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Nonlinear stability of expanding star solutions of the radially-symmetric mass-critical Euler-Poisson system

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Abstract

We prove nonlinear stability of compactly supported expanding star-solutions of the mass-critical gravitational Euler-Poisson system. These special solutions were discovered by Goldreich and Weber in 1980. The expansion rate of such solutions can be either self-similar or non-self-similar (linear), and we treat both types. An important outcome of our stability results is the existence of a new class of global-in-time radially symmetric solutions, which are not homologous and therefore not encompassed by the existing works. Using Lagrangian coordinates we reformulate the associated free-boundary problem as a degenerate quasilinear wave equation on a compact spatial domain. The problem is mass-critical with respect to an invariant rescaling and the analysis is carried out in similarity variables. © 2000 Wiley Periodicals, Inc.

Contents

1 Introduction

One of the most fundamental models of a Newtonian star is given by the compressible Euler-Poisson system. Here the star is idealized as a self-gravitating fluid/gas, kept together by the self-consistent gravitational force-field. We assume that the fluid density function $\rho : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is initially supported on a compact domain $\Omega_0 \subset \mathbb{R}^3$. As the system evolves the fluid support $\Omega(t)$ changes in time and we are naturally led to a moving (free) boundary problem. The remaining unknowns in the problem are the fluid pressure $p : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$, the fluid velocity $\mathbf{u} : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$, the gravitational potential $\Phi : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$, and the fluid support $\Omega(t)$. Coupling the fluid evolution to the Newton's gravitational theory we arrive at the Euler-Poisson system given by

$$\text{(1.1a)} \qquad \partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0 \qquad \text{in } \Omega(t)\,;$$

$$\text{(1.1b)} \qquad \rho\,(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p = -\rho \nabla \Phi \quad \text{in } \Omega(t)\,;$$

$$\text{(1.1c)} \qquad \Delta \Phi = 4\pi\rho,\ \lim_{|x|\to\infty} \Phi(t,x) = 0 \qquad \text{in } \mathbb{R}^3\,;$$

$$\text{(1.1d)} \qquad p = 0 \qquad \text{on } \partial\Omega(t)\,;$$

$$\text{(1.1e)} \qquad \mathscr{V}_{\partial\Omega(t)} = \mathbf{u} \cdot \mathbf{n}(t) \quad \text{on } \partial\Omega(t)\,;$$

$$\text{(1.1f)} \qquad (\rho(0,\cdot),\mathbf{u}(0,\cdot)) = (\rho_0,\mathbf{u}_0)\,,\ \Omega(0) = \Omega_0\,.$$

Here $\partial\Omega(t)$ denotes the boundary of the moving domain, $\mathscr{V}_{\partial\Omega(t)}$ denotes the normal velocity of $\partial\Omega(t)$, and $\mathbf{n}(t)$ denotes the outward unit normal vector to $\partial\Omega(t)$. We refer to the system of equation (1.1) as the EP-system.

Equation (1.1a) is the well-known continuity equation, while (1.1b) expresses the conservation of momentum. Equation (1.1c) is the Poisson equation equipped with a suitable asymptotic boundary condition. Boundary condition (1.1d) is the vacuum boundary condition, while (1.1e) is the kinematic boundary condition stating that the boundary movement is tangential to the fluid particles.

In this article we shall only consider ideal barotropic fluids where the pressure is a function of the fluid density, expressed through the following equation of state

$$\text{(1.2)} \qquad p = \rho^\gamma,\ \ 1 < \gamma < 2.$$

The EP-system (1.1) with the polytropic equation of state (1.2) will be referred to as the EP_γ-system.

The most famous class of special solutions of the EP_γ-system are the Lane-Emden steady states. They are spherically symmetric steady states of the form

$$(\rho,\mathbf{u}) \equiv (\rho_*,\mathbf{0}).$$

Plugging this ansatz in (1.1) the problem reduces to a single ordinary differential equation (ODE) for the enthalpy $w := \rho_*^{\gamma-1}$:

$$(1.3) \qquad \partial_{rr}w + \frac{2}{r}\partial_r w + \pi w^{\frac{1}{\gamma-1}} = 0.$$

It is well-known [5, 44, 2, 18] that for any $\gamma \in (\frac{6}{5}, 2)$ there exists a compactly supported steady solution to (1.3) with the boundary conditions $w'(0) = w(1) = 0$. The associated steady state density ρ_* has finite mass and compact support and therefore represents a steady star [5, 2, 44]. Moreover, classical linear stability arguments [26] give the following dichotomy in the stability behavior of the above family of steady stars:

$$\text{if } \frac{6}{5} < \gamma < \frac{4}{3}, \text{ the steady state } (\rho_*, \mathbf{0}) \text{ is linearly unstable;}$$

$$\text{if } \frac{4}{3} \leq \gamma < 2, \text{ the steady state } (\rho_*, \mathbf{0}) \text{ is linearly stable.}$$

The case of $\gamma = \frac{4}{3}$ admits 0 as the first eigenvalue and it is often referred to as neutrally stable. Under the assumption that a global-in-time solution exists, nonlinear stability of Lane-Emden steady stars in the range $\frac{4}{3} < \gamma < 2$ has been shown by variation arguments in an energy-based topology, see [38, 27]. Since the result is conditional upon the existence of a solution, it remains an important open problem to prove or disprove the full nonlinear stability. Nonlinear instability in the range $\frac{6}{5} \leq \gamma < \frac{4}{3}$ has been rigorously established by the second author [16, 18]. In this case, the instability is induced by the existence of a growing mode in the linearized operator, while there are no such modes when $\frac{4}{3} \leq \gamma < 2$. As we shall see in Section 2.2, Lane-Emden stars in the case $\gamma = \frac{4}{3}$ are in fact nonlinearly unstable despite the absence of growing modes in the linearized operator. For the existence and the stability theory for the Navier-Stokes-Poisson system in the vicinity of Lane-Emden stars, see [17, 24, 29, 30].

The special significance of the values $\gamma = \frac{6}{5}$ and $\gamma = \frac{4}{3}$ is better illustrated by the following scaling analysis. If $(\rho, \mathbf{u})$ is a solution of the EP$_\gamma$-system, so is the pair $(\tilde{\rho}, \tilde{\mathbf{u}})$ defined by

$$(1.4) \qquad \rho(t,x) = \lambda^{-\frac{2}{2-\gamma}} \tilde{\rho}\left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\right),$$

$$(1.5) \qquad \mathbf{u}(t,x) = \lambda^{-\frac{\gamma-1}{2-\gamma}} \tilde{\mathbf{u}}\left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\right),$$

for any $\lambda > 0$. The associated pressure $\tilde{p}$ and the gravitational potential $\tilde{\Phi}$ relate to p and Φ via:

$$p(t,x) = \lambda^{-\frac{2\gamma}{2-\gamma}} \tilde{p}\left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\right),$$

$$\Phi(t,x) = \lambda^{-\frac{2\gamma-2}{2-\gamma}} \tilde{\Phi}\left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{x}{\lambda}\right).$$

The self-similar rescaling (1.4)–(1.5) is at the heart of our analysis. Two fundamental conserved quantities associated with the EP_γ-system are the total mass

(1.6)
$$M(\rho) := \int_{\mathbb{R}^3} \rho \, dx,$$

and energy

(1.7)
$$E(\rho, \mathbf{u}) := \int_{\mathbb{R}^3} \left[\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} \rho \Phi + \frac{1}{\gamma - 1} \rho^\gamma \right] dx.$$

It is easy to examine their behavior with respect to the rescaling (1.4)–(1.5):

(1.8)
$$M(\rho) = \lambda^{\frac{4-3\gamma}{2-\gamma}} M(\tilde{\rho}), \quad E(\rho, \mathbf{u}) = \lambda^{\frac{6-5\gamma}{2-\gamma}} E(\tilde{\rho}, \tilde{\mathbf{u}}).$$

The total mass M is invariant under the self-similar rescaling (1.4)–(1.5) when $\gamma = \frac{4}{3}$ and the energy E when $\gamma = \frac{6}{5}$, so we refer to the cases $\gamma = \frac{4}{3}$ and $\gamma = \frac{6}{5}$ as the *mass-critical* and the *energy-critical* case respectively.

As all the new results in this work pertain to the case of spherical symmetry, we next formulate the EP_γ-system in radial symmetry. Let $r = |x|$, $\mathbf{u}(t,x) = v(t,r)\frac{x}{r}$, and

$$\lambda(t) := \text{ radius of the ball } B_{\lambda(t)}(\mathbf{0}) \text{ which supports the fluid.}$$

Then the system (1.1) takes the form:

(1.9a)
$$\partial_t \rho + v \partial_r \rho + \rho \left(\partial_r v + \frac{2}{r} v \right) = 0, \quad r \leq \lambda(t),$$

(1.9b)
$$\rho \left(\partial_t v + v \partial_r v \right) + \partial_r p + \rho \partial_r \phi = 0, \quad r \leq \lambda(t),$$

(1.9c)
$$\partial_{rr} \Phi + \frac{2}{r} \partial_r \Phi = 4\pi \rho, \quad \lim_{r \to \infty} \Phi(t,r) = 0, \quad r \geq 0,$$

(1.9d)
$$\rho(t, \lambda(t)) = 0,$$

(1.9e)
$$\dot{\lambda}(t) = v(t, \lambda(t)),$$

(1.9f)
$$\lambda(0) = \lambda_0, \rho(0,r) = \rho_0(r), v(0,r) = v_0(r).$$

We note that under the assumption (1.2), conditions (1.9d) and (1.1d) are equivalent.

The mass-critical regime $\gamma = \frac{4}{3}$ will be the focus of this article. The exponent $\gamma = \frac{4}{3}$ in the polytropic gas pressure law (1.2) is also referred to as the radiation case [5]. Combining the ideal gas law and the Stefan-Boltzmann radiation law the effective formula for the pressure takes the form $p \sim \rho \Theta + \Theta^4$, where Θ is the temperature of the gas. The two terms on the right-hand side are in balance when Θ is approximated by $\rho^{\frac{1}{3}}$ and this precisely leads to the radiative case $\gamma = \frac{4}{3}$.

The case $\gamma = \frac{4}{3}$ is particularly appealing in the astrophysics community as it lends itself to a simple scenario of stellar collapse/expansion: a compact fluid body can shrink or expand in a self-similar manner by cascading

between different scales $t \mapsto \lambda(t)$ thereby preserving the total mass. When $\gamma = \frac{4}{3}$ we have $\frac{1}{2-\gamma} = \frac{3}{2}$ and therefore the self-similar expansion/collapse would heuristically correspond to the rates

$$(1.10) \qquad \bar{\lambda}_{\text{expansion}}(t) \sim_{t \to \infty} k_1 t^{2/3}, \quad \bar{\lambda}_{\text{collapse}}(t) \sim_{t \to T} k_2 (T-t)^{2/3},$$

for some positive constant $k_1, k_2 > 0$ and a collapse time $T > 0$. Here $\bar{\lambda}_{\text{expansion}}(t)$ and $\bar{\lambda}_{\text{collapse}}(t)$ represent the radii of an expanding and collapsing star respectively. A remarkable feature of the $\text{EP}_{\frac{4}{3}}$-system is that special solutions exhibiting the self-similar behavior (1.10) can be explicitly constructed! They were first discovered in the work of Goldreich and Weber [14] in 1980 and their energy is exactly zero. A wider class of Type I collapsing and expanding solutions, although implicitly contained in [14], was discovered independently by Makino [33] in 1992 and Fu & Lin [13] in 1998. Solutions from [33] and [13] either expand or collapse at a *linear* rate

$$(1.11) \qquad \tilde{\lambda}_{\text{expansion}}(t) \sim_{t \to \infty} \tilde{k}_1 t, \quad \tilde{\lambda}_{\text{collapse}}(t) \sim_{t \to \tilde{T}} \tilde{k}_2 (\tilde{T}-t),$$

for some positive constant $\tilde{k}_1, \tilde{k}_2 > 0$ and a collapse time $\tilde{T} > 0$. The rates (1.11) are not self-similar as opposed to (1.10) and as is shown in Section 2.2 their energy is always different from zero. A unified treatment of the two types of special homologous solutions using the Lagrangian coordinates is given in Section 2.2.

1.1 Special expanding solutions

To describe the special homologous expanding solutions in Eulerian coordinates, we need to specify three parameters:

$$(\lambda_0, \lambda_1, \delta) \in \mathbb{R}_+ \times \mathbb{R} \times [\delta^*, \infty),$$

where $\delta^* < 0$ is a real number defined in Section 2.2. The expanding radius satisfies the ordinary differential equation:

$$(1.12) \qquad \ddot{\lambda} \lambda^2 = \delta,$$

with the initial conditions

$$(1.13) \qquad \lambda(0) = \lambda_0 > 0, \quad \dot{\lambda}(0) = \lambda_1.$$

The associated density and velocity field are given by:

$$(1.14) \qquad \rho(t,r) = \lambda(t)^{-3} w^3 \left(\frac{r}{\lambda(t)} \right), \quad v(t,r) = \frac{\dot{\lambda}(t)}{\lambda(t)} r,$$

where $w : [0,1] \to \mathbb{R}_+$ is a non-negative function called *enthalpy* solving the generalized Lane-Emden equation:

$$(1.15) \quad w'' + \frac{2}{z} w' + \pi w^3 = -\frac{3}{4} \delta, \quad \text{in} \ [0,1], \ w'(0) = 0, \ w(1) = 0.$$

The conserved energy E associated with these solutions takes the form (see Lemma 2.8):

$$(1.16) \qquad E(\lambda, \dot{\lambda}) = \left(\dot{\lambda}^2 + \frac{2\delta}{\lambda} \right) \int_0^1 2\pi w^3 z^4 dz.$$

Here we have written $E(\lambda, \dot{\lambda}) = E(\rho, v)$ where ρ and v are given in (1.14). Throughout the article, E will represent the physical energy but we will abuse the notation for the arguments in order to emphasize the dependence of the corresponding dynamical quantities for different formulations.

Self-similar solutions

It can be checked (see Section 2.2) that for any $\delta \in [\delta^*, 0)$ and any (λ_0, λ_1) satisfying

$$(1.17) \qquad E(\lambda_0, \lambda_1) = \left(\lambda_1^2 + \frac{2\delta}{\lambda_0} \right) \int_0^1 2\pi w^3 z^4 dz = 0$$

there exists a self-similar solution $\bar{\lambda}(t)$ given explicitly by:

$$(1.18) \qquad \bar{\lambda}(t) = \left(\lambda_0^{3/2} + \frac{3}{2} \lambda_0^{1/2} \lambda_1 t \right)^{2/3}$$

with the density ρ, velocity v, and the enthalpy w given by (1.14)–(1.15). Note that the cases $\lambda_1 > 0$ and $\lambda_1 < 0$ correspond to expansion and collapse respectively.

Definition 1.1 (Self-similar expanding homologous solutions)**.** The family of solutions given by (1.18) with $\bar{\rho} = \bar{\lambda}(t)^{-3} w^3(\frac{r}{\bar{\lambda}(t)})$, $v(t, r) = \frac{\dot{\bar{\lambda}}(t)}{\bar{\lambda}(t)} r$ and (1.14)–(1.15) is denoted by

$$(\bar{\lambda}, \bar{\rho}, \bar{v})_{\lambda_0, \lambda_1, \delta}$$

with $(\lambda_0, \lambda_1, \delta) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [\delta^*, 0)$. We refer to such solutions as *self-similar expanding solutions of the $EP_{\frac{4}{3}}$-system.*

Linearly expanding solutions

If we assume that either

$$(1.19) \qquad\qquad\qquad \delta > 0,$$

or

$$(1.20) \qquad\qquad\qquad \delta = 0, \ \lambda_1 > 0,$$

or

$$(1.21) \qquad\qquad \delta^* < \delta < 0, \ \lambda_1 > \lambda_1^* = \frac{\sqrt{2|\delta|}}{\lambda_0},$$

then a solution $\tilde{\lambda}$ to (1.12)–(1.13) exists globally-in-time and expands indefinitely at a linear rate, i.e. there exists a constant $c > 0$ such that

$$(1.22) \qquad \lim_{t\to\infty} \dot{\tilde{\lambda}}(t) = c.$$

The density ρ, velocity v, and the enthalpy w are given by (1.14)–(1.15). We remark that all of the solutions above have strictly positive energy, i.e. $E(\tilde{\lambda}, \dot{\tilde{\lambda}}) > 0$ (see (1.16)).

Definition 1.2 (Linearly expanding homologous solutions)**.** The family of solutions satisfying (1.22) for some $c > 0$ with $\tilde{\rho} = \tilde{\lambda}(t)^{-3} w^3(\frac{r}{\tilde{\lambda}(t)})$, $v(t,r) = \frac{\dot{\tilde{\lambda}}(t)}{\tilde{\lambda}(t)} r$ is denoted by

$$(\tilde{\lambda}, \tilde{\rho}, \tilde{v})_{\lambda_0, \lambda_1, \delta}$$

with parameters $(\lambda_0, \lambda_1, \delta)$ satisfying either (1.19), or (1.20), or (1.21). We refer to such solutions as *linearly expanding solutions of the* $EP_{\frac{4}{3}}$*-system.*

1.2 Main results

The main motivation for this article is the question of nonlinear stability of the above described family of homologous expanding solutions of the $EP_{\frac{4}{3}}$-system, given by Definitions 1.1 and 1.2. The problem of asymptotic stability of this family is a first necessary step in justifying the qualitative behavior (1.10), (1.11) as a credible scenario of stellar expansion.

Since the family of self-similar solutions $(\bar{\lambda}, \bar{\rho}, \bar{v})_{\lambda_0, \lambda_1, \delta}$ (subject to the constraint (1.17)) embeds in the family of linearly expanding ones, it is obvious that the self-similar behavior is not stable. Nevertheless, our first result is to show that if we limit our perturbations to a surface of zero-energy perturbations, then the self-similar behavior is indeed nonlinearly stable.

Theorem 1.3 (Codimension-one stability of self-similar expanding solutions – *informal statement*)**.** *The self-similar expanding solutions of the* $EP_{\frac{4}{3}}$*-system specified in Definition 1.1 are globally-in-time nonlinearly stable with respect to spherically symmetric perturbations* (ρ_0, v_0) *with vanishing energy* $E(\rho_0, v_0) = 0$. *Any such perturbation converges to a nearby self-similar expanding homologous solution.*

A rigorous version of Theorem 1.3 is stated in Theorem 2.10 using the Lagrangian coordinates.

Remark 1.4. The asymptotic attractor $(\bar{\lambda}, \bar{\rho}, \bar{v})_{\lambda_0, \lambda_1, \delta}$ is characterized by the requirements

$$(1.23) \quad E(\rho_0, v_0) = E(\bar{\rho}, \bar{v}) = 0, \ \ M(\rho_0) = M(\bar{\rho}), \ \ \mathrm{supp}(\rho_0) = [0, \lambda_0].$$

In particular, our stability result is a codimension-one stability result, as we must restrict our perturbations to the zero-energy surface of admissible

stellar configurations. It shows that only the directions transversal to the zero-energy surface are responsible for the loss of a self-similar behavior.

Our second main result concerns the stability of linearly expanding homologous solutions. Unlike self-similar solutions, we do not limit our perturbations to the same energy level as the background solution.

Theorem 1.5 (Nonlinear stability of the linearly expanding homologous solutions – *informal statement*)**.** *There exists an $\tilde{\varepsilon} > 0$ such that for any $\delta > -\tilde{\varepsilon}$ the linearly expanding homologous solutions $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})$ from Definition 1.2 are nonlinearly stable with respect to small spherically symmetric perturbations.*

A rigorous version of Theorem 1.5 is stated in Theorem 2.15 using the Lagrangian coordinates.

Remark 1.6. Theorem 1.5 states that the behavior in the light blue shaded region in Figure 2.1 is dynamically stable, as long as $\delta > -\tilde{\varepsilon}$. In contrast to Theorem 1.3, Theorem 1.5 does not specify the asymptotic attractor for the perturbation nor is it expected to behave like a nearby member of the family of homologous solutions. Instead, we show that the perturbed solution remains nearby in a suitable sense and that its support grows linearly. The question of asymptotic behavior is more subtle, see Remark 2.16.

Theorems 1.3 and 1.5 are to the best of our knowledge the first results producing a family of global-in-time non-homologous classical solutions to the $EP_{\frac{4}{3}}$-system. In particular, we show that the expansion of the background homologous solution counteracts the possibility of shock formation and produces a certain damping effect. This effect is visible only in suitably rescaled similarity variables, and the role of the critical scaling invariance is a fundamental ingredient of our proof.

An important feature of expanding solutions is that the boundary of the domain (fluid support) is moving with the fluid. A simple comparison between perturbed solutions and the asymptotic attractor in Eulerian coordinates is not feasible. The problem is better understood in material coordinates by following the fluid trajectories. We will therefore formulate the problem in suitably rescaled Lagrangian coordinates, which will allow us to pull the problem back onto a fixed domain.

2 Lagrangian formulation

2.1 Physical vacuum and local-in-time well-posedness

One of the fundamental difficulties associated with the well-posedness theory for (1.1) is tied to the concept of a *physical vacuum boundary condition*. It is clear that the homologous solutions described above satisfy $\dot{\lambda}(t) \neq 0$ and therefore the support of the fluid is moving. On the other

hand, the background enthalpy profile w satisfying (1.15) vanishes at the boundary $z = 1$ while it is strictly positive on $[0, 1)$. A simple calculation based on (1.15) reveals that the strict inequality $w'(1) < 0$ has to hold. This in turn implies that $\rho(t, \lambda(t)) = 0$ and $\partial_r \rho^{1/3}(t, \lambda(t)) < 0$. Such a boundary behavior is a typical feature of compactly supported stellar configurations:

Definition 2.1 (Physical vacuum condition). We say that the system (1.1) satisfies the physical vacuum boundary condition if

$$(2.1) \qquad\qquad -\infty < \frac{\partial c^2}{\partial n} < 0 \quad \text{on } \partial \Omega(t)$$

where

$$c^2 = \frac{dp}{d\rho} = \gamma \rho^{\gamma - 1} \quad (c \text{ is known as the speed of sound})$$

and $\frac{\partial}{\partial n}$ denotes the outward normal derivative.

We note that the physical vacuum condition is analogous to the Rayleigh-Taylor sign condition arising in the free boundary problems of the incompressible fluid dynamics (see [6] for a detailed discussion and references therein). The problem of moving vacuum boundaries characterized by (2.1) has for a long time created various analytical and conceptual difficulties in addressing the question of well-posedness due to the degeneracy caused by the physical vacuum requirement. Note that by Definition 2.1, an enthalpy profile $c^2 \sim \rho^{\gamma - 1}$ satisfying the physical vacuum condition is *not* smooth across the moving boundary and it is a priori not even clear what is the suitable functional analytic framework to address the question of local-in-time well-posedness.

In 2010 the basic advance was achieved by Coutand & Shkoller [7, 8] (see [9] for the a priori estimates) and independently by Jang & Masmoudi [20, 23] where the authors developed a well-posedness theory in high-order weighted Sobolev spaces for the compressible Euler equations satisfying the physical vacuum condition. By a straightforward extension, the same framework can be applied to the free boundary Euler-Poisson system, as the effects of the added gravitational field are of lower order from the point-of-view of well-posedness theory [18, 19, 28]. We refer to [21, 22, 31] for more detailed discussion on the physical vacuum and other vacuum states.

The crux of the approach in [8, 23, 18] is the use of Lagrangian coordinates. They allow us to pull-back the EP$_{\frac{4}{3}}$-system onto a fixed compact domain. More importantly, in spherical symmetry the Lagrangian formulation of the EP$_{\frac{4}{3}}$-system reduces to a degenerate quasilinear wave equation for the Lagrangian flow-map, wherein the initial density profile ρ_0 features as a coefficient inside the nonlinear wave operator. To see this we

introduce a Lagrangian flow map $\eta : \Omega \to \Omega(t)$ as a solution of:

$$(2.2) \qquad\qquad \dot{\eta}(t,x) = \mathbf{u}(t, \eta(t,x)),$$

$$(2.3) \qquad\qquad \eta(0,x) = \eta_0(x),$$

where $\eta_0 : \Omega \to \Omega(0)$ is a diffeomorphism with positive Jacobian determinant, and $\Omega \subset \mathbb{R}^3$ is some simply connected smooth domain. Since the problem is radially symmetric, we make the ansatz:

$$(2.4) \qquad\qquad \eta(t,x) = \chi(t,z)x, \;\; z = |x|$$

which leads to the following evolution equation for χ [18] :

$$(2.5) \qquad\qquad \chi_{tt} + F_w[\chi] = 0$$

with the initial conditions:

$$(2.6) \qquad\qquad \chi(0) = \chi_0, \;\; \dot{\chi}(0) = \chi_1.$$

Here, the nonlinear operator F_w is given by

$$(2.7) \quad F_w[\chi] := \frac{\chi^2}{w^3 z} \partial_z \left(w^4 \left[\chi^2 (\chi + z \partial_z \chi) \right]^{-\frac{4}{3}} \right) + \frac{1}{\chi^2 z^3} \int_0^z 4\pi w^3 s^2 \, ds$$

where

$$(2.8) \qquad\qquad w^3 := \rho_0 \chi_0^2 (\chi_0 + z \partial_z \chi_0).$$

A remarkable feature of the formulation (2.5)–(2.6) is that the EP system (1.1) in the new variables takes the form of a *quasi-linear degenerate wave equation* in a bounded domain.

   To formulate a well-posedness result we need weighted Sobolev spaces.

Definition 2.2 (Weighted spaces and energy)**.** For any $k \in \mathbb{N} \cup \{0\}$ and a continuous function $w : [0,1] \to \mathbb{R}_+$, we define weighted spaces $L^2_{w,k}$ as a completion of the space $C_c^\infty(0,1)$ with respect to the norm $\| \cdot \|_{w,k}$ generated by the inner product

$$(2.9) \qquad\qquad (\chi_1, \chi_2)_{w,k} := \int_0^1 \chi_1 \chi_2 w^{3+k} z^4 \, dz.$$

We will set $\| \cdot \|_{w,0} := \| \cdot \|_w$. We also define the following Hilbert space

$$(2.10) \qquad H_w^j := \{ \varphi \in L^2_{w,0} : \partial_z^k \varphi \in L^2_{w,k} \text{ for all } 0 \le k \le j \}.$$

The function space for initial data is defined by

$$(2.11) \qquad\qquad \mathscr{H}_w := \left\{ (\chi_0, \chi_1) \big| \, \| (\chi_0, \chi_1) \|_{\mathscr{H}_w} < \infty \right\},$$

where

$$\| (\chi_0, \chi_1) \|_{\mathscr{H}_w}^2 := \sum_{k=0}^{8} \| \partial_z^k \chi \|_{w,k}^2 + \sum_{k=0}^{7} \| \partial_z^{k+1} \chi_1 \|_{w,k+1}^2.$$

We also introduce the following weighted space-time norm:

(2.12)
$$\mathscr{E}(t) := \sum_{j=0}^{7} \sum_{k=0}^{j} \left(\|\partial_s^{j-k+1}\partial_z^k \chi\|_{w,k}^2 + \|\partial_s^{j-k}\partial_z^{k+1}\chi\|_{w,k+1}^2 + \|\partial_s^{j-k}\partial_z^k \chi\|_{w,k}^2 \right).$$

We are now ready to state the following local-in-time well-posedness theorem [8, 18, 23, 28]:

Theorem 2.3. *Let the initial density ρ_0 satisfy the physical vacuum boundary condition* (2.1). *Assume additionally that $(\chi_0, \chi_1) \in \mathscr{H}_w$. Then there exists a constant $C > 0$ and a unique solution to the initial value problem* (2.5)–(2.6) *on some finite time interval $[0,T], T > 0$, such that the map*

$$[0,T] \ni t \to \mathscr{E}(t)$$

is continuous, and the solution (χ, χ_t) satisfies the energy bound

$$\sup_{0 \leq t \leq T} \mathscr{E}(t) \leq C \|(\chi_0, \chi_1)\|_{\mathscr{H}_w}^2.$$

If $\mathscr{T} = \sup\{0 \leq t \leq \infty : \text{solution exists on } [0,t) \text{ and } \mathscr{E}(t) < \infty\}$ is the maximal time of existence, then $\mathscr{T} < \infty$ implies $\lim_{t \to \mathscr{T}^-} \mathscr{E}(t) = \infty$.

Remark 2.4. The norm used to state our well-posedness theorem suggests an interesting scaling structure in the problem. As the number of spatial derivatives increases the weights get more singular as it is clear from (2.12). An alternative to the norm $\mathscr{E}$ is a norm based on purely spatial derivatives. In fact, the methodology developed in [23] ensures the local-in-time well-posendess of solutions in weighted Sobolev spaces generated by spatial derivatives in which suitable a priori estimates are established. This principle applies to our problem as well: based on the a priori estimates (for instance, see (4.18)), we have a local existence theorem analogous to Theorem 2.3 by using the norm $\tilde{\mathscr{E}}$ defined in (2.63) and the corresponding function space introduced in Definition 2.14.

The conservation of mass (1.6) is embedded in the new formulation of the problem, as the continuity equation (1.1a) is used fundamentally in the derivation of (2.5), see [18]. The only surviving non-trivial conservation law is the energy conservation law. In the new variable χ, the energy (1.7) takes the form:

$$E(\chi, \partial_t \chi)(t) = 2\pi \|\partial_t \chi\|_w^2 + 12\pi \| \left(z^6 \chi^2 (\chi + z\partial_z \chi) \right)^{-1/6} \|_w^2$$

(2.13)
$$- 4\pi \left\| \left(\chi^{-1} z^{-3} \int_0^z 4\pi w^3 s^2 ds \right)^{1/2} \right\|_w^2.$$

It can be checked by a direct computation that along the smooth solutions of (2.5) the energy $E(\chi, \partial_t \chi)(t)$ remains constant. In particular, the solutions obtained by Theorem 2.3 satisfy $E(\chi, \partial_t \chi)(t) = E(\chi_0, \chi_1)$ for $0 \leq t \leq T$.

2.2 homologous solutions of the EP$_{\frac{4}{3}}$-system

In this section, we unify the treatment of the existence of self-similar expanding/collapsing solutions and show that the special solutions discovered by Goldreich and Weber [14], Makino [33], Fu and Lin [13] naturally correspond to *homologous* solutions expressed in the Lagrangian coordinates introduced above. For the more general non-isentropic flows in arbitrary dimensions the existence of such solutions was shown by Deng, Xiang, and Yang in [12].

A homologous solution of the EP$_{\frac{4}{3}}$-system possesses *by definition* a flow map η of the form

$$\eta(t,x) = \lambda(t)x.$$

Equivalently, we look for the solutions to (2.5) of the form $\chi(t,z) = \lambda(t)$, thus justifying the use of the word "homologous". The terminology stems from the physics literature [14, 40, 43]. The nonlinear operator F_w is easily checked to satisfy $F_w[\lambda] = \lambda^{-2}F_w[1]$ and therefore equation (2.38) reduces to

$$(2.14) \qquad\qquad \lambda^2\ddot{\lambda} + F_w[1] = 0.$$

Since λ is a function of t only and w depends only on z, both terms must be constant:

$$(2.15) \qquad\qquad\qquad \lambda^2\ddot{\lambda} = \delta$$

$$(2.16) \qquad F_w[1] = \frac{4w'}{z} + \frac{1}{z^3}\int_0^z 4\pi w^3 s^2\, ds = -\delta$$

for some $\delta \in \mathbb{R}$. The existence of a solution to (2.15)–(2.16) amounts to the existence of $\lambda(t)$ (the radius of the star), $w(z)$ (the star configuration) satisfying (2.15)–(2.16). In fact, the above two equations are essentially the ones obtained by Goldreich and Weber [14], Makino [33], Fu and Lin [13] starting with the ansatz $r = \lambda(t)z$, $\rho(t,r) = \lambda^{-3}(t)w^3(z)$ and $v(t,r) = \dot{\lambda}(t)z$ in Eulerian coordinates. Note that a true self-similar ansatz would have the relationship $v(t,r) = \dot{\lambda}(t)z$ replaced by $v(t,r) = \lambda^{-\frac{1}{2}}(t)z$ thus honoring the self-similar rescaling (1.4)–(1.5) when $\gamma = \frac{4}{3}$. The solvability of (2.15) and (2.16) with suitable initial, boundary conditions and the behavior of the solutions are discussed in detail in [33, 13]. In what follows, we shall summarize some of these results, and state the corresponding regularity properties necessary for the nonlinear analysis.

To describe the solutions to (2.16) it is convenient to work with the second order formulation

$$(2.17) \qquad\qquad w'' + \frac{2}{z}w' + \pi w^3 = -\frac{3\delta}{4},$$

which follows easily from (2.16). Note that $\delta = 0$ corresponds to the classical Lane-Emden equation with index 3 [5]. We refer to (2.17) as the

generalized Lane-Emden equation. The following result guarantees the existence of solutions with suitable boundary conditions, thereby showing that the value of δ cannot be arbitrary.

Proposition 2.5 ([13]). *There exists a negative constant $\delta^* < 0$ such that for any $\delta \geq \delta^*$ there exists $w = w(z)$ satisfying (2.17) such that $0 < w < \infty$ in $[0,1)$ and w satisfies the boundary conditions*

$$(2.18) \qquad\qquad w'(0) = 0, \quad w(1) = 0.$$

Proposition 2.5 is essentially contained in [13]. We point out only one difference. Equation (2.17) is customarily solved with the boundary conditions $w(0) = 1$, $w'(0) = 0$ by a shooting method. The first zero denoted by $\bar{z} = \bar{z}(\delta)$ is then the radius of a star configuration. This solution is transformed to our desired solution w in Proposition 2.5 through a similarity transformation. Namely, for any $\beta > 0$ we note that $w_\beta(z) := \beta w(\beta z)$ solves (2.17) with δ replaced by $\delta \beta^3$ and it satisfies $w_\beta(0) = \beta$, $w_\beta'(0) = 0$, and $\bar{z}_\beta = \beta^{-1} \bar{z}(\delta)$.

The following lemma concerns the regularity of w.

Lemma 2.6. *Let w be a solution obtained in Proposition 2.5. Then $w \in C^\infty(0,1)$ and w is analytic near $z = 0$ as well as $z = 1$. Moreover*

(i) $w(z) = A_1 + A_2 z^2 + O(z^4)$, $z \sim 0$ for some constants A_1 and A_2 and $w^{(2k+1)}(0) = 0$ for any nonnegative integer $k \geq 0$;

(ii) w satisfies the physical vacuum condition, i.e. strict inequalities $-\infty < w'(1) < 0$ hold.

Proof. It follows from a minor modification of Lemma 3.3 in [18]. We omit the details. □

Self-similar solutions of (2.15) are by definition the ones satisfying

$$(2.19) \qquad\qquad b := -\frac{\lambda_s}{\lambda} = \text{const.}$$

where $\frac{ds}{dt} = \lambda(t)^{-\frac{3}{2}}$. This translates in the assumption $\ddot{\lambda}(t)\lambda^{\frac{1}{2}}(t) = \text{const.}$ Upon specifying the initial conditions

$$(2.20) \qquad\qquad \lambda(0) = \lambda_0 > 0, \quad \dot{\lambda}(0) = \lambda_1,$$

it is straightforward to check that for any $\delta \in (\delta^*, 0)$ there exists a self-similar solution $\lambda(t)$ of the form:

$$(2.21) \qquad\qquad \lambda(t) = \left(\lambda_0^{3/2} + \frac{3}{2} \lambda_0^{1/2} \lambda_1 t \right)^{2/3}$$

satisfying (2.15) and (2.19) with the additional initial data constraint

$$(2.22) \qquad\qquad \lambda_1^2 + \frac{2\delta}{\lambda_0} = 0.$$

Of course there exist other non-self-similar solutions of (2.15) with more general initial data. To discuss the asymptotic behavior of homologous solutions, for any $a \in [-\infty, \infty]$ and any two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ we say that

$$f(t) \sim_{t \to a} g(t)$$

if

$$\text{(2.23)} \qquad \lim_{t \to a} \frac{f(t)}{g(t)} = 1.$$

The following result discusses the behavior of the solutions of (2.15) and (2.20).

Proposition 2.7 ([33, 13]). *Let $\lambda(t)$ be the solution of* (2.15) *and* (2.20).

(1) *If $\delta > 0$, then $\lambda(t) > 0$ for all $t > 0$, $\lim_{t \to \infty} \lambda(t) = \infty$ and moreover, there exist $c_1, c_2 > 0$ such that*

$$\text{(2.24)} \qquad \lambda(t) \sim_{t \to \infty} c_1(1 + c_2 t).$$

(2) *If $\delta = 0$, then $\lambda(t) = \lambda_0 + \lambda_1 t$.*
(3) *If $\delta < 0$ let*

$$\text{(2.25)} \qquad \lambda_1^* = \sqrt{\frac{2|\delta|}{\lambda_0}}.$$

   (a) *If $\lambda_1 = \lambda_1^*$, then $\lambda(t) > 0$ for all $t > 0$ and it is explicitly given by the formula:*

$$\text{(2.26)} \qquad \lambda(t) = \left(\lambda_0^{3/2} + \frac{3}{2} \lambda_0^{1/2} \lambda_1 t \right)^{2/3}, \quad t \geq 0.$$

   (b) *If $\lambda_1 > \lambda_1^*$ then $\lambda(t) > 0$ for all $t > 0$, $\lim_{t \to \infty} \lambda(t) = \infty$, and there exist $c_1, c_2 > 0$ such that $\lambda(t)$ satisfies the asymptotic relation* (2.24).
   (c) *If $\lambda_1 < \lambda_1^*$, then there is a time $0 < T < \infty$ such that $\lambda(t) > 0$ in $(0, T)$ and $\lambda(t) \to 0$ as $t \to T^-$. Moreover, there exist constants $k_1, k_2 > 0$ such that*

$$\text{(2.27)} \qquad \lambda(t) \sim_{t \to T^-} k_1 (T - k_2 t)^{\frac{2}{3}}.$$

Proof. The only statements not discussed in [33, 13] are the precise rates of expansion stated above. Rate (2.26) is easily obtained by explicitly solving (2.15) with the initial condition (2.25), whereas rates (2.24) and (2.27) require a little more work, but follow from classical ODE arguments. □

Propositions 2.5 and 2.7 in particular imply that for any given $(\delta, \lambda_0, \lambda_1) \in [\delta^*, \infty) \times (0, \infty) \times (-\infty, \infty)$ satisfying (2.22), $\chi \equiv \lambda(t)$ is a solution of (2.5)

with λ given by (2.26) and w a solution of (2.16) and (2.18). In Eulerian coordinates, these solutions are given by

$$(2.28) \qquad \rho(t,r) = \lambda(t)^{-3} w^3 \left(\frac{r}{\lambda(t)} \right), \quad v(t,r) = \frac{\dot{\lambda}(t)}{\lambda(t)} r.$$

We next examine the physical energy and total mass for the homologous solutions given by (2.28).

Lemma 2.8. *The energy E of the homologous solutions is given by*

$$(2.29) \qquad E = \left(\lambda_1^2 + \frac{2\delta}{\lambda_0} \right) \int_0^1 2\pi w^3 z^4 \, dz.$$

Proof. Recall the total energy:
(2.30)
$$E = \int_0^{\lambda(t)} \left(\frac{1}{2} \rho v^2 + 3\rho^{\frac{4}{3}} \right) 4\pi r^2 \, dr - \int_0^{\lambda(t)} 4\pi \rho r \left(\int_0^r 4\pi \rho s^2 ds \right) dr.$$

We first compute the mass in the ball of radius r:

$$\int_0^r 4\pi \rho s^2 ds = 4\pi \int_0^r \frac{1}{\lambda^3} w^3 \left(\frac{s}{\lambda} \right) s^2 \, ds = 4\pi \int_0^{\frac{r}{\lambda}} w^3(z) z^2 \, dz$$

$$= 4 \left(-\left(\frac{r}{\lambda} \right)^2 w'\left(\frac{r}{\lambda} \right) - \left(\frac{r}{\lambda} \right)^3 \frac{\delta}{4} \right)$$

and hence the potential energy can be written as

$$\int_0^{\lambda(t)} 4\pi \rho r \left(\int_0^r 4\pi \rho s^2 ds \right) dr = \int_0^{\lambda} \frac{16\pi}{\lambda^3} w^3 r \left(-\left(\frac{r}{\lambda} \right)^2 w'\left(\frac{r}{\lambda} \right) - \left(\frac{r}{\lambda} \right)^3 \frac{\delta}{4} \right) dr$$

$$= \frac{12\pi}{\lambda^4} \int_0^{\lambda} w^4 r^2 \, dr - \frac{4\pi\delta}{\lambda^6} \int_0^{\lambda} w^3 r^4 \, dr.$$

Therefore, the total energy is written as

$$E = \frac{2\pi\dot{\lambda}^2}{\lambda^5} \int_0^{\lambda} w^3 r^4 \, dr + \frac{12\pi}{\lambda^4} \int_0^{\lambda} w^4 r^2 \, dr - \left(\frac{12\pi}{\lambda^4} \int_0^{\lambda} w^4 r^2 \, dr - \frac{4\pi\delta}{\lambda^6} \int_0^{\lambda} w^3 r^4 \, dr \right).$$

Now we use the equation for λ to obtain the total energy

$$E = \frac{\lambda_1^2 + \frac{2\delta}{\lambda_0}}{2\lambda^5} \int_0^{\lambda} 4\pi w^3 r^4 \, dr = \frac{\lambda_1^2 + \frac{2\delta}{\lambda_0}}{2} \int_0^1 4\pi w^3 z^4 dz.$$

$\square$

Lemma 2.8 and Proposition 2.7 reveal an interesting relationship between the energy of self-similar expansions and the energy of non self-similar expansions. The self-similar stellar expansion is associated with zero-energy stellar configurations, i.e.

$$(2.31) \qquad E(\bar{\lambda}(t), \dot{\bar{\lambda}}(t)) = 0.$$

For all other linearly expanding homologous solutions, the associated physical energy is strictly positive. On the other hand, the energy of the collapsing solutions can be positive, negative, or zero.
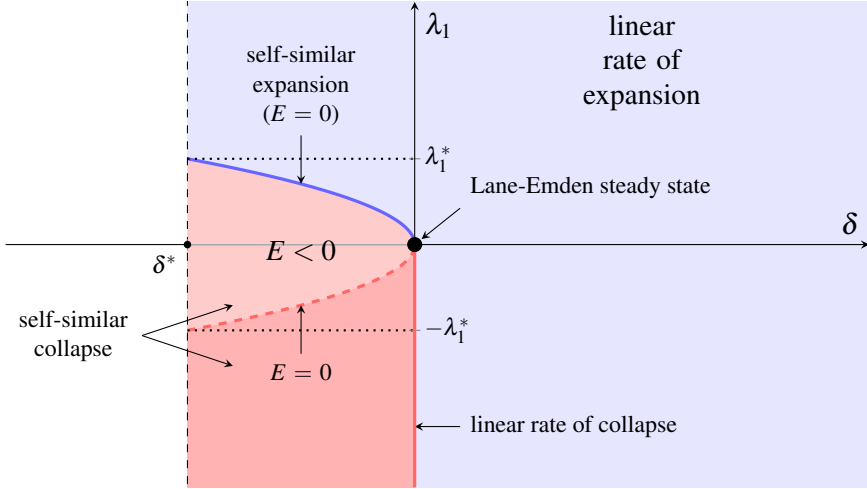

FIGURE 2.1. Bifurcation diagram, $\lambda_0 = 1$

A simple computation based on the generalized Lane-Emden equation (2.17) shows that the total mass of a homologous solution is given by

$$(2.32) \qquad M = M(\delta) = \int_0^{\lambda(t)} 4\pi \rho r^2 \, dr = -4w'(1) - \delta$$

which obviously depends only on δ. Since $\delta \in [\delta_*, \infty)$ and $-\infty < w'(1) < 0$, $M(\delta)$ cannot assume arbitrary positive values. Theorem 3.3 in [13] asserts that for any $\delta \in (\delta_*, \infty)$

$$\frac{dM(\delta)}{d\delta} < 0,$$

which implies that the admissible total mass lies in $(0, M(\delta_*)]$. This fact supports the following rather appealing interpretation of Propositions 2.7 and 2.5: if $M < M(0)$, then the star will expand and the density will eventually go to zero. If $M \geq M(0)$ and if the initial velocity is below the escape velocity, the star will collapse toward the center in finite time. We note that $M(0)$ is the mass of the steady Lane-Emden configuration. The notion of critical mass separating homologous collapse from expansion is discussed for the non-isentropic Euler-Poisson in general dimensions in [12].

Remark 2.9. It is obvious from Figure 2.1 that the Lane-Emden steady state ($\gamma = \frac{4}{3}$) at the origin of the graph is dynamically unstable. This is clear from [14, 33, 13, 12] and was specifically pointed out by Rein [38],

who remarked that positive energy configuration can lead to an indefinite expansion of the fluid support [11]. Note that Figure 2.1 shows that we can perturb the steady state even with negative energy configuration leading to an eventual collapse of the star, or with positive energy configurations that also lead to a collapse! The phase space around the steady state is intricate and the nature of instability (expansion vs. collapse) cannot be decided based solely on the sign of the energy of the perturbation.

2.3 Nonlinear stability in similarity variables

Our goal is to study the nonlinear stability of the expanding star families $(\bar{\lambda}, \bar{\rho}, \bar{v})$ and $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})$ given by Definitions 1.1 and 1.2 respectively. The general strategy will be to pass to suitable similarity coordinates, where the time-dependent expanding homologous solution transforms into a steady state of the new system of equations. Because of different rates of expansion, our stability results are presented in two separate subsections. We start with self-similar expansion.

Nonlinear stability for self-similarly expanding homologous solutions

Let $(\bar{\lambda}_*, \bar{\rho}_*, \bar{v}_*)_{\lambda_0^*, \lambda_1^*, \delta^*}$ be a given self-similar expanding homologous solution of the $\mathrm{EP}_{\frac{4}{3}}$-system given by Definition 1.2. We consider a perturbation of $(\bar{\lambda}_*, \bar{\rho}_*, \bar{v}_*)_{\lambda_0^*, \lambda_1^*, \delta^*}$

$$\rho(0,r) = \rho_0(r), \quad v(0,r) = v_0(r).$$

Without loss of generality we shall assume that $\rho_0 : [0,1] \to \mathbb{R}$ satisfies

(2.33) $\rho_0(r) > 0, \ r \in [0,1), \ \rho_0(1) = 0, \ \overline{\mathrm{supp}(\rho_0)} = [0,1].$

We further assume that

(2.34) $|M(\rho_0) - M(\bar{\rho}_*)| \ll 1, \ E(\rho_0, v_0) = E(\bar{\rho}^*, \bar{v}^*) = 0.$

The second condition means that we will restrict our stability analysis to a surface of zero-energy initial data.

In order to describe the asymptotic behavior of the solutions generated by (ρ_0, v_0) we will use the conservation-of-mass law to identify the background expanding homologous self-similar solution that we expect the perturbed solution to converge to. We accomplish this by finding a δ such that

(2.35) $$M(\delta) = M(\rho_0) = \int_{\mathbb{R}^3} \rho_0(x)\, dx.$$

In particular, this uniquely fixes a background solution $(\bar{\lambda}, \bar{\rho}, \bar{v})_{\lambda_0, \lambda_1, \delta}$ with

$$\lambda_0 = 1, \ e(\bar{\lambda}, \bar{\lambda}_t) := \lambda_1^2 + \frac{2\delta}{\lambda_0} = 0$$

where e is the coefficient appearing in E (1.16) and we call it the effective energy.

We now recall the Lagrangian formulation of the problem (2.2)–(2.4) and assume that there exists a $\chi_0 : [0,1] \to [0,1]$ such that

$$(2.36) \qquad \det D\eta_0 = \chi_0^2(\chi_0 + z\partial_z\chi_0) = \frac{\rho_0 \circ \chi_0}{\bar{\rho}(0,\cdot)}.$$

For initial density ρ_0 exhibiting the same boundary behavior as $\bar{\rho}$ so that $\rho_0/\bar{\rho}$ is a smooth positive function, the existence of such a function χ_0 follows from the Dacorogna-Moser theorem [10]. Since the energy is a dynamically conserved quantity, as long as smooth solutions to (2.5)–(2.6) exist, we have the identity

$$E(\chi, \chi_t)(t) = E(\chi_0, \chi_1).$$

By the second assumption in (2.34) we note that

$$(2.37) \qquad E(\chi_0, \chi_1) = 0.$$

To understand the nonlinear asymptotics, we introduce a new unknown ξ by setting

$$\xi(t,z) := \frac{\chi(t,z)}{\bar{\lambda}(t)}.$$

As a consequence of (2.5) we obtain an evolution equation for ξ :

$$(2.38) \qquad \bar{\lambda}^2(\bar{\lambda}\xi)_{tt} + F_{w_\delta}[\xi] = 0,$$

equipped with the initial conditions

$$(2.39) \qquad \xi(0,z) = \chi_0(z), \;\; \xi_t(0,z) = \xi_1(z), \;\; z \in [0,1],$$

where $\xi_1 = \chi_1 - \chi_0\bar{\lambda}_1$ and the nonlinear operator F_{w_δ} is given by (2.7). With respect to the new variable ξ the energy E takes the form:

$$E(\xi, \xi_t) = 2\pi\|\xi_t\bar{\lambda} + \xi\bar{\lambda}_t\|_\delta^2 + \frac{4\pi\delta}{\bar{\lambda}}\|\xi^{-1/2}\|_\delta^2$$

$$(2.40)$$

$$+ \frac{12\pi}{\bar{\lambda}}\left(\int_0^1 (\xi^2(\xi + z\xi_z))^{-1/3}w_\delta^4 z^2\, dz + \frac{1}{3}\int_0^1 \xi^{-1}\partial_z(w_\delta^4)z^3\, dz\right).$$

Motivated by the self-similar rescaling (1.4)–(1.5) we introduce a self-similar time coordinate s by setting

$$(2.41) \qquad s = s(t) = \int_0^t \frac{1}{\bar{\lambda}(\tau)^{\frac{3}{2}}}\, d\tau,$$

and the unknown ψ in new coordinates (s,z):

$$(2.42) \qquad \psi(s,z) := \xi(t,z).$$

Using (2.38), (2.41), and (2.42) a straightforward calculation gives a quasilinear wave equation for ψ

$$(2.43) \qquad \psi_{ss} - \frac{1}{2}b\psi_s + \delta\psi + F_{w_\delta}[\psi] = 0,$$

where

(2.44)
$$b = -\frac{\bar{\lambda}_s}{\bar{\lambda}} = -\sqrt{2|\delta|} < 0.$$

We supply (2.43) with initial conditions:

(2.45)
$$\psi(0) = \psi_0 = \xi_0, \quad \psi_s(0) = \psi_1 = \xi_1,$$

where we have used $\bar{\lambda}_0 = 1$. Observe that conditions (2.45) are naturally inherited from (2.39). Equation (2.43) admits a steady state

$$\bar{\psi} \equiv 1$$

which is a self-similar realization of the background expanding solution $(\bar{\lambda}, \bar{\rho}, \bar{v})_{\lambda_0, \lambda_1, \delta}$ given by (2.26). We note that $\bar{\lambda}$ in the new time coordinate s takes the following form

(2.46)
$$\bar{\lambda}(s) = e^{-bs},$$

and thus grows exponentially in s (recall that $b < 0$ corresponds to the expansion case). The energy (2.40) in the new variables takes the form

$$E(\psi, \psi_s) = \frac{2\pi}{\bar{\lambda}} \|\psi_s + \frac{\bar{\lambda}_s}{\bar{\lambda}}\psi\|_\delta^2 + \frac{4\pi\delta}{\bar{\lambda}} \|\psi^{-1/2}\|_\delta^2$$

(2.47)
$$+ \frac{12\pi}{\bar{\lambda}} \left(\int_0^1 (\psi^2(\psi + z\psi_z))^{-1/3} w_\delta^4 z^2 \, dz + \frac{1}{3} \int_0^1 \psi^{-1} \partial_z(w_\delta^4) z^3 \, dz \right).$$

To stress the dependence of the background profile w_δ on the parameter δ we shall from now on use the notation

$$(\cdot, \cdot)_{\delta, k} := (\cdot, \cdot)_{w_\delta, k}, \quad (\cdot, \cdot)_\delta := (\cdot, \cdot)_{w_\delta, 0},$$

$$\|\cdot\|_{\delta, k} := \|\cdot\|_{w_\delta, k}, \quad \|\cdot\|_\delta := \|\cdot\|_{w_\delta}, \quad \mathcal{H}_\delta := \mathcal{H}_{w_\delta},$$

where the inner product $(\cdot, \cdot)_{\delta, k}$ and the norms $\|\cdot\|_{w, k}$ have been defined in (2.9) and the high-order weighted space $\mathcal{H}_w$ is defined by (2.12). For any perturbation $\phi = \psi - 1$ from the steady state we define a high-order energy norm

(2.48)
$$\mathcal{E}(\phi, \phi_s) = \mathcal{E}(s) := \sum_{j=0}^7 \sum_{k=0}^j \|\partial_s^{j-k+1} \partial_z^k \phi\|_{\delta, k}^2 + \|\partial_s^{j-k} \partial_z^{k+1} \phi\|_{\delta, k+1}^2 + \|\partial_s^{j-k} \partial_z^k \phi\|_{\delta, k}^2.$$

Theorem 2.10 (Nonlinear stability of selfsimilar expanding stars). *There exist $\varepsilon, \mathcal{E} > 0$ such that for any $-\varepsilon < \delta < 0$ and any initial datum $(\psi(0), \psi_s(0)) = (\psi_0, \psi_1)$ with vanishing initial energy*

(2.49)
$$E(\psi_0, \psi_1) = 0,$$

satisfying the smallness condition

(2.50)
$$\mathcal{E}(\psi_0 - 1, \psi_1) \leq \varepsilon$$

there exists a unique global solution to the initial value problem (2.43)–
(2.45). *Moreover, there exists a constant* $c = c(\delta)$ *such that*

$$(2.51) \qquad\qquad \mathcal{E}(s) \lesssim \varepsilon e^{-cs}, \ s \geq 0.$$

Remark 2.11 (Initial data and compatibility conditions). Since our energy
norm contains higher s-derivatives of ψ at time $s = 0$, we use the equation (2.43) to define $\partial_s^j \psi\big|_{s=0}$, $j \geq 2$. Note that we do not have to impose
any compatibility conditions at the vacuum boundary $z = 1$.

Remark 2.12. Since Theorem 2.10 shows that $\eta(t, r) = \chi(t, r)r$ remains
close to $\bar{\eta}(t, r) = \bar{\lambda}(t)r$ for all $t > 0$, from the formula

$$\rho(t, \eta(t, r)) = \bar{\rho}(0, r)\big[\det D\eta_0\big]^{-1}$$

we may conclude that the Eulerian density $\rho(t, r)$ of the perturbed solution
remains "close" to the self-similarly rescaled Eulerian density $\bar{\lambda}^{-3}w_\delta^3(\frac{r}{\bar{\lambda}})$
of the background Eulerian density. A similar comparison holds between
the Eulerian velocity v of the perturbation and the rescaled background
velocity $\bar{\lambda}^{-\frac{1}{2}}\bar{v}(\frac{r}{\bar{\lambda}})$.

Remark 2.13 (Notation " $\lesssim$ "). We shall write $A \lesssim B$ if there exists a universal constant $C > 0$ depending only on ε such that $A \leq CB$. An analogous
definition applies to the notation $A \gtrsim B$.

Nonlinear stability of linearly expanding homologous solutions

Let $(\tilde{\lambda}_*, \tilde{\rho}_*, \tilde{v}_*)_{\lambda_0^*, \lambda_1^*, \delta^*}$ be a given linearly expanding homologous solution of the $\text{EP}_{\frac{4}{3}}$-system given by Definition 1.2. We consider a perturbation
of $(\tilde{\lambda}_*, \tilde{\rho}_*, \tilde{v}_*)_{\lambda_0^*, \lambda_1^*, \delta^*}$

$$\rho(0, r) = \rho_0(r), \ \ v(0, r) = v_0(r).$$

Without loss of generality we shall assume that $\tilde{\lambda}_0^* = 1$ and that $\rho_0 : [0, 1] \to$
$\mathbb{R}$ satisfies (2.33). We further assume that

$$|M(\rho_0) - M(\tilde{\rho}_*)| \ll 1, \ \ |E(\rho_0, v_0) - E(\tilde{\rho}^*, \tilde{v}^*)| \ll 1.$$

We may also assume that $E(\rho_0, v_0) > 0$ since by definition $E(\tilde{\rho}^*, \tilde{v}^*) > 0$.
In order to describe the asymptotic behavior of the solutions generated
by (ρ_0, v_0), just like in previous section, we will use the conservation-of-mass and conservation-of-energy laws to uniquely fix a linearly expanding
background solution $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})_{\lambda_0, \lambda_1, \delta}$ satisfying

$$\lambda_0 = 1, \ \ M(\tilde{\rho}) = M(\rho_0), \ \ E(\tilde{\rho}, \tilde{v}) = E(\rho_0, v_0).$$

Recalling the Lagrangian formulation of the problem (2.2)–(2.4) we
assume that there exists a $\chi_0 : [0, 1] \to [0, 1]$ such that (2.36) holds with $\tilde{\rho}$

instead of $\bar{\rho}$. To understand the nonlinear asymptotics, we introduce a new unknown ζ by setting

$$\zeta(t,z) := \frac{\chi(t,z)}{\tilde{\lambda}(t)}.$$

The unknown ζ solves (2.38) with ζ instead of ξ and $\tilde{\lambda}$ instead of $\bar{\lambda}$, while the associated initial conditions read

(2.52) $\zeta(0,z) = \chi_0(z), \quad \zeta_t(0,z) = \zeta_1(z), \quad z \in [0,1],$

where $\zeta_1 = \chi_1 - \chi_0 \lambda_1$. To reflect the expected linear growth rate of the perturbed solution we introduce a new time variable $\tau = \tau(t)$:

(2.53) $\tau = \tau(t) = \int_0^t \frac{1}{\tilde{\lambda}(\sigma)} \, d\sigma.$

Analogously to (2.42) we introduce an unknown θ in new coordinates (τ, z):

(2.54) $\theta(\tau, z) := \zeta(t, z).$

   Using (2.38) and (2.53) we derive an equation for the unknown $\theta(s, z) = \zeta(t, z)$

(2.55) $\tilde{\lambda}\,\theta_{\tau\tau} + \tilde{\lambda}_\tau \theta_\tau + \delta\theta + F_{w_\delta}[\theta] = 0,$

with the initial data:

(2.56) $\theta(0) = \theta_0 = \zeta_0, \quad \theta_\tau(0) = \theta_1 = \zeta_1.$

In the τ-coordinate, $\tilde{\lambda}$ satisfies the following ODE:

$$\tilde{\lambda}_\tau = \tilde{\lambda}\sqrt{\tilde{e} - \frac{2\delta}{\tilde{\lambda}}},$$

where $\tilde{e} = \lambda_1^2 + \frac{2\delta}{\lambda_0}$ is the effective energy of the linearly expanding homologous solution $(\tilde{\rho}, \tilde{u}, \tilde{\lambda})$. It follows that for large values of τ the asymptotic behavior of $\tilde{\lambda}$ is given by

(2.57) $\tilde{\lambda}(\tau) \sim_{\tau \to \infty} e^{\tilde{\beta}\tau},$

where $\tilde{\beta} := \sqrt{\tilde{e}}$. If $\delta \geq 0$ it follows that

(2.58) $\tilde{\beta} \geq \left|\frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}\right| \geq \beta'$

where

$$\beta' := \sqrt{\tilde{e} - 2\frac{\delta}{\lambda_0}}.$$

Similarly,

(2.59) $\beta' \geq \left|\frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}\right| \geq \tilde{\beta}$

if $\delta < 0$. Let $\beta_1 = \max\{\tilde{\beta}, \beta'\}$ and $\beta_2 = \min\{\tilde{\beta}, \beta'\}$. Then for all linearly expanding homologous solutions

(2.60) $e^{\beta_2\tau} \lesssim |\tilde{\lambda}(\tau)| + |\tilde{\lambda}_\tau(\tau)| \lesssim e^{\beta_1\tau}, \quad \tau \geq 0.$

Equation (2.55) admits a steady state

$$\tilde{\theta} \equiv 1$$

which corresponds to the background expanding solution $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})_{\lambda_0, \lambda_1, \delta}$ encompassed by Definition 1.2 (i.e. belonging to the light-blue region in Figure 2.1.) The total energy in the new variable takes the form

$$E(\theta, \theta_\tau) = 2\pi \|\theta_\tau + \frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}\theta\|_\delta^2 + \frac{4\pi\delta}{\tilde{\lambda}}\|\theta^{-1/2}\|_\delta^2$$

(2.61)

$$+ \frac{12\pi}{\tilde{\lambda}}\left(\int_0^1 (\theta^2(\theta + z\theta_z))^{-1/3}w_\delta^4 z^2\,dz + \frac{1}{3}\int_0^1 \theta^{-1}\partial_z(w_\delta^4)z^3\,dz\right).$$

A key differential operator in our analysis is given by

(2.62) $\mathscr{S} := \partial_z^2 + \frac{4}{z}\partial_z.$

It enters crucially in the definition of our high-order energy space.

Definition 2.14 (The space $\mathfrak{H}_{\delta,k}^j$). For any $j \in \mathbb{N}$ and $k \in \mathbb{N}$, we say that a function φ belongs to the space $\mathfrak{H}_{\delta,k}^j$ if $\mathscr{S}^\ell\varphi \in L_{\delta,2k+2\ell}^2([0,1])$ for all $0 \leq \ell \leq j$.

For any perturbation $\phi = \theta - 1$ from the steady state we define a high-order energy norm

(2.63)

$$\tilde{\mathscr{E}}(\phi, \phi_\tau) = \tilde{\mathscr{E}}(\tau) = \sum_{j=0}^4 \left(\tilde{\lambda}\|\mathscr{S}^j\phi_\tau\|_{\delta,2j}^2 + \|\mathscr{S}^j\phi\|_{\delta,2j}^2 + \|\partial_z\mathscr{S}^j\phi\|_{\delta,2j+1}^2\right)$$

Theorem 2.15 (Nonlinear stability of linearly expanding stars). *There exists $\varepsilon, \varepsilon_0 > 0$ such that for any $-\varepsilon < \delta < \infty$, any $0 < \varepsilon \leq \varepsilon_0$ and any initial data (θ_0, θ_1) satisfying*

$$\tilde{\mathscr{E}}(\theta_0 - 1, \theta_1) \leq \varepsilon$$

there exists a unique global-in-time solution to the initial value problem (4.1)–(4.2). *Moreover, there exists a constant $C > 0$ such that*

$$\sup_{\tau \geq 0} \tilde{\mathscr{E}}(\tau) \leq C\varepsilon$$

and a τ-independent function θ_∞ satisfying $\tilde{\mathscr{E}}(\theta_\infty, 0) \leq C\varepsilon$ such that

(2.64) $\lim_{\tau \to \infty} \|\theta(\tau, \cdot) - \theta_\infty(\cdot)\|_{\mathfrak{H}_{\delta,0}^4} = 0, \quad \|\theta_\tau(\tau, \cdot)\|_{\mathfrak{H}_{\delta,0}^4} \lesssim \varepsilon\tilde{\lambda}(\tau)^{-\frac{1}{2}}.$

Remark 2.16. The asymptotic attractor θ_∞ is not necessarily a steady state solution of (2.55). Unlike the self-similar case (Theorem 2.10) we do not claim that the perturbed solution converges to a nearby member of the family of linearly expanding homologous solutions. In fact we should not expect to get a decay estimate for the unknown ϕ and its spatial derivatives. A much simpler toy model ODE

$$e^\tau \partial_{\tau\tau} x + e^\tau \partial_\tau x + x = 0$$

has a continuum of possible asymptotic states, although $x \equiv 0$ is the only steady state. Due to the fast growing factors in front of $\partial_{\tau\tau} x$ and $\partial_\tau x$, the leading order decay of $\partial_\tau x$ is in general given by $\partial_\tau x = c\tau e^{-\tau} + O(e^{-\tau})$, where c is a constant depending on the initial conditions. It can be checked that a nonzero c leads to a nonzero asymptotic state for x.

Remark 2.17. An important consequence of Theorem 2.15 and the precise understanding of the asymptotic-in-time behavior of the Lagrangian map $\eta(t,r) = \chi(t,r)r = \tilde{\lambda}(t)\theta(t,r)r$ is that we can precisely track the growth of the support of the Eulerian density ρ of the perturbation. If $\lambda(t)$ denotes the radius of the support of the perturbed solution, then Theorem 2.15 implies that there exists a constant $C \geq 1$ such that

$$C^{-1}\tilde{\lambda}(t) \leq \lambda(t) \leq C\tilde{\lambda}(t), \ \ t \geq 0,$$

where $\tilde{\lambda}(t)$ is the radius of support of the background Eulerian density $\tilde{\rho}$. Just like in Remark 2.12 we can also show that the Eulerian density and velocity remain "close" to the suitably rescaled Eulerian density and velocity of the background homologous solution $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})$.

2.4 Comments and methodology

Questions of singularity formation in various critical problems have received a lot of attention in the past decades, especially in the context of nonlinear wave, Yang-Mills, nonlinear Schrödinger equations, generalized Korteweg- deVries equation, corotational wave maps and various other hyperbolic problems. An exhaustive overview with many references can be found in [36]. The possible blow-up phenomenology is rich, with rates often deviating from the predicted self-similar rates, and such non self-similar regimes can be both stable and unstable. In the context of the gravitational Euler-Poisson problem, the catalogue of potential dynamical scenarios appears to be very rich and far from being understood.

The free-boundary gravitational Euler-Poisson system is a classical example of a system of hyperbolic balance laws where we may expect shock singularities to occur in finite time. A global-in-time existence and uniqueness theory is not available even in the context of small initial data. The manifestly hyperbolic nature of the equations is clearly seen in Lagrangian coordinates where the EP$_\gamma$-system takes a form of a degenerate quasilinear wave equation on a compact spatial domain. There are however two

well-known classes of exact solutions: the Lane-Emden steady states when $\frac{6}{5} < \gamma < 2$ and a special class of homologous (also known as homologous) solutions in the case $\gamma = \frac{4}{3}$ discovered and analyzed in [14, 33, 13] (see Section 2.2). Both classes of solutions share one important common feature - they satisfy the so-called physical vacuum boundary condition at the star-vacuum interface.

In the mass-critical case $\gamma = \frac{4}{3}$ Theorems 2.10 and 2.15 confirm that the two modes of expansion (marked by light-blue and dark-blue in Figure 2.1) are nonlinearly dynamically stable. It remains open to understand the asymptotic behavior in the vicinity of collapsing solutions (marked light- and dark-red in Figure 2.1). Moreover, in the supercritical regime $\frac{6}{5} < \gamma < \frac{4}{3}$ it is not known whether there exist *any* examples of compactly supported collapsing solutions and this remains an important open question. Finally, despite the conditional stability results [38, 27] in the subcritical regime $\frac{4}{3} < \gamma < 2$, the question of nonlinear stability of Lane-Emden stars remains open, as it is not clear whether fluid shock singularities might develop from small perturbations.

When $\gamma \geq \frac{4}{3}$ it was shown in [34, 11] that *if* there exists a global-in-time strictly positive energy solution to the spherically-symmetric EP$_\gamma$-system, then the star support grows *at least* linearly-in-time. Theorem 2.15 shows that in the vicinity of a linearly expanding homologous solutions this result is optimal, i.e. the star support grows *at most* linearly-in-time. Notice that the sub linear growth of the fluid support stated in Theorem 2.10 is associated only with the zero-energy configurations and therefore does not contradict [34].

Besides the work of Goldreich & Weber [14], there exist numerous other physics works devoted to various notions of self-similar collapsing/expanding star solutions of the Euler-Poisson system, see [25, 37, 43, 3, 4, 40] and references therein. The polytropic index γ is often allowed to vary in the full range $1 \leq \gamma < 2$. All of these other solutions have an infinite support of the star density and therefore do not possess a star-vacuum free boundary.

As an important consequence of our results, we show the existence of *global-in-time large data solutions* to the EP$_\gamma$-system (with $\gamma = \frac{4}{3}$). To the best of our knowledge, this is the first work proving existence and uniqueness of global-in-time classical solutions of the free-boundary EP$_\gamma$-system that are neither homologous nor stationary. Our results provide a set of initial data and an energy space that lead to global-in-time solutions in a functional framework that allows us to track the regularity and behavior of the *free boundary*. A key mechanism that precludes finite time singularity formation is the expansion of the background star solution. The fluid density decreases sufficiently rapidly to counteract a possible shock formation. A related phenomenon, albeit in the absence of free boundaries, appears in the context of the Euler-Einstein system with positive

cosmological constant [41, 42, 15, 32], wherein the role of the expanding homologous solutions is replaced by the well-known Friedman-Lemaître-Robertson-Walker expanding spacetimes.

The presence of the physical vacuum condition, while natural from the physics point of view, causes fundamental difficulties in the analysis, as the first derivative of the fluid enthalpy is in general discontinuous across the fluid-vacuum interface. The first step in handling both the free boundary and isolating the difficulties coming from the above mentioned degeneracy is to use the Lagrangian coordinates. In these coordinates, a natural functional-analytic framework of weighted Sobolev spaces emerges as the right setting for the well-posedness analysis. The initial fluid density is a natural weight in the problem. These tools have been first designed and successfully used to overcome the above mentioned difficulties in the context of compressible Euler equations in vacuum [8, 23] and in the context of the Euler-Poisson system in [18, 19]. A similar methodology was used to prove local-in-time existence for the Euler-Poisson system [28]. In [35], Nash-Moser theory was used to establish the existence of smooth local solutions of the Euler-Poisson system approximating time periodic (linearized) profiles.

The fluid densities associated with the expanding homologous stars also satisfy the physical vacuum condition, as they solve the generalized Lane-Emden equation. To prove nonlinear stability of such solutions, we adapt the Lagrangian flow map to the expanding background, and control the deviation by suitably rescaling the time variable. In our approach, the initial density of the perturbation enters in the choice of the initial flow map.

To prove Theorem 1.3 we introduce the self-similar time coordinate s defined by

$$(2.65) \qquad \frac{ds}{dt} = \frac{1}{\overline{\lambda}(t)^{3/2}}, \ \text{ i.e. } \ s \sim \log t.$$

The self-similar expanding homologous solutions transform into steady states of the newly rescaled equation given by

$$(2.66) \qquad \phi_{ss} - \frac{1}{2} b \phi_s + 3\delta\phi + \mathscr{L}_\delta \phi = N_\delta[\phi],$$

where $\mathscr{L}_\delta$ is a suitable second order self-adjoint differential operator, N_δ a nonlinearity and $b < 0$ is a constant. Due to the scaling invariance of the problem, coefficients in front of the ∂_s-derivatives in (2.66) are *constant*. A key mechanism for our stability result is a new *damping effect* induced by the term $-\frac{1}{2} b \phi_s$ (recall that $b < 0$) in (2.66). It exists as a consequence of the expansion of the background self-similar homologous solution. However, due to a stretch effect coming from $3\delta\phi$ (note that $\delta < 0$), the linear part admits a growing mode $\phi = e^{-bs}$, which makes the stability nontrivial and interesting. It is not a priori clear that this growing mode will not

persist at the nonlinear level. However, the unstable mode is in fact a re-flection of the conservation-of-energy law and the associated eigenvector is tangential to the manifold of constant energy solutions. Therefore to deal with such an instability we crucially use a nonlinear constraint inherited from the conservation of the physical energy, thereby reducing our stability result to a codimension-one set of initial data.

The stability of the homologous self-similar expanding solutions relies on proving an exponential decay-in-s of small amplitude perturbations with zero energy. Such a decay result follows from the energy-dissipation inequalities for suitably defined high-order energy functionals $\mathscr{E}, \mathscr{D}$ taking the form

$$\mathscr{E}(s) + \int_0^s \mathscr{D}(\tau)d\tau \lesssim \mathscr{E}(0) + \int_0^s \mathscr{E}^{3/2}(\tau)d\tau, \quad s \geq 0$$

where we have the full coercivity of the dissipation

$$\mathscr{D} \gtrsim \mathscr{E}.$$

Our nonlinear estimates naturally split into two parts. First we prove the energy-dissipation inequalities for pure ∂_s-derivatives of ϕ using the standard Hardy inequalities and embeddings between weighted Sobolev spaces. In our second step we provide the bounds for the full energy (involving purely spatial and mixed space-time derivatives) and this is accomplished by using (2.66) and a bootstrap procedure allowing us to express higher spatial derivatives in terms of ∂_s-derivatives. We refer to this part of the analysis as the *elliptic estimates*. A crucial tool in the proof of the energy-dissipation bounds is the spectral gap of the linearized operator $\mathscr{L}_\delta$. Specifically $\mathscr{L}_\delta$ is a strictly positive operator as long as it acts on a space orthogonal to its null-space, i.e. as long as $(\phi, 1)_\delta = 0$. We therefore need to control $|(\phi, 1)_\delta|$ separately. This is accomplished by exploiting the nonlinear constraint $E(\phi, \phi_s) = 0$ which restricts the dynamics to the surface of zero-energy solutions. Specifically we show that the linearization of this condition yields a relationship of the schematic form

$$(\phi, 1)_\delta = \text{const.} \times (\phi_s, 1)_\delta + \text{quadratic nonlinearity}$$

which will allow us to show $|(\phi, 1)_\delta| \lesssim \mathscr{E}$.

To prove Theorem 1.5 we also rescale time. Note however that the linearly expanding homologous solutions $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})$ do not expand at a self-similar rate. Therefore, we introduce a new time coordinate τ by setting

(2.67) $$\frac{d\tau}{dt} = \frac{1}{\tilde{\lambda}(t)}, \text{ i.e. } \tau \sim \log t.$$

Like above, the homologous background solution $(\tilde{\lambda}, \tilde{\rho}, \tilde{v})$ transforms into a steady state, but unlike the self-similar case, the new equation for a perturbation ϕ in Lagrangian coordinates has τ-dependent coefficients in front

of $\partial_\tau \phi$ and $\partial_{\tau\tau}\phi$:

(2.68) $$\tilde{\lambda}\phi_{\tau\tau} + \tilde{\lambda}_\tau \phi_\tau + 3\delta\phi + \mathscr{L}_\delta \phi = N_\delta[\phi].$$

In the new coordinates $\tilde{\lambda} \sim e^{\beta\tau}$ for large values of τ where $\beta > 0$ is some constant. Although (2.68) and (2.66) look structurally similar as they appear to be damped wave equations, they are fundamentally different. By contrast to (2.66), the exponentially growing coefficients $\tilde{\lambda}$ and $\tilde{\lambda}_\tau$ in (2.68) are inevitable since the linear-in-time expansion does not honor the invariance of the system and it creates severe analytical difficulties. As a consequence, the high-order energy method developed for the self-similar case does not work since the operator ∂_τ does not commute well with (2.68) any longer. Namely, some of the commutators contain τ-dependent weights that grow faster than the natural energy and we cannot close the energy estimates. Instead we design a different high-order energy approach, based on differential operators containing only spatial derivatives. This idea goes back to the work by Jang & Masmoudi [23] on free-boundary compressible Euler equations where the number of spatial derivatives changes the weight in the energy space. In our problem a key elliptic operator

$$\mathscr{L}_{\delta,k}\psi = -\frac{4}{3w_\delta^{3+k}z^4}\partial_z(w_\delta^{4+k}z^4\partial_z\psi)$$

captures precisely the degeneracy caused by the physical vacuum. In addition to this we have to deal with the coordinate singularity at $z = 0$ because naive successive applications of the ∂_z-operators may result in unfavorable negative powers of z in our energy terms. To avoid this problem we commute the equation with high-order powers of the elliptic operator

$$\mathscr{S} := \partial_z^2 + \frac{4}{z}\partial_z$$

to derive the energy estimates. It turns out that $\mathscr{S}$ commutes well with (2.68) at both boundaries. Together with $\mathscr{L}_{\delta,k}\psi$ it enables us to construct energy spaces without commuting the equation with ∂_τ derivatives and close the energy estimates. The elliptic operators $\mathscr{L}_{\delta,k}$ and $\mathscr{S}$ have not explicitly appeared in the previous works and a careful and detailed analysis is carried out in Section 4.

The paper proceeds as follows. Section 3 is devoted to the proof of Theorem 2.10: the nonlinear stability of self-similar expanding homologous solutions. A crucial coercivity of the energy, leading to the exponential decay of perturbations, is shown in Lemma 3.4. Nonlinear energy estimates (Theorem 3.2) will be presented in detail. In Section 4, we prove Theorem 2.15: the nonlinear stability of linearly expanding homologous solutions. The $\mathscr{S}$-energy method (Theorem 4.4) will be developed based on various commutator estimates between $\mathscr{S}$ on one side and linear operator and nonlinear terms (Lemmas 4.6, 4.9, 4.13) on the other. We shall also

prove a comparison estimate between the $\mathscr{S}$ energy and the norm (Lemma 4.8). In the Appendix, we present the spectral theory of our key linearized operators, Hardy inequalities and weighted Sobolev spaces embedding results.

3 Nonlinear stability of self-similar expanding homologous solutions

To understand the stability properties of $\bar{\psi} \equiv 1$, we linearize around this steady state. Letting

$$\psi = \bar{\psi} + \phi = 1 + \phi,$$

using (2.17) we obtain

$$F_{w_\delta}[1+\phi] = \frac{(1+\phi)^2}{w_\delta^3 z} \partial_z \left(w_\delta^4 \left[1 + \frac{1}{z^2} \partial_z \left(z^3 \left[\phi + \phi^2 + \frac{\phi^3}{3} \right] \right) \right]^{-\frac{4}{3}} \right)$$

$$+ \frac{1}{(1+\phi)^2} \left(-\delta - \frac{4w_\delta'}{z} \right)$$

It is easy to check that $(1+y)^{-\frac{4}{3}} = 1 - \frac{4}{3} y + \frac{28}{9} \left(\int_0^1 (1-\theta)(1+\theta y)^{-\frac{10}{3}} d\theta \right) y^2$. Recall also that in the case of a self-similar homologous solution we have $\frac{b^2}{2} = -\delta$, where b is given by (2.19). By rearranging terms, we obtain the expansion

(3.1)

$$F_{w_\delta}[1+\phi] = -\delta + 2\delta\phi - \frac{4}{3w_\delta^3 z^4} \partial_z(w_\delta^4 z^4 \partial_z \phi)$$

$$+ \frac{b^2}{2} \left(\frac{1}{(1+\phi)^2} - 1 + 2\phi \right) + \frac{\partial_z w_\delta^4}{w^3 z} \left((1+\phi)^2 - \frac{1}{(1+\phi)^2} - 4\phi \right)$$

$$- \frac{4}{3} \frac{(1+\phi)^2 - 1}{w_\delta^3 z} \partial_z \left(w_\delta^4 \frac{1}{z^2} \partial_z(z^3 \phi) \right)$$

$$- \frac{4}{3} \frac{(1+\phi)^2}{w_\delta^3 z} \partial_z \left(w_\delta^4 \frac{1}{z^2} \partial_z(z^3(\phi^2 + \frac{\phi^3}{3})) \right)$$

$$+ \frac{28}{9} \frac{(1+\phi)^2}{w_\delta^3 z} \partial_z \left[\left(w_\delta^4 \left(\frac{1}{z^2} \partial_z \left(z^3 \left[\phi + \phi^2 + \frac{\phi^3}{3} \right] \right) \right) \right)^2 \right.$$

$$\left. \int_0^1 (1-\theta) \left[1 + \theta \left(\frac{1}{z^2} \partial_z \left(z^3 \left[\phi + \phi^2 + \frac{\phi^3}{3} \right] \right) \right) \right]^{-\frac{10}{3}} d\theta \right].$$

It follows that the linearization of $F_{w_\delta}[\cdot]$ about $\bar{\psi} \equiv 1$ is given by the operator

(3.2) $$DF_{w_\delta}(\bar{\psi})[\phi] = \mathscr{L}_\delta \phi + 2\delta\phi, \quad \mathscr{L}_\delta \phi := -\frac{4}{3w_\delta^3 z^4} \partial_z \left(w_\delta^4 z^4 \partial_z \phi \right).$$

Therefore, the perturbation ϕ satisfies the following equation:

$$(3.3) \qquad \phi_{ss} - \frac{1}{2}b\phi_s + 3\delta\phi + \mathscr{L}_\delta\phi = N_\delta[\phi],$$

equipped with the following initial conditions:

$$(3.4) \qquad \phi(0,z) = \phi_0(z) = \xi_0(z) - 1, \quad \phi_s(0,z) = \phi_1(z) = \xi_1.$$

Here $N_\delta[\phi]$ is the nonlinear remainder defined by the relationship

$$(3.5) \qquad F_{w_\delta}[1+\phi] = -\delta + \mathscr{L}_\delta\phi + 2\delta\phi - N_\delta[\phi].$$

The explicit form of N_δ is obtained from the second through sixth lines of the expansion (3.1). For the nonlinear estimates however, it will only be important that N_δ has the following structure:

$$N_\delta[\phi] = p_0(\phi) + \frac{\partial_z w_\delta}{z} p_1(\phi) + \frac{1}{w_\delta^3 z} p_2(\phi)\partial_z\left(\frac{w_\delta^4}{z^2}\partial_z\left(z^3\phi\right)\right)$$

$$+ \frac{1}{w_\delta^3 z} p_3(\phi)\partial_z\left(\frac{w_\delta^4}{z^2}\partial_z\left(z^3 p_4(\phi)\right)\right)$$

$$(3.6)$$

$$+ \frac{1}{w_\delta^3 z} p_5(\phi)\partial_z\left[w_\delta^4\left(\frac{1}{z^2}\partial_z\left(z^3 p_6(\phi)\right)\right)^2\int_0^1(1-\theta)\left(1+\frac{\theta}{z^2}\partial_z\left(z^3 p_6(\phi)\right)\right)^{\frac{10}{3}}d\theta\right],$$

where p_i, $i = 0,\ldots,6$ are rational functions such that $p_i : [-c,c] \to \mathbb{R}$ is a C^∞ function for some $c > 0$ and $i = 0,\ldots,6$. Furthermore, the polynomials p_0, p_1, p_4 are at least quadratic in ϕ, and the polynomials p_2, p_6 are at least linear in ϕ when expanded about zero. In other words

$$(3.7) \qquad p_i(0) = 0, \quad i = 0,1,2,4,6, \quad p_i'(0) = 0, \quad i = 0,1,4.$$

It is easy to see that the operator $\mathscr{L}_\delta$ defined in (3.2) is non-negative and symmetric with respect to $(\cdot,\cdot)_\delta$:

$$(\mathscr{L}_\delta\phi,\phi)_\delta = (\phi,\mathscr{L}_\delta\phi)_\delta = \frac{4}{3}\|\partial_z\phi\|_{\delta,1}^2$$

for all $\phi \in C_c^\infty(0,1)$. Moreover, following the same argument in [1, 35], we deduce that $\mathscr{L}_\delta$ has the Friedrichs extension, which is a self-adjoint operator in the weighted space $L_{\delta,0}^2$ whose spectrum is purely discrete. See Appendix A for more detail on the analysis of the linearized operator. The following coercivity properties result from the spectral theory of $\mathscr{L}_\delta$, which will be crucial for the energy estimates in Section 3.1.

Lemma 3.1 (Spectral gap for the linearized operator). *There exists a sufficiently small $\varepsilon > 0$ such that for any $-\varepsilon < \delta < 0$ the following statements hold:*

(1) *There exists a constant $\mu_1 > 0$ such that for any $\varphi \in H^1_\delta$ satisfying $(\varphi, 1)_\delta = 0$ the following bound holds:*

$$(3.8) \qquad\qquad (\mathcal{L}_\delta \varphi, \varphi)_\delta \geq \mu_1 \|\varphi\|^2_\delta$$

(2) *There exists $\mu_2 > 0$ such that for any $\varphi \in H^1_\delta$ satisfying $(\varphi, 1)_\delta = 0$ the following bound holds:*

$$(3.9) \qquad ((\mathcal{L}_\delta + 3\delta)\varphi, \varphi)_\delta \geq \mu_2 \left(\|\partial_z \varphi\|^2_{\delta,1} + \|\varphi\|^2_\delta \right)$$

Here μ_1 and μ_2 can be chosen uniformly in δ.

The proof of Lemma 3.1 will be given in Appendix A.

The conserved energy expressed in terms of the perturbation ϕ takes the form

$$E(1+\phi, \phi_s) = \frac{2\pi}{\bar{\lambda}} \|\phi_s + \frac{\bar{\lambda}_s}{\bar{\lambda}}(1+\phi)\|^2_\delta + \frac{4\pi\delta}{\bar{\lambda}} \|(1+\phi)^{-1/2}\|^2_\delta$$

$$+ \frac{12\pi}{\bar{\lambda}} \Big(\int_0^1 ((1+\phi)^2(1+\phi+z\phi_z))^{-1/3} w^4_\delta z^2 \, dz$$

$$(3.10) \qquad + \frac{1}{3} \int_0^1 (1+\phi)^{-1} \partial_z(w^4_\delta) z^3 \, dz \Big).$$

Linearizing the relation

$$(3.11) \qquad\qquad E(1+\phi, \phi_s) = E(1,0) = 0,$$

we conclude that

$$(3.12) \qquad\qquad \frac{3}{2} b(\phi, 1)_\delta - (\phi_s, 1)_\delta = \mathscr{J}[\phi],$$

where $\mathscr{J}[\phi]$ is a nonlinear functional, at least quadratic in ϕ, given by

$$\mathscr{J}[\phi] = \frac{\bar{\lambda}}{4\pi b} \left(E(1+\phi, \phi_s) - E(1,0) - \partial_\phi E(1,0)\phi \right)$$

$$= \frac{1}{2b} \Big(\|\phi_s - b\phi\|^2_\delta + 2 \int_0^1 \left(\delta + \frac{4w'_\delta}{z} \right) \frac{\phi^2}{1+\phi} w^3_\delta z^4 dz$$

$$(3.13) \quad + 6 \int_0^1 (((1+\phi)^2(1+\phi+z\phi_z))^{-1/3} - 1 + \frac{3\phi + z\phi_z}{3}) w^4_\delta z^2 dz \Big).$$

In addition to the total energy $\mathscr{E}$ defined in (2.48) we also introduce an auxiliary energy $\mathscr{E}(s)$:

$$(3.14) \quad \mathscr{E}(s) := \|\phi(s, \cdot)\|^2_\delta + \sum_{j=0}^7 \left[\|\partial_s^{j+1}\phi(s, \cdot)\|^2_\delta + \|\partial_s^j \partial_z \phi(s, \cdot)\|^2_{\delta,1} \right].$$

Theorem 3.2 (High-order energy estimate). *Let $\delta < 0$ be given and let $|\delta|$ be sufficiently small. Let ϕ solve the degenerate wave equation (3.3)–(3.4) on a time interval $[0, S]$ Assume that for a small but fixed $M > 0$ the*

following a priori bound holds:

(3.15) $$\sup_{s\in[0,S]} \mathscr{E}(s) \leq M.$$

There exists a constant ε such that for any initial data $(\phi(0), \phi_s(0)) = (\phi_0, \phi_1)$ satisfying the initial bound:

(3.16) $$\mathscr{E}(0) \leq \varepsilon,$$

and satisfying the vanishing energy assumption

$$E(1+\phi_0, \phi_1) = 0,$$

there exist constants $C_1, C_2, C_3 > 0$ such that the following energy bound holds:

(3.17)
$$\sup_{s\in[0,S]} \mathscr{E}(s) + C_1 \int_0^s \mathscr{E}(\sigma)\,d\sigma \leq C_2\mathscr{E}(0) + C_3 \int_0^s \mathscr{E}(\sigma)^{\frac{3}{2}}\,d\sigma, \ \ s \in [0, S].$$

Before we prove Theorem 3.2 we collect the following key estimates.

3.1 Nonlinear energy estimates

Proposition 3.3 (Energy equivalence)**.** *Assume the same as in Theorem 3.2. Then the total energy $\mathscr{E}(s)$ is equivalent to the instant energy $\mathcal{E}(s)$:*

(3.18) $$\mathcal{E}(s) \leq \mathscr{E}(s) \lesssim \mathcal{E}(s), \ \ s \geq 0,$$

where the notational convention for " $\lesssim$ " is explained in Remark 2.13.

Proof. Note that the bound $\mathcal{E}(s) \leq \mathscr{E}(s)$ trivially follows from the definitions of $\mathcal{E}$ and $\mathscr{E}$. The upper bound $\mathscr{E}(s) \lesssim \mathcal{E}(s)$ is a non-trivial consequence of the fact that ϕ solves the equation (3.3):

(3.19) $$-\mathscr{L}_\delta \phi = \phi_{ss} - \frac{1}{2}b\phi_s + 3\delta\phi - N_\delta[\phi].$$

By exploiting the elliptic structure of $\mathscr{L}_\delta$, we will establish the following estimates:

(3.20) $$\mathscr{E}(s) \lesssim \mathcal{E}(s) + \mathcal{E}(s)^2, \ \ s \geq 0.$$

Then the claim follows from (3.20) for sufficiently small $\mathcal{E}(s)$. The proof of (3.20) is analogous to the proof of Proposition 4.1 from [18] and it proceeds by induction. Here we briefly discuss how to obtain (3.20). We start with $\|\partial_z^2 \phi\|_{\delta,2}^2$. In order to see that $\|\partial_z^2 \phi\|_{\delta,2}^2$ is bounded by the right-hand side of (3.20), square (3.19) and take the inner product with 1:

(3.21) $$(1, (\mathscr{L}_\delta \phi)^2)_\delta \lesssim \|\phi_{ss}\|_\delta^2 + \|\phi_s\|_\delta^2 + \|\phi\|_\delta^2 + \|N_\delta[\phi]\|_\delta^2.$$

By using the L^∞ estimates and Hardy inequalities, one can deduce that

$$\|N_\delta[\phi]\|_\delta^2 \lesssim \mathscr{E}\left(\|\phi\|_\delta^2 + \|\partial_z\phi\|_{\delta,1}^2 + \|\partial_z^2\phi\|_{\delta,2}^2\right).$$

The left-hand-side of (3.21) can be rewritten as

$$
\begin{aligned}
(1, (\mathscr{L}_\delta \phi)^2)_\delta &= \frac{16}{9} \int_0^1 (w_\delta \phi_{zz} + \frac{4 w_\delta \phi_z}{z} + 4 w_\delta' \phi_z)^2 w_\delta^3 z^4 dz \\
&= \frac{16}{9} \Big[\int_0^1 (\phi_{zz})^2 w_\delta^5 z^4 dz + 4 \int_0^1 (\phi_z)^2 w_\delta^5 z^2 dz \\
&\quad - 4 \int_0^1 w_\delta''(\phi_z)^2 w_\delta^4 z^4 dz - 4 \int_0^1 (\phi_z)^2 w_\delta' w_\delta^4 z^3 dz \Big]
\end{aligned}
$$

where we have integrated by parts. Since $w_\delta' = O(z)$, the last two terms are bounded by $\|\partial_z \phi\|_{\delta,1}^2$. Hence we deduce that

$$
\text{(3.22)} \qquad\qquad \|\partial_z^2 \phi\|_{\delta,2}^2 \lesssim \mathscr{E}(s) + \mathscr{E}(s)^2
$$

The bound for $\|\partial_s^k \partial_z^2 \phi\|_{\delta,2}^2$, $1 \le k \le 6$ can be obtained from the time differentiated equation of (3.19) in the same fashion.

To illustrate the induction procedure, we now discuss the case of $\|\partial_z^3 \phi\|_{\delta,3}^2$. To this end, we multiply (3.19) by z and differentiate with respect to z:

$$
\begin{aligned}
\frac{4}{3}(w_\delta z \phi_{zzz} + 5 w_\delta' z \phi_{zz} + 5 w_\delta \phi_{zz}) &= -\frac{4}{3}(4 w_\delta'' z \phi_z + 8 w_\delta' \phi_z) + z \phi_{ssz} + \phi_{ss} \\
&\quad - \frac{1}{2} b z \phi_{sz} - \frac{1}{2} b \phi_s + 3\delta z \phi_z + 3\delta \phi - z \partial_z N_\delta[\phi] - N_\delta[\phi].
\end{aligned}
$$

Dividing this equation by z, squaring it, and then taking $(,)_{\delta,1}$ inner product, we obtain an inequality similar to (3.21). Notice that the right-hand side consists of quartic terms and quadratic terms involving the instant energy $\mathscr{E}$ and $\mathscr{E}$ containing two spatial derivatives only, which have been estimated at the previous step. The quartic terms can be estimated by the embedding inequalities and Hardy inequalities. The left-hand side gives rise to $\|\partial_z^3 \phi\|_{\delta,3}^2$ and remaining terms. The remainder is bounded by $\|\partial_z^2 \phi\|_{\delta,2}^2$. Therefore, by (3.22), we have the desired bound for $\|\partial_z^3 \phi\|_{\delta,3}^2$. The time derivatives of $\partial_z^3 \phi$ can be estimated in the same way. The estimation of higher order derivatives follows inductively.   □

We next state a technical coercivity lemma, that will be needed in the derivation of our high-order energy identity.

Lemma 3.4. *Let (ϕ, ϕ_s) be a solution to the nonlinear degenerate wave equation (3.3). Then there exists $c_0 = c_0(|b|), |\delta|$ sufficiently small and constants $C_1, C_2 > 0$ so that the following coercivity bounds hold:*

$$
\sum_{j=0}^7 \left(\|\partial_s^j \phi_s\|_\delta^2 + (\mathscr{L}_\delta \partial_s^j \phi + 3\delta \partial_s^j \phi, \partial_s^j \phi)_\delta + c_0 (\partial_s^j \phi_s, \partial_s^j \phi)_\delta - \frac{c_0 b}{4} \|\partial_s^j \phi\|_\delta^2 \right)
$$

$$
\text{(3.23)}
$$

$$
\ge C_1 \mathscr{E}(s) - C_2 \sum_{j=0}^7 \left(\partial_s^j \mathscr{J}[\phi] \right)^2,
$$

$$\sum_{j=0}^{7} \left(-\frac{1}{2}(b+c_0)\|\partial_s^j \phi_s\|_\delta^2 + \frac{c_0}{2}(\mathscr{L}_\delta \partial_s^j \phi + 3\delta \partial_s^j \phi, \partial_s^j \phi)_\delta \right)$$

$$(3.24) \qquad \geq C_1 |b| \mathscr{E}(s) - C_2 |b| \sum_{j=0}^{7} \left(\partial_s^j \mathscr{J}[\phi] \right)^2,$$

where $\mathscr{E}$ and $\mathscr{J}$ are defined by (3.14) and (3.13) respectively. Here b is defined in (2.19) and C_1 depends on $|\delta|$.

Proof. Since λ describes the self-similar homologous solution, b relates to δ via $b = -\sqrt{2|\delta|}$. Letting $g^j = \partial_s^j \phi$, $j = 0, 1, \ldots, 7$, we now want to prove the coercivity of the energy functional

$$(3.25) \quad \mathscr{F}(g^j) = \|g_s^j\|_\delta^2 + (\mathscr{L}_\delta g^j + 3\delta g^j, g^j)_\delta + c_0(g_s^j, g^j)_\delta - \frac{c_0 b}{4} \|g^j\|_\delta^2,$$

where (ϕ, ϕ_s) is a local-in-time solution to (3.3) and c_0 is to be determined.

We first decompose

$$\|g^j\|_\delta^2 = \frac{1}{\|1\|_\delta^2}(g^j, 1)_\delta^2 + \|\tilde{g}^j\|_\delta^2, \quad \tilde{g}^j = g^j - \frac{1}{\|1\|_\delta}(g^j, 1)_\delta.$$

Since

$$\mathscr{L}_\delta(1) = 0,$$

and $\mathscr{L}_\delta$ is a self-adjoint operator with respect to $(\cdot, \cdot)_\delta$, it is easy to check that

$$(\mathscr{L}_\delta g^j, g^j)_\delta = (\mathscr{L}_\delta \tilde{g}^j, \tilde{g}^j)_\delta.$$

Therefore from (3.25) it follows that

$$\mathscr{F}(g^j) = \|\tilde{g}^{j+1}\|_\delta^2 + \frac{1}{\|1\|_\delta^2}(g^{j+1}, 1)_\delta^2 + (\mathscr{L}_\delta \tilde{g}^j + 3\delta \tilde{g}^j, \tilde{g}^j)_\delta + \frac{3\delta}{\|1\|_\delta^2}(g^j, 1)_\delta^2$$

$$+ c_0(\tilde{g}^{j+1}, \tilde{g}^j)_\delta + \frac{c_0}{\|1\|_\delta^2}(g^{j+1}, 1)_\delta(g^j, 1)_\delta - \frac{c_0 b}{4}\|g^j\|_\delta^2.$$

On the other hand, applying ∂_s^j to (3.12) we obtain the identity

$$(3.26) \qquad \frac{3}{2}b(g^j, 1)_\delta - (g^{j+1}, 1)_\delta = \partial_s^j \mathscr{J}[\phi], \quad j = 0, \ldots, 7.$$

from which we infer that

$$\frac{8}{9\|1\|_\delta^2}(g^{j+1}, 1)_\delta^2 + \frac{3\delta}{\|1\|_\delta^2}(g^j, 1)_\delta^2$$

$$= \frac{1}{\|1\|_\delta^2}\left(\frac{8}{9}\left(\frac{3}{2}b(g^j, 1)_\delta - \partial_s^j \mathscr{J}[\phi] \right)^2 + 3\delta(g^j, 1)_\delta^2 \right)$$

$$= \frac{b^2}{2\|1\|_\delta^2}(g^j, 1)_\delta^2 - \frac{8}{3\|1\|_\delta^2}b(g^j, 1)_\delta \partial_s^j \mathscr{J}[\phi] + \frac{8}{9\|1\|_\delta^2}\left(\partial_s^j \mathscr{J}[\phi] \right)^2,$$

where we have used $\delta = -\frac{1}{2}b^2$. By the spectral gap property (3.9), the previous two equalities, and the fact that $g^j = \tilde{g}^j + \frac{1}{\|1\|_\delta}(g^j, 1)_\delta$, we conclude that for sufficiently small $|\delta|$ and for any $k_1, k_2, k_3 > 0$

$$
\begin{aligned}
\mathscr{F}(g^j) \geq &\|\tilde{g}^{j+1}\|_\delta^2 + \frac{1}{9\|1\|_\delta^2}(g^{j+1}, 1)_\delta^2 + \mu_2\left(\|\partial_z g^j\|_{\delta,1}^2 + \|\tilde{g}^j\|_\delta^2\right) \\
&+ \frac{c_0|b|}{4}\|\tilde{g}^j\|_\delta^2 + (\frac{c_0|b|}{4} + \frac{b^2}{2})\frac{1}{\|1\|_\delta^2}(g^j, 1)_\delta^2 \\
&- \frac{2b^2 k_1}{3\|1\|_\delta^2}(g^j, 1)_\delta^2 - \frac{8}{3k_1\|1\|_\delta^2}\left(\partial_s^j \mathscr{J}[\phi]\right)^2 + \frac{8}{9\|1\|_\delta^2}\left(\partial_s^j \mathscr{J}[\phi]\right)^2 \\
&- k_2\|\tilde{g}^{j+1}\|_\delta^2 - \frac{c_0^2}{4k_2}\|\tilde{g}^j\|_\delta^2 - \frac{k_3}{\|1\|_\delta^2}(g^{j+1}, 1)_\delta^2 - \frac{c_0^2}{4k_3\|1\|_\delta^2}(g^j, 1)_\delta^2.
\end{aligned}
\tag{3.27}
$$

We can choose $c_0, k_1, k_2, k_3 > 0$ such that the following inequality holds

$$
\mathscr{F}(g^j) \geq c_1\left(\|g^{j+1}\|_\delta^2 + \|\partial_z g^j\|_{\delta,1}^2 + \|\tilde{g}^j\|_\delta^2\right) + c_2\frac{1}{\|1\|_\delta^2}(g^j, 1)_\delta^2 - \frac{C}{\|1\|_\delta^2}\left(\partial_s^j \mathscr{J}[\phi]\right)^2
\tag{3.28}
$$

where $c_1 = c_1(\mu_2)$, $c_2 = c_2(|\delta|)$. For instance, $k_1 = k_2 = 1/2$, $k_3 = 1/18$, $c_0 = |b|/6$ will work. Summing the above bound over all $j \in \{0, \ldots, 7\}$, we obtain the estimate

$$
\sum_{j=0}^{7} \mathscr{F}(g^j) \geq C_1 \sum_{j=0}^{7}\left(\|\partial_s^{j+1}\phi\|_\delta^2 + \|\partial_s^j\phi\|_\delta^2 + \|\partial_s^j\partial_z\phi\|_{\delta,1}^2\right) - C\sum_{j=0}^{7}\left(\partial_s^j \mathscr{J}[\phi]\right)^2
\tag{3.29}
$$

where $C_1 = C_1(|\delta|)$. This concludes the proof of estimate (3.23). Bound (3.24) follows in an analogous way. $\qquad\square$

3.2 Proof of Theorem 3.2

Evaluating the inner product of (3.3) and ϕ_s we obtain the fundamental energy dissipation law:

$$
\frac{1}{2}\frac{d}{ds}\left(\|\phi_s\|_\delta^2 + (\mathscr{L}_\delta\phi + 3\delta\phi, \phi)_\delta\right) - \frac{1}{2}b\|\phi_s\|_\delta^2 = (N_\delta[\phi], \phi_s)_\delta.
\tag{3.30}
$$

We can also evaluate the inner product of (3.3) and ϕ to obtain

$$
(\phi_{ss}, \phi)_\delta - \frac{b}{4}\frac{d}{ds}\|\phi\|_\delta^2 + (\mathscr{L}_\delta\phi + 3\delta\phi, \phi)_\delta = (N_\delta[\phi], \phi)_\delta.
$$

Now we rewrite the first term as

$$
(\phi_{ss}, \phi)_\delta = \frac{d}{ds}(\phi_s, \phi)_\delta - \|\phi_s\|_\delta^2.
$$

By adding a small constant multiple of this identity to (3.30) and integrating with respect to s we obtain the following key energy identity:

$$\frac{1}{2}\left(\|\phi_s\|_\delta^2+(\mathcal{L}_\delta\phi+3\delta\phi,\phi)_\delta+c_0(\phi_s,\phi)_\delta-\frac{c_0 b}{4}\|\phi\|_\delta^2\right)\Big|_0^s$$

$$+\int_0^s\left(-\frac{1}{2}(b+c_0)\|\phi_s\|_\delta^2+\frac{c_0}{2}(\mathcal{L}_\delta\phi+3\delta\phi,\phi)_\delta\right)d\sigma$$

(3.31) $$=\int_0^s\left[(N_\delta[\phi],\phi_s)_\delta+\frac{c_0}{2}\left(N_\delta[\phi],\phi\right)_\delta\right]d\sigma.$$

Using the same calculation for the time differentiated problem we obtain

$$\left(\|\partial_s^j\phi_s\|_\delta^2+(\mathcal{L}_\delta\partial_s^j\phi+3\delta\partial_s^j\phi,\partial_s^j\phi)_\delta+c_0(\partial_s^j\phi_s,\partial_s^j\phi)_\delta-\frac{c_0 b}{4}\|\partial_s^j\phi\|_\delta^2\right)\Big|_0^s$$

$$+\int_0^s\left(-\frac{1}{2}(b+c_0)\|\partial_s^j\phi_s\|_\delta^2+\frac{c_0}{2}(\mathcal{L}_\delta\partial_s^j\phi+\delta\partial_s^j\phi,\partial_s^j\phi)_\delta\right)d\sigma$$

(3.32)

$$=\int_0^s\left[\left(\partial_s^j N_\delta[\phi],\partial_s^j\phi_s\right)_\delta+\frac{c_0}{2}\left(\partial_s^j N_\delta[\phi],\partial_s^j\phi\right)_\delta\right]d\sigma,$$

for $0\le j\le 7$.

Summing over $j\in\{0,\ldots,7\}$ and using Lemma 3.4 we conclude that there exists some positive numbers κ_0',κ_1' such that

$$\mathcal{E}(s)+\kappa_0'\int_0^s\mathcal{E}(\tau)d\sigma\le\kappa_1'\mathcal{E}(0)+C\sum_{j=0}^7\int_0^s\left[\left(\partial_s^j N_\delta[\phi],\partial_s^j\phi_s\right)_\delta+\frac{c_0}{2}\left(\partial_s^j N_\delta[\phi],\partial_s^j\phi\right)_\delta+C\right.$$

(3.33)

$$+C\left(\partial_s^j\mathcal{J}[\phi]\right)^2.$$

Estimates for the right-hand side of (3.33). We first recall the expression (3.6):

$$N_\delta[\phi]=p_0(\phi)+\frac{\partial_z w_\delta}{z}p_1(\phi)+\frac{1}{w_\delta^3 z}p_2(\phi)\partial_z\left(\frac{w_\delta^4}{z^2}\partial_z\left(z^3\phi\right)\right)$$

$$+\frac{1}{w_\delta^3 z}p_3(\phi)\partial_z\left(\frac{w_\delta^4}{z^2}\partial_z\left(z^3 p_4(\phi)\right)\right)$$

(3.34)

$$+\frac{1}{w_\delta^3 z}p_5(\phi)\partial_z\left[w_\delta^4\left(\frac{1}{z^2}\partial_z\left(z^3 p_6(\phi)\right)\right)^2\int_0^1\left(1-\theta+\frac{\theta}{z^2}\partial_z\left(z^3 p_6(\phi)\right)\right)^{\frac{10}{3}}d\theta\right],$$

where p_i, $i=0,\ldots,6$ are rational polynomials such that $p_i:[-c,c]\to\mathbb{R}$ is a C^∞ function for some $c>0$ and $i=0,\ldots,7$. Furthermore, polynomials p_0,p_1,p_4 are at least quadratic in ϕ, and polynomials p_2,p_6 are at least

linear in ϕ when expanded about zero. In other words

$$p_i(0) = 0, \ i = 0,1,2,4,6, \ p_i'(0) = 0, \ i = 0,1,4.$$

Fix $j \in \{0,\dots,7\}$. From (3.34) it is clear that

(3.35)
$$\left| \int_0^s \left(\partial_s^j \left(p_0(\phi) + \frac{\partial_z w_\delta}{z} p_1(\phi) \right), \partial_s^{j+1}\phi \right)_\delta d\sigma \right| \lesssim \int_0^s \left[M^{1/2} \mathscr{E}(\sigma) + \mathscr{E}^{3/2}(\sigma) \right] d\sigma,$$

where we used the L^∞-bounds of Lemma B.3, Cauchy-Schwarz inequality, Hardy inequality, the quadratic structure of p_0 and p_1, and the a priori assumption (4.16). The third, fourth, and the fifth term on the right-hand side of (3.34) contain two ∂_z-derivatives and are therefore rather subtle to estimate. We focus first on the third term and rewrite

(3.36)
$$\partial_s^j \left(\frac{1}{w_\delta^3 z} p_2(\phi) \partial_z \left(\frac{w_\delta^4}{z^2} \partial_z \left(z^3 \phi \right) \right) \right) = \frac{1}{w_\delta^3 z} p_2(\phi) \partial_z \left(\frac{w_\delta^4}{z^2} \partial_z \left(z^3 \partial_s^j \phi \right) \right) + \mathscr{R}^j,$$

where $\mathscr{R}^j$ is a lower order remainder arising due to the application of the Leibniz rule and it satisfies the energy bound

(3.37)
$$\left| \int_0^s (\mathscr{R}^j, \partial_s^{j+1}\phi)_\delta d\sigma \right| \lesssim \int_0^s \left[M^{1/2} \mathscr{E}(\sigma) + \mathscr{E}^{3/2}(\sigma) \right] d\sigma.$$

Note that

$$\left(\frac{1}{w_\delta^3 z} p_2(\phi) \partial_z \left(\frac{w_\delta^4}{z^2} \partial_z \left(z^3 \partial_s^j \phi \right) \right), \partial_s^{j+1}\phi \right)_\delta$$

$$= \int_0^1 p_2(\phi) \partial_z \left(\frac{w_\delta^4}{z^2} \partial_z \left(z^3 \partial_s^j(\phi) \right) \right) \partial_s^{j+1}\phi z^3 dz$$

$$= -\frac{1}{2}\partial_s \int_0^1 \frac{w_\delta^4}{z^2} p_2(\phi) |\partial_z(z^3 \partial_s^j \phi)|^2 dz + \frac{1}{2}\int_0^1 \frac{w_\delta^4}{z^2} \partial_s(p_2(\phi)) |\partial_z(z^3 \partial_s^j \phi)|^2 dz$$

(3.38)
$$- \int_0^1 \partial_z(p_2(\phi)) w_\delta^4 \partial_z \left(z^3 \partial_s^j \phi \right) \partial_s^{j+1}\phi z dz,$$

where we integrated by parts. Note that

(3.39)
$$\int_0^1 \frac{w_\delta^4}{z^2} |\partial_z(z^3 \partial_s^j \phi)|^2 \lesssim \int_0^1 w_\delta^4 z^2 (\partial_s^j \phi)^2 dz + \int_0^1 w_\delta^4 z^4 (\partial_z \partial_s^j \phi)^2 dz \lesssim \mathscr{E},$$

where we used the Hardy inequality in the last estimate to conclude that $\int_0^1 w_\delta^4 z^2 (\partial_s^j \phi)^2 \, dz \lesssim \int_0^1 w_\delta^4 z^4 (\partial_s^j \phi)^2 \, dz + \int_0^1 w_\delta^4 z^4 (\partial_z \partial_s^j \phi)^2 \, dz \leq \mathscr{E}$. Therefore from (3.38) we conclude that

$$
\left| \int_0^s \left(\frac{1}{w_\delta^3 z} p_2(\phi) \partial_z \left(\frac{w_\delta^4}{z^2} \partial_z \left(z^3 \partial_s^j \phi \right) \right), \partial_s^{j+1} \phi \right)_\delta \, d\sigma \right|
$$

$$
\lesssim \| p_2(\phi) \|_{L^\infty} \mathscr{E}(s) + \sup_{s \in [0,S]} \| p_2(\phi) \|_{L^\infty} \mathscr{E}(0) + \sup_{s \in [0,S]} \| \partial_s p_2(\phi) \|_\infty \int_0^s \mathscr{E}(\sigma) \, d\sigma
$$

$$
+ \sup_{s \in [0,S]} \| \partial_z p_2(\phi) \|_\infty \int_0^s \| \frac{1}{z^3} \partial_z (z^3 \partial_s^j \phi) \|_\delta \| \partial_s^{j+1} \phi \|_\delta \, d\sigma
$$

(3.40)

$$
\lesssim \mathscr{E}^{1/2} (\mathscr{E}(0) + \mathscr{E}(s)) + \sup_{s \in [0,S]} \mathscr{E}^{1/2} \int_0^s \mathscr{E}(\sigma) \, d\sigma.
$$

The estimate for the fourth term on the right-hand side of (3.34) is entirely analogous to the previous bound and we conclude that

$$
\left| \int_0^s \partial_s^j \left(\frac{1}{w_\delta^3 z} p_3(\phi) \partial_z \left(\frac{w_\delta^4}{z^2} \partial_z \left(z^3 p_4(\phi) \right) \right), \partial_s^{j+1} \phi \right)_\delta \, d\sigma \right|
$$

$$
\lesssim \mathscr{E}^{1/2} (\mathscr{E}(0) + \mathscr{E}(s)) + \sup_{s \in [0,S]} \mathscr{E}^{1/2} \int_0^s \mathscr{E}(\sigma) \, d\sigma.
$$

The last term on the right-hand side of (3.34) appears slightly different in structure, but the top order estimate relies on the same integration-by-parts idea as in (3.38)–(3.39). Namely, introducing a shorthand notation $q(\phi, \phi_z) = \int_0^1 (1 - \theta) \left(1 + \frac{\theta}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^{\frac{10}{3}} \, d\theta$, the term we need to estimate takes the form

$$
\left| \int_0^s \left(\partial_s^j \left(\frac{1}{w_\delta^3 z} p_5(\phi) \partial_z \left[w_\delta^4 \left(\frac{1}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^2 q(\phi, \phi_z) \right] \right), \partial_s^{j+1} \phi \right)_\delta \, d\sigma \right|
$$

Commuting the operator ∂_s^j all the way through in the above expression and recalling the definition of $(\cdot, \cdot)_\delta$, we see that the highest order term takes the form

$$
\left| \int_0^s \int_0^1 p_5(\phi) \partial_z \left[w_\delta^4 \left(\frac{1}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^2 \partial_s^j q(\phi, \phi_z) \right] \partial_s^{j+1} \phi z^3 \, dz d\sigma \right|
$$

Integrating-by-parts with respect to z we can rewrite the above term in the form

$$\left| \int_0^s \int_0^1 p_5(\phi) \partial_z \left[w_\delta^4 \left(\frac{1}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^2 \partial_s^j q(\phi, \phi_z) \right] \partial_s^{j+1} \phi z^3 \, dz \, d\sigma \right|$$

$$\leq \left| \int_0^s \int_0^1 p_5(\phi) w_\delta^4 \left(\frac{1}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^2 \partial_s^j q(\phi, \phi_z) \partial_z \left(\partial_s^{j+1} \phi z^3 \right) \, dz \, d\sigma \right|$$

(3.41)

$$+ \left| \int_0^s \int_0^1 \partial_z p_5(\phi) w_\delta^4 \left(\frac{1}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^2 \partial_s^j q(\phi, \phi_z) \partial_s^{j+1} \phi z^3 \, dz \, d\sigma \right|.$$

The key property of the nonlinearity $q(\phi, \phi_z)$ is that at the top order

$$\partial_s^j q(\phi, \phi_z) = \frac{10}{3} \int_0^1 \theta(1-\theta) \left(1 + \frac{\theta}{z^2} \partial_z \left(z^3 p_6(\phi) \right) \right)^{\frac{7}{3}} \frac{1}{z^2} \partial_z \left(z^3 \partial_s^j p_6(\phi) \right) d\theta$$

$$+ \text{ lower order terms.}$$

The top order factor $\partial_s^j \phi_z$ enters linearly in the above expression and we may integrate-by-parts with respect to s by the same reasoning as in (3.36)–(3.40). We thereby use Hardy inequalities and weighted Sobolev space embeddings to bound (3.41) by $C\mathscr{E}(0) + C \int_0^s \mathscr{E}(\sigma)^{\frac{3}{2}} \, d\sigma$. All of the remainder terms are of lower order and they are estimated again by a systematic application of Hölder inequality in conjunction with the bounds from Appendix B. Combining (3.35)–(3.41) we conclude that

$$(3.42) \qquad \sum_{j=0}^7 \left| \int_0^s \left(\partial_s^j N_\delta[\phi], \partial_s^{j+1} \phi \right)_\delta \, d\sigma \right| \lesssim \mathscr{E}(0) + \int_0^s \mathscr{E}(\sigma)^{\frac{3}{2}} \, d\sigma.$$

Analogously to (3.42) we obtain (an easier estimate)

$$(3.43) \qquad \sum_{j=0}^7 \left| \int_0^s \left(\partial_s^j N_\delta[\phi], \partial_s^j \phi \right)_\delta \, d\sigma \right| \lesssim \mathscr{E}(0) + \int_0^s \mathscr{E}(\sigma)^{\frac{3}{2}} \, d\sigma.$$

Recalling the definition (3.13) of $\mathscr{J}$ and using the Hardy inequalities and weighted Sobolev space embeddings of Appendix B, due to the quadratic structure of $\mathscr{J}$ it is straightforward to check the bound

$$(3.44) \qquad \int_0^s \left(\partial_s^j \mathscr{J}[\phi] \right)^2 \, d\sigma + \left(\partial_s^j \mathscr{J}[\phi] \right)^2 \lesssim \mathscr{E}(0) + \int_0^s \mathscr{E}(\sigma)^{\frac{3}{2}} \, d\sigma.$$

Therefore, from (3.33), the bounds (3.42)–(3.44), and Proposition 3.3 it follows that

$$\mathscr{E}(s) + \kappa_0' \int_0^s \mathscr{E}(\sigma) \, d\sigma \leq \kappa_1' \mathscr{E}(0) + C \left(M^{1/2} + \sup_{s \geq 0} \mathscr{E}^{1/2} \right) \int_0^s \mathscr{E}(\sigma) \, d\sigma.$$

By the a priori assumption (4.16), we can absorb the first term on the right-hand side into the left-hand side for sufficiently small M to finally obtain

$$(3.45) \qquad \mathscr{E}(s) + \kappa_0'' \int_0^s \mathscr{E}(\sigma)\,d\sigma \le \kappa_1'' \mathscr{E}(0).$$

for some constants $\kappa_0'', \kappa_1'' > 0$. Repeating the same argument we deduce that for any $0 \le s' \le s < \infty$ the following energy bound holds:

$$(3.46) \qquad \mathscr{E}(s) + \kappa_0'' \int_{s'}^s \mathscr{E}(\sigma)\,d\sigma \le \kappa_1'' \mathscr{E}(s').$$

Theorem 3.2 now follows from Proposition 3.3.

3.3 Proof of Theorem 2.10

Let $\varepsilon < \frac{M}{2\kappa_1''}$ be given, where κ_1'' is the constant appearing in (3.45). We define

$$\mathscr{S} := \sup_{s \ge 0} \left\{ \sup_{0 \le \sigma \le s} \mathscr{E}(\sigma) \le \varepsilon \text{ and the solution to (3.3) exists on the interval } [0,s]. \right\}$$

From the local well-posedness theorem 2.3 it follows that $\mathscr{S} > 0$. It follows from (3.45) that for a sufficiently small $\mathscr{E}(0)$ the solution to (3.3) has to exist globally-in-time, i.e. $\mathscr{S} = \infty$. Since $\psi = 1 + \phi$ this implies that the solution to (2.43)–(2.45) also exists globally-in-time. Letting $s \to \infty$ in (3.46) we conclude that

$$(3.47) \qquad \int_{s'}^\infty \mathscr{E}(\sigma)\,d\sigma \le C\mathscr{E}(s'), \quad s' \ge 0.$$

Defining $V(s') := \int_{s'}^\infty \mathscr{E}(\sigma)\,d\sigma$ we conclude from (3.47) that

$$V'(s') = -\mathscr{E}(s') \le -\kappa_2 V(s'), \quad s' \ge 0.$$

By an elementary integration we obtain the decay

$$V(s') \le CV(0)e^{-\kappa_2 s'} \le C\mathscr{E}(0)e^{-\kappa_2 s'}, \quad \text{for all } s' \ge 0.$$

Now integrate (3.46) with respect to s' over $[\frac{s}{2}, s]$ to get

$$\mathscr{E}(s)\frac{s}{2} \le CV(\frac{s}{2})$$

which in turn gives

$$\mathscr{E}(s) \le Ce^{-\kappa s}, \ s \ge 0, \quad \text{for some } 0 < \kappa < \frac{\kappa_2}{2}.$$

Together with Proposition 3.3 the last bound completes the proof of Theorem 2.10.

4 Nonlinear stability of linearly expanding homologous solutions

Linearizing (2.55) about the steady state $\tilde{\theta} = 1$ and writing $\theta = 1 + \phi$ we obtain the equation satisfied by ϕ

$$(4.1) \qquad \tilde{\lambda}\phi_{\tau\tau} + \tilde{\lambda}_\tau \phi_\tau + 3\delta\phi + \mathscr{L}_\delta \phi = N_\delta[\phi],$$

equipped with the initial conditions

$$(4.2) \qquad \phi(0,z) = \phi_0(z) = \zeta_0(z) - 1, \quad \phi_\tau(0,z) = \phi_1(z) = \zeta_1.$$

The operators $\mathscr{L}_\delta$ and N_δ are defined by (3.2) and (3.6) respectively.

With respect to the unknown ϕ the energy E takes the form

$$E(1+\phi,\phi_\tau) = 2\pi\|\phi_\tau + \frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}(1+\phi)\|_\delta^2 + \frac{4\pi\delta}{\tilde{\lambda}}\|(1+\phi)^{-1/2}\|_\delta^2$$

$$+ \frac{12\pi}{\tilde{\lambda}}\int_0^1 ((1+\phi)^2(1+\phi+z\phi_z))^{-1/3} w_\delta^4 z^2\, dz$$

$$(4.3) \qquad + \frac{4\pi}{\tilde{\lambda}}\int_0^1 (1+\phi)^{-1}\partial_z(w_\delta^4)z^3\, dz.$$

Expanding $E(1+\phi,\phi_\tau)$ around the steady state, we obtain

$$E(1+\phi,\phi_\tau) = E(1,0) + 4\pi\left(\tilde{e} - \frac{3\delta}{\tilde{\lambda}}\right)(\phi,1)_\delta + 4\pi\frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}(\phi_\tau,1)_\delta + 4\pi\tilde{\mathscr{J}}[\phi]$$

where $\tilde{e} = \lambda_1^2 + \frac{2\delta}{\lambda_0}$ and the nonlinear remainder $\tilde{\mathscr{J}}[\phi]$ is given by

$$\tilde{\mathscr{J}}[\phi] = \frac{1}{2}\left(\|\phi_\tau + \frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}\phi\|_\delta^2 + \frac{2}{\tilde{\lambda}}\int_0^1 \left(\delta + \frac{4w_\delta'}{z}\right)\frac{\phi^2}{1+\phi}w_\delta^3 z^4 dz\right.$$

$$(4.4) \qquad \left. + \frac{6}{\tilde{\lambda}}\int_0^1 \left(((1+\phi)^2(1+\phi+z\phi_z))^{-1/3} - 1 + \frac{3\phi + z\phi_z}{3}\right)w_\delta^4 z^2 dz\right).$$

Since the energy is conserved i.e. $E(1+\phi,\phi_\tau) = E(1+\phi_0,\phi_1)$, we obtain the identity

$$(4.5) \qquad \left(\tilde{e} - \frac{3\delta}{\tilde{\lambda}}\right)(\phi,1)_\delta + \frac{\tilde{\lambda}_\tau}{\tilde{\lambda}}(\phi_\tau,1)_\delta = \kappa - \tilde{\mathscr{J}}[\phi]$$

where κ is the physical energy deviation of the perturbation from the background state,

$$(4.6) \qquad \kappa = \frac{E(1+\phi_0,\phi_1) - E(1,0)}{4\pi}.$$

As mentioned in Section 2.4 the exponentially growing coefficients $\tilde{\lambda}$ and $\tilde{\lambda}_\tau$ in (4.1) force us to avoid commuting (4.1) with higher ∂_τ-derivatives as we did previously in Section 3. Instead we develop a high-order energy method based on spatial derivatives solely. Inspired by [23] we know that

the more degenerate weights are needed for higher-order spatial derivatives in order to capture the physical vacuum singularity. In addition to that, we need to handle coordinate singularities at $z = 0$ that arise with repeated application of spatial derivatives. To handle these difficulties we already introduced the operator $\mathscr{S}$ in (2.62) and we additionally define another carefully chosen elliptic operator.

Definition 4.1 (A high-order version of the operator $\mathscr{L}_\delta$). For any $k \in \mathbb{N}$ we define

$$(4.7) \qquad \mathscr{L}_{\delta,k}\psi := -\frac{4}{3w_\delta^{3+k}z^4}\partial_z(w_\delta^{4+k}z^4\partial_z\psi).$$

Note that the operator $\mathscr{L}_{\delta,k}$ defined in Definition 4.1 can be also written in the form

$$(4.8) \qquad \mathscr{L}_{\delta,k}\psi = -\frac{4}{3}\left(w_\delta\partial_z^2\psi + (4+k)w_\delta'\partial_z\psi + \frac{4w_\delta}{z}\partial_z\psi\right).$$

We first state the result for $\mathscr{L}_{\delta,k}$ analogous to Lemma 3.1 for $\mathscr{L}_\delta$.

Lemma 4.2 (Spectral gap for the linearized operator). *There exists a sufficiently small $\varepsilon > 0$ such that for any $-\varepsilon < \delta \leq 0$ the following statements hold:*

(1) There exists a constant $\mu_{1,k} > 0$ such that for any $\varphi \in H^1_{\delta,k}$ satisfying $(\varphi,1)_{\delta,k} = 0$ the following bound holds:

$$(4.9) \qquad (\mathscr{L}_{\delta,k}\varphi,\varphi)_{\delta,k} \geq \mu_{1,k}\|\varphi\|^2_{\delta,k}$$

(2) There exists $\mu_{2,k} > 0$ such that for any $\varphi \in H^1_{\delta,k}$ satisfying $(\varphi,1)_{\delta,k} = 0$ the following bound holds:

$$(4.10) \qquad ((\mathscr{L}_{\delta,k}+3\delta)\varphi,\varphi)_{\delta,k} \geq \mu_{2,k}\left(\|\partial_z\varphi\|^2_{\delta,k+1} + \|\varphi\|^2_{\delta,k}\right)$$

We remark that $\mu_{1,k}$ and $\mu_{2,k}$ can be chosen uniformly in δ. For the spectral theoretic properties of $\mathscr{L}_{\delta,k}$ and the proof Lemma 4.2 see Appendix A.

Recall the definition (2.63) of the high-order energy $\tilde{\mathscr{E}}$. If $\delta > 0$, it is clear that

$$(4.11) \qquad \sum_{j=0}^{4}\left(\tilde{\lambda}\|\mathscr{S}^j\phi_\tau\|^2_{\delta,2j} + ((L_{\delta,2j}+3\delta)\mathscr{S}^j\phi,\mathscr{S}^j\phi)_{\delta,2j}\right) \gtrsim \tilde{\mathscr{E}}.$$

However, if $\delta \leq 0$, $\mathscr{L}_{\delta,k}+3\delta$ is not positive definite any longer. Nevertheless, we will show that if ϕ solves (4.1)–(4.2) and if $|\delta|$ sufficiently small, we have a desired coercivity for nonpositive δ-s with a sufficiently small $|\delta| \ll 1$.

Lemma 4.3 (Energy coercivity). *Let $\delta \in (\delta^*, 0]$ be given and let $\delta > -\tilde{\varepsilon}$ for some $\tilde{\varepsilon} > 0$. Assume that (ϕ, ϕ_τ) solves (4.1)–(4.2) on some τ-interval $[0, T]$, $T > 0$. Then there exist constants $c_j > 0$, $0 \le j \le 4$ and $\tilde{C}_1, \tilde{C}_2 > 0$ such that*

(4.12)
$$\sum_{j=0}^{4} c_j \left(\tilde{\lambda} \| \mathscr{S}^j \phi_\tau \|_{\delta, 2j}^2 + 3\delta \| \mathscr{S}^j \phi \|_{\delta, 2j}^2 + \frac{4}{3} \| \partial_z \mathscr{S}^j \phi \|_{\delta, 2j+1}^2 \right)$$
$$\ge \tilde{C}_1 \tilde{\mathscr{E}}(\tau) - \tilde{C}_2 \left(\tilde{\mathscr{E}}(0) + \left| \tilde{\mathscr{J}}[\phi] \right|^2 \right), \quad \tau \in [0, T]$$

if $\tilde{\varepsilon} > 0$ is sufficiently small.

Proof. To prove (4.12), we first take advantage of (4.5). Since $\tilde{e} > 0$ and $\tilde{\lambda} \sim e^{\sqrt{\tilde{e}}\tau}$, for sufficiently small $|\delta|$, $\tilde{e} - \frac{3\delta}{\tilde{\lambda}} > 0$. Together with $1 \lesssim \frac{\tilde{\lambda}_\tau}{\tilde{\lambda}} \lesssim 1$, it follows that

(4.13)
$$\left| (\phi, 1)_\delta \right| \lesssim \sqrt{\tilde{\mathscr{E}}(0)} + \left| (\phi_\tau, 1)_\delta \right| + \left| \tilde{\mathscr{J}}[\phi] \right| \lesssim \sqrt{\tilde{\mathscr{E}}(0)} + \| \phi_\tau \|_\delta + \left| \tilde{\mathscr{J}}[\phi] \right|$$

where we have used $|\kappa| \lesssim \sqrt{\tilde{\mathscr{E}}(0)}$.

For any $j \ge 1$ it is easy to check that

$$(\mathscr{S}^j \phi, 1)_{\delta, 2j} = -(3 + 2j) \int_0^1 \partial_z \mathscr{S}^{j-1} \phi \, w_\delta^{2+2j} w_\delta' z^4 \, dz$$
$$= -(3 + 2j) \left(\partial_z \mathscr{S}^{j-1} \phi, w_\delta' \right)_{\delta, 2(j-1)+1}.$$

The Cauchy-Schwarz inequality and the physical vacuum condition $-\infty < w_\delta'(1) < 0$ together imply that

(4.14)
$$\frac{\left| (\mathscr{S}^j \phi, 1)_{\delta, 2j} \right|}{\| 1 \|_{\delta, 2j}} \le k_j \| \partial_z \mathscr{S}^{j-1} \phi \|_{\delta, 2(j-1)+1},$$

for some $k_j > 0$, $j \in \mathbb{N}$. For any $j \in \mathbb{N}$ we have

$$3\delta \| \mathscr{S}^j \phi \|_{\delta, 2j}^2 + \frac{4}{3} \| \partial_z \mathscr{S}^j \phi \|_{\delta, 2j+1}^2$$
$$= \frac{4}{3} \| \partial_z \mathscr{S}^j \phi \|_{\delta, 2j+1}^2 + 3\delta \| \mathscr{S}^j \phi - \frac{1}{\| 1 \|_{\delta, 2j}} (\mathscr{S}^j \phi, 1)_{\delta, 2j} \|_{\delta, 2j}^2 + 3\delta \frac{1}{\| 1 \|_{\delta, 2j}^2} (\mathscr{S}^j \phi, 1)_{\delta, 2j}^2.$$

The first two terms can be rewritten as

$$\frac{4}{3} \| \partial_z \mathscr{S}^j \phi \|_{\delta, 2j+1}^2 + 3\delta \| \mathscr{S}^j \phi - \frac{1}{\| 1 \|_{\delta, 2j}} (\mathscr{S}^j \phi, 1)_{\delta, 2j} \|_{\delta, 2j}^2 = \left((\mathscr{L}_{\delta, 2j} + 3\delta) \varphi_j, \varphi_j \right)_{\delta, 2j}$$

where $\varphi_j = \mathscr{S}^j\phi - \frac{1}{\|1\|_{\delta,2j}}(\mathscr{S}^j\phi,1)_{\delta,2j}$. Since $(\varphi_j,1)_{\delta,2j}=0$, for $|\delta|$ sufficiently small, by Lemma 4.2, there exist $\tilde{\mu}>0$ such that

$$\frac{4}{3}\|\partial_z\mathscr{S}^j\phi\|^2_{\delta,2j+1} + 3\delta\|\mathscr{S}^j\phi - \frac{1}{\|1\|_{\delta,2j}}(\mathscr{S}^j\phi,1)_{\delta,2j}\|^2_{\delta,2j}$$

$$\geq \tilde{\mu}\left(\|\partial_z\mathscr{S}^j\phi\|^2_{\delta,2j+1} + \|\mathscr{S}^j\phi - \frac{1}{\|1\|_{\delta,2j}}(\mathscr{S}^j\phi,1)_{\delta,2j}\|^2_{\delta,2j}\right), \quad j\in\{0,1,2,3,4\}.$$

Using (4.14) it then follows that

$$3\delta\|\mathscr{S}^j\phi\|^2_{\delta,2j} + \frac{4}{3}\|\partial_z\mathscr{S}^j\phi\|^2_{\delta,2j+1}$$

$$\geq \tilde{\mu}\left(\|\partial_z\mathscr{S}^j\phi\|^2_{\delta,2j+1} + \|\mathscr{S}^j\phi\|^2_{\delta,2j}\right) - (3|\delta|+\tilde{\mu})k_j^2\|\partial_z\mathscr{S}^{j-1}\phi\|^2_{\delta,2(j-1)+1}.$$

Since the last negative term is indexed at $j-1$, we can select constants $c_j>0$ so that it can be absorbed into the linear combination of the left-hand side successively. Hence we deduce that

$$\sum_{j=0}^4 c_j\left(3\delta\|\mathscr{S}^j\phi\|^2_{\delta,2j} + \frac{4}{3}\|\partial_z\mathscr{S}^j\phi\|^2_{\delta,2j+1}\right)$$

$$\geq \tilde{C}_1\sum_{j=0}^4\left(\|\partial_z\mathscr{S}^j\phi\|^2_{\delta,2j+1} + \|\mathscr{S}^j\phi\|^2_{\delta,2j+1}\right) - \tilde{C}_2|(\phi,1)_\delta|^2.$$

By using (4.13), we obtain the desired inequality (4.12). $\qquad\square$

In addition to the high-order energy $\tilde{\mathscr{E}}$ we also define a high-order dissipation functional by

$$(4.15)\qquad \tilde{\mathscr{D}} = \tilde{\mathscr{D}}(\tau) = \sum_{j=0}^4\frac{\tilde{\lambda}_\tau}{2}\|\mathscr{S}^j\phi_\tau\|^2_{\delta,2j}.$$

Theorem 4.4 (High-order energy estimate). *Let $-\tilde{\varepsilon}<\delta<\infty$ be given for a sufficiently small $\tilde{\varepsilon}>0$, where $\tilde{\varepsilon}$ appears in Lemma 4.2. Let (ϕ,ϕ_τ) be a solution to the degenerate wave equation (4.1)–(4.2) on a τ-time interval $[0,T]$. Assume that for a small but fixed $\tilde{M}>0$ the following a priori bound holds:*

$$(4.16)\qquad \sup_{\tau\in[0,T]}\tilde{\mathscr{E}}(\tau)\leq\tilde{M}.$$

There exists a constant $\varepsilon>0$ such that for any initial data $(\phi(0),\phi_s(0))=(\phi_0,\phi_1)$ satisfying the initial bound:

$$(4.17)\qquad \tilde{\mathscr{E}}(0)\leq\varepsilon,$$

there exist constants $\tilde{C}_1,\tilde{C}_2,\tilde{C}_3>0$ such that the following energy bound holds:

(4.18)

$$\sup_{\tau'\in[0,\tau]}\tilde{\mathscr{E}}(\tau') + \tilde{C}_1\int_0^\tau\tilde{\mathscr{D}}(\sigma)\,d\sigma \leq \tilde{C}_2\tilde{\mathscr{E}}(0) + \tilde{C}_3\int_0^\tau e^{-\beta_2\sigma}\tilde{\mathscr{E}}(\sigma)\,d\sigma, \quad \tau\in[0,T]$$

where β_2 is defined in Section 2.3.

Lemma 4.5 (Algebraic properties of $\mathscr{S}$). *Assuming that* $\Phi, \Psi \in H^2_{loc}([0,1])$ *and* $p : \mathbb{R} \to \mathbb{R}$ *is a* C^2 *function, the following identities hold:*

(1)

$$\mathscr{S}(c_1\Phi + c_2\Psi) = c_1\mathscr{S}\Phi + c_2\mathscr{S}\Psi \text{ for } c_1, c_2 \in \mathbb{R}. \tag{4.19}$$

(2)

$$\mathscr{S}(\Phi\Psi) = (\mathscr{S}\Phi)\Psi + \Phi(\mathscr{S}\Psi) + 2\partial_z\Phi\partial_z\Psi. \tag{4.20}$$

(3)

$$\mathscr{S}(p(\Phi)) = p'(\Phi)\mathscr{S}\Phi + p''(\Phi)|\partial_z\Phi|^2. \tag{4.21}$$

Proof. The proof of the lemma is a straightforward application of the product and the chain rule. □

Lemma 4.6 (Commutator of $\mathscr{S}$ and $\mathscr{L}_{\delta,k}$). *For any* $k \in \mathbb{N}$ *the following commutation formula holds:*

$$\mathscr{S}\mathscr{L}_{\delta,2k}\psi = \mathscr{L}_{\delta,2k+2}\mathscr{S}\psi + a_{k+1}(z)\mathscr{S}\psi + b_{k+1}(z)\partial_z\psi, \tag{4.22}$$

where

(4.23)
$$a_k(z) := -\frac{4}{3}\left[(5+4k)w_\delta''(z) + \frac{4w_\delta'(z)}{z}\right],$$

(4.24)
$$b_k(z) := -\frac{4}{3}\left((2+2k)w_\delta'''(z) - \frac{4(2+2k)w_\delta''(z)}{z} + \frac{4(2+2k)w_\delta'(z)}{z^2}\right).$$

Proof. The proof relies on a direct calculation.

(4.25)
$$\partial_z\mathscr{L}_{\delta,2k}\psi = -\frac{4}{3}\left(w_\delta\partial_z^3\psi + (5+2k)w_\delta'\partial_z^2\psi + \frac{4w_\delta}{z}\partial_z^2\psi + [(4+2k)w_\delta'' + \frac{4w_\delta'}{z} - \frac{4w_\delta}{z^2}]\partial_z\psi\right)$$

(4.26)
$$\partial_z^2\mathscr{L}_{\delta,2k}\psi = -\frac{4}{3}\left(w_\delta\partial_z^4\psi + (6+2k)w_\delta'\partial_z^3\psi + \frac{4w_\delta}{z}\partial_z^3\psi + [(9+4k)w_\delta'' + \frac{8w_\delta'}{z} - \frac{8w_\delta}{z^2}]\partial_z^2\psi \right.$$
$$\left. + [(4+2k)w_\delta''' + \frac{4w_\delta''}{z} - \frac{8w_\delta'}{z^2} + \frac{8w_\delta}{z^3}]\partial_z\psi\right)$$

Hence we obtain

(4.27)

$$\mathscr{S}\mathscr{L}_{\delta,2k}\psi = (\partial_z^2 + \frac{4}{z}\partial_z)\mathscr{L}_{\delta,2k}\psi$$

$$= -\frac{4}{3}\left(w_\delta\partial_z^4\psi + (6+2k)w_\delta'\partial_z^3\psi + \frac{8w_\delta}{z}\partial_z^3\psi + \left[(9+4k)w_\delta'' + \frac{4(7+2k)w_\delta'}{z}\right.\right.$$

$$\left.\left. + \frac{8w_\delta}{z^2}\right]\partial_z^2\psi + \left[(4+2k)w_\delta''' + \frac{4(5+2k)w_\delta''}{z} + \frac{8w_\delta'}{z^2} - \frac{8w_\delta}{z^3}\right]\partial_z\psi\right)$$

On the other hand,

(4.28)

$$\mathscr{L}_{\delta,2k+2}\mathscr{S}\psi = -\frac{4}{3w_\delta^{5+2k}z^4}\partial_z(w_\delta^{6+2k}z^4\partial_z[(\partial_z^2 + \frac{4}{z}\partial_z)\psi])$$

$$= -\frac{4}{3}\left(w_\delta\partial_z^4\psi + (6+2k)w_\delta'\partial_z^3\psi + \frac{8w_\delta}{z}\partial_z^3\psi\right.$$

$$\left. + \frac{4(6+2k)w_\delta'}{z}\partial_z^2\psi + \frac{8w_\delta}{z^2}\partial_z^2\psi - \frac{4(6+2k)w_\delta'}{z^2}\partial_z\psi - \frac{8w_\delta}{z^3}\partial_z\psi\right)$$

Therefore,

$$\mathscr{S}\mathscr{L}_{\delta,2k}\psi = \mathscr{L}_{\delta,2k+2}\mathscr{S}\psi - \frac{4}{3}\left(\left[(9+4k)w_\delta'' + \frac{4w_\delta'}{z}\right]\partial_z^2\psi\right.$$

$$\left. + \left[(4+2k)w_\delta''' + \frac{4(5+2k)w_\delta''}{z} + \frac{4(8+2k)w_\delta'}{z^2}\right]\partial_z\psi\right)$$

(4.29)

$$= \mathscr{L}_{\delta,2k+2}\mathscr{S}\psi - \frac{4}{3}\left[(9+4k)w_\delta'' + \frac{4w_\delta'}{z}\right]\mathscr{S}\psi$$

$$- \frac{4}{3}\left[(4+2k)w_\delta''' - \frac{4(4+2k)w_\delta''}{z} + \frac{4(4+2k)w_\delta'}{z^2}\right]\partial_z\psi$$

$$= \mathscr{L}_{\delta,2k+2}\mathscr{S}\psi + a_{k+1}(z)\mathscr{S}\psi + b_{k+1}(z)\partial_z\psi,$$

where $a_{k+1}(z)$ and $b_{k+1}(z)$ are given by (4.23) and (4.24) respectively. $\square$

Lemma 4.7. *For any $k \in \mathbb{N}$ the functions a_k and b_k are smooth and b_k possess the following Taylor expansions about $z = 0$: $a_k(z) = a_{k,0} + a_{k,2}z^2 + O(z^4)$, $\partial_z^{2\ell+1}a_k\big|_{z=0} = 0$, $b_k(z) = b_{k,1}z + b_{k,3}z^3 + O(z^5)$, $\partial_z^{2\ell}b_k\big|_{z=0} = 0$, $\ell \in \mathbb{N}$. Moreover, for any $i,j,k \in \mathbb{N}$ there exists a $c_{ijk} > 0$ such that*

$$\sum_{\ell=0}^{j}\left(\|\partial_z^{2\ell}\zeta\|_{L^\infty([0,1])} + \|\mathscr{S}^\ell\zeta\|_{L^\infty([0,1])}\right) \le c_{ijk}, \quad \zeta = a_k, z^{2i+1}b_k.$$

Proof. Since $w_\delta \in C^\infty(0,1)$, it suffices to check the smoothness of a_k and b_k near $z = 0$. From Lemma 2.6 we have $w_\delta(z) = A_1 + A_2 z^2 + O(z^4)$, $z \sim 0$. Then $w_\delta'/z = 2A_2 + O(z^2)$ and hence a_k is smooth. For b_k, first note that $w_\delta''/z = 2A_2/z + O(z)$ and $w_\delta'/z^2 = 2A_2/z + O(z)$. The last two

terms in (4.24) are smooth: $-4(2+2k)w''_\delta/z + 4(2+2k)w'_\delta/z^2 = O(z)$. It is straightforward to see that only odd powers of z remain in the expansion of b_k and only even powers in the expansion of a_k around $z=0$. The claimed inequality follows easily from the already established Taylor expansions of a_k and b_k. □

4.1 The energy norm and weighted Sobolev norms

The next lemma allows us to estimate the higher ∂_z-derivatives of φ in terms of the high-order $\mathscr{S}$-derivatives of φ in the weighted spaces $\mathfrak{H}^j_{\delta,k}$ (Definition 2.14).

Lemma 4.8 (From the energy to the norm). *Let $k \in \mathbb{N}$ be given.*

(1) For any $\varphi \in \mathfrak{H}^1_{\delta,k-1}$ the following bound holds:

$$\tag{4.30} \|\partial_z^2 \varphi\|^2_{\delta,2k} + \|\frac{\partial_z \varphi}{z}\|^2_{\delta,2k-1} \lesssim \|\mathscr{S}\varphi\|^2_{\delta,2k},$$

where the notational convention for " $\lesssim$ " is explained in Remark 2.13.

(2) Let $\varphi \in \mathfrak{H}^j_{\delta,k-1}$, $j \in \{1,2,3,4\}$. Then there exists a constant $C > 0$ such that

$$\tag{4.31} \sum_{\ell=1}^{j}\left(\|\partial_z^{2\ell}\varphi\|^2_{\delta,2k+2\ell-2} + \|\frac{\partial_z^{2\ell-1}\varphi}{z}\|^2_{\delta,2k+2\ell-3} + \|\frac{\partial_z^{2\ell-2}\varphi}{z^2}\|^2_{\delta,2k+2\ell-4}\right) \leq C\sum_{\ell=1}^{j}\|\mathscr{S}^\ell\varphi\|^2_{\delta,2k+2\ell-}$$

Proof. We recall that w_δ satisfies $w'_\delta < 0$, $w'_\delta \sim cz$ near $z=0$, and $w'_\delta = O(1)$ (see Lemma 2.6). We first examine the case $\ell = 1$. Note that

$$\tag{4.32} (\mathscr{S}\varphi, \mathscr{S}\varphi)_{\delta,2k} = \int_0^1 w_\delta^{3+2k} z^4 (\partial_z^2\varphi + \frac{4}{z}\partial_z\varphi)^2 dz$$
$$= \int_0^1 w_\delta^{3+2k} z^4 ((\partial_z^2\varphi)^2 + \frac{16}{z^2}(\partial_z\varphi)^2)dz + 4\int_0^1 w_\delta^{3+2k} z^3 \partial_z(\partial_z\varphi)^2 dz$$
$$= \int_0^1 w_\delta^{3+2k} z^4 ((\partial_z^2\varphi)^2 + \frac{4}{z^2}(\partial_z\varphi)^2)dz - 4(3+2k)\int_0^1 w_\delta^{2+2k} w' z^3 (\partial_z\varphi)^2 dz.$$

Each term is positive and it contains the desired square norms for $\partial_z^2\varphi$ and $\frac{\partial_z\varphi}{z}$:

$$\tag{4.33} (\mathscr{S}\varphi, \mathscr{S}\varphi)_{\delta,2k} \gtrsim \|\partial_z^2\varphi\|^2_{\delta,2k} + \|\frac{w_\delta\partial_z\varphi}{z}\|^2_{\delta,2k-2} + \|\partial_z\varphi\|^2_{\delta,2k-1}$$
$$\gtrsim \|\partial_z^2\varphi\|^2_{\delta,2k} + \|\frac{\partial_z\varphi}{z}\|^2_{\delta,2k-1},$$

which proves (4.30). By Rellich inequality or Hardy inequality, we also obtain

$$(\mathscr{S}\varphi, \mathscr{S}\varphi)_{\delta,2k} + \|\varphi\|^2_{\delta,2k-2} \gtrsim \|\frac{\varphi}{z^2}\|_{\delta,2k-2}$$

By a similar argument, we obtain

$$(4.34) \qquad (\mathscr{S}^2\varphi, \mathscr{S}^2\varphi)_{\delta,2k+2} \gtrsim \|\partial_z^2\mathscr{S}\varphi\|_{\delta,2k+2}^2 + \|\frac{\partial_z\mathscr{S}\varphi}{z}\|_{\delta,2k+1}^2,$$

as well as

$$(4.35) \qquad (\mathscr{S}^2\varphi, \mathscr{S}^2\varphi)_{\delta,2k+2} + \|\mathscr{S}\varphi\|_{\delta,2k}^2 \gtrsim \|\frac{\mathscr{S}\varphi}{z^2}\|_{\delta,2k}^2.$$

Next we will show how to recover higher order φ-norms from $\mathscr{S}\varphi$. Recall that

$$(4.36) \qquad \|\frac{\mathscr{S}\varphi}{z^2}\|_{\delta,2k}^2 = \int_0^1 w_\delta^{3+2k}(\partial_z^2\varphi + \frac{4}{z}\partial_z\varphi)^2 dz$$

The finiteness of the integral leads to $\partial_z\phi|_{z=0} = 0$ because letting $f = \partial_z^2\varphi + \frac{4}{z}\partial_z\varphi$, we have

$$|\partial_z\varphi| = |\frac{1}{z^4}\int_0^z \zeta^4 f d\zeta| \leq z^{\frac{1}{2}}(\int_0^z f^2 d\zeta)^{\frac{1}{2}}$$

which implies $\lim_{z\to 0^+}\partial_z\phi = 0$. We deduce that $\lim_{z\to 0^+}\frac{(\partial_z\phi)^2}{z} = 0$. Therefore by integration-by-parts we obtain

$$(4.37) \qquad \|\frac{\mathscr{S}\varphi}{z^2}\|_{\delta,2k}^2 \gtrsim \|\frac{\partial_z^2\varphi}{z^2}\|_{\delta,2k}^2 + \|\frac{\partial_z\varphi}{z^3}\|_{\delta,2k-1}^2.$$

Now we rewrite the second term in the right-hand side of (4.34) as

$$\|\frac{\partial_z\mathscr{S}\varphi}{z}\|_{\delta,2k+1}^2 = \int_0^1 w_\delta^{4+2k}z^2|\partial_z(\partial_z^2\varphi + \frac{4}{z}\partial_z\varphi)|^2 dz$$

$$= \int_0^1 w_\delta^{4+2k}z^2|\partial_z^3\varphi + \frac{4}{z}\partial_z^2\varphi - \frac{4}{z^2}\partial_z\varphi|^2 dz.$$

Due to (4.37), we deduce that

$$(4.38) \qquad \int_0^1 w_\delta^{4+2k}z^2|\partial_z^3\varphi|^2 dz \leq \|\frac{\partial_z\mathscr{S}\varphi}{z}\|_{\delta,2k+1}^2 + \|\frac{\mathscr{S}\varphi}{z^2}\|_{\delta,2k}^2$$

The first term of the right-hand side of (4.34) gives

$$\|\partial_z^2\mathscr{S}\varphi\|_{\delta,2k+2}^2 = \int_0^1 w_\delta^{5+2k}z^4|\partial_z^2(\partial_z^2\varphi + \frac{4}{z}\partial_z\varphi)|^2 dz$$

By previous bounds, we obtain

$$(4.39)$$
$$\int_0^1 w_\delta^{5+2k}z^4|\partial_z^4\varphi|^2 dz \leq \|\partial_z^2\mathscr{S}\varphi\|_{\delta,2k+2}^2 + \|\frac{\partial_z\mathscr{S}\varphi}{z}\|_{\delta,2k+1}^2 + \|\frac{\mathscr{S}\varphi}{z^2}\|_{\delta,2k}^2$$
$$\leq (\mathscr{S}^2\varphi, \mathscr{S}^2\varphi)_{\delta,2k+2} + (\mathscr{S}\varphi, \mathscr{S}\varphi)_{\delta,2k}$$

To sum up, we have shown that

(4.40)
$$\|\frac{\mathscr{S}\varphi}{z^2}\|_{\delta,2k}^2 + \|\frac{\partial_z\mathscr{S}\varphi}{z}\|_{\delta,2k+1}^2 + \|\partial_z^2\mathscr{S}\varphi\|_{\delta,2k+2}^2$$
$$+ \|\frac{\partial_z^2\varphi}{z^2}\|_{\delta,2k}^2 + \|\frac{\partial_z^3\varphi}{z}\|_{\delta,2k+1}^2 + \|\partial_z^4\varphi\|_{\delta,2k+2}^2 \le \|\mathscr{S}^2\varphi\|_{\delta,2k+2}^2 + \|\mathscr{S}\varphi\|_{\delta,2k}^2$$

Inductively, we can recover all the spatial norms via $\|\mathscr{S}^\ell\varphi\|_{\delta,2k+2\ell-2}^2$. To be more precise, we first apply (4.40) to $\varphi \to \mathscr{S}\varphi$ and $k \to k+1$,

(4.41)
$$\|\frac{\mathscr{S}^2\varphi}{z^2}\|_{\delta,2k+2}^2 + \|\frac{\partial_z\mathscr{S}^2\varphi}{z}\|_{\delta,2k+3}^2 + \|\partial_z^2\mathscr{S}^2\varphi\|_{\delta,2k+4}^2$$
$$+ \|\frac{\partial_z^2\mathscr{S}\varphi}{z^2}\|_{\delta,2k+2}^2 + \|\frac{\partial_z^3\mathscr{S}\varphi}{z}\|_{\delta,2k+3}^2 + \|\partial_z^4\mathscr{S}\varphi\|_{\delta,2k+4}^2 \le \|\mathscr{S}^3\varphi\|_{\delta,2k+4}^2 + \|\mathscr{S}^2\varphi\|_{\delta,2k+2}^2$$

We next note that

(4.42)
$$\partial_z\mathscr{S}\varphi = \partial_z^3\varphi + \frac{4}{z}\partial_z^2\varphi - \frac{4}{z^2}\partial_z\varphi$$
$$\partial_z^2\mathscr{S}\varphi = \partial_z^4\varphi + \frac{4}{z}\partial_z^3\varphi - \frac{8}{z^2}\partial_z^2\varphi + \frac{8}{z^3}\partial_z\varphi$$
$$\partial_z^3\mathscr{S}\varphi = \partial_z^5\varphi + \frac{4}{z}\partial_z^4\varphi - \frac{12}{z^2}\partial_z^3\varphi + \frac{24}{z^3}\partial_z^2\varphi - \frac{24}{z^4}\partial_z\varphi$$
$$\partial_z^4\mathscr{S}\varphi = \partial_z^6\varphi + \frac{4}{z}\partial_z^5\varphi - \frac{16}{z^2}\partial_z^4\varphi + \frac{48}{z^3}\partial_z^3\varphi - \frac{96}{z^4}\partial_z^2\varphi + \frac{96}{z^5}\partial_z\varphi.$$

It follows that
$$\partial_z^3\mathscr{S}\varphi = \partial_z^5\varphi + \frac{7}{z}\partial_z^4\varphi - \frac{3}{z}\partial_z^2\mathscr{S}\varphi,$$

and
$$\partial_z^4\mathscr{S}\varphi = \partial_z^6\varphi + \frac{8}{z}\partial_z^5\varphi - \frac{4}{z}\partial_z^3\mathscr{S}\varphi.$$

Then from the boundedness of $\partial_z^4\mathscr{S}\varphi$, $\frac{\partial_z^3\mathscr{S}\varphi}{z}$ and $\frac{\partial_z^2\mathscr{S}\varphi}{z^2}$ in (4.41), we deduce that

(4.43)
$$\|\partial_z^6\varphi\|_{\delta,2k+4}^2 + \|\frac{\partial_z^5\varphi}{z}\|_{\delta,2k+3}^2 + \|\frac{\partial_z^4\varphi}{z^2}\|_{\delta,2k+2}^2 \le \|\mathscr{S}^3\varphi\|_{\delta,2k+4}^2 + \|\mathscr{S}^2\varphi\|_{\delta,2k+2}^2.$$

Now applying (4.43) to $\mathscr{S}\varphi$ and $k+1$, we also deduce
(4.44)
$$\|\partial_z^8\varphi\|_{\delta,2k+6}^2 + \|\frac{\partial_z^7\varphi}{z}\|_{\delta,2k+5}^2 + \|\frac{\partial_z^6\varphi}{z^2}\|_{\delta,2k+4}^2 \le \|\mathscr{S}^4\varphi\|_{\delta,2k+6}^2 + \|\mathscr{S}^3\varphi\|_{\delta,2k+4}^2.$$

This concludes the proof. $\square$

4.2 Nonlinear energy estimates and proof of Theorem 4.4

In the estimates below we shall be freely using (2.58)–(2.60).

We commute the equation (4.1) with the operator $\mathscr{S}^j$, $j = 0, 1, 2, 3, 4$. As a result we obtain the following equation satisfied by $\mathscr{S}^j \phi$:

$$\tilde{\lambda} \partial_{\tau\tau} \mathscr{S}^j \phi + \tilde{\lambda}_\tau \partial_\tau \mathscr{S}^j \phi + 3\delta \mathscr{S}^j \phi + \mathscr{L}_{\delta,2j} \mathscr{S}^j \phi$$

$$(4.45) \qquad + \sum_{\ell=0}^{j-1} \mathscr{S}^\ell \left(a_{j-\ell} \mathscr{S}^{j-\ell} \phi + b_{j-\ell} \partial_z \mathscr{S}^{j-1-\ell} \phi \right) = \mathscr{S}^j N_\delta[\phi].$$

In the derivation of (4.45) we successively used Lemma 4.6. Taking the $(\cdot, \cdot)_{\delta,2j}$ inner product with $\mathscr{S}^j \phi_\tau$ we obtain the following identity:

$$\frac{1}{2} \frac{d}{d\tau} \left[\tilde{\lambda} (\mathscr{S}^j \phi_\tau, \mathscr{S}^j \phi_\tau)_{\delta,2j} + 3\delta (\mathscr{S}^j \phi, \mathscr{S}^j \phi)_{\delta,2j} + \frac{4}{3} (\partial_z \mathscr{S}^j \phi, \partial_z \mathscr{S}^j \phi)_{\delta,2j+1} \right]$$

$$(4.46)$$

$$+ \frac{\tilde{\lambda}_\tau}{2} (\mathscr{S}^j \phi_\tau, \mathscr{S}^j \phi_\tau)_{\delta,2j} = \mathscr{Q}_j + \mathscr{C}_j,$$

where
$$(4.47)$$

$$\mathscr{Q}_j := -\sum_{\ell=0}^{j-1} \left(\mathscr{S}^\ell \left(a_{j-\ell} \mathscr{S}^{j-\ell} \phi + b_{j-1-\ell} \partial_z \mathscr{S}^{j-1-\ell} \phi \right), \mathscr{S}^j \phi_\tau \right)_{\delta,2j}$$

and

$$(4.48) \qquad\qquad \mathscr{C}_j := (\mathscr{S}^j N_\delta[\phi], \mathscr{S}^j \phi_\tau)_{\delta,2j}$$

for any $j = 0, 1, 2, 3, 4$. We recall that the functions $a_k(\cdot), b_k(\cdot)$, $k \in \mathbb{N}$, are defined by (4.23)–(4.24).

For any $j \in \{0, 1, 2, 3, 4\}$ the quantities $\mathscr{Q}_j$ represent the quadratic error terms and they are therefore very dangerous as they are a priori of the same order of magnitude as the energy itself. The (at least) cubic error terms $\mathscr{C}_j$ are better behaved from the point of view of the order of magnitude, but they possess an intricate quasilinear structure and different ideas are needed to control them.

Energy estimates for $\mathscr{Q}_j$

We recall that for any given $\delta \geq 0$ or $\delta^* < \delta < 0$ with $\tilde{e} = \lambda_1^2 + \frac{2\delta}{\lambda_0} > 0$, the homologous solution $\tilde{\lambda}$ behaves like

$$\tilde{\lambda}(\tau) \sim_{\tau \to \infty} e^{\beta \tau}, \quad \beta = \sqrt{\tilde{e}}.$$

Lemma 4.9 (Estimate for $\mathscr{Q}_j$). *Assume the same as in Theorem 4.4. Then the following energy bound holds:*
$$(4.49)$$

$$\left| \int_0^\tau \mathscr{Q}_j \, d\sigma \right| \leq \nu \int_0^\tau \tilde{\mathscr{D}}(\sigma) \, d\sigma + C_\nu \int_0^\tau e^{-\beta_2 \sigma} \tilde{\mathscr{E}}(\sigma) \, d\sigma, \quad \tau \in [0, T]$$

for any $\nu > 0$. Here $\beta_2 > 0$ is a constant defined in (2.60).

Proof. Applying the product rule (4.20) it is easy to see that the top order error term appearing in the first sum on the right-hand side of (4.47) is of the form

$$
(4.50) \qquad -\sum_{\ell=1}^{j} \left(a_{j-\ell} \mathscr{S}^j \phi, \mathscr{S}^j \phi_\tau \right)_{\delta,2j}.
$$

We apply the Young inequality to estimate (4.50):

$$
\sum_{\ell=0}^{j-1} \left| (a_{j-\ell}(z) \mathscr{S}^j \phi, \mathscr{S}^j \phi_\tau)_{\delta,2j} \right| \leq \nu \frac{\tilde{\lambda}_\tau}{2} \| \mathscr{S}^j \phi_\tau \|_{\delta,2j}^2 + C \frac{\sum_{\ell=1}^{j} \| a_{j-\ell} \|_{L^\infty([0,1])}}{\nu \tilde{\lambda}_\tau} \| \mathscr{S}^j \phi \|_{\delta,2j}^2
$$

$$
\leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau} \| \mathscr{S}^j \phi \|_{\delta,2j}^2.
$$

The remaining terms appearing in the first sum on the right-hand side of (4.47) are all below-top order. To illustrate this, consider the case $\ell = 1$ in the first sum on the right-hand side of (4.47):

$$
\left(\mathscr{S}(a_{j-1} \mathscr{S}^{j-1} \phi), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} = \left(\mathscr{S} a_{j-1} \mathscr{S}^{j-1} \phi, \mathscr{S}^j \phi_\tau \right)_{\delta,2j}
$$

$$
(4.51) \qquad + 2 \left(\partial_z a_{j-1} \partial_z \mathscr{S}^{j-1} \phi, \mathscr{S}^j \phi_\tau \right)_{\delta,2j} + \left(a_{j-1} \mathscr{S}^j \phi, \mathscr{S}^j \phi_\tau \right)_{\delta,2}
$$

Observe that we used the product rule (4.20) on the right-hand side above. The last term on the right-hand side of (4.51) has already been bounded above. The first term on the right-hand side of (4.51) is bounded using the Hölder ineqaulity

$$
\left| \left(\mathscr{S} a_{j-1} \mathscr{S}^{j-1} \phi, \mathscr{S}^j \phi_\tau \right)_{\delta,2j} \right| \lesssim \| \mathscr{S} a_{j-1} w_\delta \|_\infty \| \mathscr{S}^{j-1} \phi \|_{\delta,2j-2} \| \mathscr{S}^j \phi_\tau \|_{\delta,2j}
$$

$$
\leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau} \tilde{\mathscr{E}},
$$

where we used Lemma 4.7 to infer that $\| \mathscr{S} a_{j-1} w_\delta \|_\infty \lesssim 1$ and the definition of $\tilde{\mathscr{E}}$ and $\tilde{\mathscr{D}}$. To bound the second term on right-hand side of (4.51), we use (4.30) with $\varphi = \mathscr{S}^{j-1} \phi$. We thus obtain

$$
\left| \left(\partial_z a_{j-1} \partial_z \mathscr{S}^{j-1} \phi, \mathscr{S}^j \phi_\tau \right)_{\delta,2j} \right| \lesssim \| \partial_z a_{j-1} z w_\delta^{1/2} \|_\infty \| \frac{\partial_z \mathscr{S}^{j-1} \phi}{z} \|_{\delta,2j-1} \| \mathscr{S}^j \phi_\tau \|_{\delta,2j}
$$

$$
\leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau} \tilde{\mathscr{E}},
$$

just like in the previous estimate. Combining the last three bounds we conclude that

$$
(4.52) \qquad \left| \left(\mathscr{S}(a_{j-1} \mathscr{S}^{j-1} \phi), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} \right| \leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau} \tilde{\mathscr{E}}.
$$

Analogously to the above estimate, using Lemmas 4.7, 4.8, the Hölder inequality and the Young inequality we finally infer that

$$
\left| \sum_{\ell=0}^{j-1} \left(\mathscr{S}^\ell \left(a_{j-\ell} \mathscr{S}^{j-\ell} \phi \right), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} \right| \leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau} \sum_{\ell=0}^{j} \| \mathscr{S}^\ell \phi \|_{\delta,2\ell}^2
$$

$$
\leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau} \tilde{\mathscr{E}}.
$$

Analogously, for any $\ell \in \{1, \ldots, j\}$ we first estimate the top-order term

$$\left|(b_{j-1-\ell}(z)\partial_z \mathscr{S}^{j-1}\phi, \mathscr{S}^j \phi_\tau)_{\delta,2j}\right| \leq \nu \frac{\tilde{\lambda}_\tau}{2} \|\mathscr{S}^j \phi_\tau\|^2_{\delta,2j} + C_\nu \frac{\|zb_{j-1-\ell}w_\delta^{1/2}\|^2_{L^\infty([0,1])}}{\tilde{\lambda}_\tau} \left\|\frac{\partial_z \mathscr{S}^{j-1}\phi}{z}\right.$$

$$\leq \nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau}\tilde{\mathscr{E}}.$$

Using the Young inequality and Lemmas 4.7, 4.8, it is straightforward to check that the remaining below-top-order terms in the first sum on the right-hand side of (4.47) are also bounded by $\nu \tilde{\mathscr{D}} + C_\nu e^{-\beta_2 \tau}\tilde{\mathscr{E}}$. Therefore, using the above bounds and integrating with respect to τ, we obtain the bound (4.49). $\qquad\qquad\square$

Convenient representation of N_δ and commutator identities

We shall rewrite the quadratic nonlinearity $N_\delta[\cdot]$ defined in (3.5) in a slightly different form, more conducive to our energy estimates. To that end, for any $k \in \mathbb{N} \cup \{0\}$ we define the following linear operator:

$$M_{\delta,k}\phi := \frac{1}{w_\delta^{3+k}z}\partial_z\left(\frac{w_\delta^{4+k}}{z^2}\partial_z\left(z^3\phi\right)\right).$$

It is straightforward to check that the operator $M_{\delta,k}$ is symmetric with respect to the inner product $(\cdot,\cdot)_{\delta,k}$:

$$\left(M_{\delta,k}\phi_1, \phi_2\right)_{\delta,k} = -\int_0^1 \frac{w_\delta^{4+k}}{z^2}\partial_z(z^3\phi_1)\partial_z(z^3\phi_2)\,dz, \quad \phi_1, \phi_2 \in H^1_{\delta,k}.$$

With the aid of $M_{\delta,k}$ the nonlinearity $N_\delta[\phi]$ (3.5) can be rewritten in the following form:

(4.53)

$$N_\delta[\phi] = p_0(\phi) + \frac{w_\delta'}{z}p_1(\phi) + p_2(\phi)M_{\delta,0}\phi + p_3(\phi)M_{\delta,0}(p_4(\phi)) + Q[\phi,\phi].$$

Here

$$Q[\phi,\phi] := \frac{1}{w_\delta^3 z}p_5(\phi)\partial_z\left[w_\delta^4\left(\frac{1}{z^2}\partial_z\left(z^3 p_6(\phi)\right)\right)^2\int_0^1\left(1-\theta+\frac{\theta}{z^2}\partial_z\left(z^3 p_6(\phi)\right)\right)^{\frac{10}{3}}d\theta\right]$$

and rational polynomials p_i, $i = 0, \ldots, 6$ are defined just before (3.7).

Lemma 4.10 (Commutator identities for $M_{k,\delta}$ and $\mathscr{S}$). *For any $k \in \mathbb{N}$ the following commutation property holds:*

(4.54) $\mathscr{S}M_{\delta,k}\phi = M_{\delta,k+2}\mathscr{S}\phi + (\alpha_k\mathscr{S} + \beta_k\partial_z + \gamma_k)\phi,$

where the functions $\alpha_k, \beta_k, \gamma_k$ are given by

$$(4.55) \qquad \alpha_k := (9+2k)w_\delta'' + 4\frac{w_\delta'}{z}$$

$$(4.56) \qquad \beta_k := -3(4+k)\pi w_\delta^2 w_\delta'$$

$$(4.57) \qquad \gamma_k := -9(4+k)\pi w_\delta^2 w_\delta'$$

Proof. The proof is a direct calculation analogous to the proof of Lemma 4.6 and the relationship (2.17). □

Lemma 4.11 (Uniform boundedness of commutator coefficients). *For any $k \in \mathbb{N}$ functions $\alpha_k, \beta_k, \gamma_k$ from the previous lemma are smooth on $[0,1]$ and for any $j \in \mathbb{N}$ there exists a $C_{jk} > 0$ such that*

$$\sum_{\ell=0}^{j} \left(\|\partial_z^{2\ell} \zeta\|_{L^\infty([0,1])} + \|\mathscr{S}^\ell \zeta\|_{L^\infty([0,1])} \right) \le C_{jk}, \quad \zeta = \alpha_k, \beta_k, \gamma_k.$$

Proof. The proof is a direct consequence of the formulas (4.55)–(4.57), the generalized Lane-Emden equation (2.17) and the smoothness properties of w_δ near 0 stated in Lemma 2.6. □

Lemma 4.12 (High-order commutators). *For any $j \in \mathbb{N}$ the following commutation property holds:*

$$(4.58)$$

$$\mathscr{S}^j M_{\delta,0} \phi = M_{\delta,2j} \mathscr{S}^j \phi + \sum_{\ell=0}^{j-1} \mathscr{S}^\ell \left[\left(\alpha_{j-\ell}\mathscr{S} + \beta_{j-\ell}\partial_z + \gamma_{j-\ell} \right) \mathscr{S}^{j-1-\ell} \phi \right]$$

where the functions $\alpha_k, \beta_k, \gamma_k$ are given by (4.55)–(4.57).

Energy estimates for $\mathscr{C}_j$

Lemma 4.13 (Estimate for $\mathscr{C}_j$). *Assume the same as in Theorem 4.4. Then the following energy bound holds:*

$$(4.59)$$

$$\left| \int_0^\tau \mathscr{C}_j \, d\sigma \right| \lesssim \tilde{\mathscr{E}}(0) + \left(\nu + \sup_{0 \le \sigma \le \tau} \sqrt{\tilde{\mathscr{E}}}(\sigma) \right) \int_0^\tau \tilde{\mathscr{D}}(\sigma) \, d\sigma + \sup_{0 \le \sigma \le \tau} \left(\tilde{\mathscr{E}}^{\frac{3}{2}}(\sigma) + \tilde{\mathscr{E}}^2(\sigma) \right), \quad \tau$$

for any $\nu > 0$.

Proof. Recall the definition (4.48) of the cubic error term $\mathscr{C}_j = (\mathscr{S}^j N_\delta[\phi], \mathscr{S}^j \phi_\tau)_{\delta,2j}$. We can rewrite $(\mathscr{S}^j N_\delta[\phi], \mathscr{S}^j \phi_\tau)_{\delta,2j}$ using (4.53) in the form

$$(\mathscr{S}^j N_\delta[\phi], \mathscr{S}^j \phi_\tau)_{\delta,2j} = \left(\mathscr{S}^j (p_0(\phi)), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} + \left(\mathscr{S}^j \left(\frac{w_\delta'}{z} p_1(\phi) \right), \mathscr{S}^j \phi_\tau \right)_{\delta,2j}$$

$$(4.60)$$

$$+ \left(\mathscr{S}^j (p_2(\phi)M_{\delta,0}\phi), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} + \left(\mathscr{S}^j (p_3(\phi)M_{\delta,0}(p_4(\phi))), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} + \left(\mathscr{S}^j Q[\phi, \phi]$$

Step 1: Estimates for the first and second term on the right-hand side of (4.60). Using (4.21) and (4.20) for any $j \geq 1$ we may rewrite

$$\mathscr{S}^j(p_0(\phi)) = \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} \left(\mathscr{S}^\ell(p_0'(\phi))\mathscr{S}^{j-\ell}\phi + \mathscr{S}^\ell\left(p_0''(\phi)\right) \mathscr{S}^{j-1-\ell}\left(|\partial_z\phi|^2\right) \right) + B_j[\phi]$$

where $B_j[\phi]$ consists of remaining terms after applying the product rule (4.20). Each factor in $B_j[\phi]$ contains at least one ∂_z derivative and in general a combination of ∂_z and $\mathscr{S}$. For instance,

$$B_1[\phi] = 0, \quad B_2[\phi] = 2\partial_z(p_0'(\phi))\partial_z\mathscr{S}\phi + 2\partial_z(p_0''(\phi))\partial_z|\partial_z\phi|^2,$$
$$B_3[\phi] = \mathscr{S}(B_2[\phi]) + 2\partial_z(p_0'(\phi))\partial_z\mathscr{S}^2\phi + 2\partial_z\mathscr{S}(p_0'(\phi))\partial_z\mathscr{S}\phi$$
$$+ 2\partial_z(p_0''(\phi))\partial_z\mathscr{S}(|\partial_z\phi|^2) + 2\partial_z\mathscr{S}(p_0''(\phi))\partial_z(|\partial_z\phi|^2)$$

and so on. Since $p_0'(0) = 0$ the above expression is at least quadratic in ϕ. A routine application of Lemma 4.8, the Hardy and Sobolev embedding inequalities of Lemmas B.1–B.3 yields the estimate

$$\|\mathscr{S}^j(p_0(\phi))\|_{\delta,2j} \lesssim \tilde{\mathscr{E}}.$$

Together with $\|\mathscr{S}^j\phi_\tau\|_{\delta,2j}^2 \lesssim e^{-\beta_2\tau}\tilde{\mathscr{D}}$ we conclude that

$$(4.61) \quad \left| \left(\mathscr{S}^j(p_0(\phi)), \mathscr{S}^j\phi_\tau\right)_{\delta,2j} \right| \lesssim e^{-\beta_2\tau/2}\tilde{\mathscr{E}}\sqrt{\tilde{\mathscr{D}}} \lesssim \nu\tilde{\mathscr{D}} + \frac{e^{-\beta_2\tau}\tilde{\mathscr{E}}}{\nu}\tilde{\mathscr{E}}.$$

From (2.17) and Lemma 2.6 it follows that $\frac{w_\delta'}{z}$ is in fact a smooth function and all of its $\mathscr{S}$-derivatives are uniformly bounded on the interval $[0,1]$. By the same argument as above we conclude that

$$(4.62)$$
$$\left| \left(\mathscr{S}^j\left(\frac{w_\delta'}{z}p_1(\phi)\right), \mathscr{S}^j\phi_\tau\right)_{\delta,2j} \right| \lesssim e^{-\beta_2\tau/2}\tilde{\mathscr{E}}\sqrt{\tilde{\mathscr{D}}} \lesssim \nu\tilde{\mathscr{D}} + \frac{e^{-\beta_2\tau}\tilde{\mathscr{E}}}{\nu}\tilde{\mathscr{E}}.$$

Step 2: Estimates for the third, fourth, and the fifth term on the right-hand side of (4.60). Applying the product rule (4.20) and the chain rule (4.21) we obtain

$$(4.63) \quad \mathscr{S}^j\left(p_2(\phi)M_{\delta,0}\phi\right) = \sum_{k=0}^{j} \binom{j}{k}\mathscr{S}^{j-k}p_2(\phi)\mathscr{S}^k M_{\delta,0}\phi + A_j[\phi]$$

where $A_j[\phi]$ consists of remaining nonlinear terms after applying the product rule (4.20) and each factor in $A_j[\phi]$ contains at least one ∂_z derivative.

We focus on the first summation. We can rewrite it as

$$\sum_{k=0}^{j} \binom{j}{k} \mathscr{S}^{j-k} p_2(\phi) \mathscr{S}^k M_{\delta,0} \phi$$

$$= \sum_{k=0}^{j} \binom{j}{k} \mathscr{S}^{j-k} p_2(\phi) \left(M_{\delta,2k} \mathscr{S}^k \phi + \sum_{\ell=0}^{k-1} \mathscr{S}^\ell \left[(\alpha_{k-\ell} \mathscr{S} + \beta_{k-\ell} \partial_z + \gamma_{k-\ell}) \mathscr{S}^{k-1-\ell} \phi \right] \right)$$

(4.64)

$$= p_2(\phi) M_{\delta,2j} \mathscr{S}^j \phi + \mathscr{R}_j,$$

where the remainder term $\mathscr{R}_j$ is given by

$$\mathscr{R}_j = p_2(\phi) \sum_{\ell=0}^{j-1} \mathscr{S}^\ell \left[(\alpha_{k-\ell} \mathscr{S} + \beta_{k-\ell} \partial_z + \gamma_{k-\ell}) \mathscr{S}^{k-1-\ell} \phi \right]$$

(4.65)

$$+ \sum_{k=0}^{j-1} \binom{j}{k} \mathscr{S}^{j-k} p_2(\phi) \left(M_{\delta,2k} \mathscr{S}^k \phi + \sum_{\ell=0}^{k-1} \mathscr{S}^\ell \left[(\alpha_{k-\ell} \mathscr{S} + \beta_{k-\ell} \partial_z + \gamma_{k-\ell}) \mathscr{S}^{k-1-\ell} \phi \right] \right).$$

We isolated the top order term $p_2(\phi) M_{\delta,2j} \mathscr{S}^j \phi$ on the right-most side of (4.64) while the lower-order remainder term $\mathscr{R}_j$ is estimated by the Sobolev and Hardy inequalities on Lemmas B.1–B.3 and Lemma 4.8. To handle the top-order term, we note that

$$(p_2(\phi) M_{\delta,2j} \mathscr{S}^j \phi, \mathscr{S}^j \phi_\tau)_{\delta,2j} = \int_0^1 p_2(\phi) \partial_z \left(w_\delta^{4+2j} \frac{1}{z^2} \partial_z (z^3 \mathscr{S}^j \phi) \right) z^3 \mathscr{S}^j \phi_\tau \, dz$$

(4.66)

$$= -\frac{1}{2} \partial_\tau \int_0^1 p_2(\phi) \frac{1}{z^2} w_\delta^{4+2j} \left| \partial_z (z^3 \mathscr{S}^j \phi) \right|^2 dz + \frac{1}{2} \int_0^1 p_2'(\phi) \phi_\tau \frac{1}{z^2} \left| \partial_z (z^3 \mathscr{S}^j \phi) \right|^2 w_\delta^{4+j} \, dz$$

Just like in (3.39) using the Hardy inequality we have

$$\int_0^1 \frac{1}{z^2} w_\delta^{4+2j} \left| \partial_z (z^3 \mathscr{S}^j \phi) \right|^2 dz \lesssim \int_0^1 w_\delta^{4+2j} z^2 (\mathscr{S}^j \phi)^2 \, dz + \int_0^1 w_\delta^{4+2j} z^4 (\partial_z \mathscr{S}^j \phi)^2 \, dz \lesssim \tilde{\mathscr{E}}.$$

Using (3.7) and Lemma B.3 it follows that $\| p_2(\phi) \|_\infty \lesssim \sqrt{\tilde{\mathscr{E}}}$. Therefore

(4.67)

$$\left| \int_0^\tau -\frac{1}{2} \partial_\tau \int_0^1 p_2(\phi) \frac{1}{z^2} w_\delta^{4+2j} \left| \partial_z (z^3 \mathscr{S}^j \phi) \right|^2 dz \, d\sigma \right| \lesssim \tilde{\mathscr{E}}(0) + \tilde{\mathscr{E}}^{\frac{3}{2}}(\tau).$$

The second term on the right-most side of (4.66) is estimated similarly:

$$\left| \int_0^\tau \int_0^1 p_2'(\phi)\phi_\tau \frac{1}{z^2}\left|\partial_z(z^3 \mathscr{S}^j \phi)\right|^2 w_\delta^{4+j} dz d\sigma \right|$$

$$\lesssim \int_0^\tau \|p_2'(\phi)\phi_\tau\|_{L^\infty([0,1])} \int_0^1 \frac{1}{z^2} w_\delta^{4+2j} \left|\partial_z(z^3 \mathscr{S}^j \phi)\right|^2 dz d\sigma$$

(4.68)

$$\lesssim \int_0^\tau e^{-\beta_2 \sigma/2}\sqrt{\tilde{\mathscr{D}}}(\sigma)\tilde{\mathscr{E}}(\sigma)\, d\sigma \lesssim \nu \int_0^\tau \tilde{\mathscr{D}}(\sigma)\, d\sigma + \sup_{0\le\sigma\le\tau} \tilde{\mathscr{E}}^2(\sigma).$$

Using Lemmas B.1–B.3 and Lemma 4.8 and the $L^\infty - L^2 - L^2$ Hölder inequality the remainder term (4.65) is easily shown to satisfy the energy bound

(4.69)

$$\left| \int_0^\tau (\mathscr{R}_j, \mathscr{S}^j \phi_\tau)_{\delta,2j}\, d\sigma \right| \lesssim \left(\nu + \sup_{0\le\sigma\le\tau}\sqrt{\tilde{\mathscr{E}}}(\sigma) \right) \int_0^\tau \tilde{\mathscr{D}}(\sigma)\, d\sigma + \sup_{0\le\sigma\le\tau} \tilde{\mathscr{E}}^2(\sigma).$$

From (4.66)–(4.69) we obtain the inequality

$$\left| \int_0^\tau \left(\mathscr{S}^j \left(p_2(\phi)M_{\delta,0}\phi \right), \mathscr{S}^j \phi_\tau \right)_{\delta,2j}\, d\sigma \right|$$

(4.70) $$\lesssim \tilde{\mathscr{E}}(0) + \left(\nu + \sup_{0\le\sigma\le\tau}\sqrt{\tilde{\mathscr{E}}}(\sigma) \right) \int_0^\tau \tilde{\mathscr{D}}(\sigma)\, d\sigma + \sup_{0\le\sigma\le\tau} \tilde{\mathscr{E}}^2(\sigma).$$

The fourth and the fifth term on the right-hand side of (4.60) are estimated in an analogous manner:

$$\left| \int_0^\tau \left[\left(\mathscr{S}^j \left(p_3(\phi)M_{\delta,0}(p_4(\phi)) \right), \mathscr{S}^j \phi_\tau \right)_{\delta,2j} + \left(\mathscr{S}^j Q[\phi,\phi], \mathscr{S}^j \phi_\tau \right)_{\delta,2j} \right] d\sigma \right|$$

(4.71)

$$\lesssim \tilde{\mathscr{E}}(0) + \left(\nu + \sup_{0\le\sigma\le\tau}\sqrt{\tilde{\mathscr{E}}}(\sigma) \right) \int_0^\tau \tilde{\mathscr{D}}(\sigma)\, d\sigma + \sup_{0\le\sigma\le\tau} \mathscr{E}^2(\sigma).$$

Summing (4.61)–(4.62) and (4.70)–(4.71) we obtain (4.59). $\qquad\square$

Proof of Theorem 4.4

Integrating the energy identity (4.46) with respect to τ we obtain

$$\frac{1}{2}\left(\tilde{\lambda}\|\mathscr{S}^j \phi_\tau\|_{\delta,2j}^2 + 3\delta\|\mathscr{S}^j \phi\|_{\delta,2j}^2 + \frac{4}{3}\|\partial_z \mathscr{S}^j \phi\|_{\delta,2j+1}^2 \right)\Big|_0^\tau + \int_0^\tau \frac{\tilde{\lambda}_\tau}{2}\|\mathscr{S}^j \phi_\tau\|_{\delta,2j}^2 d\sigma$$

(4.72)

$$= \int_0^\tau \mathscr{Q}_j d\sigma + \int_0^\tau \mathscr{C}_j d\sigma$$

for $0 \le j \le 4$. If $-\tilde{\varepsilon} < \delta \le 0$, we apply Lemma 4.3 to the first term of the linear combination of (4.72) and use the estimates for $\mathscr{Q}_j$ (4.49) and $\mathscr{C}_j$

(4.59) to obtain

$$\sup_{0\leq\tau'\leq\tau}\tilde{\mathscr{E}}(\tau')+C_1\int_0^\tau\tilde{\mathscr{D}}d\sigma\leq C_2\tilde{\mathscr{E}}(0)+C_3|\tilde{\mathscr{J}}[\phi]|^2+C_4\sup_{0\leq\sigma\leq\tau}\left(\tilde{\mathscr{E}}^{\frac{3}{2}}(\sigma)+\tilde{\mathscr{E}}^2(\sigma)\right)$$

$$+C_5\left(\nu+\sup_{0\leq\sigma\leq\tau}\sqrt{\tilde{\mathscr{E}}}(\sigma)\right)\int_0^\tau\tilde{\mathscr{D}}d\sigma+C_\nu\int_0^\tau e^{-\beta_2\sigma}\tilde{\mathscr{E}}(\sigma)d\sigma.$$

From (4.4), it is straightforward to check $|\tilde{\mathscr{J}}[\phi]|^2\lesssim\tilde{\mathscr{E}}^2$. Therefore, by choosing sufficiently small $\nu>0$ and $\tilde{M}>0$ if necessary, we deduce the energy-dissipation bound (4.18). If $\delta>0$, by using the energy positivity (4.11), the estimates for $\mathscr{Q}_j$ (4.49) and $\mathscr{C}_j$ (4.59), and the smallness of $\tilde{M}$, one can easily deduce (4.18).

4.3 Proof of Theorem 2.15

By Remark 2.4 the local-in-time well-posedness theory implies that the unique solution to the initial value problem (4.1)–(4.2) exists on a time interval $[0,T]$ where $T\sim\frac{1}{\tilde{\mathscr{E}}(0)}\gtrsim\frac{1}{\varepsilon}$. Choose $\varepsilon>0$ so small that the time of existence T satisfies

(4.73) $$e^{-\beta_2 T/2}\leq\frac{\beta_2}{2\tilde{C}_3},\quad\sup_{\tau\leq T}\tilde{\mathscr{E}}(\tau)\leq C\tilde{\mathscr{E}}(0).$$

Define

$$\mathscr{T}:=\sup_{\tau\geq0}\{\text{ solution to (4.1)–(4.2) exists on }[0,\tau)\text{ and }\sup_{0\leq\tau'\leq\tau}\tilde{\mathscr{E}}(\tau')\leq4\tilde{C}_2C\mathscr{E}(0)\}.$$

Observe that $\mathscr{T}\geq T$. From (4.18) we obtain
(4.74)
$$\sup_{\tau'\in[\frac{T}{2},\tau]}\tilde{\mathscr{E}}(\tau')+\tilde{C}_1\int_{\frac{T}{2}}^\tau\tilde{\mathscr{D}}(\tau')\,d\tau'\leq\tilde{C}_2\tilde{\mathscr{E}}(\frac{T}{2})+\tilde{C}_3\int_{\frac{T}{2}}^\tau e^{-\beta_2\tau'}\tilde{\mathscr{E}}(\tau')\,d\tau',\ \tau\in[\frac{T}{2},\mathscr{T}].$$

Therefore, using (4.73) we conclude that
(4.75)
$$\sup_{\tau'\in[\frac{T}{2},\tau]}\tilde{\mathscr{E}}(\tau')+\tilde{C}_1\int_{\frac{T}{2}}^\tau\tilde{\mathscr{D}}(\tau')\,d\tau'\leq\tilde{C}_2\tilde{\mathscr{E}}(\frac{T}{2})+\frac{\tilde{C}_3}{\beta_2}e^{-\beta_2 T/2}\sup_{\tau'\in[\frac{T}{2},\tau]}\tilde{\mathscr{E}}(\tau')\leq\tilde{C}_2\tilde{\mathscr{E}}(\frac{T}{2})+\frac{1}{2}\sup_{\tau'\in[\frac{T}{2},\tau]}$$

From this inequality we conclude that

(4.76) $$\sup_{\tau'\in[\frac{T}{2},\tau]}\tilde{\mathscr{E}}(\tau')\leq2\tilde{C}_2\tilde{\mathscr{E}}(\frac{T}{2})\leq2\tilde{C}_2C\tilde{\mathscr{E}}(0),\ \tau\in[\frac{T}{2},\mathscr{T}].$$

From the continuity of the map $\tau\mapsto\sup_{\tau'\in[T,\tau]}\tilde{\mathscr{E}}(\tau')$ and the definition of $\mathscr{T}$ we conclude that $\mathscr{T}=\infty$ and the solution to (4.1)–(4.2) exists globally-in-time. From the proved global-in-time boundedness of $\tilde{\mathscr{E}}$, the estimate $\sum_{j=0}^4\tilde{\lambda}\|\mathscr{S}^j\phi_\tau\|_{\delta,2j}^2\lesssim\tilde{\mathscr{E}}$, and the bound (2.60), the second claim of (2.64) follows.

It now remains to prove the first part of (2.64). From the global-in-time boundedness of $\tilde{\mathscr{E}}$, there exists a weak limit ϕ_∞ independent of τ such

that $\tilde{\mathscr{E}}(\phi_\infty, 0) = \sum_{j=0}^{4} \|\mathscr{S}^j \phi_\infty\|_{\delta,2j}^2 + \|\partial_z \mathscr{S}^j \phi_\infty\|_{\delta,2j+1}^2 \lesssim \tilde{\mathscr{E}} \leq C\varepsilon$, which in particular implies $\mathscr{S}^j \phi_\infty \in H_{\delta,2j}^1$ for $0 \leq j \leq 4$. To show the strong convergence of $\phi(\tau)$ as $\tau \to \infty$ in $\mathfrak{H}_{\delta,0}^4 = \{\phi : \mathscr{S}^j \phi_\infty \in L_{\delta,2j}^2, \text{ for } 0 \leq j \leq 4\}$ (low-regularity space), we observe that for any $0 < \tau_2 < \tau_1$,

$$\|\mathscr{S}^j \phi(\tau_1) - \mathscr{S}^j \phi(\tau_2)\|_{L_{\delta,2j}^2}^2 = \int_0^1 z^4 w^{3+2j} \left| \int_{\tau_2}^{\tau_1} \mathscr{S}^j \phi_\tau d\tau \right|^2 dz$$

$$\leq \left(\int_{\tau_2}^{\tau_1} \frac{1}{\tilde{\lambda}_\tau} d\tau \right) \left(\int_{\tau_2}^{\tau_1} \tilde{\lambda}_\tau \int_0^1 z^4 w^{3+2j} |\mathscr{S}^j \phi_\tau|^2 dz d\tau \right)$$

$$\lesssim (e^{-\beta_2 \tau_2} - e^{-\beta_2 \tau_1}) \int_{\tau_2}^{\tau_1} \tilde{\mathscr{D}} d\tau \lesssim (e^{-\beta_2 \tau_2} - e^{-\beta_2 \tau_1})\varepsilon.$$

Therefore, given a strictly increasing sequence $\tau_n \to \infty$, the sequence $\{\mathscr{S}^j \phi(\tau_n)\}_{n=1}^{\infty}$ is Cauchy in $\mathfrak{H}_{\delta,0}^4$. This completes the proof.

Appendix A: Spectral-theoretic properties of $\mathscr{L}_\delta$ and $\mathscr{L}_{\delta,k}$

In this section, we discuss the spectral theory of the linearized operators $\mathscr{L}_\delta$ and $\mathscr{L}_{\delta,k}$ defined in (3.2) and Definition 4.1 respectively for $-\varepsilon < \delta \leq 0$ where $\varepsilon > 0$ is sufficiently small. Recall that for any $k \in \mathbb{N} \cup \{0\}$,

$$(A.1) \qquad \mathscr{L}_{\delta,k} \psi = -\frac{4}{3w_\delta^{3+k} z^4} \partial_z (w_\delta^{4+k} z^4 \partial_z \psi).$$

Here we identify $\mathscr{L}_{\delta,0} = \mathscr{L}_\delta$. Notice that $\mathscr{L}_{\delta,k}$ is a singular Sturm-Liouville operator, and it is nonnegative and symmetric with respect to $(,)_{\delta,k}$:

$$(\mathscr{L}_{\delta,k} \psi, \psi)_{\delta,k} = (\psi, \mathscr{L}_{\delta,k} \psi) = \frac{4}{3} \|\partial_z \psi\|_{\delta,k}^2$$

for any $\psi \in C_c^\infty([0,1])$. Since $C_c^\infty([0,1])$ is dense in the weighted Sobolev spaces, it is natural to consider the Friedrichs extension in $L_{\delta,k}^2$.

When w_δ is given by the Lane-Emden solution, namely when $\delta = 0$, it was shown in [1, 35] that $\mathscr{L}_{0,0}$ has the Friedrichs extension in $L_{0,0}^2$, which is a well-defined self-adjoint operator whose spectrum is purely discrete with 0 as the smallest isolated eigenvalue. The proof relies on the qualitative boundary behavior of w: $w'(0) = 0$, $w(1) = 0$ and $w(z) \sim 1 - z$ for $z < 1$ sufficiently close to 1. Since w_δ behaves in the same way near the boundaries, the same conclusion holds for $\mathscr{L}_\delta$ for any $\delta < 0$ sufficiently close to 0. The same argument works for $\mathscr{L}_{\delta,k}$ for any $k \in \mathbb{N}$. In order to state the result, let us introduce appropriate function spaces:

$$(A.2) \qquad H_{\delta,k}^j := \{\psi \in L_{\delta,k}^2 | \partial_z^l \psi \in L_{\delta,k+l}^2, \quad \text{for all} \quad 0 \leq l \leq j\}.$$

When $k = 0$, $H_{\delta,k}^j = H_\delta^j$ where $H_\delta^j = H_{w_\delta}^j$ and $L_{\delta,k}^2$ were defined in Definition 2.2.

Proposition A.1. $\mathcal{L}_{\delta,k}$ *has the Friedrichs extension in* $L^2_{\delta,k}$ *with the domain* $H^2_{\delta,k}$. *It is a self-adjoint operator with respect to* $(\cdot,\cdot)_{\delta,k}$ *whose spectrum consists of simple eigenvalues* $\mu_0 < \mu_1 < \cdots < \mu_n < \mu_{n+1} < \cdots \to \infty$. *The smallest eigenvalue is* $\mu_0 = 0$ *and the corresponding eigenspace consists of constant functions.*

Proof. The case of $\mathcal{L}_0$ directly follows from Proposition 1 of [35] and other cases can be treated similarly. For completeness, we will describe the argument in [35] for other cases.

In order to apply the classical theory, we first perform the so-called Liouville transformation:

$$y := \int_0^z \sqrt{\frac{w_\delta^{3+k}}{\frac{4}{3}w_\delta^{4+k}}}d\tilde{z}, \quad \eta := z^2(\tfrac{4}{3}w_\delta^{7+2k})^{\frac{1}{4}}\psi.$$

Here $y \in [0,y_+]$ where $y_+ = \int_0^1 \sqrt{\frac{w_\delta^{3+k}}{\frac{4}{3}w_\delta^{4+k}}}dz < \infty$. Then

$$\mathcal{L}_{\delta,k}\psi = \frac{1}{z^2(\tfrac{4}{3}w_\delta^{7+2k})^{\frac{1}{4}}}\left(-\partial_y^2\eta + q\eta\right)$$

where

$$q = q(y) = \frac{\partial_y^2(w_\delta^{\frac{7+2k}{4}})}{w_\delta^{\frac{7+2k}{4}}} + 4\frac{\partial_y z}{z}\frac{\partial_y(w_\delta^{\frac{7+2k}{4}})}{w_\delta^{\frac{7+2k}{4}}} + 2\frac{z\partial_y^2 z + (\partial_y z)^2}{z^2}.$$

Let us consider the operator $\mathcal{T}_0\eta = -\partial_y^2\eta + q\eta$ for $\eta \in C_c^\infty(0,y_+)$. For y sufficiently small, $y \sim \sqrt{\frac{3}{4w_\delta(0)}}z$ and the last term in q dominates. Using $\partial_y z = \sqrt{\frac{4w_\delta}{3}}$, we thus obtain

$$q \sim 2\frac{\frac{4w_\delta}{3}}{z^2} \sim \frac{2}{y^2} \quad \text{as} \quad y \to 0.$$

For y close to y_+, we have $\frac{c}{3}(y_+ - y)^2 \sim 1 - z$ where $c = -\partial_z w_\delta(1)$. The biggest contribution of q in this regime comes from the first term. It is easier to compare in z variable. Using $\partial_y = \sqrt{\frac{4w_\delta}{3}}\partial_z$, we get

$$q \sim \frac{(7+2k)(5+2k)}{12}\frac{(\partial_z w_\delta)^2}{w_\delta} \sim \frac{(7+2k)(5+2k)}{4}\frac{1}{(y_+ - y)^2}.$$

Hence, the boundary points $y = 0$, y_+ are of limit point type for all $k \geq 0$ and for all δ by Theorem X.10 of [39]. Therefore, as in Proposition 1 of [35] (see also [1]), the operator $\mathcal{T}_0$ with the domain $C_c^\infty(0,y_+)$ has the Friedrichs extension $\mathcal{T}$ in $L^2(0,y_+)$, a self-adjoint operator with simple eigenvalues $\mu_0 < \mu_1 < \cdots < \mu_n < \mu_{n+1} < \cdots \to \infty$. The domain of $\mathcal{T}$ is characterized by $\{\eta \in C[0,y_+] : \eta(0) = \eta(y_+) = 0, \ -\partial_y^2\eta + q\eta \in$

$L^2(0, y_+)\}$. By unwinding these results in terms of $\mathscr{L}_{\delta,k}$ and ψ in z variable, we obtain the desired result on the spectral theory of $\mathscr{L}_{\delta,k}$ in the weighted spaces $L^2_{\delta,k}$. Notice that $\eta \in L^2(0, y_+)$ corresponds to $\psi \in L^2_{\delta,k}$. Furthermore, we deduce that the domain of $L^2_{\delta,k}$ is $H^2_{\delta,k}$ from the elliptic regularity (for instance, see Lemma 5.2 in [23]): for each $f \in L^2_{\delta,k}$, there exists a unique solution $\psi \in H^2_{\delta,k}$ solving $\mathscr{L}_{\delta,k}\psi = f$.

To finish the proof of the proposition, it suffices to show that 0 is an eigenvalue of $\mathscr{L}_{\delta,k}$. It is easy to see that the set of constant functions belongs to kernel of $\mathscr{L}_{\delta,k}$. Since $\mathscr{L}_{\delta,k}$ is non-negative and by the spectral theory, the last assertion follows. $\qquad\square$

We are now ready to prove Lemma 3.1 and Lemma 4.2. Since the proofs of two lemmas are almost identical, we only provide the proof of Lemma 3.1.

Proof of Lemma 3.1. From the spectral theory of $\mathscr{L}_\delta$, the operator

$$\mathscr{L}_\delta \phi = -\frac{4}{3}\partial_z\left(w_\delta^4 z^4 \partial_z \phi\right)$$

is completely diagonalizable and the smallest eigenvalue is precisely 0, i..e:

$$\mathscr{L}_\delta \phi_k = \mu_k w_\delta^3 z^4 \phi_k, \quad k \in \mathbb{N} \cup \{0\},$$

and

$$0 = \mu_0 < \mu_1 < \mu_2 < \dots$$

This implies the first statement. For the second claim, note that

$$((\mathscr{L}_\delta + 3\delta)\varphi, \varphi)_\delta = \frac{4}{3}\|\partial_z\varphi\|^2_{\delta,1} - \frac{3}{2}b^2\|\varphi\|^2_\delta \geq (\mu_1 - \frac{3}{2}b^2)\|\varphi\|^2_\delta$$

Hence for sufficiently small b satisfying $b^2 < \frac{2}{3}\mu_1$, we deduce that

$$((\mathscr{L}_\delta + 3\delta)\varphi, \varphi)_\delta \gtrsim \|\partial_z\varphi\|^2_{\delta,1} + \|\varphi\|^2_\delta.$$

$\qquad\square$

Appendix B: Hardy inequalities and embeddings of weighted Sobolev spaces

The L^∞ bounds on ϕ and its derivatives with suitable weights can be obtained by using the Hardy inequalities and embedding inequalities. Since our energy norms involve different weights near the origin and near the boundary, we will utilize localized Hardy inequalities as in [18, 19]. In order to state the results, we introduce suitable functions spaces. Let Z_0 be a completion of $C_c^\infty(0,1)$ with respect to the norm $\|\cdot\|_{Z_0}$ generated by the inner product $(u, v)_{Z_0} := \int_0^1 z^4 u v \, dz$. For any $i \in \mathbb{N}$ we define Z_i by

$$Z_i := \{u \in Z_0 : \partial_z^l u \in Z_0 \text{ for all } 0 \leq l \leq i\}$$

with $\|u\|_{Z_i}^2 = \sum_{l=0}^{i} \|\partial_z^l u\|_{Z_0}^2$. Similarly, we introduce X_a as a completion of $C_c^\infty(0,1)$ with respect to the norm $\|\cdot\|_{X_a}$ generated by the inner product $(u,v)_{X_a} := \int_0^1 w_\delta^a uv\,dz$ for $a > 1$. We let

$$X_a^i := \{v \in X_a : \partial_z^l v \in X_a \text{ for all } 0 \le l \le i\}$$

with $\|v\|_{X_a^1}^2 = \|v\|_{X_a}^2 + \|\partial_z v\|_{X_a}^2$.

Lemma B.1 (Hardy inequalities).   (1) *For any $u \in Z_1$, we have*

$$(B.1) \qquad \int_0^{\frac{1}{2}} z^2 |u|^2 dz \lesssim \int_0^{\frac{3}{4}} z^4 |u_z|^2 dz + \int_0^{\frac{3}{4}} z^4 |u|^2 dz.$$

(2) *For any $u \in Z_2$, we have*

$$(B.2) \qquad \int_0^{\frac{1}{2}} |u|^2 dz \lesssim \int_0^{\frac{3}{4}} z^4 |u_{zz}|^2 dz + \int_0^{\frac{3}{4}} z^4 |u_z|^2 dz + \int_0^{\frac{3}{4}} z^4 |u|^2 dz.$$

(3) *Let $a > 1$ be given. For any $v \in X_a^1$, we have*

$$(B.3) \qquad \int_{\frac{1}{2}}^1 w_\delta^{a-2} |v|^2 dz \lesssim \int_{\frac{1}{4}}^1 w_\delta^a |v_z|^2 dz + \int_{\frac{1}{4}}^1 w_\delta^a |v|^2 dz.$$

As a consequence of the previous lemma, we have Hardy embedding inequalities.

Lemma B.2 (Embedding inequalities). *Let m be any nonnegative integer.*

(1) *For any $u \in Z_2 \cap X_{2m}^m$, we have*

$$(B.4) \qquad \|u\|_{L^1}^2 \lesssim \sum_{k=0}^2 \int_0^{\frac{3}{4}} z^4 |\partial_z^k u|^2 dz + \sum_{k=0}^m \int_{\frac{1}{4}}^1 w_\delta^{2m} |\partial_z^k u|^2 dz.$$

(2) *For any $u \in Z_3 \cap X_{2m}^{m+1}$, we have*

$$(B.5) \qquad \|u\|_\infty^2 \lesssim \sum_{k=0}^3 \int_0^{\frac{3}{4}} z^4 |\partial_z^k u|^2 dz + \sum_{k=0}^{m+1} \int_{\frac{1}{4}}^1 w_\delta^{2m} |\partial_z^k u|^2 dz.$$

A direct consequence of the above lemmas are the L^∞ bounds presented in the lemma below. We shall explain how to control the L^∞ norms by two different notions of energy $\mathscr{E}$ and $\tilde{\mathscr{E}}$ introduced in (2.48) and (2.63) respectively.

Lemma B.3.   *(1) Let $\phi = \phi(s,z) \in H_\delta^8$ for each fixed $s \in [0,S]$ be given so that $\partial_s^i \phi \in H_\delta^{8-i}$ for each $0 \le i \le 8$ and that the corresponding energy $\mathscr{E}(s) < \infty$ for all $s \in [0,S]$. We recall H_δ^j is defined in Definition 2.2 and $\mathscr{E}$ is given in (2.48). Then the following estimates hold:*

(a)

$$\left|\phi\right| + \left|\partial_s\phi\right| + \left|\partial_s^2\phi\right| + \sum_{q=1}^{5}\left|z^{\delta(q)}w^{\frac{q-1}{2}}\partial_s^q\partial_s^2\phi\right| \lesssim \sqrt{\mathscr{E}}$$

where $\delta(q) = 0$ for $q \le 3$, $\delta(q) = 1$ for $q = 4$ and $\delta(q) = 2$ for $q = 5$.

(b)

$$\left|\partial_z\phi\right| + \left|\partial_{zs}\phi\right| + \sum_{q=1}^{5}\left|z^{\delta(q)}w^{\frac{q}{2}}\partial_s^q\partial_{zs}\phi\right| \lesssim \sqrt{\mathscr{E}}$$

where $\delta(q) = 0$ for $q \le 3$, $\delta(q) = 1$ for $q = 4$ and $\delta(q) = 2$ for $q = 5$.

(c)

(B.6)
$$\sum_{q=0}^{5}\left|w^{\frac{q+1}{2}}z^{\delta(q)}\partial_s^q\partial_z^2\phi\right| \lesssim \sqrt{\mathscr{E}}$$

where $\delta(q) = 0$ for $q \le 3$, $\delta(q) = 1$ for $q = 4$ and $\delta(q) = 2$ for $q = 5$.

(2) *Let* $\phi = \phi(\tau, z) \in \mathfrak{H}_{\delta,0}^4$ *for each fixed* $\tau \in [0, T]$ *be given so that* $\partial_\tau\phi \in \mathfrak{H}_{\delta,0}^4$, $\mathscr{S}^4\phi \in H_{\delta,8}^1$, *and that the corresponding energy* $\tilde{\mathscr{E}} < \infty$ *and dissipation* $\tilde{\mathscr{D}} < \infty$ *for all* $\tau \in [0, T]$. *We recall* $\mathfrak{H}_{\delta,0}^j$ *is defined in Definition 2.14,* $H_{\delta,k}^j$ *is introduced in (A.2), and* $\tilde{\mathscr{E}}$, $\tilde{\mathscr{D}}$ *in (2.63) and (4.15) respectively. Then the following estimates hold:*

(a)

$$\left|\phi\right| + \left|\partial_z\phi\right| + \left|\partial_z^2\phi\right| + \left|\mathscr{S}\phi\right| \lesssim \sqrt{\tilde{\mathscr{E}}}$$

(c)

$$\left|z^{\delta(q)}w^q\partial_z^{2+q}\phi\right| + \left|z^{\delta(q)}w^q\partial_z^q\mathscr{S}\phi\right| \lesssim \sqrt{\tilde{\mathscr{E}}}$$

for $1 \le q \le 5$. $\delta(q) = 0$ *for* $q \le 3$, $\delta(q) = 1$ *for* $q = 4$ *and* $\delta(q) = 2$ *for* $q = 5$. *We also have*

$$\left|w^2\mathscr{S}^2\phi\right| + \left|zw^4\mathscr{S}^3\phi\right| \lesssim \sqrt{\tilde{\mathscr{E}}}$$

(c)

$$\sqrt{\tilde{\lambda}}\left|\partial_\tau\phi\right| \lesssim \sqrt{\tilde{\mathscr{D}}}$$

For the proofs of the above lemmas, we refer to [18, 19].

Bibliography

[1] Beyer, H. R. The spectrum of radial adiabatic stellar oscillations. *J. Math. Phys.* **36** (1995), 4815–4825.

[2] J. Binney, S. Tremaine. *Galactic Dynamics*. Princeton University Press, Princeton, 2008.

[3] Blottiau, P., Bouquet, S., Chièze, J. P. An asymptotic self-similar solution for the gravitational collapse. *Astron. Astrophys.* **207**, (1988) 24–36.

[4] Bouquet, S., Feix, M. R., Fijalkow, E., Munier, A. Density bifurcation in a homogeneous isotropic collapsing star. *The Astrophysical Journal* **293** (1985), 494–503.

[5] Chandrasekhar, S. *An Introduction to the Study of Stellar Structures.* University of Chicago Press, Chicago, 1938.

[6] Coutand, D., Shkoller, S. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.* **20** (2007), (3), 829–930.

[7] Coutand, D., Shkoller, S. Well-posedness in smooth function spaces for the moving-boundary 1-D compressible Euler equations in physical vacuum. *Comm. Pure Appl. Math.* **64** (2011), no. 3, 328–366.

[8] Coutand, D., Shkoller, S. Well-posedness in smooth function spaces for the moving boundary three-dimensional compressible Euler equations in physical vacuum. *Arch. Ration. Mech. Anal.* **206** (2012), no. 2, 515–616.

[9] Coutand, D., Lindblad, H., Shkoller, S. A priori estimates for the free–boundary 3D compressible Euler equations in physical vacuum. *Comm. Math. Phys.* **296** (2010), 559–587.

[10] Dacorogna B., Moser, J. On a partial differential equation involving the Jacobian determinant. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7** (1990), no. 1, 1–26.

[11] Deng, Y., Liu, T.P., Yang, T., Yao Z. Solutions of Euler-Poisson equations for gaseous stars. *Arch. Ration. Mech. Anal.* **164** (2002), no. 3, 261–285.

[12] Deng, Y., Xiang, J., Yang, T. Blowup phenomena of solutions to Euler-Poisson equations. *J. Math. Anal. Appl.* **286** (2003), 295–306.

[13] Fu, C.-C.; Lin, S.-S. On the critical mass of the collapse of a gaseous star in spherically symmetric and isentropic motion. *Japan J. Indust. Appl. Math.* **15** (1998), no. 3, 461–469.

[14] Goldreich, P., Weber, S. Homologously collapsing stellar cores. *Astrophys. J.* **238** (1980), 991–997.

[15] Hadžić, M., Speck, J. The Global Future Stability of the FLRW Solutions to the Dust-Einstein System with a Positive Cosmological Constant. *Jour. Hyp. Diff. Eqns.* **12** (2015), no.1, 87–188.

[16] Jang, J. Nonlinear Instability in Gravitational Euler-Poisson system for $\gamma = 6/5$. *Arch. Ration. Mech. Anal.* **188** (2008), 265–307.

[17] Jang, J. Local well-posedness of viscous gaseous stars. *Arch. Ration. Mech. Anal.* **195** (2010), no. 3, 797–863.

[18] Jang, J. Nonlinear Instability Theory of Lane-Emden stars. *Comm. Pure Appl. Math.* **67** (2014), no. 9, 1418–1465.

[19] Jang, J. Time periodic approximations of the Euler-Poisson system near Lane-Emden stars. *Anal. PDE*. **9** (2016), no. 5, 1043–1078.

[20] Jang, J., Masmoudi, N. Well-posedness for compressible Euler equations with physical vacuum singularity, *Comm. Pure Appl. Math.* **62** (2009), 1327–1385.

[21] Jang, J., Masmoudi, N. Vacuum in Gas and Fluid dynamics. *Proceedings of the IMA summer school on Nonlinear Conservation Laws and Applications*, Springer (2011), 315–329.

[22] Jang, J., Masmoudi, N. Well and ill-posedness for compressible Euler equations with vacuum. *J. Math. Phys.* **53** (2012), 115625.

[23] Jang, J., Masmoudi, N. Well-posedness of compressible Euler equations in a physical vacuum. *Communications on Pure and Applied Mathematics* **68** (2015), no. 1, 61–111.

[24] Jang, J.; Tice, I. Instability theory of the Navier-Stokes-Poisson equations. *Anal. PDE.* **6** (2013), no. 5, 1121–1181

[25] Larson, R. B. Numerical calculations of the dynamics of a collapsing protostar. *Monthly Notices Roy. Astron. Soc.* **145** (1969), 271–295.

[26] Lin, S.-S. Stability of gaseous stars in spherically symmetric motions. *SIAM J. Math. Anal.* **28** (1997), no. 3, 539–569.

[27] Luo, T., Smoller, J. Existence and Nonlinear Stability of Rotating Star Solutions of the Compressible Euler-Poisson Equations. *Arch. Ration. Mech. Anal.* **191** (2009), 3, 447–496.

[28] Luo, T., Xin,Z., Zeng, H. Well-posedness for the motion of physical vacuum of the three-dimensional compressible Euler equations with or without self-gravitation. *Arch. Ration. Mech. Anal.* **213** (2014), no. 3, 763–831.

[29] Luo, T., Xin,Z., Zeng, H. On nonlinear asymptotic stability of the Lane-Emden solutions for the viscous gaseous star problem *Adv. Math.* **291** (2016), 90–182.

[30] Luo, T., Xin,Z., Zeng, H. Nonlinear Asymptotic Stability of the Lane-Emden Solutions for the Viscous Gaseous Star Problem with Degenerate Density Dependent Viscosities. *Comm. Math. Phys.* **347** (2016), no. 3, 657–702.

[31] LUO, T., ZENG, H.: Global Existence of Smooth Solutions and Convergence to Barenblatt Solutions for the Physical Vacuum Free Boundary Problem of Compressible Euler Equations with Damping, *Comm. Pure Appl. Math.* **69** (2016), no.7, 1354 –1396.

[32] Lübbe, C., Valiente-Kroon, J. A. A conformal approach for the analysis of the nonlinear stability of pure radiation cosmologies. *Ann. Phys.* **328** (2013), 1–25.

[33] Makino, T. Blowing up solutions of the Euler-Poisson equation for the evolution of gaseous stars. *Transport Theory Statist. Phys.* **21** (1992), 615–624.

[34] Makino, T., Perthame, B. Sur les Solution á Symétrie Sphérique de l'Equation d'Euler-Poisson pour l'Evolution d'Etoiles Gazeuses. *Japan J. Appl. Math.* **7** (1990), 165–170.

[35] Makino, T. On spherically symmetric motions of a gaseous star governed by the Euler-Poisson equations. *Osaka J. Math.* **52** (2015), no. 2, 545–580.

[36] Martel, Y., Merle, F., Raphaël, P. Blow up for the critical gKdV equation I: dynamics near the solitary wave. *Acta Math.* **212** (2014), no. 1, 59–140.

[37] Penston, M. V. Dynamics of self-gravitating gaseous spheres-III. Analytical results in the free-fall of isothermal cases. *Monthly Notices Roy. Astron. Soc.* **144** (1969), 425–448.

[38] Rein, G. Non-linear stability of gaseous stars. *Arch. Ration. Mech. Anal.* **168** (2003), no. 2, 115–130.

[39] Reed, M., Simon, B. *Methods of Modern Mathematical Physics II, Fourier Analysis, Self-Adjointness.* Academic Press, New York, 1975.

[40] Ringeval, C., Bouquet, S. Dynamical stability for the gravitational evolution of a homogeneous polytrope. *Astron. Astrophys.* **355** (2000), 564–572.

[41] Rodnianski, I., Speck, J. The Nonlinear Future Stability of the FLRW Family of Solutions to the Irrotational Euler-Einstein System with a Positive Cosmological Constant. *Journal of European Math. Soc.* **15**(2013), (6), 2369–2462.

[42] Speck, J. The nonlinear future stability of the FLRW family of solutions to the Euler-Einstein system with a positive cosmological constant. *Selecta Mathematica* **18** (2012), no. 3, 633–715.

[43] Yahil, A. Self-similar stellar collapse. *The Astrophysical Journal* **265** (1983), 1047–1055.

[44] Zel'dovich, Y. B., Novikov, I. D. *Relativistic Astrophysics Vol. 1: Stars and Relativity.* Chicago University Press, Chicago, 1971.