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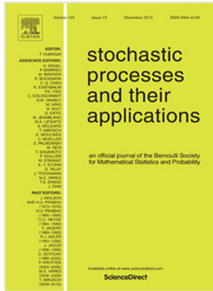

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FLUCTUATIONS OF THE TOTAL NUMBER OF CRITICAL POINTS OF RANDOM SPHERICAL HARMONICS

V. CAMMAROTA AND I. WIGMAN

Abstract. We determine the asymptotic law for the fluctuations of the total number of critical points of random Gaussian spherical harmonics in the high degree limit. Our results have implications on the sophistication degree of an appropriate percolation process for modelling nodal domains of eigenfunctions on generic compact surfaces or billiards.

1. Introduction and main results

1.1. **Critical points of random spherical harmonics.** It is well-known that the eigenvalues λ of the Laplacian Δ on the 2-dimensional round unit sphere $\mathcal{S}^2$, satisfying the Schrödinger equation

$$\Delta f + \lambda f = 0$$

are of the form $\lambda = \lambda_\ell = \ell(\ell+1)$ for some integer $\ell \geq 1$. For any given eigenvalue λ_ℓ of the above form, the corresponding eigenspace is the $(2\ell+1)$-dimensional space of spherical harmonics of degree ℓ; we can choose an arbitrary L^2-orthonormal basis $\{Y_{\ell m}(.)\}_{-\ell \leq m \leq \ell}$, and consider random eigenfunctions of the form

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \tag{1.1}$$

where the coefficients $\{a_{\ell m}\}_{-\ell \leq m \leq \ell}$ are independent, standard Gaussian variables. The random fields

$$\{f_\ell(x),\ x \in \mathcal{S}^2\}$$

are centred Gaussian and the law of f_ℓ in (1.1) is invariant with respect to the choice of $\{Y_{\ell m}\}$. Also, f_ℓ are isotropic, meaning that the probability laws of $f_\ell(\cdot)$ and $f_\ell^g(\cdot) := f_\ell(g\cdot)$ are the same for every rotation $g \in SO(3)$. Here we choose the commonly adopted basis of real valued spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \begin{cases} \sqrt{2}\,\mathcal{K}_\ell^m \cos(m\varphi) P_\ell^m(\cos\theta), & \text{if } \ m > 0, \\ \mathcal{K}_\ell^0 P_\ell^0(\cos\theta), & \text{if } \ m = 0, \\ \sqrt{2}\,\mathcal{K}_\ell^m \sin(-m\varphi) P_\ell^{-m}(\cos\theta), & \text{if } \ m < 0; \end{cases}$$

where P_ℓ^m are the associated Legendre functions and

$$\mathcal{K}_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}.$$

By the addition theorem for spherical harmonics [3, Theorem 9.6.3] the covariance function of f_ℓ is given by

$$\mathbb{E}[f_\ell(x) f_\ell(y)] = P_\ell(\cos d(x,y)),$$

where P_ℓ are the usual Legendre polynomials,

$$\cos d(x,y) = \cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y)$$

is the spherical geodesic distance between x and y, $\theta \in [0,\pi]$, $\varphi \in [0, 2\pi)$ are standard spherical coordinates and (θ_x, φ_x), (θ_y, φ_y) are the spherical coordinates of x and y respectively.

Our primary focus is the total number of critical points of f_ℓ

$$\mathcal{N}^c(f_\ell) = \#\{x \in \mathcal{S}^2 : \nabla f_\ell(x) = 0\}.$$

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It is known [19, 11] that, as $\ell \to \infty$, the expected total number of critical points $\mathcal{N}^c(f_\ell)$ is asymptotic to

$$\mathbb{E}[\mathcal{N}^c(f_\ell)] = \frac{2}{\sqrt{3}}\ell^2 + O(1).$$

An upper bound for the variance of the number of critical points $\mathcal{N}^c(f_\ell)$ was also derived [11]:

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = O(\ell^{\frac{5}{2}});$$

in fact, it is likely that the same method yields the stronger result

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = O(\ell^2 \log \ell).$$

It was conjectured [11] that the true asymptotic behaviour of the variance is

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = \mathrm{const} \cdot \ell^2 \log \ell + O(\ell^2). \tag{1.2}$$

More generally let $I \subseteq \mathbb{R}$ be any interval and $\mathcal{N}_I^c(f_\ell)$ be the number of critical points of f_ℓ with value in I:

$$\mathcal{N}_I^c(f_\ell) = \#\{x \in \mathcal{S}^2 : f_\ell(x) \in I, \nabla f_\ell(x) = 0\};$$

it was proved in [11, Theorem 1.2] that as $\ell \to \infty$ it holds that

$$\mathrm{Var}(\mathcal{N}_I^c(f_\ell)) = \ell^3 \nu^c(I) + O(\ell^{5/2}),$$

where the leading constant $\nu^c(I)$ was evaluated explicitly. For some intervals I, such as, for example $I = \mathbb{R}$ (corresponding to the total number of critical points), the leading constant $\nu^c(I)$ vanishes, and, accordingly, the order of magnitude of the variance is smaller than ℓ^3. In this paper we prove (1.2), i.e. we determine the precise asymptotic shape for the variance of the total number of critical points of f_ℓ.

1.2. Statement of the main result.

The principal result of this paper is the following:

Theorem 1.1. *As* $\ell \to \infty$

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = \frac{1}{3^3\pi^2}\ell^2 \log \ell + O(\ell^2).$$

The constant in the $O(\cdot)$ *term is universal.*

As in [11], our argument is based on an approximate version of the Kac-Rice formula for counting the number of zeros of the gradient of f_ℓ (see Section 2). It is easy to adapt the same approach to separate critical points into extrema and saddles; in fact, we have the following:

Remark 1.2. Let $\mathcal{N}^e(f_\ell)$ and $\mathcal{N}^s(f_\ell)$ be the total number of extrema and saddles of f_ℓ

$$\mathcal{N}^e(f_\ell) = \#\{x \in \mathcal{S}^2 : \nabla f_\ell(x) = 0, \det(\nabla^2 f_\ell(x)) > 0\},$$
$$\mathcal{N}^s(f_\ell) = \#\{x \in \mathcal{S}^2 : \nabla f_\ell(x) = 0, \det(\nabla^2 f_\ell(x)) < 0\}.$$

As $\ell \to \infty$ we have that

$$\mathrm{Var}(\mathcal{N}^e(f_\ell)) = \frac{1}{2^2 \cdot 3^3\pi^2}\ell^2 \log \ell + O(\ell^2), \tag{1.3}$$

$$\mathrm{Var}(\mathcal{N}^s(f_\ell)) = \frac{1}{2^2 \cdot 3^3\pi^2}\ell^2 \log \ell + O(\ell^2). \tag{1.4}$$

The asymptotic laws for the fluctuations of the total number of extrema and saddles in (1.3) and (1.4) follow immediately from Theorem 1.1 and Morse Theory. In fact,

$$\mathcal{N}^c(f_\ell) = \mathcal{N}^e(f_\ell) + \mathcal{N}^s(f_\ell)$$

and, via Morse Theory, it is possible to prove that

$$\mathcal{N}^e(f_\ell) = \frac{\mathcal{N}^c(f_\ell)}{2} + 1.$$

Remark 1.3. For the intervals $I \neq \mathbb{R}$ such that the constant $\nu^c(I)$, $\nu^e(I)$ or $\nu^s(I)$ vanish, the variance of the number of critical points, extrema and saddles in I has the following asymptotic behaviour: as $\ell \to \infty$

$$\mathrm{Var}(\mathcal{N}_I^a(f_\ell)) = [\mu^a(I)]^2 \ell^2 \log \ell + O(\ell^2), \tag{1.5}$$

where we use $a = c, e, s$ to denote critical points extrema and saddles, $\mu^a(I) = \int_I \mu^a(t)dt$, and the functions μ^a, for $a = c, e, s$ are defined in (B.3)-(B.5) and derived in Appendix B.

1.3. **Nodal domains and percolation.** The nodal domains of f_ℓ are the connected components of the complement of the nodal lines $f_\ell^{-1}(0)$, i.e. the connected components of

$$\mathcal{S}^2 \setminus f_\ell^{-1}(0).$$

Let $N(f_\ell)$ be the number of nodal domains of f_ℓ. Nazarov and Sodin [18] proved that there exists a constant $a > 0$ such that the expected number of nodal domains is asymptotic to

$$\mathbb{E}[N(f_\ell)] \sim a\ell^2. \tag{1.6}$$

Little is known about the leading constant a in (1.6). For once the nodal domains number is bounded from above by the total number of critical points; the latter inequality could be improved by a factor of 2 by separating the critical points into extrema and saddles (for example, via Morse Theory), an approach pursued by Nicolaescu [19] yielding the upper bound

$$a \leq \frac{1}{\sqrt{3}};$$

while it is possible to improve the latter bound by using other local estimates (e.g. [14]), these are far off the numerical Monte-Carlo simulations or the conjectured values of a.

To the other end, other than the Nastasescu's [17] explicating the Nazarov-Sodin "barrier" construction [18] (yielding a tiny lower bound on a), to our best knowledge, no lower bound for a is known rigorously. Bogomolny and Schmit [10] conjectured that, as $\ell \to \infty$, nodal domains of f_ℓ (more generally, deterministic Laplace eigenfunctions on generic compact surfaces or billiards) are described by the clusters in a rectangular lattice bond percolation-like process with $\approx \ell^2$ sites (called the Percolation Model), and in particular that the true value of a equals the leading constant

$$a = \frac{3\sqrt{3}-5}{\pi} \approx 0.0624$$

for the asymptotic number of connected clusters in the Percolation Model. Here we think of the maxima and minima of f_ℓ as rigidly arranged along two mutually dual percolation lattices; adjacent maxima are connected independently with probability $\frac{1}{2}$, if and only if the dual minima are disconnected.

Some recent simulations [17, 6] showed deviations of about 4.5% between the predicted constant for a and its numerical values; these cannot be attributed to numerical errors. It is then desirable to come up with a more sophisticated percolation model[1] [6, 7] that would match these constant more precisely, where, in particular, the arrangement critical points of f_ℓ would exhibit some degree of randomness, less rigid than rectangular lattice. The variance (1.5) of the total number of critical points (or the extrema) of f_ℓ is then crucial in determining the rigidity or flexibility of the (random) percolation sites.

1.4. **Cosmological applications.** It is well-known that random spherical harmonics are the Fourier components of square integrable isotropic fields on the sphere [16], i.e., for every centred Gaussian spherical random field f the following spectral representation holds:

$$f(x) = \sum_{\ell=1}^{\infty} f_\ell(x) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{C_\ell}\, a_{\ell m}\, Y_{\ell m}(x), \qquad x \in \mathcal{S}^2,$$

where equality holds in the L^2 sense and the sequence $\{C_\ell\}_{\ell=1,2,\dots}$ denotes the so-called angular power spectrum, which fully characterizes the dependence structure of f. The analysis of spherical random fields is at the heart of observational cosmology, for instance for experiments handling Cosmic Microwave Background radiation data [1, 9]: indeed CMB observations can be viewed as a realization $\hat{f}$ of an isotropic Gaussian random function f on the sphere. A natural question is whether these observed CMB maps are indeed consistent with the assumptions of Gaussianity and isotropy, in fact, departures from these assumptions, could signal physically motivated deviations from standard cosmological models.

Our results can be exploited in this setting by means of the implementation of Gaussianity and isotropy tests, for instance, by comparing the actual number of maxima for an observed component $\hat{f}_\ell$ with the expected values which we reported in the previous sections. It is natural to expect the convergence of the standardized number of maxima to a standard Gaussian limit, in the high-energy regime, under the null assumption that f_ℓ is a pure Gaussian field. In practice, it is not always easy to derive the Fourier components f_ℓ from realized maps; due to missing observations in some regions of the sky. However, very recently, a number of statistical

[1] We would like to thank Dmitry Belyaev for discussing connections between our work and percolation.

techniques have been proposed [8] to reconstruct these missing observations, so that, in principle, our results are applicable to the resulting maps.

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2. On (approximate) Kac-Rice formula for computing 2nd (factorial) moment

In this section we express the second factorial moment of $\mathcal{N}^c(f_\ell)$ via Kac-Rice formula. Let $\mathcal{E} \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$, and $g : \mathcal{E} \to \mathbb{R}^n$ a centred Gaussian random field such that all components of f, ∇f and $\nabla^2 f$ are a.s. continuous. Define the 2-point correlation function

$$K_2 = K_{2;g} : \mathcal{E}^2 \to \mathbb{R}$$

of the zeros of g as

$$K_2(x, y) = \phi_{(g(x),g(y))}(\mathbf{0}, \mathbf{0}) \cdot \mathbb{E}[|\det J_g(x)| \cdot |\det J_g(y)| \big| g(x) = g(y) = \mathbf{0}], \tag{2.1}$$

where $\phi_{(g(x),g(y))}$ is the Gaussian probability density of $(g(x), g(y)) \in \mathbb{R}^{2n}$ and $J_g(x)$, $J_g(y)$ are the Jacobian matrices of g at x and y respectively. In view of [4, Theorem 6.3] (see also [4, Proposition 1.2]) the 2nd factorial moment of $g^{-1}(0)$ is given by

$$\mathbb{E}[\#g^{-1}(0) \cdot (\#g^{-1}(0) - 1)] = \int_{\mathcal{E}^2} K_2(x, y)dxdy, \tag{2.2}$$

provided that the Gaussian distribution of $(g(x), g(y)) \in \mathbb{R}^{2n}$ is non-degenerate for all $(x, y) \in \mathcal{E}^2$. Moreover, for $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{E}$ two open disjoint domains, we have

$$\mathbb{E}[(\#g^{-1}(0) \cap \mathcal{D}_1) \cdot (\#g^{-1}(0) \cap \mathcal{D}_2)] = \iint_{\mathcal{D}_1 \times \mathcal{D}_2} K_2(x, y)dxdy, \tag{2.3}$$

under the same non-degeneracy assumption for all $(x, y) \in \mathcal{D}_1 \times \mathcal{D}_2$. To apply Kac-Rice formulas (2.3) and (2.2) in our case we will work with spherical coordinates on $\mathcal{S}^2$ and choose an explicit orthogonal frame, see (3.2) below. Counting the critical points of f_ℓ is then equivalent to counting the zeros of the map $[0, \pi] \times [0, 2\pi] \to \mathbb{R}^2$ given by $x \to \nabla f_\ell(x)$; accordingly for $x \neq \pm y$ the two-point correlation function of critical points of f_ℓ is (cf. (2.1))

$$K_{2,\ell}(x, y) = \phi_{(\nabla f_\ell(x),\nabla f_\ell(y))}(\mathbf{0}, \mathbf{0}) \cdot \mathbb{E}[|\det H_{f_\ell}(x)| \cdot |\det H_{f_\ell}(y)| \big| \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}], \tag{2.4}$$

where $H_{f_\ell}(x)$ and $H_{f_\ell}(y)$ are the Hessian matrices of f_ℓ at x and y respectively. Here [4, Theorem 6.3] (see also [2, Theorem 11.2.1]) would yield

$$\mathbb{E}[\mathcal{N}^c(f_\ell) \cdot (\mathcal{N}^c(f_\ell) - 1)] = \iint_{\mathcal{S}^2 \times \mathcal{S}^2} K_{2,\ell}(x, y)dxdy, \tag{2.5}$$

under the condition that for all $x, y \in \mathcal{S}^2$, the Gaussian distribution of $(\nabla f(x), \nabla f(y)) \in \mathbb{R}^4$ were non-degenerate. We can easily adapt the definition of the 2-point correlation in (2.4) to separate the critical points into extrema and saddles, or count critical points with values lying in I (see Appendix B.1 and [11]).

Note that the rotational invariance of f_ℓ implies that the function $K_{2,\ell}$ in (2.4) depends on the points x, y only via their geodesic distance $\phi = d(x, y)$; with a slight abuse of notations we write $K_{2,\ell}(\phi) = K_{2,\ell}(x, y)$. Also, note that $K_{2,\ell}(\phi)$ is everywhere nonnegative.

We do not validate the non-degeneracy assumption of the 4×4 covariance matrices of $(\nabla f(x), \nabla f(y))$ depending on both x and y (and ℓ); instead we prove that the precise Kac-Rice formula (2.5) holds up to an admissible error, an approach inspired by [20]. We recall here the main steps of the proof of the *approximate Kac-Rice* formula and refer to [11, Section 3] for a complete proof. The argument is based on a partitioning of the integration domain in (2.5); we apply (2.3) on the valid slices we bound the contribution of the rest.

For $x \in \mathcal{S}^2$, $r > 0$ let $\mathcal{B}(x, r) = \{y \subseteq \mathcal{S}^2 : d(x, y) \leq r\}$ be a closed spherical cap on $\mathcal{S}^2$. For $\varepsilon > 0$ we say that

$$\Xi_\varepsilon = \{\xi_{1,\varepsilon}, \ldots, \xi_{N,\varepsilon}\} \subseteq \mathcal{S}^2$$

is a maximal ε-net if for every $i \neq j$ we have $d(\xi_{i,\varepsilon}, \xi_{j,\varepsilon}) > \varepsilon$, and also every $x \in \mathcal{S}^2$ satisfies

$$d(x, \Xi_\varepsilon) \leq \varepsilon.$$

That is, informally speaking an ε-net is a collection of ε-separated points, whose ε-thickening covers the whole of $\mathcal{S}^2$. The number N of points in a ε-net on the sphere can be bounded from above and from below; indeed it satisfies the following [5, Lemma 5]:

$$\frac{4}{\varepsilon^2} \leq N \leq \frac{4}{\varepsilon^2}\pi^2. \tag{2.6}$$

Given a maximal ε-net, it is natural to partition the sphere into disjoint sets, each of them associated with a single point in the net. This task is accomplished by the Voronoi cells construction [16, Section 11.2]:

Definition 2.1. Let Ξ_ε be a maximal ε-net. For all $\xi_{i,\varepsilon} \in \Xi_\varepsilon$, the associated *family of Voronoi cells* is defined by

$$\mathcal{V}(\xi_{i,\varepsilon}, \varepsilon) = \{x \in \mathcal{S}^2 : \forall j \neq i,\ d(x, \xi_{i,\varepsilon}) \leq d(x, \xi_{j,\varepsilon})\}.$$

Each Voronoi cell is associated to a single point on the net. The Voronoi cells are disjoint, save to boundary overlaps, and cover the whole sphere.

It is possible to prove the following:

Proposition 2.2. *There exists a constant $C > 0$ sufficiently big, such that the following approximate Kac-Rice holds:*

$$\mathrm{Var}\left(\mathcal{N}^c(f_\ell)\right) = \int_{\mathcal{W}} K_{2,\ell}(x, y)\, dxdy - (\mathbb{E}[\mathcal{N}^c(f_\ell)])^2 + O(\ell^2), \tag{2.7}$$

where $\mathcal{W}$ is the union of all tuples of points belonging to Voronoi cells far from the domain of degeneracy, i.e.,

$$\mathcal{W} = \bigcup_{d(\mathcal{V}(\xi_{i,\varepsilon}),\mathcal{V}(\xi_{j,\varepsilon}))\in(C/\ell,\pi-C/\ell)} \mathcal{V}(\xi_{i,\varepsilon}) \times \mathcal{V}(\xi_{j,\varepsilon}).$$

2.1. **On the proof of Proposition 2.2.** Note that, almost surely, the summation of the critical points over the Voronoi cells equals the total number of critical points, therefore we write the variance of the total number of critical points as

$$\mathrm{Var}\left(\mathcal{N}^c(f_\ell)\right) = \sum_{\xi_{i,\varepsilon},\xi_{j,\varepsilon}\in\Xi_\varepsilon} \mathrm{Cov}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon})), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j,\varepsilon}))\right), \tag{2.8}$$

where

$$\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon)) = \#\{x \in \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon) : \nabla f_\ell(x) = 0\}.$$

The main steps of the proof of Proposition 2.2 are the following. In [11, Lemma 3.2] it was proved that there exists a constant $C > 0$ sufficiently big, such that, in the regime $d(\mathcal{V}(\xi_{i,\varepsilon}), \mathcal{V}(\xi_{j,\varepsilon})) \in (C/\ell, \pi - C/\ell)$, the covariance matrix is nonsingular and so Kac-Rice formula holds exactly. This gives the first term in (2.7).

In the regime $d(\mathcal{V}(\xi_{i,\varepsilon}), \mathcal{V}(\xi_{j,\varepsilon})) \in [0, C/\ell] \cup [\pi - C/\ell, \pi]$, using Cauchy-Schwartz inequality, we can bound the covariance as

$$\left|\mathrm{Cov}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon})), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j,\varepsilon}))\right)\right| \leq \sqrt{\mathrm{Var}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon}))\right)} \cdot \sqrt{\mathrm{Var}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j,\varepsilon}))\right)}. \tag{2.9}$$

In [11, Section 4.2] the non-degeneracy of the covariance matrix was proved for sufficiently close points x, y, i.e., it was proved that there exists a constant $c > 0$ sufficiently small such that for $\varepsilon = c/\ell$ the Kac-Rice formula holds precisely:

$$\mathrm{Var}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon,i}))\right) = \iint_{\mathcal{V}(\xi_{\varepsilon,i})\times\mathcal{V}(\xi_{\varepsilon,i})} K_{2,\ell}(x, y) dxdy + \mathbb{E}\left[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon,i}))\right] - \left(\mathbb{E}\left[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon,i}))\right]\right)^2. \tag{2.10}$$

Now, in view of [11, Lemma 3.6], there exists a constant $c > 0$ such that, for $d(x, y) < c/\ell$, one has

$$K_{2,\ell}(x, y) = O(\ell^4),$$

and since $\mathcal{B}(\xi_{i,\varepsilon}, \varepsilon/2) \subseteq \mathcal{V}(\xi_{i,\varepsilon}, \varepsilon) \subseteq \mathcal{B}(\xi_{i,\varepsilon}, \varepsilon)$, we have $\mathrm{Vol}(\mathcal{V}(\xi_{i,\varepsilon}, \varepsilon)) \approx \varepsilon^2$. Then the first term in (2.10) is bounded by

$$\iint_{\mathcal{V}(\xi_{\varepsilon,i})\times\mathcal{V}(\xi_{\varepsilon,i})} K_{2,\ell}(x, y)dxdy \le \ell^4 \cdot (\pi\varepsilon^2)^2 = O(1),$$

moreover, by [11, Proposition 1.1], for the expectation in (2.10) we have

$$\mathbb{E}\left[\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{\varepsilon,i}))\right] \le \mathbb{E}\left[\mathcal{N}^c(f_\ell; \mathcal{B}(\xi_{\varepsilon,i}))\right] \le \pi\varepsilon^2\ell^2 = O(1).$$

Then, using (2.9), we can bound the covariance as

$$|\mathrm{Cov}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon})), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j,\varepsilon}))\right)| = O(1),$$

and since by (2.6) there are $O(\ell^2)$ pairs of Voronoi cells at distance $d(x, y) \in [0, C/\ell] \cup [\pi - C/\ell, \pi]$, we finally obtain

$$\sum_{d(\mathcal{V}(\xi_{i,\varepsilon}),\mathcal{V}(\xi_{j,\varepsilon}))\in[0,C/\ell]\cup[\pi-C/\ell,\pi]} |\mathrm{Cov}\left(\mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{i,\varepsilon})), \mathcal{N}^c(f_\ell; \mathcal{V}(\xi_{j,\varepsilon}))\right)| = O(\ell^2).$$

3. Proof of Theorem 1.1

3.1. **Kac-Rice formula in coordinate system.** To study the asymptotic behaviour of the two-point correlation function we write a more explicit frame-dependent formula by using the orthogonal frames (3.2) so that, by the isotropic property of f_ℓ, $K_{2,\ell}$ depends only on the geodesic distance $\phi = d(x, y)$.

For $x, y \in \mathcal{S}^2$ we define the following random vector

$$Z_{\ell;x,y} = (\nabla f_\ell(x), \nabla f_\ell(y), \nabla^2 f_\ell(x), \nabla^2 f_\ell(y)).$$

To write the Kac-Rice formula in coordinate system, given x, $y \in \mathcal{S}^2$, we consider two local orthogonal frames $\{e^x_1, e^x_2\}$ and $\{e^y_1, e^y_2\}$ defined in some neighbourhood of x and y respectively. This gives rise to the (local) identifications

$$T_x(\mathcal{S}^2) \cong \mathbb{R}^2 \cong T_y(\mathcal{S}^2), \tag{3.1}$$

so that we do not have to work with probability densities defined on tangent planes which depend on the points x and y respectively. Under the identification (3.1) the random vector $Z_{\ell;x,y}$ is a $\mathbb{R}^{10}$ centred Gaussian random vector.

By the isotropic property of f_ℓ it is convenient to perform our computations along a specific geodesic. In particular, we focus on the equatorial line $x = (\pi/2, \phi)$, $y = (\pi/2, 0)$ and we work with the orthogonal frames

$$\left\{e^x_1 = \frac{\partial}{\partial\theta_x},\ e^x_2 = \frac{\partial}{\partial\varphi_x}\right\}, \qquad \left\{e^y_1 = \frac{\partial}{\partial\theta_y},\ e^y_2 = \frac{\partial}{\partial\varphi_y}\right\}. \tag{3.2}$$

Let $\Delta_\ell(\phi)$ be the conditional covariance matrix of the scaled Gaussian vector

$$\frac{\sqrt{8}}{\lambda_\ell}(\nabla^2 f_\ell(x), \nabla^2 f_\ell(y) \big| \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}).$$

With the choice (3.2) the covariance matrix $\Delta_\ell(\phi)$ is of the following form

$$\Delta_\ell(\phi) = \begin{pmatrix} \Delta_{1,\ell}(\phi) & \Delta_{2,\ell}(\phi) \\ \Delta_{2,\ell}(\phi) & \Delta_{1,\ell}(\phi) \end{pmatrix},$$

where

$$\Delta_{1,\ell}(\phi) = \begin{pmatrix} 3 - \frac{16\beta^2_{2,\ell}(\phi)}{\lambda_\ell(\lambda^2_\ell - 4\alpha^2_{2,\ell}(\phi))} - \frac{2}{\lambda_\ell} & 0 & 1 - \frac{16\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{\lambda_\ell(\lambda^2_\ell - 4\alpha^2_{2,\ell}(\phi))} + \frac{2}{\lambda_\ell} \\ 0 & 1 - \frac{16\beta^2_{1,\ell}(\phi)}{\lambda_\ell(\lambda^2_\ell - 4\alpha^2_{1,\ell}(\phi))} & 0 \\ 1 - \frac{16\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{\lambda_\ell(\lambda^2_\ell - 4\alpha^2_{2,\ell}(\phi))} + \frac{2}{\lambda_\ell} & 0 & 3 - \frac{16\beta^2_{3,\ell}(\phi)}{\lambda_\ell(\lambda^2_\ell - 4\alpha^2_{2,\ell}(\phi))} - \frac{2}{\lambda_\ell} \end{pmatrix}, \tag{3.3}$$

$$\Delta_{2,\ell}(\phi) = \begin{pmatrix} 8\frac{\gamma_{1,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta^2_{2,\ell}(\phi)}{4\alpha^2_{2,\ell}(\phi) - \lambda^2_\ell}}{\lambda^2_\ell} & 0 & 8\frac{\gamma_{3,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{4\alpha^2_{2,\ell}(\phi) - \lambda^2_\ell}}{\lambda^2_\ell} \\ 0 & 8\frac{\gamma_{2,\ell}(\phi) + \frac{4\alpha_{1,\ell}(\phi)\beta^2_{1,\ell}(\phi)}{4\alpha^2_{1,\ell}(\phi) - \lambda^2_\ell}}{\lambda^2_\ell} & 0 \\ 8\frac{\gamma_{3,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{4\alpha^2_{2,\ell}(\phi) - \lambda^2_\ell}}{\lambda^2_\ell} & 0 & 8\frac{\gamma_{4,\ell}(\phi) + \frac{4\alpha_{2,\ell}(\phi)\beta^2_{3,\ell}(\phi)}{4\alpha^2_{2,\ell}(\phi) - \lambda^2_\ell}}{\lambda^2_\ell} \end{pmatrix}, \tag{3.4}$$

with

$$\alpha_{1,\ell}(\phi) = P'_\ell(\cos\phi), \quad \alpha_{2,\ell}(\phi) = -\sin^2\phi P''_\ell(\cos\phi) + \cos\phi P'_\ell(\cos\phi),$$

$$\beta_{1,\ell}(\phi) = \sin\phi P''_\ell(\cos\phi), \quad \beta_{2,\ell}(\phi) = \sin\phi\cos\phi P''_\ell(\cos\phi) + \sin\phi P'_\ell(\cos\phi),$$
$$\beta_{3,\ell}(\phi) = -\sin^3\phi P'''_\ell(\cos\phi) + 3\sin\phi\cos\phi P''_\ell(\cos\phi) + \sin\phi P'_\ell(\cos\phi),$$

$$\gamma_{1,\ell}(\phi) = (2+\cos^2\phi)P''_\ell(\cos\phi) + \cos\phi P'_\ell(\cos\phi), \quad \gamma_{2,\ell}(\phi) = -\sin^2\phi P'''_\ell(\cos\phi) + \cos\phi P''_\ell(\cos\phi),$$
$$\gamma_{3,\ell}(\phi) = -\sin^2\phi\cos\phi P'''_\ell(\cos\phi) + (-2\sin^2\phi + \cos^2\phi)P''_\ell(\cos\phi) + \cos\phi P'_\ell(\cos\phi),$$
$$\gamma_{4,\ell}(\phi) = \sin^4\phi P''''_\ell(\cos\phi) - 6\sin^2\phi\cos\phi P'''_\ell(\cos\phi) + (-4\sin^2\phi + 3\cos^2\phi)P''_\ell(\cos\phi) + \cos\phi P'_\ell(\cos\phi).$$

For a proof of (3.3) and (3.4) we refer to [11, Appendix A and Appendix B]. We also introduce the vector $\mathbf{a}$ that collects the perturbing elements of the covariance matrix $\Delta_\ell(\phi)$:

$$\mathbf{a} = \mathbf{a}_\ell(\phi) = (a_{1,\ell}(\phi), a_{2,\ell}(\phi), a_{3,\ell}(\phi), a_{4,\ell}(\phi), a_{5,\ell}(\phi), a_{6,\ell}(\phi), a_{7,\ell}(\phi), a_{8,\ell}(\phi))$$

with $a_{i,\ell}(\phi)$, $i = 1,\dots,8$, defined by

$$\Delta_{1,\ell}(\phi) = \begin{pmatrix} 3 + a_{1,\ell}(\phi) & 0 & 1 + a_{4,\ell}(\phi) \\ 0 & 1 + a_{2,\ell}(\phi) & 0 \\ 1 + a_{4,\ell}(\phi) & 0 & 3 + a_{3,\ell}(\phi) \end{pmatrix}, \quad \Delta_{2,\ell}(\phi) = \begin{pmatrix} a_{5,\ell}(\phi) & 0 & a_{8,\ell}(\phi) \\ 0 & a_{6,\ell}(\phi) & 0 \\ a_{8,\ell}(\phi) & 0 & a_{7,\ell}(\phi) \end{pmatrix}.$$

In what follows, with a slight abuse of notation, we write the conditional covariance matrix $\Delta_\ell(\phi)$ as a function of $\mathbf{a}$

$$\Delta_\ell(\phi) = \Delta(\mathbf{a}_\ell(\phi)) = \Delta(\mathbf{a}) = \begin{pmatrix} \Delta_1(\mathbf{a}) & \Delta_2(\mathbf{a}) \\ \Delta_2(\mathbf{a}) & \Delta_1(\mathbf{a}) \end{pmatrix},$$

where

$$\Delta_1(\mathbf{a}) = \begin{pmatrix} 3 + a_1 & 0 & 1 + a_4 \\ 0 & 1 + a_2 & 0 \\ 1 + a_4 & 0 & 3 + a_3 \end{pmatrix}, \qquad \Delta_2(\mathbf{a}) = \begin{pmatrix} a_5 & 0 & a_8 \\ 0 & a_6 & 0 \\ a_8 & 0 & a_7 \end{pmatrix}.$$

At this point we may write the 2-point correlation function $K_{2,\ell}$ in (2.4) as a function of the perturbing elements $a_{i,\ell}(\phi)$, $i = 1,\dots,8$ of the covariance matrix:

$$K_{2,\ell}(\phi) = \frac{\lambda_\ell^4}{8^2} \frac{1}{(2\pi)^2\sqrt{\det(A_\ell(\phi))}} q(\mathbf{a}_\ell(\phi)),$$

where $A_\ell(\phi)$ is the covariance matrix of the Gaussian random vector $(\nabla f_\ell(x), \nabla f_\ell(y))$

$$A_\ell(\phi) = \begin{pmatrix} \frac{\lambda_\ell}{2} & 0 & \alpha_{1,\ell}(\phi) & 0 \\ 0 & \frac{\lambda_\ell}{2} & 0 & \alpha_{2,\ell}(\phi) \\ \alpha_{1,\ell}(\phi) & 0 & \frac{\lambda_\ell}{2} & 0 \\ 0 & \alpha_{2,\ell}(\phi) & 0 & \frac{\lambda_\ell}{2} \end{pmatrix},$$

see [11, Appendix B], and $q(\mathbf{a}_\ell(\phi))$ is the conditional expectation

$$q(\mathbf{a}_\ell(\phi)) = \frac{1}{(2\pi)^3\sqrt{\det(\Delta_\ell(\phi))}} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |z_1 z_3 - z_2^2| \cdot |w_1 w_3 - w_2^2|$$
$$\times \exp\Big\{ -\frac{1}{2}(z_1, z_2, z_3, w_1, w_2, w_3)\Delta_\ell(\phi)^{-1}(z_1, z_2, z_3, w_1, w_2, w_3)^t \Big\} dz_1 dz_2 dz_3 dw_1 dw_2 dw_3.$$

Note that there exists a constant $C > 0$ sufficiently big such that, for $C/\ell < \phi < \pi - C/\ell$, the covariance matrix $A_\ell(\phi)$ is nonsingular, in view of [4, Proposition 1.2], this guarantees the existence of the inverse $\Delta_\ell(\phi)^{-1}$ and the validity of Kac-Rice formula on this range as discussed in Section 2.1. The determinant of $A_\ell(\phi)$ can be easily computed so that we obtain

$$K_{2,\ell}(\phi) = \frac{\lambda_\ell^4}{8^2} \frac{1}{\pi^2\sqrt{(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))(\lambda_\ell^2 - 4\alpha_{1,\ell}^2(\phi))}} q(\mathbf{a}_\ell(\phi)). \tag{3.5}$$

3.2. **Taylor expansion of the two-point correlation function.** To study the asymptotic behaviour of the variance in the long-range regime, we investigate now the asymptotic behaviour of (2.7), i.e., the high energy asymptotic behaviour of

$$\frac{\lambda_\ell^2}{8}\int_{C/\ell}^{\pi-C/\ell}\frac{\sin\phi}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}q(\mathbf{a}_\ell(\phi))d\phi-(\mathbb{E}[\mathcal{N}^c(f_\ell)])^2. \tag{3.6}$$

In the range $\phi\in(C/\ell,\pi-C/\ell)$ the conditional covariance matrix $\Delta_\ell(\phi)=\Delta(\mathbf{a})$ is a small perturbation of the 6×6 matrix U where

$$U=\begin{pmatrix} U_1 & \mathbf{0}\\ \mathbf{0} & U_1\end{pmatrix},\qquad U_1=\begin{pmatrix}3&0&1\\0&1&0\\1&0&3\end{pmatrix}.$$

The elements a_i, $i=1,\dots,8$ are in fact uniformly small for $\phi\in(C/\ell,\pi-C/\ell)$, see Lemma 4.1 below. Consequently we may use perturbation theory [13, Theorem 1.5] to yield that the Gaussian expectation q is an analytic functions of the perturbing elements a_i, $i=1,\dots,8$ and we can expand it into a Taylor polynomial around $\mathbf{a}=0$ as follows:

$$\begin{aligned}
&\int_{C/\ell}^{\pi-C/\ell}\frac{\sin\phi}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}q(\mathbf{a}_\ell(\phi))d\phi\\
&=A_{0,\ell}\,q(\mathbf{0})+\sum_{i_1=1}^{8}A_{i_1,\ell}\left[\frac{\partial}{\partial a_{i_1}}q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}}+\frac{1}{2}\sum_{i_1,i_2=1}^{8}A_{i_1i_2,\ell}\left[\frac{\partial^2}{\partial a_{i_1}\partial a_{i_2}}q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}}\\
&\quad+\frac{1}{3!}\sum_{i_1,i_2,i_3=1}^{8}A_{i_1i_2i_3,\ell}\left[\frac{\partial^3}{\partial a_{i_1}\partial a_{i_2}\partial a_{i_3}}q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}}+\frac{1}{4!}\sum_{i_1,i_2,i_3,i_4=1}^{8}A_{i_1i_2i_3i_4,\ell}\left[\frac{\partial^4}{\partial a_{i_1}\partial a_{i_2}\partial a_{i_3}\partial a_{i_4}}q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}}\\
&\quad+\int_{C/\ell}^{\pi-C/\ell}\frac{\sin\phi}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}O(||\mathbf{a}||^5)\,d\phi,
\end{aligned}\tag{3.7}$$

where we adopted the following notation: for $i_1,i_2,i_3,i_4=1,\dots,8$,

$$\begin{aligned}
A_{0,\ell}&=\int_{C/\ell}^{\pi-C/\ell}\frac{1}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}\sin\phi\,d\phi,\\
A_{i_1,\ell}&=\int_{C/\ell}^{\pi-C/\ell}\frac{a_{i_1,\ell}(\phi)}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}\sin\phi\,d\phi,\\
A_{i_1,\dots i_k,\ell}&=\int_{C/\ell}^{\pi-C/\ell}\frac{a_{i_1,\ell}(\phi)\cdots a_{i_k,\ell}(\phi)}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}\sin\phi\,d\phi,\qquad k=2,3,4.
\end{aligned}$$

Note that to obtain the exact asymptotic behaviour of the variance of the total number of critical point we need to sharpen the bounds obtained in [11]; for this reason we have expanded q in (3.7) up to order four (instead of order three as in [11]).

3.3. **Asymptotics for the two-point correlation function.** We now study the decay rate of $A_{i_1,\dots i_k,\ell}$. In particular, we improve the bounds obtained in [11, Lemma 4.3 and Lemma 4.4] for the $A_{i_1,\dots i_k,\ell}$ to $O(\ell^2)$. Such refinement requires a more careful investigation of the tail decay of the perturbing elements $\mathbf{a}_\ell(\phi)$ of $\Delta_\ell(\phi)$ that are expressed in terms of the first four derivatives of Legendre polynomials as shown in (3.3)-(3.4).

The tail decay, for $\ell\to\infty$, of the first four derivatives of Legendre polynomials, is derived in Appendix A using the high degree asymptotics of the Legendre polynomials and their derivatives, i.e., Hilb asymptotics. In particular, to improve the bounds obtained in [11], we apply here a more general version of the Hilb asymptotic derived in [12, Lemma 1] (see also [21, Theorem 8.21.5]).

All the work for establishing the asymptotics of the perturbing elements $\mathbf{a}_\ell(\phi)$ (see Lemma 4.1 in the next section) leads to the high energy asymptotic behaviour of the terms $A_{i_1,\dots i_k,\ell}$ in Lemma 3.1 and Lemma 3.2. In particular we see that the main contribution to the $A_{i_1,\dots i_k,\ell}$ comes from the leading non-oscillatory terms in the Taylor expansion (3.7), so we obtain Lemma 3.1 and Lemma 3.2 by bounding the contribution of the oscillatory terms and error terms.

We first show that the first term in the expansion (3.7) cancels out the squared expectation in (3.6):

Lemma 3.1. *As* $\ell \to \infty$,

$$\frac{\lambda_\ell^2}{8} A_{0,\ell}\, q(\mathbf{0}) - (\mathbb{E}\,[\mathcal{N}^c(f_\ell)])^2 = \frac{1}{8}\left[2\ell^3 + \frac{2\cdot 3^2}{\pi^2}\ell^2 \log \ell\right] q(\mathbf{0}) + O(\ell^2).$$

Then we study the high frequency asymptotic behaviour of the other terms:

Lemma 3.2. *As* $\ell \to \infty$, *for* $i \neq 3$, *we have* $\lambda_\ell^2 A_{i,\ell} = O(\ell^2)$, *whereas for* $i = 3$, *we get*

$$\lambda_\ell^2 A_{3,\ell} = \left[-16\ell^3 - \frac{2^5\cdot 3}{\pi^2}\ell^2 \log \ell\right] + O(\ell^2),$$

for $(i,j) \neq (3,3),(7,7)$ *we have* $\lambda_\ell^2 A_{ij,\ell} = O(\ell^2)$, *instead for* $(i,j) = (3,3),(7,7)$ *we have*

$$\lambda_\ell^2 \frac{1}{2} A_{33,\ell} = \frac{3\cdot 2^7}{\pi^2}\ell^2 \log \ell + O(\ell^2), \qquad \lambda_\ell^2 \frac{1}{2} A_{77,\ell} = \left[32\ell^3 - \frac{2^6}{\pi^2}\ell^2 \log \ell\right] + O(\ell^2),$$

for $(i,j,k) \neq (3,7,7)$ *we have* $\lambda_\ell^2 A_{ijk,\ell} = O(\ell^2)$, *and*

$$\lambda_\ell^2 \frac{3}{3!} A_{377,\ell} = -\frac{2^9}{\pi^2}\ell^2 \log \ell + O(\ell^2),$$

for $(i,j,k,l) \neq (7,7,7,7)$ *we have* $\lambda_\ell^2 A_{ijkl,\ell} = O(\ell^2)$, *whereas*

$$\lambda_\ell^2 \frac{1}{4!} A_{7777,\ell} = \frac{2^9}{\pi^2}\ell^2 \log \ell + O(\ell^2).$$

The proofs of Lemma 3.1 and Lemma 3.2 are postponed to Section 4.

In view of Lemma 3.1 and Lemma 3.2, we immediately see that, as $\ell \to \infty$, (3.6) has the following leading terms

$$\begin{aligned}\mathrm{Var}(\mathcal{N}^c(f_\ell)) &= \frac{1}{8}\left[2\ell^3 + \frac{2\cdot 3^2}{\pi^2}\ell^2 \log \ell\right] q(\mathbf{0}) + \frac{1}{8}\left[-16\ell^3 - \frac{2^5\cdot 3}{\pi^2}\ell^2 \log \ell\right] \left[\frac{\partial}{\partial a_3} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} \\ &+ \frac{1}{8}\left[32\ell^3 - \frac{2^6}{\pi^2}\ell^2 \log \ell\right]\left[\frac{\partial^2}{\partial a_7 \partial a_7} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} + \frac{1}{8}\left[\frac{3\cdot 2^7}{\pi^2}\ell^2 \log \ell\right]\left[\frac{\partial^2}{\partial a_3 \partial a_3} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} \\ &+ \frac{1}{8}\left[-\frac{2^9}{\pi^2}\ell^2 \log \ell\right]\left[\frac{\partial^3}{\partial a_3 \partial a_7 \partial a_7} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} + \frac{1}{8}\left[\frac{2^9}{\pi^2}\ell^2 \log \ell\right]\left[\frac{\partial^4}{\partial a_7 \partial a_7 \partial a_7 \partial a_7} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} + O(\ell^2). \end{aligned} \tag{3.8}$$

3.4. **Evaluation of the leading constant.** Let $Y = (Y_1, Y_2, Y_3)$ be a centred jointly Gaussian random vector with covariance matrix

$$U_1 = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

and let $\mathcal{I}_r$, $r = 0, 2, 4$, be the Gaussian expectations of the form

$$\mathcal{I}_r = \mathbb{E}[|Y_1 Y_3 - Y_2^2|(Y_1 - 3Y_3)^r].$$

The relevant derivatives in (3.8) and $q(\mathbf{0})$ are evaluated in the following two lemmas. We first note that

$$q(\mathbf{0}) = (\mathbb{E}[|Y_1 Y_3 - Y_2^2|])^2 = \mathcal{I}_0^2. \tag{3.9}$$

Lemma 3.3. *One has*

$$\left[\frac{\partial}{\partial a_3} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} = \frac{1}{2^3}[-3\mathcal{I}_0^2 + \frac{1}{2^3}\mathcal{I}_0 \mathcal{I}_2], \tag{3.10}$$

$$\left[\frac{\partial^2}{\partial a_7 \partial a_7} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} = \frac{1}{2^6}[3\mathcal{I}_0 - \frac{1}{2^3}\mathcal{I}_2]^2, \tag{3.11}$$

$$\left[\frac{\partial^2}{\partial a_3 \partial a_3} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} = \frac{1}{2^{11}}[2^6 \cdot 3^2 \mathcal{I}_0^2 - 2^4 \cdot 3 \mathcal{I}_0 \mathcal{I}_2 + \frac{1}{2^2}\mathcal{I}_0\mathcal{I}_4 + \frac{1}{2^2}\mathcal{I}_2^2], \tag{3.12}$$

$$\left[\frac{\partial^3}{\partial a_3 \partial a_7 \partial a_7} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} = \frac{1}{2^{19}}[-2^{10} \cdot 3^4 \mathcal{I}_0^2 + 2^7 \cdot 3^4 \mathcal{I}_0\mathcal{I}_2 - 2^4 \cdot 3\mathcal{I}_0\mathcal{I}_4 + 2\mathcal{I}_2\mathcal{I}_4 - 2^5 \cdot 3^2 \mathcal{I}_2^2], \tag{3.13}$$

$$\left[\frac{\partial^4}{\partial a_7 \partial a_7 \partial a_7 \partial a_7} q(\mathbf{a})\right]_{\mathbf{a}=\mathbf{0}} = \frac{1}{2^{24}}[2^6 \cdot 3^3 \mathcal{I}_0 + \mathcal{I}_4 - 2^4 \cdot 3^2 \mathcal{I}_2]^2. \tag{3.14}$$

Substituting (3.9) and (3.10)-(3.14) into (3.8) we obtain the following simple form for the variance

$$\mathrm{Var}(\mathcal{N}^c(f_\ell)) = \frac{[\mathcal{I}_2 - 2^3 \cdot 5\, \mathcal{I}_0]^2}{2^{10}}\ell^3 + \frac{[2^6 \cdot 3 \cdot 17\, \mathcal{I}_0 - 2^4 \cdot 11\, \mathcal{I}_2 + \mathcal{I}_4]^2}{2^{18}\,\pi^2}\ell^2 \log \ell + O(\ell^2). \tag{3.15}$$

In the next lemma we compute the Gaussian expectations $\mathcal{I}_r$, $r = 0, 2, 4$.

Lemma 3.4. *One has*

$$\mathcal{I}_0 = \frac{2^2}{\sqrt{3}}, \qquad \mathcal{I}_2 = \frac{2^5 \cdot 5}{\sqrt{3}}, \qquad \mathcal{I}_4 = \frac{2^8 \cdot 5^2 \cdot 7}{3\sqrt{3}}.$$

The proofs of Lemma 3.3 and Lemma 3.4 are postponed to the next section.

The statement of Theorem 1.1 now follows upon substituting the values of $\mathcal{I}_r$, obtained in Lemma 3.4, into (3.15).

4. Proofs of auxiliary lemmas

To prove Lemma 3.1 and Lemma 3.2 we first derive, in the next lemma, the asymptotic behaviour of the terms appearing in the perturbing elements of the covariance matrix $\Delta_\ell(\phi)$.

Lemma 4.1. *Let $h_0(0)$, $h_0(1)$ and $h_1(0)$ be the constants $h_0(0) = \sqrt{\frac{2}{\pi}}$, $h_0(1) = -\frac{1}{8}\sqrt{\frac{2}{\pi}}$, $h_1(0) = \sqrt{\frac{2}{\pi}}$ and let $\psi_{n,\ell+u}$ be the functions $\psi_{n,\ell+u} = (\ell + u + 1/2)\phi - n\pi/2 - \pi/4$ where $\ell \geq 1$, $n, u = 0, 1$, $\phi \in [C/\ell, \pi/2]$ and C be any positive constant. We have the following estimates.*

For $\alpha_{i,\ell}(\phi)$, $i = 1, 2$, we get

$$\frac{\alpha_{1,\ell}^2(\phi)}{\ell^2(\ell+1)^2} = \phi^{-3}O(\ell^{-3}), \tag{4.1}$$

$$\begin{aligned}\frac{\alpha_{2,\ell}^2(\phi)}{\ell^2(\ell+1)^2} &= h_0^2(0)\cos^2\psi_{0,\ell}\frac{1}{\ell\sin\phi} + 2h_0^2(0)\sin\psi_{0,\ell+1}\cos\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} \\ &\quad - h_0(0)h_0(1)\sin\psi_{0,\ell}\cos\psi_{0,\ell}\frac{1}{\ell^2\phi\sin\phi} - h_0^2(0)\sin\psi_{0,\ell-1}\cos\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} \\ &\quad + h_0^2(0)\cos\phi\cos\psi_{0,\ell}\sin\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} + \phi^{-1}O(\ell^{-2}),\end{aligned} \tag{4.2}$$

$$\frac{\alpha_{2,\ell}^4(\phi)}{\ell^4(\ell+1)^4} = h_0^4(0)\cos^4\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} + \phi^{-3}O(\ell^{-3}). \tag{4.3}$$

For $\beta_{i,\ell}(\phi)$, $i = 1, 2, 3$, we have

$$\frac{\beta_{1,\ell}^2(\phi)}{\ell^3(\ell+1)^3} = \phi^{-3}O(\ell^{-3}), \tag{4.4}$$

$$\frac{\beta_{2,\ell}(\phi)}{\ell^{3/2}(\ell+1)^{3/2}} = -h_0(0)\cos\psi_{0,\ell}\cos\phi\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}), \tag{4.5}$$

$$\frac{\beta_{2,\ell}^2(\phi)}{\ell^3(\ell+1)^3} = \phi^{-3}O(\ell^{-3}), \tag{4.6}$$

$$\begin{aligned}\frac{\beta_{3,\ell}(\phi)}{\ell^{3/2}(\ell+1)^{3/2}} &= -h_0(0)\sin\psi_{0,\ell}\frac{1}{\ell^{1/2}\sin^{1/2}\phi}\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j} + \frac{3}{2}h_0(0)\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi} \\ &\quad + 3h_0(0)\cos\psi_{0,\ell+1}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} - h_0(0)\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1/2}} \\ &\quad - h_0(1)\cos\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\phi\sin^{1/2}\phi} + h_1(0)\sin\psi_{1,\ell}A_1(\phi)\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi}\end{aligned} \tag{4.7}$$

$$- \frac{1}{2} h_0(0)(\cos\psi_{0,\ell+1} + 5\cos\psi_{0,\ell-1}) \frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi}$$
$$- 3h_0(0)\cos\phi\cos\psi_{0,\ell} \frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \ell^{-2-1/2}O(\phi^{-2-1/2}),$$

$$\frac{\beta^2_{3,\ell}(\phi)}{\ell^3(\ell+1)^3} = h_0^2(0)\sin^2\psi_{0,\ell}\frac{1}{\ell\sin\phi} - 6h_0^2(0)\sin\psi_{0,\ell}\cos\psi_{0,\ell+1}\frac{1}{\ell^2\sin^2\phi} \tag{4.8}$$
$$+ 2h_0(0)h_0(1)\sin\psi_{0,\ell}\cos\psi_{0,\ell}\frac{1}{\ell^2\phi\sin\phi}$$
$$- \frac{1}{4}h_0(0)h_1(0)\sin\psi_{0,\ell}\sin\psi_{1,\ell}\Big(\frac{1}{\ell^2\sin^2\phi}\cos\phi - \frac{1}{\ell^2\phi\sin\phi}\Big)$$
$$+ h_0^2(0)\sin\psi_{0,\ell}(\cos\psi_{0,\ell+1} + 5\cos\psi_{0,\ell-1})\frac{1}{\ell^2\sin^2\phi}$$
$$+ 6h_0^2(0)\sin\psi_{0,\ell}\cos\phi\cos\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} + \phi^{-1}O(\ell^{-2}),$$

$$\frac{\beta^4_{3,\ell}(\phi)}{\ell^6(\ell+1)^6} = h_0^4(0)\sin^4\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} + \phi^{-3}O(\ell^{-3}), \tag{4.9}$$

where

$$A_1(\phi) = \Big[(\alpha+1)^2 - \frac{1}{4}\Big]\Big(\frac{1-\phi\cot\phi}{2\phi}\Big) - \frac{(\alpha+1)^2-\beta^2}{4}\tan\frac{\phi}{2}.$$

And finally for $\gamma_{i,\ell}(\phi)$, $i = 1,2,3,4$, *we have*

$$\frac{\gamma_{1,\ell}(\phi)}{\ell^2(\ell+1)^2} = \phi^{-2-1/2}O(\ell^{-2-1/2}), \tag{4.10}$$

$$\frac{\gamma_{2,\ell}(\phi)}{\ell^2(\ell+1)^2} = -h_0(0)\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}), \tag{4.11}$$

$$\frac{\gamma_{3,\ell}(\phi)}{\ell^2(\ell+1)^2} = -h_0(0)\sin\psi_{0,\ell}\cos\phi\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}), \tag{4.12}$$

$$\frac{\gamma_{4,\ell}(\phi)}{\ell^2(\ell+1)^2} = h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1/2}\sin^{1/2}\phi}\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j} - 2h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi} \tag{4.13}$$
$$+ 4\cos\psi_{1,\ell+1}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi}$$
$$- h_0(1)\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\phi\sin^{1/2}\phi} + h_1(0)\cos\psi_{1,\ell}A_1(\phi)\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi}$$
$$- \frac{3}{2}h_0(0)(\sin\psi_{0,\ell+1} + 3\sin\psi_{0,\ell-1})\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi}$$
$$- 6h_0(0)\cos\phi\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}),$$

$$\frac{\gamma^2_{4,\ell}(\phi)}{\ell^4(\ell+1)^4} = h_0^2(0)\cos^2\psi_{0,\ell}\frac{1}{\ell\sin\phi} + 4h_0(0)\cos\psi_{1,\ell+1}\cos\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} \tag{4.14}$$
$$- h_0(0)h_0(1)\sin\psi_{0,\ell}\cos\psi_{0,\ell}\frac{1}{\ell^2\phi\sin\phi} - \frac{3}{2}h_0^2(0)(\sin\psi_{0,\ell+1} + 3\sin\psi_{0,\ell-1})\cos\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi}$$
$$- 6h_0^2(0)\cos\phi\sin\psi_{0,\ell}\cos\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} + \phi^{-1}O(\ell^{-2}),$$

$$\frac{\gamma^3_{4,\ell}(\phi)}{\ell^6(\ell+1)^6} = h_0^3(0)\cos^3\psi_{0,\ell}\frac{1}{\ell^{3/2}\sin^{3/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}), \tag{4.15}$$

$$\frac{\gamma^4_{4,\ell}(\phi)}{\ell^8(\ell+1)^8} = h_0^4(0)\cos^4\psi_{0,\ell}\frac{1}{\ell^2\sin^2\phi} + \phi^{-3}O(\ell^{-3}). \tag{4.16}$$

Proof. The proof follows immediately from the tail decay of the derivatives of Legendre polynomials derived in Appendix A. Recalling that $\alpha_{1,\ell}(\phi) = P'_\ell(\cos\phi)$ and in view of (A.7) we obtain

$$\frac{\alpha_{1,\ell}(\phi)}{\ell(\ell+1)} = h_0(0)\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}) + \phi^{-1}O(\ell^{-2}).$$

Similarly, plugging (A.7) and (A.8) into $\alpha_{2,\ell}(\phi)$, we have

$$\begin{aligned}\frac{\alpha_{2,\ell}(\phi)}{\ell(\ell+1)} &= h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1/2}\sin^{1/2}\phi}\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j} - h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi}\\ &+ 2h_0(0)\sin\psi_{0,\ell+1}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1/2}} - h_0(1)\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\phi\sin^{1/2}\phi}\\ &+ h_1(0)\cos\psi_{1,\ell}A_1(\phi)\frac{1}{\ell^{1+1/2}\sin^{1/2}\phi} - h_0(0)\sin\psi_{0,\ell-1}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi}\\ &+ h_0(0)\cos\phi\sin\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}) + O(\ell^{-2}),\end{aligned}$$

$$\frac{\beta_{1,\ell}(\phi)}{\ell^{3/2}(\ell+1)^{3/2}} = -h_0(0)\cos\psi_{0,\ell}\frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} + \phi^{-2-1/2}O(\ell^{-2-1/2}),$$

and the asymptotic behaviour of $\frac{\beta_{2,\ell}(\phi)}{\ell^{3/2}(\ell+1)^{3/2}}$ and $\frac{\gamma_{1,\ell}(\phi)}{\ell^2(\ell+1)^2}$ as in the statement. From (A.7), (A.8) and (A.9) we obtain the decay rate of $\frac{\beta_{3,\ell}(\phi)}{\ell^{3/2}(\ell+1)^{3/2}}$, $\frac{\gamma_{2,\ell}(\phi)}{\ell^2(\ell+1)^2}$ and $\frac{\gamma_{3,\ell}(\phi)}{\ell^2(\ell+1)^2}$. Finally, in view of (A.7), (A.8), (A.9) and (A.10), we obtain the asymptotic behaviour of $\frac{\gamma_{4,\ell}(\phi)}{\ell^2(\ell+1)^2}$. □

We exploit now Lemma 4.1 to obtain the bounds for the terms $A_{0,\ell}$, $A_{i_1,\ell}$ and $A_{i_1,\dots i_k,\ell}$ for $k = 2,3,4$, $i_1,\dots i_k = 1,\dots,8$.

Proof of Lemma 3.1. In view of (4.1) and (4.2) we first obtain

$$\begin{aligned}A_{0,\ell} &= \int_{C/\ell}^{\pi-C/\ell}\frac{\sin\phi}{\sqrt{(1-4\alpha_{2,\ell}^2(\phi)/\lambda_\ell^2)(1-4\alpha_{1,\ell}^2(\phi)/\lambda_\ell^2)}}d\phi\\ &= \int_{C/\ell}^{\pi-C/\ell}\left(1+2\frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2}+2\cdot 3\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\right)\sin\phi d\phi + O(\ell^{-2})\\ &= \cos(C/\ell) + 2\int_{C/\ell}^{\pi-C/\ell}\frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2}\sin\phi d\phi + 2\cdot 3\int_{C/\ell}^{\pi-C/\ell}\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\sin\phi d\phi + O(\ell^{-2}).\end{aligned}$$

Now, from (4.2), we have

$$\begin{aligned}2\int_{C/\ell}^{\pi-C/\ell}\frac{\alpha_{2,\ell}^2(\phi)}{\lambda_\ell^2}\sin\phi d\phi &= 2\int_{C/\ell}^{\pi-C/\ell}\left[\frac{h_0^2(0)\cos^2\psi_{0,\ell}}{\ell\sin\phi} + 2\binom{-1/2}{1}h_0^2(0)\frac{\sin\psi_{0,\ell+1}\cos\psi_{0,\ell}}{\ell^2\sin^2\phi} - h_0(0)h_0(1)\frac{\cos\psi_{0,\ell}\sin\psi_{0,\ell}}{\ell^2\phi\sin\phi}\right.\\ &\left.- h_0^2(0)\frac{\cos\psi_{0,\ell}\sin\psi_{0,\ell-1}}{\ell^2\sin^2\phi} + h_0^2(0)\frac{\cos\psi_{0,\ell}\cos\phi\sin\psi_{0,\ell}}{\ell^2\sin^2\phi}\right]\sin\phi d\phi + O(\ell^{-2})\\ &= 2\int_{C/\ell}^{\pi-C/\ell}\left[\frac{h_0^2(0)\cos^2\psi_{0,\ell}}{\ell\sin\phi}\right]\sin\phi d\phi + O(\ell^{-2})\\ &= \frac{4}{\pi}\int_{C/\ell}^{\pi-C/\ell}\frac{1}{\ell\sin\phi}\cos^2[(\ell+1/2)\phi-\pi/4]\sin\phi d\phi + O(\ell^{-2})\\ &= \frac{2}{\pi}\frac{1}{\ell}\int_{C/\ell}^{\pi-C/\ell}d\phi + \frac{2}{\pi}\frac{1}{\ell}\int_{C/\ell}^{\pi-C/\ell}\cos[2(\ell+1/2)\phi-2\pi/4]d\phi + O(\ell^{-2})\\ &= \frac{2}{\ell} + O(\ell^{-2}),\end{aligned}$$

and from (4.3), and the equality $\cos^4\psi_{0,\ell} = \frac{3}{8}+\frac{1}{8}[-\cos(2\phi(2\ell+1))+4\sin(\phi(2\ell+1))]$, we have

$$2\cdot 3\int_{C/\ell}^{\pi-C/\ell}\frac{\alpha_{2,\ell}^4(\phi)}{\lambda_\ell^4}\sin\phi d\phi = 2\cdot 3\frac{1}{\ell^2}\int_{C/\ell}^{\pi-C/\ell}h_0^4(0)\frac{1}{\sin\phi}\cos^4\psi_{0,\ell}d\phi + O(\ell^{-2})$$

$$= 2 \cdot 3 \frac{3}{8} \frac{2^2}{\pi^2} \frac{1}{\ell^2} \int_{C/\ell}^{\pi - C/\ell} \frac{1}{\sin \phi} d\phi + O(\ell^{-2}) = \frac{2 \cdot 3^2}{\pi^2} \frac{\log \ell}{\ell^2} + O(\ell^{-2}).$$

Therefore the statement follows since we obtain

$$\lambda_\ell^2 A_{0,\ell}\, q(\mathbf{0}) - (\mathbb{E}\,[\mathcal{N}^c(f_\ell)])^2 = \lambda_\ell^2 \left[\frac{2}{\ell} + \frac{2 \cdot 3^2}{\pi^2} \frac{\log \ell}{\ell^2} + O(\ell^{-2}) \right] q(\mathbf{0}) = \left[2\ell^3 + \frac{2 \cdot 3^2}{\pi^2} \ell^2 \log \ell \right] q(\mathbf{0}) + O(\ell^2).$$

□

Proof of Lemma 3.2. In view of (4.1), (4.2), (4.4) and (4.6), we immediately obtain that $A_{1,\ell}, A_{2,\ell} = O(\ell^{-2})$:

$$A_{1,\ell} = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{2,\ell}^2(\phi)}{(1 - 4\alpha_2^2/\lambda_\ell^2)^{3/2} (1 - 4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin \phi d\phi = -16 \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{2,\ell}^2(\phi)}{\ell^6} \sin \phi d\phi + O(\ell^{-2}) = O(\ell^{-2}),$$

$$A_{2,\ell} = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{1,\ell}^2(\phi)}{(1 - 4\alpha_2^2/\lambda_\ell^2)^{1/2} (1 - 4\alpha_1^2/\lambda_\ell^2)^{3/2}} \sin \phi d\phi = -16 \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{1,\ell}^2(\phi)}{\ell^6} \sin \phi d\phi + O(\ell^{-2}) = O(\ell^{-2}).$$

Form (4.1), we have

$$A_{3,\ell} = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{3,\ell}^2(\phi)}{(1 - 4\alpha_2^2/\lambda_\ell^2)^{3/2} (1 - 4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin \phi d\phi = -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi - C/\ell} \beta_{3,\ell}^2(\phi) \left[1 + \frac{3}{2} \frac{4\alpha_2^2}{\lambda_\ell^2} \right] \sin \phi d\phi + O(\ell^{-2}),$$

where, in view of (4.8), we obtain a leading non-oscillatory term in

$$\begin{aligned} -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi - C/\ell} \beta_{3,\ell}^2(\phi) \sin \phi d\phi &= -16 \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{3,\ell}^2(\phi)}{\ell^6} \sin \phi d\phi + O(\ell^{-2}) \\ &= -16 \int_{C/\ell}^{\pi - C/\ell} \left[h_0^2(0) \frac{\sin^2 \psi_{0,\ell}}{\ell \sin \phi} \right] \sin \phi d\phi + O(\ell^{-2}) \\ &= -16\, h_0^2(0)\, \frac{1}{\ell} \int_{C/\ell}^{\pi - C/\ell} \left[\frac{1}{2} - \frac{1}{2} \cos(2\psi_{0,\ell}) \right] d\phi + O(\ell^{-2}) \\ &= -16\, h_0^2(0)\, \frac{1}{\ell}\, \frac{1}{2}\, \pi + O(\ell^{-2}) = -\frac{16}{\ell} + O(\ell^{-2}) \end{aligned}$$

and, from (4.2), (4.8) and the equality $\sin^2 \psi_{0,\ell} \cos^2 \psi_{0,\ell} = \frac{1}{4} \cos^2[(2\ell + 1)\phi] = \frac{1}{8} + \frac{1}{8} \cos[2\phi(2\ell + 1)]$, we have

$$\begin{aligned} -\frac{16}{\lambda_\ell^3} \frac{3 \cdot 4}{2} \int_{C/\ell}^{\pi - C/\ell} \beta_{3,\ell}^2(\phi) \frac{\alpha_2^2}{\lambda_\ell^2} \sin \phi d\phi &= -16 \frac{3 \cdot 4}{2} \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{3,\ell}^2(\phi)}{\ell^6} \frac{\alpha_2^2}{\ell^4} \sin \phi d\phi + O(\ell^{-2}) \\ &= -16 \frac{3 \cdot 4}{2} h_0^4(0) \frac{1}{\ell^2} \int_{C/\ell}^{\pi - C/\ell} \left[\frac{\sin^2 \psi_{0,\ell} \cos^2 \psi_{0,\ell}}{\sin^2 \phi} \right] \sin \phi d\phi + O(\ell^{-2}) \\ &= -16 \frac{3 \cdot 4}{2} h_0^4(0) \frac{1}{8} \frac{1}{\ell^2} \int_{C/\ell}^{\pi - C/\ell} \frac{1}{\sin \phi} d\phi + O(\ell^{-2}) \\ &= -\frac{3 \cdot 4^2}{\pi^2} \frac{2}{\ell^2} \left[\log \left(\cos \left(\frac{C}{2\ell} \right) \right) - \log \left(\sin \left(\frac{C}{2\ell} \right) \right) \right] + O(\ell^{-2}) \\ &= -\frac{3 \cdot 4^2}{\pi^2} \frac{2}{\ell^2} \log \ell + O(\ell^{-2}). \end{aligned}$$

Then

$$\lambda_\ell^2 A_{3,\ell} = -16\ell^3 - \frac{2^5 \cdot 3}{\pi^2} \ell^2 \log \ell + O(\ell^2).$$

The terms $A_{4,\ell}, A_{5,\ell}, A_{6,\ell}, A_{7,\ell}, A_{8,\ell}$ are $O(\ell^{-2})$. In fact, for $A_{4,\ell}$, using (4.1), (4.2), (4.5), (4.7) and the trigonometric equality $-\cos \psi_{0,\ell} \sin \psi_{0,\ell} = 1/2 \cos[(2\ell + 1)\phi]$, we have

$$\begin{aligned} A_{4,\ell} &= -\frac{16}{\lambda_\ell^3} \int_{C/\ell}^{\pi - C/\ell} \frac{\beta_{2,\ell}(\phi) \beta_{3,\ell}(\phi)}{(1 - 4\alpha_2^2/\lambda_\ell^2)^{3/2} (1 - 4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin \phi d\phi \\ &= -\frac{16}{\ell^6} \int_{C/\ell}^{\pi - C/\ell} \beta_{2,\ell}(\phi) \beta_{3,\ell}(\phi) \sin \phi d\phi + O(\ell^{-2}) \end{aligned}$$

$$= -16 \int_{C/\ell}^{\pi-C/\ell} \left[-h_0(0) \frac{\cos\psi_{0,\ell}\cos\phi}{\ell^{1+1/2}\sin^{1+1/2}\phi} \right] \left[-h_0(0) \frac{\sin\psi_{0,\ell}}{\ell^{1/2}\sin^{1/2}\phi} \right] \sin\phi d\phi + O(\ell^{-2}) = O(\ell^{-2}),$$

for $A_{5,\ell}$, form (4.1), (4.2), (4.6) and (4.10), we immediately have

$$A_{5,\ell}$$
$$= \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi-C/\ell} \frac{\gamma_{1,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi - \frac{8\cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi-C/\ell} \frac{\alpha_{2,\ell}(\phi)\beta_{2,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{3/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi$$
$$= 8 \int_{C/\ell}^{\pi-C/\ell} \frac{\gamma_{1,\ell}(\phi)}{\ell^4} \sin\phi d\phi - 8\cdot 4 \int_{C/\ell}^{\pi-C/\ell} \frac{\alpha_{2,\ell}(\phi)\beta_{2,\ell}^2(\phi)}{\ell^8} \sin\phi d\phi = O(\ell^{-2}),$$

the asymptotic behaviour of $A_{6,\ell}$ follows from (4.1), (4.2), (4.4) and (4.11)

$$A_{6,\ell}$$
$$= \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi-C/\ell} \frac{\gamma_{2,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi - \frac{8\cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi-C/\ell} \frac{\alpha_{1,\ell}(\phi)\beta_{1,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{3/2}} \sin\phi d\phi$$
$$= 8 \int_{C/\ell}^{\pi-C/\ell} \frac{\gamma_{2,\ell}(\phi)}{\ell^4} \sin\phi d\phi + O(\ell^{-2}) = -8 \int_{C/\ell}^{\pi-C/\ell} \frac{1}{\ell^{1+1/2}\sin^{1+1/2}\phi} \sin\psi_{0,\ell} h_0(0) \sin\phi d\phi + O(\ell^{-2}) = O(\ell^{-2}),$$

the asymptotic behaviour of $A_{7,\ell}$ follows from (4.1), (4.2), (4.8) and (4.13), in fact we have

$$A_{7,\ell}$$
$$= \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi-C/\ell} \frac{\gamma_{4,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi - \frac{8\cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi-C/\ell} \frac{\alpha_{2,\ell}(\phi)\beta_{3,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{3/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi$$
$$= \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi-C/\ell} \gamma_{4,\ell}(\phi) \sin\phi d\phi + \frac{8}{\lambda_\ell^2} \frac{4}{2} \frac{1}{\lambda_\ell^2} \int_{C/\ell}^{\pi-C/\ell} \gamma_{4,\ell}(\phi)\alpha_{2,\ell}^2(\phi) \sin\phi d\phi - \frac{8\cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi/2} \alpha_{2,\ell}(\phi)\beta_{3,\ell}^2(\phi) \sin\phi d\phi + O(\ell^{-2})$$
$$= \frac{8}{\ell^4} \left(1 - \frac{2}{\ell}\right) \int_{C/\ell}^{\pi-C/\ell} \gamma_{4,\ell}(\phi) \sin\phi d\phi + \frac{16}{\ell^8} \int_{C/\ell}^{\pi-C/\ell} \gamma_{4,\ell}(\phi)\alpha_{2,\ell}^2(\phi) \sin\phi d\phi$$
$$- \frac{8\cdot 4}{\ell^8} \int_{C/\ell}^{\pi-C/\ell} \alpha_{2,\ell}(\phi)\beta_{3,\ell}^2(\phi) \sin\phi d\phi + O(\ell^{-2}) = \frac{8}{\ell^4} \int_{C/\ell}^{\pi-C/\ell} \gamma_{4,\ell}(\phi) \sin\phi d\phi + O(\ell^{-2}),$$

where, using integration by parts,

$$\frac{8}{\ell^4} \int_{C/\ell}^{\pi-C/\ell} \gamma_{4,\ell}(\phi) \sin\phi d\phi$$
$$= 8h_0(0) \frac{1}{\ell^{1/2}} \int_{C/\ell}^{\pi-C/\ell} \frac{\cos\psi_{0,\ell}}{\sin^{1/2}\phi} \sin\phi \, d\phi$$
$$= 8h_0(0) \frac{1}{\ell^{1/2}} \int_{C/\ell}^{\pi-C/\ell} \cos[(\ell+1/2)\phi - \pi/4] \sin^{1/2}\phi \, d\phi$$
$$= 8h_0(0) \frac{1}{\ell^{1/2}(\ell+1/2)} \left\{ \sin[(\ell+1/2)\phi - \pi/4] \sin^{1/2}\phi \Big|_{C/\ell}^{\pi-C/\ell} - \frac{1}{2} \int_{C/\ell}^{\pi-C/\ell} \frac{\sin[(\ell+1/2)\phi - \pi/4]}{\sin^{1/2}\phi} d\phi \right\}$$
$$= 8\sqrt{\frac{2}{\pi}} \frac{1}{\ell^{3/2}} \Big[\sin[(\ell+1/2)\pi/2 - \pi/4] \sin^{1/2}(\pi - C/\ell) - \sin[(\ell+1/2)C/\ell - \pi/4] \sin^{1/2}(C/\ell) \Big] + O(\ell^{-2}),$$

and finally $A_{8,\ell}$ follows from (4.1), (4.2), (4.5), (4.7) and (4.12):

$$A_{8,\ell}$$
$$= \frac{8}{\lambda_\ell^2} \int_{C/\ell}^{\pi-C/\ell} \frac{\gamma_{3,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi - \frac{8\cdot 4}{\lambda_\ell^4} \int_{C/\ell}^{\pi-C/\ell} \frac{\alpha_{2,\ell}(\phi)\beta_{2,\ell}(\phi)\beta_{3,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{3/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi$$
$$= \frac{8}{\ell^4} \int_{C/\ell}^{\pi-C/\ell} \gamma_{3,\ell}(\phi) \sin\phi d\phi + O(\ell^{-2}) = -8h_0(0) \frac{1}{\ell^{1+1/2}} \int_{C/\ell}^{\pi-C/\ell} \frac{\sin\psi_{0,\ell}\cos\phi}{\sin^{1/2}\phi} d\phi + O(\ell^{-2}) = O(\ell^{-2}).$$

We study now the asymptotic behaviour of the higher order terms of the form $A_{i_1,\dots i_k,\ell}$ with $k = 2, 3, 4$. Note that each term of the form

$$A_{i_1,\dots i_k,\ell} \text{ with } (i_1, \dots i_k) \neq (3,3), (3,7), (7,7), (3,7,7), (7,7,7), (7,7,7,7),$$

is of order $O(\ell^{-2})$. This implies that, to prove the statement, it is enough to analyse the high energy asymptotic behaviour of the following terms.

We first note that $A_{77,\ell}$ produces a leading non-oscillating term, in fact

$$A_{77,\ell} = \int_{C/\ell}^{\pi-C/\ell} \frac{a_{7,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi$$

where

$$a_{7,\ell}^2(\phi) = \frac{8^2}{\lambda_\ell^4}\left[\gamma_4 - \frac{4}{\lambda_\ell^2}\frac{\alpha_2\,\beta_3^2}{(1-4\alpha_2^2/\lambda_\ell^2)}\right]^2 = \frac{8^2}{\lambda_\ell^4}\left[\gamma_4^2 + \frac{16}{\lambda_\ell^4}\frac{\alpha_2^2\,\beta_3^4}{(1-4\alpha_2^2/\lambda_\ell^2)^2} - 2\gamma_4\frac{4}{\lambda_\ell^2}\frac{\alpha_2\,\beta_3^2}{(1-4\alpha_2^2/\lambda_\ell^2)}\right],$$

so that

$$A_{77,\ell} = 8^2\int_{C/\ell}^{\pi-C/\ell}\frac{\gamma_4^2}{\ell^8}\left(1+2\frac{\alpha_2^2}{\ell^4}\right)\sin\phi d\phi - \frac{8^3}{\ell^{12}}\int_{C/\ell}^{\pi-C/\ell}\gamma_4\alpha_2\beta_3^2\sin\phi d\phi + O(\ell^{-2}). \tag{4.17}$$

Now, in view of (4.14) and (4.2), we obtain

$$8^2\int_{C/\ell}^{\pi-C/\ell}\frac{\gamma_4^2}{\ell^8}\sin\phi d\phi = 8^2 h_0^2(0)\frac{1}{\ell}\int_{C/\ell}^{\pi-C/\ell}\cos^2\psi_{0,\ell}d\phi + O(\ell^{-2}) = \frac{2\cdot 32}{\ell} + O(\ell^{-2}),$$

$$\begin{aligned}
2\cdot 8^2\int_{C/\ell}^{\pi-C/\ell}\frac{\gamma_4^2}{\ell^8}\frac{\alpha_2^2}{\ell^4}\sin\phi d\phi &= 2\cdot 8^2\int_{C/\ell}^{\pi-C/\ell}\left(h_0^2(0)\frac{\cos^2\psi_{0,\ell}}{\ell\sin\phi}\right)\left(h_0^2(0)\frac{\cos^2\psi_{0,\ell}}{\ell\sin\phi}\right)\sin\phi d\phi + O(\ell^{-2})\\
&= 2\cdot 8^2 h_0^4(0)\frac{1}{\ell^2}\int_{C/\ell}^{\pi-C/\ell}\left[\frac{3}{8}+\frac{1}{8}\cos(4\psi_{0,\ell})+\frac{1}{2}\cos(2\psi_{0,\ell})\right]\frac{1}{\sin\phi}d\phi + O(\ell^{-2})\\
&= 2\cdot 8^2 h_0^4(0)\frac{3}{8}\frac{1}{\ell^2}\int_{C/\ell}^{\pi-C/\ell}\frac{1}{\sin\phi}d\phi + O(\ell^{-2})\\
&= 2\cdot 2\cdot 8^2\frac{2^2}{\pi^2}\frac{3}{8}\frac{1}{\ell^2}\log\ell + O(\ell^{-2}) = \frac{2\cdot 3\cdot 2^6}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2}),
\end{aligned}$$

for the last term in (4.17) we use (4.8), (4.13) and

$$\cos^2\psi_{0,\ell}\sin^2\psi_{0,\ell} = \frac{1}{8} + \frac{1}{4}\cos[2(\ell+1/2)\phi+\pi/2] + \frac{1}{4}\cos[2(\ell+1/2)\phi-\pi/2] + \frac{1}{8}\cos[2(2\ell+1)\phi]$$

to obtain

$$\begin{aligned}
&-\frac{8^3}{\ell^{12}}\int_{C/\ell}^{\pi-C/\ell}\gamma_4\alpha_2\beta_3^2\sin\phi d\phi\\
&= -8^3\int_{C/\ell}^{\pi-C/\ell}\left(h_0(0)\frac{\cos\psi_{0,\ell}}{\ell^{1/2}\sin^{1/2}\phi}\right)\left(h_0(0)\frac{\cos\psi_{0,\ell}}{\ell^{1/2}\sin^{1/2}\phi}\right)\left(h_0^2(0)\frac{\sin^2\psi_{0,\ell}}{\ell\sin\phi}\right)\sin\phi d\phi + O(\ell^{-2})\\
&= -8^3 h_0^4(0)\frac{1}{\ell^2}\int_{C/\ell}^{\pi-C/\ell}\frac{\cos^2\psi_{0,\ell}\sin^2\psi_{0,\ell}}{\sin\phi}d\phi + O(\ell^{-2})\\
&= -8^2 h_0^4(0)\frac{1}{\ell^2}\int_{C/\ell}^{\pi-C/\ell}\frac{1}{\sin\phi}d\phi + O(\ell^{-2}) = -\frac{2\cdot 2^8}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2}).
\end{aligned}$$

Therefore

$$\lambda_\ell^2\frac{1}{2}A_{77,\ell} = \lambda_\ell^2\left[\frac{32}{\ell} + \frac{3\cdot 2^6}{\pi^2}\frac{1}{\ell^2}\log\ell - \frac{2^8}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2})\right] = 32\ell^3 - \frac{2^6}{\pi^2}\ell^2\log\ell + O(\ell^2).$$

We apply now (4.1), (4.2) and (4.9) to study the asymptotic behaviour of $A_{33,\ell}$:

$$A_{33,\ell} = \int_{C/\ell}^{\pi-C/\ell}\frac{a_{3,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}}\sin\phi d\phi$$

where

$$a_{3,\ell}^2(\phi) = 16^2 \frac{\beta_3^4}{\lambda_\ell^6(1-4\alpha_2^2/\lambda_\ell^2)^2},$$

that is

$$A_{33,\ell} = 16^2 \int_{C/\ell}^{\pi-C/\ell} \frac{\beta_3^4}{\lambda_\ell^6(1-4\alpha_2^2/\lambda_\ell^2)^{5/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi = 16^2 \int_{C/\ell}^{\pi-C/\ell} \frac{\beta_3^4}{\lambda_\ell^6} \sin\phi d\phi + O(\ell^{-2})$$
$$= 16^2 \frac{1}{\ell^2} \int_{C/\ell}^{\pi-C/\ell} h_0^4(0) \frac{\sin^4\psi_{0,\ell}}{\sin\phi} d\phi + O(\ell^{-2}) = 16^2 \frac{2}{\ell^2} \frac{3}{8} \frac{2^2}{\pi^2} \log\ell + O(\ell^{-2}) = \frac{2\cdot 3\cdot 2^7}{\pi^2} \frac{1}{\ell^2} \log\ell + O(\ell^{-2})$$

since $\sin^4\psi_{0,\ell} = \frac{3}{8} - \frac{1}{8}\cos[2\phi(2\ell+1)] - \frac{1}{2}\sin[\phi(2\ell+1)]$. Therefore

$$\lambda_\ell^2 \frac{1}{2} A_{33,\ell} = \lambda_\ell^2 \left[\frac{3\cdot 2^7}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2})\right] = \frac{3\cdot 2^7}{\pi^2}\ell^2\log\ell + O(\ell^2).$$

The terms $A_{37,\ell}$ and $A_{777,\ell}$ are both $O(\ell^{-2})$ since their leading non-constant term are oscillating. One has

$$A_{37,\ell} = \int_{C/\ell}^{\pi-C/\ell} \frac{a_{3,\ell}(\phi)a_{7,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi$$

where, by (4.1) and (4.2),

$$\frac{a_{3,\ell}(\phi)a_{7,\ell}(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} = -16 \frac{\frac{\beta_3^2}{\lambda_\ell^3(1-4\alpha_2^2/\lambda_\ell^2)} 8 \frac{\gamma_4 + \frac{4\alpha_2\beta_3^2}{4\alpha_2^2-\lambda_\ell^2}}{\lambda_\ell^2}}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} = -16\cdot 8 \frac{\beta_3^2}{\ell^6}\frac{\gamma_4}{\ell^4} + O(\frac{1}{\ell^3\phi^3}).$$

Then, in view of (4.8) and (4.13), we have

$$A_{37,\ell} = -16\cdot 8 \int_{C/\ell}^{\pi-C/\ell} \frac{\beta_3^2}{\ell^6}\frac{\gamma_4}{\ell^4} \sin\phi d\phi + O(\ell^{-2})$$
$$= -16\cdot 8 \int_{C/\ell}^{\pi-C/\ell} \left[h_0^2(0)\frac{\sin^2\psi_{0,\ell}}{\ell\sin\phi}\right]\left[h_0(0)\frac{\cos\psi_{0,\ell}}{\ell^{1/2}\sin^{1/2}\phi}\right] \sin\phi d\phi = O(\ell^{-2})$$

since

$$\int_{C/\ell}^{\pi-C/\ell} \frac{\sin^2\psi_{0,\ell}\cos\psi_{0,\ell}}{\sin^{1/2}\phi} d\phi = O(\ell^{-1/2}).$$

And for $A_{777,\ell}$ we write

$$A_{777,\ell} = \int_{C/\ell}^{\pi-C/\ell} \frac{a_{7,\ell}^3(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi$$

where

$$\frac{a_{7,\ell}^3(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} = 8^3 \frac{\gamma_4^3}{\ell^{4\cdot 3}} + O(\frac{1}{\ell^3\phi^3})$$

and, from (4.15),

$$A_{777,\ell} = 8^3 h_0^3(0) \frac{1}{\ell^{1+1/2}} \int_{C/\ell}^{\pi-C/\ell} \frac{\cos^3\psi_{0,\ell}}{\sin^{1/2}\phi} d\phi = O(\ell^{-2}).$$

The last two terms we need to study are $A_{377,\ell}$ and $A_{7777,\ell}$; $A_{377,\ell}$ is defined by

$$A_{377,\ell} = \int_{C/\ell}^{\pi-C/\ell} \frac{a_{3,\ell}(\phi)a_{7,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} \sin\phi d\phi,$$

where

$$\frac{a_{3,\ell}(\phi)a_{7,\ell}^2(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} = -16 \frac{\frac{\beta_3^2}{\lambda_\ell^3(1-4\alpha_2^2/\lambda_\ell^2)} 8^2 \frac{\left[\gamma_4 + \frac{4\alpha_2\beta_3^2}{4\alpha_2^2-\lambda_\ell^2}\right]^2}{\lambda_\ell^4}}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}} = -16\cdot 8^2 \frac{\beta_3^2}{\ell^6}\frac{\gamma_4^2}{\ell^8} + O(\frac{1}{\ell^3\phi^3}).$$

Now, by applying (4.1), (4.2), (4.8) and (4.14), we have

$$\begin{aligned}A_{377,\ell} &= -16\cdot 8^2 \int_{C/\ell}^{\pi-C/\ell} \frac{\beta_3^2}{\ell^6}\frac{\gamma_4^2}{\ell^8}\sin\phi d\phi + O(\ell^{-2})\\ &= -16\cdot 8^2 \int_{C/\ell}^{\pi-C/\ell}\left[h_0^2(0)\frac{\sin^2\psi_{0,\ell}}{\ell\sin\phi}\right]\left[h_0^2(0)\frac{\cos^2\psi_{0,\ell}}{\ell\sin\phi}\right]\sin\phi d\phi + O(\ell^{-2})\\ &= -2\cdot 16\cdot 8^2\frac{1}{\ell^2}\frac{2^2}{\pi^2}\frac{1}{8}\log\ell + O(\ell^{-2}) = -\frac{2\cdot 2^9}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2}).\end{aligned}$$

Therefore

$$\lambda_\ell^2\frac{3}{3!}A_{377,\ell} = -\frac{2^9}{\pi^2}\ell^2\log\ell + O(\ell^2).$$

Finally for $A_{7777,\ell}$ we apply (4.1), (4.2) and (4.16), so that

$$\begin{aligned}A_{7777,\ell} &= \int_{C/\ell}^{\pi-C/\ell}\frac{a_{7,\ell}^4(\phi)}{(1-4\alpha_2^2/\lambda_\ell^2)^{1/2}(1-4\alpha_1^2/\lambda_\ell^2)^{1/2}}\sin\phi d\phi = 8^4\int_{C/\ell}^{\pi-C/\ell}\frac{\gamma_4^4}{\ell^{4\cdot 4}}\sin\phi d\phi\\ &= 8^4h_0^4(0)\frac{1}{\ell^2}\int_{C/\ell}^{\pi-C/\ell}\frac{\cos^4\psi_{0,\ell}}{\sin\phi}d\phi + O(\ell^{-2}) = \frac{2\cdot 3\cdot 2^{11}}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2}),\end{aligned}$$

and

$$\lambda_\ell^2\frac{1}{4!}A_{7777,\ell} = \lambda_\ell^2\frac{1}{3\cdot 4}\left[\frac{3\cdot 2^{11}}{\pi^2}\frac{1}{\ell^2}\log\ell + O(\ell^{-2})\right] = \frac{2^9}{\pi^2}\ell^2\log\ell + O(\ell^2).$$

□

We prove now Lemma 3.3 and Lemma 3.4 stated in Section 3.4.

Proof of Lemma 3.3. Let

$$\hat{q}(\mathbf{a}, z_1, z_2, z_3, w_1, w_2, w_3) = \frac{1}{\sqrt{\det(\Delta(\mathbf{a}))}}\exp\Big\{-\frac{1}{2}(z_1, z_2, z_3, w_1, w_2, w_3)\Delta(\mathbf{a})^{-1}(z_1, z_2, z_3, w_1, w_2, w_3)^t\Big\},$$

and

$$\mathbf{a}_i = (0, \dots, 0, a_i, 0, \dots, 0), \qquad i = 1, \dots, 8,$$

where a_i is the ith perturbing element of $\mathbf{a}$. Since $\hat{q}(\mathbf{a}, z_1, z_2, z_3, w_1, w_2, w_3)$ is an analytic function of the elements of the vector $\mathbf{a}$ [13, Theorem 1.5], to simplify the calculations note that, for example for the jth derivative with respect to a_i, we have

$$\Big[\frac{\partial^j}{\partial a_i^j}\hat{q}(\mathbf{a}; t_1, t_2; z_1, z_2, w_1, w_2)\Big]_{\mathbf{a}=\mathbf{0}} = \Big[\frac{\partial^j}{\partial a_i^j}\hat{q}(\mathbf{a}_i; t_1, t_2; z_1, z_2, w_1, w_2)\Big]_{\mathbf{a}_i=\mathbf{0}}, \quad i = 1, \dots, 8.$$

Now, using Leibniz integral rule and a computer-oriented computation to evaluate the derivates of $\hat{q}$, we obtain the statement of Lemma 3.3. □

Proof of Lemma 3.4. To prove the lemma it is convenient to introduce the transformation $W_1 = Y_1$, $W_2 = Y_2$ and $W_3 = Y_1 + Y_3$, so that

$$\mathcal{I}_r = \mathbb{E}[|Y_1Y_3 - Y_2^2|(Y_1 - 3Y_3)^r] = \mathbb{E}[|W_1(W_3 - W_1) - W_2^2|\,(W_1 - 3(W_3 - W_1))^r].$$

We write now $\mathcal{I}_r$ in terms of a conditional expectation as follows:

$$\mathcal{I}_r = \mathbb{E}_{W_3}[\mathbb{E}[|W_1(W_3 - W_1) - W_2^2|\,(W_1 - 3(W_3 - W_1))^r|W_3 = t]]$$

and note that

$$\mathbb{E}[|W_1(W_3 - W_1) - W_2^2|\,(W_1 - 3(W_3 - W_1))^r|W_3 = t] = \mathbb{E}\left[\left|\frac{t^2}{4} - Z_1^2 - Z_2^2\right|(4Z_1 - t)^r\right]$$

where Z_1, Z_2 denote standard independent Gaussian variables.

In the case $r = 0$ we only have the chi-squared random variable $\zeta = Z_1^2 + Z_2^2$ with density

$$f_\zeta(v) = \frac{1}{2}e^{-\frac{v}{2}}, \qquad v \in \mathbb{R},$$

so that we immediately have

$$\mathbb{E}\left[\left|\frac{t^2}{4}-Z_1^2-Z_2^2\right|\right]=\mathbb{E}\left[\left|\frac{t^2}{4}-\zeta\right|\right]=-2+4e^{-\frac{t^2}{8}}+\frac{t^2}{4},$$

and, since W_3 is a centred Gaussian with density

$$f_{W_3}(t)=\frac{1}{4\sqrt{\pi}}e^{-\frac{t^2}{16}},$$

we obtain

$$\mathcal{I}_0=\frac{1}{4\sqrt{\pi}}\int_{\mathbb{R}}e^{-\frac{t^2}{16}}\left(-2+4e^{-\frac{t^2}{8}}+\frac{t^2}{4}\right)dt=\frac{2^2}{\sqrt{3}}.$$

For $r=2,4$ the proof is similar with the only difference that now we need to compute the joint density function of $\xi=Z_1$ and $\zeta=Z_1^2+Z_2^2$ that is given by

$$f_{(\xi,\zeta)}(u,v)=\frac{\partial^2}{\partial u\partial v}\mathbb{P}[Z_1<u,Z_1^2+Z_2^2<v]=\frac{1}{2\pi}\frac{\partial^2}{\partial u\partial v}\iint\limits_{\substack{Z_1<u\\ 0\le Z_1^2+Z_2^2<v}}e^{-\frac{z_1^2+z_2^2}{2}}dz_1dz_2,$$

i.e.,

$$f_{(\xi,\zeta)}(u,v)=\begin{cases}0 & u\le -\sqrt{v},\\ \frac{1}{2\pi}\frac{\partial^2}{\partial u\partial v}\int_{-\sqrt{v}}^{u}dz_1\int_{-\sqrt{v-z_1^2}}^{\sqrt{v-z_1^2}}e^{-\frac{z_1^2+z_2^2}{2}}dz_2 & -\sqrt{v}<u\le 0,\\ \frac{1}{2\pi}\frac{\partial^2}{\partial u\partial v}\int_{-\sqrt{v}}^{u}dz_1\int_{-\sqrt{v-z_1^2}}^{\sqrt{v-z_1^2}}e^{-\frac{z_1^2+z_2^2}{2}}dz_2 & 0<u<\sqrt{v},\\ 0 & u\ge\sqrt{v},\end{cases}$$

$$=\frac{1}{2\pi}\frac{e^{-\frac{v}{2}}}{\sqrt{v-u^2}}\mathbb{1}_{\{v\ge0,\ u\in(-\sqrt{v},\sqrt{v})\}},$$

so that, for $r=2,4$, we have

$$\mathbb{E}\left[\left|\frac{t^2}{4}-Z_1^2-Z_2^2\right|(4Z_1-t)^r\right]=\mathbb{E}\left[\left|\frac{t^2}{4}-\zeta\right|(4\xi-t)^r\right]=\frac{1}{2\pi}\int_0^\infty\left|\frac{t^2}{4}-v\right|e^{-\frac{v}{2}}dv\int_{-\sqrt{v}}^{\sqrt{v}}\frac{(4u-t)^r}{\sqrt{v-u^2}}du.$$

□

Appendix A. Estimates for the first four derivatives of Legendre polynomials

We start with the following lemma:

Lemma A.1. *For $\alpha+1>-1/2$ and $\alpha+\beta+1\ge-1$, we have*

$$\left(\sin\frac{\phi}{2}\right)^{\alpha+1}\left(\cos\frac{\phi}{2}\right)^{\beta}P_\ell^{(\alpha+1,\beta)}(\cos\phi)=\frac{\Gamma(\ell+\alpha+2)}{\ell!}\left(\frac{\phi}{\sin\phi}\right)^{1/2}\Big[\sum_{n=0}^{m-1}A_n(\phi)\frac{J_{\alpha+n+1}(N\phi)}{N^{\alpha+n+1}}+\sigma_m\Big]$$

where

$$N=\ell+\frac{1}{2}(\alpha+\beta+2)$$

and

$$\sigma_m=\phi^m O(N^{-m-\alpha-1})$$

the O-therm being uniform with respect to $\theta\in[0,\pi-\varepsilon]$, $\varepsilon>0$. The coefficients $A_n(\phi)$ are analytic functions in $0\le\phi\le\pi-\varepsilon$, and are $O(\phi^n)$ in that interval. In particular, $A_0(\phi)=1$ and

$$A_1(\phi)=\Big[(\alpha+1)^2-\frac{1}{4}\Big]\Big(\frac{1-\phi\cot\phi}{2\phi}\Big)-\frac{(\alpha+1)^2-\beta^2}{4}\tan\frac{\phi}{2}.$$

For a proof of Lemma A.1 see [12, Lemma 1]. We will apply Lemma A.1 with $\alpha=-1$, $\beta=0$ and $m=1,2,3$, i.e.,

$$P_{\ell+u}(\cos\phi)=\Big(\frac{\phi}{\sin\phi}\Big)^{1/2}\Big[\sum_{n=0}^{m-1}A_n(\phi)\frac{J_n((\ell+u+1/2)\phi)}{(\ell+u+1/2)^n}+\phi^m O((\ell+u+1/2)^{-m})\Big],$$

with $u=0,1,2\dots$

Lemma A.2. *The following asymptotic representation for the Bessel functions of the first kind holds:*

$$J_n(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - n\pi/2 - \pi/4) \sum_{k=0}^{\infty} (-1)^k (n, 2k)\ (2x)^{-2k}$$
$$- \left(\frac{2}{\pi x}\right)^{1/2} \sin(x - n\pi/2 - \pi/4) \sum_{k=0}^{\infty} (-1)^k (n, 2k+1)\ (2x)^{-2k-1},$$

where $\varepsilon > 0$, $|\arg x| \le \pi - \varepsilon$, $(n, 0) = 1$, *and*

$$(n, k) = \frac{(4n^2 - 1)(4n^2 - 3^2) \cdots (4n^2 - (2k-1)^2)}{2^{2k} k!}.$$

For a proof of Lemma A.2 see [15, Section 5.11].

We will use the following notation: for $n = 0, \ldots, m-1$ and $u = 0, 1, 2, \ldots$

$$p_{n,\ell+u}(\phi) = \Big(\frac{\phi}{\sin \phi}\Big)^{1/2} A_n(\phi) \frac{J_n((\ell + u + 1/2)\phi)}{(\ell + u + 1/2)^n},$$

so that we have

$$P_{\ell+u}(\cos \phi) = \sum_{n=0}^{m-1} p_{n,\ell+u}(\phi) + \phi^m O(\ell^{-m}). \tag{A.1}$$

Let

$$h_n(k) = \left(\frac{2}{\pi}\right)^{1/2} (n, k) \frac{1}{2^k}, \qquad \psi_{n,\ell+u} = (\ell + u + 1/2)\phi - n\pi/2 - \pi/4, \qquad s_{n,k}(\ell, \phi) = \frac{1}{\sqrt{\sin \phi}} \frac{A_n(\phi)}{\phi^k} \frac{1}{\ell^{k+n+1/2}}.$$

In view of Lemma A.2 we can rewrite $p_{n,\ell+u}$ as follows

$$p_{n,\ell+u}(\phi) = p_{n,r,\ell+u}(\phi) + \phi^{n-1/2} O(\ell^{-r-n-3/2}),$$

where

$$\begin{aligned} p_{n,r,\ell+u}(\phi) &= \cos \psi_{n,\ell+u} \sum_{k=0}^{\infty} (-1)^k h_n(2k) s_{n,2k}(\ell, \phi) \sum_{i=0}^{r} \sum_{j=i}^{r} \binom{-2k - n - 1/2}{j} \binom{j}{i} \frac{1}{\ell^j} u^i 2^{i-j} \\ &\quad - \sin \psi_{n,\ell+u} \sum_{k=0}^{\infty} (-1)^k h_n(2k+1) s_{n,2k+1}(\ell, \phi) \sum_{i=0}^{r} \sum_{j=i}^{r} \binom{-2k - n - 3/2}{j} \binom{j}{i} \frac{1}{\ell^j} u^i 2^{i-j} \\ &\quad + \phi^{n-1/2} O(\ell^{-r-n-3/2}); \end{aligned} \tag{A.2}$$

in particular for $u = 0$ we have

$$\begin{aligned} p_{n,\ell}(\phi) &= p_{n,r,\ell}(\phi) + \phi^{n-1/2} O(\ell^{-r-n-3/2}) \\ &= \cos \psi_{n,\ell} \sum_{k=0}^{\infty} (-1)^k h_n(2k) s_{n,2k}(\ell, \phi) \sum_{j=0}^{r} \binom{-2k - n - 1/2}{j} \frac{1}{\ell^j} \frac{1}{2^j} \\ &\quad - \sin \psi_{n,\ell} \sum_{k=0}^{\infty} (-1)^k h_n(2k+1) s_{n,2k+1}(\ell, \phi) \sum_{j=0}^{r} \binom{-2k - n - 3/2}{j} \frac{1}{\ell^j} \frac{1}{2^j} \\ &\quad + \phi^{n-1/2} O(\ell^{-r-n-3/2}). \end{aligned}$$

We will use the following recurrence relations to express the first four derivatives of Legendre polynomials in terms of $P_{\ell+u}$, for $u = 0, 1, 2, 3, 4$. We have [15, Section 4.3]:

Lemma A.3. *For* $\ell = 0, 1, 2 \ldots$

$$P'_\ell(x) = \frac{\ell + 1}{(x^2 - 1)} [x P_\ell(x) - P_{\ell+1}(x)], \tag{A.3}$$

(A.4)

$$P''_\ell(x) = \frac{\ell(\ell+1)}{(x^2-1)^2} [x^2 P_\ell(x) - 2x P_{\ell+1}(x) + P_{\ell+2}(x)] + \frac{\ell + 1}{(x^2 - 1)^2} [(1 + 2x^2) P_\ell(x) - 5x P_{\ell+1}(x) + 2P_{\ell+2}(x)]$$

$$P_\ell'''(x) = \frac{\ell+1}{(x^2-1)^3}\sum_{u=0}^{2}\ell^u\sum_{v=0}^{3}{}_3\omega_{u,v}(x)P_{\ell+v}(x), \tag{A.5}$$

where

$${}_3\omega_{2,0}(x)=-x^3,\quad {}_3\omega_{2,1}(x)=3x^2,\quad {}_3\omega_{2,2}(x)=-3x,\quad {}_3\omega_{2,3}(x)=1,$$
$${}_3\omega_{1,0}(x)=-(3x+5x^3),\quad {}_3\omega_{1,1}(x)=(3+18x^2),\quad {}_3\omega_{1,2}(x)=-18x,\quad {}_3\omega_{1,3}(x)=5,$$
$${}_3\omega_{0,0}(x)=-(9x+6x^3),\quad {}_3\omega_{0,1}(x)=(6+27x^2),\quad {}_3\omega_{0,2}(x)=-24x,\quad {}_3\omega_{0,3}(x)=6,$$

$$P_\ell''''(x) = \frac{\ell+1}{(x^2-1)^4}\sum_{u=0}^{3}\ell^u\sum_{v=0}^{4}{}_4\omega_{u,v}(x)P_{\ell+v}(x), \tag{A.6}$$

where

$${}_4\omega_{3,0}(x)=x^4,\ {}_4\omega_{3,1}(x)=-4x^3,\ {}_4\omega_{3,2}(x)=6x^2,\ {}_4\omega_{3,3}(x)=-4x,\ {}_4\omega_{3,4}(x)=1,$$
$${}_4\omega_{2,0}(x)=9x^4+6x^2,\ {}_4\omega_{2,1}(x)=-(42x^3+12x),\ {}_4\omega_{2,2}(x)=66x^2+6,\ {}_4\omega_{2,3}(x)=-42x,\ {}_4\omega_{3,4}(x)=9,$$
$${}_4\omega_{1,0}(x)=26x^4+42x^2+3,\ {}_4\omega_{1,1}(x)=-(146x^3+78x),\ {}_4\omega_{1,2}(x)=231x^2+30,\ {}_4\omega_{1,3}(x)=-134x,\ {}_4\omega_{3,4}(x)=26,$$
$${}_4\omega_{0,0}(x)=24x^4+72x^2+9,\ {}_4\omega_{0,1}(x)=-(168x^3+111x),\ {}_4\omega_{0,2}(x)=246x^2+36,\ {}_4\omega_{0,3}(x)=-132x,\ {}_4\omega_{3,4}(x)=24.$$

We state now the main result of the section:

Lemma A.4. *For any constant $C>0$, we have, uniformly for $\ell\geq 1$ and $\phi\in[C/\ell,\pi/2]$*

$$P_\ell'(\cos\phi)=\frac{\ell}{\sin^2\phi}\big[h_0(0)\ \sin\phi\sin\psi_{0,\ell}\, s_{0,0}(\ell,\phi)\big]+\phi^{-2-1/2}O(\ell^{-1/2})+O(\phi^{-1}), \tag{A.7}$$

(A.8)

$$\begin{aligned}P_\ell''(\cos\phi)&=\frac{\ell^2}{\sin^4\phi}\Big[-h_0(0)\sin^2\phi\ \cos\psi_{0,\ell}s_{0,0}(\ell,\phi)\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j}-2h_0(0)\sin\phi\ \sin\psi_{0,\ell+1}s_{0,0}(\ell,\phi)\frac{1}{\ell}\Big]\\&+\frac{\ell}{\sin^4\phi}\big[-h_0(0)\sin^2\phi\ \cos\psi_{0,\ell}\, s_{0,0}(\ell,\phi)\big]+\frac{\ell^2}{\sin^4\phi}\big[h_0(1)\sin^2\phi\ \sin\psi_{0,\ell}\, s_{0,1}(\ell,\phi)\big]\\&+\frac{\ell^2}{\sin^4\phi}[-h_1(0)\sin^2\phi\cos\psi_{1,\ell}s_{1,0}(\ell,\phi)]+\frac{\ell}{\sin^4\phi}\big[h_0(0)\sin\phi\ \sin\psi_{0,\ell-1}s_{0,0}(\ell,\phi)\big]\\&+\phi^{-4-1/2}O(\ell^{-1/2})+O(\phi^{-2}),\end{aligned}$$

$$\begin{aligned}P_\ell'''(\cos\phi)&=\frac{\ell^3}{\sin^6\phi}\big[h_0(0)\sin^3\phi\sin\psi_{0,\ell}s_{0,0}(\ell,\phi)\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j}+\sin^3\phi s_{0,2}(\ell,\phi)f_b(\phi)\\&-3h_0(0)\sin^2\phi\cos\psi_{0,\ell+1}s_{0,0}(\ell,\phi)\frac{1}{\ell}+\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}\big]\\&+\frac{\ell^2}{\sin^6\phi}\big[h_0(0)\sin^3\phi\sin\psi_{0,\ell}s_{0,0}(\ell,\phi)\big]\\&+\frac{\ell^3}{\sin^6\phi}\big[h_0(1)\sin^3\phi\cos\psi_{0,\ell}s_{0,1}(\ell,\phi)+\sin^2\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\big]\\&+\frac{\ell^3}{\sin^6\phi}\big[-h_1(0)\sin^3\phi\sin\psi_{1,\ell}s_{1,0}(\ell,\phi)\big]\\&+\frac{\ell^2}{\sin^6\phi}\big[\frac{1}{2}h_0(0)\sin^2\phi(\cos\psi_{0,\ell+1}+5\cos\psi_{0,\ell-1})s_{0,0}(\ell,\phi)+\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\big]\\&+\frac{\ell^2}{\sin^6\phi}\big[\sin^2\phi s_{0,1}(\ell,\phi)f_b(\phi)\big]+\frac{\ell}{\sin\phi}\big[\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\big]+\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell),\end{aligned} \tag{A.9}$$

(A.10)

$$P_\ell''''(\cos\phi)=\frac{\ell^4}{\sin^8\phi}\Big[\sin^4\phi\cos\psi_{0,\ell}\sum_{k=0}^{1}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi)\sum_{j=0}^{1}\binom{-2k-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j}$$

$$
\begin{aligned}
&+\sin^4\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}+4\sin^3\phi\,\cos\psi_{1,\ell+1}\sum_{k=0}^{1}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi)\frac{1}{\ell}\\
&+\sin^3\phi\,s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}+\sin^2\phi s_{0,0}(\ell,\phi)f_b(\phi)\sum_{j=2}^{3}\frac{1}{\ell^j}+\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell^3}\Big]\\
&+\frac{\ell^3}{\sin^8\phi}\Big[\sin^4\phi\cos\psi_{0,\ell}\sum_{k=0}^{1}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi)+\sin^4\phi s_{0,0}(\ell,\phi)\frac{1}{\ell}\\
&+\sin^3\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell}+\sin^2\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}\Big]\\
&+\frac{\ell^4}{\sin^8\phi}\Big[-\sin^4\phi\sin\psi_{0,\ell}\sum_{k=0}^{1}(-1)^k h_0(2k+1)s_{0,2k+1}(\ell,\phi)-\sin^4\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\\
&+\sin^3\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell}+\sin^3\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}+\sin^2\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}\Big]\\
&+\frac{\ell^3}{\sin^8\phi}\Big[\sin^4\phi s_{0,1}(\ell,\phi)f_b(\phi)+\sin^3\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\Big]\\
&+\frac{\ell^4}{\sin^8\phi}\Big[h_1(0)\sin^4\phi\cos\psi_{1,\ell}s_{1,0}(\ell,\phi)\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j}+\sin^3\phi s_{1,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\Big]\\
&+\frac{\ell^3}{\sin^8\phi}\Big[\sin^4\phi s_{1,0}(\ell,\phi)f_b(\phi)\Big]+\frac{\ell^4}{\sin^8\phi}\Big[\sin^4\phi s_{1,1}(\ell,\phi)f_b(\phi)\Big]+\frac{\ell^4}{\sin^8\phi}\Big[\sin^4\phi s_{2,0}(\ell,\phi)f_b(\phi)\Big]\\
&+\frac{\ell^3}{\sin^8\phi}\Big[-\frac{3}{2}h_0(0)\sin^3\phi(\sin\psi_{0,\ell+1}+3\sin\psi_{0,\ell-1})s_{0,0}(\ell,\phi)\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{2^j}\frac{1}{\ell^j}\\
&+\sin^3\phi s_{0,2}(\ell,\phi)f_b(\phi)+\sin^2\phi s_{0,0}(\ell,\phi)f_b(\phi)\sum_{j=1}^{2}\frac{1}{\ell^j}+\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell^2}\Big]\\
&+\frac{\ell^2}{\sin^8\phi}\Big[-\frac{3}{2}h_0(0)\sin^3\phi(\sin\psi_{0,\ell+1}+3\sin\psi_{0,\ell-1})s_{0,0}(\ell,\phi)+\sin^2\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\Big]\\
&+\frac{\ell^3}{\sin^8\phi}\Big[\sin^3\phi s_{0,1}(\ell,\phi)f_b(\phi)\sum_{j=0}^{1}\frac{1}{\ell^j}+\sin^2\phi s_{0,1}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\Big]+\frac{\ell^2}{\sin^8\phi}\Big[\sin^3\phi s_{0,1}(\ell,\phi)f_b(\phi)\Big]\\
&+\frac{\ell^3}{\sin^8\phi}\Big[\sin^3\phi s_{1,0}(\ell,\phi)f_b(\phi)\Big]+\frac{\ell^2}{\sin^8\phi}\Big[\sin^2\phi s_{0,0}(\ell,\phi)f_b(\phi)\sum_{j=0}^{1}\frac{1}{\ell^j}+\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\frac{1}{\ell}\Big]\\
&+\frac{\ell}{\sin^8\phi}\Big[\sin^2\phi s_{0,0}(\ell,\phi)f_b(\phi)\Big]+\frac{\ell^2}{\sin^8\phi}\Big[\sin^2\phi s_{0,1}(\ell,\phi)f_b(\phi)\Big]+\frac{\ell}{\sin^8\phi}\Big[\sin\phi s_{0,0}(\ell,\phi)f_b(\phi)\Big]\\
&+\phi^{-8-1/2}O(\ell^{-1/2})+\phi^{-5}O(\ell),
\end{aligned}
$$

where f_b denotes a bounded function on $(0,\pi/2]$.

Proof. First derivative

To prove (A.7) we start from (A.3) and we rewrite P_ℓ and $P_{\ell+1}$ as in (A.1) with $m=1$, i.e.,

$$
\begin{aligned}
P'_\ell(\cos\phi) &= \frac{\ell+1}{\sin^2\phi}\big[\cos\phi\,P_\ell(\cos\phi)-P_{\ell+1}(\cos\phi)\big]\\
&= \frac{\ell+1}{\sin^2\phi}\big[\cos\phi\,p_{0,\ell}(\phi)-p_{0,\ell+1}(\phi)\big]\\
&\quad+O(\phi^{-1}).
\end{aligned}
\tag{A.11}
$$

We rewrite now (A.11) in the form (A.2) with $r=0$, i.e.

$\cos\phi\,p_{0,\ell}(\phi)-p_{0,\ell+1}(\phi)$

$$
\begin{aligned}
&= \cos\phi\, p_{0,0,\ell}(\phi) - p_{0,0,\ell+1}(\phi) + \phi^{-1/2}O(\ell^{-3/2})\\
&= \cos\phi \Big[\cos\psi_{0,\ell} \sum_{k=0}^{\infty}(-1)^k h_0(2k) s_{0,2k}(\ell,\phi) - \sin\psi_{0,\ell} \sum_{k=0}^{\infty}(-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi)\Big]\\
&- \Big[\cos\psi_{0,\ell+1} \sum_{k=0}^{\infty}(-1)^k h_0(2k) s_{0,2k}(\ell,\phi) - \sin\psi_{0,\ell+1} \sum_{k=0}^{\infty}(-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi)\Big]\\
&+ \phi^{-1/2}O(\ell^{-3/2});
\end{aligned}
$$

now note that

$$
\begin{cases} \cos\phi\cos\psi_{0,\ell} - \cos\psi_{0,\ell+1} = \sin\phi\sin\psi_{0,\ell}, \\ -\cos\phi\sin\psi_{0,\ell} + \sin\psi_{0,\ell+1} = \sin\phi\cos\psi_{0,\ell}, \end{cases} \tag{A.12}
$$

(A.12) implies that

$$
\begin{aligned}
\cos\phi\, p_{0,\ell}(\phi) - p_{0,\ell+1}(\phi) &= \sin\phi\sin\psi_{0,\ell} \sum_{k=0}^{\infty}(-1)^k h_0(2k) s_{0,2k}(\ell,\phi)\\
&+ \sin\phi\cos\psi_{0,\ell} \sum_{k=0}^{\infty}(-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi)\\
&+ \phi^{-1/2}O(\ell^{-3/2})
\end{aligned}
$$

and we obtain the estimate in the statement, in fact (A.11) is such that

$$
\begin{aligned}
&\frac{\ell+1}{\sin^2\phi}[\cos\phi\, p_{0,\ell}(\phi) - p_{0,\ell+1}(\phi)]\\
&= \frac{\ell}{\sin^2\phi}\sin\phi\sin\psi_{0,\ell} h_0(0) s_{0,0}(\ell,\phi) + \phi^{-2-1/2}O(\ell^{-1/2}) + O(\phi^{-1}). \qquad \text{(A.13)}
\end{aligned}
$$

Second derivative

We prove now (A.8). We start from (A.4) and we rewrite $P_{\ell+u}$ for $u = 0,1,2$ in the form (A.1) with $m = 2$, i.e.,

$$
\begin{aligned}
P''_\ell(\cos\phi) &= \frac{\ell(\ell+1)}{\sin^4\phi}\big[\cos^2\phi P_\ell(\cos\phi) - 2\cos\phi P_{\ell+1}(\cos\phi) + P_{\ell+2}(\cos\phi)\big]\\
&+ \frac{\ell+1}{\sin^4\phi}\big[(1+2\cos^2\phi)P_\ell(\cos\phi) - 5\cos\phi P_{\ell+1}(\cos\phi) + 2P_{\ell+2}(\cos\phi)\big]\\
&= \frac{\ell(\ell+1)}{\sin^4\phi}\big[\cos^2\phi\, p_{0,\ell}(\phi) - 2\cos\phi\, p_{0,\ell+1}(\phi) + p_{0,\ell+2}(\phi)\big] \qquad \text{(A.14)}\\
&+ \frac{\ell(\ell+1)}{\sin^4\phi}\big[\cos^2\phi\, p_{1,\ell}(\phi) - 2\cos\phi\, p_{1,\ell+1}(\phi) + p_{1,\ell+2}(\phi)\big] \qquad \text{(A.15)}\\
&+ \frac{\ell+1}{\sin^4\phi}\big[(1+2\cos^2\phi)\, p_{0,\ell}(\phi) - 5\cos\phi\, p_{0,\ell+1}(\phi) + 2p_{0,\ell+2}(\phi)\big] \qquad \text{(A.16)}\\
&+ \frac{\ell+1}{\sin^4\phi}\big[(1+2\cos^2\phi)\, p_{1,\ell}(\phi) - 5\cos\phi\, p_{1,\ell+1}(\phi) + 2p_{1,\ell+2}(\phi)\big] + O(\phi^{-2}). \qquad \text{(A.17)}
\end{aligned}
$$

We first consider the terms (A.14) and (A.15); we rewrite them in the form (A.2) with $r = 1$. For (A.14) we obtain:

$$
\begin{aligned}
&\cos^2\phi\, p_{0,\ell}(\phi) - 2\cos\phi\, p_{0,\ell+1}(\phi) + p_{0,\ell+2}(\phi)\\
&= \cos^2\phi\, p_{0,1,\ell}(\phi) - 2\cos\phi\, p_{0,1,\ell+1}(\phi) + p_{0,1,\ell+2}(\phi) + \phi^{-1/2}O(\ell^{-2-1/2})\\
&= \cos^2\phi\Big[\cos\psi_{0,\ell} \sum_{k=0}^{\infty}(-1)^k h_0(2k) s_{0,2k}(\ell,\phi) \sum_{j=0}^{1} \binom{-2k-1/2}{j} \frac{1}{\ell^j}\frac{1}{2^j}\\
&\quad - \sin\psi_{0,\ell} \sum_{k=0}^{\infty}(-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi) \sum_{j=0}^{1} \binom{-2k-3/2}{j} \frac{1}{\ell^j}\frac{1}{2^j}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- 2\cos\phi \Big[\cos\psi_{0,\ell+1} \sum_{k=0}^{\infty} (-1)^k h_0(2k) s_{0,2k}(\ell,\phi) \sum_{i=0}^{1}\sum_{j=i}^{1} \binom{-2k-1/2}{j} \frac{1}{\ell^j} \binom{j}{i} 2^{i-j} \\
&- \sin\psi_{0,\ell+1} \sum_{k=0}^{\infty} (-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi) \sum_{i=0}^{1}\sum_{j=i}^{1} \binom{-2k-3/2}{j} \frac{1}{\ell^j} \binom{j}{i} 2^{i-j} \Big] \\
&+ \Big[\cos\psi_{0,\ell+2} \sum_{k=0}^{\infty} (-1)^k h_0(2k) s_{0,2k}(\ell,\phi) \sum_{i=0}^{1}\sum_{j=i}^{1} \binom{-2k-1/2}{j} \frac{1}{\ell^j} \binom{j}{i} 2^{i} 2^{i-j} \\
&- \sin\psi_{0,\ell+2} \sum_{k=0}^{\infty} (-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi) \sum_{i=0}^{1}\sum_{j=i}^{1} \binom{-2k-3/2}{j} \frac{1}{\ell^j} \binom{j}{i} 2^{i} 2^{i-j} \Big] \\
&+ \phi^{-1/2} O(\ell^{-2-1/2}),
\end{aligned}
$$

now, since

$$
\left\{
\begin{array}{c}
\cos^2\phi \, \cos\psi_{0,\ell} - 2\cos\phi\cos\psi_{0,\ell+1} + \cos\psi_{0,\ell+2} = -\sin^2\phi \, \cos\psi_{0,\ell}, \\
-\cos^2\phi \, \sin\psi_{0,\ell} + 2\cos\phi\sin\psi_{0,\ell+1} - \sin\psi_{0,\ell+2} = \sin^2\phi \, \sin\psi_{0,\ell}, \\
-2\cos\phi\cos\psi_{0,\ell+1} + 2\cos\psi_{0,\ell+2} = -2\sin\phi \, \sin\psi_{0,\ell+1}, \\
2\cos\phi\sin\psi_{0,\ell+1} - 2\sin\psi_{0,\ell+2} = -2\sin\phi \, \cos\psi_{0,\ell+1},
\end{array}
\right. \tag{A.18}
$$

in view of (A.18), we obtain

$$
\begin{aligned}
&\cos^2\phi \, p_{0,\ell}(\phi) - 2\cos\phi \, p_{0,\ell+1}(\phi) + p_{0,\ell+2}(\phi) \\
&= -\sin^2\phi \, \cos\psi_{0,\ell} \sum_{k=0}^{\infty} (-1)^k h_0(2k) s_{0,2k}(\ell,\phi) \sum_{j=0}^{1} \binom{-2k-1/2}{j} \frac{1}{\ell^j} \frac{1}{2^j} \\
&\quad - 2\sin\phi \, \sin\psi_{0,\ell+1} \sum_{k=0}^{\infty} (-1)^k h_0(2k) s_{0,2k}(\ell,\phi) \binom{-2k-1/2}{1} \frac{1}{\ell} \\
&\quad + \sin^2\phi \, \sin\psi_{0,\ell} \sum_{k=0}^{\infty} (-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi) \sum_{j=0}^{1} \binom{-2k-3/2}{j} \frac{1}{\ell^j} \frac{1}{2^j} \\
&\quad - 2\sin\phi \, \cos\psi_{0,\ell+1} \sum_{k=0}^{\infty} (-1)^k h_0(2k+1) s_{0,2k+1}(\ell,\phi) \binom{-2k-3/2}{1} \frac{1}{\ell} \\
&\quad + \phi^{-1/2} O(\ell^{-2-1/2})
\end{aligned}
$$

and then the term (A.14) has the following asymptotic behaviour:

$$
\begin{aligned}
&\frac{\ell(\ell+1)}{\sin^4\phi} [\cos^2\phi \, p_{0,\ell}(\phi) - 2\cos\phi \, p_{0,\ell+1}(\phi) + p_{0,\ell+2}(\phi)] \\
&= \frac{\ell^2}{\sin^4\phi} \Big[-\sin^2\phi \, \cos\psi_{0,\ell} h_0(0) s_{0,0}(\ell,\phi) \sum_{j=0}^{1} \binom{-1/2}{j} \frac{1}{\ell^j} \frac{1}{2^j} - 2\sin\phi \, \sin\psi_{0,\ell+1} h_0(0) s_{0,0}(\ell,\phi) \frac{1}{\ell} \Big] \\
&\quad + \frac{\ell}{\sin^4\phi} \Big[-\sin^2\phi \, \cos\psi_{0,\ell} \, h_0(0) s_{0,0}(\ell,\phi) \Big] \\
&\quad + \frac{\ell^2}{\sin^4\phi} \Big[\sin^2\phi \, \sin\psi_{0,\ell} \, h_0(1) s_{0,1}(\ell,\phi) \Big] + \phi^{-4-1/2} O(\ell^{-1/2}) + O(\phi^{-2}).
\end{aligned} \tag{A.19}
$$

For (A.15) we get

$$
\begin{aligned}
&\cos^2\phi \, p_{1,\ell}(\phi) - 2\cos\phi \, p_{1,\ell+1}(\phi) + p_{1,\ell+2}(\phi) \\
&= \cos^2\phi \, p_{1,1,\ell}(\phi) - 2\cos\phi \, p_{1,1,\ell+1}(\phi) + p_{1,1,\ell+2}(\phi) + \phi^{1/2} O(\ell^{-3-1/2});
\end{aligned}
$$

now, note that

$$
\left\{
\begin{array}{c}
\cos^2\phi \, \cos\psi_{1,\ell} - 2\cos\phi\cos\psi_{1,\ell+1} + \cos\psi_{1,\ell+2} = -\sin^2\phi \, \cos\psi_{1,\ell}, \\
-\cos^2\phi \, \sin\psi_{1,\ell} + 2\cos\phi\sin\psi_{1,\ell+1} - \sin\psi_{1,\ell+2} = \sin^2\phi \, \sin\psi_{1,\ell}, \\
-2\cos\phi\cos\psi_{1,\ell+1} + 2\cos\psi_{1,\ell+2} = \sin\phi \, f_b(\phi), \\
2\cos\phi\sin\psi_{1,\ell+1} - 2\sin\psi_{1,\ell+2} = \sin\phi \, f_b(\phi),
\end{array}
\right. \tag{A.20}
$$

where f_b is a bounded function of $\phi \in (0, \pi/2]$. Exploiting as before the trigonometric relations in (A.20) we obtain that (A.15) is such that

$$\begin{aligned}&\frac{\ell(\ell+1)}{\sin^4\phi}[\cos^2\phi\, p_{1,\ell}(\phi) - 2\cos\phi\, p_{1,\ell+1}(\phi) + p_{1,\ell+2}(\phi)]\\ &= \frac{\ell^2}{\sin^4\phi}[-\sin^2\phi\cos\psi_{1,\ell}h_1(0)s_{1,0}(\ell,\phi)] + \phi^{-4-1/2}O(\ell^{-1/2}) + O(\phi^{-2}).\end{aligned} \tag{A.21}$$

We apply the same procedure to obtain the asymptotic behaviour of the terms (A.16) and (A.17). We rewrite them in the form (A.2) but in this case it is enough to choose $r = 0$. For (A.16) we get

$$\begin{aligned}&(1+2\cos^2\phi)\, p_{0,\ell}(\phi) - 5\cos\phi\, p_{0,\ell+1}(\phi) + 2p_{0,\ell+2}(\phi)\\ &= (1+2\cos^2\phi)\; p_{0,0,\ell}(\phi) - 5\cos\phi\, p_{0,0,\ell+1}(\phi) + 2p_{0,0,\ell+2}(\phi) + \phi^{-1/2}O(\ell^{-1-1/2})\\ &= (1+2\cos^2\phi)\Big[\cos\psi_{0,\ell}\sum_{k=0}^{\infty}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi) - \sin\psi_{0,\ell}\sum_{k=0}^{\infty}(-1)^k h_0(2k+1)s_{0,2k+1}(\ell,\phi)\Big]\\ &\quad - 5\cos\phi\Big[\cos\psi_{0,\ell+1}\sum_{k=0}^{\infty}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi) - \sin\psi_{0,\ell+1}\sum_{k=0}^{\infty}(-1)^k h_0(2k+1)s_{0,2k+1}(\ell,\phi)\Big]\\ &\quad + 2\Big[\cos\psi_{0,\ell+2}\sum_{k=0}^{\infty}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi) - \sin\psi_{0,\ell+2}\sum_{k=0}^{\infty}(-1)^k h_0(2k+1)s_{0,2k+1}(\ell,\phi)\Big]\\ &\quad + \phi^{-1/2}O(\ell^{-1-1/2}),\end{aligned}$$

and since

$$\begin{cases}(1+2\cos^2\phi)\;\cos\psi_{0,\ell} - 5\cos\phi\cos\psi_{0,\ell+1} + 2\cos\psi_{0,\ell+2} = \sin\phi\;\sin\psi_{0,\ell-1},\\ -(1+2\cos^2\phi)\;\sin\psi_{0,\ell} + 5\cos\phi\sin\psi_{0,\ell+1} - 2\sin\psi_{0,\ell+2} = \sin\phi\;\cos\psi_{0,\ell-1},\end{cases} \tag{A.22}$$

in view of (A.22) we obtain that

$$\begin{aligned}&(1+2\cos^2\phi)\, p_{0,\ell}(\phi) - 5\cos\phi\, p_{0,\ell+1}(\phi) + 2p_{0,\ell+2}(\phi)\\ &= \sin\phi\;\sin\psi_{0,\ell-1}\sum_{k=0}^{\infty}(-1)^k h_0(2k)s_{0,2k}(\ell,\phi) + \sin\phi\;\cos\psi_{0,\ell-1}\sum_{k=0}^{\infty}(-1)^k h_0(2k+1)s_{0,2k+1}(\ell,\phi)\\ &\quad + \phi^{-1/2}O(\ell^{-1-1/2}),\end{aligned}$$

and then for (A.16) we obtain:

$$\begin{aligned}&\frac{\ell+1}{\sin^4\phi}\big[(1+2\cos^2\phi)\, p_{0,\ell}(\phi) - 5\cos\phi\, p_{0,\ell+1}(\phi) + 2p_{0,\ell+2}(\phi)\big]\\ &= \frac{\ell}{\sin^4\phi}\big[\sin\phi\;\sin\psi_{0,\ell-1}h_0(0)s_{0,0}(\ell,\phi)\big] + \phi^{-4-1/2}O(\ell^{-1/2}) + O(\phi^{-2}).\end{aligned} \tag{A.23}$$

Finally for (A.17) one has

$$\begin{aligned}&(1+2\cos^2\phi)\, p_{1,\ell}(\phi) - 5\cos\phi\, p_{1,\ell+1}(\phi) + 2p_{1,\ell+2}(\phi)\\ &= (1+2\cos^2\phi)\; p_{1,0,\ell}(\phi) - 5\cos\phi\, p_{1,0,\ell+1}(\phi) + 2p_{1,0,\ell+2}(\phi) + \phi^{1/2}O(\ell^{-2-1/2}),\end{aligned} \tag{A.24}$$

and since

$$\begin{cases}(1+2\cos^2\phi)\;\cos\psi_{1,\ell} - 5\cos\phi\cos\psi_{1,\ell+1} + 2\cos\psi_{1,\ell+2} = \sin\phi f_b(\phi),\\ -(1+2\cos^2\phi)\;\sin\psi_{1,\ell} + 5\cos\phi\sin\psi_{1,\ell+1} - 2\sin\psi_{1,\ell+2} = \sin\phi f_b(\phi),\end{cases} \tag{A.25}$$

exploiting, as before, (A.25) in (A.24) one obtain that (A.17) is of order

$$\begin{aligned}&\frac{\ell+1}{\sin^4\phi}\big[(1+2\cos^2\phi)\, p_{1,\ell}(\phi) - 5\cos\phi\, p_{1,\ell+1}(\phi) + 2p_{1,\ell+2}(\phi)\big]\\ &= \phi^{-4-1/2}O(\ell^{-1/2}) + O(\phi^{-2}).\end{aligned} \tag{A.26}$$

By summing up the terms in (A.19), (A.21), (A.23) and (A.26) we obtain the asymptotic expression (A.8) in the statement. The main steps in the proof of (A.7) and (A.8) are summarised in Table 1.

Third derivative

$P'_\ell(\cos\phi)$	$P''_\ell(\cos\phi)$
fix $m = 1$ in (A.1)	fix $m = 2$ in (A.1)
(A.11) with $r = 0$ and (A.12) $\Longrightarrow$ (A.13)	(A.14) with $r = 1$ and (A.18) $\Longrightarrow$ (A.19)
	(A.15) with $r = 1$ and (A.20) $\Longrightarrow$ (A.21)
	(A.16) with $r = 0$ and (A.22) $\Longrightarrow$ (A.23)
	(A.17) with $r = 0$ and (A.25) $\Longrightarrow$ (A.26)

TABLE 1.

We move now to the proof of the asymptotic behaviour of the third and fourth derivative given in formula (A.9) and formula (A.10) of the statement. For brevity sake we do not give, as before, all details of the proof; the main steps of the proof are summarised in Table 2 and the related formulas written below.

To prove (A.9) we start form (A.5) and we write $P_{\ell+u}$, $u = 0, 1, 2, 3$ in the form (A.1) with $m = 2$:

$$P'''_\ell(\cos\phi) = \frac{\ell+1}{\sin^6\phi}\sum_{u=0}^{2}\ell^u\sum_{n=0}^{1}\sum_{v=0}^{3}{}_3\omega_{u,v}(\cos\phi)p_{n,\ell+v}(\phi) + \phi^{-4}O(\ell). \tag{A.27}$$

Now, as described in Table 2, one can rewrite the $p_{n,\ell+u}$'s in the form (A.2) with the value of the parameter r chosen so that the error term is small enough (see Table 2). By exploiting the simplifications produced by the following trigonometric relations:

$$\begin{cases} \sum_{v=0}^{3} {}_3\omega_{2,v}(\cos\phi)\cos\psi_{0,\ell+v} = \sin^3\phi\sin\psi_{0,\ell}, \\ \sum_{v=1}^{3} {}_3\omega_{2,v}(\cos\phi)\, v\cos\psi_{0,\ell+v} = -3\sin^2\phi\cos\psi_{0,\ell+1}, \\ \sum_{v=1}^{3} {}_3\omega_{2,v}(\cos\phi)\, v^2\cos\psi_{0,\ell+v} = \sin\phi\, f_b(\phi), \\ -\sum_{v=0}^{3} {}_3\omega_{2,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin^3\phi\cos\psi_{0,\ell}, \\ -\sum_{v=1}^{3} {}_3\omega_{2,v}(\cos\phi)\, v^i\sin\psi_{0,\ell+v} = \sin^{3-i}\phi\, f_b(\phi), \quad i = 1, 2, \end{cases} \tag{A.28}$$

$$\begin{cases} \sum_{v=0}^{3} {}_3\omega_{2,v}(\cos\phi)\cos\psi_{1,\ell+v} = -\sin^3\phi\sin\psi_{1,\ell}, \\ \sum_{v=1}^{3} {}_3\omega_{2,v}(\cos\phi)\, v^i\cos\psi_{1,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i = 1, 2, \\ -\sum_{v=0}^{3} {}_3\omega_{2,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin^3\phi f_b(\phi), \\ -\sum_{v=1}^{3} {}_3\omega_{2,v}(\cos\phi)\, v^i\sin\psi_{1,\ell+v} = \sin^{3-i}\phi\, f_b(\phi), \quad i = 1, 2, \end{cases} \tag{A.29}$$

$$\begin{cases} \sum_{v=0}^{3} {}_3\omega_{1,v}(\cos\phi)\cos\psi_{0,\ell+v} = 1/2\sin^2\phi(\cos\psi_{0,\ell+1} + 5\cos\psi_{0,\ell-1}), \\ \sum_{v=1}^{3} {}_3\omega_{1,v}(\cos\phi)v\cos\psi_{0,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{v=0}^{3} {}_3\omega_{1,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin^2\phi f_b(\phi), \\ -\sum_{v=1}^{3} {}_3\omega_{1,v}(\cos\phi)v\sin\psi_{0,\ell+v} = \sin\phi f_b(\phi), \end{cases} \tag{A.30}$$

$$\begin{cases} \sum_{v=0}^{3} {}_3\omega_{1,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin^2\phi f_b(\phi), \\ \sum_{v=1}^{3} {}_3\omega_{1,v}(\cos\phi)v\cos\psi_{1,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{v=0}^{3} {}_3\omega_{1,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin^2\phi f_b(\phi), \\ -\sum_{v=1}^{3} {}_3\omega_{1,v}(\cos\phi)v\sin\psi_{1,\ell+v} = \sin\phi f_b(\phi), \end{cases} \tag{A.31}$$

$$\begin{cases} \sum_{v=0}^{3} {}_3\omega_{0,v}(\cos\phi)\cos\psi_{0,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{v=0}^{3} {}_3\omega_{0,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin\phi f_b(\phi), \end{cases} \tag{A.32}$$

$$\begin{cases} \sum_{v=0}^{3} {}_3\omega_{0,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{v=0}^{3} {}_3\omega_{0,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin\phi f_b(\phi), \end{cases} \tag{A.33}$$

we can rewrite the terms (A.27) as follows:

$$\frac{\ell+1}{\sin^6\phi}\ell^2\sum_{v=0}^{3}{}_3\omega_{2,v}(\cos\phi)p_{0,\ell+v}(\phi) = \frac{\ell^3}{\sin^6\phi}\big[\sin^3\phi\sin\psi_{0,\ell}h_0(0)s_{0,0}(\ell,\phi)\sum_{j=0}^{1}\binom{-1/2}{j}\frac{1}{\ell^j}\frac{1}{2^j} \tag{A.34}$$

$$+\sin^3\phi\sin\psi_{0,\ell}h_0(2)s_{0,2}(\ell,\phi)-3\sin^2\phi\cos\psi_{0,\ell+1}h_0(0)s_{0,0}(\ell,\phi)\frac{1}{\ell}$$
$$+f_b(\phi)\sin\phi s_{0,0}(\ell,\phi)\frac{1}{\ell^2}\big]+\frac{\ell^2}{\sin^6\phi}\big[\sin^3\phi\sin\psi_{0,\ell}h_0(0)s_{0,0}(\ell,\phi)\big]$$
$$+\frac{\ell^3}{\sin^6\phi}\big[\sin^3\phi\cos\psi_{0,\ell}h_0(1)s_{0,1}(\ell,\phi)+\sin^2\phi f_b(\phi)s_{0,1}(\ell,\phi)\frac{1}{\ell}\big]$$
$$+\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell),$$

(A.35)
$$\frac{\ell+1}{\sin^6\phi}\ell^2\sum_{v=0}^{3}{}_3\omega_{2,v}(\cos\phi)p_{1,\ell+v}(\phi)=\frac{\ell^3}{\sin^6\phi}\big[-\sin^3\phi\sin\psi_{1,\ell}h_1(0)s_{1,0}(\ell,\phi)\big]$$
$$+\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell),$$

(A.36)
$$\frac{\ell+1}{\sin^6\phi}\ell\sum_{v=0}^{3}{}_3\omega_{1,v}(\cos\phi)p_{0,\ell+v}(\phi)=\frac{\ell^2}{\sin^6\phi}\big[\frac{1}{2}\sin^2\phi(\cos\psi_{0,\ell+1}+5\cos\psi_{0,\ell-1})h_0(0)s_{0,0}(\ell,\phi)+\sin\phi f_b(\phi)s_{0,0}(\ell,\phi)\frac{1}{\ell}\big]$$
$$+\frac{\ell^2}{\sin^6\phi}\big[-\frac{1}{2}\sin^2\phi(\sin\psi_{0,\ell+1}+5\sin\psi_{0,\ell-1})h_0(1)s_{0,1}(\ell,\phi)\big]$$
$$+\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell),$$

(A.37)
$$\frac{\ell+1}{\sin^6\phi}\ell\sum_{v=0}^{3}{}_3\omega_{1,v}(\cos\phi)p_{1,\ell+v}(\phi)=\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell),$$

(A.38)
$$\frac{\ell+1}{\sin^6\phi}\sum_{v=0}^{3}{}_3\omega_{0,v}(\cos\phi)p_{0,\ell+v}(\phi)=\frac{\ell}{\sin\phi}\big[\sin\phi f_b(\phi)s_{0,0}(\ell,\phi)\big]+\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell),$$

(A.39)
$$\frac{\ell+1}{\sin^6\phi}\sum_{v=0}^{3}{}_3\omega_{0,v}(\cos\phi)p_{1,\ell+v}(\phi)=\phi^{-6-1/2}O(\ell^{-1/2})+\phi^{-4}O(\ell).$$

Formula (A.9) in the statement is obtained by summing up the terms (A.34)-(A.39).

Fourth derivative

The proof of formula (A.10) goes along the same lines. In view of (A.6) and by applying (A.1), where we fix $m=3$, we have:

(A.40)
$$P_\ell''''(\cos\phi)=\frac{\ell+1}{\sin^8\phi}\sum_{u=0}^{3}\ell^u\sum_{n=0}^{2}\sum_{v=0}^{4}{}_4\omega_{u,v}(\cos\phi)p_{n,\ell+v}(\phi)+\phi^{-5}O(\ell).$$

We can simplify each term in (A.40) by observing that:

(A.41)
$$\begin{cases}\sum_{v=0}^4 {}_4\omega_{3,v}(\cos\phi)\cos\psi_{0,\ell+v}=\sin^4\phi\cos\psi_{0,\ell},\\ \sum_{u=1}^4 u\,{}_4\omega_{3,v}(\cos\phi)\cos\psi_{0,\ell+v}=4\sin^3\phi\cos\psi_{1,\ell+1},\\ \sum_{u=1}^4 u^i\,{}_4\omega_{3,v}(\cos\phi)\cos\psi_{0,\ell+v}=\sin^{4-i}\phi f_b(\phi),\qquad i=2,3,\\ -\sum_{u=0}^4 {}_4\omega_{3,v}(\cos\phi)\sin\psi_{0,\ell+v}=-\sin^4\phi\sin\psi_{0,\ell},\\ -\sum_{u=1}^4 u^i\,{}_4\omega_{3,v}(\cos\phi)\sin\psi_{0,\ell+v}=\sin^{4-i}\phi f_b(\phi),\qquad i=1,2,3,\end{cases}$$

(A.42)
$$\begin{cases}\sum_{u=0}^4 {}_4\omega_{3,v}(\cos\phi)\cos\psi_{1,\ell+v}=\sin^4\phi\cos\psi_{1,\ell},\\ \sum_{u=1}^4 u^i\,{}_4\omega_{3,v}(\cos\phi)\cos\psi_{1,\ell+v}=\sin^{4-i}\phi f_b(\phi),\qquad i=1,2,3,\\ -\sum_{u=0}^4 {}_4\omega_{3,v}(\cos\phi)\sin\psi_{1,\ell+v}=\sin^4\phi f_b(\phi),\\ -\sum_{u=1}^4 u^i\,{}_4\omega_{3,v}(\cos\phi)\sin\psi_{1,\ell+v}=\sin^{4-i}\phi f_b(\phi),\qquad i=1,2,3,\end{cases}$$

$P_\ell'''(\cos\phi)$				$P_\ell''''(\cos\phi)$	
fix $m=2$ in (A.1)				fix $m=3$ in (A.1)	
(A.27) with $u=2, n=0$ $r=2$	and (A.28)	$\Longrightarrow$	(A.34)	(A.40) $u=3, n=0$ $r=3$	and (A.41)
(A.27) with $u=2, n=1$ $r=2$	and (A.29)	$\Longrightarrow$	(A.35)	(A.40) $u=3, n=1$ $r=3$	and (A.42)
(A.27) with $u=1, n=0$ $r=1$	and (A.30)	$\Longrightarrow$	(A.36)	(A.40) $u=3, n=2$ $r=3$	and (A.43)
(A.27) with $u=1, n=1$ $r=1$	and (A.31)	$\Longrightarrow$	(A.37)	(A.40) $u=2, n=0$ $r=2$	and (A.44)
(A.27) with $u=0, n=0$ $r=0$	and (A.32)	$\Longrightarrow$	(A.38)	(A.40) $u=2, n=1$ $r=2$	and (A.45)
(A.27) with $u=0, n=1$ $r=0$	and (A.33)	$\Longrightarrow$	(A.39)	(A.40) $u=2, n=2$ $r=2$	and (A.46)
				(A.40) $u=1, n=0$ $r=1$	and (A.47)
				(A.40) $u=1, n=1$ $r=1$	and (A.48)
				(A.40) $u=1, n=2$ $r=1$	and (A.49)
				(A.40) $u=0, n=0$ $r=0$	and (A.50)
				(A.40) $u=0, n=1$ $r=0$	and (A.51)
				(A.40) $u=0, n=2$ $r=0$	and (A.52)

TABLE 2.

$$\text{(A.43)}\quad \begin{cases} \sum_{u=0}^4 {}_4\omega_{3,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin^4\phi f_b(\phi), \\ \sum_{u=1}^4 u^i\,{}_4\omega_{3,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin^{4-i}\phi f_b(\phi), \quad i=1,2,3, \\ -\sum_{u=0}^4 {}_4\omega_{3,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin^4\phi f_b(\phi), \\ -\sum_{u=1}^4 u^i\,{}_4\omega_{3,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin^{4-i}\phi f_b(\phi), \quad i=1,2,3, \end{cases}$$

$$\text{(A.44)}\quad \begin{cases} \sum_{u=0}^4 {}_4\omega_{2,v}(\cos\phi)\cos\psi_{0,\ell+v} = -3/2\sin^3\phi(\sin\psi_{0,\ell+1}+3\sin\psi_{0,\ell-1}), \\ \sum_{u=1}^4 u^i\,{}_4\omega_{2,v}(\cos\phi)\cos\psi_{0,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i=1,2 \\ -\sum_{u=0}^4 {}_4\omega_{2,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin^3\phi f_b(\phi), \\ -\sum_{u=1}^4 u^i\,{}_4\omega_{2,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i=1,2, \end{cases}$$

$$\text{(A.45)}\quad \begin{cases} \sum_{u=0}^4 {}_4\omega_{2,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin^3\phi f_b(\phi), \\ \sum_{u=1}^4 u^i\,{}_4\omega_{2,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i=1,2, \\ -\sum_{u=0}^4 {}_4\omega_{2,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin^3\phi f_b(\phi), \\ -\sum_{u=1}^4 u^i\,{}_4\omega_{2,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i=1,2, \end{cases}$$

$$\text{(A.46)}\quad \begin{cases} \sum_{u=0}^4 {}_4\omega_{2,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin^3\phi f_b(\phi), \\ \sum_{u=1}^4 u^i\,{}_4\omega_{2,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i=1,2, \\ -\sum_{u=0}^4 {}_4\omega_{2,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin^3\phi f_b(\phi), \\ -\sum_{u=1}^4 u^i\,{}_4\omega_{2,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin^{3-i}\phi f_b(\phi), \quad i=1,2, \end{cases}$$

$$\text{(A.47)}\quad \begin{cases} \sum_{u=0}^4 {}_4\omega_{1,v}(\cos\phi)\cos\psi_{0,\ell+v} = \sin^2\phi f_b(\phi), \\ \sum_{u=1}^4 u\,{}_4\omega_{1,v}(\cos\phi)\cos\psi_{0,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{u=0}^4 {}_4\omega_{1,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin^2\phi f_b(\phi), \\ -\sum_{u=1}^4 u\,{}_4\omega_{1,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin\phi f_b(\phi), \end{cases}$$

$$\text{(A.48)}\quad \begin{cases} \sum_{u=0}^4 {}_4\omega_{1,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin^2\phi f_b(\phi), \\ \sum_{u=1}^4 u\,{}_4\omega_{1,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{u=0}^4 {}_4\omega_{1,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin^2\phi f_b(\phi), \\ -\sum_{u=1}^4 u\,{}_4\omega_{1,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin\phi f_b(\phi), \end{cases}$$

(A.49)
$$\begin{cases} \sum_{u=0}^{4} {}_4\omega_{1,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin^2\phi f_b(\phi), \\ \sum_{u=1}^{4} u\, {}_4\omega_{1,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{u=0}^{4} {}_4\omega_{1,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin^2\phi f_b(\phi), \\ -\sum_{u=1}^{4} u\, {}_4\omega_{1,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin\phi f_b(\phi), \end{cases}$$

(A.50)
$$\begin{cases} \sum_{u=0}^{4} {}_4\omega_{0,v}(\cos\phi)\cos\psi_{0,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{u=0}^{4} {}_4\omega_{0,v}(\cos\phi)\sin\psi_{0,\ell+v} = \sin\phi f_b(\phi), \end{cases}$$

(A.51)
$$\begin{cases} \sum_{u=0}^{4} {}_4\omega_{0,v}(\cos\phi)\cos\psi_{1,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{u=0}^{4} {}_4\omega_{0,v}(\cos\phi)\sin\psi_{1,\ell+v} = \sin\phi f_b(\phi), \end{cases}$$

(A.52)
$$\begin{cases} \sum_{u=0}^{4} {}_4\omega_{0,v}(\cos\phi)\cos\psi_{2,\ell+v} = \sin\phi f_b(\phi), \\ -\sum_{u=0}^{4} {}_4\omega_{0,v}(\cos\phi)\sin\psi_{2,\ell+v} = \sin\phi f_b(\phi), \end{cases}$$

Now combining (A.40) with (A.41)-(A.52) as described in Table 2 one obtain the asymptotic behaviour of the fourth order derivative (A.10).

□

Appendix B. Proof of formula (1.5)

B.1. **Approximate Kac-Rice formula for counting critical points with value in $I \subseteq \mathbb{R}$.** For counting the number of critical points with corresponding value lying in any interval I in the real line, we define, for $x \neq \pm y$, the two-point correlation function $K_{2,\ell}(x, y)$ as:

$$K_{2,\ell}(x,y;t_1,t_2) = \mathbb{E}\left[\left|\nabla^2 f_\ell(x)\right| \cdot \left|\nabla^2 f_\ell(y)\right| \Big| \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}, f_\ell(x) = t_1, f_\ell(y) = t_2\right] \cdot \varphi_{x,y,\ell}(t_1,t_2,\mathbf{0},\mathbf{0}),$$

where $\varphi_{x,y,\ell}(t_1,t_2,\mathbf{0},\mathbf{0})$ denotes the density of the 6-dimensional vector

$$(f_\ell(x), f_\ell(y), \nabla f_\ell(x), \nabla f_\ell(y))$$

in $f_\ell(x) = t_1, f_\ell(y) = t_2, \nabla f_\ell(x) = \nabla f_\ell(y) = \mathbf{0}$. In [11] the following *approximate Kac-Rice* formula is derived:

(B.1)
$$\mathrm{Var}\left(\mathcal{N}_I^c(f_\ell)\right) = \int_{\mathcal{W}} \iint_{I\times I} K_{2,\ell}(x,y;t_1,t_2)\, dt_1 dt_2 dx dy - (\mathbb{E}[\mathcal{N}_I^c(f_\ell)])^2 + O(\ell^2).$$

Now, exploiting isotropy and observing that the level field f_ℓ is a linear combination of gradient and second order derivatives, we have [11, Section 4.1.2]:

$$K_{2,\ell}(\phi;t_1,t_2) = \frac{\lambda_\ell^4}{8} \frac{1}{\pi^2\sqrt{(\lambda_\ell^2 - 4\alpha_{2,\ell}^2(\phi))(\lambda_\ell^2 - 4\alpha_{1,\ell}^2(\phi))}} q(\mathbf{a}_\ell(\phi);t_1,t_2),$$

where

$$\begin{aligned} q(\mathbf{a};t_1,t_2) = &\frac{1}{(2\pi)^3\sqrt{\det(\Delta(\mathbf{a}))}} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left|z_1\sqrt{8}t_1 - z_1^2 - z_2^2\right| \cdot \left|w_1\sqrt{8}t_2 - w_1^2 - w_2^2\right| \\ &\times \exp\left\{-\frac{1}{2}(z_1,z_2,\sqrt{8}t_1 - z_1, w_1, w_2, \sqrt{8}t_2 - w_1)\Delta(\mathbf{a})^{-1}(z_1,z_2,\sqrt{8}t_1 - z_1, w_1, w_2, \sqrt{8}t_2 - w_1)^t\right\} \\ &\times dz_1 dz_2 dw_1 dw_2. \end{aligned}$$

B.2. **Taylor expansion and asymptotics for the two-point correlation function.** By performing the Taylor expansion as in Section 3.2 and by applying Lemma 3.1 and Lemma 3.2 as in Section 3.3, one obtain that, in the high energy limit,

$$\begin{aligned} \mathrm{Var}(\mathcal{N}^c(f_\ell)) = &\frac{1}{8}\left[2\ell^3 + \frac{2\cdot 3^2}{\pi^2}\ell^2\log\ell\right] \iint_{I\times I} q(\mathbf{0};t_1,t_2)dt_1dt_2 \\ &+ \frac{1}{8}\left[-16\ell^3 - \frac{2^5\cdot 3}{\pi^2}\ell^2\log\ell\right] \iint_{I\times I} \left[\frac{\partial}{\partial a_3} q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}} dt_1dt_2 \\ &+ \frac{1}{8}\left[32\ell^3 - \frac{2^6}{\pi^2}\ell^2\log\ell\right] \iint_{I\times I} \left[\frac{\partial^2}{\partial a_7\partial a_7} q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}} dt_1dt_2 \\ &+ \frac{1}{8}\left[\frac{3\cdot 2^7}{\pi^2}\ell^2\log\ell\right] \iint_{I\times I} \left[\frac{\partial^2}{\partial a_3\partial a_3} q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}} dt_1dt_2 \end{aligned}$$

$$+\frac{1}{8}\left[-\frac{2^9}{\pi^2}\ell^2\log\ell\right]\iint_{I\times I}\left[\frac{\partial^3}{\partial a_3\partial a_7\partial a_7}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2$$

$$+\frac{1}{8}\left[\frac{2^9}{\pi^2}\ell^2\log\ell\right]\iint_{I\times I}\left[\frac{\partial^4}{\partial a_7\partial a_7\partial a_7\partial a_7}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2+O(\ell^2). \tag{B.2}$$

B.3. **Evaluation of the leading constant.** Let $\mathcal{I}_{I,r}$, $r=0,2,4$, be the integrals $\mathcal{I}_{I,r}=\int_I p_r(t)dt$ where the functions p_r for $r=0,2,4$ are defined by

$$\begin{aligned}
p_0(t)&=\sqrt{8}\cdot\mathbb{E}\Big[|Y_1Y_3-Y_2^2|\Big|Y_1+Y_3=\sqrt{8}t\Big]\cdot\phi_{Y_1+Y_3}(\sqrt{8}t)=\sqrt{\frac{2}{\pi}}[2e^{-t^2}+t^2-1]e^{-\frac{t^2}{2}},\\
p_2(t)&=\sqrt{8}\cdot\mathbb{E}\Big[(3t-\sqrt{2}Y_1)^2|Y_1Y_3-Y_2^2|\Big|Y_1+Y_3=\sqrt{8}t\Big]\cdot\phi_{Y_1+Y_3}(\sqrt{8}t)\\
&=\sqrt{\frac{2}{\pi}}[-4+t^2+t^4+e^{-t^2}2(4+3t^2)]e^{-\frac{t^2}{2}},\\
p_4(t)&=\sqrt{8}\cdot\mathbb{E}\Big[(3t-\sqrt{2}Y_1)^4|Y_1Y_3-Y_2^2|\Big|Y_1+Y_3=\sqrt{8}t\Big]\cdot\phi_{Y_1+Y_3}(\sqrt{8}t)\\
&=\sqrt{\frac{2}{\pi}}[(72+96t^2+38t^4)e^{-t^2}-36-12t^2+11t^4+t^6]e^{-\frac{t^2}{2}},
\end{aligned}$$

and $Y=(Y_1,Y_2,Y_3)$ is the centred jointly Gaussian random vector defined in Section 3.4. Using Leibniz integral rule and some mechanical computations, one has

$$\begin{aligned}
&\iint_{I\times I}q(\mathbf{0};t_1,t_2)dt_1dt_2=\frac{1}{2^3}\mathcal{I}_{I,0}^2,\\
&\iint_{I\times I}\left[\frac{\partial}{\partial a_3}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2=\frac{1}{2^6}[-3\mathcal{I}_{I,0}^2+\mathcal{I}_{I,0}\mathcal{I}_{I,2}],\\
&\iint_{I\times I}\left[\frac{\partial^2}{\partial a_7\partial a_7}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2=\frac{1}{2^9}[3\mathcal{I}_{I,0}-\mathcal{I}_{I,2}]^2,\\
&\iint_{I\times I}\left[\frac{\partial^2}{\partial a_3\partial a_3}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2=\frac{1}{2^{11}}[2^3\cdot 3^2\mathcal{I}_{I,0}^2-2^4\cdot 3\mathcal{I}_{I,0}\mathcal{I}_{I,2}+2\mathcal{I}_{I,0}\mathcal{I}_{I,4}+2\mathcal{I}_{I,2}^2],\\
&\iint_{I\times I}\left[\frac{\partial^3}{\partial a_3\partial a_7\partial a_7}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2=\frac{1}{2^{13}}[-2\cdot 3^4\mathcal{I}_{I,0}^2+2\cdot 3^4\mathcal{I}_{I,0}\mathcal{I}_{I,2}-2\cdot 3\mathcal{I}_{I,0}\mathcal{I}_{I,4}+2\mathcal{I}_{I,4}\mathcal{I}_{I,2}-2^2\cdot 3^2\mathcal{I}_{I,2}^2],\\
&\iint_{I\times I}\left[\frac{\partial^4}{\partial a_7\partial a_7\partial a_7\partial a_7}q(\mathbf{a};t_1,t_2)\right]_{\mathbf{a}=\mathbf{0}}dt_1dt_2=\frac{1}{2^{15}}[3^3\mathcal{I}_{I,0}-2\cdot 3^2\mathcal{I}_{I,2}+\mathcal{I}_{I,4}]^2.
\end{aligned}$$

so that the variance (B.2) can be rewritten as

$$\mathrm{Var}(\mathcal{N}_I^c(f_\ell))=\frac{1}{2^4}[5\mathcal{I}_{I,0}-\mathcal{I}_{I,2}]^2\ \ell^3+\frac{1}{\pi^2 2^6}[51\mathcal{I}_{I,0}-2\cdot 11\ \mathcal{I}_{I,2}+\mathcal{I}_{I,4}]^2\ \ell^2\log\ell+O(\ell^2).$$

Formula (1.5) follows by observing that

(B.3)

$$\mu^c(t)=\frac{1}{\pi 2^3}[51\,p_0(t)-2\cdot 11\,p_2(t)+p_4(t)]=\frac{1}{\pi 2^3}\sqrt{\frac{2}{\pi}}[(-2-36t^2+38t^4)e^{-t^2}+1+17t^2-11t^4+t^6]e^{-\frac{t^2}{2}}.$$

The proof of (1.5) for extrema and saddles is analogous and we obtain that μ^e and μ^s in (1.5) are given by

$$\mu^e(t)=\frac{1}{2^3\pi}\sqrt{\frac{2}{\pi}}[(-1-18t^2+19t^4)e^{-t^2}+1+17t^2-11t^4+t^6]e^{-\frac{t^2}{2}}, \tag{B.4}$$

$$\mu^s(t)=\frac{1}{2^3\pi}\sqrt{\frac{2}{\pi}}(-1-18t^2+19t^4)e^{-\frac{3t^2}{2}}. \tag{B.5}$$

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