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New construction of eigenstates and separation of variables for $SU(N)$ quantum spin chains

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ABSTRACT: We conjecture a new way to construct eigenstates of integrable XXX quantum spin chains with $SU(N)$ symmetry. The states are built by repeatedly acting on the vacuum with a single operator $B^{\text{good}}(u)$ evaluated at the Bethe roots. Our proposal serves as a compact alternative to the usual nested algebraic Bethe ansatz. Furthermore, the roots of this operator give the separated variables of the model, explicitly generalizing Sklyanin's approach to the $SU(N)$ case. We present many tests of the conjecture and prove it in several special cases. We focus on rational spin chains with fundamental representation at each site, but expect many of the results to be valid more generally.

KEYWORDS: Bethe Ansatz, Lattice Integrable Models

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1 Introduction

Integrable quantum spin chains are among the most famous exactly solvable models in mathematical physics. In addition to exhibiting rich mathematical structures, they have found diverse applications ranging from condensed matter physics to supersymmetric gauge theories and string theory. The key feature of these spin chains is a powerful hidden

symmetry which in particular allows one to obtain eigenvalues of the spin chain Hamiltonian in terms of the Bethe ansatz equations [1] (see e.g. [2–6] for pedagogical reviews). At the same time, the problem of efficiently describing the spin chain eigenstates is much more difficult and has been the subject of active investigation over many years. A renewed interest in this problem stems from the appearance of higher rank (super-)spin chains in the context of computing correlation functions in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with the use of integrability (see e.g. [7–11] and the review [12]). Only for the simplest spin chains, which are based on the $SU(2)$ -invariant R-matrix, can the states be obtained in a direct and compact way. In this case the algebraic Bethe ansatz approach allows one to build the eigenstates by acting on a reference state with a “creation operator” $B(u)$. We found that similar operators $B^{\text{good}}(u)$ can be constructed for any $SU(N)$ spin chain, so that the eigenstates are given by

$$|\Psi\rangle = B^{\text{good}}(u_1)B^{\text{good}}(u_2)\dots B^{\text{good}}(u_M)|0\rangle \quad (1.1)$$

where u_k are the Bethe roots. Furthermore, in contrast with the standard $SU(2)$ construction for $B(u)$, our $B^{\text{good}}(u)$ is diagonalizable and is suitable for immediate application of the separation of variables (SoV) approach as we describe below. We provide an explicit expression for this operator as a polynomial in the monodromy matrix entries.

Our formula (1.1) provides a concise and compact alternative to all the existing techniques for construction of the eigenstates. The most transparent method to obtain the states available in the literature is the *nested* algebraic Bethe ansatz approach, in which the eigenvectors are built recursively based on the solution of a lower rank spin chain [13–15]. Other known constructions include an explicit representation via sums over partitions of Bethe roots [16], as well as the rather sophisticated trace formulas of [17] and the Drinfeld current construction [18–23] (see also [24–27]). These methods have been explored in-depth and have facilitated numerous calculations of various observables such as form factors and scalar products [23, 28–37]. Nevertheless in all of these approaches the expression for the states has a rather involved structure as well as being hard to implement computationally in many situations.¹ In contrast to the nested Bethe ansatz, our construction involves no recursion and is also free from many of the complications inherent in other methods.

Although in the $SU(N)$ Bethe ansatz there are several types of Bethe roots, in the examples we considered it is just the momentum-carrying Bethe roots which should be plugged into the B^{good} operators in (1.1) to construct the states. The auxiliary Bethe roots enter the construction only indirectly, via the usual Bethe equations which determine the positions of the momentum-carrying roots.² Thus our construction is also free from the ambiguity associated with different possible choices of one set of Bethe ansatz equations out of the $N!$ equivalent possibilities (corresponding to different paths in the Hasse diagram).

¹In particular, the result usually involves sums over partitions of Bethe roots or sums over states of an auxiliary spin chain, leading to exponentially many terms for highly excited states.

²In fact the Bethe ansatz equations are not necessarily the best way of finding Bethe roots. In many situations it is much more convenient to solve directly the N -th order Baxter equation involving no auxiliary Bethe roots. This method gives directly the Baxter polynomial containing the momentum-carrying roots only.

Our construction is closely related to the Separation of Variables (SoV) approach. At the classical level, SoV is a fundamental method of solving completely integrable models which reduces the dynamics to a set of decoupled 1-particle problems. While implementation of the SoV at the quantum level is more subtle (in particular due to operator ordering issues), its efficiency has also been demonstrated in many settings. Broadly speaking, the idea of the SoV method is to find variables in which the wavefunction of the system factorizes into 1-particle blocks. The SoV in application to quantum integrable spin chains was pioneered by Sklyanin in [38, 39]. Our proposal for the form of the operator $B^{\text{good}}(u)$ is based on the Sklyanin's approach.

To ensure non-degeneracy of the construction we consider spin chains with a generic diagonal twist and generic inhomogeneities. Another important element of our proposal is that one should add an extra similarity transformation in the auxiliary space in order to construct the eigenstates of the original spin chain.

Given the highly compact form of our representation for the states, we hope that our results should be useful in various contexts and may also help to approach the longstanding problem of generalizing Slavnov's determinant expression for the scalar product beyond the $SU(2)$ -type models. Exploring potential applications for spin chains with $PSU(2, 2|4)$ symmetry would be especially interesting in view of their relation with the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

Although the construction itself has a rather simple form, we found it is complicated to prove it in general. Instead, we check the conjecture extensively on numerous examples and numerically, and prove it in several particular cases. In this paper we focus on the canonical example of closed spin chains with rational R-matrix and a fundamental representation of $SU(N)$ at each site.³ It would be interesting to find a general proof algebraically.

This paper is organized as follows. In section 2 we introduce the relevant notation and the key definitions related to algebraic Bethe ansatz. In section 3 we discuss the $SU(2)$ case in detail and prove some novel nontrivial aspects of the construction involving our operator B^{good} . In section 4 we describe the SoV in the $SU(3)$ case and present our conjecture for the new construction of eigenstates in this setting. In section 5 we propose an extension of the SoV approach beyond the well-studied $SU(2)$ and $SU(3)$ cases, and describe our proposal which should provide eigenstates in any $SU(N)$ spin chain. Along the way we discuss many tests of the conjecture. In section 6 we present conclusions and outline future directions. Lastly, appendices contain some more technical details.

2 Notation and basic definitions

As the results we present in this paper are based on the algebraic Bethe ansatz framework, in this section we will describe its basic components for $SU(N)$ spin chains, and also introduce relevant notation.

³The usual R-matrix on which the spin chain is based is of course $GL(N)$ -invariant, but we refer to the spin chain as having $SU(N)$ symmetry to emphasize that in the examples we consider we have a finite-dimensional representation at each site.

In the algebraic Bethe ansatz approach the key object defining the integrable model is the R-matrix. The $SU(N)$ spin chains we study are based on the rational R-matrix which acts in $\mathbb{C}^N \otimes \mathbb{C}^N$ and has the form

$$R_{12}(u) = (u - i/2) + iP_{12} \tag{2.1}$$

where P is the permutation operator and u is the spectral parameter. We will concentrate on the case of spin chains with a fundamental representation of $SU(N)$ at each site. As usual we introduce an auxiliary space \mathbb{C}^N supplementing the physical Hilbert space \mathcal{H} which is a tensor product of L copies of \mathbb{C}^N ,

$$\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N \otimes \dots \otimes \mathbb{C}^N, \tag{2.2}$$

and construct the monodromy matrix as the product of R-matrices,

$$T(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2) \dots R_{0L}(u - \theta_L)g, \tag{2.3}$$

where the R-matrix R_{ak} acts in the tensor product of the auxiliary space V_a and the k -th copy of \mathbb{C}^N inside the Hilbert space. The complex parameters θ_k are known as inhomogeneities, while g is the twist matrix acting in the auxiliary space which we take to be diagonal,

$$g = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N). \tag{2.4}$$

The complex twists λ_i and inhomogeneities θ_k serve as important regulators in our construction, and we assume they are all distinct and in generic position.

It is convenient to understand $T(u)$ as an $N \times N$ matrix,

$$T(u) = \begin{pmatrix} T_{11}(u) & \dots & T_{1N}(u) \\ \vdots & \ddots & \vdots \\ T_{N1}(u) & \dots & T_{NN}(u) \end{pmatrix}, \tag{2.5}$$

whose elements $T_{ij}(u)$ are operators acting only in the Hilbert space of the spin chain. These operators T_{ij} satisfy nontrivial commutation relations that can be deduced from the RTT relation, which reads

$$R_{ab}(u - v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u - v), \tag{2.6}$$

where we introduce two auxiliary spaces $V_a, V_b \simeq \mathbb{C}^N$ so the R-matrix R_{ab} acts in their tensor product, while $T_k(u)$ acts in $V_k \otimes \mathcal{H}$.

The trace of $T(u)$ over the auxiliary space can be written as

$$t_1(u) = \text{tr}_0 T(u) = \sum_{n=1}^N T_{nn}(u), \tag{2.7}$$

and is an operator in the physical space \mathcal{H} known as the transfer matrix. It commutes with itself at different values of the spectral parameter,

$$[t_1(u), t_1(v)] = 0, \tag{2.8}$$

so that coefficients of the expansion of the transfer matrix in powers of u form a commutative family of operators, a particular combination of which is in fact the Hamiltonian of the spin chain. These operators can therefore be diagonalized simultaneously. They also have a particularly simple eigenvector given by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.9}$$

which is used as a reference/vacuum state in the algebraic Bethe ansatz approach we will discuss below. Let us finally mention that we will often use brief notation for shifts in the spectral parameter, namely

$$f^\pm \equiv f(u \pm i/2), \quad f^{[+a]} \equiv f(u + ia/2). \tag{2.10}$$

The main problem we focus on in this paper is constructing the common eigenvectors of the commuting operators $t_1(u)$, which are automatically also eigenstates of the spin chain Hamiltonian. In the next section we describe how this problem is solved for the SU(2) spin chains.

3 Eigenstates and SoV for the SU(2) spin chain

In this section we describe in detail the well-known construction of eigenstates and separated variables for the simplest SU(2) spin chains. This case also illustrates several key elements of our approach which we will use later for higher rank spin chains.

3.1 Algebraic Bethe ansatz and SoV in the SU(2) case

For SU(2) spin chains the monodromy matrix $T(u)$ from (2.5) is a 2×2 matrix whose entries act on the physical Hilbert space, and we denote them as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{3.1}$$

The transfer matrix whose eigenstates and eigenvalues we would like to obtain is then given by $A(u) + D(u)$, while the eigenstates can be built using $B(u)$ as

$$|\Psi\rangle = B(u_1) \dots B(u_M)|0\rangle. \tag{3.2}$$

Here $|0\rangle$ is the vacuum state,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.3}$$

while the Bethe roots u_j are determined by the Bethe equations

$$\prod_{n=1}^L \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} = \frac{\lambda_2}{\lambda_1} \prod_{k \neq j}^M \frac{u_j - u_k + i}{u_j - u_k - i}. \tag{3.4}$$

Let us also note that

$$[B(u), B(v)] = 0, \tag{3.5}$$

so the states are symmetric in the Bethe roots.

Remarkably, in addition to generating the eigenstates the operator $B(u)$ has another important property — namely, it provides the basis of separated variables within Sklyanin’s SoV framework [39]. Let us first summarize in general the SoV construction in the $SU(2)$ case, and then discuss several important subtleties regarding its implementation for our spin chain.

In order to construct the full set of separated variables, one assumes that $B(u)$ is a polynomial of degree L . Making explicit its zeros, one can write

$$B(u) = B_0 \prod_{n=1}^L (u - x_n), \tag{3.6}$$

where B_0 and x_n are operators which commute due to (3.5).⁴ At least in simplest examples, the operator B_0 is proportional to the identity operator on the Hilbert space so we can treat B_0 as a number. Assuming further that one can construct a complete set of left eigenvectors of $B(u)$ which form a basis in the Hilbert space, and labelling them as $\langle x_1, \dots, x_L |$ to indicate the eigenvalues of all x_n , we find from (3.2)⁵

$$\langle x_1, \dots, x_L | \Psi \rangle = \langle x_1, \dots, x_L | 0 \rangle B_0 \prod_{j=1}^M \prod_{n=1}^L (u_j - x_n). \tag{3.7}$$

Then normalizing the eigenvectors so that $\langle x_1, \dots, x_L | 0 \rangle B_0 = 1$ we finally see that

$$\langle x_1, \dots, x_L | \Psi \rangle = \prod_{n=1}^L (-1)^M Q_1(x_n), \tag{3.8}$$

where $Q_1(u)$ is the Baxter Q-function,

$$Q_1(u) = \prod_{j=1}^M (u - u_j) \tag{3.9}$$

and the extra sign $(-1)^M$ comes from rearranging the product. In other words, (3.8) means that the wavefunction in this basis is factorized, realizing the key goal of the SoV approach. This simple form of the wavefunction greatly facilitates the computation of various observables such as form factors or scalar products (see [40–44, 47, 48] for some interesting recent examples).

We see that the factorization of the wavefunction in the eigenbasis of $B(u)$ is clear almost at once from the construction of eigenstates (3.2). In the next section we will discuss details of the SoV approach in an explicit example.

⁴Several subtleties in defining the individual operators x_n as operator zeros of B are discussed in [38, 39].

⁵As discussed in [39], algebraically one can deduce various important properties of these variables even without using the construction of eigenstates (3.2) (that construction in particular assumes the existence of a good reference state $|0\rangle$ which is not true for some models).

3.2 Implementation for the SU(2) spin chain

While the above construction of separated variables should apply in principle to a wide variety of integrable models based on the rational SU(2) R-matrix, it is not completely trivial to implement it in practice for the spin chain we consider. The main problem is that the operator $B(u)$ is nilpotent and not diagonalizable,⁶ even with generic twists λ_j and inhomogeneities θ_k . This degeneracy spoils the construction, since it is then not possible to construct an eigenbasis for the operators x_n . Moreover $B(u)$ is a polynomial of degree less than L , making it even more problematic to define L nontrivial operators x_n .

To circumvent this problem it is convenient to introduce an extra similarity transformation in the auxiliary space. Namely, instead of $T(u)$ we will consider a new matrix⁷

$$T^{\text{good}}(u) = K^{-1}T(u)K, \tag{3.10}$$

where the 2×2 constant matrix K acts only in the auxiliary space. This extra twisting will also be important for the SU(N) spin chains that we discuss below. Notice that this transformation leaves unchanged the trace of $T(u)$, i.e. the transfer matrix which we want to diagonalize. Moreover, due to GL(2) invariance of the R-matrix the new monodromy matrix $T^{\text{good}}(u)$ will satisfy the same RTT relations (2.6) as $T(u)$, so commutation relations between the elements $T_{ij}^{\text{good}}(u)$ will be the same as before. We will label the entries of T^{good} similarly to (3.1),

$$T^{\text{good}}(u) = \begin{pmatrix} A^{\text{good}}(u) & B^{\text{good}}(u) \\ C^{\text{good}}(u) & D^{\text{good}}(u) \end{pmatrix}. \tag{3.11}$$

The key point is that now one can use $B^{\text{good}}(u)$ to construct both the separated variables and the eigenstates of the original spin chain. With a generic choice of the matrix K , the operator $B^{\text{good}}(u)$ will be diagonalizable and also of degree L as a polynomial in u , as needed. As the commutation relations are unchanged, we also have

$$[B^{\text{good}}(u), B^{\text{good}}(v)] = 0. \tag{3.12}$$

As for the eigenstates, let us first focus on a simple example when K is upper triangular,⁸

$$K = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \tag{3.13}$$

(note that a particularly simple choice would be $a = b = 1$). This already guarantees that if a, b are nonzero then $B^{\text{good}}(u)$ is of degree L in u as one can easily check. Moreover, the matrix T^{good} then reads

$$T^{\text{good}}(u) = \begin{pmatrix} A(u) - abC(u) & \frac{b}{a}(A(u) - D(u)) + \frac{1}{a^2}B(u) - b^2C(u) \\ a^2C(u) & D(u) + abC(u) \end{pmatrix}, \tag{3.14}$$

⁶In fact it's intuitively clear that B is nilpotent in this case, as one can use it repeatedly to create all the states starting from the vacuum $|0\rangle$ which is a 'highest weight' state, and eventually reaching the maximally excited 'lowest weight' state which is annihilated by B .

⁷A similar transformation was considered independently in [45], and we would like to thank I. Kostov and D. Serban for illuminating discussions related to this approach. Parts of the construction were also discussed in [49].

⁸Without loss of generality we can assume $\det K = 1$ as the scalar part of K drops out of (3.10).

and as the vacuum state $|0\rangle$ is annihilated by $C(u)$, we see that

$$A^{\text{good}}(u)|0\rangle = A(u)|0\rangle, \quad D^{\text{good}}(u)|0\rangle = D(u)|0\rangle. \quad (3.15)$$

Let us recall that to prove that (3.2) are eigenstates of the transfer matrix it is enough to use the commutation relations between the A, B, C, D operators, together with the fact that the vacuum is an eigenstate of $A(u)$ and $D(u)$ with eigenvalues of a prescribed form. As we discussed, $T^{\text{good}}(u)$ satisfies the same RTT relations as $T(u)$, which in combination with relations (3.15) guarantees that B^{good} can be used instead of B to build eigenstates as in (3.2), i.e.⁹

$$|\Psi\rangle \propto B^{\text{good}}(u_1) \dots B^{\text{good}}(u_M)|0\rangle \quad (3.16)$$

Quite surprisingly, the operator B^{good} turns out to generate the eigenstates even for a generic matrix K . The proof of this fact is considerably more involved, the main difficulty being that in this case $|0\rangle$ is no longer an eigenstate of A^{good} and D^{good} . Nevertheless we found a proof and present it in appendix A. The main idea is to apply the SoV approach and make use of the variables conjugated to the separated variables x_n . Our only assumption is that the spectrum of x_n is $\theta_n \pm i/2$ as we discuss below.

Having constructed the operator B^{good} we can study in detail the implementation of the SoV program. As expected, we observe that the eigenvalues of each x_n are $\theta_n \pm i/2$ (see e.g. [39]).¹⁰ Then we can label the common eigenbasis of all x_n by the choice of signs in the eigenvalues, e.g. the state $\langle ++ \dots + |$ corresponds to all signs chosen as $+$. As we need to choose L signs, we get a complete basis of 2^L states. As an example, from (3.8) we get

$$\begin{aligned} \langle +++ \dots + | \Psi \rangle &= (-1)^{LM} Q_1 \left(\theta_1 + \frac{i}{2} \right) Q_1 \left(\theta_2 + \frac{i}{2} \right) Q_1 \left(\theta_3 + \frac{i}{2} \right) \dots Q_1 \left(\theta_L + \frac{i}{2} \right), \\ \langle +-+ \dots + | \Psi \rangle &= (-1)^{LM} Q_1 \left(\theta_1 + \frac{i}{2} \right) Q_1 \left(\theta_2 - \frac{i}{2} \right) Q_1 \left(\theta_3 + \frac{i}{2} \right) \dots Q_1 \left(\theta_L + \frac{i}{2} \right), \end{aligned} \quad (3.17)$$

and so on.

A curious feature of the construction is that one can also build eigenstates with the same operator B^{good} but using a *dual* set of Bethe roots and acting on a different reference state. This property is directly related to the fact that one can use an arbitrary matrix K when building the operator B^{good} via (3.10). As this feature will also hold for the $SU(N)$ chains we describe below, let us discuss it in detail here for the $SU(2)$ case. Recall that in the usual $SU(2)$ algebraic Bethe ansatz one can alternatively build the states by starting with an alternative reference state

$$|0'\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.18)$$

⁹We use the \propto sign to indicate that normalization of the states (3.2) and (3.16) may be different.

¹⁰Strictly speaking there is an ambiguity in the definition of individual x_n operators, as the coefficients of the polynomial $B(u)$ involve only their symmetric combinations. It is natural to define them in such a way that each x_n is associated with a particular site of the chain and has eigenvalues $\theta_n \pm i/2$ determined by the inhomogeneity at that site. This choice fixes the operators x_n uniquely.

and acting on it with $C(u)$ rather than $B(u)$,

$$|\Psi'\rangle = C(v_1)\dots C(v_{L-M})|0'\rangle \tag{3.19}$$

with operators C evaluated at the dual Bethe roots v_i which are zeros of the polynomial Q_2 defined by

$$\lambda_2 Q_1\left(u + \frac{i}{2}\right) Q_2\left(u - \frac{i}{2}\right) - \lambda_1 Q_1\left(u - \frac{i}{2}\right) Q_2\left(u + \frac{i}{2}\right) = (\lambda_2 - \lambda_1) \prod_{n=1}^L (u - \theta_n). \tag{3.20}$$

In our construction there is no need to switch between the operators B and C , instead one can use the *same* operator B^{good} and act on the different reference state using the dual set of Bethe roots,

$$|\Psi'\rangle \propto B^{\text{good}}(v_1)\dots B^{\text{good}}(v_{L-M})|0'\rangle. \tag{3.21}$$

This can be proven using the same arguments as for the construction with usual Bethe roots and a generic matrix K which we discussed above (see appendix A).

3.3 The scalar product in the SoV representation

In this section we discuss briefly the scalar products of two ‘off-shell’ states. For convenience we pick a one-parametric family of B^{good} parameterized with an upper triangular matrix K in (3.10) of the form

$$K = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & -\frac{1}{\sqrt{\alpha}} \\ 0 & \sqrt{\alpha} \end{pmatrix}. \tag{3.22}$$

Here α is an unfixed parameter which we retain for convenience. To make the results more compact we also choose to multiply B^{good} by an overall factor $1/(\lambda_2 - \lambda_1)$, so that explicitly we have

$$B_{\alpha}^{\text{good}} = \frac{1}{\lambda_2 - \lambda_1} (\alpha B - 1/\alpha C - A + D). \tag{3.23}$$

In this subsection we also assume that the twist matrix is an $SU(2)$ element, i.e.

$$\lambda_1 = 1/\lambda_2 = 1/\lambda_1^*. \tag{3.24}$$

We can define two Bethe vectors by:

$$|\psi\rangle = B_{\alpha}^{\text{good}}(u_1)\dots B_{\alpha}^{\text{good}}(u_N)|0\rangle, \quad |\phi\rangle = B_{-\frac{1}{\alpha}}^{\text{good}}(v_1)\dots B_{-\frac{1}{\alpha}}^{\text{good}}(v_{L-N})|0'\rangle. \tag{3.25}$$

Let us note that in the second equation here we use B^{good} evaluated with α in (3.23) replaced by $-1/\bar{\alpha}$. If the Bethe roots u_i satisfy the corresponding Bethe equations (and dual Bethe equations for v 's) we get two eigenstates of the Hamiltonian. We can define off-shell Bethe states when this requirement is relaxed. We will see that it is not required for the computation of the scalar products of these two vectors. Using that $(B_{\alpha}^{\text{good}}(u))^{\dagger} = B_{-1/\bar{\alpha}}^{\text{good}}(\bar{u})$ (i.e. Hermitian conjugation corresponds to replacing α by $-1/\bar{\alpha}$) we find that

$$\langle\phi|\psi\rangle = \langle 0'|B_{\alpha}^{\text{good}}(u_1)\dots B_{\alpha}^{\text{good}}(u_M)|0\rangle \tag{3.26}$$

where u_i are the original Bethe roots for $i = 1 \dots N$ and $u_i = v_{i-N}$ for $i > N$.

The main idea is to write a representation of the identity operator in terms of left and right eigenvectors of B^{good} (in the remainder of this section we omit the subscript α). We already considered the left eigenvectors $\langle x_1, \dots, x_L |$ which we will denote for brevity as $\langle x |$, so we have¹¹

$$\langle x | B^{\text{good}}(u) = \langle x | \prod_{n=1}^L (u - x_n). \quad (3.27)$$

One can easily see (at least for the first few L 's) that the right eigenvalues of B^{good} are the same as the left ones. We will denote the right eigenstates as $|x\rangle$, although they are not Hermitian conjugates of the left eigenvectors $\langle x |$. Then we have

$$B^{\text{good}}(u) |x\rangle = \prod_{n=1}^L (u - x_n) |x\rangle. \quad (3.28)$$

While the left eigenstates are normalized such that

$$\langle x | 0 \rangle = 1, \quad (3.29)$$

it is convenient to normalize the right states using the dual vacuum $|0'\rangle$ so that

$$\langle 0' | x \rangle = 1. \quad (3.30)$$

Then the key observation is that the scalar product of the left and the right states is simple and reads¹²

$$\langle x | y \rangle = \frac{1}{\mu(x)} \delta_{x,y} \quad (3.31)$$

with the measure μ given by¹³

$$\mu(x) = \left(\frac{\alpha \lambda_2}{\lambda_1 - \lambda_2} \right)^L \prod_{n=1}^L \left(\frac{x_n - \theta_n}{i/2} \right) \frac{\prod_{m < n}^L (\bar{x}_m - \bar{x}_n)}{\prod_{m < n}^L (\theta_m - \theta_n)}, \quad (3.32)$$

where we denoted

$$\bar{x}_n = \begin{cases} \theta_n - i/2, & \text{for } x_n = \theta_n + i/2 \\ \theta_n + i/2, & \text{for } x_n = \theta_n - i/2 \end{cases}. \quad (3.33)$$

This means that we can write a resolution of the identity as

$$\mathbf{1} = \sum_{x_n = \theta_n \pm i/2} |x\rangle \langle x| \mu(x). \quad (3.34)$$

¹¹With the definition (3.23) that we are using, the leading coefficient of B^{good} as a polynomial in u is 1.

¹²We have checked this explicitly for the first few L 's with high numerical precision, but a complete proof should also not be too difficult.

¹³Similar expressions for the measure already appeared in [41, 45].

Inserting this representation of the identity operator immediately in front of the vacuum $|0\rangle$ in the scalar product (3.26), we get at once

$$\langle 0' | B^{\text{good}}(u_1) \dots B^{\text{good}}(u_L) | 0 \rangle = \sum_{x_n = \theta_n \pm i/2} \mu(x) \prod_{m=1}^M \prod_{n=1}^L (u_m - x_n). \quad (3.35)$$

The left-hand side of this equation is actually zero for $M < L$, but it still holds for any M . Thus as expected we can write this scalar product as a sum over all values of separated variables, with a simple measure (3.32).

In the appendix B we also link the usual Gaudin formula for the norm of a transfer matrix eigenstate with the SoV approach and also with the variation of the Q-functions with respect to the inhomogeneity parameters θ_i . In particular we rewrite the Gaudin formula in terms of Q-functions only, rather than in terms of individual Bethe roots (the result is given by (B.41)). Similar results already appeared in [40, 41, 47], but our derivation is different and utilizes several tricks which may be useful in other contexts.

4 Eigenstates for SU(3) spin chains from SoV

In this section we review how the SoV approach works for SU(3) spin chains. We then discuss in detail our main conjecture for this case, which states that spin chain eigenstates can be generated using the same operator that provides separated variables.

4.1 The B operator

The operator $B(u)$ which should provide separated variables for quantum SU(3)-type models was constructed by Sklyanin in [50] (following the construction in the classical case [51]). It is built from entries of the monodromy matrix which in this case is a 3×3 matrix

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix} \quad (4.1)$$

whose entries act on the physical Hilbert space. To build the operator B it is convenient to introduce the 2×2 quantum minors of this matrix as

$$T_{j_1 j_2 | k_1 k_2}(u) = T_{j_1 k_1}(u) T_{j_2 k_2}(u + i) - T_{j_2 k_1}(u) T_{j_1 k_2}(u + i). \quad (4.2)$$

In the SU(3) case it is natural to assemble these minors into a new matrix U as

$$U_{jk}(u) = (-1)^{j+k} T_{\bar{j} | \bar{k}}(u) \quad (4.3)$$

where $\bar{j} = \{1, 2, 3\} \setminus \{j\}$. In fact U is also the monodromy matrix constructed for the case when the auxiliary space carries the antisymmetric representation of SU(3) rather than the fundamental representation (in particular, U satisfies the same RTT relations (2.6) as the monodromy matrix T). With these definitions, the operator $B(u)$ of [50] is given by

$$B(u) = T_{23}(u) U_{31}(u - i) - T_{13}(u) U_{32}(u - i), \quad (4.4)$$

or explicitly

$$\begin{aligned}
 B(u) = & T_{23}(u)T_{12}(u-i)T_{23}(u) - T_{23}(u)T_{22}(u-i)T_{13}(u) \\
 & + T_{13}(u)T_{11}(u-i)T_{23}(u) - T_{13}(u)T_{21}(u-i)T_{13}(u).
 \end{aligned}
 \tag{4.5}$$

As an indication of the complexity of the SU(3) models, let us note that $B(u)$ is a polynomial of the 3rd degree in the monodromy matrix entries, while in the SU(2) case it was simply the element T_{12} .

From the commutation relations between elements of $T(u)$ it follows that, remarkably,

$$[B(u), B(v)] = 0 \tag{4.6}$$

so that similarly to the SU(2) case illustrated in (3.6) one can introduce new commuting operators as zeros of $B(u)$. Importantly, in the SU(3) case $B(u)$ is expected to be a polynomial of degree $3L$ in u as discussed in [50], so we have three zeros for each site of the chain. We will see below that based on the eigenvalue spectrum of $B(u)$ it is natural to split the new operators into L groups with three operators in each group, so we can write

$$B(u) = B_0 \prod_{n=1}^L \prod_{a=1}^3 (u - x_{n,a}), \tag{4.7}$$

where the index n labels the sites, while a enumerates the three x operators associated with this site. The leading coefficient B_0 also commutes with all x_n .

4.2 Application to spin chains and construction of eigenstates

Let us discuss how to apply the above construction to our main example, the spin chain with a fundamental representation at each of the L sites. Calculating $B(u)$ explicitly for low values of L we immediately see that as in the SU(2) case it is nilpotent and not diagonalizable. In fact it annihilates almost the entire Hilbert space (for instance, when $L = 1$ all its matrix elements are simply zero). This is true even with generic twists λ_i and inhomogeneities θ_n . To remedy this problem we again apply a similarity transformation to the monodromy matrix as we described in (3.10) for the SU(2) case, so we introduce

$$T^{\text{good}}(u) = K^{-1}T(u)K \tag{4.8}$$

where K is a constant matrix for which a particularly simple choice is

$$K = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

As we will discuss below, we observed that even for a generic K all the key features of our construction are preserved.

Now we build a new operator $B^{\text{good}}(u)$ by using the expression (4.5) for $B(u)$ in which we replace the elements $T_{ij}(u)$ by $T_{ij}^{\text{good}}(u)$. All algebraic properties of the original $B(u)$ will

be inherited by B^{good} as the commutation relations between elements of T are preserved by the transformation (4.8), in particular we have

$$[B^{\text{good}}(u), B^{\text{good}}(v)] = 0. \quad (4.10)$$

At the same time, computing B^{good} explicitly for various L 's we observe that it now can be diagonalized and moreover is a polynomial of degree $3L$ as required, with a nonvanishing leading coefficient given by

$$B_0 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3). \quad (4.11)$$

Then the separated variables x_n will be defined as zeros of B^{good} ,

$$B^{\text{good}}(u) = B_0 \prod_{n=1}^L \prod_{a=1}^3 (u - x_{n,a}). \quad (4.12)$$

Our key conjecture is that in addition to providing separated variables, the operator $B^{\text{good}}(u)$ also generates the eigenstates of the transfer matrix, which can be built exactly as in the $SU(2)$ case, namely

$$|\Psi\rangle = B^{\text{good}}(u_1) \dots B^{\text{good}}(u_M)|0\rangle, \quad (4.13)$$

where $|0\rangle$ is the usual reference state

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (4.14)$$

In this formula the parameters u_k are the momentum-carrying Bethe roots. They are determined by the usual Bethe equations which describe the spectrum of the transfer matrix eigenvalues and are well-known from the nested Bethe ansatz approach. These equations read

$$\begin{aligned} \prod_{n=1}^L \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} &= \frac{\lambda_2}{\lambda_1} \prod_{k \neq j}^M \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^R \frac{u_j - v_k - i/2}{u_j - v_k + i/2}, \\ \prod_{n=1}^M \frac{v_j - u_n + i/2}{v_j - u_n - i/2} &= \frac{\lambda_3}{\lambda_2} \prod_{k \neq j}^R \frac{v_j - v_k + i}{v_j - v_k - i}. \end{aligned} \quad (4.15)$$

The parameters v_k are known as the auxiliary Bethe roots and they do not enter directly our construction of the states. Their only role is that they appear in the equations above which fix the positions¹⁴ of the momentum-carrying roots u_n .

In summary, we expect that to build the full set of states one should go through all solutions of the Bethe equations, and for each of them plug the corresponding set of u_j into the formula (4.13).

¹⁴Let us also note that for a fixed number of magnons $M \leq L$, there can be several states, parameterized in particular by the number of auxiliary roots which should satisfy $R \leq M$.

For comparison let us briefly recall the well-known construction of the states via the nested algebraic Bethe ansatz. In this approach they are given by

$$|\Psi\rangle = \sum_{a_i=2,3} F^{a_1 a_2 \dots a_M} T_{1a_1}(u_1) T_{1a_2}(u_2) \dots T_{1a_M}(u_M) |0\rangle \quad (4.16)$$

where $F^{a_1 a_2 \dots a_M}$ is the wavefunction of an auxiliary $SU(2)$ spin chain with M sites in which u_j are the inhomogenities. The auxiliary Bethe roots v_k should be understood as the rapidities of magnons propagating on this spin chain. Notice that two different operators act on the vacuum here (T_{12} and T_{13}), which moreover do not commute with each other. In contrast, our conjecture (4.13) features only one operator $B^{\text{good}}(u)$ which also commutes with itself at different values of u . Furthermore, (4.16) is a sum of 2^M nontrivial terms, with coefficients $F^{a_1 a_2 \dots a_M}$ that are quite complicated and are determined recursively from a lower rank spin chain. Our formula instead is much more compact and has the same form as for the simplest $SU(2)$ spin chain.

We have checked our conjecture in a multitude of cases, leaving little doubt in its validity. At the same time, obtaining a complete analytic proof is likely to be quite non-trivial. In the next section we describe various tests as well as presenting a proof for some special cases.

4.3 Tests of the proposal for eigenstates

We have numerically checked our conjecture extensively using Mathematica for values of L up to about $L \sim 5$. We considered many states, with various numbers of roots of each type, and found that our proposal (4.13) works perfectly. In order to easily solve the Bethe equations, we typically considered a configuration with generic real twists $\lambda_i \sim 100$ and large real inhomogenities $\theta_i \sim 100$ which are also well separated from each other, i.e. $|\theta_i - \theta_j| \sim 100$. In this regime the values of all the Bethe roots are close to the inhomogenities, so having these good starting points it is simple to find very accurate numerical solutions of the Bethe equations. Then we checked that our proposal provides eigenstates with very high precision (typically 50 – 100 digits). In the $SU(N)$ cases which will be discussed below in this paper we followed a similar numerical approach.

In addition, we have proven the conjecture analytically for *arbitrary* L for states with 1 and 2 momentum-carrying roots. The proof is done for the case when the matrix K which we use to build B^{good} via (4.8) is taken to be the upper triangular matrix (4.9). The key point is that in this case $|0\rangle$ is an eigenstate for most of the elements T_{ij} (see e.g. [16]), namely

$$T_{11}(u)|0\rangle = \lambda_1 Q_\theta^+(u)|0\rangle, \quad T_{22}(u)|0\rangle = \lambda_2 Q_\theta^-(u)|0\rangle, \quad (4.17)$$

$$T_{33}(u)|0\rangle = \lambda_3 Q_\theta^-(u)|0\rangle, \quad (4.18)$$

and

$$T_{21}(u)|0\rangle = T_{31}(u)|0\rangle = T_{32}(u)|0\rangle = T_{23}(u)|0\rangle = 0. \quad (4.19)$$

Here we denoted

$$Q_\theta = \prod_{n=1}^L (u - \theta_n) \quad (4.20)$$

and we recall the notation from (2.10),

$$f^\pm \equiv f(u \pm i/2), \quad f^{[+a]} \equiv f(u + ia/2). \quad (4.21)$$

The operator B^{good} which we use is constructed out of elements of T^{good} , for which we have precisely the same relations (4.17), (4.18), (4.19) with the only exception being

$$T_{23}^{\text{good}}(u)|0\rangle = (\lambda_2 Q_\theta^- - \lambda_3 Q_\theta^-)|0\rangle. \quad (4.22)$$

Then our strategy is to act on the state (4.13) with the transfer matrix, and commute all the elements of T^{good} (except T_{12}^{good} and T_{13}^{good}) to the right until they hit the vacuum $|0\rangle$ which is an eigenstate for them. To do this we use the commutation relations between elements of T^{good} which follow from the RTT relation (2.6). As a result the expression greatly simplifies, and one can also see that if the Bethe equations are satisfied several terms will cancel. As an example, for a state with one momentum-carrying root built as

$$|\Psi\rangle = B^{\text{good}}(u)|0\rangle \quad (4.23)$$

we find

$$\begin{aligned} & \left(T_{11}^{\text{good}}(w) + T_{22}^{\text{good}}(w) + T_{33}^{\text{good}}(w) \right) B^{\text{good}}(u)|0\rangle \\ &= f_1 T_{12}^{\text{good}}(u)|0\rangle + f_2 T_{13}^{\text{good}}(u)|0\rangle + f_3 T_{13}^{\text{good}}(w)|0\rangle, \end{aligned} \quad (4.24)$$

where f_i are some lengthy coefficients. We see that the last term here is unwanted, as it includes $T_{13}^{\text{good}}(w)$ which clearly cannot appear in the expression for the state itself. However, its coefficient f_3 is rather simple,

$$f_3 = -\frac{i(\lambda_3 - \lambda_2)Q_\theta^{[-3]}(u)}{w - u} (\lambda_2 Q_\theta^-(u) - \lambda_1 Q_\theta^+(u)) (\lambda_3 Q_\theta^-(u) - \lambda_1 Q_\theta^+(u)). \quad (4.25)$$

It's easy to see that on the solution of Bethe equations without auxiliary roots the next-to-last factor in f_3 will be zero, while for the solution with one auxiliary root the last factor will vanish. This means that the unwanted term will disappear from (4.24). One can then check that the remaining part is proportional to the original state given by (4.23) which is therefore indeed an eigenstate, as expected.

In a similar way we have proven that the construction works for states with 2 momentum-carrying roots at any L . This calculation involves even more lengthy expressions and we do not give them here. In principle, the same approach should allow to prove the conjecture for any specified number of magnons. We hope however that a more algebraic proof can be found which would apply for any L and any number of excitations at once.

4.4 Construction with dual roots

Another surprising feature of our construction is that one can use dual Bethe roots¹⁵ as arguments of the operators B^{good} in (4.13), provided they also act on the corresponding

¹⁵See e.g. [52, 53] for a pedagogical discussion of dualities in the Bethe ansatz.

different reference state. This is in complete analogy with the SU(2) case discussed above (see (3.21)) where B^{good} can be used either with the usual roots or with the dual roots. The main difference with the SU(2) case is that now we have three natural reference states (defined as in (4.14) but with the 1's in the second row or in the third row), and accordingly to each eigenstate of the transfer matrix correspond three dual sets of momentum-carrying Bethe roots.¹⁶ However, we observed that one and the same operator B^{good} allows to build the eigenstates starting from either reference state, as long as one uses the corresponding set of Bethe roots as arguments of B 's in (4.13). We have checked this in many examples, but it would be interesting to find a rigorous proof.

This property is also directly linked with the fact that our construction works (as we observed in various examples) for a generic matrix K in (4.8), as with arbitrary K the reference state $|0\rangle$ is no longer distinguished. The possibility of taking K generic also means that we have a whole family of operators B^{good} which all work well, and we found that this family is three-parametric in the SU(3) case (a more detailed discussion is given in section 5.3).

4.5 The spectrum of separated variables

To characterize the states in the SoV language, it is important to understand the eigenvalue spectrum of the operators $x_{n,a}$, which are the zeros of $B^{\text{good}}(u)$ introduced in (4.12),

$$B^{\text{good}}(u) = B_0 \prod_{n=1}^L \prod_{a=1}^3 (u - x_{n,a}). \quad (4.26)$$

These operators define separated variables and we label their eigenstates according to their eigenvalues, as $|x_{1,1}, \dots, x_{L,3}\rangle$. In direct analogy (3.8) for the SU(2) case, our construction of states immediately guarantees that the wavefunction in the eigenbasis of B^{good} will take a factorized form,

$$\langle x_{1,1}, \dots, x_{L,3} | \Psi \rangle = \prod_{n=1}^L \prod_{a=1}^3 (-1)^M Q_1(x_{n,a}) \quad (4.27)$$

where $Q_1(u) = \prod_{j=1}^M (u - u_j)$.

In the SU(2) case there were as many x_n 's as sites, while here we have three variables $x_{n,a}$ for each site n of the chain, since B is a polynomial of degree $3L$. When $L = 1$ the Hilbert space is \mathbb{C}^3 and we found that the three left eigenvalues¹⁷ of B^{good} read¹⁸

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \left(u - \left(\theta_1 + \frac{3i}{2} \right) \right) \left(u - \left(\theta_1 - \frac{i}{2} \right) \right)^k \left(u - \left(\theta_1 + \frac{i}{2} \right) \right)^{2-k} \quad (4.28)$$

with $k = 0, 1, 2$. We have verified this expression numerically with a high precision after setting λ_i and θ_i to numerical values. There is an ambiguity in choosing which of the

¹⁶In the Q-system language (see e.g. [54] and references therein), these are the zeros of $Q_1(u), Q_2(u)$ and $Q_3(u)$.

¹⁷It is the *left* eigenvalues and eigenvectors of B^{good} which define the basis of separated variables, see the discussion for SU(2) in the previous section.

¹⁸The overall prefactor given here is the one obtained if one uses the upper triangular matrix K from (4.9).

eigenvalues should be attributed to which of the x 's (as B^{good} is symmetric in all the $x_{n,a}$), but for the purpose of labelling the states we may choose e.g. the operator $x_{1,1}$ to act as a scalar equal to $\theta_1 + \frac{3i}{2}$. Then there are three options for the remaining eigenvalues $\{x_{1,2}, x_{1,3}\}$, namely

$$\left\{ \theta_1 + \frac{i}{2}, \theta_1 + \frac{i}{2} \right\}, \left\{ \theta_1 + \frac{i}{2}, \theta_1 - \frac{i}{2} \right\}, \left\{ \theta_1 - \frac{i}{2}, \theta_1 - \frac{i}{2} \right\}. \quad (4.29)$$

We can label the corresponding three eigenstates according to the \pm signs appearing in these eigenvalues, so we get three states

$$|((++)), |((+-)), |((-)) \quad (4.30)$$

which form a basis in the Hilbert space \mathbb{C}^3 . This is precisely the basis in which the wavefunction factorizes. Note that it is just the unordered set of eigenvalues which identifies the state uniquely, so we do not have an additional state $|((-+))$.

Furthermore, we observed that the pattern of eigenvalues for higher L is obtained trivially from this one. Namely, B^{good} will contain a trivial overall scalar factor

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \prod_{n=1}^L \left(u - \left(\theta_n + \frac{3i}{2} \right) \right) \quad (4.31)$$

so one can choose the L operators $x_{k,1}$ with $k = 1, \dots, L$ to act as scalars (with eigenvalues $\theta_k + \frac{3i}{2}$). The remaining nontrivial matrix part of B^{good} determines eigenvalues of the other x 's, which are equal to

$$\left\{ \theta_1 \pm \frac{i}{2}, \theta_1 \pm \frac{i}{2}, \theta_2 \pm \frac{i}{2}, \theta_2 \pm \frac{i}{2}, \dots, \theta_L \pm \frac{i}{2}, \theta_L \pm \frac{i}{2} \right\}. \quad (4.32)$$

All combinations of signs are allowed here, and the unordered set of these eigenvalues labels the eigenstates of B^{good} giving 3^L possibilities which precisely corresponds to the dimension of the Hilbert space.

In other words, with each site k we associate three operators $x_{k,1}, x_{k,2}, x_{k,3}$, one of which acts trivially on the whole space as $\theta_k + \frac{3i}{2}$ and the two others have eigenvalues $\theta_k \pm \frac{i}{2}$. At each site we can choose one of the three distinct combinations of signs in these eigenvalues, namely $++$, $+-$ and $--$. To indicate all the nontrivial eigenvalues we can then label a state as e.g.

$$|((++)(+-)(+-)\dots) \quad (4.33)$$

where we group in brackets the labels corresponding to the same site.

5 Extension to the $SU(N)$ case

In this section we demonstrate how to extend the compact construction of eigenstates to the $SU(N)$ spin chains. We first discuss the $SU(4)$ case and present the operator $B^{\text{good}}(u)$, which we obtained by making an ansatz for it as a combination of quantum minors and then fixing the free parameters by several constraints. Then we extend the construction to the $SU(N)$ setting and discuss its features.

5.1 The B operator for $SU(4)$

While the SoV program has been thoroughly studied for $SU(2)$ -type models and to a lesser degree for the $SU(3)$ case (see e.g. [26, 55]), its extension to the case of higher rank groups presents a challenge. One of the key required ingredients is the operator $B(u)$ providing the separated variables. Let us motivate and present our conjecture for the form of this operator in the $SU(4)$ case. We will first discuss constructing the operator $B(u)$ itself and then, as before, we will apply an extra similarity transformation which removes degeneracies and provides the operator $B^{\text{good}}(u)$ which generates the eigenstates of the spin chain.

The main inspiration comes from the form of B given for $SU(3)$ by (4.4), where it is written in terms of 2×2 quantum minors of the monodromy matrix. For higher rank groups one can also construct $n \times n$ quantum minors which are known to satisfy various identities and frequently appear in the study of integrable models (see e.g [56]). This suggests to make an ansatz for $B(u)$ using these building blocks. The $n \times n$ quantum minors are defined by a sum over permutations,

$$T_{j_1, \dots, j_n | k_1, \dots, k_n}(u) = \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} T_{j_{\sigma(1)} k_1}(u) T_{j_{\sigma(2)} k_2}(u+i) \dots T_{j_{\sigma(n)} k_n}(u+(n-1)i). \quad (5.1)$$

In this notation $T_{i|j}(u)$ stands for the monodromy matrix element $T_{ij}(u)$ itself. The formula (5.1) is similar to a determinant but includes extra shifts in u . We can write the $SU(3)$ result explicitly as

$$B(u) = T_{1|3}(u)T_{12|13}(u-i) + T_{2|3}(u)T_{12|23}(u-i), \quad (5.2)$$

so for $SU(4)$ it would be natural to include also 3×3 minors.

Another piece of information is the known form of $B(u)$ for *classical* $SU(N)$ spin chains. It was found in [57, 58] (see also [59, 60] and the more recent work [61]) and reads¹⁹

$$B = \sum_{i_1, \dots, i_{N-1}=1}^{N-1} \epsilon_{i_1 \dots i_{N-1}} T_{i_1 N}(T^2)_{i_2 N} \dots (T^{N-1})_{i_{N-1} N}. \quad (5.3)$$

We see that B is a polynomial of degree $N(N-1)/2$ in the entries of the monodromy matrix, which here are all evaluated at the same value u of the spectral parameter. However it is highly nontrivial to generalize this expression to the quantum case, as the classical limit corresponds to treating T_{ij} as commuting elements so the operator ordering is lost. In addition, in this limit one removes all the shifts of the spectral parameter by multiples of i , even though e.g. in the quantum $SU(3)$ result (5.2) the shifts play a key role.

One possible quantization of the classical result was proposed in [62] in the context of models associated to $U_q(\widehat{\mathfrak{sl}}_N)$ symmetry, i.e. with a trigonometric R-matrix, although the implementation of this approach for concrete models was not discussed in that paper. While the result there is given in an implicit form, we independently derived an explicit expression for the B operator which we will present shortly (and we also identified the role of B for creating the transfer matrix eigenstates). It would be interesting though rather

¹⁹To be clear, in (5.3) we denote by T^n the product of n matrices T_{ij} .

nontrivial to compare our results with [62], in particular we expect that an extra twist similar to K we use to build T^{good} will be crucial for applying the construction of [62] in explicit examples.

Motivated by the structure of (5.3) and (5.2), we made the following ansatz for $B(u)$ in the $SU(4)$ case:

$$B(u) = \sum s_{i_1 i_2 \dots i_6} T_{i_1|4}(u) T_{i_2 i_3 | i_4 4}(u + c_1) T_{123 | i_5 i_6 4}(u + c_2) \tag{5.4}$$

where the sum runs over the values of indices

$$i_1, i_2, \dots, i_6 = 1, 2, 3 \text{ with } i_2 < i_3, i_5 < i_6. \tag{5.5}$$

Here $s_{i_1 \dots i_6}$ and c_1, c_2 are unfixed parameters. Note that due to the product of minors of increasing sizes in each term, B is a polynomial of degree $1 + 2 + 3 = 6$ in the entries of T_{ij} , as required by the classical formula (5.3). We also ensured that 4 appears among the column indices of each minor, to reflect the structure of the classical result.

To fix the unknowns in this ansatz we specialize to the case of the $SU(4)$ spin chain with fundamental representation at each site. Also, as before we construct from B the operator B^{good} , using instead of T_{ij} the elements of T^{good} which is obtained by an upper triangular similarity transformation as in (4.8), (4.9):

$$T^{\text{good}} = K^{-1}TK, \quad K = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.6}$$

We use this concrete choice of K when fixing the parameters in our ansatz, but we also found that once they are fixed the construction continues to work just as well with generic K .

Then in order to determine uniquely the unknown parameters $s_{i_1 \dots i_6}$ and $c_{1,2}$ it is enough to impose that:

- In the classical limit B reduces to (5.3)²⁰
- $[B^{\text{good}}(u), B^{\text{good}}(v)] = 0$ for $L = 1, 2$
- For $L = 1$ the operator $B^{\text{good}}(u)$ generates all the three eigenstates with one momentum-carrying Bethe root by acting on the usual reference state $|0\rangle$, similarly to (4.13)

Remarkably, this fixes most of the coefficients $s_{i_1 \dots i_6}$ to zero, while the rest are set to be equal to 1! Another nice feature is that the shifts $c_{1,2}$ are fixed to successive multiples of i , namely

$$c_1 = -i, \quad c_2 = -2i. \tag{5.7}$$

²⁰In fact we impose that it should reduce to minus the classical result, this overall sign is irrelevant but gives a nicer expression for the quantum B operator.

Thus the expression for B can be written simply as

$$B(u) = \sum_{j,k} T_{j|4}(u) T_{k|j4}(u-i) T_{123|k4}(u-2i) \quad (5.8)$$

where $k = \{k_1, k_2\}$ and the sum runs over the values of indices j, k_1, k_2 from 1 to 3 with $k_1 < k_2$.

We have checked this result extensively. In particular we find that the property

$$[B^{\text{good}}(u), B^{\text{good}}(v)] = 0, \quad (5.9)$$

which we ensured for $L = 1, 2$, continues to hold for $L = 3, 4$. Most importantly, we found that eigenstates of the transfer matrix can be generated as before,

$$|\Psi\rangle \propto B^{\text{good}}(u_1) \dots B^{\text{good}}(u_M) |0\rangle \quad (5.10)$$

where u_k are the momentum-carrying Bethe roots.

Thus we have extended our conjecture for construction of eigenstates to the $SU(4)$ case. We have numerically²¹ checked it thoroughly for values of L up to $L = 4$ and various numbers of the Bethe roots of different types excited.²² In particular, for $L = 2$ we verified numerically that the construction gives *all* the 16 states, both for a generic matrix K and for the upper triangular K from (5.6). We expect that the same operator B should also provide the separated variables as its operator zeros, similarly to the $SU(2)$ and $SU(3)$ cases.

Let us also mention that another expression for the operator $B(u)$ which should provide separated variables for rational $SU(N)$ models was proposed in [63], motivated by considerations related to so-called Manin matrices. In the $SU(4)$ case we observed that, as expected, without the extra twist matrix K that operator is nilpotent and cannot be diagonalized (at least for $L = 1, 2$). Using a nontrivial K matrix as in (5.6) cures this problem, but we found that already for $L = 2$ the resulting operator does not satisfy the commutativity property (5.9) and therefore cannot be used to define separated variables.²³ The structure of our result is nevertheless rather similar to that in [63], and it would be interesting to look for possible connections.

5.2 Generalization to any $SU(N)$

Let us now present the conjecture for the operator B generalizing the above results to any $SU(N)$ group. To see the structure let us write again the $SU(3)$ and $SU(4)$ results from (5.2), (5.8), which read

$$B^{\text{SU}(3)}(u) = \sum_{j=1}^2 T_{j|3}(u) T_{12|j3}(u-i), \quad (5.11)$$

$$B^{\text{SU}(4)}(u) = \sum_{j,k}^3 T_{j|4}(u) T_{k|j4}(u-i) T_{123|k4}(u-2i).$$

²¹See the beginning of section 4.3 for a brief summary of our numerical strategy.

²²Note that for $L = 4$ we already have to deal with rather large matrices having $256 \times 256 = 65536$ elements.

²³Accordingly, it also does not generate the two-magnon eigenstates of the transfer matrix already for $L = 2$.

Our conjecture for $SU(N)$ is

$$B(u) = \sum_{j,k,\dots,p} T_{j|N}(u) T_{k|jN}(u-i) T_{l|kN}(u-2i) \dots T_{123\dots(N-1)|pN}(u-(N-2)i) \quad (5.12)$$

where

$$k = \{k_1, k_2\}, \quad l = \{l_1, l_2, l_3\}, \quad \dots, \quad p = \{p_1, p_2, \dots, p_{N-2}\} \quad (5.13)$$

and the sum runs over the values of indices j, k_n, l_n, \dots, p_n from 1 to $N-1$ with

$$k_1 < k_2, \quad l_1 < l_2 < l_3, \quad \dots, \quad p_1 < p_2 < \dots < p_{N-2}. \quad (5.14)$$

The main ingredient here is the pattern of indices in the expression under the sum, which for $SU(3)$, $SU(4)$, $SU(5)$ reads

$$jj, jkjk, jkjlkl \quad (5.15)$$

and so on — we see that each time a new index appears both before and after the last index of the previous expression. This pattern ensures in particular that all minors in (5.12) have the appropriate number of indices. In appendix C we present explicit expressions for B corresponding to $N \leq 5$.

We also see that B is a polynomial of degree $1 + 2 + \dots + (N-1) = \frac{N(N-1)}{2}$ in the elements T_{ij} , as it should be according to the known classical result (5.3). For $N \leq 6$ we have also checked that in the classical limit our result precisely reproduces the classical expression (5.3) (up to an overall sign which is irrelevant). This matching involves various cancellations which are certainly not obvious from the form of our result, and is already a highly nontrivial test of our conjecture.

Like in the lower rank cases, it is important to remove degeneracies by constructing the operator B^{good} from this operator B . We again use the similarity transformation (5.6) with a $N \times N$ matrix K which we can take to be generic or specialize to an upper triangular K with all elements equal to 0 or 1, like in (5.6). Then we expect that the operator $B^{\text{good}}(u)$ will provide a commutative family,

$$[B^{\text{good}}(u), B^{\text{good}}(v)] = 0, \quad (5.16)$$

and will again generate the eigenstates of the transfer matrix by acting on the vacuum,

$$|\Psi\rangle = B^{\text{good}}(u_1) \dots B^{\text{good}}(u_M)|0\rangle \quad (5.17)$$

where u_k are the momentum-carrying roots.

Moreover, we expect that like in the $SU(3)$ and $SU(2)$ cases one can perform any duality transformation in the nested Bethe ansatz equations and then use the dual sets of momentum-carrying roots as arguments of B^{good} to build the states (and acting with B^{good} on the corresponding reference state instead of $|0\rangle$).

We have checked commutativity (5.16) and the construction of various states in the $SU(5)$ case for the first several values of L . Moreover for $L = 2$ we explicitly verified with generic K that our conjecture provides all the 25 eigenstates, going one by one through each solution of the rather involved Bethe equations which in particular contain four types of Bethe roots. In combination with the compact form of our operator B , these confirmations are hardly an accident and we believe they provide strong evidence for our proposal.

5.3 Free parameters in the construction

It is important to reiterate that rather than just one operator B^{good} , we can build a family of such operators by taking a generic $N \times N$ matrix K in the expression (5.6) for the improved monodromy matrix T^{good} , which reads

$$T^{\text{good}} = K^{-1}TK. \tag{5.18}$$

Even with generic K , we observed in many examples that the operator B^{good} constructed from this T^{good} retains all the key properties such as commutativity and the ability to create the states. Without loss of generality we can impose $\det K = 1$, as a scalar K will not change T at all. This leaves $N^2 - 1$ complex parameters in K . However, not all K 's lead to distinct operators B^{good} . Based on explicit examples we considered, we expect that B^{good} will generically contain only N free parameters (one of which corresponds to a trivial rescaling $B^{\text{good}} \rightarrow \text{const} \cdot B^{\text{good}}$). We have checked this for the first several values of L when $N \leq 4$ by considering the effect on B^{good} of small variation of the matrix K around some generic matrix. This extra freedom may be useful in some applications and it would be interesting to better understand its role.

Let us mention that at the classical level one can easily introduce N parameters into the expression for B given in (5.3), namely can consider (see e.g. [60])²⁴

$$B = \sum_{i_1, \dots, i_N=1}^N \epsilon_{i_1 \dots i_N} \alpha_{i_1} (T\alpha)_{i_2} (T^2\alpha)_{i_3} \dots (T^{N-1}\alpha)_{i_N}, \tag{5.19}$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a constant vector. The original expression (5.3) is recovered by setting $\alpha_i = \delta_{iN}$. Different choices of α correspond to different normalizations of the Baker-Akhiezer function which plays an important role in the SoV program for classical models (see [39, 60] for details). Nicely, at the classical level the family of functions (5.19) parameterized by α_i is exactly the same as the family obtained in our approach which involves free parameters in the matrix K . Namely, by starting from the expression (5.3) and replacing in it the matrix T by $T^{\text{good}} = K^{-1}TK$ one obtains precisely (5.19) in which the parameters α_i are given by²⁵

$$\alpha_i = K_{iN}, \quad i = 1, \dots, N \tag{5.20}$$

and we also assume that $\det K = 1$. Thus at the classical level the N parameters in our operator B^{good} correspond to the N -parametric freedom of choosing the normalization of the Baker-Akhiezer function. At the same time, this identification is only valid classically, and it would be important to clarify the algebraic meaning of the free parameters in our construction in the quantum case.

5.4 Eigenvalues of separated variables

It is natural to conjecture that the operator B^{good} we have constructed should also provide separated variables for the $SU(N)$ case. We expect all the main features to be in

²⁴We are grateful to D. Medina and M. Heinze for related discussions.

²⁵We have checked this for $N = 2, 3, 4$.

direct analogy with the SU(2) and SU(3) cases, and here we will briefly describe how they generalize to SU(N).

In the SU(N) case the operator B^{good} is a polynomial of degree $N(N-1)/2$ in u . We again define the operators $x_{n,a}$ as its zeros, with $n = 1, \dots, L$ and $a = 1, \dots, N(N-1)/2$. Then the construction of eigenstates (5.17) implies that in the appropriately normalized left eigenbasis of $x_{n,a}$ (whose eigenvalues we denote as $x_{n,a}$) the wavefunction will factorize as

$$\langle x_{1,1}, \dots, x_{L, N(N-1)/2} | \Psi \rangle = \prod_{n=1}^L \prod_{a=1}^{N(N-1)/2} (-1)^M Q_1(x_{n,a}) \quad (5.21)$$

where $Q_1(u) = \prod_{j=1}^M (u - u_j)$ is the Baxter function associated to momentum-carrying Bethe roots. Thus it is important to understand the spectrum of the $x_{n,a}$ operators, which can be found from the zeros of eigenvalues of $B^{\text{good}}(u)$.

It is natural to associate $x_{n,a}$ with site n of the chain. As before, we found that B^{good} contains a simple scalar factor multiplying a nontrivial matrix piece. For $L = 1$ the scalar factor is²⁶

$$\left[u - \left(\theta_1 + \frac{3i}{2} \right) \right]^{N-2} \left[u - \left(\theta_1 + \frac{5i}{2} \right) \right]^{N-3} \dots \left[u - \left(\theta_1 + \frac{(2N-3)i}{2} \right) \right] \quad (5.22)$$

so we can choose all of the $x_{1,a}$ except $N-1$ of them to act as scalars with eigenvalues corresponding to zeros of (5.22). The remaining $N-1$ of the $x_{1,a}$ have eigenvalues $\theta_1 \pm \frac{i}{2}$, and the *unordered* set of these eigenvalues identifies the eigenstate. We can therefore label the eigenstates by the pattern of \pm signs in these eigenvalues, and all possible patterns are realized, giving N states

$$|(+ + + \dots +)\rangle, |(- + + \dots +)\rangle, |(- - + \dots +)\rangle, \dots, |(- - - \dots -)\rangle \quad (5.23)$$

which form a basis in the Hilbert space. For higher L we expect that again eigenvalues are trivially combined between different sites, like in the SU(3) case (see (4.33)).

We have checked that the spectrum of B has the form described above at least for the first several L 's in the SU(4) and SU(5) cases. This simple pattern of eigenvalues can also be viewed as yet another nontrivial test of our proposal (5.12) for the B operator in the SU(N) case.

6 Conclusions and future directions

In this paper we put forward a new construction of eigenstates for SU(N) integrable spin chains. It has a highly compact form which closely parallels the SU(2) case, involving only a single operator $B(u)$ which is directly related to Sklyanin's SoV approach. We would like to emphasize that even for an arbitrary SU(N) group the states are built using just one operator, rather than going through many levels of nesting in the usual algebraic Bethe ansatz.

²⁶There is also a u -independent prefactor which contains elements of the K matrix, we do not write it here for brevity.

Our proposal is supported by highly nontrivial analytic and numerical tests, leaving little doubt in its correctness. The simple pattern of eigenvalues of the separated variables that we observed also gives extra support to our conjecture. Our results also provide one of the very few concrete examples of the SoV program at work beyond the most-studied $SU(2)$ and $SU(3)$ cases.

Let us list several open problems and interesting directions for future work.

- One of the main motivations for us was the potential to apply the SoV program for computing 3-point correlators in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) and its dual string theory. The SoV has been already used successfully in this AdS/CFT context [45, 64–68] (see also [69]), but its application has been restricted to essentially rank one sectors.²⁷ We hope that our results will facilitate further progress, especially given that the Q -functions in $\mathcal{N} = 4$ SYM are available *at any coupling* via the Quantum Spectral Curve proposed in [71]. In view of the rather mysterious simplicity we observed for $SU(N)$ spin chains, one can hope for hidden simplifications in the $\mathcal{N} = 4$ theory and perhaps eventually obtain a framework allowing to access arbitrary correlators at finite coupling (despite impressive recent progress [11], this goal is far from having been accomplished). Let us also mention that the power of SoV in quantum field theory has already been demonstrated in various settings, see e.g. [72].
- It would be highly interesting to extend our construction to more general integrable models. This includes models based on trigonometric or elliptic R-matrices, spin chains with arbitrary representation of $SU(N)$ at each site and Gaudin-type models. We hope that it should also apply to various boundary problems, for example it would be interesting to study the interplay of our methods with the off-diagonal Bethe ansatz [73, 74].
- In many ways our construction is much simpler than the conventional nested Bethe ansatz, so it would be very interesting to prove it rigorously. One way to achieve this would be to derive concise commutation relations between the transfer matrix and the operator B^{good} which creates the states (even though it is a challenging task, similar calculations were done in e.g. [75]). Knowing the commutation relations would also open the way to computing form factors for which only partial results are available in higher rank cases. It would also be interesting to try to obtain a proof using the variables canonically conjugated to the separated variables x_n , following an approach similar to the one we used to prove the construction for $SU(2)$ with a generic matrix K (see appendix A).
- Deriving a compact expression for Sklyanin’s measure in separated variables would further facilitate calculation of various observables for higher rank spin chains. This problem is particularly interesting for non-compact $sl(n)$ spin chains with infinite dimensional representations at each site, where partial results for the measure and expressions for Q -operators are available [76, 77].

²⁷See [70] for some direct calculations in the $SU(3)$ sector.

- As our construction has a rather compact form compared to many of the other approaches, we hope it could be useful in attacking the challenging question of extending the celebrated Slavnov’s determinant result for scalar products beyond the $SU(2)$ case.
- It is curious that the *same* operator B^{good} allows to build the states just as well using a *dual* set of momentum-carrying Bethe roots, obtained via a duality transformation in the nested Bethe equations. Together with the presence of free parameters in B^{good} (as discussed in the end of section 5.2), this fact shows a surprising versatility of our construction whose implications remain to be understood.
- An interesting question is to better understand the algebraic structure of our operator B^{good} for $SU(N)$, e.g. it might be possible to write it in terms of monodromy matrices in antisymmetric representations like in the $SU(3)$ case (see (4.4)). Moreover, one can consider the operator built like B^{good} but using the elements of such a monodromy matrix instead of those of the usual $T(u)$. We hope that the resulting operator may allow to generalize our construction to spin chains with other representations at each site.
- While we presented an explicit result for the B operator, it would be interesting to also build for $SU(N)$ the A operator of Sklyanin [50], which provides variables canonically conjugated to the separated variables x_n .
- In our construction it is important that all twists and inhomogeneities are switched on. Taking the limit corresponding to the homogenous periodic XXX chain is non-trivial, but should be possible to accomplish.
- As the Q-functions feature prominently in our approach, it would be natural to look for links with the explicit construction of Q-operators from [78].
- Surprisingly, the SoV approach is completely undeveloped even for the simplest *super-symmetric* chains based on the $su(1|1)$ superalgebra. It would be of great interest to generalize our conjecture to super-spin chains, for which various ways of constructing the eigenstates have been explored recently in [15, 79] (see also [75, 80, 81]).
- Finally, we hope that our results for the $SU(N)$ case may help to shed light on algebraic structures underlying the SoV approach, which has close links with deep subjects in mathematics such as the Langlands correspondence [82–84]. It would be also interesting to explore possible relations with Talalaev’s quantum spectral curve and the Manin matrices approach [63, 85] as well as with classical/quantum and spectral dualities in higher rank integrable models (see e.g. [86–91]).

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A Proving the construction of eigenstates for a generic matrix K in the $SU(2)$ case

In this section we present a proof of the fact (discussed in section 3.2) that for an $SU(2)$ spin chain the operator B^{good} generates the eigenstates as

$$|\Psi\rangle = B^{\text{good}}(u_1) \dots B^{\text{good}}(u_M)|0\rangle \tag{A.1}$$

even if it is constructed using a generic matrix K . Let us remind that this matrix is used to build an improved monodromy matrix T^{good} from the original T (see (3.10)),

$$T^{\text{good}} = K^{-1}TK. \tag{A.2}$$

We will focus on the spin chain with fundamental representation at each site. The only assumption we make is that the spectrum of eigenvalues of the separated variables x_k is $\theta_k \pm i/2$ as discussed in the main text,²⁸ and their eigenstates form a basis in the Hilbert space labelled as $|x_1, \dots, x_L\rangle$. For other representations of $SU(2)$ the spectrum would be different but we expect a similar proof should work.

The main problem for generic K is that the vacuum $|0\rangle$ is not an eigenstate of $A^{\text{good}}(u)$ and $D^{\text{good}}(u)$, so the usual proof breaks down. Our strategy is to follow instead the SOV approach and use the variables canonically conjugated to x_k . Eventually we will arrive at the Baxter equation which we know is satisfied by the Q-function

$$Q_1(u) = \prod_{k=1}^M (u - u_k), \tag{A.3}$$

which will complete the proof.

Roughly speaking the variables conjugated to x_k are given by $A^{\text{good}}(u)$ and $D^{\text{good}}(u)$ evaluated at $u = x_k$, but one should be careful with operator ordering [39]. Expanding

²⁸This statement likely can also be proven rigorously along the lines of appendix A in [41].

these operators as

$$A^{\text{good}}(u) = \sum_{n=0}^L A_n u^n, \quad D^{\text{good}}(u) = \sum_{n=0}^L D_n u^n, \quad (\text{A.4})$$

we define

$$A^{\text{good}}(x_k) = \sum_{n=0}^L (x_k)^n A_n, \quad D^{\text{good}}(x_k) = \sum_{n=0}^L (x_k)^n D_n. \quad (\text{A.5})$$

Then from the RTT commutation relations it follows that [39]

$$A^{\text{good}}(x_k) x_n = (x_n - i\delta_{kn}) A^{\text{good}}(x_k), \quad D^{\text{good}}(x_k) x_n = (x_n + i\delta_{kn}) D^{\text{good}}(x_k) \quad (\text{A.6})$$

(note that T^{good} satisfies the same RTT algebra as the original T). This means that $A^{\text{good}}(x_k)$ and $D^{\text{good}}(x_k)$ act as raising and lowering operators,²⁹ so we have

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_L | A^{\text{good}}(x_k) = F_{1k}(\mathbf{x}) \langle \mathbf{x}_1, \dots, \mathbf{x}_k - i, \dots, \mathbf{x}_L |, \quad (\text{A.7})$$

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_k, \dots, \mathbf{x}_L | D^{\text{good}}(x_k) = F_{2k}(\mathbf{x}) \langle \mathbf{x}_1, \dots, \mathbf{x}_k + i, \dots, \mathbf{x}_L |, \quad (\text{A.8})$$

where $F_{1k}(\mathbf{x}), F_{2k}(\mathbf{x})$ are some scalar coefficients. Let us explain how to fix them.

First, since there is no state with eigenvalue of x_k equal to $\theta_k - 3i/2$, we must have³⁰

$$F_{1k}(\mathbf{x}) = 0 \quad \text{when} \quad \mathbf{x}_k = \theta_k - i/2. \quad (\text{A.9})$$

Similarly,

$$F_{2k}(\mathbf{x}) = 0 \quad \text{when} \quad \mathbf{x}_k = \theta_k + i/2. \quad (\text{A.10})$$

To compute the remaining nonzero coefficients let us consider the matrix element

$$\mathcal{M} = \langle \mathbf{x}_1, \dots, \theta_k + i/2, \dots, \mathbf{x}_L | \left(A^{\text{good}}(x_k) + D^{\text{good}}(x_k) \right) | 0 \rangle \quad (\text{A.11})$$

i.e. we chose $\mathbf{x}_k = \theta_k + i/2$. On the one hand, $D^{\text{good}}(x_k)$ annihilates this bra vector, so using (A.7) and our normalization

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_L | 0 \rangle = 1, \quad (\text{A.12})$$

we see that the matrix element (A.11) is given by

$$\mathcal{M} = F_{1k}(\mathbf{x}_1, \dots, \theta_k + i/2, \dots, \mathbf{x}_L). \quad (\text{A.13})$$

On the other hand, due to the ordering in (A.5) we can act with operators x_k directly on the bra state, so the operator x_k in the argument of A^{good} and D^{good} can be replaced by the constant $\theta_k + i/2$. While $|0\rangle$ is not an eigenvector of A^{good} or D^{good} , it is still an

²⁹Therefore e.g. $\log(A^{\text{good}}(x_k))$ would have canonical commutation relations with the x_n operators.

³⁰For spin chains with other representations of SU(2) it may be possible to lower the eigenvalue to $\theta_k - 3i/2$ and even beyond. However at some point this process should terminate as otherwise we would get an infinite-dimensional Hilbert space. We hope therefore that a similar proof should work for other SU(2) representations as well.

eigenvector of $A^{\text{good}}(\theta_k + i/2) + D^{\text{good}}(\theta_k + i/2)$, because the transformation (A.2) does not change the trace of the monodromy matrix. The eigenvalue is given by

$$\left(A^{\text{good}}(u) + D^{\text{good}}(u) \right) |0\rangle = (\lambda_1 Q_\theta^+(u) + \lambda_2 Q_\theta^-(u)) |0\rangle, \quad (\text{A.14})$$

and substituting $u = \theta_k + i/2$ into this expression we see that the term with λ_2 drops out. This means that the matrix element (A.11) is equal to $\lambda_1 Q_\theta^+(\theta_k + i/2)$. In combination with (A.13) this finally gives

$$F_{1k}(x_1, \dots, \theta_k + i/2, \dots, x_L) = \lambda_1 Q_\theta^+(\theta_k + i/2). \quad (\text{A.15})$$

We can write this result and (A.9) in a uniform way as

$$F_{1k}(x) = \lambda_1 Q_\theta^+(x_k). \quad (\text{A.16})$$

Similarly, one can show that

$$F_{2k}(x) = \lambda_2 Q_\theta^-(x_k). \quad (\text{A.17})$$

Having found the coefficients F_{1k}, F_{2k} it is now rather easy to complete the proof. As the transfer matrix is a polynomial in u , it is enough to show that $|\Psi\rangle$ is its eigenstate at L distinct values of u . We will consider $u = \theta_k \pm i/2$ for all k which gives more than enough points.

Namely, let us evaluate the scalar product

$$\mathcal{S} = \langle x | \left(A^{\text{good}}(x_k) + D^{\text{good}}(x_k) \right) |\Psi\rangle \quad (\text{A.18})$$

in two different ways like we just did for the case when $|\Psi\rangle$ is simply $|0\rangle$. First, due to the ordering in (A.5) we can again replace the operator x_k by its eigenvalue x_k , so that

$$\mathcal{S} = \langle x | \left(A^{\text{good}}(x_k) + D^{\text{good}}(x_k) \right) |\Psi\rangle. \quad (\text{A.19})$$

The operator appearing here is precisely the transfer matrix at $u = \theta_k \pm i/2$. Alternatively, we can use the fact that $A^{\text{good}}(x_k)$ and $D^{\text{good}}(x_k)$ act as shift operator on $\langle x |$, so that

$$\mathcal{S} = \lambda_1 Q_\theta^+(x_k) \langle x_1, \dots, x_k - i, \dots, x_L | \Psi \rangle + \lambda_2 Q_\theta^-(x_k) \langle x_1, \dots, x_k + i, \dots, x_L | \Psi \rangle \quad (\text{A.20})$$

where we used (A.7), (A.8), (A.16), (A.17). The scalar products appearing here are just products of Q-functions, so using (3.8) we get

$$\mathcal{S} = [\lambda_1 Q_\theta^+(x_k) Q_1^{--}(x_k) + \lambda_2 Q_\theta^-(x_k) Q_1^{++}(x_k)] (-1)^{ML} \prod_{j \neq k}^L Q_j(x_j). \quad (\text{A.21})$$

Inside the square brackets we recognize part of the Baxter equation! The equation itself reads

$$\tau(u) Q_1(u) = \lambda_1 Q_\theta^+(u) Q_1^{--}(u) + \lambda_2 Q_\theta^-(u) Q_1^{++}(u), \quad (\text{A.22})$$

where $\tau(u)$ is the transfer matrix eigenvalue. This means that

$$\mathcal{S} = \tau(x_k) Q_1(x_k) (-1)^{ML} \prod_{j \neq k}^L Q_j(x_j), \quad (\text{A.23})$$

which can also be written as

$$\mathcal{S} = \langle x | \tau(x_k) | \Psi \rangle. \tag{A.24}$$

Comparing the last expression with (A.19) we see that for all $\langle x |$ we have

$$\langle x | \left(A^{\text{good}}(x_k) + D^{\text{good}}(x_k) \right) | \Psi \rangle = \langle x | \tau(x_k) | \Psi \rangle. \tag{A.25}$$

Since the states $\langle x |$ form a complete basis, we get

$$\left(A^{\text{good}}(x_k) + D^{\text{good}}(x_k) \right) | \Psi \rangle = \tau(x_k) | \Psi \rangle. \tag{A.26}$$

In other words, $|\Psi\rangle$ is an eigenstate of the transfer matrix when $u = \theta_k \pm i/2$. The polynomiality of the transfer matrix now guarantees that $|\Psi\rangle$ is its eigenstate for any u , thus completing the proof.

B Derivation of the norm for SU(2) spin chains

Here we give technical details on the derivation of the norm of a spin chain eigenstate in the SU(2) case, mentioned in the main text in section 3.3.

In order to find the norm of a state it is convenient to consider it together with another one, corresponding to the dual set of Bethe roots. Below we use the following notation:

$$\{u_{k=1\dots M}\} \quad \text{and} \quad \{v_{k=1\dots L-M}\}, \tag{B.1}$$

are the original set of Bethe roots and the dual roots, satisfying the Bethe equations, while

$$|\Psi_1^{\text{std}}\rangle = B(u_1) \dots B(u_M) |0\rangle \quad \text{and} \quad |\Psi_2^{\text{std}}\rangle = C(v_1) \dots C(v_{L-M}) |0'\rangle \tag{B.2}$$

are the two states we consider, with $|0\rangle$ and $|0'\rangle$ being the usual vacuum and the dual vacuum (defined in (2.9), (3.18)). By N_1, N_2 we denote the norms of the states (B.2). We will study the case when the twists satisfy

$$\lambda_1 = 1/\lambda_2 = 1/\lambda_1^*, \tag{B.3}$$

and accordingly the set of Bethe roots $\{u_k\}$ is invariant under complex conjugation (the same is true for dual roots $\{v_k\}$).

In the first subsection below we derive an expression for the ratio of norms of these two states via SoV, and in the second one we obtain an expression for their product via the usual Gaudin's formula. In the second part we compute as an intermediate step the variation of Q-functions with respect to the inhomogeneities. Combining the ratio and the product of the norms, we obtain the final result for the individual norm N_1 , given in (B.41).

B.1 Ratio of the norms

To derive the ratio of the norms we will make use of construction of the states via the operator B^{good} which provides separated variables for our spin chain. When building this operator it is convenient to choose the matrix K in (3.10) to be of the form

$$K = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & -\frac{1}{\sqrt{\alpha}} \\ 0 & \sqrt{\alpha} \end{pmatrix} \tag{B.4}$$

and to multiply B^{good} by an extra overall factor of $1/(\lambda_2 - \lambda_1)$ as in (3.23), so that we get

$$B^{\text{good}} = \frac{\alpha B - 1/\alpha C - A + D}{\lambda_2 - \lambda_1}. \quad (\text{B.5})$$

Here α is an unfixed parameter which we retain for convenience. Consider the left eigenvector of B^{good} with all eigenvalues of the form $\theta_k + i/2$, denoted as $\langle + \cdots + + |$. We have observed that

$$\langle + \cdots + + | 0' \rangle = \left(\frac{\lambda_1}{\lambda_2 \alpha} \right)^L \langle + \cdots + + | 0 \rangle. \quad (\text{B.6})$$

We have checked this curious fact for the first several L 's, and leave a general proof for the future.

Let us introduce two states analogous to $|\Psi_1^{\text{std}}\rangle$ and $|\Psi_2^{\text{std}}\rangle$ but created with the operator B^{good} instead of B and C :

$$|\Psi_1\rangle = B^{\text{good}}(u_1) \dots B^{\text{good}}(u_L) |0\rangle, \quad |\Psi_2\rangle = B^{\text{good}}(v_1) \dots B^{\text{good}}(v_{L-M}) |0'\rangle. \quad (\text{B.7})$$

As we explained in section 3.2, the same operator $B^{\text{good}}(u)$ allows to construct the states using either usual Bethe roots or dual Bethe roots. This means that the two states (B.7) are both eigenvectors of $T(u)$ with the same eigenvalue, so they should be collinear,³¹ i.e. for some c

$$|\Psi_1\rangle = c |\Psi_2\rangle. \quad (\text{B.8})$$

Let us find the constant c . To this end, consider the scalar product $\langle + \cdots + + | \Psi_1 \rangle$. Since $|\Psi\rangle$ was created with the operator $B^{\text{good}}(u)$ and $\langle + \cdots + + |$ is an eigenstate of $B^{\text{good}}(u)$, we get

$$\langle + \cdots + + | \Psi_1 \rangle = \langle + \cdots + + | 0 \rangle \prod_{k=1}^L (-1)^M Q_1^+(\theta_k). \quad (\text{B.9})$$

Similarly,

$$\langle + \cdots + + | \Psi_2 \rangle = \langle + \cdots + + | 0' \rangle \prod_{k=1}^L (-1)^{L-M} Q_2^+(\theta_k). \quad (\text{B.10})$$

Using the relation (B.6) we get

$$\langle + \cdots + + | \Psi_2 \rangle = \langle + \cdots + + | 0 \rangle \left(\frac{\lambda_1}{\lambda_2 \alpha} \right)^L \prod_{k=1}^L (-1)^{L-M} Q_2^+(\theta_k), \quad (\text{B.11})$$

and thus

$$c = \left(\frac{\alpha \lambda_2}{\lambda_1} \right)^L \prod_{k=1}^L (-1)^L \frac{Q_1^+(\theta_k)}{Q_2^+(\theta_k)}. \quad (\text{B.12})$$

We also observed that

$$|\Psi_1\rangle = \left(\frac{\alpha}{\lambda_2 - \lambda_1} \right)^M |\Psi_1^{\text{std}}\rangle, \quad |\Psi_2\rangle = \left(-\frac{1/\alpha}{\lambda_2 - \lambda_1} \right)^{L-M} |\Psi_2^{\text{std}}\rangle. \quad (\text{B.13})$$

³¹We recall that with for nontrivial twists λ_1, λ_2 all eigenvalues of the transfer matrix have multiplicity one.

We have checked this identity for the first several values of L and postpone a rigorous proof to later work. These relations allow us to express the ratio of norms of $|\Psi_1^{std}\rangle$ and $|\Psi_2^{std}\rangle$ through the ratio of norms of $|\Psi_1\rangle$ and $|\Psi_2\rangle$, which is equal to $|c|$, so finally we get the simple result

$$N_1/N_2 = \left| (\lambda_2 - \lambda_1)^{2M-L} \prod_{k=1}^L \frac{Q_1^+(\theta_k)}{Q_2^+(\theta_k)} \right|. \quad (\text{B.14})$$

B.2 Product of the norms

Let us now calculate the product of norms of the states $|\Psi_1^{std}\rangle$ and $|\Psi_2^{std}\rangle$ in terms of the Q-functions. Each of the norms is given by the Gaudin formula [46],

$$N_1^2 = (i\lambda_1\lambda_2)^M \prod_{\substack{k=1\dots M \\ n=1\dots L}} \left((u_k - \theta_n)^2 + \frac{1}{4} \right) \prod_{j < k} \left(1 + \frac{1}{(u_j - u_k)^2} \right) \det \frac{\partial f_k^{(1)}}{\partial u_l}, \quad (\text{B.15})$$

$$N_2^2 = (i\lambda_1\lambda_2)^{L-M} \prod_{\substack{k=1\dots L-M \\ n=1\dots L}} \left((v_k - \theta_n)^2 + \frac{1}{4} \right) \prod_{j < k} \left(1 + \frac{1}{(v_j - v_k)^2} \right) \det \frac{\partial f_k^{(2)}}{\partial v_l}, \quad (\text{B.16})$$

where

$$f_k^{(1)} = \sum_{i=1}^L \ln \left(\frac{u_k - \theta_i + i/2}{u_k - \theta_i - i/2} \right) + \sum_{i=1}^M \ln \left(\frac{u_k - u_i - i}{u_k - u_i + i} \right), \quad (\text{B.17})$$

$$f_k^{(2)} = \sum_{i=1}^L \ln \left(\frac{v_k - \theta_i + i/2}{v_k - \theta_i - i/2} \right) + \sum_{i=1}^{L-M} \ln \left(\frac{v_k - v_i - i}{v_k - v_i + i} \right). \quad (\text{B.18})$$

We see that in the product of these two norms, a product of the Jacobians will appear. The trick we will use is rewriting this product in a simpler form by switching between variables $\{u_j, v_j\}$ and $\{\theta_n\}$.

Namely, let us define a function

$$F = \ln \frac{z^+}{z^-}, \quad (\text{B.19})$$

with

$$z = \frac{\lambda_2 Q_1^+ Q_2^- - \lambda_1 Q_1^- Q_2^+}{(\lambda_2 - \lambda_1) Q_\theta}. \quad (\text{B.20})$$

We will pack $\{u_i\}$ and $\{v_j\}$ into one vector $\{U_k\} = \{u_1, \dots, u_M, v_1, \dots, v_{L-M}\}$ and denote $F_k \equiv F(U_k)$. Then

$$F_k = -f_k^{(1)} + \ln \left(-\frac{\lambda_2}{\lambda_1} \right), \quad k = 1, \dots, M, \quad (\text{B.21})$$

$$F_k = -f_k^{(2)} + \ln \left(-\frac{\lambda_1}{\lambda_2} \right), \quad k = M + 1, \dots, L. \quad (\text{B.22})$$

Consider the Jacobian $\frac{\partial F_k}{\partial U_n}$. On the one hand, this matrix is block-diagonal, so

$$\det \frac{\partial F_k}{\partial U_n} = (-1)^L \det \frac{\partial f^{(1)}(u_k)}{\partial u_n} \det \frac{\partial f^{(2)}(v_k)}{\partial v_n}. \quad (\text{B.23})$$

On the other hand, using the chain rule we have³²

$$\det \frac{\partial F_k}{\partial U_n} = - \det \frac{\partial F_k}{\partial \theta_p} \det \frac{\partial \theta_p}{\partial U_n}. \quad (\text{B.24})$$

The first Jacobian in this product is easy to compute, and we get

$$\frac{\partial F_k}{\partial \theta_p} = R_k \frac{i}{(U_k - \theta_p)^2 + \frac{1}{4}}, \quad (\text{B.25})$$

where

$$R_k = \begin{cases} 1, & k = 1 \dots M \\ -1, & k = M + 1 \dots L \end{cases}. \quad (\text{B.26})$$

The second Jacobian in (B.24) can be calculated using the QQ-relation

$$\lambda_2 Q_1^+ Q_2^- - \lambda_1 Q_1^- Q_2^+ = (\lambda_2 - \lambda_1) Q_\theta. \quad (\text{B.27})$$

Indeed, let us vary one θ_l in (B.27) infinitesimally, then the solutions of Bethe equations u_i and v_i will vary as well, but the equation should still hold, so we get

$$- \sum_{k=1}^M \left(\lambda_2 \frac{Q_1^+ Q_2^-}{u - u_k + i/2} - \lambda_1 \frac{Q_1^- Q_2^+}{u - u_k - i/2} \right) \delta u_k \quad (\text{B.28})$$

$$- \sum_{k=1}^{L-M} \left(\lambda_2 \frac{Q_1^+ Q_2^-}{u - v_k - i/2} - \lambda_1 \frac{Q_1^- Q_2^+}{u - v_k + i/2} \right) \delta v_k = (\lambda_2 - \lambda_1) \frac{\partial Q_\theta}{\partial \theta_l} \delta \theta_l. \quad (\text{B.29})$$

Evaluating this equation on $u = \theta_n$ and noticing that (B.27) implies that

$$\lambda_2 Q_1^+(\theta_n) Q_2^-(\theta_n) = \lambda_1 Q_1^-(\theta_n) Q_2^+(\theta_n), \quad (\text{B.30})$$

we get

$$\sum_{k=1}^L R_k \frac{\delta U_k}{\delta \theta_l} i \frac{\lambda_2}{\lambda_2 - \lambda_1} \frac{\partial Q_\theta(\theta_n)}{\partial \theta_n} \frac{Q_1^+(\theta_n) Q_2^-(\theta_n)}{(\theta_n - U_k)^2 + 1/4} = \delta_{ln}. \quad (\text{B.31})$$

Therefore,

$$\frac{\partial \theta_m}{\partial U_k} = R_k \frac{i \lambda_2}{\lambda_2 - \lambda_1} \frac{Q_1^+(\theta_m) Q_2^-(\theta_m)}{\prod_{n \neq m} (\theta_n - \theta_m)} \frac{i}{(U_k - \theta_m)^2 + \frac{1}{4}}. \quad (\text{B.32})$$

Plugging this back into (B.24) gives

$$\det \frac{\partial F_k}{\partial U_n} = \left(\frac{i \lambda_2}{\lambda_2 - \lambda_1} \right)^L \frac{\prod_m Q_1^+(\theta_m) Q_2^-(\theta_m)}{\prod_{n < m} (\theta_n - \theta_m)^2} \left(\det \frac{i}{(U_k - \theta_m)^2 + \frac{1}{4}} \right)^2. \quad (\text{B.33})$$

We can now multiply the starting expressions (B.15), (B.16) for the norms and express the product $\det \frac{\partial f^{(1)}(u_k)}{\partial u_n} \det \frac{\partial f^{(2)}(v_k)}{\partial v_n}$ through (B.23) and (B.24). Finally we use (B.33) to get

³²Notice that if Bethe equations are satisfied, $F_k = 0$.

an important intermediate result,

$$(N_1 N_2)^2 = (-1)^L \left(\frac{i\lambda_2}{\lambda_2 - \lambda_1} \right)^L \prod_n Q_1^+(\theta_n) Q_1^-(\theta_n) Q_2^+(\theta_n) Q_2^-(\theta_n) \quad (\text{B.34})$$

$$\times \prod_{n < m} \left(1 + \frac{1}{(u_n - u_m)^2} \right) \prod_{i < j} \left(1 + \frac{1}{(v_i - v_j)^2} \right) \quad (\text{B.35})$$

$$\times \frac{\prod_m Q_1^+(\theta_m) Q_2^-(\theta_m)}{\prod_{n < m} (\theta_n - \theta_m)^2} \left(\det \frac{i}{(U_k - \theta_m)^2 + \frac{1}{4}} \right)^2. \quad (\text{B.36})$$

Now let us simplify this expression. First we can expand the determinant using the identity³³

$$\det \frac{i}{(U_k - \theta_m)^2 + \frac{1}{4}} = \sum_{\{s_i = \pm 1\}} \prod_{i=1}^L s_i \det \frac{i}{U_k - \theta_m + \frac{is_m}{2}}. \quad (\text{B.37})$$

Applying the Cauchy determinant formula to each of the individual determinants, we obtain

$$|N_1 N_2|^2 = \left| \left(\frac{1}{\lambda_2 - \lambda_1} \right)^L \prod_{n < m} (1 + (u_n - u_m)^2) \prod_{i < j} (1 + (v_i - v_j)^2) \prod_{m, n} (u_m - v_n)^2 \right| \\ \times \left| \frac{\left(\prod_m Q_1^+(\theta_m) Q_2^-(\theta_m) \right)^3}{\prod_{n < m} (\theta_n - \theta_m)^2} \left(\sum_{\{s_i = \pm 1\}} \prod_{i=1}^L s_i \frac{\prod_{m < n} [\theta_n - \theta_m + i \frac{s_n}{2} - i \frac{s_m}{2}]}{\prod_m Q_1^{[s_m]}(\theta_m) Q_2^{[s_m]}(\theta_m)} \right)^2 \right|.$$

Next, we notice that for any sets $\{u_k\}, \{v_k\}$ (regardless of Bethe equations)

$$\prod_{n < m} (1 + (u_n - u_m)^2) \prod_{i < j} (1 + (v_i - v_j)^2) \prod_{m, n} (u_m - v_n)^2 \\ = C \prod_n Q_1^{++}(u_n) Q_2(u_n) \prod_m Q_1(v_m) Q_2^-(v_m),$$

where the irrelevant constant C is i to some integer power. Using that as a consequence of (B.27) we have

$$\lambda_2 Q_1^{++}(u_k) Q_2(u_k) = (\lambda_2 - \lambda_1) Q_\theta^+(u_k), \quad (\text{B.38})$$

$$\lambda_2 Q_1(v_k) Q_2^-(v_k) = (\lambda_2 - \lambda_1) Q_\theta^-(v_k), \quad (\text{B.39})$$

we get

$$\left| \prod_{n < m} (1 + (u_n - u_m)^2) \prod_{i < j} (1 + (v_i - v_j)^2) \prod_{m, n} (u_m - v_n)^2 \right| \\ = \left| (\lambda_1 - \lambda_2)^L \prod_{k=1}^M Q_\theta^+(u_k) \prod_{k=1}^{L-M} Q_\theta^-(v_k) \right|,$$

³³This identity can be easily proven by writing the determinant as a sum over permutations and representing each matrix element as a sum of two simple fractions. Then after expansion of each product we get a sum over 2^n combination of signs. The order of the two sums (over permutations and over signs) can be switched, so the sums over permutations are reassembled in 2^n determinants.

which after some cancellations and application of (B.30) results in

$$|N_1 N_2| = \left| \sum_{\{s_i = \pm 1\}} \prod_{i=1}^L s_i \prod_{m < n} \frac{[\theta_n - \theta_m + i \frac{s_n}{2} - i \frac{s_m}{2}]}{\theta_n - \theta_m} \prod_m Q_1^{[-s_m]}(\theta_m) Q_2^{[-s_m]}(\theta_m) \right|, \quad (\text{B.40})$$

where we have taken into account that $\lambda_1 = 1/\lambda_2$. This is the final result for the product of the norms.

As a last step we combine this result with the formula (B.14) for the ratio of the norms, which gives the result for the norm of the usual transfer matrix eigenstate $|\Psi_1^{std}\rangle$ from (B.2) in terms of Q-functions only,

$$N_1 = \left| \sum_{\{s_i = \pm 1\}} \prod_{i=1}^L s_i \prod_{m < n} \frac{[\theta_n - \theta_m + i \frac{s_n}{2} - i \frac{s_m}{2}]}{\theta_n - \theta_m} \prod_{m=1}^L Q_1^{[-s_m]}(\theta_m) Q_2^{[-s_m]}(\theta_m) \right|^{1/2} \\ \times \left| (\lambda_1 - 1/\lambda_1)^{2M-L} \prod_{m=1}^L \frac{Q_1^+(\theta_m)}{Q_2^+(\theta_m)} \right|^{1/2}. \quad (\text{B.41})$$

C The B operator for $SU(N)$ with $N = 2, 3, 4, 5$

Here we explicitly write the B operator for spin chains with $SU(N)$ symmetry with $2 \leq N \leq 5$. The general result for any $SU(N)$ is given in (5.12) and explicit expressions are:

$$\begin{aligned} \text{SU}(2) : \quad B(u) &= T_{1|2}(u) \\ \text{SU}(3) : \quad B(u) &= \sum_j T_{j|3}(u) T_{12|j3}(u-i) \\ \text{SU}(4) : \quad B(u) &= \sum_{j,k} T_{j|4}(u) T_{k|j4}(u-i) T_{123|k4}(u-2i) \\ \text{SU}(5) : \quad B(u) &= \sum_{j,k,l} T_{j|5}(u) T_{k|j5}(u-i) T_{l|k5}(u-2i) T_{1234|l5}(u-3i) \end{aligned}$$

These formulas are written in terms of quantum minors defined in (5.1). We used a shorthand notation

$$k = \{k_1, k_2\}, \quad l = \{l_1, l_2, l_3\} \quad (\text{C.1})$$

and for $SU(N)$ the sum runs over the values of indices j, k_n, l_n from 1 to $N - 1$ with

$$k_1 < k_2, \quad l_1 < l_2 < l_3. \quad (\text{C.2})$$

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