(m, \overline{S}_p \setminus \{\bar{v}\})$.
- For $\bar{v}'' \in \overline{S}_p \setminus \{\bar{v}, \bar{v}'\}$, $\lambda_{\bar{v}''} = 0$, and $\tau_{\bar{v}''}$ is trivial.
- For each $\iota : F_v \hookrightarrow E$, $-\lambda_{\iota c,1} - \lambda_{\iota,1} \geq 0$, and $\tilde{\lambda} = (-w_0^{\mathrm{GL}_n} \lambda_{v^c}, \lambda_v)_{\bar{v} \in \overline{S}_p}$ is CTG.[9]

---

[7] Here the convention is that $n_0 := 0$.

[8] This will be used in the various upcoming twisting arguments.

[9] See Remark 6.3.2 for the role of this condition.

In particular, by setting $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^{\infty})$ to be a good subgroup satisfying

- $\widetilde{K}^{\bar{v}} \cap G(\mathbf{A}_{F^+}^{\infty}) = K^{\bar{v}}$;
- $\widetilde{K}^T = \widetilde{G}(\widehat{\mathcal{O}}_{F^+}^T)$;
- For $\bar{v}'' \in \overline{S}_p \setminus \{\bar{v}\}$, $U(\mathcal{O}_{F_{\bar{v}}^+}) \subset \widetilde{K}_{\bar{v}''}$, $\widetilde{K} = \widetilde{K}(m, \overline{S}_p \setminus \{\bar{v}\})$;
- $\widetilde{K}_{\bar{v}} = \widetilde{\mathcal{Q}}_{\bar{v}}^{w_0}(0, c_p)$ (associated with $Q_{\bar{v}}$);

and assuming that $q \in [\lfloor \frac{d}{2} \rfloor, d-1]$, we are in the situation of Proposition 6.1.1.

We briefly explain what we mean by "twisting argument" as it will be also used in the next step of the proof. Assume that we have a continuous character $\overline{\chi} : G_F \to k^{\times}$. Since for the upcoming step this is the only relevant case, we also assume that $\overline{\chi}$ is unramified at S_p. We set $\chi : G_F \to \mathcal{O}^{\times}$ to be its Teichmüller lift. Choose a finite set of finite places $T \subset T'$ of F that is closed under complex conjugation, containing all places at which χ is ramified, and a good normal subgroup $K' \subset K$ satisfying:

- $(K')^{T' \setminus T} = K^{T' \setminus T}$.
- K/K' is abelian of order prime to p.
- For each finite place v of F, $\chi|_{G_{F_v}} \circ \mathrm{Art}_{F_v}$ is trivial on $\det(K'_v)$.
- T' again satisfies assumption iii) from the Proposition.

Then set $\mathfrak{m}(\chi) \subset \mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}}$ to be $f_{\chi}(\mathfrak{m})$, where $f_{\chi} : \mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}} \xrightarrow{\sim} \mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}}$ is the map defined in the discussion preceding Lemma 2.3.2. Note that $\overline{\rho}_{\mathfrak{m}(\chi)} = \overline{\rho}_{\mathfrak{m}} \otimes \chi$. Set

$$A(K', \lambda, \underline{\tau}, q, \bar{v}, m, \chi) := \mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}} \left(H^q(X_{K'}, \mathcal{V}_{(\lambda, \underline{\tau_{\bar{v}}})}^{Q_p} / \varpi^m)_{\mathfrak{m}(\chi)}^{Q_{\bar{v}}\text{-ord}} \right).$$

<u>Claim:</u> Verifying the Proposition for any of the $\mathfrak{m}(\chi)$'s with corresponding level K' will imply it for $\mathfrak{m}$ with level K.

Proof of Claim. Note that we have a surjection

$$A(K', \lambda, \underline{\tau}, q, \bar{v}, m) \to A(K, \lambda, \underline{\tau}, q, \bar{v}, m)$$

of $\mathbf{T}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}}$-algebras so we can and do assume that $K = K'$. Use the abbreviation $A(\chi) := A(K, \lambda, \underline{\tau}, q, \bar{v}, m, \chi)$. Assume that we found an integer N_{χ}, a nilpotent ideal $I_{\chi} \subset A(\chi)$, a continuous representation $\rho_{\mathfrak{m}(\chi)} : G_{F,T} \to \mathrm{GL}_n(A(\chi)/I_{\chi})$, a surjection $\widetilde{A}(\chi) \twoheadrightarrow A(\chi)/I_{\chi}$ of $\mathfrak{z}_{\lambda_v, \underline{\tau_v}}^{\circ}$-algebras, and a continuous representation $\widetilde{\rho}_{v,\chi} : G_{F_v} \to \mathrm{GL}_n(\widetilde{A})$ satisfying the conditions of the Proposition. Set $N := N_{\chi}$, $I := f_{\chi^{-1}}(I_{\chi})$, $\rho_{\mathfrak{m}} := (f_{\chi^{-1}} \circ \rho_{\mathfrak{m}(\chi)}) \otimes \chi^{-1}$, and $\widetilde{A} := f_{\chi^{-1}}^{*} \widetilde{A}(\chi)$ as a $\mathfrak{z}_{\lambda_v, \underline{\tau_v}}^{\circ}$-algebra. Define the surjection $\widetilde{A} \to A/I$ to be

$\widetilde{A}(\chi) \to A(\chi)/I_\chi \cong^{f_{\chi^{-1}}} A/I$ and set $\widetilde{\rho}_v := (f_{\chi^{-1}} \circ \widetilde{\rho}_{v,\chi}) \otimes \chi^{-1} : G_{F_v} \to \mathrm{GL}_n(\widetilde{A})$. Then, as already observed at the end of the proof of [All+23], Corollary 4.4.8, $\rho_{\mathfrak{m}}$ satisfies the first condition of the Proposition. An easy diagram chase shows that condition ii) is satisfied. Conditions iii) and iv) are clearly preserved under twisting by χ, so they are also satisfied. To check the last condition, pick $x : \widetilde{A} \to E$, and consider one of the subquotients $\widetilde{\rho}_{v,\chi,i}$ of $\widetilde{\rho}_{v,\chi}[1/p]$ from condition iii) for $\mathfrak{m}(\chi)$. Then the corresponding subquotient of $\widetilde{\rho}_v[1/p]$ is $\widetilde{\rho}_{v,\chi,i} \otimes \chi^{-1}$, and the morphism $\mathfrak{z}_{\Omega_{v,i}} \to E$ is induced by $\mathfrak{z}_{\Omega_{v,i}} \to \widetilde{A}(= \mathfrak{z}_{\Omega_{v,i}} \xrightarrow{f_\chi} \mathfrak{z}_{\Omega_{v,i}} \to \widetilde{A}(\chi))$. Then condition v) follows from the computation

$$\mathrm{rec}^{T,-1}(\mathrm{WD}(x \circ \widetilde{\rho}_{v,i})) = \mathrm{rec}^{T,-1}(\mathrm{WD}(x \circ \widetilde{\rho}_{v,\chi,i}) \otimes \chi^{-1}) =$$

$$\mathrm{rec}^{T,-1}(\mathrm{WD}(x \circ \widetilde{\rho}_{v,\chi,i})) \otimes (\chi^{-1} \circ \mathrm{Art}_{F_v})$$

and the definition of $f_\chi : \mathfrak{z}_{\Omega_{v,i}} \to \mathfrak{z}_{\Omega_{v,i}}$. □

For now we assume that $q \in [\lfloor \frac{d}{2} \rfloor, d-1]$. Then, according to Proposition 6.1.1, for every choice of $\overline{\chi}$ such that $\widetilde{\mathfrak{m}(\chi)}$ is decomposed generic, we obtain an integer N_χ, a nilpotent ideal I_χ, a flat $\mathcal{O}$-algebra $\widetilde{A}(\chi)$, and a map $\widetilde{A}(\chi) \to A(K', \lambda, \underline{\tau}, q, \bar{v}, m, \chi)/I_\chi$. Write $\widetilde{A}(\chi) = \prod_x E$ using Proposition 6.2.2. Applying Theorem 2.9.2, and Theorem 4.3.4 to each $x : \widetilde{A} \to E$ (keeping in mind Proposition 6.2.2), we obtain a continuous representation

$$\widetilde{\rho}_{\mathfrak{m}(\chi)} := \prod_x \widetilde{\rho}_{\mathfrak{m}(\chi),x} : G_F \to \mathrm{GL}_{2n}(\widetilde{A}(\chi)[1/p]),$$

admitting a block upper-triangular shape with blocks of size $(n_1, ..., n_k, m_{k^c}, ..., m_1)$. In particular, for each x, we have an isomorphism

$$\widetilde{\rho}_{\mathfrak{m}(\chi),x}|_{G_{F_v}} \sim \begin{pmatrix} r_{1,\chi,x} & * \\ 0 & r_{2,\chi,x} \end{pmatrix}.$$

As shown in Sub-Lemma 1 in the proof of [CN23], Proposition 4.2.13, we can find $\overline{\chi}$ such that

- the set of isomorphism classes of the Jordan-Hölder constituents of $\overline{\rho}_{\mathfrak{m}(\chi)}|_{G_{F_v}}$ and $\overline{\rho}^{\vee,c}_{\mathfrak{m}(\chi)}(1-2n)|_{G_{F_v}}$ are disjoint;
- and, for all $x : \widetilde{A} \to E$, the isomorphism classes of the Jordan-Hölder constituents of $\overline{r}_{1,\chi,x}$ coincide with those of $\overline{\rho}_{\mathfrak{m}(\chi)}|_{G_{F_v}}$.

For such a χ, we can apply [CN23], Proposition 3.2.4 to obtain a $\widetilde{A}(\chi)$-valued lift $\widetilde{\rho}_{v,\chi} : G_{F_v} \to \mathrm{GL}_n(\widetilde{A}(\chi))$ of $\rho_{\mathfrak{m}(\chi)}|_{G_{F_v}}$ such that $\widetilde{\rho}_{v,\chi}[1/p]$ becomes isomorphic to $\prod_x r_{1,\chi,x}$. In particular, by Theorem 4.3.4 and Proposition 6.2.2, conditions iii), iv), and v) are all satisfied for $\widetilde{\rho}_{v,\chi}$. This finishes the proof in the case of $q \in [\lfloor \frac{d}{2} \rfloor, d-1]$.

To treat the case of $q < \lfloor \frac{d}{2} \rfloor$, one uses the Poincaré duality isomorphisms

$$\iota : A(K, \lambda, \underline{\tau}, q, \bar{v}, m) \cong A^{\vee}(K, \lambda, \underline{\tau}, d-1-q, \bar{v}, m),$$

$$\widetilde{\iota} : \widetilde{\mathbf{T}}^{T, \lambda_{\bar{v}}, \underline{\tau_{\bar{v}}}} \left(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}_{\bar{v}}, w_0^P}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})}[1/p])^{\widetilde{Q}_{\bar{v}}\text{-ord}}_{\tilde{\iota}^* \mathcal{S}^*(\mathfrak{m}^{\vee})} \right) \cong \widetilde{A}^{\vee}(\widetilde{K}, \tilde{\lambda}, \underline{\tau_{\bar{v}}}, \bar{v})[1/p]$$

provided by Proposition 3.3.5. Then an analogous argument using the dual degree shifting (Proposition 6.1.3), and a version of Proposition 6.2.2 for the cohomology group $H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^{\widetilde{Q}_{\bar{v}}, w_0^P}_{(\tilde{\lambda}, \underline{\tau_{\bar{v}}})}[1/p])^{\widetilde{Q}_{\bar{v}}\text{-ord}}_{\tilde{\iota}^* \mathcal{S}^*(\mathfrak{m}^{\vee})}$ proves the Proposition also for q. For more details, see [CN23], Proposition 4.2.13. □

Remark 6.3.2. We emphasise the role of reserving the place $\bar{v}'$ in the degree shifting argument (cf. Proposition 6.1.1). Namely, combined with the assumption that the level is deep enough at $\bar{v}'$, it allows us to freely change the weight $\lambda_{\bar{v}}$ without changing the faithful Hecke algebra $A(K, \lambda, \underline{\tau}, q, \bar{v}, m)$. This lets us assume in the proof of Proposition 6.3.1 that $\widetilde{\lambda}$ is CTG (cf. [All+23], Lemma 4.3.6). This is crucial for us in order to have access to Proposition 6.2.2.

Nevertheless, as was pointed out to the author by James Newton, once Theorem 5.3.4, in particular, Theorem 1.3.1 is proved, one could try and run the proof of Theorem 5.3.4 with a version of Proposition 6.2.2 where we drop the CTG assumption. This way the role of $\bar{v}'$ would not be relevant anymore, leading to a strengthening of Theorem 5.3.4 where condition i) is weakened to asking for

$$\sum_{\bar{v}'' \in \overline{S}_p, \bar{v}'' \neq \bar{v}} [F^+_{\bar{v}''} : \mathbf{Q}_p] \geq \frac{1}{2}[F^+ : \mathbf{Q}] \tag{6.3.1}$$

instead. The author is planning to revisit this strategy in future work.

Proof of Theorem 5.3.4. Pick a collection of data $(F, K, \lambda, \tau, \mathfrak{m}, \bar{v})$ as in the statement of the Theorem. Pick a choice of lift $v \mid \bar{v}$ in $S_p(F)$. We will apply Proposition 6.3.1 with the choice $(Q_v, \lambda_v, \underline{\tau_v})_{v \in S_p(F)} := (\mathrm{GL}_n, \lambda_v, \tau_v)$.

Note that it suffices to prove the statement for Galois representations with coefficients in $\mathbf{T}^{T, \lambda, \tau}(R\Gamma(X_K, \mathcal{V}_{(\lambda, \tau)}/\varpi^m)_{\mathfrak{m}})$ for integers $m \in \mathbf{Z}_{\geq 1}$. This is because, by [NT16], Lemma 3.11, we have an isomorphism

$$\mathbf{T}^{T, \lambda, \tau}(R\Gamma(X_K, \mathcal{V}_{(\lambda, \tau)})_{\mathfrak{m}}) \xrightarrow{\sim} \varprojlim_{m} \mathbf{T}^{T, \lambda, \tau}(R\Gamma(X_K, \mathcal{V}_{(\lambda, \tau)}/\varpi^m)_{\mathfrak{m}})$$

of $\mathbf{T}^{T, \lambda, \tau}$-algebras. Moreover, by an argument using [KT17], Lemma 2.5, and Carayol's lemma (cf. the proof of [All+23] Corollary 4.4.8, and Theorem 4.5.1), it suffices to verify the Theorem one cohomological degree at a time. Therefore, fix $q \in [0, d-1]$, and set $\mathfrak{m}_{\bar{v}} \subset \mathbf{T}^{T, \lambda_{\bar{v}}, \tau_{\bar{v}}}$ to be the maximal ideal obtained by pulling back $\mathfrak{m}$ along $\mathbf{T}^{T, \lambda_{\bar{v}}, \tau_{\bar{v}}} \hookrightarrow \mathbf{T}^{T, \lambda, \tau}$. Set $A_{\bar{v}} := \mathbf{T}^{T, \lambda_{\bar{v}}, \tau_{\bar{v}}}(H^q(X_K, \mathcal{V}_{(\lambda, \tau)}/\varpi^m)_{\mathfrak{m}_{\bar{v}}})$. It is then clearly sufficient to verify the statement for a Galois representation with coefficients in $A_{\bar{v}}$.

Applying Proposition 6.3.1 gives a nilpotent ideal $I_{\bar{v}} \subset A_{\bar{v}}$, a continuous representation $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(A_{\bar{v}}/I_{\bar{v}})$, a finite flat local $\mathcal{O}$-algebra $\widetilde{A}$, a surjective homomorphism $\overline{\mathcal{S}}^v : \widetilde{A} \to A_{\bar{v}}/I_{\bar{v}}$ of $\mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$-algebras, and a lift $\widetilde{\rho}_v : G_{F_v} \to \mathrm{GL}_n(\widetilde{A})$ of $\rho_v := \rho_{\mathfrak{m}}|_{G_{F_v}}$ under $\overline{\mathcal{S}}^v$ satisfying the five listed conditions. In particular, by the first condition, it satisfies local-global compatibility outside T, and the only thing left to check is the existence of a dotted arrow making the diagram in the statement of the Theorem commutative.

We first note that an easy unravelling of the definitions (using condition iii), iv), and v) from Proposition 6.3.1) shows that we have a (necessarily unique) dotted arrow making the diagram

$$\begin{tikzcd}
R^{\square}_{\overline{\rho}_v}[1/p] \arrow[r, "\widetilde{\rho}_v{[1/p]}"] \arrow[d, two heads] & \widetilde{A}[1/p] \\
R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}[1/p] \arrow[ur, dotted] & \mathfrak{z}\Omega_v \arrow[l, "\eta"'] \arrow[u, "\mathrm{nat}_{\widetilde{A}}"']
\end{tikzcd} \tag{6.3.2}$$

commutative. Here $\mathrm{nat}_{\widetilde{A}}$ denotes the natural map towards the faithful Hecke algebra and η denotes the interpolation of local Langlands. We then obtain the following diagram

$$\begin{tikzcd}
\mathfrak{z}\Omega_v \arrow[r, "\eta"] & R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}[1/p] \arrow[r, "\widetilde{\rho}_v{[1/p]}"] & \widetilde{A}[1/p] & \\
 & R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v} \arrow[u, hook] \arrow[r, "\widetilde{\rho}_v"] & \widetilde{A} \arrow[u, hook] \arrow[r, two heads, "\overline{\mathcal{S}}^v"] & A_{\bar{v}}/I_{\bar{v}} \\
 & \mathfrak{z}^{\circ}_{\lambda_v,\tau_v} \arrow[uul, hook] \arrow[ur, "\mathrm{nat}_{\widetilde{A}}"] \arrow[urr, "\mathrm{nat}_{A_{\bar{v}}}"'] & &
\end{tikzcd}$$

where all the inclusions are the natural ones induced by inverting p. Note that, by abuse of notation, we write Kisin's local deformation rings as the source of $\widetilde{\rho}_v[1/p]$ resp. $\widetilde{\rho}_v$. This is justified by 6.3.2. Moreover, by the very definition of $\widetilde{\rho}_v$, we have $\overline{\mathcal{S}}^v \circ \widetilde{\rho}_v = \rho_{\mathfrak{m}}|_{G_{F_v}}$. In particular, we see that in the local-global compatibility diagram of Theorem 5.3.4 we indeed have a dotted arrow making the upper triangle commutative. To see that the obtained arrow also commutes with the lower triangle, we need to prove the following.
<u>Claim:</u> If $z \in \mathfrak{z}^{\circ}_{\lambda_v,\tau_v}$ such that $\eta(z)$ lies in $R^{\lambda_v,\preceq\tau_v}_{\overline{\rho}_v}$, then $\mathrm{nat}_{A_{\bar{v}}}(z) = \rho_v(z)$.

Proof of Claim. We have

$$\mathrm{nat}_{A_{\bar{v}}}(z) = \overline{\mathcal{S}}^v(\mathrm{nat}_{\widetilde{A}}(z)) = \overline{\mathcal{S}}^v(\widetilde{\rho}_v[1/p](\eta(z))) =$$

$$\overline{\mathcal{S}}^v(\widetilde{\rho}_v(\eta(z))) = \rho_v(\eta(z)),$$

where the first equality is by the definition of the degree shifting map, the second equality is the content of 6.3.2, and the last equality follows from the definition of $\widetilde{\rho}_v$. □

□

Proof of Theorem 1.3.1. Consider an imaginary CM field F, an identification $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$, and a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ as in the statement of the Theorem. Fix also a p-adic place v in F where we wish to prove local-global compatibility. We can then find a cyclic CM field extension F'/F such that v and v^c split completely in F', F' is linearly disjoint from $\overline{F}^{\ker \overline{r_t(\pi)}}$, $\overline{r_t(\pi)}|_{G_{F'}}$ stays decomposed generic, and F' and v satisfy the conditions of Theorem 5.3.4. In particular, it suffices to prove local-global compatibility for the cyclic base change $\pi' := \mathrm{BC}_{F'/F}(\pi)$ for any place $v'|v$ in F'.

Let τ be the Weil–Deligne inertial type of π', set λ' to be its weight, and note that $\lambda_{v'} = \lambda_v$. Let T be a suitable finite set of places containing $S_p(F')$ such that $(\pi')^{T\cup\{\infty\}}$ is unramified. Pick a good subgroup $K^p \subset \mathrm{GL}_n(\mathbf{A}_{F'}^{\infty,p})$ such that $((\pi')^p)^{K^p} \neq 0$. Set $K = K^pK_p^0$ where $K_p^0 := \prod_{v\in S_p(F')} \mathrm{GL}_n(\mathcal{O}_{F'_v})$. Then, by Theorem 2.6.4,

$$\mathrm{Hom}_{K_p^0}(\sigma(\tau), (\pi')^{K^p}) \neq 0,$$

and $\mathbf{T}^{T,\lambda',\tau}[1/p]$ acts via scalars, inducing a map $x : \mathbf{T}^{T,\lambda',\tau}[1/p] \to \overline{\mathbf{Q}}_p$. For a large enough field extension $E/\mathbf{Q}_p$, [Fra98], and [FS98] show (cf. [All+23], Theorem 2.4.10) that $(\pi')^{K^pK'_p}$ can be found in

$$H^*(X_{K^pK'_p}, \mathcal{V}_{(\lambda',\tau)}[1/p])$$

as a $\mathbf{T}^T$-equivariant direct summand for any compact open normal subgroup $K'_p \subset K_p^0$ such that $\sigma(\tau)|_{K_p^0}$ is trivial. Since finite group cohomology is torsion, an argument with Hocschild–Serre spectral sequence shows that $\mathrm{Hom}_{K_p^0}(\sigma(\tau), (\pi')^{K^p})$ can be found in

$$H^*(X_K, \mathcal{V}_{(\lambda',\tau)}[1/p])$$

as a $\mathbf{T}^T$-equivariant direct summand. To see that this direct summand is also $\mathbf{T}^{T,\lambda',\tau}[1/p]$-equivariant one has to compare the natural action of $\mathbf{T}^{T,\lambda',\tau}[1/p]$ (cf. §2.6) on

$$\mathrm{Hom}_{K_p^0}(\sigma(\tau), (\pi')^{K^p})$$

with the one on

$$H^*(X_K, \mathcal{V}_{(\lambda',\tau)}[1/p])$$

given by Lemma 2.3.1. The two induced actions on the direct summand can be seen to coincide by writing both actions in terms of correspondences as in §2.4.

In the previous paragraph we proved that the map x factors through $\mathbf{T}^{T,\lambda',\tau}(K^p)_{\mathfrak{m}}$ where $\mathfrak{m} \subset \mathbf{T}^{T,\lambda',\tau}$ is the (non-Eisenstein and decomposed generic) maximal ideal so that $\overline{r_t(\pi')} \cong \overline{\rho}_{\mathfrak{m}}$. The Theorem then follows from applying Theorem 5.3.4 to $\mathfrak{m}$ and specialising the diagram in the statement at x. □

Appendix A

Bernstein–Zelevinsky and Langlands classifications

Since the theory of Bernstein–Zelevinsky and its relation to the Langlands classification is used at several points in §4.2, we recollect the necessary results of the theory. Throughout this section, we fix a finite field extension $L/\mathbf{Q}_p$ with a choice of uniformiser ϖ_L and, for any integer $m \in \mathbf{Z}_{\geq 1}$, we set $G_m := \mathrm{GL}_m(L)$. All of our representations will have $\overline{\mathbf{Q}}_p$-coefficients and recall that we fix an identification $t : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. For a smooth representation π of G_m, we set $\deg(\pi) := m$.

For an integer $n \in \mathbf{Z}_{\geq 1}$ with a partition $n = n_1 + ... + n_s$, consider the corresponding standard parabolic subgroup $Q \subset G_n$ with its Levi decomposition $Q = M \ltimes N$. Recall that for any smooth representations π_i of G_{n_i} ($i = 1, ..., s$) we denoted by $\text{n-Ind}_Q^{G_n} \pi_1 \otimes ... \otimes \pi_s$ the normalised parabolic induction. For this chapter, we will abbreviate this by writing

$$\pi_1 \times ... \times \pi_s := \text{n-Ind}_Q^{G_n} \pi_1 \otimes ... \otimes \pi_s.$$

Moreover, for any smooth representation π, we set $J_Q(\pi)$ to be the (unnormalised) Jacquet module of π with respect to Q. By [Zel80], Proposition 1.4, both operations carry finite length representations to finite length ones. (See [Zel80], Proposition 1.1 for some of the properties of these functors.)

For any smooth irreducible representation π of G_n, there is a unique multiset $\{\pi_1, ..., \pi_s\}$ of supercuspidal representations of auxiliary general linear groups G_{n_i} such that π is a subquotient of $\pi_1 \times ... \times \pi_s$. We will refer to this multi-set as the *supercuspidal* support of π.[1] Moreover, the supercuspidal support can be ordered in a way so that π is actually a subrepresentation of $\pi_1 \times ... \times \pi_s$ (cf. [BZ77], Theorem 2.5, Theorem 2.9).

Given a smooth representation π of G_n, for some $n \geq 1$, and a real number $a \in \mathbf{R}$, set $\pi(a)$ to be $\pi \otimes |\det|_L^a$. Zelevinsky introduced the notion of a

---

[1] We warn the reader that this is slightly unconventional as in the literature it is the pair $(\pi_1 \otimes ... \otimes \pi_s, G_{n_1} \times ... \times G_{n_s})$ that is referred to as the supercuspidal support of π.

segment, a set of isomorphism classes of irreducible supercuspidal representations of the form

$$\Delta(\pi, r) := \{\pi, \pi(1), ..., \pi(r-1)\}$$

where $r \geq 1$ is some integer. We also use the notation $[\pi, \pi(r-1)]$ for the set $\Delta(\pi, r)$. We further say that two segments Δ_1 and Δ_2 are linked if neither contains the other and $\Delta_1 \cup \Delta_2$ is also a segment. Finally, for two linked segments $\Delta_1 = [\pi(r_1), \pi(r_2)]$ and $\Delta_2 = [\pi(u_1), \pi(u_2)]$, respectively, we say that Δ_1 precedes Δ_2 if $r_1 \leq u_1$. Given a segment $\Delta = \Delta(\pi, r)$, we write

$$\pi(\Delta) := \pi \times ... \times \pi(r-1).$$

Then the Bernstein–Zelevinsky and Langlands classifications read as follows (cf. [Zel80], Theorem 6.1, [Rod82], Theorem 3).

Theorem A.0.1. *i. Given a segment $\Delta = \Delta(\pi, m)$, $\pi(\Delta)$ has length 2^{m-1} and admits a unique irreducible subrepresentation $Z(\Delta)$ and a unique irreducible quotient $L(\Delta)$ (called the Langlands quotient of the segment).*

ii. Call an ordered multiset of segments $(\Delta_1, ..., \Delta_l)$ well-ordered, if it is ordered in a way that, for $i < j$, Δ_i doesn't precede Δ_j. Then, for such a well-ordered multiset of segments, the representation

$$Z(\Delta_1) \times ... \times Z(\Delta_l)$$

admits a unique irreducible subrepresentation $Z(\Delta_1, ..., \Delta_l)$. Similarly, the representation

$$L(\Delta_1) \times ... \times L(\Delta_l)$$

admits a unique irreducible quotient $L(\Delta_1, ..., \Delta_l)$ (called the Langlands quotient of the multiset of segments $(\Delta_1, ..., \Delta_l)$). Moreover, in both cases, the isomorphism class of the obtained representation does not depend on the chosen order.

iii. For any smooth irreducible representation π of G_n, up to reordering, there is a unique well-ordered multiset of segments $(\Delta_1, ..., \Delta_l)$ resp. $(\Delta'_1, ..., \Delta'_h)$ such that $\pi \cong Z(\Delta_1, ..., \Delta_l)$ resp. $\pi \cong L(\Delta'_1, ..., \Delta'_h)$.

The relation between the Bernstein–Zelevinsky and Langlands classifications is slightly subtle. Nevertheless, Zelevinsky introduced an involution which allows one to pass between the two to some extent. Namely, for every $n \geq 1$, one can look at the Grothendieck group $\mathcal{R}_n$ of finite length smooth representations of G_n. Then the operation of normalised parabolic induction makes $\mathcal{R} := \oplus_{n \geq 0} \mathcal{R}_n$ into a graded commutative ring (cf. [Zel80], 1.9). Moreover, according to [Zel80], Corollary 7.5, $\mathcal{R}$ is in fact a polynomial algebra over $\mathbf{Z}$ with indeterminates given by $Z(\Delta)$. Therefore, one can define a ring endomorphism

$$\mathcal{D} : \mathcal{R} \to \mathcal{R}$$

by sending, for any segment Δ, the element $Z(\Delta)$ to $L(\Delta)$ and linearly extending it. One observes that $\mathcal{D}$ is in fact an involution which sends $Z(\Delta(\pi, r))$ to $Z(\pi, ..., \pi(r-1))$ (cf. [Zel80], 9.15). Moreover, Zelevinsky conjectured that $\mathcal{D}$ sends irreducible representations to irreducible representations ([Zel80], 9.17). Assuming Zelevinsky's conjecture, Rodier deduced that, for a (well-ordered) multiset of segments $(\Delta_1, ..., \Delta_l)$, $\mathcal{D}$ sends $Z(\Delta_1, ..., \Delta_l)$ resp. $L(\Delta_1, ..., \Delta_l)$ to $L(\Delta_1, ..., \Delta_l)$ resp. $Z(\Delta_1, ..., \Delta_l)$. Bernstein proposed a proof of the conjecture which stayed unpublished. The first written up proofs can be found in [Aub95] and [Pro98], respectively. Another property of the involution which will be useful for us is the fact that it commutes with the operation of twisting by the determinant character. This can be easily seen by first verifying it on the generators of $\mathcal{R}$ as a polynomial algebra over $\mathbf{Z}$ corresponding to segments.

Given the Langlands classification, we can explain the reduction of the local Langlands correspondence for GL_n to a correspondence between the set of supercuspidal representations and irreducible Weil–Deligne representations. Namely, given a supercuspidal representation π of G_n, [HT01; Hen00] attaches to it an n-dimensional irreducible Weil–Deligne representation $\mathrm{rec}^T(\pi)$. Moreover, if π is only essentially square integrable i.e., according to Bernstein, it is of the form $L(\Delta(\pi', r))$ ([Zel80], Theorem 9.3), we set $\mathrm{rec}^T(\pi) = \mathrm{rec}^T(\pi') \otimes \mathrm{Sp}(r)$ where $\mathrm{Sp}(r)$ is the Steinberg representation (see page 213 of [Rod82]). Finally, if π is a general smooth irreducible representation of the form $L(\Delta_1, ..., \Delta_l)$, we set $\mathrm{rec}^T(\pi) = \oplus_{i=1}^{l} \mathrm{rec}^T(L(\Delta_i))$.

Since we will mainly be working with smooth irreducible representations of G_n coming from cuspidal automorphic representations, we can always assume that our representations π are so that $t\pi \otimes |\det|^{-s}$ is unitary for some $s \in \mathbf{R}$. We will refer to such representations as *t-preunitary* and if $s = 0$, then we further call it *t-unitary*. In particular, it will be useful for us to recall Tadic's classification of unitary irreducible smooth representations of G_n [Tad86] which reveals that, in the case of unitary representations, the relation between the Bernstein–Zelevinsky and Langlands classifications becomes explicit. To state the classification, we first need to introduce some further notation. Given a t-unitary supercuspidal representation π of G_m for some $m \in \mathbf{Z}_{\geq 1}$ and an integer $d \in \mathbf{Z}_{\geq 1}$, we define the unitary segment

$$\Delta^u(d, \pi) := [\pi(\frac{1-d}{2}), ..., \pi(\frac{d-1}{2})].$$

Then the building blocks of the classification will be the following two types of representations.

i. Given $d, n \in \mathbf{Z}_{\geq 1}$ and π a t-unitary supercuspidal representation of G_m

for some $m \in \mathbf{Z}_{\geq 1}$, we set

$$a(d,n,\pi) = Z\left(\Delta^u(d,\pi)(\frac{n-1}{2}), ..., \Delta^u(d,\pi)(\frac{1-n}{2})\right).$$

ii. Given $d, n \in \mathbf{Z}_{\geq 1}$, $\alpha \in (0, \frac{1}{2})$ and π a t-unitary supercuspidal representation of G_m for some $m \in \mathbf{Z}_{\geq 1}$, we set

$$a(d,n,\pi,\alpha) = a(d,n,\pi)(\alpha) \times a(d,n,\pi)(-\alpha).$$

The classification is as follows ([Tad86], Theorem A, Theorem B).

Theorem A.0.2. *Given an integer $m \geq 1$, all smooth representations of G_m obtained as normalised parabolic induction of type i) and type ii) representations are t-unitary and irreducible.*

Moreover, any smooth irreducible t-unitary representation of G_m can be obtained this way and the associated multiset of type i) and type ii) representations is well-defined (i.e., the associated ordered multiset is unique up to permutation).

Finally, under the Zelevinsky involution a type i) representation $a(d,n,\pi)$, respectively a type ii) representation $a(d,n,\pi,\alpha)$ is sent to $a(n,d,\pi)$, respectively $a(n,d,\pi,\alpha)$. In particular, we have

$$a(d,n,\pi) = a^L(n,d,\pi) := L\left(\Delta^u(n,\pi)(\frac{d-1}{2}), ..., \Delta^u(n,\pi)(\frac{1-d}{2})\right),$$

respectively

$$a(d,n,\pi,\alpha) = a^L(d,n,\pi)(\alpha) \times a^L(d,n,\pi)(-\alpha).$$

We conclude the appendix with two technical lemmas, the first of which relies on Tadic's classification. The role of the first computation is to understand the monodromy under local Langlands of the ordinary support in §4.2 (see Corollary 4.2.17).

Lemma A.0.3. *Let $\{\Delta_1, ..., \Delta_l\}$ be a multiset of segments, $r \in \mathbf{Z}_{\geq 2}$ and $0 = k_0 < k_1 < ... < k_r = l$ be integers such that*

$$\frac{\mathrm{val}_p(\pi_1'(\langle\varpi\rangle))}{\deg \pi_1'} \leq \frac{\mathrm{val}_p(\pi_2'(\langle\varpi\rangle))}{\deg \pi_2'} + \mathrm{val}_p(|\varpi|_K)$$

whenever π_1' resp. π_2' lies in the underlying supercuspidal support of $\{\Delta_{k_{i-1}+1}, ..., \Delta_{k_i}\}$ resp. $\{\Delta_{k_{j-1}+1}, ..., \Delta_{k_j}\}$ for $i < j$. In particular, if π_1' happens to be of the form $\pi_2' \otimes |\det|^s$, then $s \geq 1$. Assume that for each $1 \leq i \leq r$, $(\Delta_{k_{i-1}+1}, ..., \Delta_{k_i})$ is well-ordered.[2] *Finally, assume that*

$$\widetilde{\pi} := Z(\Delta_1, ..., \Delta_l) = L(\Delta_1', ..., \Delta_h')$$

---

[2] Note that this ensures that already $(\Delta_1, ..., \Delta_l)$ is well-ordered.

is t-preunitary and, for $i = 1, ..., r$, set

$$\pi_i := Z(\Delta_{k_{i-1}+1}, ..., \Delta_{k_i}) = L(\Delta''_{t_{i-1}+1}, ..., \Delta''_{t_i})$$

where $0 = t_0 < t_1 < ... < t_r$ are the appropriate integers. Then every segment Δ'_i is of the form

$$\bigcup_{j=1}^{r} \Delta'''_j$$

with $\Delta'''_j \in \{\emptyset, \Delta''_{k_{j-1}+1}, ..., \Delta''_{k_j}\}$.

Proof. Note that we can assume that $\widetilde{\pi}$ is t-unitary. This follows from the fact that rec^T is compatible with character twists and that the involution $\mathcal{D}$ commutes with twisting by the determinant character. Arguing by induction on r, we can further restrict ourselves to the case when $r = 2$. Therefore, we do assume the above restrictions and set $k := k_1$. In particular, we have access to Tadic's classification. This means that, since we know the shape of $(\Delta'_1, ..., \Delta'_h)$ by Theorem A.0.2, we need to compute the shape of the segments $(\Delta''_1, ..., \Delta''_{t_2})$.

First, we pin down the possible unitary representations appearing in Tadic's classification as building blocks which have underlying multiset of segments strictly separated by

$$\{(\Delta_1, ..., \Delta_k), (\Delta_{k+1}, ..., \Delta_l)\}. \tag{A.0.1}$$

By the assumption on the central characters of the supercuspidal support of the partition A.0.1, the representations of the second type appearing in $\widetilde{\pi} = Z(\Delta_1, ..., \Delta_l)$'s description via Tadic's classification have associated multiset of segments lying either in $(\Delta_1, ..., \Delta_k)$ or $(\Delta_{k+1}, ..., \Delta_l)$, respectively. Indeed, this is because all the neighbours of the supercuspidal support of such a representation differ by a power of the determinant character with exponent strictly smaller than 1. Now assume that a representation of the first type appears in $\widetilde{\pi}$ and its associated multiset of segments is strictly separated by the partition $\{(\Delta_1, ..., \Delta_k), (\Delta_{k+1}, ..., \Delta_l)\}$. Say it is given by $a(d, n, \pi)$ for some integers $d, n \geq 1$ and t-unitary supercuspidal representation π of G_m for some integer $m \geq 1$. Note that in the case when $n = 1$, we have only 1 segment so n must be at least 2. On the other hand, notice that d must be equal to 1. Indeed, if we assume that d is at least 2, then all the neighbours of

$$(\Delta^u(d, \pi)(\frac{n-1}{2}), ..., \Delta^u(d, \pi)(\frac{1-n}{2}))$$

have overlapping supercuspidal supports. In particular, by the assumption on the central characters, all the segments must be contained in $(\Delta_1, ..., \Delta_k)$ or $(\Delta_{k+1}, ..., \Delta_l)$, respectively. This leads to a contradiction. Therefore, d necessarily equals 1 and

$$a(d, n, \pi) = Z(\pi(\frac{n-1}{2}), ..., \pi(\frac{1-n}{2})).$$

We now have the following claim.
Claim: $Z(\Delta_1, ...\Delta_k)$ is obtained by applying normalised parabolic induction to a product of representations of type *i)*, type *ii)* and representations of the form

$$Z(\pi(\frac{n-1}{2}), ..., \pi(j)) = L([\pi(j), ..., \pi(\frac{n-1}{2})])$$

for some t-unitary supercuspidal representation π of G_m, $n \in \mathbf{Z}_{\geq 0}$ and $\frac{1-n}{2} \leq j \leq \frac{n-1}{2}$.

Proof of Claim. For a choice of t-unitary supercuspidal representation of G_m, set $S_\pi \subset (\Delta_1, ..., \Delta_k)$ to be the ordered subset of segments with supercuspidal support lying in $\{\pi \otimes |\det|^s\}_{s\in\mathbf{R}}$. By [Zel80], Proposition 8.5, $Z(\Delta_1, ..., \Delta_k)$ is the parabolic induction of the representations $Z(S_\pi)$ for all π such that $S_\pi \neq \emptyset$.[3] Moreover, S_π decomposes into S_π^{int} and S_π^{nonint} where the first ordered subset consists of the segments with supercuspidal support lying in $\{\pi \otimes |\det|^s\}_{s\in\frac{1}{2}\mathbf{Z}}$ and define the latter to be its complement. Again, $Z(S_\pi)$ is the parabolic induction of $Z(S_\pi^{\text{int}})$ and $Z(S_\pi^{\text{nonint}})$.

By looking at Tadic's classification, we see that S_π^{nonint} consists of all the segments corresponding to type *ii)* representations and nothing else. In particular, using Tadic's classification and [Zel80], Proposition 8.4, we see that it must be the parabolic induction of type *ii)* representations. We are left with spelling out $Z(S_\pi^{\text{int}})$.

We further divide S_π^{int} into the ordered multiset of segments S_π^{nonres} given by the collection of segments which correspond to a type i) representations which are not separated by the partition A.0.1 and S_π^{res} consisting of segments that are part of the multiset of segments of a type *i)* representation which is strictly separated by the partition A.0.1. Then, by [Zel80], Proposition 8.4, $Z(S_\pi^{\text{int}})$ occurs with multiplicity 1 in the set of Jordan-Hölder factors of

$$Z(S_\pi^{\text{nonres}}) \times Z(S_\pi^{\text{res}}). \tag{A.0.2}$$

Therefore, in order to prove the claim, it suffices to prove that A.0.2 is irreducible and that both $Z(S_\pi^{\text{nonres}})$ and $Z(S_\pi^{\text{res}})$ have the claimed shape.

Note that by our previous discussion $Z(S_\pi^{\text{res}})$ is of the form

$$Z(\{\pi(\frac{n_1-1}{2}), ..., \pi(j), \pi(\frac{n_2-1}{2}), ..., \pi(j), ..., \pi(\frac{n_f-1}{2}), ..., \pi(j)\}^{\text{ord}})$$

where $\{-\}^{\text{ord}}$ means that we chose any allowable ordering of the underlying multiset of segments, $f \in \mathbf{Z}_{\geq 0}$, $n_1, ..., n_f \in \mathbf{Z}_{\geq 1}$ and j is some integer between $\frac{1-n_i}{2}$ and $\frac{n_i-1}{2}$ for any $1 \leq i \leq f$. This is simply given by

$$Z(\pi(\frac{n_1-1}{2}), ..., \pi(j)) \times .... \times Z(\pi(\frac{n_f-1}{2}), ..., \pi(j)). \tag{A.0.3}$$

---

[3] Note that the order does not matter.

To see this, note that A.0.3 can be rewritten as

$$L([\pi(j), ..., \pi(\frac{n_1-1}{2})]) \times .... \times L([\pi(j), ..., \pi(\frac{n_f-1}{2})]).$$

This is irreducible by [Zel80], Proposition 9.7 so it must be $Z(S_\pi^{\text{res}})$ by [Zel80], Proposition 8.4.

By combining Tadic's classification with [Zel80], Proposition 8.4, we also see that $Z(S_\pi^{\text{nonres}})$ is of the form

$$a(\tilde{d}_1, \tilde{n}_1, \pi) \times ... \times a(\tilde{d}_h, \tilde{n}_h, \pi) = a^L(\tilde{n}_1, \tilde{d}_1, \pi) \times ... \times a^L(\tilde{n}_h, \tilde{d}_h, \pi)$$

for some $h \in \mathbf{Z}_{\geq 0}$, $\tilde{n}_1, ..., \tilde{n}_h, \tilde{d}_1, ..., \tilde{d}_h \in \mathbf{Z}_{\geq 1}$.

We are left with proving that A.0.2 is irreducible. Note that if we apply Zelevinsky duality to A.0.2 bearing in mind the previous discussion and Theorem A.0.2, we get

$$\Big(a(\tilde{n}_1, \tilde{d}_1, \pi) \times ... \times a(\tilde{n}_h, \tilde{d}_h, \pi)\Big) \times$$

$$\Big(Z([\pi(j), ..., \pi(\frac{n_1-1}{2})]) \times .... \times Z([\pi(j), ..., \pi(\frac{n_f-1}{2})])\Big).$$

By Proposition 8.5 of [Zel80], it suffices to see that for any $1 \leq a \leq h$, $\frac{1-\tilde{d}_a}{2} \leq b \leq \frac{\tilde{d}_a-1}{2}$ and $1 \leq c \leq f$, the segments

$$[\pi(\frac{1-\tilde{n}_a}{2}+b), ..., \pi(\frac{\tilde{n}_a-1}{2}+b)] \text{ and } [\pi(j), ..., \pi(\frac{n_c-1}{2})]$$

are not linked. Note that by assumption $j \in \frac{1}{2}\mathbf{Z}$ is the lowest such that $\pi(j)$ appears as a supercuspidal support of some segment in S_π^{int}. Therefore, we have

$$j \leq \frac{1-\tilde{n}_a}{2} + \frac{1-\tilde{d}_a}{2} \leq \frac{1-\tilde{n}_a}{2} + b.$$

On the other hand, if the two segments were linked, we would need to have

$$\frac{n_c-1}{2} < \frac{\tilde{n}_a-1}{2} + b \leq \frac{\tilde{n}_a-1}{2} + \frac{\tilde{d}_a-1}{2}.$$

So, by multiplying by -1, we would get

$$\frac{1-\tilde{n}_a}{2} + \frac{1-\tilde{d}_a}{2} \leq \frac{1-n_c}{2} < j,$$

a contradiction. Therefore, A.0.2 is indeed irreducible and we proved the claim. $\square$

The same observations apply to $Z(\Delta_{k+1}, ..., \Delta_l)$ and the lemma now follows. To be more precise, from the claim we see what can be the shape of a segment Δ''_{i_1} appearing in $Z(\Delta_1, ..., \Delta_k) = L(\Delta''_1, ..., \Delta''_{t_1})$. Moreover, the proof of the claim shows that if any such segment is of the form $[\pi(j), ..., \pi(\frac{n-1}{2})]$ with $\frac{1-n}{2} < j$, then there must be a corresponding segment Δ_{i_2} with $t_1+1 \leq i_2 \leq t_2$ such that they are disjoint and linked and their union is $[\pi(\frac{1-n}{2}), ..., \pi(\frac{n-1}{2})]$. Such pairs then build up to segments in $L(\Delta'_1, ..., \Delta'_h)$ and any other segments of the latter were already a segment of $(\Delta''_1, .., \Delta''_{t_2})$. □

Lemma A.0.4. *Let* $(\Delta_1, ..., \Delta_l)$ *be a well-ordered multiset of segments such that* $Z(\Delta_1, ..., \Delta_l)$ *has degree* n *and set* $Q_{\text{sc}} = M_{\text{sc}} \ltimes N_{\text{sc}} \subset G_n$ *to be the standard parabolic subgroup corresponding to the underlying supercuspidal support with the induced ordering. Then the Jacquet module* $J_{Q_{\text{sc}}}(Z(\Delta_1, ..., \Delta_l))$ *admits* $\delta_{Q_{\text{sc}}}^{1/2}\Delta_1 \otimes ... \otimes \Delta_l$ *as a quotient where, by abuse of notation, for a segment* $\Delta = [\pi, ..., \pi(r-1)]$ *we also denote by* Δ *the representation* $\pi \otimes ... \otimes \pi(r-1)$.

Proof. Let $Q = M \ltimes N \subset G_n$ be the parabolic subgroup corresponding to $Z(\Delta_1) \otimes ... \otimes Z(\Delta_l)$. Then Frobenius reciprocity applied to Theorem A.0.1 gives a surjection $J_Q(Z(\Delta_1, ..., \Delta_l)) \to \delta_Q^{1/2} Z(\Delta_1) \otimes ... \otimes Z(\Delta_l)$. Then [Zel80] Proposition 1.1 parts a) and c) combined with *loc. cit.* 3.1 implies that taking Jacquet module with respect to Q_{sc} gives the desired result. □

Bibliography

[ACa23] L. A'Campo. *Rigidity of automorphic Galois representations over CM fields.* preprint, arXiv:2202.14022, to appear in IMRN, 2023.

[AHV19] N. Abe, G. Henniart, and M.-F. Vigneras. *Modulo p representations of reductive p-adic groups: Functoriality properties.* Transactions of the A.M.S. 371, 8297-8337, 2019.

[All+23] P. B. Allen et al. *Potential automorphy over CM fields.* Ann. of Math. (2) 197, no. 3, 897-1113. MR 4564261, 2023.

[AN21] P. Allen and J. Newton. *Monodromy for some rank two Galois representations over CM fields.* preprint, arXiv:1901.05490, 2021.

[Aub95] A.-M. Aubert. *Dualité dans le groupe de Grothendieck de la catégorie des repr ésentations lisses de longueur finie dún groupe réductif p-adique.* Trans. Amer. Math. Soc. 347, no. 6, 2179-2189, 1995.

[Bar+11] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor. *Local-global compatibility for $l = p$, II.* Annales de la faculté des sciences de Toulouse Mathématiques 47(1), 2011.

[Bar+12] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor. *Local-global compatibility for $l = p$, I.* Annales de la Faculté des sciences de Toulouse : Mathématiques, Serie 6, Volume 21 no. 1, pp. 57-92., 2012.

[Bar+14] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor. *Potential automorphy and change of weight.* Ann. of Math. (2) 179, no. 2, 501-609, 2014.

[BC09] J. Bellaïche and G. Chenevier. *Families of Galois representations and Selmer groups.* Astérisque 324, pp. xii+314, 2009.

[BD20] C. Breuil and Y. Ding. *Higher $\mathcal{L}$-invariants for* $\mathrm{GL}_r(\mathbf{Q}_p)$ *and local-global compatibility.* Cambridge J. of Math. 8, 775-951, 2020.

[BD84] J. Bernstein and P. Deligne. *Le "centre" de Bernstein.* In "Representations des groups redutifs sur un corps local, Traveaux en cours" (P.Deligne ed.), Hermann, Paris, 1-32, 1984.

[BG14] K. Buzzard and T. Gee. *The conjectural connections between automorphic representations and Galois representations.* In Automorphic forms and Galois representations. Vol. 1, volume 414 of London Math. Soc. Lecture Note Ser., pages 135-187. Cambridge Univ. Press, Cambridge, 2014.

[BG19] R. Bellovin and T. Gee. *G-valued local deformation rings and global lifts.* Algebra and Number Theory 13.2, pp. 333-378, 2019.

[BK98] C. J. Bushnell and P. C. Kutzko. *Smooth representations of reductive p-adic groups: structure theory via types.* Proc. London Math. Soc. (3) 77, no. 3, 582-634., 1998.

[BK99] C. J. Bushnell and P. C. Kutzko. *Semisimple types in* GL_n. Compositio Math. 119.1 (1999), pp. 53-97, 1999.

[Bre+01] C. Breuil, B. Conrad, F. Diamond, and R. Taylor. *On the modularity of elliptic curves over Q: wild 3-adic exercises.* J. Amer. Math. Soc. 14, 843–939. MR 2002d:11058 Zbl 0982.1103, 2001.

[BS73] A. Borel and J.-P. Serre. *Corners and arithmetic groups.* Comment. Math. Helv. 48, 436-491, Avec un appendice: Arrondissement des variétés á coins, par A. Douady et L. Hérault, 1973.

[BV13] N. Bergeron and A. Venkatesh. *The asymptotic growth of torsion homology for arithmetic groups.* J. Inst. Math. Jussieu, 12(2):391-447, 2013.

[BZ77] I.N. Bernstein and A.V. Zelevinsky. *Induced representations of reductive* $\mathfrak{p}$*-adic groups. I.* Ann.Sci.École Norm. Sup. (4) 10(4), 441-472, 1977.

[Cab84] M. Cabanes. *Irreducible modules and Levi supplements.* J. Algebra, 90(1):84–97, 1984.

[Car+16a] A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paskunas, and S. W. Shin. *Patching and the p-adic Langlands program for* $GL_2(Q_p)$. Compositio Math. 154 (2018), no. 3, 503-548., 2016.

[Car+16b] A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paskunas, and S. W. Shin. *Patching and the p-adic local Langlands correspondence.* Cambridge Journal of Math. 4, no. 2, 197-287., 2016.

[Car+21] A. Caraiani, M. Emerton, T. Gee, and D. Savitt. *Components of moduli stacks of two-dimensional Galois representations.* preprint, 2021.

[Car12] A. Caraiani. *Local-global compatibility and the action of monodromy on nearby cycles.* Duke Math. J. 161(12): 2311-2413, 2012.

[Car14] A. Caraiani. *Monodromy and local-global compatibility for* $l = p$. Algebra Number Theory 8, no. 7, 1597-1646, 2014.

[Car79] P. Cartier. *Representations of p-adic groups: a survey.* In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

[Car86] H. Carayol. *Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert.* Ann.Sci.ENS 19, 409±468, 1986.

[Cas95] W. Casselman. *Introduction to the theory of admissible representations of p-adic reductive groups.* preprint, 1995.

[CDT99] B. Conrad, F. Diamond, and R. Taylor. *Modularity of certain potentially Barsotti–Tate Galois representations.* J. Amer. Math. Soc. 12, 521–567. MR 99i:11037 Zbl 0923.11085, 1999.

[CG18] F. Calegari and D. Geraghty. *Modularity lifting beyond the Taylor-Wiles method.* Invent. Math. 211, no. 1, 297-433, 2018.

[CH13] G. Chenevier and M. Harris. *Construction of automorphic Galois representations II.* Cambridge Journal of Mathematics 1, 2013.

[Che09] G. Chenevier. *Une application des variétés de Hecke des groupes unitaires.* preprint, 2009.

[CHT08] L. Clozel, M. Harris, and R. Taylor. "Automorphy for some l-adic lifts of automorphic mod l Galois representations". In: *Publ. Math. Inst. Hautes Etudes Sci., (108):1-181. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras* (2008).

[Clo06] L. Clozel. *Motives and automorphic representations.* ffhal-01019707f, 2006.

[Clo13] L. Clozel. *Purity reigns supreme.* International Mathematics Research Notices, Volume 2013, Issue 2, 328-346, 2013.

[Clo22] L. Clozel. "On the Eisenstein functoriality in cohomology for maximal parabolic subgroups". In: *Selecta Mathematica* 28 (2022). DOI: `10.1007/s00029-022-00794-y`.

[Clo90] L. Clozel. *Motifs et formes automorphes: applications du principe de fonctorialité.* In Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), volume 10 of Perspect. Math., pages 77-159. Academic Press, Boston, MA, 1990.

[CN23] A. Caraiani and J. Newton. *On the modularity of elliptic curves over imaginary quadratic fields.* preprint, 2023.

[Con10] B. Conrad. *The Shimura construction for weight 2, Appendix to Lectures on Serre's conjectures by K. A. Ribet and W. A. Stein.* 2010. URL: `https://api.semanticscholar.org/CorpusID:125304959`.

[CS19] A. Caraiani and P. Scholze. *On the generic part of the cohomology of non-compact unitary Shimura varieties.* preprint, 2019.

[CS23] A. Caraiani and S. W. Shin. *Recent progress on Langlands reciprocity for GLn: Shimura varieties and beyond.* preprint, 2023.

[Del71] P. Deligne. *Travaux de Shimura.* Séminaire Bourbaki, 23 éme année (1970/71), Exp. No. 389, Springer, Berlin, pp. 123-165. Lecture Notes in Math., Vol. 244., 1971.

[DFG04] F. Diamond, M. Flach, and L. Guo. *The Tamagawa number conjecture for adjoint motives of modular forms.* Ann. Sci. Ec. Norm. Sup 37, 663-727, 2004.

[Dia96] F. Diamond. *On deformation rings and Hecke rings.* Ann. Math. 144, 137–166. MR 97d:11172, 1996.

[Dia97] F. Diamond. *The Taylor-Wiles construction and multiplicity one.* Invent. Math. 128, 379–391. MR 98c:11047 Zbl 0916.11037, 1997.

[Dot21] A. Dotto. *Breuil–Mézard conjectures for central division algebras.* arXiv:1808.06851, 2021.

[DS74] P. Deligne and J.-P. Serre. *Formes modulaires de poids 1.* Ann. Sci. Ecole Norm. Sup. (4) 7, 507-530 (1975). MR 379379, 1974.

[Eme06a] M. Emerton. *Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties.* Ann. Sci. Ecole Norm. Sup. 39, p. 775-839., 2006.

[Eme06b] M. Emerton. *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms.* Invent. Math. 164, no. 1, 1-84. MR 2207783, 2006.

[Eme10a] M. Emerton. *Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties.* Astérisque, no. 331, 355-402, 2010.

[Eme10b] M. Emerton. *Ordinary parts of admissible representations of p-adic reductive groups II.* Astérisque, no. 331, 403-459, 2010.

[Eme11] M. Emerton. *Local-global compatibility in the p-adic Langlands programme for* $\mathrm{GL}_2/\mathbf{Q}$. preprint, 2011.

[Eme17] M. Emerton. *Locally analytic vectors in representations of locally p-adic analytic groups.* Mem. Amer. Math. Soc. 248, no. 1175, iv+158 pp, 2017.

[Fal87] G. Faltings. *Hodge-Tate structures and modular forms.* Math. Ann., 278(1-4):133-149, 1987.

[Far11] L. Fargues. *Motives and automorphic forms: the (potentially) abelian case.* IHES-Orsay summer school on motives proceedings, 2011.

[Fla79] D. Flath. *Decomposition of representations into tensor products.* In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pages 179–183. Amer. Math. Soc., Providence, R.I., 1979.

[Fon94] J. M. Fontaine. *Représentations p-adiques semi-stables.* Astérisque, no. 223, 113-184, With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988). MR MR1293972 (95g:14024), 1994.

[Fra98] J. Franke. *Harmonic analysis in weighted L2-spaces.* Ann. Sci. Ecolé Norm. Sup. (4) 31, no. 2, 181-279., 1998.

[FS98] J. Franke and J. Schwermer. *A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups.* Math. Ann., 311(4):765-790, 1998.

[Gel75] Stephen S. Gelbart. *Automorphic Forms on Adele Groups.* . (AM-83), Volume 83, Princeton University Press, 1975, 1975.

[Ger18] D. Geraghty. *Modularity lifting theorems for ordinary Galois representations.* Mathematische Annalen, 2018.

[GN20] T. Gee and J. Newton. *Patching and the completed homology of locally symmetric spaces.* Journal of the Institute of Mathematics of Jussieu, 1-64. doi:10.1017/S1474748020000158, 2020.

[Har+16] M. Harris, K.-W. Lan, R. Taylor, and J. Thorne. *On the rigid cohomology of certain Shimura varieties.* Res. Math. Sci. 3, 3:37, 2016.

[Hau16] J. Hauseux. *Extensions entre séries principales p-adiques et modulo p.* J. Inst. Math. Jussieu 15, no. 2, 225-270, 2016.

[Hau18] J. Hauseux. *Parabolic induction and extensions.* Algebra Number Theory 12, no. 4, 779-831., 2018.

[Hel16] D. Helm. *The Bernstein center of the category of smooth* $W(k)[GL_n(F)]$*-modules.* Forum Math. Sigma 4, e11, 2016.

[Hen00] G. Henniart. *Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique.* Invent. Math. 139.2 (2000), pp. 439-455., 2000.

[HT01] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties.* Vol. 151. Annals of Mathematics Studies. With an appendix by Vladimir G. Berkovich. Princeton, NJ: Princeton University Press, pp. viii+276., 2001.

[Hu09] Y. Hu. *Normes invariantes et existence de filtrations admissibles.* J. Reine Angew. Math. 634, 107-141., 2009.

[Hun+18] B. Le Hung, D. Le, B. Levin, and S. Morra. *Local models for Galois deformation rings and applications.* to appear in Inventiones Mathematicae, 2018.

[JT] C. Johansson and J. Thorne. *On subquotients of the étale cohomology of Shimura varieties.* In T. Haines I& M. Harris (Eds.), Shimura Varieties (London Mathematical Society Lecture Note Series), pp. 306-334.

[Kis07] M. Kisin. *Potentially semi-stable deformation rings.* J. Amer. Math. Soc. 21, no. 2, 513-547, 2007.

[Kis09a] M. Kisin. *Moduli of finite flat group schemes, and modularity.* Ann. of Math. (2) 170, no. 3, 1085-1180, 2009.

[Kis09b] M. Kisin. *The Fontaine-Mazur conjecture for GL2.* J. Amer. Math. Soc. 22, no. 3, 641– 690. MR 2505297, 2009.

[Kos21] T. Koshikawa. *On the generic part of the cohomology of local and global Shimura varieties.* 2021. URL: `https://arxiv.org/pdf/2106.10602.pdf`.

[KS94] M. Kashiwara and P. Schapira. *Sheaves on manifolds.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original, 1994.

[KT17] C. B. Khare and J. A. Thorne. *Potential automorphy and the Leopoldt conjecture.* Amer. J. Math. 139, no. 5, 1205-1273, 2017.

[Lan79] R. P. Langlands. *Automorphic representations, Shimura varieties, and motives.* M¨archen, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 205-246. MR 546619, 1979.

[Lan89] R. P. Langlands. *On the classification of irreducible representations of real algebraic groups.* In Representation theory and harmonic analysis on semisimple Lie groups, volume 31 of Math. Surveys Monogr., pages 101-170. Amer. Math. Soc., Providence, RI., 1989.

[Mat24] K. Matsumoto. *On the potential automorphy and the local-global compatibility for the monodromy operators at pl over CM fields.* arXiv:2312.01551 [math.NT], 2024.

[MT22] K. Miagkov and J. Thorne. *Automorphy lifting with adequate image.* preprint, 2022.

[NT16] J. Newton and J. A. Thorne. *Torsion Galois representations over CM fields and Hecke algebras in the derived category.* Forum Math. Sigma 4, Paper No. e21, 88, 2016.

[Pan20] L. Pan. *On locally analytic vectors of the completed cohomology of modular curves.* arXiv:2008.07099, 2020.

[Pan22] L. Pan. *The Fontaine-Mazur conjecture in the residually reducible case.* J. Amer. Math. Soc. 35 (2022), 1031-1169, 2022.

[Pan23] L. Pan. *On locally analytic vectors of the completed cohomology of modular curves II.* arXiv:2209.06366, 2023.

[Pro98] K. Procter. *Parabolic induction via Hecke algebras and the Zelevinsky duality conjecture.* Proc. London Math. Soc. (3) 77, no. 1, 79-116, 1998.

[Pyv20a] A. Pyvovarov. *Generic smooth representations.* Documenta Math. 25, pp. 2473- 2485, 2020.

[Pyv20b] A. Pyvovarov. *On the Breuil-Schneider conjecture: generic case.* Algebra and Number Theory 15, Issue 2, pp. 309-339, 2020.

[Rib77] K. A. Ribet. *Galois representations attached to eigenforms with nebentypus.* In Modular functions of one variable V, pages 18–52. Springer, 1977, 1977.

[Rod82] F. Rodier. *Représentations de GL(n,k) oú k est un corps p-adique.* Séminaire Bourbaki exp. 587 (1981-1982), Astérisque 92-93, 201-218., 1982.

[Sai97] T. Saito. *Modular forms and p-adic Hodge theory.* Invent math 129, 607–620. https://doi.org/10.1007/s002220050175, 1997.

[Sch13] P. Scholze. *The Langlands-Kottwitz approach for the modular curve.* unpublished preprint, 2013.

[Sch15] P. Scholze. *On torsion in the cohomology of locally symmetric varieties.* Ann. of Math. (2) 182, no. 3, 945-1066. MR 3418533, 2015.

[Sch88] N. Schappacher. *Periods of Hecke Characters.* Astronomy and Astrophysics Library. Springer-Verlag, 1988. ISBN: 9783540189152. URL: `https://books.google.co.uk/books?id=sQbvAAAAMAAJ`.

[Sch98] P. Schneider. *Equivariant homology for totally disconnected groups.* J. Algebra 203, no. 1, 50-68, 1998.

[Ser69] J.-P. Serre. *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures).* Séminaire Delange-Pisot-Poitou. Théorie des nombres, Volume 11 (1969-1970) no. 2, Talk no. 19, 15 p., 1969.

[Shi11] S. W. Shin. *Galois representations arising from some compact Shimura varieties.* Ann. of Math. (2) 173, 1645-1741. MR 2800722 (2312, 2314, 2316, 2322, 2382, 2387, 2389, 2391, 2395, 2396, 2400, 2401, 2402, 2403, 2404, 2405, 2409)., 2011.

[Shi14] S. W. Shin. *On the cohomological base change for unitary similitude groups.* Compos. Math. 150, no. 2, 220-225, Appendix to Galois representations associated to holomorphic limits of discrete series, by Wushi Goldring., 2014.

[ST06] P. Schneider and J. Teitelbaum. *Banach-Hecke algebras and p-adic Galois representations.* Doc. Math., no. Extra Vol., 631-684. MR 2290601 (2008b:11126), 2006.

[Sta24] The Stacks project authors. *The Stacks project.* `https://stacks.math.columbia.edu`. 2024.

[SW99] C. Skinner and A. Wiles. *Residually reducible representations and modular forms.* Inst. Hautes Etudes Sci. Publ. Math. IHES ´ 89, 5-126., 1999.

[SZ99] P. Schneider and E.-W. Zink. *K-types for the tempered components of a p-adic general linear group.* J. Reine Angew. Math. 517. With an appendix by Schneider and U. Stuhler, pp. 161-208, 1999.

[Tad86] M. Tadic. *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case).* Ann. Sci. Éc. Norm. Supér. (4) 19 ,335-382.MR0870688 (2383)., 1986.

[Tho12] J. Thorne. *On the automorphy of l-adic Galois representations with small residual image.* Journal of the Institute of Mathematics of Jussieu, 1-66, 2012.

[Tho15] J. A. Thorne. *Automorphy lifting for residually reducible l-adic Galois representations.* J. Amer. Math. Soc. 28, no. 3, 785-870, 2015.

[TW95] R. Taylor and A. Wiles. *Ring-theoretic properties of certain Hecke algebras.* Ann. of Math. 141, 553–572. MR 96d:11072 Zbl 0823.11030, 1995.

[TY07] R. Taylor and T. Yoshida. *Compatibility of local and global Langlands correspondences.* J. Amer. Math. Soc. 20, 467-493. MR 2276777. Zbl 05120650. doi: 10.1090/S0894-0347-06-00542-X., 2007.

[Var14] I. Varma. *Local-global compatibility for regular algebraic cuspidal automorphic representation when $\ell \neq p$.* Forum Math. Sigma, to appear, arXiv:1411.2520., 2014.

[Vig04] M.-F. Vignéras. *Representations modulo p of the p-adic group GL(2, F)*. Compos. Math., 140(2):333-358, 2004.

[Wei94] C. A. Weibel. *An introduction to homological algebra*. volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.

[Wil95] A. Wiles. *Modular elliptic curves and Fermat's last theorem.* Ann. of Math. 141, 443–551. MR 96d:11071 Zbl 0823.11029, 1995.

[Yan21] Y. Yang. *An ordinary rank-two case of local-global compatibility for automorphic representations of arbitrary weight over CM fields*. arXiv: 2111.00318 [math.NT], 2021.

[Zel80] A. V. Zelevinsky. *Induced representations of reductive p-adic groups. II. On irreducible representations of GL(n).* Ann. Sci. École Norm. Sup. (4) 13.2, pp. 165-210., 1980.

[Zhu20] Y. Zhu. *Introduction to the Langlands–Kottwitz method.* A chapter in the Paris Book Project Shimura Varieties, London Mathematical Society Lecture Note Series: 457 (edited by T. Haines and M. Harris), 2020.