E_{11} and exceptional field theory

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Abstract

We argue that exceptional field theory is a truncation of the non-linear realisation of the semi-direct product of E_{11} and its first fundamental as proposed in 2003. Evaluating the simple equations of the E_{11} approach, and using the commutators of the E_{11} algebra, we find the local variations of the fields of exceptional field theory after making a radical truncation. This proceedure does not respect any of the higher level E_{11} symmetries and so these are lost. We suggest that the need for the section condition in exceptional field theory could be a consequence of the truncation.

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1 Introduction

The scalars that are part of a supergravity multiplet occur in the corresponding supergravity theory as a coset, or non-linear realisation, and the full supergravity theory possess the symmetries of the coset. This phenomenon was first observed in the context of the four dimensional $N = 4$ supergravity [1]. However, it was a surprise when it was found that the maximal supergravity theory in four dimensions possessed seventy scalars which belonged to a coset of E_7 with subgroup $SU(8)$ [2]. The other maximal supergravities in D dimensions for $D \leq 9$ were then found to possess a corresponding E_{11-D} symmetry [3]. While the IIB supergravity theory posses a SL(2,R) symmetry [4].

It was almost universally thought that the E_n symmetries found in the maximal supergravity theories were a result of the dimensional reduction process and did not lift to the ten or eleven dimensional theories. As a result it was not thought that the symmetries were part of any underlying theory. However in 2001 it was proposed that the underlying theory of strings and branes had an E_{11} symmetry regardless of which dimension it was formulated in [5]. In particular, it was conjectured that the E_{11} symmetry was encoded in a non-linear realisation which extended the known supergravity theories. This proposal was motivated by the fact [6] that the maximal supergravity theories could be formulated as non-linear realisations of an algebra which contained traces of the E_{11} algebra. This conjecture can be put another way; it was proposed that the coset, or non-linear realisation, that the scalars belonged to generalised to include the other fields of the supergravity theory and that the underlying group had the features we are familiar in physics, that is, it is a Kac-Moody group, then as a consequence one that one had include in the non-linear realisation many more fields in the theory than just those found in the usual supergravity theories. This procedure involved introducing generators that carried space-time indices, that is, Lorentz indices corresponding to the fields that were not scalars. An account of the literature on symmetries in supergravity theories can be found in reference [5] and an even more detailed account in [7].

The paper of reference [5] put to one side the question of how to systematically incorporate space-time into the theory, however, this question was addressed in 2003 when it was proposed that the one should consider the non-linear realisation of the semi-direct product of E_{11} with generators that belonged to the first fundamental representation of E_{11}, called the l_1 representation [8]. While the E_{11} part of the non-linear realisation lead to an infinite number of fields the first few of which were the fields of the maximal supergravity theories, the l_1 part lead to an infinite number of coordinates that were in a one to one relation with the generators of the l_1 realisation. To be more precise, taking into account the local symmetries of the non-linear realisation, for each generator in the Borel subgroup of E_{11} the theory possessed a corresponding field, while for each generator of the l_1 representation one found a coordinate. The fields then depended on the coordinates of this generalised space-time.

The theories in the different dimensions emerge by carrying out the $E_{11} \otimes_s l_1$ non-linear realisation for different decompositions of the algebra [9,10,11,12]. The theory in D dimensions comes from taking the decomposition into $GL(D) \times E_{11-D}$. This choice is unique except for ten dimensions where there are two possible choices of GL(10), or A_9 and so two possible theories in ten dimensions [5,6]. These are, of course, the IIA and

IIB theories. In each dimension the generators of E_{11} (fields) can be classified by a level which is the number of up (down) minus down (up) indices. The two exceptions are in eleven dimensions where this number must be divided by three and the ten dimensional IIA theory where the counting is straightforward, but more subtle. The same is true for the generators (coordinates) of the l_1 representation but one must add (subtract) one.

There is very good evidence that the l_1 representation contains all brane charges [8,13,14,15]. Consequently, there is a one to one correspondence between the coordinates of the generalised space-time and the brane charges. Indeed one can think of each coordinate as associated with a given type of brane probe. Furthermore, for every generator in the Borel subalgebra of E_{11} there is at least one element in the l_1 representation [13], and as a result for every field at low level in the non-linear realisation one finds at least one corresponding coordinate. The reader will easily see this correspondence between the coordinates and fields at low levels. Papers [15,14,13] discussed the appearance in the l_1 representation of "exotic" representations of GL(D), that is, representations that are not just labelled by a single set of antisymmetrised indices. The appearance of "exotic" brane charges implies the existence of correspondingly exotic generalised space-time coordinates.

In any dimension, with the exception of the IIA theory, the level zero coordinates of the l_1 representation are just the usual coordinates of space-time, x^a. In eleven dimensions the l_1 representation leads to the coordinates [8,13]

$$x^a(0), x_{ab}(1), x_{a_1...a_5}(2), x_{a_1...a_7,b}(3), x_{a_1...a_8}(3),$$

$$x_{\hat{b}_1\hat{b}_2\hat{b}_3,\hat{a}_1...\hat{a}_8}\ (4),\ \ x_{(\hat{c}\hat{d}),\hat{a}_1...\hat{a}_9}\ (4),\ \ x_{\hat{c}\hat{d},\hat{a}_1...\hat{a}_9}\ (4),\ \ x_{\hat{c},\hat{a}_1...\hat{a}_{10}}\ (4),\ \ \ldots \qquad (1.1)$$

where the number in brackets gives the level. The coordinates up to and including level seven were given in reference [13]. An analysis of the equations determining the generalised coordinates in lower dimensions was given in reference [15]. At level one in the l_1 representation one finds coordinates which are scalars under the Lorentz group but transform as [15,16]

$$10,\quad \overline{16},\quad \overline{27},\quad 56,\quad \text{and} \quad 248 \oplus 1,\quad \text{of} \quad SL(5),\quad SO(5,5),\quad E_6,\quad E_7 \quad \text{and } E_8 \quad (1.2)$$

for $D = 7, 6, 5, 4$ and 3 dimensions respectively.

The generators of the l_1 representation in D dimensions that are forms, that is, carry a single set of completely anti-symmetrised space-time indices, are listed [12,15,16,14] in the table below. The coordinates of the generalised space-time, being in one to one correspondence with the generators of the l_1 representation, it is trivial to read them off from the table. In the first column we find the Lorentz scalar coordinates mentioned just above.

A not well appreciated point is that the non-linear realisation does not just specify the representations of the fields and coordinates contained in the theory, it also specifies the dynamics. By its definition the non-linear realisation possess specified symmetries which act on the group element. The fields and coordinates of the theory are the parameters in the group element and as the symmetries determine the way the group element appears in the theory they also determine the way in which the fields appear.

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Non-linear realisations played a crucial part in the development of particle physics. The group of interest to researchers at that time was $SU(2) \times SU(2)$ with local subgroup being the diagonal subgroup SU(2).The dynamics which the non-linear realisation determined was found to be that of the pions and as a result researchers begun to understand the role symmetry played in the underlying theory.

Table. The generators in the l_1 representation in D dimensions

D	G	Z	Z^a	$Z^{a_1 a_2}$	$Z^{a_1...a_3}$	$Z^{a_1...a_4}$	$Z^{a_1...a_5}$	$Z^{a_1...a_6}$	$Z^{a_1...a_7}$
8	$SL(3) \otimes SL(2)$	$(\mathbf{3,2})$	$(\mathbf{\bar{3},1})$	$(\mathbf{1,2})$	$(\mathbf{3,1})$	$(\mathbf{\bar{3},2})$	$(\mathbf{1,3})$ $(\mathbf{8,1})$ $(\mathbf{1,1})$	$(\mathbf{3,2})$ $(\mathbf{6,2})$	$(\mathbf{6,1})$ $(\mathbf{18,1})$ $(\mathbf{3,1})$ $(\mathbf{6,1})$ $(\mathbf{3,3})$
7	$SL(5)$	$\mathbf{10}$	$\mathbf{\bar{5}}$	$\mathbf{5}$	$\mathbf{\overline{10}}$	$\mathbf{24}$ $\mathbf{1}$	$\mathbf{40}$ $\mathbf{15}$ $\mathbf{10}$	$\mathbf{70}$ $\mathbf{50}$ $\mathbf{45}$ $\mathbf{5}$	- - - -
6	$SO(5,5)$	$\mathbf{\overline{16}}$	$\mathbf{10}$	$\mathbf{16}$	$\mathbf{45}$ $\mathbf{1}$	$\mathbf{\overline{144}}$ $\mathbf{16}$	$\mathbf{320}$ $\mathbf{126}$ $\mathbf{120}$	- - -	- - -
5	E_6	$\mathbf{\overline{27}}$	$\mathbf{27}$	$\mathbf{78}$ $\mathbf{1}$	$\mathbf{\overline{351}}$ $\mathbf{\overline{27}}$	$\mathbf{1728}$ $\mathbf{351}$ $\mathbf{27}$	- - -	- - -	- - -
4	E_7	$\mathbf{56}$	$\mathbf{133}$ $\mathbf{1}$	$\mathbf{912}$ $\mathbf{56}$	$\mathbf{8645}$ $\mathbf{1539}$ $\mathbf{133}$ $\mathbf{1}$	- - - -	- - - -	- - - -	- - - -
3	E_8	$\mathbf{248}$ $\mathbf{1}$	$\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{30380}$ $\mathbf{3875}$ $\mathbf{248}$ $\mathbf{1}$	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -

The construction of the $E_{11} \otimes_s l_1$ non-linear realisation was computed in many of the early E_{11} papers in various dimensions for the fields at low levels, but keeping only the usual coordinates of space-time and taking the local subgroup to be just the Lorentz group. However, the very unfamiliar nature of the many of the higher level fields and especially the coordinates inhibited the construction for a number of years. Nonetheless, in 2007 all gauged maximal supergravities in five dimensions were constructed by taking the fields to depend on some of the generalised coordinates [12]. In 2009 the papers [17,18] computed the dynamics of the $E_{11} \otimes_s l_1$ non-linear realisation in four dimension keeping

the 56 Lorentz scalar coordinates in addition to the usual four coordinates of space-time, but only the fields at level zero, that is, the metric and the scalar fields.

Subsequently the $E_{11} \otimes_s l_1$ non-linear realisations was constructed in dimensions four to seven and corresponding Lagrangians were constructed but keeping only the scalar fields and only the level one coordinates, that is, the coordinates of equation (1.2) [19]. This meant discarding the graviton and and even the usual coordinates of space-time as well as all fields at all levels greater than zero, except the scalar fields, and all coordinates at level greater than one [19]. In fact these Lagrangians in six and seven dimensions had previously been constructed [20], but the results of [19] make it clear that they were just the non-linear realisation of $E_{11} \otimes_s l_1$ radically truncated in the way described.

A more systematic construction of the $E_{11} \otimes_s l_1$ non-linear realisation was undertaken more recently in eleven dimensions [21] and in four dimensions [22]. In the former paper the eleven-dimensional equations of motion involving the fields at levels up to and including the dual graviton (level 3) and the generalised coordinates x^a, $x_{a_1 a_2}$ and $x_{a_1 \dots a_5}$ were given. While in reference [22], the four dimensional equations involving the scalars and vectors and the coordinates at levels zero and one, that is, the usual coordinates of space-time and the Lorentz scalar coordinates in the 56-dimensional representation of E_7, were given. It was found that once one specified the Lorentz, and in the case of four dimensions also the SU(8), character of the equations they were unique. Furthermore these equations when restricted to the usual space-time and fields did indeed agree with those of maximal supergravity, although the equation relating the usual graviton to the dual graviton required further study to be sure of its precise form.

The non-linear realisation of $E_{11} \otimes_s l_1$ when evaluated at low levels for the fields and coordinates provides detailed equations of motion that involve the usual supergravity fields and derivatives with respect to the space-time coordinates as well as some of the higher level coordinates. It also provides a geometry for the generalised space-time which has a specified tangent space equipped with a tangent group and a generalised vielbein which is determined and indeed easily computed at low levels.

The E_{11} conjecture can also be stated as follows; the complete low energy effective action of strings and branes is the non-linear realisation of $E_{11} \otimes_s l_1$. It has been thought that the complete low energy effective actions were the maximal supergravity theories, however, it has become clear that there are fields in the $E_{11} \otimes_s l_1$ non-linear realisation that are needed in the complete theory and these fields are not contained in the usual formulations of supergravity. Hence if it is shown that the $E_{11} \otimes_s l_1$ non-linear realisation does contain the dynamics of the supergravity theories when one truncates the higher level fields and coordinates then one has in effect provided overwhelming evidence for the conjecture. It may be hoped that the existence of an E_{11} symmetry in the low energy effective action can provide some hints on what is the theory that underlies strings and branes. Clearly the E_{11} proposal is more ambitious than to just reformulate the supergravity theories.

As we have mentioned the symmetries of the $E_{11} \otimes_s l_1$ non-linear realisation essentially determine the dynamics, but these symmetries do not include the familiar general coordinate and gauge transformations. However, the dynamics, at least in the most recent calculations of references [21,22] , do seem to possess these symmetries. In the early papers on E_{11} it was hoped that one could carry out a simultaneous non-linear realisation with

the conformal group as was done in reference [23] for gravity and diffeomorphism transformations, however, this has not been possible to implement for the full E_{11} symmetry. Recently, by assuming compatibility with the E_{11} symmetries, a simple formula for the local transformations, that is, the diffeomorphisms and gauge transformations and their higher level analogues has been given [24]. One finds that the local transformations are labelled by the l_1 representation and so there is one local transformation for each coordinate in the generalised space-time.

A separate development that originated from the realisation that strings could wrap around circles and as a result the string fields possessed not only the usual space-time momenta but also momenta associated with the wrapping of the circle. The resulting theories were found to be invariant under T duality. Corresponding to the doubling of momenta a corresponding doubling of coordinates was then introduced into the first quantised string theory [25] in such a way as to obtain a theory that was manifestly T duality invariant. In 1993 a manifestly T duality field theory that contained the massless NS-NS fields of the superstring which depended on doubled coordinates [26,27] was found. In this work a section condition that restricted the dependence of the fields on the doubled space-time was introduced and the field equations were shown to be a consequence of a generalised Riemann curvature condition.

Beginning in 2009 a field theory, called doubled field theory, was given. It contained the massless NS-NS fields of the superstring which depended on the doubled coordinates and a had section condition [28,29,30,31]. It was eventually revealed that doubled field theory was equivalent [32] to the theory introduced by Siegel in references [26,27]. As a result we refer to this theory as Siegel theory. A new development of Siegel theory was its extension to contain the massless fields of the R-R sector of the superstring [34]. This result was found by constructing the $E_{11} \otimes_s l_1$ non-linear realisation in the decomposition appropriate to the IIA theory in ten dimensions. The dynamics at level zero was just Siegel theory [33], but the dynamics at level one contained the massless fields of the R-R sector of the superstring [34]. We note that this work required no section condition.

A more recent development has been the construction of the so called exceptional field theory [35-39]. The starting point for this development was that
 - the E_n symmetries can be lifted to the eleven dimensional theory. [5]
 - The usual space-time is generalised to include the coordinates given in equation (1.2) [8].

The fields content of the exceptional field theories are essentially the usual supergravity fields and so it has largely the same field content as the fields of the $E_{11} \otimes_s l_1$ non-linear realisation at low levels. Hence the main ingredients of exceptional field theory are just those found previously in the non-linear realisation of $E_{11} \otimes_s l_1$ at low levels. Indeed this latter construction was explained in detail to one of the subsequent authors of exceptional field theory [40].

To understand why the reader, and particularly the younger researcher, could be forgiven for thinking that the E_{11} approach was only remotely connected to exceptional field theory it is useful to note that the reference to the E_{11} approach in the first paper on exceptional field theory [35] was "......(For more ambitious proposals see [12-[14]).....".

We resist the temptation to further comment on the misleading account given in

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reference [35], but we note that it did not reference paper [22] which gave the field equations of the gauge and scalar fields in the $E_{11} \otimes_s l_1$ non-linear realisation in four dimensions, that is, the dynamics of these field which depended on the generalised spacetime of equation (1.2).

One aspect in which exceptional field theory differs from the E_{11} approach is that in the latter the equations of motion are derived from the symmetries of the non-linear realisation, however, in the former one uses gauge and diffeomorphism transformations which were suggested by the notion of a generalised Lie derivative [41] in the enlarged space-time of equation (1.2) [8]. However, diffeomorphism and generalised gauge transformations were formulated in paper [24] within the context of the $E_{11} \otimes_s l_1$ non-linear realisation. Their form was in fact determined by the symmetries of the latter.

As mentioned above the equations of motion that result from the non-linear realisation $E_{11} \otimes_s l_1$ have been found in eleven [21] and four dimensions [22] for all the fields except for those in the gravity sector which have not been found completely. In this paper we concentrate on the equations of exceptional field theory that are its starting point, namely the the local variations of the fields. Indeed, we will show that exceptional field theory not only has the generic features of the $E_{11} \otimes_s l_1$ non-linear realisation at low levels, but that the equations which give the local variations of the fields are contained within it if one makes a radical truncations.

In section two we review the parts of the $E_{11} \otimes_s l_1$ non-linear realisation that we require. In section three we evaluate the non-linear gauge transformations of reference [24] for the low level fields using a parameter that carries a tangent space index. In section four we give the generalised field strengths that appear in the field equations of the $E_{11} \otimes_s l_1$ non-linear realisation. While in section five we introduce a matter field in the context of the $E_{11} \otimes_s l_1$ non-linear realisation, which belongs to the l_1 representation, and construct its covariant derivatives which are evaluated at low levels. In section six we transfer the results of sections three and five to a world volume point of view. We find an alternative formulation of the transformation of the vielbein which is given by

$$E_A{}^\Sigma \delta E_\Pi{}^A = (D^{\underline{\alpha}})_\Pi{}^\Sigma (D^{\underline{\alpha}})_\Sigma{}^\Theta \partial_\Theta \Lambda^\Sigma + \Lambda^\Gamma \partial_\Gamma E_\Pi{}^A E_A{}^\Sigma$$

where the parameter Λ^Σ has a world index, $E_\Pi{}^A$ denotes the vielbein that lives on the generalised spacetime and $(D^{\underline{\alpha}})_\Pi{}^\Sigma$ are the representation matrices of the l_1 representation. We also find the transformation of a world volume matter field V^Π which belongs to the l_1 representation; the result is given by

$$\delta V^\Pi = - V^\Lambda \left(D^{\underline{\alpha}}\right)_\Lambda{}^\Pi \left(D_{\underline{\alpha}}\right)_\Sigma{}^\Xi \partial_\Xi \Lambda^\Sigma + \Lambda^\Sigma \partial_\Sigma V^\Pi + \mu \, V^\Pi \partial_\Sigma \Lambda^\Sigma.$$

Evaluating the two results at low levels, and brutally truncating, we find variations for the low level fields that agree with those found in exceptional field theory [35,36]. The same applies to the generalised field strengths found in section four.

2 Review of the non-linear realisation of $E_{11} \otimes_s l_1$.

In view of the results of this paper it is conceivable that readers may, and indeed perhaps should, want to learn the E_{11} approach. As such we take the liberty of briefly reviewing this approach.

It has been conjectured that the low energy effective action of strings and branes is the non-linear realisation of semi-direct product of E_{11} with its first fundamental representation denoted l_1 [5,8]. The commutators of this algebra, which is denoted by $E_{11} \otimes_s l_1$, can be written in the form

$$[R^{\underline{\alpha}}, R^{\underline{\beta}}] = f^{\underline{\alpha}\underline{\beta}}{}_{\underline{\gamma}} R^{\underline{\gamma}}, \quad [R^{\underline{\alpha}}, l_A] = -(D^{\underline{\alpha}})_A{}^B l_B \tag{2.1}$$

where $R^{\underline{\alpha}}$ are the generators of E_{11} and l_A are the generators belonging to the l_1 representation, as is encoded in the second commutator. The Jacobi identities imply that the matrices $(D^{\underline{\alpha}})_A{}^B$ are a representation of the E_{11} algebra,

$$[D^{\underline{\alpha}}, D^{\underline{\beta}}] = f^{\underline{\alpha}\underline{\beta}}{}_{\underline{\gamma}} D^{\underline{\gamma}} \tag{2.2}$$

We use the indices $\underline{\alpha}, \underline{\beta}, \dots$ to label the generators of E_{11}. A review of the E_{11} algebra and indeed Kac-Moody algebras in general can be found in the book of reference [42]. Indeed in this reference the reader can find the $E_{11} \otimes_s l_1$ algebra when decomposed into representations of SL(11) is given up to level four.

The non-linear realisation of $E_{11} \otimes_s l_1$ is constructed from the group element $g \in E_{11} \otimes_s l_1$ that can be written as

$$g = g_l g_E \tag{2.3}$$

In this equation g_E is a group element of E_{11} and so can be written in the form $g_E = e^{A_{\underline{\alpha}} R^{\underline{\alpha}}}$ and $A_{\underline{\alpha}}$ are the fields in the non-realisation. The group element g_l is formed from the generators of the l_1 representation and so has the form $e^{z^A l_A}$ where z^A are the coordinates of the generalised space-time. The fields $A_{\underline{\alpha}}$ depend on the coordinates z^A. The explicit form of these group elements can be found in earlier papers on E_{11}, such as in five dimensions in [12], and more recently in eleven dimensions in [21] and four dimensions in [22].

The non-linear realisation is, by definition, invariant under the transformations

$$g \to g_0 g, \quad g_0 \in E_{11} \otimes_s l_1, \quad \text{as well as} \quad g \to gh, \quad h \in I_c(E_{11}) \tag{2.4}$$

The group element $g_0 \in E_{11}$ is a rigid transformation and so it is a constant, while the group element h belongs to the Cartan involution subalgebra of E_{11}, denoted $I_c(E_{11})$; it is a local transformation, that is, it depends on the generalised space-time. As the generators in g_l form a representation of E_{11} the above transformations for $g_0 \in E_{11}$ can be written as

$$g_l \to g_0 g_l g_0^{-1}, \quad g_E \to g_0 g_E \quad \text{and} \quad g_E \to g_E h \tag{2.5}$$

The Cartan involution is an automorphism of the algebra, that is $I_c(AB) = I_c(A)I_c(B)$ for any two elements of the Lie algebra A and B and an involution, that is, $I_c^2(A) = A$. It takes positive root generators to negative root generators and its action can be taken to be

$$I_c(R^{\underline{\alpha}}) = -R^{-\underline{\alpha}} \tag{2.6}$$

for any root $\underline{\alpha}$. The Cartan Involution subalgebra is generated by $R^{\underline{\alpha}} - R^{-\underline{\alpha}}$. For the familiar finite dimensional semi-simple Lie groups the Cartan Involution invariant subalgebra

is the maximal compact of subalgebra; for example for SL(2,R) the Cartan Involution subalgebra is SO(2), while for E_8 it is SO(16).

Equation (2.6) leads to a theory that lives in a space with Euclidean signature. However, one can adopt an alternative Cartan Involution by scattering some minus signs and one finds that the theory has a Minkowski signature [43]. As it is tiresome to keep track of these minus signs, it is easier to work with the Cartan Involution that leads to the Euclidean signature and then change to Minkowski signature at the end. This is the course of action we will take in this paper.

It follows from equation (2.5) that the coordinates are inert under the local transformations but transform under the rigid transformations as

$$z^A l_A \to z^{A\prime} l_A = g_0 z^A l_A g_0^{-1} = z^\Pi D(g_0^{-1})_\Pi{}^A l_A \tag{2.7}$$

When written in matrix form the differential transformations act as $dz^T \to dz^{T\prime} = dz^T D(g_0^{-1})$. As a result the derivative $\partial_\Pi \equiv \frac{\partial}{\partial z^\Pi}$ in the generalised space-time transforms as $\partial'_\Pi = D(g_0)_\Pi{}^\Lambda \partial_\Lambda$.

The l_1 representation of E_{11} is, by definition, given by

$$U(k)(l_A) \equiv k^{-1} l_A k = D(k)_A{}^B l_B, \quad k \in E_{11} \tag{2.8}$$

where $D(k)_A{}^B$ is the matrix representative. For an infinitesimal transformation $k = 1 + a_{\underline{\alpha}} R^{\underline{\alpha}}$ and the matrix takes the form $D(k)_A{}^B = \delta_A^B + (D^{\underline{\alpha}})_A{}^B$. As a result, we recognise the matrix $D(g_0^{-1})$, that appears in equation (2.7) as the matrix representation of the l_1 representation, although the indices are labelled according to the role which they will play later in the physical theory that emerges from the non-linear realisation.

The dynamics of the non-linear realisation is just an action, or set of equations of motion, that are invariant under the transformations of equation (2.4). We now recall how to construct the dynamics of the the $E_{11} \otimes_s l_1$ non-linear realisation using the Cartan forms which are given by

$$\mathcal{V} \equiv g^{-1} dg = \mathcal{V}_E + \mathcal{V}_l, \tag{2.9}$$

where

$$\mathcal{V}_E = g_E^{-1} dg_E \equiv dz^\Pi G_{\Pi,\underline{\alpha}} R^{\underline{\alpha}}, \tag{2.10}$$

belongs to the E_{11} algebra and are the Cartan form for E_{11}, while the part in the space of generators of the l_1 representation is given by

$$\mathcal{V}_l = g_E^{-1}(g_l^{-1} dg_l) g_E = g_E^{-1} dz \cdot l g_E \equiv dz^\Pi E_\Pi{}^A l_A \tag{2.11}$$

While both $\mathcal{V}_E$ and $\mathcal{V}_l$ are invariant under rigid transformations, but under local transformations of equation (2.5) they change as the

$$\mathcal{V}_E \to h^{-1} \mathcal{V}_E h + h^{-1} dh \quad \text{and} \quad \mathcal{V}_l \to h^{-1} \mathcal{V}_l h \tag{2.12}$$

Although the Cartan form is inert under rigid transformations, the rigid transformations do act on the coordinate differentials contained in the Cartan form and this action

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induces a corresponding rigid E_{11} transformation on the lower index of $E_{\Pi}{}^A$. On the other hand, a local $I_c(E_{11})$ transformation acts on the $\underline{\alpha}$ index of $G_{\Pi,\underline{\alpha}}$ and on the A index of $E_{\Pi}{}^A$ as governed by equation (2.12). We may summarise these two results as

$$E_{\Pi}{}^{A\prime} = D(g_0)_{\Pi}{}^{\Lambda} E_{\Lambda}{}^B D(h)_B{}^A \quad \text{or} \quad (E^{-1})_A{}^{\Pi\prime} = D(h^{-1})_A{}^B (E^{-1})_B{}^{\Lambda} D(g_0^{-1})_{\Lambda}{}^{\Pi} \quad (2.13)$$

Thus $E_{\Pi}{}^A$ transforms under a local $I_c(E_{11})$ transformation on its A index and by a rigid E_{11} induced coordinate transformation of the generalised space-time on its Π index. These transformations mean that we can interpretation of $E_{\Pi}{}^A$ as a generalised vielbein of the generalised space-time which possess a generalised tangent space that has the tangent group $I_c(E_{11})$.

Examining equation (2.11) and recalling equation (2.8), we recognise $E_{\Pi}{}^A$ as the representation matrix $D(g_E)_{\Pi}{}^A$, and so $E_{\Pi}{}^A = D(g_E)_{\Pi}{}^A = (e^{A_{\underline{\alpha}} D^{\underline{\alpha}}})_{\Pi}{}^A$. It is useful to take the Cartan form to be in the l_1 representation, taking the commutator of $G_{\Pi,\underline{\alpha}} R^{\underline{\alpha}}$ of equation (2.10) with l_A we find that

$$G_{\Pi,A}{}^B l_B \equiv G_{\Pi,\underline{\alpha}} (D^{\underline{\alpha}})_A{}^B l_B = (E^{-1})_A{}^{\Lambda} \partial_{\Pi} E_{\Lambda}{}^B l_B = -[G_{\Pi\underline{\alpha}} R^{\underline{\alpha}}, l_A] \quad (2.14)$$

Converting the first index on the Cartan form $G_{\Pi,A}{}^B$ to be a tangent index and using the generalised vielbein we can write The Cartan form in tangent space as

$$G_{A,B}{}^C \equiv (E^{-1})_A{}^{\Pi} G_{\Pi B}{}^C = (E^{-1})_A{}^{\Pi} (E^{-1})_B{}^{\Lambda} \partial_{\Pi} E_{\Lambda}{}^C \quad (2.15)$$

Under the $I_C(E_{11})$ local transformation of equation (2.12) we find, using equation (2.14), that

$$G'_{\Pi,A}{}^B = D(h^{-1})_A{}^C G_{\Pi,C}{}^D D(h)_D{}^B + D(h^{-1})_A{}^C \partial_{\Pi} D(h)_C{}^B \quad (2.16)$$

Thus if one chooses to construct the dynamics using the Cartan forms $G_{A,B}{}^C$ one just has to ensure that the equations of motion, or the action, are invariant under the local $I_C(E_{11})$ transformations. Furthermore the non-linear realisation automatically comes equipped with a generalised geometry in that it has a generalised tangent space, with tangent group $I_c(E_{11})$, and a generalised vielbein $E_A{}^{\Pi}$.

As the dynamics is just that which is invariant under the transformation of equation (2.5), or equivalently (2.12), it follows that this is, for a general non-linear realisation, determined by the group theory of the non-linear realisation. The worst that can happen is that there are several constants whose values are not fixed, but it is very often the case that the dynamics is uniquely fixed.

In this paper we will consider the $E_{11} \otimes_s l_1$ non-linear realisation in five dimensions. The algebra, fields, coordinates, generalised vielbein and all the data need for the construction are given in appendix A. Here, for convenience, we record the fields which are given by

$$h_a{}^b, \ \varphi_{\alpha}, \ A_{aM}, \ A_{a_1 a_2}{}^N, \ A_{a_1 a_2 a_3, \alpha}, \ A_{a_1 a_2, b}, \ \ldots \quad (2.17)$$

and the coordinates of the generalised space-time which are

$$x^a, \ x_N, \ x_a{}^N, \ x_{a_1 a_2, \alpha}, \ x_{ab}, \ \ldots \quad (2.18)$$

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3 The gauge transformations

The $E_{11} \otimes_s l_1$ non-linear realisation does not obviously possess diffeomorphisms and gauge transformations as one of its symmetries. However, it was shown in reference [24] how the E_{11} symmetries of the non-linear realisation could be used to determine the transformation laws of these, and other, local symmetries; the result was given by

$$E^{-1}{}_A{}^\Pi \delta E_\Pi{}^B = (D^{\underline{\alpha}})_A{}^B (D_{\underline{\alpha}})_C{}^D D_D \Lambda^C \tag{3.1}$$

where Λ^C is the parameter of the gauge transformation, which depends on the generalised space-time, and $D_D \Lambda^C = E_D{}^\Pi (\partial_\Pi \Lambda^C - V^E \Omega_{\Pi,E}{}^C)$ and $\Omega_{\Pi,E}{}^C$ is the connection which will be discussed in sections five and six. The repeated $\underline{\alpha}$ index means a sum over all such E_{11} indices. The parameter Λ^C carries the indices of the l_1 representation which leads to the generalised space-time. Thus there is a one to one correspondence between local transformations given in equation (3.1) and the coordinates of space-time as well as the brane charges. Despite its unfamiliar appearance, equation (3.1) does contains the usual general coordinate and gauge transformations [24].

At the linearised level the transformations of equation (3.1) reads [24]

$$\delta \mathcal{A}_{\underline{\alpha}} = (D_{\underline{\alpha}})_D{}^C \partial_C \Lambda^D, \quad \text{or equivalently} \quad \delta \mathcal{A}_A{}^B = (D^{\underline{\alpha}})_A{}^B (D_{\underline{\alpha}})_D{}^C \partial_C \Lambda^D \tag{3.2}$$

where at this level $E_\Pi{}^A = \delta_\Pi^A + \mathcal{A}_{\underline{\alpha}} (D^{\underline{\alpha}})_\Pi{}^A = \delta_\Pi^A + \mathcal{A}_\Pi{}^A$. These lowest level variations were evaluated at low levels in four, five and eleven dimensions in reference [24].

We now evaluate the gauge transformations of equation (3.1) in five dimensions at low levels, but including the non-linear terms. The $E_{11} \otimes_S l_1$ algebra, the fields and coordinates of the non-linear realisation and generalised vielbein and the other quantities we need are reviewed in appendices A and B. Taking $A = a$ and $B = N$ in equation (3.1) we find that the only non-vanishing $(D^{\underline{\alpha}})_A{}^B$ is for $\underline{\alpha} = aN$. Using that the Cartan-Killing metric is given by $g^{aN}{}_{bM} = \delta_b^a \delta_M^N$, we find that

$$(D_{aN})_C{}^D = g^{-1}_{AN\underline{\alpha}} (D^{\underline{\alpha}})_C{}^D = (D_{aN})_C{}^D \tag{3.3}$$

Examining the content of the l_1 representation of equation (A.3), and using that $(D^{aN})_{bM} = -\delta_b^a \delta_M^N$, we find that equation (3.1), for our chosen values of A and B, is given by

$$E^{-1}{}_a{}^\Pi \delta E_{\Pi N} = -(D_{aN})_D{}^C D_C \Lambda^D$$

$$= -(D_{aN})^P{}_{\|}{}^b D_b \Lambda_P - (D_{aN})^b{}_{P\|Q} D^Q \Lambda_b{}^R - (D_{aN})^{c_1 c_2 \beta}{}_{\|b}{}^Q D^b{}_Q \Lambda_{c_1 c_2 \beta} + \dots$$

$$= D_a \Lambda_N - 10 \, d_{NMP} \, D^M \Lambda_a{}^P - 12 \, (D^\alpha)_N{}^M \, D^b{}_M \Lambda_{ab,\,\alpha} + \dots \tag{3.4}$$

Care has to be taken with the position of the indices, for example taking $B = N$ results in an N index which is downstairs, as it should be when it appears on the vielbein, rather than the inverse vielbein. Looking at the definition of the vielbein of equation (2.11) we indeed see that it should be in the opposite position to the N index on Z^N. Given the subtleties with the index placements we will often use a $\|$ to make clear the separation of the A and B indices on $D^{\underline{\alpha}}$ and the connection.

We now evaluate the left-hand side of equation (3.4). Looking at the five dimensional vielbein of equation (A.28) and its inverse of equation (A.32) we find that it is equal to

$$E^{-1}{}_{a\|}{}^{\Pi}\delta E_{\Pi\|N} = E^{-1}{}_{a\|}{}^{\nu}\delta E_{\nu\|N} + E^{-1}{}_{a\|}{}^{\dot{M}}\delta E^{\dot{M}}{}_{\|N} + E^{-1}{}_{a\|}{}^{\dot{M}}\delta E^{\nu}_{\dot{M}\|N}$$

$$= -e_a^{-1\nu}\delta(e_\nu{}^b A_{bN}) + A_{aP}d_{\dot{M}}^P\delta(d_N{}^{\dot{M}}) = -e_a^{-1\nu}\delta A_{b\dot{M}}(d_N{}^{\dot{M}}) \tag{3.5}$$

We note that the vielbein has a universal factor of $(\det e)^{-\frac{1}{2}}$ but this, and its derivative, drop out of the above expression . Taking this equation together with the right-hand side of equation (3.4) we find that

$$\delta A_{\mu\dot{N}} = -e_\mu{}^a d_{\dot{N}}^M(D_a\Lambda_M - 10\, d_{MRP}\, D^R\Lambda_a{}^P - 12\,(D^\alpha)_M{}^P\, D^b{}_P\Lambda_{ab,\,\alpha} + \ldots) \tag{3.6}$$

We now consider the right-hand side of equation (3.1) when $A = a_1$ and $B = {}^N{}_{a_2}$ whereupon it takes the form

$$E_{a_1\|}{}^{\Pi}\,\delta E_{\Pi\|a_2}{}^N = (D^{\underline{\alpha}})_{a_1\|a_2}{}^N(D^{\underline{\alpha}})_C{}^D D_D\Lambda^C$$

$$= \frac{1}{10}(D_{a_1a_2}{}^N)_C{}^D D_D\Lambda^C = \frac{1}{10}\{(D_{a_1a_2}{}^N)^{b_1}{}_{M\|}{}^{b_2}\, D_{b_2}\Lambda_{b_1}{}^M$$

$$+ \left(D_{a_1a_2}{}^N\right)^{b_1b_2,\,\alpha}{}_{\|M}\, D^M\Lambda_{b_1b_2,\,\alpha} + \left(D_{a_1a_2}{}^N\right)^{b_1b_2}{}_{\|M}\, D^M\Lambda_{b_1b_2} + \ldots\}$$

$$= 2D_{[a_1}\Lambda_{a_2]}{}^N - 12\,(D^\alpha)_M{}^N D^M\Lambda_{a_1a_2,\,\alpha} + \frac{2}{3}\,(D^\alpha)_M{}^N D^M\Lambda_{a_1a_2} + \ldots \tag{3.7}$$

To find the variation of the field we must evaluate the left-hand side of this equation;

$$E_{a_1\|}{}^{\Pi}\,\delta E_{\Pi\|a_2}{}^N = -2\delta(A_{\mu\nu}{}^{\dot{M}})d_{\dot{M}}^N e_a{}^\mu e_b{}^\nu + d^{\dot{N}\dot{R}\dot{S}}d_{\dot{N}}^N A_{[\mu|\dot{R}}\delta A_{|\nu]\dot{S}}e_a{}^\mu e_b{}^\nu \tag{3.8}$$

In deriving this equation we have used the fact that d^{NMP} is an E_6 invariant tensor and that $d_P^{\dot{M}}$ can be identified with an E_6 group element in the relevant representation. Taking these results for the left and right-hand sides of equation (3.1), for our chosen indices, we find that

$$\delta A_{\mu\nu}{}^{\dot{M}} = e_\mu{}^a e_\nu{}^b d_P^{\dot{M}}\left(-D_{[a}\Lambda_{b]}{}^P + 6\,(D^\beta)_R{}^P D^R\Lambda_{a_1a_2,\,\beta} - \frac{1}{3}\,(D^\alpha)_R{}^P D^R\Lambda_{a_1a_2} + \ldots\right)$$

$$+\frac{1}{2}e_\mu{}^a e_\nu{}^b d^{\dot{N}\dot{R}\dot{S}}d_{\dot{N}}^N A_{[\mu|\dot{R}}\delta A_{|\nu]\dot{S}} \tag{3.9}$$

Finally, in equation (3.1), we take the values $A = a_1$ and $B = {}_{a_1a_2\alpha}$ to find that the right-hand side of equation (3.1) reads

$$-\frac{3}{360}\,(D_{a_1a_2a_3}{}^\alpha)^{b_1b_2,\,\beta\|b_3}\, D_{b_3}\Lambda_{b_1b_2,\,\beta} = -3D_{[a_1}\Lambda_{a_2a_3]}{}^\alpha + \ldots \tag{3.10}$$

while the left-hand side is given by

$$E_{a_1\parallel}{}^{\Pi}\, \delta E_{\Pi\parallel a_2 a_3}{}^{\alpha} = -3 e_{a_1}{}^{\mu} \delta(e_{\mu}{}^{b} A_{b a_1 a_2}) + \dots$$

and as a result

$$\delta A_{\mu_1\mu_2\mu_3} = -D_{[\mu_1}\Lambda_{\mu_2\mu_3]\alpha} + \dots \tag{3.11}$$

We will now evaluate the above gauge transformations at very low levels. To do this we will need to evaluate the covariant derivatives acting on the gauge parameters Λ^A which belong to the l_1 representation. In section five, with the help of appendix B, the covariant derivatives of the an object V^A which transforms as the l_1 representation does under $I_c(E_{11})$ are introduced and evaluated at low levels. The covariant derivatives that appear in the gauge transformations have the same form as the the covariant derivatives acting on V^A and the connections act on the parameters Λ^A in the same way. Nonetheless the detailed expressions for the connections may not be the same in the two different applications.

We may write the gauge transformation of the field A_{aN} as

$$\delta A_{\mu\dot{N}} = -e_{\mu}{}^{a} d_{\dot{N}}^{M} \mathcal{D}_a \Lambda_M = e_{\mu}{}^{a} d_{\dot{N}}^{M} (D_{aM})_C{}^{D} D_D \Lambda^C \tag{3.12}$$

In section five this covariant derivative on V^A was evaluated at low levels and in equation (5.14) truncated to contain only the component V_N, which corresponds here to the component Λ_N, using this result we find that

$$\delta A_{\mu\dot{N}} = -(\det e)^{\frac{1}{2}} d_{\dot{N}}^{N} \{ \partial_{\mu}\Lambda_N + \Lambda_P \Omega_{\mu,\alpha}(D^{\alpha} - D^{-\alpha})_N{}^{P}$$

$$+ A_{\mu P} d_{\dot{P}}^{P}(\partial^{\dot{P}}\Lambda_N + \Lambda_M \Omega^{\dot{P}}{}_{,\alpha}(D^{\alpha} - D^{-\alpha})_N{}^{M})$$

$$+ 10 d_{NMP} d_{\dot{R}}^{M}(\partial^{\dot{R}}\Lambda_a{}^{P} + \Lambda_L \Omega^{\dot{R}}{}_{,\mu Q} d^{QLP} + \dots \} \tag{3.13}$$

where we have also keep one term which contains the level two parameter $\Lambda_a{}^{P}$ but discarded all other such terms.

Using the covariant derivative introduced in section five we can write the gauge variation of the field $A_{\mu_1\mu_2}{}^{\dot{M}}$, given in equation (3.9), in the form

$$\delta A_{\mu_1\mu_2}{}^{\dot{M}} = -\frac{1}{2} e_{\mu_1}{}^{a} e_{\mu_2}{}^{b} d_{p}^{\dot{M}} \mathcal{D}_{[a}\Lambda_{b]}{}^{P} + \frac{1}{2} d^{\dot{N}\dot{R}\dot{S}} A_{[\mu_1|\dot{R}} \delta A_{|\mu_2]\dot{S}} \tag{3.14}$$

This can be further evaluated using the formulae in section five, especially equation (5.17).

We close this section by finding, at low levels, the gauge transformations of the level zero fields, namely the vielbein and the scalars. To find the variation of the vielbein $e_{\mu}{}^{a}$ we take $A = a$ and $B = b$ in equation (3.1) whereupon, using the vielbein and generalised vielbein of Appendix A, that is, equations (A.28) and (A.32), we find that the left-hand side of this equation becomes

$$E_a{}^{\Pi}\, \delta E_{\Pi}{}^{b} = (\det e)^{\frac{1}{2}} e_a{}^{\mu}\, \delta((\det e)^{-\frac{1}{2}} e_{\mu}{}^{b}) = e_a{}^{\mu} \delta e_{\mu}{}^{b} - \frac{1}{2} \delta_a^b e_c{}^{\lambda} \delta e_{\lambda}{}^{c} \tag{3.15}$$

While the right-hand side of this equation is given by

$$E_a{}^\Pi \delta E_\Pi{}^b = \mathcal{D}_a \Lambda^b = D_a \Lambda^b - D^b{}_N \Lambda_a{}^N - \frac{1}{2}\delta_a^b \left(D_c \Lambda^c + D^N \Lambda_N + D^c{}_N \Lambda_c{}^N\right) + \dots \quad (3.16)$$

where we have used equation (B.14). As a result we conclude that

$$e_a{}^\mu \delta e_\mu{}^b = D_a \Lambda^b - D^b{}_N \Lambda_a{}^N + \delta_a^b(\frac{1}{3}D^N \Lambda_N + \frac{2}{3}D^c{}_N \Lambda_c{}^N) + \dots. \quad (3.17)$$

Keeping only terms involving the field $e_\mu{}^a$ and the parameter Λ^a we recognise that this field has the standard transformation under general coordinate transformations. This justifies the choice of $e_\mu{}^a = (e^h)_\mu{}^a$ to be the correct vielbein rather than the combination $\tilde{e}_\mu{}^a = (\det e)^{-\frac{1}{2}}e_\mu{}^a$ that appears in the generalised vielbein given in equation (A.28).

To find the variation of the scalars, that are contained in their vielbein $\delta d_N{}^{\dot{N}}$ we choose $A = N$ and $B = M$ in equation (3.1). Using the vielbein and inverse vielbein given in equations (A.28) and (A.32), we find that the left-hand side of equation (3.1) is given by

$$E^{N\Pi} \delta E_{\Pi M} = (\det e)^{\frac{1}{2}} d_{\dot{N}}{}^N \delta((\det e)^{-\frac{1}{2}}) d_M{}^{\dot{N}}) = d_{\dot{N}}{}^N \delta d_M{}^{\dot{N}} - \frac{1}{2}\delta_M^N e_c{}^\lambda \delta e_\lambda{}^c \quad (3.18)$$

While, using equation (B.13) the righthand side of equation (3.1) is given by

$$E^{N\Pi} \delta E_{\Pi M} = \mathcal{D}^N \Lambda_M = (D^{\underline{\alpha}})^N{}_M \left(D_{\underline{\alpha}}\right)_A{}^B D_B \Lambda^A$$

$$= 6 \left(D^\alpha\right)_M{}^N \left(D_\alpha\right)_P{}^Q D^P \Lambda_Q - 6 \left(D^\alpha\right)_M{}^N \left(D_\alpha\right)_P{}^Q D^a{}_Q \Lambda_a{}^P$$

$$+ \left(D^a{}_b\right)^N{}_M \left(D_a{}^b\right)_c{}^d D_d \Lambda^c + \left(D^a{}_b\right)^N{}_M \left(D_a{}^b\right)^P{}_Q D^Q \Lambda_P + \left(D^a{}_b\right)^N{}_M \left(D_a{}^b\right)_c{}^{P,d}{}_Q D^d{}_Q \Lambda_c{}^P$$

$$- \frac{1}{3}\left(\left(D^a{}_a\right)^N{}_M \left(D^b{}_b\right)_c{}^d D_d \Lambda^c + \left(D^a{}_a\right)^N{}_M \left(D^b{}_b\right)^P{}_Q D^Q \Lambda_P + \left(D^a{}_a\right)^N{}_M (D^b{}_b)_c{}^{P,d}{}_Q D^d{}_Q \Lambda_c{}^P\right)$$

$$= D^N \Lambda_M - D^a{}_M \Lambda_a{}^N - 10\, d^{NPR} d_{MQR} \left(D^Q \Lambda_P - D^a{}_P \Lambda_a{}^Q\right)$$

$$- \frac{1}{2}\delta_M^N(D_c\Lambda^c + D^p \Lambda_P + 3D^c{}_P \Lambda_c{}^P) + \dots. \quad (3.19)$$

In this last equation we have used the expression of the Cartan-Killing metric of equation (B.13) and the identity of equation (A.14). The variation of $d_N{}^{\dot{N}}$ can be found by using equation (3.18) and equation (3.19) to be given by

$$d^N{}_{\dot{P}}\delta d_M{}^{\dot{P}} = D^N \Lambda_M - D^c{}_M \Lambda_c{}^N - 10 d^{MPR} d_{NQR} \left(D^Q \Lambda_P - D^c{}_P \Lambda_c{}^Q\right)$$

$$+ \frac{1}{3}\delta_M^N D^p \Lambda_P - \frac{1}{3}\delta_M^N D^c{}_P \Lambda_c{}^P$$

$$= 6 \left(D^\alpha\right)_M{}^N \left(D_\alpha\right)_P{}^Q D^P \Lambda_Q - 6 \left(D^\alpha\right)_M{}^N \left(D_\alpha\right)_P{}^Q D^a{}_Q \Lambda_a{}^P + \dots \quad (3.20)$$

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In section six we compute the gauge variations of the fields with a parameter that is a world vector rather than a tangent vector. After a radical truncation we make contact with the analogous quantities found in exceptional field theory.

We close this section by noting the relation between the gauge transformations of equation (3.1) and local $I_c(E_{11})$ transformations. We can write equation (3.1) in the form

$$E^{-1}{}_A{}^\Pi \delta E_\Pi{}^B = \frac{1}{2}(D^{\underline{\alpha}} + D^{-\underline{\alpha}})_A{}^B(D_{\underline{\alpha}})_C{}^D D_D \Lambda^C + \frac{1}{2}(D^{\underline{\alpha}} - D^{-\underline{\alpha}})_A{}^B(D_{\underline{\alpha}})_C{}^D D_D \Lambda^C \quad (3.21)$$

We recognise the last term as an $I_c(E_{11})$ transformation of $E_\Pi{}^A$ and this we can discard from the gauge transformation as it is symmetry of the theory. We also note that the gauge transformations also contain the rigid E_{11} transformations [24].

4 Construction of the generalised field strengths.

As explained in section two, if one uses the Cartan forms to construct the dynamics then the invariance under the rigid E_{11} symmetry of the non-linear realisation is automatically encoded and one just has to ensure invariance under the local subalgebra $I_C(E_{11})$. This has been the strategy adopted in most of the previous E_{11} papers. Indeed, in [21] and [22] a systematic effort was made to find the equations of motion of the $E_{11} \otimes l_1$ non-linear realisation in eleven and four dimensions respectively. The dynamics consists of a series of duality relations that are first order in derivatives with respect to the generalised coordinates and, at least at low levels, the dynamics appears to be uniquely determined by the symmetries of the non-linear realisation. As such the dynamics consists of equations which were linear in the Cartan forms which, we recall from section two, have the form $G_{A,\underline{\alpha}}$ where the first index A arises from the l_1 part of the non-linear realisation and the second index $\underline{\alpha}$ is from the E_{11} part. At the linearised level $G_{A,\underline{\alpha}} = \delta_A^\Pi \partial_\Pi A_{\underline{\alpha}}$ and as such if $A = a$ we get a lead term in the Cartan form that contains the usual space-time derivative of a field while if A is a higher level index we get a derivative with respect to one of the higher level coordinates.

Examining the resulting equations of motion of references [21] and [22], it becomes apparent that the Cartan forms that contain a leading term that has a usual space-time derivative, that is, $G_{a,\underline{\alpha}}$ came together with a Cartan form whose leading term contains the derivative with respect to one of the higher level generalised coordinates. The reader may look at equations (4.10), (4.11) and (4.14-17) of reference [22] to see how this works in four dimensions using just the symmetries of the $E_{11} \otimes_s l_1$ non-linear realisation.

In reference [24] the same phenomenon was observe in four and five dimensions, but instead using the rigid E_{11} symmetry instead of the local $I_c(E_{11})$ symmetry. Since we are considering five dimensions in this paper, we now recall how this goes in this dimension. The equation of motion of the field A_{aN} must contain the Cartan form $G_{[a_1,a_2]N}$ as at the linearised order it is of the form $G_{[a_1,a_2]N} \sim \partial_{[a_1} A_{a_2]N}$. It was found that such a term in the equation of motion occurs in the combination

$$\partial_{[a_1} A_{a_2]N} + 10 d_{NRS} \partial^R A_{a_1 a_2}{}^S \quad (4.1)$$

In terms of Cartan forms this means that the equations of motion involve the combination

$$\mathcal{G}_{a_1 a_2 N} = G_{[a_1,a_2]N} + 10 d_{NRS} G^R{}_{,a_1 a_2}{}^S \quad (4.2)$$

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The equation of motion of the field A_{aN} implies that its field strength, which is contained in $G_{[a_1,a_2]N}$, is dual to the field strength of the field $A_{a_1a_2}{}^N$ which is continued in the Cartan form $G_{[a_1,a_2a_3]}{}^N$. At the linearised level the latter is of the form $G_{[a_1,a_2a_3]}{}^N \sim \partial_{[a_1}A_{a_2a_3]}{}^N$

Carrying out the same procedure for the field $A_{a_1a_2}{}^S$ we find that it occurs in the linearised combination

$$\partial_{[a_1}A_{a_2a_3]}{}^N + 6(D^\beta)_R{}^N\partial^R A_{,a_1a_2a_3\beta} \tag{4.3}$$

and as a result the associated Cartan forms must occur in the combination

$$\mathcal{G}_{a_1a_2a_3}{}^S = G_{[a_1,a_2a_3]}{}^N + 6(D^\beta)_R{}^N G^R{}_{,a_1a_2a_3\beta} \tag{4.4}$$

We refer to the modified field strengths, such as in equations (4.2) and (4.4), as generalised field strengths. Consequently, the equation of motion for the vector field should relate the generalised field strength of equation (4.2) to the dual of the generalised field strength of equation (4.4).

An alternative way to find these combinations is to use the gauge transformations of equation (3.1). It suffices to do this using the linearise expressions for the fields in the Cartan forms and their lowest order gauge transformations. One finds that $\mathcal{G}_{[a_1}A_{a_2]N}$ and $\mathcal{G}_{a_1a_2a_3}{}^S$ are invariant under the lowest level gauge transformations of equation (3.1), that is, those given in equation (3.2). These latter transformations for the lowest level fields are [24]

$$\delta A_{aM} = -\partial_a\Lambda_M + 10d_{MPQ}\partial^Q\Lambda_a^Q + 12(D^\beta)_M{}^N\partial_N^b\Lambda_{ba\beta} + \ldots$$

$$\delta A_{a_1a_2}{}^M = -\partial_{[a_1}\Lambda_{a_2]}^M + 6(D^\beta)_N{}^M\partial^N\Lambda_{ba_1a_2\beta} + \ldots$$

$$\delta h_a{}^b = \partial_a\xi^b + \frac{1}{3}\delta_a^b\partial^M\Lambda_M - \partial_N^a\Lambda_b{}^N + \frac{2}{3}\delta_a^b\partial_N^c\Lambda_c{}^N + \ldots,$$

$$\delta\phi^\alpha = -6(D^\alpha)_P{}^R\partial^P\Lambda_R + 6(D^\alpha)_P{}^R\partial_R^b\Lambda_b{}^P + \ldots \tag{4.5}$$

The last equation can also be written as

$$\delta\phi_M{}^N = 6(D^\alpha)_M{}^N(D^\alpha)_P{}^R(\partial^P\Lambda_R - \partial_R^b\Lambda_b{}^P) + \ldots$$

where $\delta\phi_M{}^N = -(D^\alpha)_M{}^N\phi_\alpha$. It is instructive to recall that the scalar vielbein is defined by the equation $e^{-\phi_\alpha R^\alpha}Z^{\dot N}e^{\phi_\alpha R^\alpha} = Z^M(d^{-1})_M{}^{\dot N}$. One then finds that $(d^{-1})_M{}^{\dot N} = (\exp(-D^\alpha\phi_\alpha))_M{}^{\dot N}$.

Applying these arguments to the scalars we find that the quantity

$$\partial_a\phi_\alpha - 6(D^\alpha)_P{}^R\partial^P A_{aR} \tag{4.6}$$

is gauge invariant at lowest level under Λ_N transformations and so the non-linear quantity that should appear in the equations of motion is given by

$$\mathcal{G}_{a,\alpha} = G_{a,\alpha} - 6(D_\alpha)_P{}^R G^P{}_{,aR} \tag{4.7}$$

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For the graviton it is the quantity

$$\partial_a h_b{}^c + \frac{1}{3}\delta_b^c \partial^P A_{aP} \tag{4.8}$$

which is invariant under under lowest order Λ_N transformations and this in turn implies one should in the equations of motion use the quantity

$$\mathcal{G}_{a,b}{}^c = G_{a,b}{}^c + \frac{1}{3}\delta_b^c G^P{}_{,aP} \tag{4.9}$$

Of course for the scalars one must also take into account the standard local symmetry of the non-linear realisation E_6 with local subgroup $USp(8)$ which requires one to use the part of the Cartan form that is in the coset and so transforms homogeneously. While for the vielbein we must also ensure general invariance. In fact the unique object composed from the vielbein and which has only one derivative that is invariant under general coordinate transformations at lowest order is the spin connection. The above discussion suggests that this should be modified by replacing derivative of the vielbein by the combination of equation (4.9).

We now give explicit expressions for the objects in equations (4.2) and (4.4). The Cartan forms $G_{\Pi,\underline{\alpha}}$ are defined in equation (2.11). They are very simple to evaluate using their definition in equation (2.10) and they have been extensively computed in previous E_{11} papers, for example in five dimensions in the [12,44]. They are given by

$$G_{\Pi,aN} = \partial_\Pi A_{aN} \tag{4.10}$$

$$G_{\Pi,a_1 a_2}{}^N = \partial_\Pi A_{a_1 a_2}{}^N + \frac{1}{2}d^{NPQ}\partial_\Pi A_{[a_1|P}A_{|a_2]Q} \tag{4.11}$$

$$G_{\Pi,a_1 a_2 a_3 \beta} = \partial_\Pi A_{a_1 a_2 a_3 \beta} - \frac{1}{6}\partial_\Pi A_{[a_1|M}A_{|a_2|N}A_{|a_3]P}d^{MNQ}(D^\alpha)_Q{}^P - \partial_\Pi A_{[a_1 a_2}{}^M A_{a_3]N}(D^\alpha)_M{}^N \tag{4.12}$$

The Cartan forms of equations (4.2) and (4.4) have as their first index a tangent index rather than the world index that they possess in equations (4.10-12). The quantities are related by $G_{A,\underline{\alpha}} = E^{-1}{}_A{}^\Pi G_{\Pi,\underline{\alpha}}$ where the inverse vielbein $E^{-1}{}_A{}^\Pi$ is given in appendix A in equation (A.32). Evaluating the effect of the inverse vielbein we find for example that

$$G_{a,\underline{\alpha}} = e_a{}^\mu G_{\mu,\underline{\alpha}} + A_{aM}d^M_{\dot{N}}G^{\dot{N}}{}_{,\underline{\alpha}} + (2A_{a\nu}{}^M d^{\dot{P}}_M - \frac{1}{2}d^{\dot{P}}_M d^{MRS}A_{aR}A_{\nu S})G^\nu{}_{\dot{P},\underline{\alpha}} + \dots \tag{4.13}$$

We now truncate the quantities of equations (4.2) and (4.4), keeping only the fields A_{aN} and $A_{a_1 a_2}{}^N$ and the derivative ∂_μ and $\partial^{\dot{N}}$, and discarding all higher level terms to find the expressions

$$\mathcal{G}_{a_1 a_2 N} = e_{[a_1}{}^\mu \partial_\mu A_{\mu a_2]N} + A_{[a_1 \dot{N}}\partial^{\dot{N}}A_{a_2]Q} + 10d_{NRS}d^R_{\dot{R}}(\partial^{\dot{R}}A_{a_1 a_2}{}^S + \frac{1}{2}d^{SPQ}\partial^{\dot{R}}A_{[a_1|P}A_{|a_2]Q}) \tag{4.14}$$

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and

$$\mathcal{G}_{a_1 a_2 a_3}{}^N = e_{[a_1}{}^\mu (\partial_\mu A_{a_2 a_3]}{}^N + \frac{1}{2} d^{NPQ} \partial_\mu A_{|a_2|P} A_{|a_3]Q}) + A_{[a_1|P} d^P_{\dot{P}} \partial^{\dot{P}} A_{|a_2 a_3]}{}^N$$

$$+ \frac{1}{3} d^{NRQ} d^P_{\dot{P}} A_{[a_1|P} \partial^{\dot{P}} A_{|a_2|R} A_{|a_3]Q} - \frac{5}{3} d^{NQW} d^{MPL} d^R_{\dot{R}} \partial^{\dot{R}} A_{[a_1|M} A_{|a_2|P} A_{|a_3]Q} d_{LRW}$$

$$+ A_{[a_1 a_2}{}^N d^S_{\dot{T}} \partial^{\dot{T}} A_{a_3]S} - \frac{1}{3} d_{\dot{T}}{}^N \partial^{\dot{T}} A_{[a_1 a_2}{}^S A_{a_3]S} - 10 d^{NTW} d_{RSW} d^R_{\dot{M}} A_{[a_1 a_2}{}^S \partial^{\dot{M}} A_{a_3]T} \quad (4.15)$$

In deriving this result we have used the identity of equation (A.14) and we have redefined $A_{a_1 a_2 a_3 \alpha}$ before discarding it. After a suitable field redefinition the generalised field strengths of equation (4.14) and (4.15) agree with the generalised field strengths given in the papers [35,36] on exceptional field theory in five dimensions. The possible exception is the coefficient of one term whose value in the papers on exceptional field theory we are not sure of. We will assume that this coefficient does in fact agree on closer examination.

The equation of motion for the vector was deduced using the symmetries of the non-linear realisation $E_{11} \otimes_s l_1$ in four dimensions [22]. It essentially stated that the generalised field strength were self-dual. It is inevitable that a similar calculation in five dimensions will lead to a condition self-duality condition relating the generalised field strength in equation (4.14) and (4.15).

5 Matter representations

Matter representations for non-linear realisations were introduced in the classic papers of reference [45] for the case when the group did not include space-time generators. These are objects which transform linearly under the local subgroup. In previous papers on E_{11} such matter representations were not discussed as they are not needed for the development of the theory, but it is straight forward to introduce them. We recall that the generators of the l_1 representation by definition belong to a representation of E_{11}, a fact that is encoded in the second commutator of equation (2.1), and they transform under a finite E_{11} transformation as in equation (2.7). As a result they must transform under the local subgroup, $I_c(E_{11})$ as follows

$$h^{-1} l_A h = D(h)_A{}^B l_B \quad (5.1)$$

where $h \in I_c(E_{11})$. Such an infinitesimal transformation is of the form $h = 1 + a_{\underline{\alpha}}(R^{\underline{\alpha}} - R^{-\underline{\alpha}})$.

We can introduce a l_1 matter representation V^A whose transformation is such that $V^{A\prime} l_A = h^{-1} V^A l_A h$ and as a result it transforms as

$$V^{A\prime} = V^B D(h)_B{}^A = V^A + V^B (D^{\underline{\alpha}} - D^{-\underline{\alpha}})_B{}^A a_{\underline{\alpha}} + \ldots \quad (5.2)$$

We observe that this is the same way as the A index on the generalised vielbein $E_\Pi{}^A$ transforms. Examining the components of the generators of the l_1 representation given in appendix A we find that the representation V^A has the components

$$V^A = \{V^a, V_N, V_a^N, V_{c_1 c_2 \alpha}, V_{c_1 c_2}, \ldots\} \quad (5.3)$$

We define the covariant derivative of V^A to be

$$\mathcal{D}_A V^B = (D^{\underline{\alpha}})_A{}^B (D_{\underline{\alpha}})_C{}^D D_D V^C \tag{5.4}$$

where the sum is over all indices $\underline{\alpha}$. In this equation we define

$$D_{\underline{\alpha}} = g_{\underline{\alpha}\beta} D^{\underline{\beta}} \tag{5.5}$$

where $g_{\underline{\alpha}\beta}$ is the inverse of $g^{\underline{\alpha}\beta}$, which is the Cartan-Killing metric of E_{11} which is given in appendix A. We also take

$$D_D V^C = E_D{}^\Pi (\partial_\Pi V^C - V^D \Omega_{\Pi,D}{}^C) \tag{5.6}$$

Where $\Omega_{\Pi,D}{}^C$ is the connection which changes under a local $I_c(E_{11})$ transformation as

$$\Omega_\Pi{}' = D(h^{-1}) \Omega_\Pi D(h) + D(h^{-1}) \partial_\Pi D(h) \tag{5.7}$$

where in this last equation we have viewed $\Omega_{\Pi,D}{}^C$ as a matrix and $D(h)$ is the matrix $D(h)_A{}^B$ for $h \in I_c(E_{11})$. The connection is to be constructed from the Cartan forms of equation (2.9). Examining equation (2.16) we recognise that these Cartan forms have the correct transformation law of equation (5.7). In fact the part of the Cartan form that belongs to the local subalgebra has precisely this transform and so from this viewpoint alone one could just take this to be the connection. However, it will turn out that the connection is the $I_c(E_{11})$ part of the Cartan form plus a specific combination of the Cartan forms that belong to the coset part which transform covariantly under $I_C(E_{11})$. This is apparent when one considers general relativity from this viewpoint. It is usual to take the connection, no matter how it is constructed, to belong in the local subalgebra which would imply that $\Omega_{\Pi,D}{}^C = \Omega_{\Pi,\underline{\alpha}}(D^{\underline{\alpha}} - D^{-\underline{\alpha}})_D{}^C$, however, in this paper we will take a more liberal perspective.

The derivative $D_D V^C$ transforms covariantly under a local $I_C(E_{11})$ as one might expect

$$(D_D V^C)' = D(h^{-1})_D{}^E D_E V^F D(h)_F{}^C \tag{5.8}$$

This is not an irreducible representation of $I_c(E_{11})$. We can project it onto the adjoint representation of $I_c(E_{11})$ by acting on it with $(D_{\underline{\alpha}})_C{}^D$ and consider the quantity $(D^{\underline{\alpha}})_C{}^D D_D V^C$ which, using equation (5.8), transforms as

$$(D^{\underline{\alpha}})_C{}^D (D_D V^C)' = (D^{\underline{\alpha}})_C{}^D (D_D V^C) + a_{\underline{\beta}} (f^{\underline{\beta}\underline{\alpha}}{}_{\underline{\gamma}} - f^{-\underline{\beta}\underline{\alpha}}{}_{\underline{\gamma}})(D^{\underline{\gamma}})_C{}^D D_D V^C \tag{5.9}$$

from which it is clear that it does indeed transform in the adjoint representation of $I_C(E_{11})$. In evaluating this equation we have used the result

$$D(h)_D{}^F (D^{\underline{\alpha}})_F{}^E D(h^{-1})_E{}^C = D(hR^{\underline{\alpha}}h^{-1})_D{}^E = D^{\underline{\alpha}} + a_{\underline{\beta}}(f^{\underline{\beta}\underline{\alpha}}{}_{\underline{\gamma}} - f^{-\underline{\beta}\underline{\alpha}}{}_{\underline{\gamma}})D^{\underline{\gamma}} \tag{5.10}$$

where h is the infinitesimal transformation given below equation (5.1) and that by definition $1 + D^{\underline{\alpha}} = D(I + R^{\underline{\alpha}})$.

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Examining equation (5.4) we find an additional factor of $(D^{\underline{\alpha}})_A{}^B$ in the definition of the covariant derivative. However, when the indices A and B are specified the value of $\underline{\alpha}$ is determined. As such the effect of multiplying by this factor is just an equivalent way to writing the quantity $(D_{\underline{\alpha}})_C{}^D D_D V^C$. Using the fact that the transformation of the adjoint representation simply rearranges the sum over $\underline{\alpha}$ we find that the covariant derivative defined in equation (5.4) transforms as

$$(\mathcal{D}_A V^B)' = D(h^{-1})_A{}^E \mathcal{D}_E V^F D(h)_F{}^B \tag{5.11}$$

When examining this formula it is important to remember that the factors of $D(h)$ are projected by the $(D^{\underline{\alpha}})_A{}^B$ to lie in the adjoint representation.

The covariant derivative discussed in this section has the same form as the one which appears in the gauge transformations acting on the gauge parameters. This motivates our use of the projector onto the adjoint representation when constructing the covariant derivative, however, as is apparent from equation (5.8) even if we do not project the covariant derivative of equation (5.6) is indeed covariant. The steps below, in conjunction with appendix B, can be used to evaluate either covariant derivative.

We will now evaluate in more explicit form the covariant derivative of V^A taking particular components. To do this we will use the equations of appendix B and in particular we repeat equation (B.16) which is given by

$$\mathcal{D}_a V_N = D_a V_N - 10\, d_{NMP}\, D^M V_a{}^P - 12\, (D^{\alpha})_N{}^M D^b{}_M V_{ab,\,\alpha} - \frac{2}{3}\, D^b{}_N V_{ab} - \frac{1}{3}\, D^b{}_N V_{ba} + \dots \tag{5.12}$$

Discarding the covariant derivatives of the level two components of V^A and all higher level components, that is $D^c{}_M V_{ab,\,\alpha} = 0 = D^c{}_N V_{ab} = \dots$ and using the expression for the inverse vielbein of equation (A.32), and equations (B.3), and (B.10) of appendix B, we find that

$$\mathcal{D}_a V_N = (\det e)^{\frac{1}{2}} \{ e_a{}^{\mu}(\partial_{\mu} V_N + V^c \Omega_{\mu,cN} + V_P \Omega_{\mu,\alpha}(D^{\alpha} - D^{-\alpha})_N{}^P + 10 d_{PMN} V_e^M \Omega_{\mu,}{}^{eP} \dots)$$

$$+ A_{aP} d_{\dot{P}}^P (\partial^{\dot{P}} V_N + V^c \Omega^{\dot{P}}{}_{,cN} + V_M \Omega^{\dot{P}}{}_{,\alpha}(D^{\alpha} - D^{-\alpha})_N{}^M + \dots)$$

$$+ (2 A_{ab}{}^M - \frac{1}{2} d^{MST} A_{aS} A_{bT}) e_{\mu}{}^b d_M^{\dot{Q}} (\partial^{\mu}{}_{\dot{Q}} V_N + V^c \Omega^{\mu}{}_{\dot{Q},cN} + \dots)$$

$$- 10 d_{NMP} d_{\dot{R}}^M (\partial^{\dot{R}} V_a{}^P + 2 V^c \Omega^{\dot{R}}{}_{,ca}{}^P - V_L \Omega^{\dot{R}}{}_{,aQ} d^{QLP}$$

$$- V_a^L \Omega^{\dot{R}}{}_{,\alpha}(D^{\alpha} - D^{-\alpha})_L{}^P + 2 V_d^P \Omega^{\dot{R}}{}_{,a}{}^d + \dots) + \dots \} \tag{5.13}$$

The appearance of the $(\det e)^{\frac{1}{2}}$ factors will be compensated by the change from a tangent to a world index on V^A which requires the vielbein and so leads to the factor of $(\det e)^{-\frac{1}{2}}$.

In this last formula we now further truncate and set all components of V^A to zero except for V_N to find that

$$\mathcal{D}_a V_N = (\det e)^{\frac{1}{2}} \{ e_a{}^{\mu}(\partial_{\mu} V_N + V_P \Omega_{\mu,\alpha}(D^{\alpha} - D^{-\alpha})_N{}^P$$

$$+A_{aP}d_{\dot{P}}^P(\partial^{\dot{P}}V_N + V_M\Omega^{\dot{P}},_\alpha(D^\alpha - D^{-\alpha})_N{}^M) + 10d_{NMP}d_{\dot{R}}^M V_L\Omega^{\dot{R}},_{aQ}d^{QLP} + ...\} \quad (5.14)$$

We can repeat the above process for the derivative of the component V^b. The expansion of equation (5.4) when we take the values $A = a$ and $B = b$ is given in appendix B in particular in equation (B.14). We then carry out truncations in the same spirit as above to find the expression

$$\mathcal{D}_a V^b = (\det e)^{\frac{1}{2}}\left[e_a{}^\mu\left(\partial_\mu V^b - 2\Omega_{\mu,[c}{}^{b]}V^c - \Omega_\mu{}^{bM}V_M + ...\right)\right.$$

$$+ A_{aN}d_{\dot{N}}{}^N\left(\partial^{\dot{N}}V^b - 2\Omega^{\dot{N}}{}_{[c}{}^{b]}V^c - \Omega^{\dot{N}\,bM}V_M + ...\right)$$

$$+ \left(2A_{ac}{}^N - \frac{1}{2}d^{NPR}A_{aP}A_{cR}\right)e_\mu{}^c d_N{}^{\dot{N}}\left(\partial^\mu{}_{\dot{N}}V^b - 2\Omega^\mu{}_{\dot{N},[c}{}^{b]}V^c - \Omega^\mu{}_{\dot{N}}{}^{bM}V_M + ...\right)$$

$$\left.- e_\mu{}^b d_N{}^{\dot{N}}\left(\partial^\mu{}_{\dot{N}}V_a{}^N + 2\Omega^\mu{}_{\dot{N},ca}{}^N V^c - d^{NMP}\Omega^\mu{}_{\dot{N},aM}V_P + ...\right)\right]$$

$$- \frac{1}{2}(\det e)^{\frac{1}{2}}\delta_a^b\left[e_d{}^\mu\left(\partial_\mu V^d - 2\Omega_{\mu,[c}{}^{d]}V^c - \Omega_\mu{}^{dM}V_M + ...\right)\right.$$

$$+ A_{dN}d_{\dot{N}}{}^N\left(\partial^{\dot{N}}V^d - 2\Omega^{\dot{N}}{}_{[c}{}^{d]}V^c - \Omega^{\dot{N}\,dM}V_M + ...\right)$$

$$+ \left(2A_{dc}{}^N - \frac{1}{2}d^{NPR}A_{dP}A_{cR}\right)e_\mu{}^c d_N{}^{\dot{N}}\left(\partial^\mu{}_{\dot{N}}V^d - 2\Omega^\mu{}_{\dot{N},[c}{}^{d]}V^c - \Omega^\mu{}_{\dot{N}}{}^{dM}V_M + ...\right)$$

$$+ d_{\dot{N}}{}^N\left(\partial^{\dot{N}}V_N + \left(D^\alpha - D^{-\alpha}\right)_N{}^M\Omega^{\dot{N}}{}_\alpha + 10d_{NMP}\Omega^{\dot{N},dM}V_d{}^P + ...\right)$$

$$- d^{NRS}d_R{}^{\dot{N}}A_{cS}\left(\partial^c{}_{\dot{N}}V_N + \left(D^\alpha - D^{-\alpha}\right)_N{}^M\Omega^c{}_{\dot{N},\alpha} + 10d_{NMP}\Omega^c{}_{\dot{N}}{}^{dM}V_d{}^P + ...\right)$$

$$\left.+ e_\mu{}^d d_N{}^{\dot{N}}\left(\partial^\mu{}_{\dot{N}}V_d{}^N + 2\Omega^\mu{}_{\dot{N},cd}{}^N V^c - d^{NMP}\Omega^\mu{}_{\dot{N},dM}V_P + ...\right) + ...\right] \quad (5.16)$$

Proceeding in the same way we find that the covariant derivative of $V_a{}^N$ is given by

$$\mathcal{D}_a V_b{}^N = -2(\det e)^{\frac{1}{2}}e_{[a}{}^\mu\left(\partial_\mu V_{b]}{}^N - d^{NMP}\Omega_{\mu,b]M}V_P\right.$$

$$\left.- \left(D^\alpha - D^{-\alpha}\right)_M{}^N\Omega_{\mu,\alpha}V_{b]}{}^M + 2\Omega_{\mu,b]}{}^d V_d{}^N + 2V^e\Omega_{\mu,eb}{}^N...\right) + ... \quad (5.17)$$

6 The world volume viewpoint

The deliberations of the paper so far have largely been based on tangent space quantities. In this section we will transfer our results to an approach based on world volume quantities. We begin by reformulating the gauge transformations of equation (3.1) by replacing the parameter of gauge transformations Λ^A by the world volume object Λ^Π, the two quantities being related by $\Lambda^A E_A{}^\Pi = \Lambda^\Pi$. We begin by processing the gauge transformation of equation (3.1) which can be written as

$$\delta E_\Pi{}^A = (D^{\underline{\alpha}})_B{}^A E_\Pi{}^B (D^{\underline{\alpha}})_C{}^D E_D{}^\Sigma E_\Delta{}^C E_F{}^\Delta D_\Sigma\Lambda^F \quad (6.1)$$

As explained above equation (2.14), the vielbein can be written as $E_\Pi{}^A = D(g_E)_\Pi{}^A$. Substituting this into the above equation, using arguments similar to those given below equation (5.10) and the fact that $\underline{\alpha}$ is a dummy index we find that

$$\delta E_\Pi{}^A = (D^{\underline{\alpha}})_\Pi{}^\Sigma E_\Sigma{}^A (D^{\underline{\alpha}})_\Sigma{}^\Theta E_C{}^\Sigma D_\Theta \Lambda^C \tag{6.2}$$

Changing to the world volume parameter this equation becomes

$$\delta E_\Pi{}^A = (D^{\underline{\alpha}})_\Pi{}^\Sigma E_\Sigma{}^A (D^{\underline{\alpha}})_\Sigma{}^\Theta (\partial_\Theta \Lambda^\Sigma + E_F{}^\Sigma (\partial_\Theta E_\Gamma{}^F - E_\Gamma{}^D \Omega_{\Theta,D}{}^F) \Lambda^\Gamma) \tag{6.3}$$

Using the fact that the torsion tensor is given by

$$T_{\Theta\Delta}{}^C = \partial_\Theta E_\Delta{}^C - \partial_\Delta E_\Theta{}^C - E_\Delta{}^D \Omega_{\Theta,D}{}^C + E_\Theta{}^D \Omega_{\Delta,D}{}^C \tag{6.4}$$

we may rewrite equation (6.3) as

$$\delta E_\Pi{}^A = (D^{\underline{\alpha}})_\Pi{}^\Sigma E_\Sigma{}^A (D^{\underline{\alpha}})_\Sigma{}^\Theta D_\Theta \Lambda^\Sigma + \Lambda^\Gamma \partial_\Gamma E_\Pi{}^A \tag{6.5}$$

where

$$D_\Theta \Lambda^\Sigma = \partial_\Theta \Lambda^\Sigma + T_{\Theta\Gamma}{}^\Sigma \Lambda^\Gamma - E_C{}^\Sigma \Omega_{\Gamma,\Theta}{}^C \Lambda^\Gamma \tag{6.6}$$

In evaluating equation (6.3) we have used the identity

$$(D^{\underline{\alpha}})_\Pi{}^\Delta E_\Delta{}^A (D^{\underline{\alpha}})_\Sigma{}^\Theta E_C{}^\Sigma \Lambda^\Gamma \partial_\Gamma E_\Theta{}^C = (D^{\underline{\alpha}})_F{}^A E_\Pi{}^F (D_{\underline{\alpha}})_C{}^D \Lambda^\Gamma G_{\Gamma,D}{}^C$$

$$= \Lambda^\Gamma \partial_\Gamma E_\Pi{}^A \tag{6.7}$$

In the last step we have adopted the normalisation

$$(D^{\underline{\alpha}})_C{}^D (D_{\underline{\beta}})_D{}^C = \delta_{\underline{\beta}}^{\underline{\alpha}} \tag{6.8}$$

We note that the part of the connection in equation (6.3) that is in $I_c(E_{11})$ results in a local $I_c(E_{11})$ transformation of $E_\Pi{}^A$ of the form

$$-(D^{\underline{\alpha}})_F{}^A E_\Pi{}^F (D_{\underline{\alpha}})_C{}^D \Lambda^\Gamma \Omega_{\Gamma,D}{}^C = -(D^{\underline{\alpha}} - D^{-\underline{\alpha}})_F{}^A E_\Pi{}^F \Lambda^\Gamma \Omega_{\Gamma,\underline{\alpha}} \tag{6.9}$$

and so it can be dropped form the variation of $\delta E_\Pi{}^A$ if desired.

We now demand that the derivative that appears in the transformation of the vielbein of equation (6.5) is just an ordinary derivative, that is, it comes with no connection. The condition for this to be true we require that

$$(D^{\underline{\alpha}} + D^{-\underline{\alpha}})_\Sigma{}^\Theta (T_{\Theta\Gamma}{}^\Sigma - E_C{}^\Sigma \Omega_{\Gamma,\Theta}{}^C) = 0 \tag{6.10}$$

That we choose the particular sum over the values of $\underline{\alpha}$ in the above expression is related to the fact, noted above, that the same sum with a minus sign is an $I_c(E_{11})$ transformation.

As a result the transformation of the vielbein is given by

$$E_A{}^\Sigma \delta E_\Pi{}^A = (D^{\underline{\alpha}})_\Pi{}^\Sigma (D^{\underline{\alpha}})_\Sigma{}^\Theta \partial_\Theta \Lambda^\Sigma + \Lambda^\Gamma \partial_\Gamma E_\Pi{}^A E_A{}^\Sigma \tag{6.11}$$

This is the same pattern that appears in general relativity; the general coordinate transformation of the vielbein $e_\mu{}^a$ contains space-time derivatives of a parameter that appear with a connection when the parameter has a tangent index, but with no connection when the parameter has a world index. As a result the definition of the covariant derivative depends on whether it is acting on a world parameter or a tangent parameter. The same is true for the generalised geometry we are constructing in this paper.

In section five we introduced the l_1 representation V^A which carried the tangent space index A. We now consider the corresponding world volume object $V^\Pi = V^A E_A{}^\Pi$. Pursuing the analogy with general relativity further we take the vector V^A to transform under the local transformations as

$$\delta V^A = \Lambda^\Gamma \partial_\Gamma V^A \tag{6.12}$$

Using the transformation of the vielbein of equation (6.5), we then find that under a gauge transformation V^Π transforms as

$$\delta V^\Pi = - V^\Lambda \left(D^{\underline{\alpha}}\right)_\Lambda{}^\Pi \left(D_{\underline{\alpha}}\right)_\Sigma{}^\Xi \partial_\Xi \Lambda^\Sigma + \Lambda^\Sigma \partial_\Sigma V^\Pi + \mu\, V^\Pi \partial_\Sigma \Lambda^\Sigma . \tag{6.13}$$

We have added by hand the last term to take into account of the fact that V^Π could be a density. We note that had we not insisted that the transformation of the vielbein contain an ordinary derivative, rather than a covariant derivative, then the transformation of V^Π would not transform into itself alone.

If we take $\Pi = \dot{N}$ then equation (6.13) becomes

$$\delta V_{\dot{N}} = - 6\, V^{\dot{M}} \left(D^\alpha\right)_{\dot{N}}{}^{\dot{M}} \left(D_\alpha\right)_{\dot{P}}{}^{\dot{Q}} \partial^{\dot{P}} \Lambda_{\dot{Q}} + \Lambda^\Sigma \partial_\Sigma V_{\dot{N}} + \mu\, V_{\dot{N}} \partial^{\dot{P}} \Lambda_{\dot{P}} . \tag{6.14}$$

provided we brutally truncate it to keep only the components $V_{\dot{N}}$ and $\Lambda^{\dot{N}}$ of the vectors V^Π and Λ^Π respectively. This agrees with the transformation of this object given in the papers [35,36] on exceptional field theory.

We now evaluate the gauge transformations of equation (6.11) for the level zero and one fields. We begin with the transformation of the vielbein which occurs at the lowest level of the generalised vielbein, given in equation (A.28), in the combination $\tilde{e}_\mu{}^a = (\det e)^{-\frac{1}{2}} e_\mu{}^a$. Taking $\Pi = \mu$ and $\Lambda = \nu$ in equation (6.11) we find that the right-hand side becomes

$$\delta \tilde{e}_\mu{}^a \, \tilde{e}_a{}^\nu =$$

$$= (D^{\underline{\alpha}})_\mu{}^\nu (D^{\underline{\alpha}})_\Sigma{}^\Theta \partial_\Theta \Lambda^\Sigma + \Lambda^\Pi \partial_\Pi \tilde{e}_\mu{}^a \, \tilde{e}_a{}^\nu$$

$$= \left(D^\lambda{}_\sigma\right)_\mu{}^\nu \left(\delta^\eta_\lambda \delta^\sigma_\rho - \frac{1}{3} \delta^\sigma_\lambda \delta^\eta_\rho\right) \left[\left(D^\rho{}_\eta\right)_\tau{}^\kappa \partial_\kappa \Lambda^\tau + \left(D^\rho{}_\eta\right)^{\dot{N}}{}_{\dot{M}} \partial^{\dot{M}} \Lambda_{\dot{N}} + \left(D^\rho{}_\eta\right)_\tau{}^{\dot{N},\,\kappa}{}_{\dot{M}} \partial^\tau{}_{\dot{N}} \Lambda_\kappa{}^{\dot{M}}\right]$$

$$+ \Lambda^\Pi \partial_\Pi \tilde{e}_\mu{}^a \, \tilde{e}_a{}^\nu$$

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$$= \left(D^{\nu}{}_{\mu}\right)_{\tau}{}^{\kappa} \partial_{\kappa}\Lambda^{\tau} + \left(D^{\nu}{}_{\mu}\right)^{\dot{N}}{}_{\dot{M}} \partial^{\dot{M}}\Lambda_{\dot{N}} + \left(D^{\nu}{}_{\mu}\right)_{\tau}{}^{\dot{N},\kappa}{}_{\dot{M}} \partial^{\tau}{}_{\dot{N}}\Lambda_{\kappa}{}^{\dot{M}} + \Lambda^{\Pi} \partial_{\Pi}\tilde{e}_{\mu}{}^{a} \tilde{e}_{a}{}^{\nu}$$

$$= \partial_{\mu}\Lambda^{\nu} - \partial^{\nu}{}_{\dot{N}} \Lambda_{\mu}{}^{\dot{N}} - \frac{1}{2} \delta^{\nu}_{\mu} \left(\partial_{\sigma}\Lambda^{\sigma} + \partial^{\dot{N}}\Lambda_{\dot{N}} + \partial^{\sigma}{}_{\dot{N}} \Lambda_{\sigma}{}^{\dot{N}}\right) + \Lambda^{\Pi} \partial_{\Pi}\tilde{e}_{\mu}{}^{a} \tilde{e}_{a}{}^{\nu}, \qquad (6.15)$$

where we have only included the components of the parameter Λ^{Π} at levels zero, one and two and discarded the rest. On the other hand the left-hand side of equation (6.11) is given by

$$\delta\tilde{e}_{\mu}{}^{a} \tilde{e}_{a}{}^{\nu} = \delta e_{\mu}{}^{a} e_{a}{}^{\nu} - \frac{1}{2} \delta^{\nu}_{\mu} \left(e_{a}{}^{\lambda} \delta e_{\lambda}{}^{a}\right). \qquad (6.16)$$

and so

$$\tilde{e}_{a}{}^{\lambda} \delta\tilde{e}_{\lambda}{}^{a} = -\frac{3}{2} e_{a}{}^{\lambda} \delta e_{\lambda}{}^{a}. \qquad (6.17)$$

Using this result, and equation (6.15), we find that

$$\delta e_{\mu}{}^{a} e_{a}{}^{\nu} = \delta\tilde{e}_{\mu}{}^{a} \tilde{e}_{a}{}^{\nu} - \frac{1}{3} \delta^{\nu}_{\mu} \left(\tilde{e}_{a}{}^{\lambda} \delta\tilde{e}_{\lambda}{}^{a}\right)$$

$$= \partial_{\mu}\Lambda^{\nu} - \partial^{\nu}{}_{\dot{N}} \Lambda_{\mu}{}^{\dot{N}} + \frac{1}{3} \delta^{\nu}_{\mu} \partial^{\dot{N}}\Lambda_{\dot{N}} + \frac{2}{3} \delta^{\nu}_{\mu} \partial^{\sigma}{}_{\dot{N}} \Lambda_{\sigma}{}^{\dot{N}} + \Lambda^{\Pi} \partial_{\Pi}e_{\mu}{}^{a} e_{a}{}^{\nu}. \qquad (6.18)$$

We now consider the gauge transformations of the level zero scalar fields which are contained in the scalar vielbein $d_{\dot{N}}{}^{N}$, however, the scalar vielbein, given in equation (A.28), contains the combination $\tilde{d}_{\dot{N}}{}^{M} = (\det e)^{-\frac{1}{2}} d_{\dot{N}}{}^{M}$. Taking $\Pi = \dot{N}$ and $\Sigma = \dot{M}$ in equation (6.11) give the equation

$$\tilde{d}^{P}{}_{\dot{M}} \delta\tilde{d}^{\dot{N}}{}_{P} = (D^{\underline{\alpha}})^{\dot{N}}{}_{\dot{M}} (D_{\underline{\alpha}})_{\Sigma}{}^{\Theta} \partial_{\Theta}\Lambda^{\Sigma} + \tilde{d}^{P}{}_{\dot{M}}\Lambda^{\Pi} \partial_{\Pi}\tilde{d}^{\dot{N}}{}_{P}$$

$$= 6 \left(D^{\alpha}\right)_{\dot{M}}{}^{\dot{N}} \left(D_{\alpha}\right)_{\dot{Q}}{}^{\dot{P}} \left(\partial^{\dot{Q}}\Lambda_{\dot{P}} - \partial^{\mu}{}_{\dot{P}}\Lambda_{\mu}{}^{\dot{Q}}\right) + \tilde{d}^{P}{}_{\dot{M}}\Lambda^{\Pi} \partial_{\Pi}\tilde{d}^{\dot{N}}{}_{P}$$

$$+ \left(D^{\lambda}{}_{\sigma}\right)_{\dot{M}}{}^{\dot{N}} \left(\delta^{\eta}_{\lambda} \delta^{\sigma}_{\rho} - \frac{1}{3} \delta^{\sigma}_{\lambda} \delta^{\eta}_{\rho}\right) \left[\left(D^{\rho}{}_{\eta}\right)_{\tau}{}^{\kappa} \partial_{\kappa}\Lambda^{\tau} + \left(D^{\rho}{}_{\eta}\right)^{\dot{P}}{}_{\dot{Q}} \partial^{\dot{Q}}\Lambda_{\dot{P}} + \left(D^{\rho}{}_{\eta}\right)_{\tau}{}^{\dot{P},\kappa}{}_{\dot{Q}} \partial^{\tau}{}_{\dot{P}}\Lambda_{\kappa}{}^{\dot{Q}}\right]$$

$$= 6 \left(D^{\alpha}\right)_{\dot{M}}{}^{\dot{N}} \left(D_{\alpha}\right)_{\dot{Q}}{}^{\dot{P}} \left(\partial^{\dot{Q}}\Lambda_{\dot{P}} - \partial^{\mu}{}_{\dot{P}}\Lambda_{\mu}{}^{\dot{Q}}\right) - \frac{1}{2} \delta^{\dot{N}}_{\dot{M}} \left(\partial_{\mu}\Lambda^{\mu} + \frac{5}{3} \partial^{\dot{P}}\Lambda_{\dot{P}} + \frac{7}{3} \partial^{\mu}{}_{\dot{P}}\Lambda_{\mu}{}^{\dot{P}}\right)$$

$$+ \tilde{d}^{P}{}_{\dot{M}}\Lambda^{\Pi} \partial_{\Pi}\tilde{d}^{\dot{N}}{}_{P} \qquad (6.19)$$

While the left-hand side is given by

$$\tilde{d}^{P}{}_{\dot{M}} \delta\tilde{d}^{\dot{N}}{}_{P} = d^{P}{}_{\dot{M}} \delta d^{\dot{N}}{}_{P} - \frac{1}{2} \delta^{\dot{N}}_{\dot{M}} \left(e_{a}{}^{\lambda} \delta e_{\lambda}{}^{a}\right). \qquad (6.20)$$

As a result, truncating as before, we find that

$$\delta d_{N}{}^{\dot{N}} d_{\dot{M}}{}^{N} = 6 \left(D^{\alpha}\right)_{\dot{M}}{}^{\dot{N}} \left(D_{\alpha}\right)_{\dot{Q}}{}^{\dot{P}} \left(\partial^{\dot{Q}}\Lambda_{\dot{P}} - \partial^{\mu}{}_{\dot{P}}\Lambda_{\mu}{}^{\dot{Q}}\right) + d^{P}{}_{\dot{M}}\Lambda^{\Pi} \partial_{\Pi}d^{\dot{N}}{}_{P}. \qquad (6.21)$$

In order to find the transformation of level one field A_{aN} we take $\Pi = \mu$ and $\Sigma = \dot{N}$. The left-hand side of equation (6.11) is then given by

$$E_{A\dot{N}}\,\delta E_\mu{}^A = E_{a\dot{N}}\,\delta E_\mu{}^a + E^P{}_{\dot{N}}\,\delta E_{\mu P} = (\det e)^{\frac{1}{2}}\,A_{a\dot{N}}\,\delta\left[(\det e)^{-\frac{1}{2}}\,e_\mu{}^a\right] - d^P{}_{\dot{N}}\delta A_{\mu P}$$

$$= -\delta A_{\mu\dot{N}} - \left(\delta d_P{}^{\dot{M}}\,d_{\dot{N}}{}^P\right)A_{\mu\dot{M}} + (e_a{}^\nu\,\delta e_\mu{}^a)\,A_{\nu\dot{N}}, \tag{6.22}$$

where $\left(\delta d_P{}^{\dot{M}}\,d_{\dot{N}}{}^P\right)$ and $(e_a{}^\nu\,\delta e_\mu{}^a)$ were found in equations (6.21) and (6.18). While the right-hand side of equation (6.11) is given by

$$E_{A\dot{N}}\,\delta E_\mu{}^A = (D^{\underline{\alpha}})_{\mu\dot{N}}\,(D_{\underline{\alpha}})_\Pi{}^\Lambda\,\partial_\Lambda\Lambda^\Pi + \Lambda^\Pi\left(E_{A\dot{N}}\,\partial_\Pi E_\mu{}^A\right)$$

$$= \left(D_{\mu\dot{N}}\right)^{\dot{M}\nu}\partial_\nu\Lambda_{\dot{M}} + \left(D_{\mu\dot{N}}\right)^\nu{}_{\dot{M},}{}^{\dot{P}}\partial_{\dot{P}}\Lambda_\nu{}^{\dot{M}} + \Lambda^\Pi\left(E_{A\dot{N}}\,\partial_\Pi E_\mu{}^A\right) + \ldots$$

$$= \partial_\mu\Lambda_{\dot{N}} - 10\,d_{\dot{N}\dot{M}\dot{P}}\,\partial^{\dot{M}}\Lambda_\mu{}^{\dot{P}} - \Lambda^\Pi\,\partial_\Pi A_{\mu\dot{N}} - \Lambda^\Pi\left(d_{\dot{N}}{}^N\,\partial_\Pi d_N{}^{\dot{M}}\right)A_{\mu\dot{M}} + \Lambda^\Pi\,(e_a{}^\nu\,\partial_\Pi e_\mu{}^a)\,A_{\nu\dot{N}} + \ldots\,$$
$$\tag{6.23}$$

Combining these last two equations we find that

$$\delta A_{\mu\dot{N}} = \Lambda^\Pi\,\partial_\Pi A_{\mu\dot{N}} - \partial_\mu\Lambda_{\dot{N}} + 10\,d_{\dot{N}\dot{M}\dot{P}}\,\partial^{\dot{M}}\tilde{\Lambda}_\mu{}^{\dot{P}} + \partial_\mu\Lambda^\nu A_{\nu\dot{N}}$$

$$-A_{\mu\dot{M}}\partial^{\dot{M}}\Lambda_{\dot{N}} - 10\,d_{\dot{N}\dot{P}\dot{R}}\,d^{\dot{M}\dot{Q}\dot{R}}\Lambda_{\dot{Q}}\,\partial^{\dot{P}}A_{\mu\dot{M}} + \ldots, \tag{6.24}$$

In deriving this result we have used the following identity

$$10\,d_{\dot{N}\dot{P}\dot{R}}\,d^{\dot{M}\dot{Q}\dot{R}}\,A_{\mu\dot{M}}\,\partial^{\dot{P}}\Lambda_{\dot{Q}} = 10\,d_{\dot{N}\dot{P}\dot{R}}\,d^{\dot{M}\dot{Q}\dot{R}}\,\partial^{\dot{P}}\left(\Lambda_{\dot{Q}}\,A_{\mu\dot{M}}\right) - 10\,d_{\dot{N}\dot{P}\dot{R}}\,d^{\dot{M}\dot{Q}\dot{R}}\,\Lambda_{\dot{Q}}\,\partial^{\dot{P}}A_{\mu\dot{M}},$$
$$\tag{6.25}$$

the definition

$$\tilde{\Lambda}_\mu{}^{\dot{P}} = \Lambda_\mu{}^{\dot{P}} + d^{\dot{P}\dot{Q}\dot{R}}\,\Lambda_{\dot{Q}}\,A_{\mu\dot{R}} \tag{6.26}$$

and we have discarded level two derivatives of the level two parameter, that is, terms of the form $\partial^\nu{}_{\dot{M}}\bullet$ where $\bullet$ is any parameter. Keeping only the parameter $\Lambda_{\dot{N}}$, except for one term with the parameter $\Lambda_\mu{}^{\dot{P}}$, we find that the variation of the level one field is given by

$$\delta A_{\mu\dot{N}} = -\partial_\mu\Lambda_{\dot{N}} + \Lambda^{\dot{N}}\,\partial_{\dot{N}}A_{\mu\dot{N}} - A_{\mu\dot{M}}\partial^{\dot{M}}\Lambda_{\dot{N}} - 10\,d_{\dot{N}\dot{P}\dot{R}}\,d^{\dot{M}\dot{Q}\dot{R}}\Lambda_{\dot{Q}}\,\partial^{\dot{P}}A_{\mu\dot{M}}$$

$$+10\,d_{\dot{N}\dot{M}\dot{P}}\,\partial^{\dot{M}}\tilde{\Lambda}_\mu{}^{\dot{P}} + \ldots \tag{6.27}$$

We note that in carrying out this last step we have, in equation (6.26), redefined the level two parameter before we have discarded it. The similarly truncated results for the vielbeins in space-time and the scalar sector of equations (6.18) and (6.21) are given respectively by

$$\delta e_\mu{}^a\,e_a{}^\nu = \partial_\mu\Lambda^\nu + \frac{1}{3}\,\delta_\mu^\nu\,\partial^{\dot{N}}\Lambda_{\dot{N}} + \Lambda^{\dot{P}}\,\partial_{\dot{P}}e_\mu{}^a\,e_a{}^\nu. \tag{6.28}$$

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and

$$\delta d_N{}^{\dot{N}} d_{\dot{M}}{}^N = 6 \left(D^\alpha\right)_{\dot{M}}{}^{\dot{N}} \left(D_\alpha\right)_{\dot{Q}}{}^{\dot{P}} \partial^{\dot{Q}} \Lambda_{\dot{P}} + d^P{}_{\dot{M}} \Lambda^{\dot{P}} \partial_{\dot{P}} d^{\dot{N}}{}_P. \qquad (6.29)$$

Equations (6.27-29) are indeed the the transformations of the gauge fields found in the papers [35,36] on the five dimensional exceptional field theory. However, to get these formulae we have carried out a brutal from the E_{11} perspective. In particular, we have set all components of Λ^Π to zero except $\Lambda_{\dot{N}}$, something which is only compatible with the symmetry $GL(5) \otimes E_6$ part of the E_{11} symmetry.

7. Conclusion

In this paper we have started from the E_{11} approach in five dimensions and evaluated, at the low levels, the corresponding gauge transforms and generalised field strengths for the fields which are defined on a generalised space-time whose coordinates are the usual coordinates of space-time as well as Lorentz scalar coordinates that belong to the $\overline{27}$-dimensional representation of E_6. We also introduced matter fields transforming in the l_1 representation into the non-linear realisation and gave their transformation rules. After a radical truncation the resulting formulae agree with those proposed in exceptional field theory. As we explained in the introduction the main ideas that underlie exceptional field theory are those proposed much earlier in the context of the $E_{11} \otimes_s l_1$ non-linear realisation and, as one can choose to work at low levels in the later, it should not come as too great a surprise to find that one does indeed recover exceptional field theory after a truncation. While we have not computed all possible quantities the reader can be left in no doubt that exceptional field theory is a truncation of the $E_{11} \otimes_s l_1$ non-linear realisation.

The truncation required to find exceptional field theory only preserves the E_6 symmetry and breaks all the higher level E_{11} symmetries. However, if one would like to extend exceptional field theory to include larger symmetries one only has to include the higher level fields and coordinates in the $E_{11} \otimes_s l_1$ non-linear realisation, that is, adopt the $E_{11} \otimes_s l_1$ non-linear realisation as ones starting point. Seen in its proper context exceptional field theory provides evidence for the E_{11} approach.

As we have mentioned in the E_{11} approach the dynamics is determined using the symmetries of the $E_{11} \otimes_s l_1$ non-linear realisation. This was carried out in four dimensions for the gauge fields and scalars [22], although the calculation of the gravity-dual gravity equation was not fully worked out. Analogous result were also found in eleven dimensions [21]. It would be straightforward to find the equivalent results in five dimensions and it would be very surprising if these, when radically truncated, do not agree with those of exceptional field theory. We note that we do not apparently need the gauge transformations studied in this paper and certainly not the matter fields. However, it could be that a knowledge of the gauge transformations could shorten the calculation.

One fact that emerged in references [22,21] was that the $E_{11} \otimes_s l_1$ invariant equations of motion were invariant under the usually understood gauge transformations of form fields and we see in reference [24] and section four that, at the linearised level, they are also invariant under the low level gauge transformations associated with the generalised space-time. Thus it might appear that the symmetries of the $E_{11} \otimes_s l_1$ imply the presence of the diffeomorphism and gauge symmetries. This possibility requires further study.

To find the dynamics of the $E_{11} \otimes_s l_1$ non-linear realisation at higher levels requires us to find equations of motion for the fields which are beyond the usual supergravity fields. However, these fields have in general mixed space-time indices, that is they do not just possess one block of totally antisymmetrised indices. As such they possess gauge transformations that are much more complicated, but are correctly given by the gauge transformations proposed in reference [24] where the gauge parameters are in one to one correspondence with the l_1 representation. As a result the gauge invariant field equations for these higher and higher level fields will involve more and more space-time derivatives, indeed one more space-time derivative for each additional index block. It follows that the equations of motion cannot be given by a Ricci tensor equal zero condition as this has only two derivatives. The gauge transformations contain the rigid E_{11} transformations [24], but these symmetries will rotate the low level supergravity fields into the higher level mixed index fields and as a result one might not expect the Riemann tensor to be gauge invariant. As such the geometry that the non-linear realisation corresponds to must be quite different to that which encodes Einstein's theory and more similar to that found in higher spin theories.

The dynamics of $E_{11} \otimes_s l_1$ non-linear realisation of the four and eleven dimensional theories given in references [22,21] is formulated in terms of duality equations which are first order in derivatives. However, these equations only hold modulo gauge transformations and to eliminate these one must act on the equations with more and more derivatives as the number of gauge transformations grows. This is consistent with the remarks made in the previous paragraph.

Exceptional field is defined so as to possess a section condition which restricts the dependence of the fields on the generalised space-time. Indeed this condition is required from the very outset in order to perform calculations in this theory. However, in the E_{11} approach one does not, at least up to now, need such a condition to find the equations of motion. As such it is possible that the need for the section condition in exceptional field theory is a result of the radical truncation required to find exceptional field theory in the E_{11} approach. As was pointed out in reference [47] the section condition of Siegel theory is a BPS condition and it is not usual to impose such a condition from the beginning. How the restrictions on the field dependence on the generalised space-time might arise was addressed from the viewpoint of the first quantise string theory in reference [48]. One can think of the generalised space-time introduced in the $E_{11} \otimes_s l_1$ non-linear realisation as an effective spacetime just as one thinks about a field theory as an effective field theory. From this perspective how to recover the spacetime we are used to is a physical question whose resolution might require us to understand the physics of spacetime in a more fundamental way.

To find the different theories in the E_{11} approach one decomposes the algebra into different subalgebras. As such the theories in the different dimensions are the same but the quantities are rearranged [10]. Hence it is obvious from this perspective that the theory in five, or any other, dimensions will lift, or descend, to the theory in the new dimension.

As we mentioned in the introduction the E_{11} conjecture is that the effective theory of string and branes is given by the dynamics encoded in the $E_{11} \otimes_s l_1$ non-linear realisation. If one has could show that the dynamics, when truncated to the low level fields which

depend on the usual spacetime only, are just those of maximal supergravity theory then one would, in effect, have to believe in the conjecture. This follows as some of the fields present at higher levels in the E_{11} theory are known to present in the underlying theory. In reference [22] this result was shown for the vector and scalar fields and the graviton equation was also correct at linearised order [49]. The technical problem that arose for the graviton equation is now understood [51], using the ideas put forward in reference [50], and it is to be hoped that this calculation can now be completed.

Acknowledgements

Appendix A: The five dimensional theory

In this appendix we review the $E_{11} \otimes_s l_1$ non-linear realisation in five dimensions. The five dimensional theory is obtained by deleting node 5 from the E_{11} Dynkin diagram to find the algebra $Gl(5) \otimes E_6$. The $E_{11} \otimes_s l_1$ algebra is then decomposed into representations of this subalgebra of E_{11} [12,44].

The generators of E_{11} are denoted by $R^{\underline{\alpha}}$. The level of an E_{11} generators is just the number of up minus down $GL(5)$ indices. The positive, including zero, level generators of the E_{11} up to level 3 are

$$K^a{}_b, \ R^\alpha, \ R^{aM}, \ R^{a_1 a_2}{}_N, \ R^{a_1 a_2 a_3, \alpha}, \ R^{a_1 a_2, b} \ldots \qquad (A.1)$$

where $R^{[a_1 a_2, b]} = 0$. The negative level generators are given by

$$R_{aM}, \ R_{a_1 a_2}{}^N, \ R_{a_1 a_2 a_3}{}^\alpha, \ R_{a_1 a_2, b}, \ldots \qquad (A.2)$$

The l_1 representation decomposes into representations of $Gl(5) \otimes E_6$ as follows

$$l_A = \{P_a, \quad Z^N, \ Z^a{}_N, \ Z^{a_1 a_2, \alpha}, \ Z^{ab} \ldots\} \qquad (A.3)$$

The fourth generator Z^{ab} has no symmetries on its indices. The level is the number of up minus down GL(5) indices plus one. For all these objects the lower case Latin indexes correspond to 5-dimensional fundamental representation of $GL(5)$ ($a, b, c, \ldots = 1, \ldots, 5$). Lower case Greek indexes correspond to 78-dimensional adjoint representation of E_6 ($\alpha, \beta, \gamma, \ldots = 1, \ldots, 78$). We note that the generators of E_6 are denoted by R^α which has no underline on the α. to distinguish it from the generators of E_{11}. Upper and lower case Latin indexes correspond to $\overline{27}$-dimensional and 27-dimensional representations of E_6 ($N, M, P, \ldots = 1, \ldots, 27$) respectively. An arbitrary group element of $E_{11} \otimes_s l_1$ can be written as $g = g_l g_E$ where

$$g_l = \exp\left(x^a P_a + x_N Z^N + x_a{}^N Z^a{}_N + x_{a_1 a_2, \alpha} Z^{a_1 a_2, \alpha} + x_{ab} Z^{ab} + \ldots\right),$$

$$g_E = \exp\left(h_a{}^b\, K^a{}_b\right)\, \exp\left(\varphi_\alpha\, R^\alpha\right)\, \exp\left(A_{a_1 a_2 a_3,\,\alpha}\, R^{a_1 a_2 a_3,\,\alpha}\right)$$

$$\times \exp\left(A_{a_1 a_2,\,b}\, R^{a_1 a_2,\,b}\right)\, \exp\left(A_{a_1 a_2}{}^N\, R^{a_1 a_2}{}_N\right)\, \exp\left(A_{aN}\, R^{aN}\right)\dots. \qquad (A.4)$$

In writing this group element we have used the local symmetry of the non-linear realisation $g \to gh$ to gauge away all terms that involve negative level generators. The group element g_l is labelled by the quantities

$$x^a,\ x_N,\ x_a{}^N,\ x_{a_1 a_2,\,\alpha},\ x_{ab},\ \dots \qquad (A.5)$$

which will be identified with the generalised coordinates of the generalised space-time, while the fields

$$h_a{}^b,\ \varphi_\alpha,\ A_{aM},\ A_{a_1 a_2}{}^N,\ A_{a_1 a_2 a_3,\,\alpha},\ A_{a_1 a_2,\,b},\ \dots \qquad (A.6)$$

which depend on the generalised space-time.

We now give the E_{11} algebra when written in terms of the above generators. The commutators of the E_{11} generators with the $GL(4)$ generators $K^a{}_b$ are

$$[K^a{}_b,\, K^c{}_d] = \delta^c_b\, K^a{}_d - \delta^a_d\, K^c{}_b, \quad [K^a{}_b,\, R^\alpha] = 0,$$

$$[K^a{}_b,\, R^{cN}] = \delta^c_b\, R^{aN}, \quad [K^a{}_b,\, R_{cN}] = -\,\delta^a_c\, R_{bN},$$

$$[K^a{}_b,\, R^{a_1 a_2}{}_N] = 2\,\delta^{[a_1}_b\, R^{|a|a_2]}{}_N, \quad [K^a{}_b,\, R_{a_1 a_2}{}^N] = -\,2\,\delta^a_{[a_1}\, R_{|b|a_2]}{}^N,$$

$$[K^a{}_b,\, R^{a_1 a_2 a_3,\,\alpha}] = 3\,\delta^{[a_1}_b\, R^{|a|a_2 a_3],\,\alpha}, \quad [K^a{}_b,\, R_{a_1 a_2 a_3}{}^\alpha] = -\,3\,\delta^a_{[a_1}\, R_{|b|a_2 a_3]}{}^\alpha,$$

$$[K^a{}_b,\, R^{a_1 a_2,\,c}] = 2\,\delta^{[a_1}_b\, R^{|a|a_2],\,c} + \delta^c_b\, R^{a_1 a_2,\,a}, \quad [K^a{}_b,\, R_{a_1 a_2,\,c}] = -\,2\,\delta^a_{[a_1}\, R_{|b|a_2],\,c} - \delta^a_c\, R_{a_1 a_2,\,b}.$$

$$[K^a{}_b,\, R^{a_1 \dots a_4}{}_{N_1 N_2}] = 4\delta^{[a_1}_b\, R^{|a|a_2 a_3 a_4]}{}_{N_1 N_2}, \quad [K^a{}_b R_{a_1 \dots a_4}{}^{N_1 N_2}] = -\,4\,\delta^a_{[a_1}\, R_{|b|a_2 a_3 a_4]}{}^{N_1 N_2},$$

$$\left[K^a{}_b,\, R^{a_1 a_2 a_3,\,cN}\right] = 3\,\delta^{[a_1}_b\, R^{|a|a_2 a_3],\,cN} + \delta^c_b\, R^{a_1 a_2 a_3,\,aN},$$

$$[K^a{}_b,\, R_{a_1 a_2 a_3,\,cN}] = -\,3\,\delta^a_{[a_1}\, R_{|b|a_2 a_3],\,cN} - \delta^a_c\, R_{a_1 a_2 a_3,\,bN}. \qquad (A.7)$$

The commutators of the E_{11} generators with E_6 generators R^α are determined by the representation of E_6 that this generator belongs to. They are given by

$$\left[R^\alpha,\, R^\beta\right] = f^{\alpha\beta}{}_\gamma\, R^\gamma, \quad \left[R^\alpha,\, R^{aM}\right] = (D^\alpha)_N{}^M R^{aN}, \quad \left[R^\alpha,\, R_{aM}\right] = -\,(D^\alpha)_M{}^N R_{aN},$$

$$\left[R^\alpha,\, R^{a_1 a_2}{}_M\right] = -\,(D^\alpha)_M{}^N R^{a_1 a_2}{}_N, \quad \left[R^\alpha,\, R_{a_1 a_2}{}^N\right] = (D^\alpha)_M{}^N R_{a_1 a_2}{}^M,$$

$$\left[R^\alpha,\, R^{a_1 a_2 a_3,\,\beta}\right] = f^{\alpha\beta}{}_\gamma\, R^{a_1 a_2 a_3,\,\gamma}, \quad \left[R^\alpha,\, R_{a_1 a_2 a_3}{}^\beta\right] = f^{\alpha\beta}{}_\gamma\, R_{a_1 a_2 a_3}{}^\gamma,$$

$$\left[R^\alpha,\, R^{a_1 a_2,\,b}\right] = 0, \quad [R^\alpha,\, R_{a_1 a_2,\,b}] = 0,$$

$$\left[R^\alpha,\, R^{abcd}{}_{MN}\right] = -\,(D^\alpha)_M{}^P R^{abcd}{}_{PN} - (D^\alpha)_N{}^P R^{abcd}{}_{MP},$$

$$\left[R^\alpha,\, R_{abcd}{}^{MN}\right] = (D^\alpha)_P{}^M R^{abcdPN} + (D^\alpha)_P{}^N R_{abcd}{}^{MP}, \qquad (A.8)$$

$$\left[R^\alpha,\, R^{a_1 a_2 a_3,\,bM}\right] = (D^\alpha)_N{}^M R^{a_1 a_2 a_3,\,bN}, \quad \left[R^\alpha,\, R_{a_1 a_2 a_3,\,bM}\right] = -\,(D^\alpha)_M{}^N R_{a_1 a_2 a_3,\,bN}.$$

where $f^{\alpha\beta}{}_{\gamma}$ are the structure constants of E_6, normalised by

$$f_{\alpha\beta\gamma}\, f^{\alpha\beta\delta} = -\, 4\, \delta^{\delta}_{\gamma} \qquad (A.9)$$

We lower and raise indices on the E_6 generators with the Cartan-Killing metric $g_{\alpha\beta}$ of E_6. The matrices $(D^{\alpha})_N{}^M$ are the generators of E_6 in $\mathbf{27}$ representation and so obey the relation

$$[D^{\alpha}, D^{\beta}]_M{}^N = f^{\alpha\beta}{}_{\gamma}(D^{\gamma})_M{}^N \qquad (A.10)$$

They are normalised so that

$$(D^{\alpha})_M{}^N (D^{\beta})_N{}^M = g^{\alpha\beta} \qquad (A.11)$$

The commutation relations of the positive level E_{11} generators are given by

$$\left[R^{aM},\, R^{bN}\right] = d^{MNP}\, R^{ab}{}_P, \quad \left[R^{aN},\, R^{bc}{}_M\right] = (D_{\alpha})_M{}^N R^{abc,\,\alpha} + \delta^N_M\, R^{bc,a},$$

$$\left[R^{ab}{}_M,\, R^{cd}{}_N\right] = R^{abcd}{}_{MN} - 20\, d_{MNP}\, R^{ab[c,\,d]P}, \quad \left[R^{aN},\, R^{bc,\,d}\right] = R^{abc,\,dN} - \frac{1}{3}\, R^{bcd,\,aN},$$

$$\left[R^{aN},\, R^{bcd,\,\alpha}\right] = 3\, d^{NMP}\, (D^{\alpha})_P{}^R R^{abcd}{}_{MR} + 6\, (D^{\alpha})_M{}^N R^{bcd,\,aM}. \qquad (A.12)$$

where d^{MNP} is the completely symmetric invariant tensor of E_6 formed from the product of three $\overline{\mathbf{27}}$ representations. The invariant tensor d_{MNP}, which has its indices down, is completely symmetric product of three $\mathbf{27}$ indices and satisfies the relation

$$d^{MNP} d_{MNQ} = \delta^P_Q \qquad (A.13)$$

We follow the conventions of reference [44] rather than reference [12], the difference being a rescaling of d by $\sqrt{5}$. We will also use the relation [44]

$$g_{\alpha\beta} D^{\alpha}_M{}^N D^{\beta}_P{}^Q = \frac{1}{6}\delta^N_P\delta^Q_M + \frac{1}{18}\delta^N_M\delta^Q_P - \frac{5}{3}d^{NQR}d_{MPR} \qquad (A.14)$$

The commutators of negative-level E_{11} generators are

$$[R_{aN},\, R_{bM}] = d_{NMP}\, R_{ab}{}^P, \quad [R_{aN},\, R_{bc}{}^M] = (D_{\alpha})_N{}^M R_{abc}{}^{\alpha} + \delta^M_N\, R_{bc,a}.$$

$$\left[R_{ab}{}^M,\, R_{cd}{}^N\right] = R_{abcd}{}^{MN} - 20\, d^{MNP}\, R_{ab[c,\,d]P}, \quad [R_{aN},\, R_{bc,\,d}] = R_{abc,\,dN} - \frac{1}{3}\, R_{bcd,\,aN},$$

$$[R_{aN},\, R_{bcd,\,\alpha}] = 3\, d_{NMP}\, (D_{\alpha})_R{}^P R_{abcd}{}^{MR} + 6\, (D_{\alpha})_N{}^M R_{bcd,\,aM}. \qquad (A.15)$$

The commutators between the positive and negative level generators of E_{11} up to level 4 are given by

$$\left[R^{aN},\, R_{bM}\right] = 6\, \delta^b_a\, (D_{\alpha})_M{}^N R^{\alpha} + \delta^N_M\, K^a{}_b - \frac{1}{3}\, \delta^N_M\, \delta^a_b\, K^c{}_c,$$

$$\left[R_{aN},\, R^{bc}{}_M\right] = 20\, d_{NMP}\, \delta_a^{[b}\, R^{c]P}, \quad \left[R^{aN},\, R_{bc}{}^M\right] = 20\, d^{NMP}\, \delta_{[b}^{a}\, R_{c]P},$$

$$\left[R_{aN},\, R^{a_1 a_2 a_3,\, \alpha}\right] = 18\, (D^\alpha)_N{}^M\, \delta_a^{[a_1} R^{a_2 a_3]}{}_M, \quad \left[R^{aN},\, R_{a_1 a_2 a_3}{}^\alpha\right] = 18\, (D^\alpha)_M{}^N \delta_{[a_1}^{a} R_{a_2 a_3]}{}^M,$$

$$\left[R_{aN},\, R^{a_1 a_2,\, b}\right] = \delta_a^b\, R^{a_1 a_2}{}_N - \delta_a^{[b}\, R^{a_1 a_2]}{}_N, \quad \left[R^{aN},\, R_{a_1 a_2,\, b}\right] = \delta_b^a\, R_{a_1 a_2}{}^N - \delta_{[b}^a\, R_{a_1 a_2]}{}^N,$$

$$\left[R_{aN},\, R^{a_1 \ldots a_4}{}_{N_1 N_2}\right] = 40\, d_{N[N_1|M|}\, (D_\alpha)_{N_2]}{}^M \delta_a^{[a_1}\, R^{a_2 a_3 a_4],\, \alpha},$$

$$\left[R^{aN},\, R_{a_1 \ldots a_4}{}^{N_1 N_2}\right] = 40\, d^{N[N_1|M|}\, (D_\alpha)_M{}^{N_2]} \delta_{[a_1}^{a}\, R_{a_2 a_3 a_4}{}^\alpha, \tag{A.16}$$

$$\left[R_{aN},\, R^{a_1 a_2 a_3,\, bM}\right] = (D_\alpha)_N{}^M \delta_a^b\, R^{a_1 a_2 a_3,\, \alpha} - (D_\alpha)_N{}^M \delta_a^{[b}\, R^{a_1 a_2 a_3],\, \alpha} + 3\, \delta_N^M\, \delta_a^{[a_1}\, R^{a_2 a_3],\, b},$$

$$\left[R^{aN},\, R_{a_1 a_2 a_3,\, bM}\right] = (D_\alpha)_M{}^N \delta_b^a\, R_{a_1 a_2 a_3}{}^\alpha - (D_\alpha)_M{}^N \delta_{[b}^a\, R_{a_1 a_2 a_3]}{}^\alpha + 3\, \delta_M^N\, \delta_{[a_1}^{a}\, R_{a_2 a_3],\, b}.$$

The Cartan involution acts on the generators of E_{11} as follows

$$I_c\left(K^a{}_b\right) = -\, K^b{}_a, \quad I_c\left(R^\alpha\right) = -\, R^{-\alpha}, \quad I_c\left(R^{aN}\right) = -\, J^{MN}\, R_{aM},$$

$$I_c\left(R^{ab}{}_M\right) = J_{MN}^{-1}\, R_{ab}{}^N, \quad I_c\left(R^{abc,\, \alpha}\right) = -\, R_{abc,\, -\alpha}, \quad I_c\left(R^{a_1 a_2,\, c}\right) = -\, R_{a_1 a_2,\, c},$$

$$I_c\left(R^{abcd}{}_{MN}\right) = J_{MP}^{-1}\, J_{NQ}^{-1}\, R_{abcd}{}^{PQ}, \quad I_c\left(R^{abc,\, dN}\right) = J^{NM}\, R_{abc,\, dM}. \tag{A.17}$$

where the constant J_{NM} relates generators with the **27** and $\bar{\mathbf{27}}$ representation under the Cartan involution.

The scalar product on the E_{11} algebra $(R^{\underline{\alpha}}, R^{\underline{\beta}}) \equiv g^{\underline{\alpha\beta}}$ defines the Cartan-Killing metric $g^{\underline{\alpha\beta}}$. It is straightforward to calculate using its invariance property, $([R^{\underline{\gamma}}, R^{\underline{\alpha}}], R^{\underline{\beta}}) + (R^{\underline{\alpha}}, [R^{\underline{\gamma}}, R^{\underline{\beta}}]) = 0$ together with the above commutation relations.

We find that the E_{11} Killing metric in the decomposition suitable to five dimensions is found to be given up to level 2 by

$$g^{\underline{\alpha\beta}} = \begin{pmatrix} \frac{1}{6}\, g^{\alpha\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_d^a \delta_b^c - \frac{1}{2}\, \delta_b^a \delta_d^c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_b^a \delta_M^N & 0 & 0 \\ 0 & 0 & \delta_a^b \delta_N^M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20\, \delta_N^M\, \delta_{b_1 b_2}^{a_1 a_2} \\ 0 & 0 & 0 & 0 & -20\, \delta_M^N\, \delta_{a_1 a_2}^{b_1 b_2} & 0 \end{pmatrix}, \tag{A.18}$$

where the generators in the scalar product are labelled by

$$R^{\underline{\alpha}} = \{R^\alpha,\, K^a{}_b,\, R^{aN},\, R_{aN},\, R^{a_1 a_2}{}_N,\, R_{a_1 a_2}{}^N,\, \ldots\} \tag{A.19}$$

and

$$R^{\underline{\beta}} = \{R^\beta,\, K^c{}_d,\, R^{cM},\, R_{cM},\, R^{b_1 b_2}{}_M,\, R_{b_1 b_2}{}^M,\, \ldots\}$$

. The next entries for the metric are more easily given by listing the scalar product

$$\left(R^{a_1 a_2 a_3,\, \alpha},\, R_{b_1 b_2 b_3}{}^\beta\right) = 360\, g^{\alpha\beta}\, \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3}, \quad \left(R^{a_1 a_2,\, a},\, R_{b_1 b_2,\, b}\right) = 20\, \delta_b^a\, \delta_{b_1 b_2}^{a_1 a_2} - 20\, \delta_b^{a\, a_1 a_2}{}_{b_1 b_2}. \tag{A.20}$$

To avoid confusion of notation we denote $g^{\alpha\beta}$ to be the Killing metric on E_6 and we never write the α, β components of the E_{11} Cartan-Killing metric, although of course we do use it.

We now give the commutators between the generators of E_{11} and those of the l_1 representation. The commutation relations between the later and the generators of GL(5) are given by

$$[K^a{}_b,\, P_c] = -\delta^a_c\, P_b + \frac{1}{2}\, \delta^a_b\, P_c, \quad [K^a{}_b,\, Z^N] = \frac{1}{2}\, \delta^a_b\, Z^N, \quad [K^a{}_b,\, Z^c{}_N] = \delta^c_b\, Z^a_N + \frac{1}{2}\, \delta^a_b\, Z^c_N,$$

$$[K^a{}_b,\, Z^{a_1 a_2,\, \alpha}] = 2\,\delta^{[a_1}_b\, Z^{|a|a_2],\, \alpha} + \frac{1}{2}\, \delta^a_b\, Z^{a_1 a_2,\, \alpha}, \quad [K^a{}_b,\, Z^{cd}] = \delta^c_b\, Z^{ad} + \delta^d_b\, Z^{ca} + \frac{1}{2}\, \delta^a_b\, Z^{cd}. \tag{A.21}$$

while with the generators of E_6 we have

$$[R^\alpha,\, P_a] = 0, \quad [R^\alpha,\, Z^M] = (D^\alpha)_N{}^M Z^N, \quad [R^\alpha,\, Z^a{}_N] = -(D^\alpha)_N{}^M Z^a{}_M,$$

$$[R^\alpha,\, Z^{a_1 a_2,\, \beta}] = f^{\alpha\beta}{}_\gamma\, Z^{a_1 a_2,\, \gamma}, \quad [R^\alpha,\, Z^{ab}] = 0. \tag{A.22}$$

The elements of the l_1 representation at a given level can be introduced into the algebra by taking the commutators of suitable E_{11} generators of the same level with P_a, namely

$$[R^{aN},\, P_b] = \delta^a_b\, Z^N, \quad [R^{a_1 a_2}{}_N,\, P_a] = 2\,\delta^{[a_1}_a\, Z^{a_2]}{}_N,$$

$$[R^{a_1 a_2 a_3,\, \alpha},\, P_a] = 3\,\delta^{[a_1}_a\, Z^{a_2 a_3],\, \alpha}, \quad [R^{a_1 a_2,\, b},\, P_a] = -2\,\delta^b_a\, Z^{[a_1 a_2]} - 2\,\delta^{[a_1}_a\, Z^{|b|a_2]}, \tag{A.23}$$

The commutators of the remaining positive level generators of E_{11} with the l_1 generators is determined by the Jacobi identities and they are found to be given by

$$[R^{aM},\, Z^N] = -d^{MNP}\, Z^a{}_P, \quad [R^{aN},\, Z^b{}_M] = -(D_\alpha)_M{}^N Z^{ab,\,\alpha} - \delta^N_M\, Z^{ab},$$

$$[R^{a_1 a_2}{}_N,\, Z^M] = -(D_\alpha)_N{}^M Z^{a_1 a_2,\,\alpha} + 2\,\delta^N_M\, Z^{[a_1 a_2]}. \tag{A.24}$$

Commutators between the level -1 generators of E_{11} and those of the l_1 representation are also determined by the Jacobi identities to be given by

$$[R_{aN},\, P_b] = 0, \quad [R_{aN},\, Z^M] = \delta^M_N\, P_a, \quad [R_{aN},\, Z^b{}_M] = -10\,d_{NMP}\,\delta^b_a\, Z^P,$$

$$[R_{aN},\, Z^{a_1 a_2,\,\alpha}] = -12\,(D^\alpha)_N{}^M \delta^{[a_1}_a\, Z^{a_2]}{}_M, \quad [R_{aN},\, Z^{bc}] = -\frac{2}{3}\,\delta^b_a\, Z^c{}_N - \frac{1}{3}\,\delta^c_a\, Z^b{}_N. \tag{A.25}$$

Commutators with level -2 generators are

$$[R_{a_1 a_2}{}^N,\, P_b] = 0, \quad [R_{a_1 a_2}{}^N,\, Z^M] = 0, \quad [R_{a_1 a_2}{}^N,\, Z^b{}_M] = 20\,\delta^N_M\,\delta^b_{[a_1}\, P_{a_2]},$$

$$[R_{a_1 a_2}{}^N,\, Z^{b_1 b_2,\,\alpha}] = 120\,\delta^{b_1 b_2}_{a_1 a_2}\,(D^\alpha)_M{}^N Z^M, \quad [R_{a_1 a_2}{}^N,\, Z^{b_1 b_2}] = -\frac{20}{3}\,\delta^{b_1 b_2}_{a_1 a_2}\, Z^N. \tag{A.26}$$

Commutators with level -3 generator $R_{a_1a_2a_3}{}^\alpha$ are

$$[R_{a_1a_2a_3}{}^\alpha,\, P_b] = 0, \quad [R_{a_1a_2a_3}{}^\alpha,\, Z^M] = 0, \quad [R_{a_1a_2a_3}{}^\alpha,\, Z^b{}_M] = 0,$$

$$[R_{a_1a_2a_3}{}^\alpha,\, Z^{b_1b_2,\beta}] = 360\, g^{\alpha\beta}\, \delta^{b_1b_2}_{[a_1a_2}\, P_{a_3]}, \quad [R_{a_1a_2a_3}{}^\alpha,\, Z^{bc}] = 0. \qquad (A.27)$$

The generalised vielbein is defined in equation (2.11) and it was found in references [12,46] to be given by

$$E_\Pi{}^A = (\det e)^{-\frac{1}{2}}$$

$$\begin{pmatrix}
e_\mu{}^a & e_\mu{}^b\,\alpha_{b|M} & e_\mu{}^b\,\alpha_{b|a}{}^M & e_\mu{}^b\,\alpha_{b|a_1a_2,\alpha} & e_\mu{}^b\,\alpha_{b|cd} \\
0 & (d^{-1})_M{}^{\dot N} & (d^{-1})_P{}^{\dot N}\beta^P{}_a{}^M & (d^{-1})_P{}^{\dot N}\beta^P{}_{a_1a_2,\alpha} & (d^{-1})_P{}^{\dot N}\beta^P{}_{cd} \\
0 & 0 & d_{\dot N}{}^M (e^{-1})_a{}^\mu & d_{\dot N}{}^P (e^{-1})_b{}^\mu\gamma^b{}_{P|a_1a_2,\alpha} & d_{\dot N}{}^P (e^{-1})_b{}^\mu\gamma^b{}_{P|cd} \\
0 & 0 & 0 & (e^{-1})_{a_1a_2}{}^{\mu_1\mu_2} (f^{-1})_\beta{}^\alpha & 0 \\
0 & 0 & 0 & 0 & (e^{-1})_c{}^\mu (e^{-1})_d{}^\nu
\end{pmatrix}, \quad (A.28)$$

where in the first line

$$\alpha_{a|N} = -A_{aN}, \quad \alpha_{a|b}{}^N = -2\,A_{ab}{}^N - \frac{1}{2}\,d^{NMP}A_{aM}A_{bP},$$

$$\alpha_{a|a_1a_2,\alpha} = -3\,A_{aa_1a_2,\alpha} + 2\,A_{a[a_1}{}^N A_{a_2]M}\,(D_\alpha)_N{}^M + \frac{1}{6}\,A_{aN}A_{[a_1M}A_{a_2]P}\,d^{NMS}(D_\alpha)_S{}^P,$$

$$\alpha_{a|cd} = -4\,A_{d(a,c)} - 2\,A_{ad}{}^N A_{cN} - \frac{1}{6}\,A_{aN}A_{bM}A_{cP}\,d^{NMP}, \qquad (A.29)$$

in the second line

$$\beta^N{}_a{}^M = A_{aP}\,d^{NMP}, \quad \beta^N{}_{a_1a_2,\alpha} = A_{a_1a_2}{}^M(D_\alpha)_M{}^N - \frac{1}{2}\,A_{[a_1M}A_{a_2]R}\,d^{NMP}(D_\alpha)_P{}^R,$$

$$\beta^N{}_{ab} = -2\,A_{ab}{}^N + \frac{1}{2}\,A_{aM}A_{bP}\,d^{NMP}, \qquad (A.30)$$

and in the third line

$$\gamma^a{}_{N|a_1a_2,\alpha} = -\delta^a_{[a_1}A_{a_2]M}\,(D_\alpha)_N{}^M, \quad \gamma^a{}_{N|cd} = \delta^a_d\,A_{cN}. \qquad (A.31)$$

The indices $\dot N, \dot M, \ldots$ are curved E_6 indices while the indices $N, M, \ldots$ are tangent E_6 indices. The field $e_\mu{}^a = (e^h)_\mu{}^a$, but it will turn out that the actual vielbein is given by $e_\mu{}^a = (\det e)^{-\frac{1}{2}}e_\mu{}^a$ and we will in the main text of the paper use $d_M{}^{\dot N} = (\det e)^{-\frac{1}{2}}d_M{}^{\dot N}$

The inverse generalised veibein is given by

$$E_A{}^\Pi = (\det e)^{\frac{1}{2}}$$

$$\begin{pmatrix}
e_a{}^\mu & d_{\dot N}{}^N\alpha_{a|N} & (d^{-1})_M{}^{\dot M}e_\mu{}^b\alpha_{a|b}{}^M & e_{\mu_1\mu_2}^{b_1b_2}f_{\dot\alpha}{}^\beta\alpha_{a|b_1b_2,\beta} & e_\mu{}^e e_\nu{}^f\alpha_{a|ef} \\
0 & d_{\dot N}{}^N & (d^{-1})_M{}^{\dot N}e_\mu{}^b\beta^N{}_b{}^M & e_{\mu_1\mu_2}^{b_1b_2}f_{\dot\alpha}{}^\beta\beta^N{}_{a_1a_2,\alpha} & e_\mu{}^e e_\nu{}^f\beta^N{}_{cd} \\
0 & 0 & (d^{-1})_N{}^{\dot N}e_\mu{}^a & e_{\mu_1\mu_2}^{b_1b_2}f_{\dot\alpha}{}^\beta\gamma^a{}_{N|b_1b_2,\beta} & e_\mu{}^e e_\nu{}^f\gamma^a{}_{N|ef} \\
0 & 0 & 0 & e_{\mu_1\mu_2}^{a_1a_2}f_{\dot\alpha}{}^\alpha & 0 \\
0 & 0 & 0 & 0 & e_\mu{}^c e_\nu{}^d
\end{pmatrix}, \qquad (A.32)$$

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where in the first line

$$\alpha_{a|N} = A_{aN}, \quad \alpha_{a|b}{}^N = 2\,A_{ab}{}^N - \frac{1}{2}\,d^{NMP}\,A_{aM}\,A_{bP}, \qquad (A.33)$$

$$\alpha_{a|a_1 a_2,\,\alpha} = 3\,A_{aa_1 a_2,\,\alpha} + 2\,A_{a[a_1}{}^N A_{a_2]M}\,(D_\alpha)_N{}^M - \frac{1}{6}\,A_{aN}\,A_{[a_1 M}\,A_{a_2]P}\,d^{NMS}\,(D_\alpha)_S{}^P, \tag{A.34}$$

$$\alpha_{a|cd} = 4\,A_{d(a,\,c)} - 2\,A_{ad}{}^N A_{cN} + \frac{1}{6}\,A_{aN}\,A_{bM}\,A_{cP}\,d^{NMP}, \qquad (A.35)$$

in the second line

$$\beta^N{}_a{}^M = -\,A_{aP}\,d^{NMP}, \quad \beta^N{}_{a_1 a_2,\,\alpha} = -\,A_{a_1 a_2}{}^M\,(D_\alpha)_M{}^N + \frac{1}{2}\,A_{[a_1 M}\,A_{a_2]R}\,d^{NMP}\,(D_\alpha)_P{}^R, \tag{A.36}$$

$$\beta^N{}_{ab} = 2\,A_{ab}{}^N + \frac{1}{2}\,A_{aM}\,A_{bP}\,d^{NMP}, \qquad (A.37)$$

and in the third line

$$\gamma^a{}_{N|a_1 a_2,\,\alpha} = \delta^a_{[a_1}\,A_{a_2]M}\,(D_\alpha)_N{}^M, \quad \gamma^a{}_{N|cd} = -\,\delta^a_d\,A_{cN}. \qquad (A.38)$$

Appendix B: Calculation of the covariant derivatives

The gauge transformations of section three and the matter fields of section four both involve covariant derivatives given in equations (3.1) and (5.4) respectively. Although these formulae are very elegant to evaluate them for specific cases can be rather and in this appendix we show in detail how do this is done. The covariant derivative in these sections acts on an object, denoted here by $\bullet$, that transforms under $I_c(E_{11})$. It consists of a projector acting on a more familiar type of derivative which is of the generic form

$$D_A \bullet = E^{-1}{}_A{}^\Pi \{\partial_\Pi - \Omega_{\Pi,\underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}})\}\bullet \qquad (B.1)$$

where $\Omega_{\Pi,\underline{\alpha}}$ is the $I_C(E_{11})$ valued connection and $D(R^{\underline{\alpha}})$ is the representation of $R^{\underline{\alpha}}$ appropriate to the object on which it acts. We recall that by definition $D(1+R^{\underline{\alpha}}) = 1+D^{\underline{\alpha}}$ and so we adopt the shorthand notation $D(R^{\underline{\alpha}}) = D^{\underline{\alpha}}$ as this makes it clearer what generators are involved.

As explained in section five for the object V^A, which transforms in the l_1 representation restricted to $I_c(E_{11})$, the covariant derivative before it is project into the adjoint representation is given by

$$D_A V^B = E^{-1}{}_A{}^\Pi (\partial_\Pi V^B - V^E \Omega_{\Pi,E}{}^B) \qquad (B.2)$$

where $\Omega_{\Pi,E}{}^B = \Omega_{\Pi,\underline{\alpha}}(D^{\underline{\alpha}} - D^{-\underline{\alpha}})_E{}^B = \Omega_{\Pi,\underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}}))_E{}^B$.

Let us begin by evaluating the inverse generalised vielbein. Using equation (A.32) we find that

$$E^{-1}{}_{a\|}{}^\Pi\,\partial_\Pi = (\det e)^{\frac{1}{2}}\left[e_a{}^\mu\,\partial_\mu + A_{aN}\,d_{\dot N}{}^N\,\partial^{\dot N} + \left(2\,A_{ab}{}^N - \frac{1}{2}\,d^{NMP}\,A_{aR}\,A_{bS}\right)e_\mu{}^b\,d_M{}^{\dot M}\,\partial^\mu{}_{\dot M} + \ldots\right],$$

$$(B.3)$$

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$$E^{-1 N \| \Pi} \, \partial_\Pi = (\det e)^{\frac{1}{2}} \left[d_{\dot N}{}^N \, \partial^{\dot N} - d^{NMP} A_{aP} \, e_\mu{}^a \, d_M{}^{\dot M} \, \partial^\mu{}_{\dot M} + \dots \right], \qquad (B.4)$$

and

$$E^{-1 a}{}_N{}^\Pi \, \partial_\Pi = (\det e)^{\frac{1}{2}} \, d_N{}^{\dot N} \, e_\mu{}^a \, \partial^\mu{}_{\dot N} + \dots, \qquad (B.5)$$

The evaluation of the inverse generalised vielbein in front of the term in the covariant derivative which contains the connection takes an analogous form, but with the partial derivative exchanged for the connection.

We now consider the evaluation of the connection which has the form

$$\Omega_{\Pi, E}{}^B = \Omega_{\Pi, \underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}}))_E{}^B = \Omega_{\Pi, \underline{\alpha}}(D^{\underline{\alpha}} - D^{-\underline{\alpha}})_E{}^B$$

$$= \Omega_{\Pi, e}{}^f (D(K^e{}_f) - D(K^f{}_e))_E{}^B + \Omega_{\Pi, \alpha}(D(R^\alpha) - D(R^{-\alpha}))_E{}^B$$

$$+ \Omega_{\Pi, cP}(D(R^{cP}) - D(R_{cP}))_E{}^B + \Omega_{\Pi, e_1 e_2}{}^P (D(R^{e_1 e_2}{}_P) - D(R_{e_1 e_2}{}^P))_E{}^B + \dots \qquad (B.6)$$

Contracting with V^E and evaluating using the components of the V^A multiplet of equation (5.3) we find that

$$V^E \Omega_{\Pi, E}{}^B = V^a \Omega_{\Pi, a \|}{}^B + V_M \Omega_{\Pi,}{}^M{}_\|{}^B + V_c{}^M \Omega_{\Pi, M \|}{}^c{}^B + \dots$$

$$= V^a \Omega_{\Pi, \underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}}))_{a \|}{}^B + V_M \Omega_{\Pi, \underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}}))^M{}_\|{}^B$$

$$+ V_c{}^M \Omega_{\Pi, \underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}}))^c{}_{M \|}{}^B + V_{c_1 c_2 \alpha} \Omega_{\Pi, \underline{\beta}}(D(R^{\underline{\beta}}) - D(R^{-\underline{\beta}}))^{c_1 c_2 \alpha}{}_\|{}^B$$

$$+ V_{c_1 c_2} \Omega_{\Pi, \underline{\alpha}}(D(R^{\underline{\alpha}}) - D(R^{-\underline{\alpha}}))^{c_1 c_2}{}_\|{}^B + \dots \qquad (B.7)$$

When B takes the value b to find that equation (B.7) is given by

$$V^E \Omega_{\Pi, E}{}^b = V^c \Omega_{\Pi, e}{}^f (D(K^e{}_f) - D(K^f{}_e))_{c \|}{}^b + V_M \Omega_{\Pi, cP}(D(R^{cP}) - D(R_{cP}))^{M \| b}$$

$$+ V_{f \|}{}^M \Omega_{\Pi, e_1 e_2}{}^P (D(R^{e_1 e_2}{}_P) - D(R_{e_1 e_2}{}^P))^f{}_{M \|}{}^b + \dots \qquad (B.8)$$

and as a result we find, using the algebra of appendix A, that

$$-V^E \Omega_{\Pi, E}{}^b = -2V^c \Omega_{\Pi, c}{}^b - V_P \Omega_{\Pi,}{}^{bP} - 20 V_e{}^M \Omega_{\Pi,}{}^{eb}{}_M + \dots \qquad (B.9)$$

When B takes the value N we find that equation (B.7) takes the form

$$-V^E \Omega_{\Pi, E \| N} = V^c \Omega_{\Pi, cN} + V_M \Omega_{\Pi, \alpha}(D^\alpha - D^{-\alpha})_N{}^M + 10 d_{PMN} V_e{}^M \Omega_{\Pi,}{}^{eP}$$

$$-120 V_{c_1 c_2 \beta} \Omega_{\Pi,}{}^{c_1 c_2}{}_P (D^\beta)_N{}^P + \frac{20}{3} V_{c_1 c_2} \Omega_{\Pi,}{}^{c_1 c_2}{}_N + \dots \qquad (B.10)$$

While when B takes the value $_a{}^N$ we find that equation (B.7) takes the form

$$-V^E \Omega_{\Pi, E \| a}{}^N = 2V^c \Omega_{\Pi, ca}{}^N - V_M \Omega_{\Pi, aP} d^{PMN} - V_a{}^M \Omega_{\Pi, \alpha}(D^\alpha - D^{-\alpha})_M{}^N$$

$$+ 2V_d{}^N \Omega_{\Pi, a}{}^d + \dots \qquad (B.11)$$

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The covariant derivative of V^A of equation (5.4) and the gauge transformation of equation (3.1) has a projector onto the adjoint representation of $I_C(E_{11})$. We will now show how to evaluate this projector. We called equation (5.4) for the covariant derivative of the vector V^A;

$$\mathcal{D}_a V^b = (D^{\underline{\alpha}})_a{}^b \left(D_{\underline{\alpha}}\right)_C{}^D D_D V^C = (D^e{}_f)_a{}^b (D_e{}^f)_C{}^D D_D V^C. \quad (B.12)$$

The first step is to evaluate the projector which using the Cartan-Killing form of equation (A.18), we find that

$$(D^e{}_f)_A{}^B (g^{-1})_f{}^{e,m}{}_n (D^n{}_m)_C{}^D = (D^e{}_f)_A{}^B (D^f{}_e)_C{}^D - \frac{1}{3} (D^e{}_e)_A{}^B (D^f{}_f)_C{}^D. \quad (B.13)$$

Substituting this into equation (B.12) we find that it becomes

$$\mathcal{D}_a V^b = (D^e{}_f)_a{}^b (D^f{}_e)_C{}^D D_D V^C - \frac{1}{3} (D^e{}_e)_a{}^b (D^f{}_f)_C{}^D D_D V^C = (D^b{}_a)_C{}^D D_D V^C$$

$$= D_a V^b - D^b{}_N V_a{}^N - \frac{1}{2} \delta_b^a \left(D_c V^c + D^N V_N + D^c{}_N V_c{}^N + \dots\right). \quad (B.14)$$

Finally, we consider the covariant derivative of the V_N component of V^A which is equal to

$$\mathcal{D}_a V_N = (D^{\underline{\alpha}})_{a\|N} \left(D_{\underline{\alpha}}\right)_C{}^D D_D V^C = \left(D^{cP}\right)_{a\|N} (D_{cP})_C{}^D D_D V^C \quad (B.15)$$

since the only value of $\underline{\alpha}$ for which $(D^{\underline{\alpha}})_{aN}$ is non-zero is $\underline{\alpha} = bM$. Using that $\left(D^{bM}\right)_{aM} = -\delta_a^b \delta_N^M$ and the Cartan-Killing metric of equation (A.) we find

$$\mathcal{D}_a V_N = - (D_{aN})_C{}^D D_D V^C$$

$$= D_a V_N - 10 \, d_{NMP} \, D^M V_a{}^P - 12 \, (D^\alpha)_N{}^M D^b{}_M V_{ab,\,\alpha} - \frac{2}{3} D^b{}_N V_{ab} - \frac{1}{3} D^b{}_N V_{ba}. \quad (B.16)$$

Applying the same procedures to the covariant derivative of the $V_a{}^N$ we find that

$$\mathcal{D}_a V_b{}^N = (D^{\underline{\alpha}})_{a,\,b}{}^N \left(D_{\underline{\alpha}}\right)_C{}^D D_D V^C = -\frac{1}{20} (D^{a_1 a_2}{}_M)_{a,\,b}{}^N \left(D_{a_1 a_2}{}^M\right)_C{}^D D_D V^C,$$

$$= \frac{1}{10} (D_{ab}{}^N)_C{}^D D_D V^C \quad (B.17)$$

where ${}^{a_1 a_2}{}_M$ is the only value of $\underline{\alpha}$ that gives a non-zero contribution and $-\frac{1}{20}$ factor comes from the Cartan-Killing metric. This in turn is equal to

$$\mathcal{D}_a V_b{}^N = +2 \, D_{[a} V_{b]}{}^N - 12 \, (D^\alpha)_M{}^N D^M V_{ab,\,\alpha} + \frac{2}{3} D^N V_{[ab]}. \quad (B.18)$$

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